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The Supremum of a Gaussian Process over a Random Interval

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ABSTRACT

The paper is concerned with the supremum of a centered Gaussian process with stationary increments over a random interval. The main result provides the exact asymptotics of the tail of the supremum distribution in the case that the length of the interval is regularly varying. In addition, we obtain explicit lower and upper bounds for the prefactor.

1 Introduction

Let $\{X(s) : s \geq 0\}$ be a centered Gaussian process with stationary increments. A classical problem in the extreme-value theory of Gaussian processes is to determine the tail behavior of

$$\mathbb{P}(\sup_{s \in [0,T]} X(s) > u), \tag{1.1}$$

for some fixed $T > 0$.

Since the exact form of (1.1) is only known for a few special cases, recent analysis has focused on finding the asymptotic behavior of (1.1) as $u \to \infty$. In particular, Theorem D.3 in [9] (see also Piterbarg & Prisyazhnyuk [10]) immediately implies that if the variance function $\sigma_X^2(s) = \mathbb{V}ar(X(s))$ of the process $X(s)$ is strictly increasing on $[0,T]$, then under some regularity conditions for $\sigma_X(s)$ and the covariance $\mathbb{C}ov(X(s), X(t))$,

$$\mathbb{P}(\sup_{s \in [0,T]} X(s) > u) \sim \text{Const } u^\beta \ \Psi(u/\sigma_X(T)) \tag{1.2}$$

as $u \to \infty$, where $\Psi(x) = \mathbb{P}(\mathcal{N} > x)$ denotes the tail distribution of a standard normal random variable $\mathcal{N}$. For further results on the asymptotic behavior of (1.1), see e.g. Berman [2].

In the present paper, we focus on the case where T is a *random* variable, independent of $X(s)$. Besides extending the classical theory, our motivation comes from a problem encountered in obtaining asymptotics for some fluid models, see [5].

We show that the qualitative tail behavior of the supremum may crucially depend on the variability of T. In particular, we focus on the case where T has a heavy (i.e. regularly varying) tail. In this case, it is essentially the randomness in T that determines the asymptotics, which are no longer of the form (1.2).

It is illuminating to show what happens if we take $X(s) = B_\alpha(s)$, where $B_\alpha(s)$ is a fractional Brownian motion (FBM) with Hurst parameter $\alpha \in (0, 1]$, i.e., a centered Gaussian process with stationary increments, continuous sample paths, $B_\alpha(0) = 0$, and variance function $\mathbb{V}ar(B_\alpha(t)) = t^{2\alpha}$. In this case, we can use the self-similarity of FBM and a classical result of Breiman [6] (see also Cline & Samorodnitsky [7]) to get the following result.

Proposition 1.1 *Let T be a nonnegative random variable with regularly varying tail distribution at ∞ with index $\nu > 0$. Then*

$$\mathbb{P}(\sup_{s \in [0,T]} B_\alpha(s) > u) = \mathbb{P}(T^\alpha \sup_{s \in [0,1]} B_\alpha(s) > u) \sim \mathbb{E}\left(\sup_{s \in [0,1]} B_\alpha(s)\right)^{\nu/\alpha} \mathbb{P}\left(T > u^{\frac{1}{\alpha}}\right)$$

as $u \to \infty$.

The main goal of this note is to show that the asymptotics presented in Proposition 1.1 essentially remain the same for a much larger class of Gaussian processes than FBM; this result is stated in Theorem 2.1. The proof of Theorem 2.1, presented in Section 4, relies on the fact that (under some regularity conditions), a properly scaled Gaussian process converges to FBM, see Lemma 4.2. Section 2 also provides lower and upper bounds for the prefactor $\mathbb{E}\left(\sup_{s \in [0,1]} B_\alpha(s)\right)^{\nu/\alpha}$. In Section 3 we apply Theorem 2.1 to the analysis of the class of Gaussian processes with stationary increments and differentiable sample paths a.s.

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2 Main result

Before stating our main result, we first introduce some notation.

Throughout the paper, $X(t)$ is a centered Gaussian process with stationary increments, a.s. continuous sample paths, $X(0) = 0$, and such that the standard deviation function $\sigma_X(t) = \sqrt{\mathbb{V}ar(X(t))}$ satisfies the conditions

C1 $\sigma_X(t) \in C([0, \infty))$ is ultimately strictly increasing;
C2 $\sigma_X(t)$ is regularly varying at ∞ with index $\alpha \in (0, 1]$;
C3 $\sigma_X(t)$ is regularly varying at 0 with index $\lambda \in (0, 1]$.

We denote the generalized inverse function to $\sigma_X(t)$ by $\sigma_X^{-1}(t) = \inf\{s : \sigma_X(s) \geq t\}$. Moreover, let $L(\cdot)$ denote a function slowly varying at ∞. We assume that T is a nonnegative random variable with regularly varying tail distribution at ∞ with index $\nu > 0$, i.e., $\mathbb{P}(T > x) = L(x)x^{-\nu}$. For conciseness, we write $T \in \mathbf{RV}(\nu)$.

For any two real functions $f(\cdot)$ and $g(\cdot)$, we use the notational convention $f(u) \sim g(u)$ as $u \to \infty$ to denote that $f(u) = g(u)(1 + o(1))$ as $u \to \infty$.

Throughout the paper, we assume that T and $X(t)$ are independent.

The following constant plays a crucial role

$$\mathcal{E}(\alpha, \gamma) = \mathbb{E}\left(\sup_{s \in [0,1]} B_\alpha(s)\right)^\gamma,$$

where $\alpha \in (0, 1]$, $\gamma > 0$.

Theorem 2.1 *Let $T \in \mathbf{RV}(\nu)$. If $X(t)$ is a centered Gaussian process with stationary increments and variance function $\sigma_X^2(t)$ such that conditions **C1-C3** are satisfied, then*

$$\mathbb{P}(\sup_{s \in [0,T]} X(s) > u) \sim \mathcal{E}(\alpha, \nu/\alpha)\mathbb{P}\left(T > \sigma_X^{-1}(u)\right) \quad \text{as } u \to \infty.$$

Proof
The complete proof is presented in Section 4.

$\square$

Remark 2.1 *Note that Theorem 2.1 implies that $\sup_{s \in [0,T]} X(s)$ is regularly varying at ∞ with index ν/α.*

In the following proposition we calculate the exact value of the constant $\mathcal{E}(\alpha, \gamma)$ for two special cases of the parameter α, that is for $\alpha = 1/2$ and $\alpha = 1$.

Proposition 2.1 *If $\gamma > 0$, then*
(i) $\mathcal{E}(1/2, \gamma) = \frac{1}{\sqrt{\pi}} 2^{(2+\gamma)/2} \Gamma\left(\frac{\gamma+1}{2}\right)$;
(ii) $\mathcal{E}(1, \gamma) = \frac{1}{\sqrt{\pi}} 2^{\gamma/2} \Gamma\left(\frac{\gamma+1}{2}\right)$.

Proof
Let $\alpha = 1/2$. Then $\mathbb{P}(\sup_{s \in [0,1]} B_{1/2}(s) > x) = 2\Psi(x)$, which combined with the fact that

$$\mathcal{E}(1/2, \gamma) = \gamma \int_0^\infty x^{\gamma-1} \mathbb{P}(\sup_{s \in [0,1]} B_{1/2}(s) > x)dx,$$

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completes the proof of (i).

In order to prove (ii), it is enough to note that $B_1(t) = \mathcal{N}t$, where $\mathcal{N}$ is a standard normal random variable. Thus $\mathbb{P}(\sup_{s \in [0,1]} B_1(s) > x) = \Psi(x)$. Hence, a similar argumentation as in the proof of (i) completes the proof.

$\square$

In the following proposition we show that for every fixed $\gamma > 0$ the function $\mathcal{E}(\cdot, \gamma)$ is decreasing on $[0, 1]$.

Proposition 2.2 *Let $\gamma > 0$. If $0 < \alpha_1 \leq \alpha_2 \leq 2$, then*

$$\mathcal{E}(\alpha_1, \gamma) \geq \mathcal{E}(\alpha_2, \gamma).$$

Proof
Let $\alpha_1, \alpha_2 \in (0, 2]$ and $\alpha_1 \leq \alpha_2$. Moreover, let $Y(t) = B_{\alpha_1}\left(t^{\frac{\alpha_2}{\alpha_1}}\right)$. Clearly, we have

$$\mathbb{P}(\sup_{s \in [0,1]} B_{\alpha_1}(s) > x) = \mathbb{P}(\sup_{s \in [0,1]} Y(s) > x) \tag{2.1}$$

for all $x \geq 0$. Additionally, $\mathbb{V}ar(Y(t)) = \mathbb{V}ar(B_{\alpha_2}(t)) = t^{2\alpha_2}$ for every $t \in [0, 1]$ and

$$\mathbb{V}ar(Y(t) - Y(s)) = \left| t^{\frac{\alpha_2}{\alpha_1}} - s^{\frac{\alpha_2}{\alpha_1}} \right|^{2\alpha_1} \geq |t - s|^{2\alpha_2} = \mathbb{V}ar(B_{\alpha_2}(t) - B_{\alpha_2}(s))$$

for all $s, t \in [0, 1]$.
Thus, using the Slepian inequality (e.g. Corollary 2.4 in [1]),

$$\mathbb{P}(\sup_{s \in [0,1]} Y(s) > x) \geq \mathbb{P}(\sup_{s \in [0,1]} B_{\alpha_2}(s) > x)$$

for all $x \geq 0$, which in view of (2.1) completes the proof.

$\square$

Combining Proposition 2.1 with Proposition 2.2, we obtain the following bounds for $\mathcal{E}(\alpha, \gamma)$.

Corollary 2.1 *Let $\gamma > 0$.*
(i) If $\alpha \in (0, 1/2)$, then $\mathcal{E}(\alpha, \gamma) \geq \frac{1}{\sqrt{\pi}} 2^{(2+\gamma)/2} \Gamma\left(\frac{\gamma+1}{2}\right)$.
(ii) If $\alpha \in (1/2, 1)$, then $\frac{1}{\sqrt{\pi}} 2^{\gamma/2} \Gamma\left(\frac{\gamma+1}{2}\right) \leq \mathcal{E}(\alpha, \gamma) \leq \frac{1}{\sqrt{\pi}} 2^{1+(\gamma/2)} \Gamma\left(\frac{\gamma+1}{2}\right)$.

3 Application to Gaussian integrated processes

In this section we focus on the class of Gaussian processes with stationary increments that have, in contrast to the fractional Brownian motion analyzed in Proposition 1.1, differentiable sample paths a.s. In particular, we consider Gaussian processes $X(t)$ such that

$$X(t) = \int_0^t Z(s)ds,$$

where $Z(t)$ is a centered stationary Gaussian process with continuous covariance function $R(t) = \mathbb{C}ov(Z(s + t), Z(s))$.

Proposition 3.1 *Let $T \in \mathbf{RV}(\nu)$ and $f_R(t)$ be the spectral density function of the covariance function $R(t)$ of a centered stationary Gaussian process $Z(t)$.*
If $0 < f_R(0) < \infty$ and $R(t)$ is continuous, then for $X(t) = \int_0^t Z(s)ds$,

$$\mathbb{P}(\sup_{t \in [0,T]} X(t) > u) \sim \frac{1}{\sqrt{\pi}} 2^{1+\nu} \Gamma\left(\nu + \frac{1}{2}\right) \mathbb{P}(T > \sigma_X^{-1}(u)) \text{ as } u \to \infty.$$

Proof
The idea of the proof is to check that conditions **C1-C3** hold and then use Theorem 2.1. Note that $\sigma_X^2(t) = 2 \int_0^t ds \int_0^s R(v)dv$. Hence $(\sigma_X^2)'(t) = 2 \int_0^t R(v)dv$, which combined with the fact that

$$\int_0^\infty R(t)dt = \pi f_R(0) \tag{3.1}$$

immediately yields that **C1** is satisfied.
Moreover, (3.1) implies that $\int_0^t R(v)dv$ is slowly varying at ∞. Hence, by Karamata's theorem, $\sigma_X^2(t)$ is regularly varying at ∞ with index 1. Thus **C2** holds with $\alpha = 1/2$.
Since $\lim_{t \to 0} \frac{\sigma_X^2(t)}{t^2} = R(0)$, assumption **C3** is satisfied with $\lambda = 1$.
This completes the proof.
$\square$

Proposition 3.2 *Let $T \in \mathbf{RV}(\nu)$ and $Z(t)$ be a centered stationary process with continuous covariance function $R(t)$ such that $R(t) > 0$ and is regularly varying at ∞ with index $2\alpha - 2$, for $\alpha \in (1/2, 1)$. Then for $X(t) = \int_0^t Z(s)ds$,*

$$\mathbb{P}(\sup_{t \in [0,T]} X(t) > u) \sim \mathcal{E}(\alpha, \nu/\alpha)\mathbb{P}(T > \sigma_X^{-1}(u)) \text{ as } u \to \infty.$$

Proof
Since the proof is analogous to the proof of Proposition 3.1, we only note that due to Karamata's theorem $\sigma_X^2(t) = 2 \int_0^t ds \int_0^s R(v)dv$ is regularly varying at ∞ with index 2α. Hence following Theorem 2.1, the proof is completed.
$\square$

4 Proof of Theorem 2.1

Before presenting the proof of Theorem 2.1, we first state some lemmas that may be of independent interest.
Our first result is a property for regularly varying functions which is a variation of Potter's theorem, see Theorem 1.5.6 in [4].

Lemma 4.1 *If **C2-C3** hold, then there exist $\beta, B > 0$ such that for sufficiently large u and all $x \in [0, 1]$,*

$$\sigma_X(ux) \leq Bx^\beta \sigma_X(u). \tag{4.1}$$

Proof
Take $\delta < \alpha$ and let $x^* \geq 1$ be such that Theorem 1.5.6 (iii) of [4] is satisfied. This theorem implies that there exists an A such that, for $x > y \geq x^*$, we have

$$\frac{\sigma_X(y)}{\sigma_X(x)} \leq A(y/x)^{\beta_1}, \tag{4.2}$$

with $\beta_1 = \alpha - \delta$. Next, note that for $y < x^*$,

$$\frac{\sigma_X(y)}{\sigma_X(x)} = \frac{\sigma_X(y)}{\sigma_X(x^*)} \frac{\sigma_X(x^*)}{\sigma_X(x)} \leq A' y^{\beta_2} A(x^*/x)^{\beta_1}, \tag{4.3}$$

which follows from Theorem 1.5.6 (iii) of [4] and the fact that $\sigma_X(x)$ is regularly varying at 0. Now, take $\beta = \min\{\beta_1, \beta_2\}$. Combining (4.2) with (4.3), we conclude that for $x > x^*$ and $y > 0$, there exists a constant B such that

$$\frac{\sigma_X(y)}{\sigma_X(x)} \leq B(y/x)^\beta,$$

which implies the lemma.

$\square$

The next lemma states the well-known fact that a suitably scaled version of $X(s)$ converges to FBM (see e.g Proposition 7.2.11 in [11] or [8]). We give a short proof under our specific assumptions.
For a given function $f(u)$ define $X_{f(u)}(s) = \frac{X(f(u)s)}{\sigma_X(f(u))}$.

Lemma 4.2 *Let $X(t)$ be a centered Gaussian process with stationary increments such that* **C1-C3** *are satisfied and let $f(u) > 0$ be a function such that $\lim_{u \to \infty} f(u) = \infty$. Then the process $X_{f(u)}(s)$ weakly converges in $C([0,1])$, as $u \to \infty$, to $B_\alpha(s)$.*

Proof
In order to prove Lemma 4.2, it suffices to show the convergence of appropriate finite-dimensional distributions of $X_{f(u)}(s)$ as $u \to \infty$ and tightness.
Note that for every $u > 0$, $X_{f(u)}(s)$ is a centered Gaussian stochastic process with stationary increments. Moreover, for every $s \in [0,1]$,

$$\lim_{u \to \infty} \mathbb{V}ar\left(X_{f(u)}(s)\right) = s^{2\alpha},$$

and

$$\lim_{u \to \infty} \mathbb{V}ar(X_{f(u)}(s_1) - X_{f(u)}(s_2)) = \lim_{u \to \infty} \mathbb{V}ar(X_{f(u)}(s_1 - s_2)) = |s_1 - s_2|^{2\alpha}.$$

Hence, we conclude the convergence of all finite-dimensional distributions.
In order to show tightness, it suffices to prove that for sufficiently large u

$$\mathbb{E}|X_{f(u)}(s_1) - X_{f(u)}(s_2)|^2 \leq C|s_1 - s_2|^\delta$$

for some $C, \delta > 0$ and all $s_1, s_2 \in [0,1]$ (see e.g. Theorem 12.3 in Billingsley [3]). Now note that

$$\mathbb{E}|X_{f(u)}(s_1) - X_{f(u)}(s_2)|^2 = \mathbb{V}ar(X_{f(u)}(s_1 - s_2))$$
$$= \left(\frac{\sigma_X(f(u)(s_1 - s_2))}{\sigma_X(f(u))}\right)^2 \leq (B|s_1 - s_2|^\beta)^2,$$

where the last inequality follows from Lemma 4.1. Thus, the proof is completed.

$\square$

Let $X_{u,t}(s) = \frac{X(\sigma_X^{-1}(u)ts)}{\sigma_X(\sigma_X^{-1}(u)t)}$.
The following corollary will play a crucial role in the proof of Theorem 2.1.

Corollary 4.1 *Let $\epsilon > 0$ be such that $\lim_{u \to \infty} \sigma_X^{-1}(u)u^{-\epsilon} = \infty$. Then for every $t \in [u^{-\epsilon}, A]$, the process $X_{u,t}(s)$ weakly converges in $C([0,1])$, as $u \to \infty$, to $B_\alpha(s)$.*

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Proof
The proof follows from Lemma 4.2 applied to $X_{u,t}(s)$ with $f(u) = \sigma_X^{-1}(u)t$.

$\square$

We now state a final preparatory lemma.

Lemma 4.3 *Let $X(t)$ be a centered Gaussian process with stationary increments such that conditions* **C1-C3** *are satisfied. Then for every $\epsilon > 0$ and $\gamma > 0$,*

$$\mathbb{P}(\sup_{s \in [0, \sigma_X^{-1}(u)u^{-\epsilon}]} X(s) > u) = o(u^{-\gamma}) \quad \text{as} \quad u \to \infty.$$

Proof
Since $\sigma_X^2(t)$ is regularly varying at 0 with index 2λ, there exists a $T > 0$ such that $\sigma_X^2(t) \leq \sigma_X^2(1)t^\lambda$, and $\exp\left(-t^\lambda\right) \leq 1 - \frac{t^\lambda}{2}$, for every $t \in [0, T]$. Moreover, let $k_0 \in \mathbb{N}$ be such that $\sigma_X^2(t) \geq \sigma_X^2(1)$ for all $t \geq k_0 T$.
Now, for sufficiently large $u > 0$,

$$\mathbb{P}(\sup_{s \in [0, \sigma_X^{-1}(u)u^{-\epsilon}]} X(s) > u) \leq$$

$$\leq \quad \mathbb{P}(\sup_{s \in [0, k_0 T]} X(s) > u) + \sum_{k=k_0}^{\left[\frac{\sigma_X^{-1}(u)u^{-\epsilon}}{T}\right]} \mathbb{P}(\sup_{s \in [kT, (k+1)T]} X(s) > u)$$

$$\leq \quad \mathbb{P}(\sup_{s \in [0, k_0 T]} X(s) > u) + \sum_{k=k_0}^{\left[\frac{\sigma_X^{-1}(u)u^{-\epsilon}}{T}\right]} \mathbb{P}\left(\sup_{s \in [kT, (k+1)T]} \frac{X(s)}{\sigma_X(s)} > \frac{u}{\sigma_X(\sigma_X^{-1}(u)u^{-\epsilon})}\right)$$

$$\leq \quad \mathbb{P}(\sup_{s \in [0, k_0 T]} X(s) > u) + \sum_{k=k_0}^{\left[\frac{\sigma_X^{-1}(u)u^{-\epsilon}}{T}\right]} \mathbb{P}\left(\sup_{s \in [kT, (k+1)T]} \frac{X(s)}{\sigma_X(s)} > u^{\epsilon\alpha/2}\right), \qquad (4.4)$$

where the last inequality follows from the fact that $u/\sigma_X(\sigma_X^{-1}(u)u^{-\epsilon})$ is regularly varying with index $\epsilon\alpha$.
Due to the Borell inequality (see Theorem 2.1 in [1]), the first term in (4.4) may be bounded by

$$\mathbb{P}(\sup_{t \in [0, k_0 T]} X(t) > u) \leq 2 \exp\left(-\frac{(u - \mathbb{E}\sup_{t \in [0, k_0 T]} X(t))^2}{2}\right) \quad \text{as} \quad u \to \infty.$$

Thus it suffices to find a bound for the sum in (4.4).
Let $Z(t)$ be a centered stationary Gaussian process with covariance function

$$\mathbb{C}ov(Z(s+t), Z(s)) = \exp\left(-t^\lambda\right).$$

Note that for every $s, t \in [kT, (k+1)T]$ $(k \geq k_0)$,

$$\mathbb{C}ov\left(\frac{X(t)}{\sigma_X(t)}, \frac{X(s)}{\sigma_X(s)}\right) = \frac{\sigma_X^2(t) + \sigma_X^2(s) - \sigma_X^2(|t-s|)}{2\sigma_X(t)\sigma_X(s)} \geq$$

$$\geq 1 - \frac{\sigma_X^2(|t-s|)}{2\sigma_X(t)\sigma_X(s)} \geq 1 - \frac{\sigma_X^2(|t-s|)}{2\sigma_X^2(1)}$$

$$\geq 1 - \frac{|t-s|^\lambda}{2}$$

$$\geq \exp\left(-|t-s|^\lambda\right) = \mathbb{C}ov(Z(s), Z(t)).$$

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Thus for every $k \geq k_0$ and sufficiently large u,

$$\mathbb{P}\left(\sup_{t\in[kT,(k+1)T]} \frac{X(t)}{\sigma_X(t)} > u^{\epsilon\alpha/2}\right) \leq \mathbb{P}\left(\sup_{t\in[0,T]} Z(t) > u^{\epsilon\alpha/2}\right) \tag{4.5}$$

$$\leq 2\exp\left(\frac{-\left(u^{\epsilon\alpha/2} - \mathbb{E}(\sup_{t\in[0,T]} Z(t))\right)^2}{2}\right), \tag{4.6}$$

where (4.5) follows from the Slepian inequality (see Theorem C.1 in Piterbarg [9]) and (4.6) is due to the Borell inequality. Combining (4.6) with (4.4), we obtain

$$\sum_{k=k_0}^{\left[\frac{\sigma_X^{-1}(u)u^{-\epsilon}}{T}\right]} \mathbb{P}\left(\sup_{t\in[kT,(k+1)T]} \frac{X(t)}{\sigma_X(t)} > u^{\epsilon\alpha/2}\right) \leq$$

$$\leq 2\left[\frac{\sigma_X^{-1}(u)u^{-\epsilon}}{T}\right]\exp\left(-\frac{\left(u^{\epsilon\alpha/2} - \mathbb{E}(\sup_{t\in[0,T]} Z(t))\right)^2}{2}\right) = o(u^{-\gamma})$$

as $u \to \infty$ for every $\gamma > 0$. This completes the proof.

$\square$

Proof of Theorem 2.1
Let $G(x) = \mathbb{P}(T \leq x)$ be the distribution function of T. Let $a, A > 0$ be given and $\epsilon > 0$ be such that $\sigma_X^{-1}(u)u^{-\epsilon} \to \infty$ as $u \to \infty$. We have, with $u^{-\epsilon} \leq a \leq A$,

$$\mathbb{P}(\sup_{s\in[0,T]} X(s) > u) = \int_0^\infty \mathbb{P}(\sup_{s\in[0,t]} X(s) > u) dG(t)$$

$$= \int_0^\infty \mathbb{P}(\sup_{s\in[0,\sigma_X^{-1}(u)t]} X(s) > u) dG(\sigma_X^{-1}(u)t)$$

$$= \int_0^{u^{-\epsilon}} \mathbb{P}(\sup_{s\in[0,\sigma_X^{-1}(u)t]} X(s) > u) dG(\sigma_X^{-1}(u)t)$$

$$+ \int_{u^{-\epsilon}}^a \mathbb{P}(\sup_{s\in[0,\sigma_X^{-1}(u)t]} X(s) > u) dG(\sigma_X^{-1}(u)t)$$

$$+ \int_a^A \mathbb{P}(\sup_{s\in[0,\sigma_X^{-1}(u)t]} X(s) > u) dG(\sigma_X^{-1}(u)t)$$

$$+ \int_A^\infty \mathbb{P}(\sup_{s\in[0,\sigma_X^{-1}(u)t]} X(s) > u) dG(\sigma_X^{-1}(u)t)$$

$$= I_1 + I_2 + I_3 + I_4.$$

We analyze each of the integrals I_1, I_2, I_3, I_4 separately.

Integral I_1
According to Lemma 4.3, we have

$$\int_0^{u^{-\epsilon}} \mathbb{P}(\sup_{s\in[0,\sigma_X^{-1}(u)t]} X(s) > u) dG(\sigma_X^{-1}(u)t) \leq \mathbb{P}(\sup_{s\in[0,\sigma_X^{-1}(u)u^{-\epsilon}]} X(s) > u)$$

$$= o(\mathbb{P}(T > \sigma_X^{-1}(u))) \text{ as } u \to \infty.$$

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Integral I_2

According to (4.1), we have $\sigma_X(\sigma_X^{-1}(u)t) \leq Bt^\beta u$ and thus

$$\frac{u}{\sigma_X(\sigma_X^{-1}(u)t)} \geq \frac{1}{Bt^\beta}.$$

Hence,

$$
\begin{aligned}
I_2 &= \int_{u^{-\epsilon}}^{a} \mathbb{P}(\sup_{s \in [0, \sigma_X^{-1}(u)t]} X(s) > u) dG(\sigma_X^{-1}(u)t) & (4.7) \\
&\leq \int_{u^{-\epsilon}}^{a} \mathbb{P}\left(\sup_{s \in [0,1]} \frac{X(\sigma_X^{-1}(u)ts)}{\sigma_X(\sigma_X^{-1}(u)t)} > \frac{u}{\sigma_X(\sigma_X^{-1}(u)t)} \right) dG(\sigma_X^{-1}(u)t) \\
&\leq \int_{u^{-\epsilon}}^{a} \mathbb{P}\left(\sup_{s \in [0,1]} X_{u,t}(s) > \frac{1}{Bt^\beta} \right) dG(\sigma_X^{-1}(u)t). & (4.8)
\end{aligned}
$$

Note that according to Lemma 4.2, as $u \to \infty$, the process $X_{u,t}(s)$ weakly converges to $B_\alpha(s)$ in $C([0, 1])$ for $t \in [u^{-\epsilon}, a]$. Since the functional $\sup_{s \in [0,1]} f(s)$ is continuous in the uniform metric, there exists a constant $C_0 > 0$ such that

$$\mathbb{P}\left(\sup_{s \in [0,1]} X_{u,t}(s) > C_0 \right) \leq 1/2$$

for sufficiently large u and $t \in [u^{-\epsilon}, a]$. Hence, from the Borell inequality (see e.g. Theorem D.1 in [9])

$$\mathbb{P}\left(\sup_{s \in [0,1]} X_{u,t}(s) > x \right) \leq 2\Psi(x - C_0) \tag{4.9}$$

for every x. Combining (4.8) with (4.9),

$$
\begin{aligned}
\int_{u^{-\epsilon}}^{a} \mathbb{P}\left(\sup_{s \in [0,1]} X_{u,t}(s) > \frac{1}{Bt^\beta} \right) dG(\sigma_X^{-1}(u)t) &\leq 2\int_{u^{-\epsilon}}^{a} \Psi((t^{-\beta}/B) - C_0) dG(\sigma_X^{-1}(u)t) \\
&\leq 2\int_{u^{-\epsilon}}^{a} \Psi(t^{-\beta/2}) dG(\sigma_X^{-1}(u)t) \quad (4.10)
\end{aligned}
$$

for sufficiently large u. Now, using integration by parts, we have

$$
\begin{aligned}
2\int_{u^{-\epsilon}}^{a} \Psi(t^{-\beta/2}) dG(\sigma_X^{-1}(u)t) &= -2\int_{u^{-\epsilon}}^{a} \Psi(t^{-\beta/2}) d\bar{G}(\sigma_X^{-1}(u)t) \\
&= 2\Psi(u^{\epsilon\beta/2})\bar{G}(\sigma_X^{-1}(u)u^{-\epsilon}) - 2\Psi(a^{-\beta/2})\bar{G}(\sigma_X^{-1}(u)a) + 2\int_{u^{-\epsilon}}^{a} \bar{G}(\sigma_X^{-1}(u)t) d\Psi(t^{-\beta/2}) \\
&\leq 2\Psi(u^{\epsilon\beta/2}) + 2\int_{u^{-\epsilon}}^{a} \bar{G}(\sigma_X^{-1}(u)t) d\Psi(t^{-\beta/2}) \\
&\leq 2\Psi(u^{\epsilon\beta/2}) + 2\bar{G}(\sigma_X^{-1}(u))) \int_{u^{-\epsilon}}^{a} t^{-\nu-\mu} d\Psi(t^{-\beta/2}), & (4.11)
\end{aligned}
$$

where (4.11) follows from Potter's theorem and the fact that $\mu > 0$. Moreover,

$$
\begin{aligned}
\int_{u^{-\epsilon}}^{a} t^{-\nu-\mu} d\Psi(t^{-\beta/2}) &= \frac{\beta}{2\sqrt{2\pi}} \int_{u^{-\epsilon}}^{a} t^{-\nu-\mu} t^{-(\beta/2)-1} \exp\left(-\frac{t^{-\beta}}{2} \right) dt \\
&= \frac{1}{\sqrt{2\pi}} \int_{a^{-1}}^{u^\epsilon} y^{v+\mu+\beta/2-1} \exp\left(-\frac{y^\beta}{2} \right) dy & (4.12) \\
&= o(1) & (4.13)
\end{aligned}
$$

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as $u \to \infty$ and $a \to \infty$, where (4.12) follows from the change of variable $y \equiv 1/t$.
Thus, combining (4.8) with (4.10), (4.11) and (4.13), we conclude that $I_2 = o(\bar{G}(\sigma_X^{-1}(u)))$
as $u \to \infty$.

Integral I_3

We have

$$
\begin{aligned}
I_3 &= \int_a^A \mathbb{P}(\sup_{s \in [0, \sigma_X^{-1}(u)t]} X(s) > u) dG(\sigma_X^{-1}(u)t) \\
&= \int_a^A \mathbb{P}\left(\sup_{s \in [0,1]} X_{u,t}(s) > \frac{u}{\sigma_X(\sigma_X^{-1}(u)t)}\right) dG(\sigma_X^{-1}(u)t).
\end{aligned}
$$

According to Theorem 1.5.2 in [4],

$$
\frac{u}{\sigma_X(\sigma_X^{-1}(u)t)} \to t^{-H} \quad \text{as} \quad u \to \infty,
$$

uniformly for $t \in [a, A]$. Moreover, according to Lemma 4.2, the process $X_{u,t}(s)$, converges
weakly in $C([0,1])$ to $B_\alpha(s)$, uniformly in $t \in [a, A]$. Hence, for sufficiently large u,

$$
\begin{aligned}
I_3 &\sim \int_a^A \mathbb{P}\left(\sup_{s \in [0,1]} B_\alpha(s) > t^{-\alpha}\right) dG(\sigma_X^{-1}(u)t) \\
&= \mathbb{P}\left(\left(\sup_{s \in [0,1]} B_\alpha(s)\right)^{1/H} T > \sigma^{-1}(u); a\sigma_X^{-1}(u) \leq T \leq A\sigma_X^{-1}(u)\right).
\end{aligned}
$$

Integral I_4

We have

$$
\int_A^\infty \mathbb{P}(\sup_{s \in [0, \sigma_X^{-1}(u)t]} X(s) > u) dG(\sigma_X^{-1}(u)t) \leq \bar{G}(A\sigma_X^{-1}(u)) \sim A^{-\nu} \bar{G}(\sigma_X^{-1}(u)) \quad \text{as} \quad u \to \infty.
$$

Now letting $a \to 0$ and $A \to \infty$, we obtain that $I_1 + I_2 + I_4 = o(\bar{G}(\sigma_X^{-1}(u)))$. Thus the
main contribution to the asymptotics comes from I_3. In order to complete the proof, we
need to show that

$$
\lim_{a \downarrow 0, A \to \infty} \liminf_{u \to \infty} \frac{I_3}{\bar{G}(\sigma_X^{-1}(u))} = \mathbb{E}\left(\left(\sup_{s \in [0,1]} B_\alpha(s)\right)^{\nu/H}\right).
$$

(The lim sup is trivial: take $a = 0, A = \infty$, and use the computation in the Introduction.)
Define $\bar{B} = \left(\sup_{s \in [0,1]} B_\alpha(s)\right)^{1/H}$. Note that I_3 is asymptotically larger than

$$
\mathbb{P}(\bar{B}T > \sigma_X^{-1}(u)) - \mathbb{P}(\bar{B}T > \sigma_X^{-1}(u); T \leq a\sigma_X^{-1}(u)) - \mathbb{P}(T \geq A\sigma_X^{-1}(u)).
$$

The last term can be handled in a similar fashion as I_4. This leaves us with showing that

$$
\lim_{a \downarrow 0} \limsup_{u \to \infty} \frac{\mathbb{P}(\bar{B}T > \sigma_X^{-1}(u); T \leq a\sigma_X^{-1}(u))}{\mathbb{P}(T > \sigma_X^{-1}(u))} = 0. \tag{4.14}
$$

Set $x = \sigma_X^{-1}(u)$ and write, using Markov's inequality and applying partial integration,

$$
\begin{aligned}
\mathbb{P}(\bar{B}T > x; T < ax) &= \int_0^{ax} dG(t) \mathbb{P}(\bar{B} > x/t) \leq \\
&\leq \int_0^{ax} dG(t) C(x/t)^{-\nu-\epsilon} = -\int_0^{ax} C(x/t)^{-\nu-\epsilon} d\bar{G}(t) \\
&= C\mathbb{P}(T > ax) a^{\nu+\epsilon} + (\nu + \epsilon) C x^{-\nu-\epsilon} \int_0^{ax} \bar{G}(t) t^{\nu+\epsilon-1} dt \\
&\sim C\mathbb{P}(T > ax) a^{\nu+\epsilon} + C'L(ax) x^{-\nu} a^\epsilon
\end{aligned}
$$

as $x \to \infty$, where C, C' are finite constants independent of a. The last equivalence follows from Karamata's Theorem (see Theorem 1.5.11 in [4]). The above computation gives the desired property (4.14), which completes the proof.

$\square$

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