Centrum voor Wiskunde en Informatica

Modelling, Analysis and Simulation

MAS The hexagonal versus the square lattice

P. Moree, H.J.J. te Riele

CWI is the National Research Institute for Mathematics and Computer Science. It is sponsored by the Netherlands Organization for Scientific Research (NWO).
CWI is a founding member of ERCIM, the European Research Consortium for Informatics and Mathematics.

CWI's research has a theme-oriented structure and is grouped into four clusters. Listed below are the names of the clusters and in parentheses their acronyms.

Probability, Networks and Algorithms (PNA)

Software Engineering (SEN)

Modelling, Analysis and Simulation (MAS)

Information Systems (INS)

The Hexagonal Versus the Square Lattice

Pieter Moree

KdV Institute, University of Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands
moree@science.uva.nl

Herman J.J. te Riele

CWI, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands
herman@cwi.nl

ABSTRACT

A conjecture of Schmutz Schaller [17, p. 201] regarding the lengths of the hexagonal versus the lengths of the square lattice is shown to be true. The proof uses results from (computational) prime number theory and from [10]. Using an identity due to Selberg, it is shown that the conjecture can in principle be also resolved without using computational prime number theory. By our approach, however, this would require a huge amount of computation.

1. INTRODUCTION

In [17, p. 201] Schmutz Schaller, motivated by considerations from hyperbolic geometry, makes the conjecture that in dimensions 2 to 8 the best known lattice sphere packings have 'maximal lengths' and goes on to write: "In dimension 2 the conjecture means in particular that the hexagonal lattice is 'better' than the square lattice. More precisely, let $0 < h_1 < h_2 < \cdots$ be the positive integers, listed in ascending order, which can be written as $h_i = x^2 + 3y^2$ for integers x and y. Let $0 < q_1 < q_2 < \cdots$ be the positive integers, listed in ascending order, which can be written as $q_i = x^2 + y^2$ for integers x and y. Then the conjecture is that $q_i \leq h_i$ for $i = 1, 2, 3, \cdots$." For an introductory account to these matters see [18]. For some progress regarding Schmutz Schaller's general conjecture in dimension 2 see [8] (this case of the conjecture is also mentioned in [4, p. xxx]).

For $j \geq 1$ let $b_j(n) = 1$ if n is represented by the quadratic form $X^2 + jY^2$ and $b_j(n) = 0$ otherwise. The characteristic functions b_1 and b_3 are well understood. The following result was already known to Fermat.

Lemma 1 *A positive integer n is represented by the form $X^2 + Y^2$ if and only if every prime factor p of n of the form $p \equiv 3 \pmod 4$ occurs to an even power. A positive integer n is represented by the form $X^2 + 3Y^2$ if and only if every prime factor p of n of the form $p \equiv 2 \pmod 3$ occurs to an even power.*

(In general the natural numbers n that are represented by a quadratic form $X^2 + mY^2$ are rather more difficult to describe, cf. the beautiful book of D. Cox [5].) Lemma 1 implies that b_1 and b_3 are multiplicative functions.

Let $B_i(x) = \sum_{n \leq x} b_i(n)$ for $i = 1$ and $i = 3$. Schmutz Schaller's conjecture regarding the square versus the hexagonal lattice can be reformulated as follows in terms of B_1 and B_3.

Conjecture 1 *We have $B_1(x) \geq B_3(x)$ for every x.*

The first asymptotic result on $B_1(x)$ goes back to Landau [9], who proved in 1908 that

$$B_1(x) \sim C_{b_1} \frac{x}{\sqrt{\log x}}, \tag{1.1}$$

where

$$C_{b_1} = \frac{1}{\sqrt{2}} \prod_{p \equiv 3 (\mathrm{mod}\ 4)} (1 - p^{-2})^{-1/2} = \frac{\pi}{4} \prod_{p \equiv 1 (\mathrm{mod}\ 4)} (1 - p^{-2})^{1/2} \approx 0.764. \tag{1.2}$$

(Here and in the sequel the letter p is used to indicate primes.) Landau's proof uses contour integration. It is not difficult to use his method to show, cf. [20], that for every $k \geq 2$ there exist constants $C_{b_1}(2), \cdots, C_{b_1}(k)$ such that

$$B_1(x) = C_{b_1} \frac{x}{\sqrt{\log x}} \left(1 + \frac{C_{b_1}(2)}{\log x} + \cdots + \frac{C_{b_1}(k)}{\log^{k-1} x} + O\left(\frac{1}{\log^k x} \right) \right). \tag{1.3}$$

This result can also be established by methods not using complex analysis, cf. [14, p. 288]. At the beginning of 1913 a then unknown Hindu clerk by the name of Ramanujan wrote in his first letter to Hardy [2] that he could prove that

$$B_1(x) = C_{b_1} \int_2^x \frac{dt}{\sqrt{\log t}} + O(x^{1-\varepsilon}), \tag{1.4}$$

for some $\varepsilon > 0$. (For a reconstruction of Ramanujan's speculative argument see [2, pp. 60-66].) Note the similarity of Ramanujan's claim with the prime number theorem. Ramanujan's conjecture implies by partial integration that $C_{b_1}(2) = 1/2$. Shanks [21] showed that $C_{b_1}(2) \neq 1/2$, however, thus disproving Ramanujan's claim. (Ramanujan gave the correct formula and numerical approximation for C_{b_1} though.) The constants C_{b_1} and $C_{b_1}(2)$ are known as the *Landau-Ramanujan constant* and the *second order Landau-Ramanujan constant*, respectively. For more on the evaluation of these constants see Section 5. For more on mathematical constants in general see e.g. [7].

Ramanujan [3] in his 'unpublished' manuscript on the partition and tau functions, made various claims similar to (1.4), see Section 6. All of them are disproved in [11]. It can be shown, however, that in each case Ramanujan's claims give the correct asymptotic main term.

Similarly to (1.3) it can be shown that for arbitrary $k \geq 2$ there exist constants $C_{b_3}(2), \cdots, C_{b_3}(k)$ such that

$$B_3(x) = C_{b_3} \frac{x}{\sqrt{\log x}} \left(1 + \frac{C_{b_3}(2)}{\log x} + \cdots + \frac{C_{b_3}(k)}{\log^{k-1} x} + O\left(\frac{1}{\log^k x} \right) \right), \tag{1.5}$$

where

$$C_{b_3} = \frac{1}{\sqrt{2}} \frac{1}{3^{1/4}} \prod_{p \equiv 2 (\mathrm{mod}\ 3)} (1 - p^{-2})^{-1/2} = \frac{\pi 3^{1/4} \sqrt{2}}{9} \prod_{p \equiv 1 (\mathrm{mod}\ 3)} (1 - p^{-2})^{1/2} \approx 0.639.$$

We thus arrive at the following conclusion.

Proposition 1 *Conjecture 1 is asymptotically true.*

Table 1 (copied from [22] and verified by the second author) suggests that Conjecture 1 is true for small x as well. The literature thus provides us with good indications that Conjecture 1 is true. The purpose of this paper is going beyond this and to prove that Conjecture 1 is indeed true.

Theorem 1 *We have $B_1(x) \geq B_3(x)$ for every x, that is Schmutz Schaller's conjecture that the hexagonal lattice is 'better' than the square lattice is true.*

Landau's classical result (1.1) has been generalised in many directions, see [12] for a survey with over 50 references. Despite this rich history, nobody but the first of the present authors seems to have been concerned (in [10]) with proving effective results in this area, which is precisely what is needed to establish Theorem 1.

x	$B_1(x)$	$B_3(x)$	x	$B_1(x)$	$B_3(x)$
2^1	2	1	2^{14}	4357	3645
2^2	3	3	2^{15}	8363	6993
2^3	5	4	2^{16}	16096	13456
2^4	9	8	2^{17}	31064	25978
2^5	16	14	2^{18}	60108	50248
2^6	29	25	2^{19}	116555	97446
2^7	54	45	2^{20}	226419	189291
2^8	97	82	2^{21}	440616	368338
2^9	180	151	2^{22}	858696	717804
2^{10}	337	282	2^{23}	1675603	1400699
2^{11}	633	531	2^{24}	3273643	2736534
2^{12}	1197	1003	2^{25}	6402706	5352182
2^{13}	2280	1907	2^{26}	12534812	10478044

Table 1: $B_1(x)$ versus $B_3(x)$

2. PRELIMINARIES

Let f be a multiplicative function from the natural numbers to $\mathbb{R}_{\geq 0}$. We define $M_f(x) = \sum_{n \leq x} f(n)$, $\mu_f(x) = \sum_{n \leq x} f(n)/n$ and $\lambda_f(x) = \sum_{n \leq x} f(n) \log n$. We denote the formal Dirichlet series $\sum_{n=1}^{\infty} f(n)n^{-s}$ associated to f by $L_f(s)$. We define $\Lambda_f(n)$ by

$$-\frac{L_f'(s)}{L_f(s)} = \sum_{n=1}^{\infty} \frac{\Lambda_f(n)}{n^s}.$$

Notice that

$$f(n) \log n = \sum_{d|n} f(d) \Lambda_f\left(\frac{n}{d}\right). \tag{2.1}$$

If f is the characteristic function of a multiplicative subsemigroup of the natural integers with $(1 <)q_1 < q_2 < \cdots$ as generators, then it can be shown that $\Lambda_f(n) = \log q_i$ if n equals a positive power of a generator q_i and $\Lambda_f(n) = 0$ otherwise. For $f = b_1$ we thus find, using Lemma 1,

$$\Lambda_{b_1}(n) = \begin{cases} 2 \log p & \text{if } n = p^{2r}, \ r \geq 1 \text{ and } p \equiv 3 (\text{mod } 4); \\ \log p & \text{if } n = p^r, \ r \geq 1 \text{ and } p \equiv 1 (\text{mod } 4) \text{ or } p = 2; \\ 0 & \text{otherwise.} \end{cases}$$

For $f = b_3$ we find

$$\Lambda_{b_3}(n) = \begin{cases} 2 \log p & \text{if } n = p^{2r}, \ r \geq 1 \text{ and } p \equiv 2 (\text{mod } 3); \\ \log p & \text{if } n = p^r, \ r \geq 1 \text{ and } p \equiv 1 (\text{mod } 3) \text{ or } p = 3; \\ 0 & \text{otherwise.} \end{cases}$$

From property (2.1) of $\Lambda_f(n)$, we easily infer that

$$\lambda_f(x) = \sum_{n \leq x} f(n) \psi_f\left(\frac{x}{n}\right), \tag{2.2}$$

where $\psi_f(x) = \sum_{n \leq x} \Lambda_f(n)$. The function Λ_f and ψ_f are analogues of, respectively, the Von Mangoldt and the Chebyshev ψ-function.

3. SOME RELATED CONJECTURES

Unfortunately it seems that M_f is not a very 'natural' mathematical object, μ_f is on the other hand (as is amply demonstrated by browsing through the literature). For this reason we consider two additional conjectures:

Conjecture 2 *We have $\lambda_{b_1}(x) \geq \lambda_{b_3}(x)$ for $x \geq 8$.*

Conjecture 3 *We have $\mu_{b_1}(x) \geq \mu_{b_3}(x)$ for every x.*

Note that $\exp(\lambda_{b_1}(x)/2)$ is the product of all different lengths in the square lattice not exceeding $\sqrt{x}$. Thus Conjecture 2 can be reformulated as stating that the product of the different distances not exceeding x occuring in the square lattice always exceeds the product of the different distances not exceeding x in the hexagonal lattice, provided that $x \geq 2\sqrt{2}$.

Conjecture 1 clearly implies Conjecture 3. Furthermore we have:

Proposition 2 *Conjecture 2 implies Conjecture 1.*

Proof. We have, for $x \geq 2$,

$$M_f(x) = \int_{2-}^{x} \frac{d\lambda_f(t)}{\log t} = \frac{\lambda_f(x)}{\log x} + \int_2^x \frac{\lambda_f(t)}{t \log^2 t} dt. \tag{3.1}$$

Denote the latter integral by $I_f(x)$. It is not difficult to show that $I_{b_1}(x) \geq I_{b_3}(x)$ for $x \leq 8$. Conjecture 2 then implies that the latter inequality holds for every x. The truth of Conjecture 2 together with (3.1) then implies that $B_1(x) \geq B_3(x)$ for $x \geq 8$. By direct computation we then infer that the latter inequality holds for every x. □

Thus in order to establish Theorem 1, it suffices to establish Conjecture 2. From (2.2) and $\psi_{b_i}(x) \sim x/2$ as x tends to infinity it follows that $\lambda_{b_i}(x) \sim \mu_{b_i}(x)/2$, as x tends to infinity. An effective form of this relationship, together with an effective estimate for μ_{b_i} (provided by Lemma 2), then allows us to prove the main result of this paper:

Theorem 2 *The Conjectures 1, 2 and 3 are all true.*

Complications arise due to the fact that $\lim_{x \to \infty} B_1(x)/B_3(x) = 1.1961377420\cdots$, which is rather close to 1 and that $\psi_{b_i}(y)$ is not so close to $y/2$ for various ranges of small y (the convolutional nature of (2.2) forces us to take the small y range into account).

4. THE TOOLBOX

The following result from [10] will play a crucial rôle. It is in essence an effective version of Theorem A of [25].

Lemma 2 *Let f be a multiplicative function from the natural numbers to $\mathbb{R}_{\geq 0}$. Suppose that there exist constants D_-, D_+ and τ, with $\tau > 0$, such that for every $x \geq x_0$,*

$$D_- \mu_f(x) \leq \sum_{n \leq x} \frac{f(n)}{n} \left\{ \sum_{m \leq \frac{x}{n}} \frac{\Lambda_f(m)}{m} - \tau \log \frac{x}{n} \right\} \leq D_+ \mu_f(x). \tag{4.1}$$

Then we have, for $x > \max\{x_0, \exp(D_+)\}$,

$$\frac{C_f}{\tau} \log^\tau x \frac{\left(1 - \frac{D_+}{\log x}\right)^{\tau+1}}{1 - \frac{D_-}{\log x}} \leq \mu_f(x) \leq \frac{C_f}{\tau} \log^\tau x \frac{\left(1 - \frac{D_-}{\log x}\right)^{\tau+1}}{1 - \frac{D_+}{\log x}}, \tag{4.2}$$

where

$$C_f := \frac{1}{\Gamma(\tau)} \lim_{s \to 1+0} (s-1)^\tau L_f(s). \tag{4.3}$$

In particular, if there exist constants C_- and C_+ such that

$$C_- \leq \sum_{n \leq x} \frac{\Lambda_f(n)}{n} - \tau \log x \leq C_+ \quad \text{for } x \geq 1, \tag{4.4}$$

then (4.2) holds true, for $x > \exp(C_+)$, with $D_- = C_-$ and $D_+ = C_+$.

Remark 1. From the proof of this lemma, $\int_1^x \mu_f(t)dt/t$ appears as a more easily estimated function than $\mu_f(x)$. Interestingly, Landau [9] in his proof of (1.1) using contour integration, estimates $\int_1^x \mu_{b_1}(t)dt/t$ rather than $B_1(x)$ itself.

Remark 2. If $\lim_{x\to\infty}(\sum_{n\le x}(\Lambda_f(n)/n - \tau\log x))$ exists, we denote this by B_f.

Let us put

$$L(x,\tau,D_-,D_+) = \frac{(\log x - D_+)^{\tau+1}}{\log x - D_-} \text{ and } U(x,\tau,D_-,D_+) = \frac{(\log x - D_-)^{\tau+1}}{\log x - D_+}.$$

Thus we can write (4.2) as $C_f L(x,\tau,D_-,D_+)/\tau \le \mu_f(x) \le C_f U(x,\tau,D_-,D_+)/\tau$.

Let r,s and c_1 be given. At a few instances in the sequel we want to show that for every $x \ge x_2$, with x_2 some explicit constant, we have $\mu_f(x/r) \ge c_1\mu_g(x/s)$. By Lemma 2 this leads us to consider inequalities of the form

$$L\left(\frac{x}{r},\tau,D_-,D_+\right) \ge c_2 U\left(\frac{x}{s},\tau,D'_-,D'_+\right) \tag{4.5}$$

where all variables and constants are real numbers with τ, r, s and c_2 positive, $D_- \le D_+$, $D'_- \le D'_+$ and $x \ge x_0 := \max\{\exp(D'_+)s, \exp(D_+)r\}$. We recall the following lemma from [10]:

Lemma 3 *If $\log s + D'_- \le D_+ + \log r$ and (4.5) is satisfied for some $x_1 > x_0$, then (4.5) is satisfied for every $x \ge x_1$. If $\log s + D'_- > D_+ + \log r$, and*

$$c_2\left(1 + \frac{D'_+ - D'_-}{\log(x_1/s) - D'_+}\right) \le 1 + \frac{D_- - D_+}{\log(x_1/r) - D_-},$$

for some $x_1 > x_0$, then (4.5) is satisfied for every $x \ge x_1$.

We also need the following result about the difference between $U(\frac{x}{r},\frac{1}{2},D_-,D_+)$ and $L(\frac{x}{s},\frac{1}{2},D_-,D_+)$.

Lemma 4 *Assume that $D_+ > D_-$ and $s \ge r \ge 1$. The difference*

$$U\left(\frac{x}{r},\frac{1}{2},D_-,D_+\right) - L\left(\frac{x}{s},\frac{1}{2},D_-,D_+\right)$$

is monotonically decreasing for $x \ge s\exp(1.01D_+ - 0.01D_-)$.

The difference in the latter lemma multiplied by C_f appears if we try to estimate $\mu_{b_i}(x/r) - \mu_{b_i}(x/s)$. Notice that the latter difference is not monotonically decreasing from any x onwards, although it is bounded above by a function that is.

Our proof of Lemma 4 uses the following lemma.

Lemma 5 *Let y and δ be non-negative real numbers. Then the inequality*

$$\sqrt{y+1+\delta}(y+\delta-2)(y+1)^2 \le \sqrt{y}(y+3)(y+\delta)^2, \tag{4.6}$$

holds if either $\delta \le 2$ or $y \ge 0.0099945$.

Proof. On replacing the inequality sign in (4.6) with the equality sign and squaring both sides, we obtain an equation of an algebraic curve. Using continuity and e.g. Maple's function `fsolve` (for numerically determining roots of polynomial equations), the result can then be deduced. □

Remark. For $y = 0.0099944$ and $\delta \approx 5.4$ inequality (4.6) is not satisfied. Indeed if we square both sides of the inequality and take the difference then, considered as a polynomial in y, the discriminant has $27\delta^5 - 198\delta^4 + 410\delta^3 - 936\delta^2 + 1299\delta - 730$ as a factor, which has $5.44694735\cdots$ as its largest real root. Considered as a polynomial in d, we find

$$27y^8 - 72y^7 - 2380y^6 - 12792y^5 - 33822y^4 - 48888y^3 - 32076y^2 - 2376y + 27$$

as a factor of the discriminant, which has $0.00999445028\cdots$ as its one but second largest real root.

6

We can now prove Lemma 4.

Proof of Lemma 4. Differentiating $U(x/r, \frac{1}{2}, D_-, D_+) - L(x/s, \frac{1}{2}, D_-, D_+)$ yields, after some tedious calculations, that the derivative is non-positive provided that (4.6) is satisfied with $y = (\log(x/s) - D_+)/(D_+ - D_-)$ and $\delta = \log(s/r)/(D_+ - D_-)$. The result then follows on invoking Lemma 5. □

5. NUMERICAL EVALUATION OF CERTAIN CONSTANTS

For our proof of Theorem 1 we need to evaluate the constants $C_{b_1}, C_{b_3}, B_{b_1}$ and B_{b_3} with enough numerical precision. The purpose of this section is to achieve this. (Recall that $B_{b_i} = \lim_{x \to \infty} (\sum_{n \le x} \Lambda_{b_i}(n)/n - (\log x)/2).)$

We first consider the evaluation of C_{b_3} and C_{b_1} (defined by (4.3)). We have, for $\mathrm{Re}(s) > 1$,

$$L_{b_3}(s) = (1 - 3^{-s})^{-1} \prod_{p \equiv 1 (\mathrm{mod}\ 3)} (1 - p^{-s})^{-1} \prod_{p \equiv 2 (\mathrm{mod}\ 3)} (1 - p^{-2s})^{-1}$$

and

$$L_{b_3}(s)^2 = \zeta(s) L(s, \chi_3)(1 - 3^{-s})^{-1} \prod_{p \equiv 2 (\mathrm{mod}\ 3)} (1 - p^{-2s})^{-1}. \tag{5.1}$$

From this, (4.3), $\lim_{s \to 1+0}(s - 1)\zeta(s) = 1$ and the fact that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, we obtain

$$C_{b_3}^2 = \frac{3L(1, \chi_3)}{2\pi} \prod_{p \equiv 2 (\mathrm{mod}\ 3)} (1 - p^{-2})^{-1}.$$

If χ is a real primitive character modulo k and $\chi(-1) = -1$, then

$$L(1, \chi) = -\frac{\pi}{k^{3/2}} \sum_{n=1}^{k} n\chi(n),$$

by Dirichlet's celebrated class number formula (cf. equation (17) of [6, Chapter 6]). We infer that $L(1, \chi_3) = \pi/\sqrt{27}$. Using that C_{b_3} must be positive and $\zeta(2) = \pi^2/6$, we then infer that

$$C_{b_3} = \frac{1}{\sqrt{2}} \frac{1}{3^{1/4}} \prod_{p \equiv 2 (\mathrm{mod}\ 3)} (1 - p^{-2})^{-1/2} = \frac{\pi 3^{1/4}\sqrt{2}}{9} \prod_{p \equiv 1 (\mathrm{mod}\ 3)} (1 - p^{-2})^{1/2}.$$

Likewise, using that $L(1, \chi_4) = \pi/4$, we find formula (1.2) for C_{b_1}.

Note that, for $\Re(s) > 1/2$,

$$\prod_{p \equiv 3 (\mathrm{mod}\ 4)} (1 - p^{-2s})^{-2} = \frac{\zeta(2s)(1 - 2^{-2s})}{L(2s, \chi_4)} \prod_{p \equiv 3 (\mathrm{mod}\ 4)} (1 - p^{-4s})^{-1}. \tag{5.2}$$

By recursion we then find from (1.2) and (5.2) the following formula,

$$C_{b_1} = \frac{1}{\sqrt{2}} \prod_{n=1}^{\infty} \left((1 - 2^{-2^n}) \frac{\zeta(2^n)}{L(2^n, \chi_4)} \right)^{1/2^{n+1}},$$

which was already known to Ramanujan [1, pp. 60-66] and by Shanks [21, p. 78]. Using this expression one computes that $C_{b_1} = 0.76422365358922066299\cdots$ Similarly one can show that

$$C_{b_3} = \frac{1}{\sqrt{2}} \frac{1}{3^{1/4}} \prod_{n=1}^{\infty} \left((1 - 3^{-2^n}) \frac{\zeta(2^n)}{L(2^n, \chi_3)} \right)^{1/2^{n+1}},$$

and use it to compute that $C_{b_3} = 0.63890940544534388225\cdots$ which is in agreement with the first seven (out of eight) decimals computed for C_{b_3} by Shanks and Schmid [22].

On noting that, for $\mathrm{Re}(s) \ge 1$,

$$\sum_{n=1}^{\infty} \frac{\Lambda(n) - 1}{n^s} = -\frac{\zeta'(s)}{\zeta(s)} - \zeta(s),$$

and using that $\zeta(s) = 1/(s-1) + \gamma + O(s-1)$, where γ denotes Euler's constant, is the Taylor series for $\zeta(s)$ around $s = 1$, one infers that

$$\sum_{n \leq x} \frac{\Lambda(n)}{n} = \sum_{n \leq x} \frac{1}{n} - 2\gamma + o(1) = \log x - \gamma + o(1). \tag{5.3}$$

Taking the logarithmic derivative of (5.1) one obtains that

$$-2 \frac{L'_{b_3}(s)}{L_{b_3}(s)} = -\frac{\zeta'(s)}{\zeta(s)} - \frac{L'(s, \chi_3)}{L(s, \chi_3)} + \frac{\log 3}{3^s - 1} + 2 \sum_{p \equiv 2 (\mathrm{mod}\ 3)} \frac{\log p}{p^{2s} - 1},$$

from which one easily infers that

$$2 \sum_{n \leq x} \frac{\Lambda_{b_3}(n)}{n} = \sum_{n \leq x} \frac{\Lambda(n)}{n} - \frac{L'(1, \chi_3)}{L(1, \chi_3)} + \frac{\log 3}{2} + 2 \sum_{p \equiv 2 (\mathrm{mod}\ 3)} \frac{\log p}{p^2 - 1} + o(1),$$

which yields, on invoking (5.3),

$$2 B_{b_3} = -\gamma - \frac{L'(1, \chi_3)}{L(1, \chi_3)} + \frac{\log 3}{2} + 2 \sum_{p \equiv 2 (\mathrm{mod}\ 3)} \frac{\log p}{p^2 - 1}.$$

Similarly we deduce that

$$2 B_{b_1} = -\gamma - \frac{L'(1, \chi_4)}{L(1, \chi_4)} + \log 2 + 2 \sum_{p \equiv 3 (\mathrm{mod}\ 4)} \frac{\log p}{p^2 - 1}.$$

Note that the argument above yielded that

$$B_{b_i} = -\lim_{s \to 1+0} \left(\frac{L'_{b_i}(s)}{L_{b_i}(s)} + \frac{1}{2(s-1)} \right).$$

This can be alternatively deduced from Serre's [20] proof of (1.3), cf. [11].

As to the numerical evaluation of, for example, the latter prime sum we note that

$$2 \sum_{p \equiv 3 (\mathrm{mod}\ 4)} \frac{\log p}{p^2 - 1} = -\frac{d}{ds} \log \prod_{p \equiv 3 (\mathrm{mod}\ 4)} \left(\frac{1}{1 - p^{-2s}} \right) \Big|_{s=1}.$$

Then, applying (5.2) m times, we obtain

$$\sum_{p \equiv 3 (\mathrm{mod}\ 4)} \frac{\log p}{p^2 - 1} = \sum_{p \equiv 3 (\mathrm{mod}\ 4)} \frac{\log p}{p^{2^{m+1}} - 1} + \frac{1}{2} \sum_{n=1}^{m} \left\{ \frac{L'(2^m, \chi_4)}{L(2^m, \chi_4)} - \frac{\zeta'(2^m)}{\zeta(2^m)} - \frac{\log 2}{2^{2^m} - 1} \right\}.$$

Similarly we have

$$\sum_{p \equiv 2 (\mathrm{mod}\ 3)} \frac{\log p}{p^2 - 1} = \sum_{p \equiv 2 (\mathrm{mod}\ 3)} \frac{\log p}{p^{2^{m+1}} - 1} + \frac{1}{2} \sum_{n=1}^{m} \left\{ \frac{L'(2^m, \chi_3)}{L(2^m, \chi_4)} - \frac{\zeta'(2^m)}{\zeta(2^m)} - \frac{\log 3}{3^{2^m} - 1} \right\}.$$

Using these expressions, one computes that $B_{b_1} = 0.16389731863458159585 \cdots$ and similarly $B_{b_3} = 0.15355224499499582724 \cdots$.

Now we can invoke [10, Theorem 4] to compute the constants $C_{b_1}(2)$ and $C_{b_3}(2)$. They are given by $C_f(2) = (1 + B_f)/2$ for $f \in \{b_1, b_3\}$. We thus find

$$C_{b_1}(2) = 0.58194865931729079792 8 \cdots, \quad C_{b_3}(2) = 0.57677612249749791362 2 \cdots$$

In [22] the authors write (in our notation) '$B_3(x)$ remains so closely proportional to $B_1(x)$ that it is not clear from this data whether $C_{b_3}(2) > C_{b_1}(2)$ or $C_{b_1}(2) < C_{b_3}(2)$. It would be unlikely that they are exactly equal.' We thus have resolved this matter.

The numerical data from Table 1 in conjunction with the values of C_{b_1}, $C_{b_1}(2)$ and (1.3) suggest that $C_{b_1}(3) > 0$ and $C_{b_1}(4) < 0$. Similarly it seems plausible that $C_{b_3}(3) > 0$ and $C_{b_3}(4) < 0$.

6. Intermezzo: On a claim of Ramanujan

In the previous section we have seen that $B_{b_3} < \log\sqrt{3}$. This knowledge suffices to disprove a claim that was made in a celebrated, hitherto unpublished, manuscript of Ramanujan [3] on the partition and tau-functions.

Let τ denote Ramanujan's tau-function. Put $T_n = 0$ if $3|\tau(n)$ and $T_n = 1$ otherwise. In Ramanujan's manuscript we read [3, p. 64]: 'We can show by transcendental methods that

$$\sum_{k=1}^{n} T_k = \frac{C}{3}\int_1^n \frac{dx}{\sqrt{\log x}} + O\left(\frac{n}{(\log n)^r}\right),$$

where r is any positive number and

$$C = \frac{2^{1/2}}{3^{1/4}}\cdot\frac{1-7^{-2}}{1-7^{-3}}\cdot\frac{1-13^{-2}}{1-13^{-3}}\cdot\frac{1-19^{-2}}{1-19^{-3}}\cdots\frac{1}{\{(1-2^{-2})(1-5^{-2})(1-11^{-2})\cdots\}^{1/2}},$$

$2, 5, 11, \cdots$ being primes of the form $3k - 1$ and $7, 13, 19, \cdots$ being primes of the form $3k + 1$'. This implies that for almost all n, $\tau(n)$ is divisible by 3.

Using that $\tau(n) \equiv n\sigma_1(n)\pmod 3$, where $\sigma_1(n)$ denotes the sum of the positive divisors of n, it is easy to see that T_n is multiplicative and that

$$\sum_{n=1}^{\infty} \frac{T_n}{n^s} = \prod_{p\equiv 2(\mathrm{mod}\ 3)} \frac{1}{1-p^{-2s}} \prod_{p\equiv 1(\mathrm{mod}\ 3)} \frac{1+p^{-s}}{1-p^{-3s}}. \tag{6.1}$$

From (6.1) it is not difficult to verify Ramanujan's claim regarding the value of C. By logarithmic differentiation we obtain from (6.1) that

$$\sum_{n=1}^{\infty} \frac{\Lambda_T(n)}{n^s} = \sum_{p\equiv 2(\mathrm{mod}\ 3)} \frac{2\log p}{p^{2s}-1} + \sum_{p\equiv 1(\mathrm{mod}\ 3)} \left[\frac{\log p}{p^s+1} + \frac{3\log p}{p^{3s}-1}\right].$$

On comparing this series with that for $\sum_{n=1}^{\infty}\Lambda_{b_3}(n)n^{-s}$, it is easily seen, on using that $B_{b_3} < \log\sqrt{3}$, that

$$B_T = B_{b_3} - \sum_{p\equiv 1(\mathrm{mod}\ 3)} \frac{(2p+1)\log p}{(p^2+p+1)(p+1)} - \log\sqrt{3} < B_{b_3} - \log\sqrt{3} < 0,$$

indeed we have $B_T = -0.53\cdots$. This shows that

$$\sum_{k=1}^{n} T_k = \frac{C}{3}\frac{n}{\sqrt{\log n}}\left(1 + \frac{0.23\cdots}{\log n} + O\left(\frac{1}{(\log n)^{1+\epsilon}}\right)\right),$$

where $0.23\cdots = (1 + B_T)/2 \neq 0.5$ (here we invoked Theorem 4 of [10]) and $\epsilon > 0$. Thus the above claim of Ramanujan is false for every $r > 3/2$ and true for $r \leq 3/2$.

Note that if it would be true that $B_T = 0$, an amazing identity for Euler's constant would result. The manuscript [3] contains several further assertions of the above format (with 3 replaced by various other primes), all of which are disproved for $r > 3/2$ in [11].
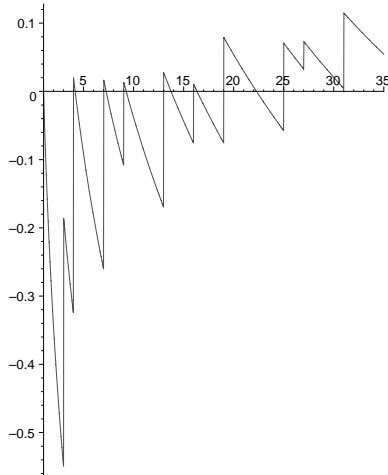
7. On the behaviour of $\sum_{n\leq x} \frac{\Lambda_{b_i}(n)}{n} - \frac{\log x}{2}$

Put $H_i(x) = \sum_{n\leq x}\Lambda_{b_i}(n)/n - \log\sqrt{x}$, for $i = 1$ and $i = 3$. A good understanding of the behaviour of H_i is needed in order to apply our key lemma, Lemma 2. Let us define for $i = 1$ and $i = 3$, $C_+(b_i) = \sup_{x\geq 1} H_i(x)$ and $C_-(b_i) = \inf_{x\geq 1} H_i(x)$.
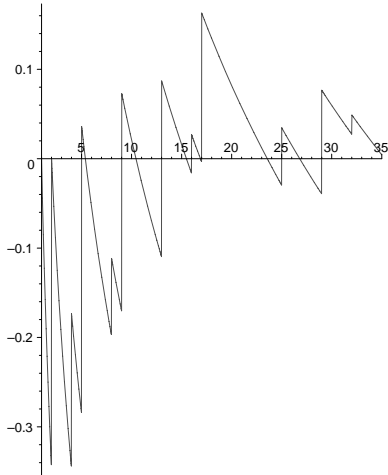
Figure 1

Plot of $H_1(x)$ for $1 \leq x \leq 35$

Figure 2

Plot of $H_3(x)$ for $1 \leq x \leq 35$

As in [15] we define $V(x; d, a) = \sum_{\substack{n \leq x \\ n \equiv a \pmod{d}}} \Lambda(n)/n$. It can be shown that as x tends to infinity $V(x; d, a) - \log x/\varphi(d)$ tends to a limit $C(d, a)$. Ramaré [15] has established the following result.

Theorem 3 [15]. *For $x \geq 68$ we have $|V(x; 3, 1) - \frac{1}{2} \log x - C(3, 1)| \leq 0.1205$ and $|V(x; 4, 1) - \frac{1}{2} \log x - C(4, 1)| \leq 0.0961$.*

We recall from [10] that

$$\sum_{\substack{p^r > \sqrt{x} \\ p \equiv a \pmod{d}}} \frac{\log p}{p^{2r}} \leq \frac{1.3}{\sqrt{x}} \text{ for } x \geq 289, \tag{7.1}$$

and that, for every fixed $v > 1$ and every $x > 0$,

$$\frac{\log v}{v - 1}\left(1 - \frac{v}{x}\right) \leq \sum_{r=1}^{[\log x/\log v]} \frac{\log v}{v^r} \leq \frac{\log v}{v - 1}. \tag{7.2}$$

Theorem 4 *We have*
a) $C_-(b_1) = -\log\sqrt{2}$ *and* $C_+(b_1) < 0.2663$
b) $C_-(b_3) = -\log\sqrt{3}$ *and* $C_+(b_3) < 0.276$

Proof. After some computation for the interval $[1, 68]$ we infer, from Theorem 3, that $C_+(b_1) \leq B_{b_1} + 0.0961$ and similarly $C_+(b_3) \leq B_{b_3} + 0.1205$. For the determination of $C_-(b_i)$ we use (7.1) and (7.2) in addition to Ramaré's inequalities, which yields, for $x \geq 289$, that $H_1(x) \geq \frac{1}{2} \log x + B_{b_1} - 1.3/\sqrt{x} - (\log 4)/x$ and $H_3(x) \geq \frac{1}{2} \log x + B_{b_3} - 1.3/\sqrt{x} - (\log 27)/(2x)$. □

Let $\mathrm{RH}(d)$ be the hypothesis that for every character χ mod d every non-trivial zero of $L(s, \chi)$ is on the critical line.

Theorem 5 *We have*
a) $C_+(b_1) = H_1(461) = 0.1701069880305239 \cdots$, *under* $\mathrm{RH}(4)$.
b) $C_+(b_3) = H_3(3739) = 0.1554480047272349 \cdots$, *under* $\mathrm{RH}(3)$.

Proof. (cf. [10, Theorem 6]). We recall from [10] that for $d \leq 432$ and $(a, d) = 1$, there exists a constant $c_{d,a}$ such that for $x \geq 224$ we have, on $\mathrm{RH}(d)$, that

$$\left| \sum_{\substack{n \leq x \\ n \equiv a \pmod{d}}} \frac{\Lambda(n)}{n} - \frac{\log x}{\varphi(d)} - c_{d,a} \right| \leq \frac{11}{32\pi\sqrt{x}}\{3\log^2 x + 8\log x + 16\}.$$

Under $\mathrm{RH}(4)$ it follows from this that $C_+(b_1) = \max_{v_i \leq 6.15 \times 10^8} H_1(v_i)$, where $2 = v_1 < v_2 < \cdots$ are the consecutive prime powers that can be written as a sum of two squares. Similarly under

RH(3) we deduce that $C_+(b_3) = \max_{w_i \leq 1.083 \times 10^{10}} H_3(w_i)$, where $3 = w_1 < w_2 < \cdots$ are the consecutive prime powers that can be represented by the form $X^2 + 3Y^2$. On computing these maxima (for details see Section 9), the proof is then completed. $\qquad\square$

The reason that, even on GRH, it requires a lot of computation to determine $C_+(b_1)$ and $C_+(b_3)$ is that these values are so close to B_{b_1}, respectively B_{b_3}. A similar phenomenon occurs in [10] for some of the functions considered there (cf. Theorem 6).

Using Theorem 4 and Lemma 2 together with good enough approximations for C_{b_1} and C_{b_3}, one infers that $\mu_{b_1}(x) \geq \mu_{b_3}(x)$ for $x \geq 27500$. After some computation we then deduce that Conjecture 3 holds true.

Unfortunately establishing Conjecture 2 requires quite a bit more work. In particular we need values for D_- and D_+ in Lemma 2 that are more closely together than those coming from Theorem 4. Without improvement of Theorem 3, the upper bounds in Theorem 4 cannot be improved. The lower bounds, however, are amenable to improvement.

Let $\Delta_f(x)$ denote the quantity that is sandwiched between $D_-\mu_f(x)$ and $D_+\mu_f(x)$ in (4.1). Using the lower bound for $H_3(x)$ appearing in the proof of Theorem 4, we deduce that $H_3(x) \geq 0$ for $x \geq 25$. We infer that

$$\Delta_{b_3}(x) \geq -\log\sqrt{3}\{\mu_{b_3}(x) - \mu_{b_3}\big(\frac{x}{25}\big)\}.$$

On applying Lemma 3 with $D_- = -\log\sqrt{3}$ and $D_+ = 0.276$, we deduce that $\Delta_{b_3}(x)/\mu_{b_3}(x) \geq -0.09586\cdots$ for $x \geq 10^9$. Taking $D_- = -0.09586\cdots$ as new value and repeating the procedure, we obtain $D_- = -0.06890\cdots$. Iterating twice more, we see that for $x \geq 10^9$ we can take $D_- = -0.0672$ in Lemma 2.

For any x satisfying the conditions of Lemma 2, we can proceed as above. If the first iteration yields an improved value of our initial D_- (which we take to be $-\log\sqrt{3}$), then it is not difficult to see that every further iteration yields a value of D_- not less than the previous one (this is so since, for given $r \geq 1$, $L(x/r, \frac{1}{2}, D_-, D_+)/U(x, \frac{1}{2}, D_-, D_+)$ is increasing, considered as a function of D_-). On the other hand the value cannot be improved beyond zero and hence the iteration process must converge. If the first iteration does not yield an improved value for D_- (which is initially taken as $-\log\sqrt{3}$), we put $\tilde{w}_i(x) = -\log\sqrt{3}$ for every $i \geq 0$, otherwise we put $\tilde{w}_0(x) = -\log\sqrt{3}$ and define

$$\tilde{w}_{i+1}(x) = \left(\frac{L\big(\frac{x}{25}\cdot\frac{1}{2}, \tilde{w}_i(x), 0.276\big)}{U(x, \frac{1}{2}, \tilde{w}_i(x), 0.276)} - 1 \right) \frac{\log 3}{2}.$$

Empirically it seems that after n iterations we can expect to have approached the limit value $\lim_{i\to\infty} \tilde{w}_i(x)$ with $O(n)$ decimal precision.

For b_1 we proceed similarly. After some computation using the lower bound for $H_1(x)$ given in Theorem 4, we find that $H_1(x) \geq 0.065$ for $x \geq 97$. Hence

$$H_1(x) \geq \left(-\log\sqrt{2} - 0.065 \right) \{\mu_{b_1}(x) - \mu_{b_1}\big(\frac{x}{97}\big)\} + 0.065\mu_{b_1}(x).$$

If the first iteration does not yield an improved value for D_- (which is initially taken as $-\log\sqrt{2}$), we put $\tilde{v}_i(x) = -\log\sqrt{2}$ for every $i \geq 0$, otherwise we put $\tilde{v}_0(x) = -\log\sqrt{2}$ and define

$$\tilde{v}_{i+1}(x) = \left(\frac{L\big(\frac{x}{97}\cdot\frac{1}{2}, \tilde{v}_i(x), 0.2663\big)}{U(x, \frac{1}{2}, \tilde{v}_i(x), 0.2663)} - 1 \right) (\log\sqrt{2} + 0.065) + 0.065.$$

To sum up, we have established:

Lemma 6 *Suppose that $x \geq x_0 \geq 2$ and $i \geq 0$. Then (4.1) holds true with $f = b_1$, $D_- = \tilde{v}_i(x_0)$, $D_+ = 0.276$. It also holds true with $f = b_3$, $D_- = \tilde{w}_i(x_0)$ and $D_+ = 0.2663$.*

This lemma, although amenable to further improvement, is sufficiently sharp for our purposes.

8. THE PROOF OF THEOREM 2

Before proving Theorem 2, we will need two more lemmas. From prime number theory we recall that $\psi(x; d, a) = \sum_{n \leq x,\ n \equiv a \,(\mathrm{mod}\ d)} \Lambda(n)$.

Lemma 7 *We have*
a) $\psi_{b_1}(x) \geq 0.4924x$ *for* $x \geq 37$.
b) $\psi_{b_3}(x) \leq 0.5176x$ *for* $x \geq 3793$.

Proof. Let $d \leq 13$ and $(a,d) = 1$. Then $|\psi(x;d,a) - x/\varphi(d)| \leq \sqrt{x}$ for $224 \leq x \leq 10^{10}$ by [16, Theorem 1] and $|\psi(x;d,a) - x/\varphi(d)| < 0.004560x/\varphi(d)$ for $x \geq 10^{10}$ by [16, Theorem 5.2.1]. From these inequalities the lemma follows after some computation. □

For $y \geq 3$ we define $S_{b_3}(y)$ by $0.5176y$, except for the intervals $[3,49), [49,181), [181,487), [487,1369), [1699,1933), [2287.243$ respectively $[3733,3793)$, where we define $S_{b_3}(y)$ to be (respectively) $0.653954y, 0.605778y, 0.557372y, 0.534528y, 0.526579y, 0.521825y$ and $0.51996y$.
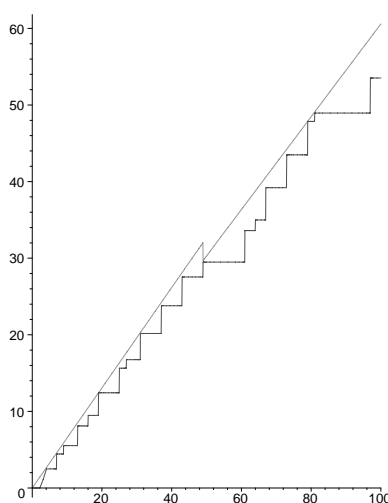
Lemma 8 *For* $y \geq 2$ *we have* $\psi_{b_3}(y) \leq S_{b_3}(y)$.

Proof. The points where ψ_{b_3} and S_{b_3} change value occur only at prime powers representable by $X^2 + 3Y^2$, which we denoted by $3 = w_1 < w_2 < \cdots$. We now check that $\psi_{b_3}(w_i) \leq S_{b_3}(w_i)$ for every $w_i \leq 3793$. For $w_i \geq 3793$ the result follows by Lemma 7. □
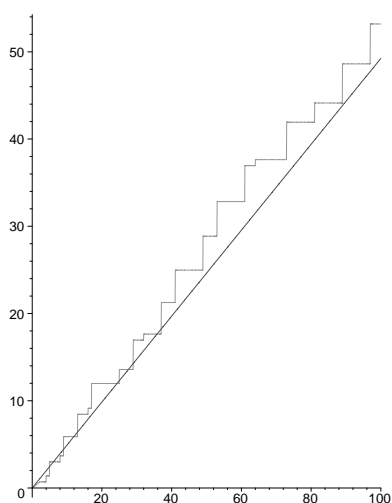
Figure 3
Plot of $\psi_{b_1}(x)$ versus $0.4924x$
for $0 \leq x \leq 100$

Figure 4
Plot of $\psi_{b_3}(x)$ versus $S_{b_3}(x)$
for $0 \leq x \leq 100$

At last we are in the position to prove Theorem 2.

Proof of Theorem 2. As we have shown in Section 2, it suffices to establish Conjecture 2. To this end we have to prove that, for $x \geq 8$,

$$\lambda_{b_1}(x) = \sum_{n \leq \frac{x}{2}} b_1(n)\psi_{b_1}\left(\frac{x}{n}\right) \geq \sum_{n \leq \frac{x}{3}} b_3(n)\psi_{b_3}\left(\frac{x}{n}\right) = \lambda_{b_3}(x).$$

Let us denote the 6 intervals in the definition of $S_{b_3}(y)$ by $[r_i, s_i]$ for $i = 1, \cdots, 6$ and put $\alpha_i = S_{b_3}(r_i)/r_i - 0.5176$ (note that $\alpha_i > 0$). From Lemma 8 we infer that

$$\lambda_{b_3}(x) \leq \sum_{n \leq \frac{x}{3}} b_3(n)S_{b_3}\left(\frac{x}{n}\right) = 0.5176\mu_{b_3}\left(\frac{x}{3}\right) + \sum_{i=1}^{6} \alpha_i\left\{\mu_{b_3}\left(\frac{x}{r_i}\right) - \mu_{b_3}\left(\frac{x}{s_i}\right)\right\}.$$

Put $x_0 = 1.5 \times 10^{11}$. Using a computer (see Section 9) Conjecture 2 can be established for $x < x_0$. Hence now assume that $x \geq x_0$. For notational convenience we shorten $U(x/r, \frac{1}{2}, \tilde{w}_8(x_0/r), 0.276)$ to $U_3(x/r)$, $L(x/r, \frac{1}{2}, \tilde{w}_8(x_0/r), 0.276)$ to $L_3(x/r)$ and $L(x/r, \frac{1}{2}, \tilde{v}_8(x_0/r), 0.2663)$ to $L_1(x/r)$, where r is some fixed number. On applying Lemma 6, we deduce that

$$\frac{\lambda_{b_3}(x)}{2C_{b_3}} \leq 0.5176U_3\left(\frac{x}{3}\right) + \sum_{i=1}^{6} \alpha_i\left\{U_3\left(\frac{x}{r_i}\right) - L_3\left(\frac{x}{s_i}\right)\right\}.$$

By Lemma 4 each of the six terms in the above sum is non-increasing for $x \geq x_0$ and thus the sum is bounded above by its value in x_0, which on its turn is less than $0.0224 U_3(x_0/3)$. One easily checks that $U_3(x/3) \geq U_3(x_0/3)$ for $x \geq x_0$ (on noting that $U(y, \tau, D_-, D_+)$, considered as a function of τ, is increasing for $y > \exp(D_+ + (D_+ - D_-)/\tau)$). We thus obtain that $\lambda_{b_3}(x) \leq 1.08 C_{b_3} U_3(x/3)$. Using Lemma 6 and the lower bound for ψ_{b_1} given in Lemma 7, we infer that $\lambda_{b_1}(x) \geq \sum_{n \leq x/37} b_1(n) \psi_{b_1}(x/n) \geq 0.4924 \mu_{b_1}(x/37) \geq 0.9848 C_{b_1} L_1(x/37)$. A computation shows that $0.9848 C_{b_1} L_1(x_0/37) > 1.08 C_{b_3} U_3(x_0/3)$. By Lemma 3 we then have $0.9848 C_{b_1} L_{b_1}(x/37) > 1.08 C_{b_3} U_3(x/3)$ for every $x \geq x_0$. We thus obtain that for every $x \geq x_0$,

$$\lambda_{b_1}(x) \geq 0.9848 C_{b_1} L_1\left(\frac{x}{37}\right) \geq 1.08 C_{b_3} U_3\left(\frac{x}{3}\right) \geq \lambda_{b_3}(x),$$

completing the proof. $\qquad\square$

9. Computations of results used in Theorems 2 and 5

In the proof of Theorem 2 we have used that Conjecture 2 is true for $x \leq x_0$ with $x_0 = 1.5 \times 10^{11}$. We have established that result as follows.

Checking Conjecture 2 requires the computation and comparison of the sums

$$\lambda_{b_i}(x) = \sum_{n \leq x} b_i(n) \log n, \quad i = 1, 3,$$

and, consequently, the computation of the characteristic functions $b_1(n)$ and $b_3(n)$ for all positive integers $n \leq x_0$. Because of the size of x_0, the range of x-values for which Conjecture 2 had to be checked, was split up in sub-intervals of length 10^6, large enough for efficiency, and small enough to avoid so-called cache misses during the computations.

We first describe the case $\lambda_{b_1}(n)$. For a given interval, say, $[A, B]$, an integer array $b(j), j = 1, 2, \ldots, B - A + 1$ of length $B - A + 1$ is initialized to 0. Here, $b(j)$ corresponds to $b_1(j + A - 1)$. Next, all the possible sums of squares $x^2 + y^2$ of integers $0 \leq x \leq y$, with $A \leq x^2 + y^2 \leq B$, hence $x \in \left[0, \sqrt{B/2}\right]$, $y \in \left[\sqrt{A/2}, \sqrt{B}\right]$, are computed as follows. First, the sequence of all the squares $y^2 \in [A/2, B]$ is precomputed and stored. Next, for each $x = 0, 1, \ldots, \left\lfloor\sqrt{B/2}\right\rfloor$, the sums $x^2 + y^2$ are computed

$$\text{for all} \quad y^2 \in \left[\max(A - x^2, A/2), B - x^2\right] \subset [A/2, B].$$

For all the sums $x^2 + y^2 =: n$ obtained in this way, $b(n - A + 1)$ is set to 1.

The case $\lambda_{b_3}(n)$ is treated similarly: the same initialization of array b is carried out. Next, all the possible sums $x^2 + 3y^2$ of integers x, y, with $A \leq x^2 + 3y^2 \leq B$, hence $x \in \left[0, \sqrt{B}\right]$, $y \in \left[0, \sqrt{B/3}\right]$ are computed as follows. First, the sequence of all the triples of squares $3y^2 \in [0, B]$ is precomputed and stored. Next, for each $x = 0, 1, \ldots, \left\lfloor\sqrt{B}\right\rfloor$, the sums $x^2 + 3y^2$ are computed

$$\text{for all} \quad 3y^2 \in \left[\max(0, A - x^2), B - x^2\right] \subset [0, B]$$

and for all the sums $x^2 + 3y^2 =: n$ obtained in this way, $b(n - A + 1)$ is set to 1. This corresponds to $b_3(n)$.

We have implemented these algorithms for $b_1(n)$ and $b_3(n)$ in Fortran and used them to compute $\lambda_{b_i}(x)$ for $i = 1, 3$, and to verify Conjecture 2 for $x = 8, 9, \ldots, 1.5 \times 10^{11}$ on one 250 MHZ processor of CWI's SGI Origin 2000 computing system. Computing time was 7.6 CPU hours. We also used our program to check the values of $B_1(x) = \sum_{n \leq x} b_1(n)$, given for $x = 10^i, i = 1, \ldots, 12$, by Shiu in Table 1 of [23] (where $B_1(x)$ is called $W(x)$). Computing time to extend our results from 1.5×10^{11} to 10^{12} was 77 CPU hours. We found agreement with Shiu for $i = 1, \ldots, 10$, but differences for $i = 11$ and $i = 12$: $B_1(10^{11}) = 15\,570\,512\,744$ and $B_1(10^{12}) = 148\,736\,628\,858$, whereas Shiu gave $W(10^{11}) = 15\,570\,523\,346$ and $W(10^{12}) = 148\,736\,629\,005$. Shiu used a different, more efficient method than ours, but he has confirmed our value of $B_1(10^{11})$ after checking and correcting his program [24].

We have spot-checked our program for computing $b_1(n)$ and $b_3(n)$ on various intervals of length 10^6 with the help of Lemma 1. This requires the decomposition in primes of each n

$x/10^{11}$	$\lambda_{b_1}(x)$	$\lambda_{b_3}(x)$	$B_1(x)$	$B_3(x)$
1	378458908590.818	316358774044.179	15570512744	13015595425
1.5	572353849423.260	478438468735.511	23160971166	19360573686
2	767521856517.400	641582406621.494	30700929088	25663340448
3	1160486988190.213	970068358550.987	45678037444	38182949191
4	1555965223692.576	1300655152892.098	60558145064	50621477125
5	1953301629004.525	1632795521743.015	75367348255	63000746043
6	2352112868630.901	1966168966371.294	90120785046	75333407591
7	2752146230205.959	2300563843364.554	104828319151	87627692348
8	3153223047545.408	2635831188875.970	119496904413	99889427349
9	3555209733889.339	2971859287714.156	134131682979	112122909167
10	3958003171956.632	3308561817015.470	148736628858	124331455166

Table 2: $\lambda_{b_1}(x), B_1(x)$ versus $\lambda_{b_3}(x), B_3(x)$

for which we wish to compute $b_1(n)$, which is extremely expensive, compared with composing all integers in a given long interval $[A, B]$ as a sum of integer squares. However, we found agreement for all the checks we did, in particular for those in the neighboorhood of $x = 10^{12}$. In Table 2, we list, for $i = 1, 3$, the values we found of $\lambda_{b_i}(x)$ and $B_i(x)$ for $x = j \times 10^{11}$, $j = 1, 1.5, 2, \cdots, 10$.

In the proof of Theorems 5a and 5b, we have used that

$$\max_{v_i \leq 6.15 \times 10^8} H_1(v_i) = H_1(461) = 0.170106..., \text{ respectively,} \tag{9.1}$$

$$\max_{w_i \leq 1.083 \times 10^{10}} H_3(w_i) = H_3(3739) = 0.155448..., \tag{9.2}$$

We have established these results as follows.

Let $x = 6.15 \times 10^8$. We first generated the primes $\leq \sqrt{x}$ with the sieve of Eratosthenes, and stored the following pairs $(n, \Lambda_{b_1}(n))$:

$$(2^k, \log 2), k = 1, 2, \ldots, \lfloor \log_2 x \rfloor,$$

$$(p^{2k}, 2 \log p), k = 1, 2, \ldots, \left\lfloor \frac{1}{2} \log_p x \right\rfloor, \text{ for the primes } p \equiv 3 \bmod 4 \leq \sqrt{x},$$

$$(p^k, \log p), k = 1, 2, \ldots, \lfloor \log_p x \rfloor, \text{ for the primes } p \equiv 1 \bmod 4 \leq \sqrt{x},$$

into an array, sorted increasingly according to the first element of the pairs. The set of numbers n in these pairs in fact contains as a subset all the prime powers $v_1, v_2, \cdots \leq \sqrt{x}$ which can be written as a sum of two squares. For these $(n, \Lambda_{b_1}(n))$-pairs, we computed $H_1(n)$ and we verified that

$$\max_{v_i \leq \lfloor \sqrt{6.15 \times 10^8} \rfloor} H_1(v_i) = H_1(461) = 0.170106...$$

The remaining interval $\left[\lfloor \sqrt{6.15 \times 10^8} \rfloor + 1, x \right]$ was split up in pieces of length 10^7, and for each of these intervals, $[A, B]$, say, the primes $p \equiv 1 \bmod 4$ were generated with the sieve of Eratosthenes, together with $\log p$. These pairs $(p, \log p)$ were mixed with the $(n, \Lambda_{b_1}(n))$-pairs generated above for which $n \in [A, B]$ and then it was verified that $\max_{v_i \in [A, B]} H_1(v_i) < H_1(461)$. This proved (9.1). Computing time was 81 CPU seconds. Relation (9.2) was proved in a similar way at the expense of 1340 CPU seconds.

10. AN ALTERNATIVE APPROACH

In the previous sections we have made essential use of asymptotic information regarding the distribution of primes. Some of the results we used depend eventually on RH(3) and RH(4) to be true up to some finite height. It might come as a surprise then that it is possible to show that $B_1(x) \geq B_3(x)$ for $x \geq 10^{9111}$, without invoking any result from computational prime number theory (one only needs the ability to compute some successive primes...).

14

Our method of establishing this is inspired on Selberg's [19, pp. 183-185] method of obtaining an asymptotic evaluation for $N(x; 4, 1)$, where $N(x; d, a)$ denotes the number of integers $n \leq x$ that have no prime factor p with $p \not\equiv a(\text{mod } d)$. Unfortunately Selberg's method does not seem to generalise well; for example we have no idea how to generalise it so as to show that $N(x; 4, 3) \geq N(x; 4, 1)$ for $x \geq x_0$, with x_0 some effectively computable constant. Vide [13] for generalisations of Selberg's method.

Lemma 9 *a) For $x \geq 2$ we have*

$$|B_1(x) - C_{b_1}\frac{x}{\sqrt{\log x}}| \leq 9.62\frac{x}{\log x},$$

b) For $x \geq 2$ we have

$$|B_3(x) - C_{b_3}\frac{x}{\sqrt{\log x}}| \leq 8.53\frac{x}{\log x},$$

Corollary 1 *For $x \geq 10^{9111}$ we have $B_1(x) \geq B_3(x)$.*

In the proof of Lemma 9 we will make use of the following result.

Lemma 10 *Let $c_2 = 2e^\gamma$ and $c_3 = \sqrt{3}e^\gamma$. For $z \geq 1$ we put*

$$f(z) = z \sum_{n \leq z, \ 2\nmid n} \frac{1}{n} - \frac{z}{2}\log(c_2 z) \text{ and } g(z) = z \sum_{n \leq z, \ 3\nmid n} \frac{1}{n} - \frac{2}{3}z\log(c_3 z).$$

We have

$$\sup_{z \geq 1}|f(z)| = -f(3^-) = \frac{3}{2}\{\log 6 + \gamma\} - 3 = 0.55346270119438 \cdots$$

and

$$\sup_{z \geq 1}|g(z)| = -g(4^-) = \frac{8}{3}\log(4\sqrt{3}e^\gamma) - 6 = 0.70084312094794 \cdots$$

Proof. We only prove the statement concerning $f(z)$, the statement regarding $g(z)$ can be proved in a similar way.

Since $z \log(c_2 z)$ is monotonically increasing, we obtain, cf. [10, Lemma 4], that $\sup_{z \geq 1}|f(z)| = \sup\{|f(1)|, |f(3^-)|, |f(3)|, |f(5^-)|, |f(5)|, \cdots\}$. Using the Euler-MacLaurin summation formula, cf. [26, p. 6], one finds that for integers $n \geq 1$,

$$\sum_{m \leq n} \frac{1}{m} = \log n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{\theta_1(n)}{60n^4}, \tag{10.1}$$

where $\theta_1(n) \in [0, 1]$. Clearly

$$\sum_{m \leq n, \ 2\nmid m} \frac{1}{m} = \sum_{m \leq n} \frac{1}{m} - \frac{1}{2}\sum_{m \leq n/2} \frac{1}{m}, \tag{10.2}$$

Let $n \geq 3$ be an odd integer. Notice that $f(n^-) = f(n) - 1$. Using (10.1) and (10.2) it is not difficult to deduce that

$$-\frac{1}{2} \leq f(n) \leq \frac{n}{2}\log(\frac{n}{n-1}) + \frac{n}{6(n-1)^2} + \frac{1}{60n^3}$$

and

$$-\frac{1}{2} \leq -f(n^-) \leq \frac{1}{2} + \frac{1}{2(n-1)} + \frac{1}{12n} + \frac{2}{15(n-1)^4}.$$

Using that the latter two right hand sides are monotonically decreasing in n and noting that $\sup_{z \geq 1}|f(z)| \geq |f(3^-)|$, we see that $\sup_{z \geq 1}|f(z)| = \sup_{1 \leq z \leq 11}|f(z)| = |f(3^-)|$. $\square$

Using (10.1) it can be shown that the functions f and g are almost periodic in the sense that they converge uniformly to the periodic functions $\frac{1}{2} - \{\frac{z-1}{2}\}$, respectively $1 - \{\frac{z-1}{3}\} - \{\frac{z+1}{3}\}$ (cf. Figure 5).
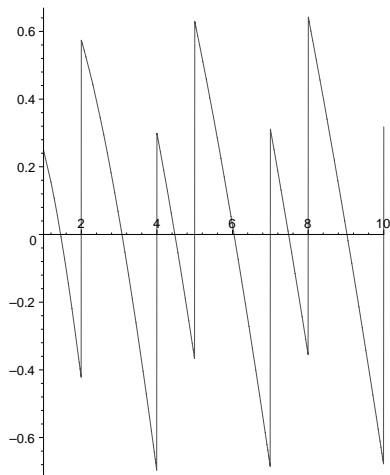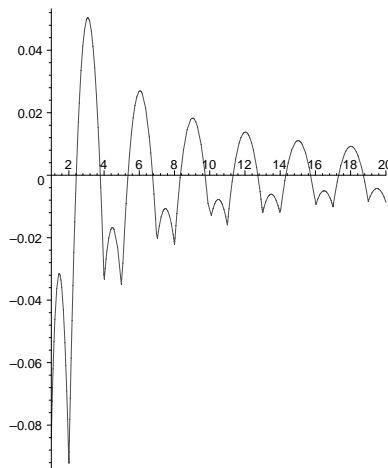
Figure 5
Plot of $g(x)$ for $1 \leq x \leq 10$

Figure 6
Plot of $g(x)$ minus its limit function
for $1 \leq x \leq 20$

11. Proof of Lemma 9

Let P_2, P_3 denote the set of primes p that satisfy $p \equiv 2 \pmod 3$, respectively $p \equiv 3 \pmod 4$. Let $(P_2), (P_3)$ denote the set of natural numbers that have no prime divisor p with $p \not\equiv 2 \pmod 3$, respectively $p \not\equiv 3 \pmod 4$. Let $\psi_3(x), \psi_4(x)$ denote the number of integers $1 \leq n \leq x$ that have no prime divisor p with $p \not\equiv 2 \pmod 3$, respectively $p \not\equiv 3 \pmod 4$.

Proof of part a. Put $c_2 = 2e^\gamma$. We consider the expression

$$\sum_{j=0}^{\infty} \sum_{m \in (P_3)} \sum_{\substack{1 \leq n \leq x/(2^j m^2) \\ n \equiv 1 \pmod 4}} \sum_{\substack{d \mid n \\ d \in (P_3)}} \mu(d) \log \frac{c_2 x}{3^j m^2 d} =$$

$$\sum_{j=0}^{\infty} \sum_{m \in (P_3)} \sum_{\substack{d \in (P_3) \\ d \leq x/(2^j m^2)}} \mu(d) \log \frac{c_2 x}{2^j m^2 d} \sum_{\substack{d \mid n, \ 1 \leq n \leq x/(2^j m^2) \\ n \equiv 1 \pmod 4}} 1. \tag{11.1}$$

By approximating both sides of this equation in terms of the function B_1, we will arrive at an approximate functional equation, (11.9), for B_1 which on solving will yield an explicit lower bound for B_1.

On recalling that $\sum_{d \mid n} \mu(d) \log d = -\Lambda(n)$, one sees that, when $n \equiv 1 \pmod 4$ and $d \leq x/(2^j m^2)$, we have

$$\sum_{\substack{d \mid n \\ d \in (P_3)}} \mu(d) \log \frac{c_2 x}{2^j m^2 d} = \begin{cases} \log \frac{c_2 x}{2^j m^2} & \text{if } n \text{ has no divisor } p \in P_3; \\ \log p & \text{if } n \text{ is divisible by exactly one } p \text{ from } P_3; \\ 0 & \text{if } n \text{ has } \geq 2 \text{ distinct prime factors from } P_3. \end{cases}$$

On noting that

$$B_1(x) = \sum_{j=0}^{\infty} \sum_{m \in (P_3)} \psi_4\left(\frac{x}{2^j m^2}\right),$$

we see that the left hand side of (11.1) equals

$$B_1(x) \log c_2 x + \sum_{\substack{p \in P_3 \\ r \geq 1}} \log p \ B_1\left(\frac{x}{p^{2r}}\right) - \sum_{j=0}^{\infty} \sum_{m \in (P_3)} \log(2^j m^2) \psi_4\left(\frac{x}{2^j m^2}\right), \tag{11.2}$$

which we write as

$$B_1(x) \log c_2 x + I_1(x) - I_2(x). \tag{11.3}$$

We write $z = x/2^j$ and consider the expression formed by the three inner sums in (11.1), that is

$$\sum_{\substack{m \in (P_3) \\ d \leq z/m^2}} \sum_{\substack{d \in (P_3) \\ d \leq z/m^2}} \mu(d) \log \frac{c_2 z}{m^2 d} \sum_{\substack{d \mid n, \ 1 \leq n \leq z/m^2 \\ n \equiv 1 (\bmod 4)}} 1. \tag{11.4}$$

Given an integer k, let $\sigma_0(k)$ denote the product of the distinct primes that occur to an odd power in the prime factorisation of k. We put $\sigma_0(k) = 1$ if there is no prime that occurs to an odd power in k. Note that

$$\sum_{m^2 d = k} \mu(d) = \mu(\sigma_0(k)), \tag{11.5}$$

where the sum is over all integers m and d such that $m^2 d = k$. On writing $m^2 d = k$ in (11.4), and invoking (11.5), we deduce that the triple sum in (11.4) equals

$$\sum_{\substack{k \in (P_3) \\ k \leq z}} \mu(\sigma_0(k)) \log \frac{c_2 z}{k} \sum_{\substack{k_1 \leq z/k \\ k_1 \equiv k (\bmod 4)}} 1.$$

The right hand side of (11.1) is thus seen to equal

$$\sum_{j=0}^{\infty} \sum_{\substack{k \in (P_3) \\ k \leq x/2^j}} \mu(\sigma_0(k)) \log \frac{c_2 x}{2^j k} \sum_{\substack{k_1 \leq x/(2^j k) \\ k_1 \equiv k (\bmod 4)}} 1,$$

which simplifies to

$$\sum_{\substack{d \in (P_3') \\ d \leq x}} \mu(\sigma_0(d')) \log \frac{c_2 x}{d} \sum_{\substack{k_1 \leq x/d \\ k_1 \equiv d' (\bmod 4)}} 1,$$

where d' denotes the largest odd divisor of d and $P_3' = P_3 \cup \{2\}$ and (P_3') is defined as (P_3) but where now no prime divisor p with $p \equiv 1 (\bmod 4)$ is allowed. The right hand side of (11.1) is thus seen to equal

$$\sum_{\substack{d \in (P_3') \\ d \leq x}} \left[\frac{x}{4d} + \frac{2 + (-1)^{\frac{d'-1}{2}}}{4} \right] \mu(\sigma_0(d')) \log \frac{c_2 x}{d} = \frac{x}{4} \sum_{\substack{d \in (P_3') \\ d \leq x}} \frac{\mu(\sigma_0(d'))}{d} \log \frac{c_2 x}{d} + I_3(x),$$

$$\tag{11.6}$$

where

$$|I_3(x)| \leq \frac{3}{4} \sum_{\substack{d \in (P_3') \\ d \leq x}} \log \frac{c_2 x}{d}. \tag{11.7}$$

For $1 \leq d \leq x$ we have, recalling the definition of f (made in Lemma 10),

$$\frac{1}{2} \log \frac{c_2 x}{d} = \sum_{n \leq x/d, \ 2 \nmid n} \frac{1}{n} - f\left(\frac{x}{d}\right) \frac{d}{x}.$$

Combining the latter equation with the sum in the right hand side of (11.6) yields

$$\frac{x}{4} \sum_{\substack{d \in (P_3') \\ d \leq x}} \frac{\mu(\sigma_0(d'))}{d} \log \frac{c_2 x}{d} = \frac{x}{2} \sum_{\substack{dn \leq x, \ d \in (P_3') \\ 2 \nmid n}} \frac{\mu(\sigma_0(d'))}{dn} + I_4(x),$$

where

$$|I_4(x)| \leq \frac{1}{2} \sum_{\substack{d \in (P_3') \\ d \leq x}} |f\left(\frac{x}{d}\right)|. \tag{11.8}$$

Note that

$$\frac{x}{2} \sum_{\substack{dn \le x,\ d \in (P_3')\\ 2 \nmid n}} \frac{\mu(\sigma_0(d'))}{dn} = \frac{x}{2} \sum_{k \le x} \frac{1}{k} \sum_{\substack{dn = k,\ d \in (P_3')\\ 2 \nmid n}} \mu(\sigma_0(d')).$$

Denote the latter inner sum by $h(k)$. We claim that $h(k) = b_1(k)$. First let us consider the case where k is odd. Then

$$h(k) = \sum_{dn = k,\ d \in (P_3)} \mu(\sigma_0(d)) = \sum_{dn = k} \mu(\sigma_0(d)) 2^{-\omega(d)} \prod_{p | d} \left(1 - (-1)^{\frac{p-1}{2}}\right),$$

where $\omega(d)$ denotes the number of distinct primes dividing d. We see that for odd k, h is the Dirichlet convolution of two multiplicative functions and is thus itself a multiplicative function. For arbitrary k we note that $h(k) = h(k')$, where k' is the largest odd divisor of k. Thus h is a multiplicative function. An easy computation shows that for every prime power q we have $h(q) = b_1(q)$. Since both h and b are multiplicative, this completes the proof of the claim. We thus infer that

$$\frac{x}{2} \sum_{\substack{dn \le x,\ d \in (P_3')\\ 2 \nmid n}} \frac{\mu(\sigma_0(d'))}{dn} = \frac{x}{2} \sum_{m \le x} \frac{b_1(m)}{m} = \frac{x}{2} \int_1^x \frac{dB_1(t)}{t} = \frac{B_1(x)}{2} + \frac{x}{2} \int_1^x \frac{B_1(t)}{t^2} dt.$$

Thus the right hand side of (11.1) equals

$$\frac{B_1(x)}{2} + \frac{x}{2} \int_1^x \frac{B_1(t)}{t^2} dt + I_3(x) + I_4(x).$$

Equating it with the expression (11.3) for the left hand side of (11.1), we get

$$B_1(x) \log c_2 x - \frac{x}{2} \int_1^x \frac{B_1(t)}{t^2} dt = -I_1(x) + I_2(x) + I_3(x) + I_4(x) + \frac{B_1(x)}{2}. \tag{11.9}$$

Next we will consider effective estimates for $I_j(x)$ for $1 \le j \le 4$. Using the trivial estimate $B_1(x) \le x$, we obtain that

$$0 \le I_1(x) \le x \sum_{p \in P_3} \frac{\log p}{p^2 - 1} < .23x.$$

On noting that

$$\sum_{j=0}^{\infty} \sum_{m \in (P_3)} \frac{\log(2^j m^2)}{2^j m^2} = 2 \sum_{m \in (P_3)} \frac{\log(2m^2)}{m^2} < 2.7$$

and $\psi_4(x) \le x$, we deduce that $0 \le I_2(x) < 2.7x$. Using (11.7) we deduce that

$$|I_3(x)| \le \frac{3}{4} \sum_{d \in (P_3'),\ d \le x} \log \frac{cx}{d} \le \frac{3}{4} \sum_{1 \le d \le x} \int_d^{cx} \frac{dt}{t} \le \frac{3}{4} cx < 2.68x.$$

For $I_4(x)$ we have, by (11.8) and Lemma 10, $|I_4(x)| \le 0.277x$. An easy calculation (divide by $x^2 \log^{3/2} x$ and integrate) now shows that if

$$-\alpha_- x \le x^2 \log x A'(x) - \frac{x}{2} A(x) \le \alpha_+ x,\ \text{for } x \ge x_0, \tag{11.10}$$

then there exists a constant c_0 such that

$$c_0 \sqrt{\log x} - 2\alpha_+ \le A(x) \le c_0 \sqrt{\log x} + 2\alpha_-,\ \text{for } x \ge x_0.$$

On inserting the latter estimate in (11.10) and invoking (1.1), it then follows that

$$\left| B_1(x) - C_{b_1} \frac{x}{\sqrt{\log x}} \right| \le (\alpha_- + \alpha_+) \frac{x}{\log x},\ \text{for } x \ge x_0. \tag{11.11}$$

18

From our estimates for $I_j(x)$ with $j = 1, \cdots, 4$, we see that we can take $\alpha_- = 3.96$, $\alpha_+ = 5.66$ and $x_0 = 2$. $\qquad\square$

Proof of part b. Making the obvious modifications in the proof of part a, we deduce that

$$B_3(x) \log c_3 x - \frac{x}{2} \int_1^x \frac{B_3(t)}{t^2} dt = -J_1(x) + J_2(x) + J_3(x) + J_4(x) + \frac{B_3(x)}{2}, \qquad (11.12)$$

where $J_1(x) = \sum_{p \in P_2, \ r \geq 1} \log p \ B_3(\frac{x}{p^{2r}})$, $J_2(x) = \sum_{j=0}^{\infty} \sum_{m \in (P_2)} \log(3^j m^2) \psi_3(\frac{x}{3^j m^2})$,

$$|J_3(x)| \leq \frac{2}{3} \sum_{d \in (P_2'), \ d \leq x} \log \frac{c_3 x}{d} \text{ and } |J_4(x)| \leq \frac{1}{2} \sum_{d \in (P_2'), \ d \leq x} |g(\frac{x}{d})|,$$

with $P_2' = P_2 \cup \{3\}$ and (P_2') defined as (P_2), but where now no prime divisor p with $p \equiv 1 \pmod 3$ is allowed. Reasoning as before we find

$$0 \leq J_1(x) < x \sum_{p \in P_2} \frac{\log p}{p^2 - 1} < 0.36 x \text{ and } 0 \leq J_2(x) < \frac{3}{4} x \sum_{m \in (P_2)} \frac{\log(3m^4)}{m^2} < 2.7 x.$$

Furthermore we find that $|J_3(x)| \leq 2xc_3/3 < 2.06x$ and $|J_4(x)| \leq 0.36x$. From these estimates and (11.12) we infer that we can take $\alpha_+ = 5.12$, $\alpha_- = 3.41$ and $x_0 = 2$ in the analogue of (11.11). $\qquad\square$

Remark. In the proof of part a we have used the trivial estimates $\psi_4(x) \leq x$ and $B_1(x) \leq x$. Using that the integers n counted by $\psi_4(x)$ satisfy $n \equiv 1 \pmod 4$ and $3 \nmid n$, we obtain the sharper estimate

$$\psi_4(x) \leq [\frac{x+11}{12}] + [\frac{x+7}{12}] \leq \frac{x}{6} + \frac{7}{6}. \qquad (11.13)$$

Similarly, some computation yields that $B_1(x) \leq x/2 + 2$. In this way the value 9.62 appearing in Lemma 9 a can still be further decreased, but we have not carried this out. Similarly the estimates in the proof of part b can be improved.

References

1. B.C. Berndt, *Ramanujan's notebooks*. Part IV, Springer-Verlag, New York, 1994.

2. B.C. Berndt and R.A. Rankin, *Ramanujan: Letters and commentary*, AMS, Rhode Island, 1995.

3. B.C. Berndt and K. Ono, Ramanujan's unpublished manuscript on the partition and tau functions with proofs and commentary, The Andrews Festschrift (Maratea, 1998), (Eds.) D. Foata, 2001, 39-110.

4. J.H. Conway and N.J.A. Sloane, *Sphere packings, lattices and groups*, Third edition, Grundlehren der Mathematischen Wissenschaften **290**, Springer-Verlag, New York, 1999.

5. D.A. Cox, *Primes of the form $x^2 + ny^2$. Fermat, class field theory and complex multiplication*, Wiley and Sons, Inc., New York, 1989.

6. H. Davenport, *Multiplicative number theory*, Third revised edition, Springer-Verlag, New York, 2000.

7. S. Finch, Mathematical constant web pages, http://www.mathcad.com/library/constants/index.htm

8. S. Kühnlein, Partial solution of a conjecture of Schmutz. *Arch. Math. (Basel)* **67** (1996), 164-172.

9. E. Landau, Über die Einteilung der positiven ganzen Zahlen in vier Klassen nach der Mindestzahl der zur ihrer additiven Zusammensetzung erforderlichen Quadrate, *Archiv der Math. und Physik* **13** (1908), 305-312.

10. P. Moree, Chebyshev's bias for composite numbers with restricted prime divisors, arXiv:math.NT/0112100, to appear in Mathematics of Computation.

11. P. Moree, On some claims in Ramanujan's 'unpublished' manuscript on the partition and tau functions, arXiv:math.NT/0201265, submitted for publication.

12. P. Moree and J. Cazaran, On a claim of Ramanujan in his first letter to Hardy, *Exposition. Math.* **17** (1999), 289-311.

13. M.R. Murty and N. Saradha, An asymptotic formula by a method of Selberg, *C. R. Math. Rep. Acad. Sci. Canada* **15** (1993), 273-277.

14. A.G. Postnikov, *Introduction to analytic number theory*, AMS translations of mathematical monographs 68, AMS, Providence, Rhode Island, 1988.

15. O. Ramaré, Sur un théorème de Mertens, submitted.

16. O. Ramaré and R. Rumely, Primes in arithmetic progressions, *Math. Comp.* **65** (1996), 397-425.

17. P. Schmutz Schaller, Geometry of Riemann surfaces based on closed geodesics, *Bull. Amer.*

     Math. Soc. (N.S.) **35** (1998), 193-214.

18. P. Schmutz Schaller, Platonische Körper, Kugelpackungen und hyperbolische Geometrie, *Math. Semesterber.* **47** (2000), 75-87.

19. A. Selberg, *Collected papers*, Vol. II, Springer-Verlag, Berlin, 1991.

20. J.-P. Serre, Divisibilité de certaines fonctions arithmétiques, *Enseignement Math.* **22** (1976), 227-260.

21. D. Shanks, The second-order term in the asymptotic expansion of $B(x)$, *Math. Comp.* **18** (1964), 75-86.

22. D. Shanks and L.P. Schmid, Variations on a theorem of Landau, I. *Math. Comp.* **20** (1966), 551-569.

23. P. Shiu, Counting sums of two squares: the Meissel-Lehmer method, *Math. Comp.* **47** (1986), 351-360.

24. P. Shiu, Private communication, February 2002.

25. J.M. Song, Sums over nonnegative multiplicative functions over integers without large prime factors. I, *Acta Arith.* **97** (2001), 329-351.

26. G. Tenenbaum, *Introduction to analytic and probabilistic number theory*, Cambridge University Press, Cambridge, 1995.