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Saddlepoint Approximations to the Trimmed Mean

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ABSTRACT
Saddlepoint approximations for the trimmed mean and the studentized trimmed mean are established. Some numerical evidence on the quality of our saddlepoint approximations is also included. These approximations can be applied to the bootstrap for the studentized trimmed mean, to provide very fast and accurate approximations to the bootstrap without the need for extensive resampling.

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1. Introduction

The centre of a distribution is often estimated by the sample mean or the sample median. However, it is well known that the sample mean is sensitive to outliers and thus not robust. On the other hand, the sample median is robust against outliers but it is not very efficient if the underlying distribution is, for instance, normal. An estimator providing intermediate behavior, and which includes both the sample mean and sample median, is the trimmed (sample) mean. Compared with robust M-estimates of maximum likelihood type, the trimmed mean not only has the same asymptotic variance but is also easy to compute.

The asymptotic normality of the trimmed mean is derived by Stigler (1973) under minimal conditions, while Bjerve (1974) and Helmers (1982) derive Edgeworth expansions under general conditions. Easton and Ronchetti(1986) obtained approximations to the density of trimmed means. The validity of the Edgeworth expansion for the studentized trimmed mean was established by Hall and Padmanabhan (1992), while a simple explicit form of the Edgeworth expansion was obtained in Gribkova and Helmers (2002). It is well known that Edgeworth expansions generally provide accurate approximation near the center of the distribution, but the relative error can become unacceptably large in the far tail of the distribution. On the other hand, saddlepoint approximation will offer an approximation whose relative error is controlled both near the center and in the far tail of the distribution. Therefore, in this paper, we derive saddlepoint approximations to the densities and tail probabilities of the trimmed mean and its studentized version. To do this, we shall exploit the special structure of the trimmed mean and employ a simple conditioning argument in the same way as Bjerve (1974) does in his derivation of an Edgeworth expansion for the trimmed mean. Conditionally given the values of the two extreme order statistics appearing in the trimmed mean, the conditional distribution of a trimmed mean reduces to a sum of i.i.d. r.v.'s, to which we an apply a saddlepoint approximation. Finally we integrate out these two extreme order statistics by a Laplace approximation. The tail probabilities of the Lugannani-Rice type are derived by another Laplace approximation of Temme type (see Temme, 1982), as was done in Daniels and Young (1991) and Jing and Robinson (1994). For a rigorous account of saddlepoint approximations, the reader is referred to a recent monograph by Jensen (1995). A general approach dealing with saddlepoint approximations for L-estimators was presented by Easton and Ronchetti (1986).

One important application of the saddlepoint approximations has been in the area of the bootstrap analysis. Davison and Hinkley (1988) were the first to apply the idea to the bootstrap with the sample mean. This was later extended to the studentized mean by Daniels and Young (1991). A major advantage of using the saddlepoint approximation is that it can completely eliminate the need for resampling and yet provides a very fast and accurate approximation to the distribution of interest. In this paper, we shall discuss how to apply the idea to the trimmed sample mean.

The layout of the paper is as follows. Some basic notation will be introduced in Section 2. In Section 3, we shall derive saddlepoint approximations to the density and tail probabilities for the trimmed mean. In Section 4, we shall carry out the same analysis for the studentized trimmed mean. Some numerical evidence on the quality of our saddlepoint approximation is given in Section 5. Application to the bootstrap with the trimmed mean is discussed in Section 6. Finally, most of the technical details are given in the Appendix.

2. Some preliminaries

Let $X_1, ..., X_n$ be a random sample from a population with distribution function $F(\cdot)$ and density $f(\cdot)$, respectively. Let $X_{1:n} \le ... \le X_{n:n}$ be the corresponding order statistics. Define the trimmed mean by

$$\overline{X}_{\alpha\beta} = \frac{1}{m}\sum_{i=r}^{s} X_{i:n},$$

where

$$r = [n\alpha] + 1, \qquad s = n - [n\beta], \qquad m = n - [n\alpha] - [n\beta],$$

and $0 \le \alpha < \frac{1}{2}$, $0 \le \beta < \frac{1}{2}$ and $[x]$ is the largest integer less than or equal to x. That is, we throw out the smallest $[n\alpha]$ and the largest $[n\beta]$ observations and take the average of the rest of data in the middle. (In particular, if we suspect that the underlying distribution is symmetric, we can take $\alpha = \beta$.) For any $0 < p < 1$, we define

$$\xi_p = F^{-1}(p) = \inf\{x : F(x) \ge p\}.$$

It is well known that the asymptotic mean and variance of $\overline{X}_{\alpha\beta}$ are given by, respectively, (e.g., see Stigler (1973)).

$$\begin{aligned} \mu &= \frac{1}{1-\alpha-\beta}\int_{\xi_\alpha}^{\xi_{1-\beta}} x dF(x), \\ \tau_{\alpha\beta}^2 &= \frac{1}{(1-\beta-\alpha)^2}\big((1-\beta-\alpha)\sigma^2 + \beta(1-\beta)(\xi_{1-\beta}-\mu)^2 \\ &\qquad -2\alpha\beta(\xi_\alpha-\mu)(\xi_{1-\beta}-\mu) + \alpha(1-\alpha)(\xi_\alpha-\mu)^2\big), \end{aligned}$$

where

$$\sigma^2 = \frac{1}{(1-\beta-\alpha)}\int_{\xi_\alpha}^{\xi_{1-\beta}} x^2 dF(x) - \mu^2.$$

We shall need the joint distribution of two order statistics. Define $q_{r,s:n}(x,y)$ to be the joint density function of order statistics $(X_{r:n}, X_{s:n})$. From David (1981)

$$q_{r,s:n}(x,y) = D_{n\alpha\beta}[F(x)]^{r-1}[F(y)-F(x)]^{s-r-1}[1-F(y)]^{n-s}f(x)f(y)\ I\{x<y\},$$

where $D_{n\alpha\beta} = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}$ and $I\{\cdot\}$ is the indicator function.

Finally, for fixed values x and y with $x < y$, let $F_{x,y}(t)$ denote the conditional df of X, given that $x \le X \le y$, that is,

$$F_{x,y}(t) = \begin{cases} 0, & t \le x \\ \frac{F(t)-F(x)}{F(y)-F(x)}, & x \le t \le y \\ 1, & t \ge y. \end{cases}$$

Also, let $Y_1, ..., Y_m$ be a random sample from a distribution $F_{x,y}(t)$. Let $Y_{1:m} \le ... \le Y_{m:m}$ be the order statistics of $Y_1, ..., Y_m$. Write $\overline{Y} = m^{-1}\sum_{i=1}^{m} Y_i$, and denote its density and distribution functions by $f_{\overline{Y}}(\cdot)$ and $F_{\overline{Y}}(\cdot)$, respectively.

3. Saddlepoint approximation to the trimmed mean

In this section, we shall derive the saddlepoint approximation to the distribution and density function of $\overline{X}_{\alpha\beta}$ defined by

$$G(t) = P(\overline{X}_{\alpha\beta} \le t), \qquad g(t) = G'(t).$$

For any t, denote $\Omega = \{(x,y) \in R^2 : x < y\}$ and $\Omega(t) = \{(x,y) \in R^2 : x < t < y\}$. First we note that

$$\begin{aligned} G(t) &= P\left(\overline{X}_{\alpha\beta} \le t\right) \\ &= \iint_{\Omega} P\left(m^{-1}\sum_{i=r}^{s} X_{i:n} \le t \,\middle|\, X_{r-1:n} = x, X_{s+1:n} = y\right) q_{r-1,s+1:n}(x,y)\ dxdy \\ &= \iint_{\Omega} P\left(m^{-1}\sum_{i=1}^{m} Y_{i:m} \le t\right) q_{r-1,s+1:n}(x,y)\ dxdy \\ &= \iint_{\Omega} P\left(\overline{Y} \le t\right)\ q_{r-1,s+1:n}(x,y)\ dxdy \\ &= \iint_{\Omega} F_{\overline{Y}}(t)\ q_{r-1,s+1:n}(x,y)\ dxdy, \end{aligned}$$

and

$$g(t) = \iint_{\Omega(t)} f_{\overline{Y}}(t)\ q_{r-1,s+1:n}(x,y)\ dxdy. \tag{3.1}$$

The first step in obtaining saddlepoint approximation to $g(t)$ is to replace $f_{\overline{Y}}(t)$ in (3.1) by its saddlepoint approximation. To do this, define the cumulant generating function of Y_1 by

$$K_{Y_1}(\lambda) = \log E\exp\{\lambda Y_1\} = \log\left(\frac{\int_x^y e^{\lambda z} dF(z)}{F(y)-F(x)}\right).$$

It follows that

$$\begin{aligned} K'_{Y_1}(\lambda) &= \frac{dK_{Y_1}(\lambda)}{d\lambda} = \frac{\int_x^y z e^{\lambda z} dF(z)}{\int_x^y e^{\lambda z} dF(z)} \\ K''_{Y_1}(\lambda) &= \frac{d^2K_{Y_1}(\lambda)}{d\lambda^2} = \frac{\int_x^y z^2 e^{\lambda z} dF(z)}{\int_x^y e^{\lambda z} dF(z)} - \left(K'_{Y_1}(\lambda)\right)^2. \end{aligned}$$

Therefore, the saddlepoint approximation to $f_{\overline{Y}}(t)$ is (see Daniels (1954), for instance)

$$f_{\overline{Y}}(t) = \sqrt{\frac{m}{2\pi K''_{Y_1}(\widetilde{\lambda})}}\exp\left\{-m[\widetilde{\lambda}t - K_{Y_1}(\widetilde{\lambda})]\right\}\left\{1 + m^{-1}r_m(x,y,t)\right\}, \tag{3.2}$$

where $\widetilde{\lambda} = \widetilde{\lambda}(t)$ satisfies the saddlepoint equation

$$K'_{Y_1}(\widetilde{\lambda}(t)) = t, \tag{3.3}$$

and $r_m(x,y,t)$ is an error term.

3.1 Saddlepoint approximation to the density of the trimmed mean

We shall now derive a saddlepoint approximation to the density of the trimmed mean. Substituting (3.2) into (3.1), we get

$$\begin{aligned} g(t) &= \iint_{\Omega(t)} \sqrt{\frac{m}{2\pi K''_{Y_1}(\widetilde{\lambda})}} \exp\{-m[\widetilde{\lambda}t - K_{Y_1}(\widetilde{\lambda})]\} q_{r-1,s-1:n}(x,y) \\ &\qquad \times\{1 + m^{-1} r_m(x,y,t)\} dxdy \\ &= \iint_{\Omega(t)} \sqrt{\frac{m}{2\pi K''_{Y_1}(\widetilde{\lambda})}} f(x)f(y)\exp[-m\Lambda(x,y,t)] \\ &\qquad \times\{1 + m^{-1} r_m(x,y,t)\} dxdy, \end{aligned} \tag{3.4}$$

where $\widetilde{\lambda}$ is the solution to $K'_{Y_1}(\widetilde{\lambda}) = t$ and

$$\begin{aligned} \Lambda_1(x,y,t) &= \widetilde{\lambda}t - K_{Y_1}\left(\widetilde{\lambda}\right), \\ \Lambda_2(x,y) &= -m^{-1}\log\left(C_{n\alpha\beta}[F(x)]^{r-2}[F(y)-F(x)]^m[1-F(y)]^{n-s-1}\right), \\ \Lambda(x,y,t) &= \Lambda_1(x,y,t) + \Lambda_2(x,y). \end{aligned}$$

where $C_{n\alpha\beta} = \frac{n!}{(r-2)!(s-r+1)!(n-s-1)!}$ Define

$$\Delta(x,y,t) = (f(x)f(y))^{-1} \begin{pmatrix} \frac{\partial^2\Lambda(x,y,t)}{\partial x^2} & \frac{\partial^2\Lambda(x,y,t)}{\partial x\partial y} \\ \frac{\partial^2\Lambda(x,y,t)}{\partial y\partial x} & \frac{\partial^2\Lambda(x,y,t)}{\partial x^2} \end{pmatrix}.$$

For each t, let $x_0 = x_0(t)$, $y_0 = y_0(t)$, $\widetilde{\lambda}_0 = \widetilde{\lambda}_0(t)$ be the solution to

$$\begin{cases} \frac{\partial\Lambda(x_0,y_0,t)}{\partial x} &= 0 \\ \frac{\partial\Lambda(x_0,y_0,t)}{\partial y} &= 0 \\ K'_{Y_1}(\widetilde{\lambda}_0) &= t. \end{cases} \tag{3.5}$$

If the density function is non-zero in the support of X, then equation (3.5) can be reduced to

$$\begin{cases} \frac{m\exp\{\widetilde{\lambda}_0 x_0\}}{\int_{x_0}^{y_0}\exp\{\widetilde{\lambda}_0 z\}dF(z)} &= \frac{r-2}{F(x_0)} \\ \frac{m\exp\{\widetilde{\lambda}_0 y_0\}}{\int_{x_0}^{y_0}\exp\{\widetilde{\lambda}_0 z\}dF(z)} &= \frac{n-s-1}{1-F(y_0)} \\ K'_{Y_1}(\widetilde{\lambda}_0) &= t. \end{cases} \tag{3.6}$$

That is, $(x_0(t), y_0(t))$ is a stationary point of $\Lambda(x,y,t)$ for fixed t. For simplicity, we write $\Lambda_0(t) = \Lambda(x_0, y_0, t)$.

Proposition 1: *Let t belong to the support of X. Then, for any n satisfying $[n\alpha] \geq 2$, $[n\beta] \geq 2$ and $n - [n\alpha] - [n\beta] \geq 1$, $\Lambda(x,y,t)$ attains its global minimum at some finite point (x_0, y_0) which satisfies equation (3.5).*

Remark 3.1. Proposition 1 will ensure that the saddlepoint equation (3.5) always has a solution under Condition (i) of Theorem 1. The conditions $[n\alpha] \geq 2$, $[n\beta] \geq 2$ and $n - [n\alpha] - [n\beta] \geq 1$ are imposed to guarantee that the exponents in $q_{r-1,s+1:n}(x,y)$ are greater than 0.

The following theorem gives the saddlepoint approximations to the density of the trimmed mean.

Theorem 1: *Let t belong to the support of X. Suppose* *(i)* $f(x) = F'(x)$ *and* $f''(x)$ *exists,*

(ii) for any n *satisfying* $[n\alpha] \geq 2$, $[n\beta] \geq 2$ *and* $n - [n\alpha] - [n\beta] \geq 1$, (x_0, y_0) *is unique i.e.* $\Lambda(x, y, t) > \Lambda(x_0, y_0, t)$ *for each* $(x, y) \neq (x_0, y_0)$, *and* $\Delta(x_0, y_0, t)$ *is positive definite,*

(iii) $|Ee^{i\eta X}| \in L^v(R)$ *for some* $v > 0$.

Then we have

$$g(t) = g_{sp}(t)\Big(1 + m^{-1}R_n(t)\Big), \tag{3.7}$$

where

$$g_{sp}(t) = \sqrt{\frac{2\pi}{m}} \frac{\exp\Big(-m\Lambda(x_0(t), y_0(t), t)\Big)}{\sqrt{K''_{Y_1}(\widetilde{\lambda}_0(t))|_{x=x_0(t),y=y_0(t)}|\Delta_0(t)|}}, \tag{3.8}$$

the error term $R_n(t)$ *in (3.7) is bounded when* t *is in some compact set and* $x_0(t)$, $y_0(t)$ *and* $\widetilde{\lambda}_0(t)$ *are solutions to (3.5).*

Remark 3.2. Condition (i) is a natural smoothness condition, which we need to validate Laplace approximation. Also note that (3.8) involves $f'(x)$. Since $\Lambda(x, y, t)$ attains its minimum at (x_0, y_0), $\Delta(x_0, y_0, t)$ is nonnegative definite. The purpose of Condition (ii) is to simplify the proof. Conditon (iii) ensures that we may apply the Fourier inversion theorem.

The proof of Proposition 1 and Theorem 1 is postponed to the appendix.

3.2 Saddlepoint approximation to the tail probability of trimmed mean

One way to obtain an approximation to the tail probability $1 - G(t) = P(\overline{X}_{\alpha\beta} \geq t)$ is to integrate the saddlepoint approximation $g_{sp}(t)$ numerically. Since $\int_{-\infty}^{\infty} g_{sp}(t)dt$ may not be equal to one in general, renormalization will usually improve the accuracy of the resulting saddlepoint approximation. The resulting approximation to $1 - G(t)$ will be denoted by $1 - G_{ss}(t)$. However, it would be more convenient to have a simple explicit approximation formula for the tail probability. Theorem 2 below will give a saddlepoint approximation to the tail probability $1 - G(t)$ of the trimmed mean. For ease of notation, let

$$\begin{cases} a(t) &= (2\pi/m)^{1/2}\left(K''_{Y_1}(\widetilde{\lambda}_0(t))\Big|_{x=x_0(t),y=y_0(t)} \cdot |\Delta_0(t)|\right)^{-1/2} \\ h(t) &= \Lambda(x_0(t), y_0(t), t). \end{cases}$$

Then (3.8) can be rewritten as $g_{sp}(t) = a(t)\exp\{-mh(t)\}$. Let $\widehat{t}$ be the solution to $h'(\widehat{t}) = 0$. Also define

$$w = \sqrt{2(h(t) - h(\widehat{t}))}\ sgn(t - \widehat{t}), \tag{3.9}$$

$$\psi(w) = (2\pi/m)^{1/2} a(t(w)) \exp\{-mh(\widehat{t})\}\left|\frac{dt}{dw}\right|. \tag{3.10}$$

Then we have

Theorem 2: *Under the conditions of Theorem 1, we have*

$$P(\overline{X}_{\alpha\beta} \geq t) = 1 - \Phi(w\sqrt{m}) - \frac{\phi(w\sqrt{m})}{\sqrt{m}}\left(\frac{\psi(0) - \psi(w)}{w\psi(0)} + O(m^{-1})\right),$$

where w *and* $\psi(w)$ *are given in (3.9) and (3.10).*

The proof of the theorem is similar to but simpler than that of Theorem 4 in Section 4, hence omitted here.

4. Saddlepoint approximation to studentized trimmed mean

4.1 Introduction

In Section 3, we have derived saddlepoint approximations to the density and tail probabilities of the trimmed mean. In this section, we shall carry out the same derivations for the studentized trimmed mean. This will have greater practical relevance if we are interested in constructing confidence intervals or hypothesis testing concerning the center of the distribution.

To studentize the trimmed mean, we employ the plug-in estimate of the variance, which is given by

$$\begin{aligned}\widehat{\tau}^2_{\alpha\beta} &= \frac{1}{(1-\beta-\alpha)^2}\Big((1-\beta-\alpha)\widehat{\sigma}^2_{\alpha\beta} + \beta(1-\beta)(\widehat{\xi}_{1-\beta} - \overline{X}_{\alpha\beta})^2 \\ &\qquad -2\alpha\beta(\widehat{\xi}_{\alpha} - \overline{X}_{\alpha\beta})(\widehat{\xi}_{1-\beta} - \overline{X}_{\alpha\beta}) + \alpha(1-\alpha)(\widehat{\xi}_{\alpha} - \overline{X}_{\alpha\beta})^2\Big),\end{aligned}$$

where $\widehat{\xi}_p = \inf\{x : \widehat{F}(x) \geq p\}$ for any $0 < p < 1$ and $\widehat{F}(x)$ denotes the empirical distribution of the X_i's $(1 \leq i \leq n)$, and

$$\widehat{\sigma}^2_{\alpha\beta} = \frac{1}{(1-\beta-\alpha)}\int_{\widehat{\xi}_{\alpha}}^{\widehat{\xi}_{1-\beta}} x^2 d\widehat{F}(x) - \overline{X}^2_{\alpha\beta},$$

(e.g., see Hall and Padmanabhan (1992)). Because of its complicated form, we shall restrict our attention from now on to the special case where we assume that

(C1) $f(x)$ is symmetric around the origin, i.e., $f(x) = f(-x)$,

(C2) the trimming proportions are the same, i.e., $\alpha = \beta$.

Clearly, (C1) and (C2) imply that $\mu = 0$. The case for the nonzero mean can be dealt with by a simple mean shift.

Under the assumptions (C1) and (C2), $\widehat{\tau}^2_{\alpha\beta}$ above reduces to

$$\widehat{\tau}^2_{\alpha\alpha} = \frac{1}{n(1-2\alpha)^2}\left(\sum_{i=r+1}^{s-1}\left(X_{i:n} - \overline{X}_{\alpha\alpha}\right)^2 + r\left[\left(X_{r:n} - \overline{X}_{\alpha\alpha}\right)^2 + \left(X_{s:n} - \overline{X}_{\alpha\alpha}\right)^2\right]\right).$$

Therefore, we can define the studentized trimmed mean by

$$T = \frac{\overline{X}_{\alpha\alpha}}{\widehat{\tau}_{\alpha\alpha}}.$$

The purpose of this section is to derive saddlepoint approximations to the density and tail probability for the studentized trimmed mean T, denoted by

$$\widetilde{G}(t) = P\left(T \leq t\right), \qquad \widetilde{g}(t) = \widetilde{G}'(t).$$

As in Section 2, let $Y_1, \cdots, Y_{m-2}$ be a random sample from the truncated distribution $F_{x,y}(t)$. We further define $Z_i = Y_i^2$ and

$$\overline{Y} = \frac{1}{m-2}\sum_{i=1}^{m-2} Y_i, \qquad \overline{Z} = \frac{1}{m-2}\sum_{i=1}^{m-2} Y_i^2.$$

Now, for fixed x and y, define

$$\begin{aligned}s(\overline{Y}, \overline{Z}) &= n^{-1/2}\left(1-2\alpha\right)^{-1}\left[(m-2)\overline{Z} - (m-2r+2)\left(\frac{m-2}{m}\overline{Y} + \frac{x+y}{m}\right)^2\right. \\ &\qquad \left. + r(x+y)^2 - 2(r-1)(x+y)\left(\frac{m-2}{m}\overline{Y} + \frac{x+y}{m}\right)\right]^{1/2}\end{aligned}$$

and

$$\begin{cases} b \equiv b(\overline{Y}, \overline{Z}) = \left(\frac{m-2}{m}\overline{Y} + \frac{x+y}{m}\right), \\ a \equiv a(\overline{Y}, \overline{Z}) = \frac{b(\overline{Y},\overline{Z})}{s(\overline{Y},\overline{Z})}. \end{cases} \tag{4.1}$$

Then conditional on $X_{r:n} = x$ and $X_{s:n} = y$, we can show, after some simple algebra, that

$$\begin{aligned} \overline{X}_{\alpha\alpha} &= b\left(\overline{Y}, \overline{Z}\right) \\ \widehat{\tau}_{\alpha\alpha} &= s\left(\overline{Y}, \overline{Z}\right). \end{aligned}$$

Therefore, we have

$$\begin{aligned} \widetilde{G}(t) &\equiv P(T < t) \\ &= \int\int_{\Omega} P(T \le t \mid X_{r:n} = x, X_{s:n} = y) q_{r,s:n}(x,y)dxdy \\ &= \int\int_{\Omega} P\left(a(\overline{Y}, \overline{Z}) \le t\right)\ q_{r,s:n}(x,y)dxdy \end{aligned}$$

and

$$\widetilde{g}(t) \equiv \widetilde{G}'(t) = \int\int_{\Omega(t)} f_{a(\overline{Y},\overline{Z})}(t)\ q_{r,s:n}(x,y)dxdy, \tag{4.2}$$

where $a(\cdot,\cdot)$ is defined in (4.1).

Similarly to Section 3, we shall obtain a saddlepoint approximation to $\widetilde{g}(t)$ and $\widetilde{G}(t)$ by first getting a saddlepoint approximation to the density of $a(\overline{Y}, \overline{Z})$ and then substituting that into the above to obtain saddlepoint approximations to $\widetilde{g}(t)$ and $\widetilde{G}(t)$. For that purpose, we shall need the joint cumulant generating function of $(Y_i, Z_i) = (Y_i, Y_i^2)$,

$$K(d,u) = \log E\{\exp(dY + uY^2)\} = \log \frac{\int_x^y \exp(dz + uz^2)dF(z)}{F(y) - F(x)}.$$

Note that $K(d,u)$ is also functions of x and y, and their derivatives with respect to x and y are given by

$$\begin{aligned} \frac{\partial K(d,u)}{\partial x} &= \left(1 - \exp(dx + ux^2 - K(d,u))\right)\frac{f(x)}{F(y)-F(x)}, \\ \frac{\partial K(d,u)}{\partial y} &= \left(\exp(dy + uy^2 - K(d,u)) - 1\right)\frac{f(y)}{F(y)-F(x)}, \\ \frac{\partial^2 K(d,u)}{\partial^2 x} &= -\frac{(d+2ux)f(x)}{F(y)-F(x)} + \frac{\partial K(d,u)}{\partial x} \\ &\quad \times \left(d + 2ux + \frac{f'(x)}{f(x)} + \frac{2f(x)}{F(y)-F(x)}\right) - \left(\frac{\partial K(d,u)}{\partial x}\right)^2, \\ \frac{\partial^2 K(d,u)}{\partial^2 y} &= \frac{(d+2uy)f(y)}{F(y)-F(x)} + \frac{\partial K(d,u)}{\partial y} \\ &\quad \times \left(d + 2uy + \frac{f'(y)}{f(y)} - \frac{2f(y)}{F(y)-F(x)}\right) - \left(\frac{\partial K(d,u)}{\partial y}\right)^2, \\ \frac{\partial^2 K(d,u)}{\partial x \partial y} &= \frac{f(x)\frac{\partial K(d,u)}{\partial y} - f(y)\frac{\partial K(d,u)}{\partial x}}{F(y)-F(x)} - \frac{\partial K(d,u)}{\partial x}\frac{\partial K(d,u)}{\partial y}. \end{aligned}$$

4.2 Saddlepoint approximation to the density of the studentized trimmed mean
Note that the inverse transformation of (4.1) is

$$\begin{cases} \overline{Y} & \equiv \overline{Y}(a,b) = (m-2)^{-1}\,(mb - x - y) \\ \overline{Z} & \equiv \overline{Z}(a,b) = (m-2)^{-1}\left(\frac{n(1-2\alpha)^2 b^2}{a^2} + (m-2r+2)b^2 - r(x+y)^2 + 2(r-1)(x+y)b\right), \end{cases}$$

whose Jacobian is given by

$$J \equiv J(a,b) = \begin{vmatrix} \frac{\partial \overline{Y}}{\partial a} & \frac{\partial \overline{Y}}{\partial b} \\ \frac{\partial \overline{Z}}{\partial a} & \frac{\partial \overline{Z}}{\partial b} \end{vmatrix} = \frac{2n(1-2\alpha)^2 m b^2}{(m-2)^2 a^3}.$$

Define

$$\begin{aligned} \Lambda_s(a,b) &= d\overline{Y} + u\overline{Z} - K(d,u), \\ \Delta_s(a,b) &= \begin{vmatrix} \frac{\partial^2 K(d,u)}{\partial^2 d} & \frac{\partial^2 K(d,u)}{\partial d \partial u} \\ \frac{\partial^2 K(d,u)}{\partial u \partial d} & \frac{\partial^2 K(d,u)}{\partial^2 u} \end{vmatrix}, \\ G(a,b) &= |\Delta_s(a,b)| \frac{\partial^2 \Lambda_s(a,b)}{\partial b^2}. \end{aligned}$$

Similarly to Daniels and Young (1991) and Jing and Robinson (1994), a saddlepoint approximation to the density of $a(\overline{Y},\overline{Z})$ is given by

$$\begin{aligned} f_{a(\overline{Y},\overline{Z})}(t) &= \sqrt{\frac{m-2}{2\pi}}\; J(t,b_0(t)) G^{-1/2}(t,b_0(t))\; \exp[-(m-2)\Lambda_s(t,b_0(t))] \\ &\quad \times\{1 + m^{-1}\widetilde{r}_m(x,y,t)\}, \end{aligned}$$

where $\widetilde{r}_m(x,y,t)$ is the error term that will not be given here explicitly. Substituting this into (4.2), we get

$$\begin{aligned} \widetilde{g}(t) &= \iint_{\Omega(t)} \sqrt{\frac{m-2}{2\pi}}\; J(t,b_0(t)) G^{-1/2}(t,b_0(t))\; \exp[-(m-2)\Lambda_s(t,b_0(t))] \\ &\quad \times q_{r,s:n}(x,y)\; \{1 + m^{-1}\widetilde{r}_m(x,y,t)\}\; dxdy \\ &= \iint_{\Omega(t)} \sqrt{\frac{m-2}{2\pi}}\; \frac{J(t,b_0(t))}{G^{1/2}(t,b_0(t))}\; \exp[-(m-2)\widetilde{\Lambda}(x,y,t)] \\ &\quad \times f(x)f(y)\; \{1 + m^{-1}\widetilde{r}_m(x,y,t)\}\; dxdy, \end{aligned} \tag{4.3}$$

where $d_0 = d_0(t)$, $u_0 = u_0(t)$ and $b_0 = b_0(t)$ are solutions to the following three equations

$$\begin{cases} \frac{\partial \Lambda_s(a,b_0(t))}{\partial b}\Big|_{d=d_0(t),u=u_0(t)} & = 0 \\ \frac{\partial K(d_0(t),u_0(t))}{\partial d} & = \overline{Y}(t,b_0(t)) \\ \frac{\partial K(d_0(t),u_0(t))}{\partial u} & = \overline{Z}(t,b_0(t)) \end{cases}$$

and further

$$\begin{aligned} \widetilde{\Lambda}_1(x,y,t) &= d_0(t)\overline{Y}(t,b_0(t)) + u_0(t)\overline{Z}(t,b_0(t)) - K(d_0(t),u_0(t)), \\ \widetilde{\Lambda}_2(x,y,t) &= -(m-2)^{-1}\log\left(D_{n\alpha\beta}[F(x)]^{r-1}[F(y)-F(x)]^{m-2}[1-F(y)]^{n-s}\right) \\ \widetilde{\Lambda}(x,y,t) &= \widetilde{\Lambda}_1(x,y,t) + \widetilde{\Lambda}_2(x,y,t), \end{aligned}$$

with $D_{n\alpha\beta} = \frac{n!}{(r-1)!(m-2)!(n-s)!}$.

Note that $\overline{Y}(t, b_0(t))$, $\overline{Z}(t, b_0(t))$ and $K(d_0(t), u_0(t))$ are also functions of x and y. So we can find their partial derivatives with respect to x and y. Some simple algebra yields

$$\begin{aligned}
\frac{\partial \widetilde{\Lambda}(x,y,t)}{\partial x} &= (m-2)^{-1}\left(-d_0 - 2ru_0(x+y) + 2(r-1)u_0 b_0\right) \\
&\quad + \left(\frac{\exp(d_0 x + u_0 x^2 - K(d_0,u_0))}{F(y)-F(x)} - \frac{r-1}{(m-2)F(x)}\right) f(x),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial \widetilde{\Lambda}(x,y,t)}{\partial y} &= (m-2)^{-1}\left(-d_0 - 2ru_0(x+y) + 2(r-1)u_0 b_0\right) \\
&\quad - \left(\frac{\exp(d_0 y + u_0 y^2 - K(d_0,u_0))}{F(y)-F(x)} - \frac{n-s}{(m-2)(1-F(y))}\right) f(y),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 \widetilde{\Lambda}(x,y,t)}{\partial x^2} &= \left(\frac{1}{(F(y)-F(x))^2} + \frac{r-1}{(m-2)F^2(x)}\right) f^2(x) \\
&\quad + \left(\frac{1}{F(y)-F(x)} - \frac{r-1}{(m-2)F(x)}\right) f'(x) \\
&\quad - \left(\frac{2ru_0}{m-2} + \frac{\partial^2 K(d_0,u_0)}{\partial^2 x}\right), \\
\frac{\partial^2 \widetilde{\Lambda}(x,y,t)}{\partial y^2} &= \left(\frac{1}{(F(y)-F(x))^2} + \frac{n-s}{(m-2)(1-F(y))^2}\right) f^2(y) \\
&\quad + \left(-\frac{1}{F(y)-F(x)} + \frac{n-s}{(m-2)(1-F(y))}\right) f'(y) \\
&\quad - \left(\frac{2ru_0}{m-2} + \frac{\partial^2 K(d_0,u_0)}{\partial^2 y}\right), \\
\frac{\partial^2 \widetilde{\Lambda}(x,y,t)}{\partial x \partial y} &= -\frac{f(x)f(y)}{[F(y)-F(x)]^2} - \left(\frac{2ru_0}{m-2} + \frac{\partial^2 K(d_0,u_0)}{\partial x \partial y}\right).
\end{aligned}$$

Define

$$\widetilde{\Delta}(t) \equiv \widetilde{\Delta}(x,y,t) = (f(x)f(y))^{-1} \begin{pmatrix} \frac{\partial^2 \widetilde{\Lambda}(x,y,t)}{\partial x^2} & \frac{\partial^2 \widetilde{\Lambda}(x,y,t)}{\partial x \partial y} \\ \frac{\partial^2 \widetilde{\Lambda}(x,y,t)}{\partial y \partial x} & \frac{\partial^2 \widetilde{\Lambda}(x,y,t)}{\partial x^2} \end{pmatrix}.$$

For each t, let $\widetilde{x}_0 = \widetilde{x}_0(t)$, $\widetilde{y}_0 = \widetilde{y}_0(t)$, $\widetilde{d}_0 = \widetilde{d}_0(t)$, $\widetilde{u}_0 = \widetilde{u}_0(t)$ and $\widetilde{b}_0 = \widetilde{b}_0(t)$ be the solutions to

$$\begin{cases} \frac{\partial \Lambda_s(a,\widetilde{b}_0)}{\partial b}\Big|_{d=\widetilde{d}_0, u=\widetilde{u}_0} &= 0 \\ \frac{\partial K(\widetilde{d}_0,\widetilde{u}_0)}{\partial d} &= \overline{Y}(t,\widetilde{b}_0) \\ \frac{\partial K(\widetilde{d}_0,\widetilde{u}_0)}{\partial u} &= \overline{Z}(t,\widetilde{b}_0) \\ \frac{\partial \widetilde{\Lambda}(\widetilde{x}_0,\widetilde{y}_0,t)}{\partial x} &= 0 \\ \frac{\partial \widetilde{\Lambda}(\widetilde{x}_0,\widetilde{y}_0,t)}{\partial y} &= 0. \end{cases} \tag{4.4}$$

We now present the following proposition and theorem whose proofs are given in the Appendix.

Proposition 2: *Let t belong to the support of X. Suppose $t \neq 0$. and let the conditions (C1)–(C2) be satisfied. Then, for any n satisfying $[n\alpha] \geq 1$ and $n - 2[n\alpha] \geq 3$, $\Lambda_s(t,b)$ attains its minimum at some interior point $b_0(t)$ and $\tilde{\Lambda}(x,y,t)$ attains its global minimum at finite point $(\tilde{x}_0, \tilde{y}_0)$ which satisfies equation (4.4).*

Remark 4.1. Proposition 2 ensures that the saddlepoint equation (4.4) always has a solution under Conditions (C1)–(C2). The conditions $[n\alpha] \geq 1$, $n-2[n\alpha] \geq 3$ guarantee that $q_{r,s:n}(x,y)$ is meaningful.

Theorem 3: *Let t belong to the support of X. Suppose $t \neq 0$. In addition to the conditions (C1)–(C2), we assume that*

(i) $f(x) = F^{'}(x)$ and $f^{''}(x)$ exists.

(ii) For any n satisfying $[n\alpha] \geq 1$ and $n-2[n\alpha] \geq 3$,, $(\tilde{x}_0, \tilde{y}_0)$ is unique, i.e. $\tilde{\Lambda}(x,y,t) > \tilde{\Lambda}(\tilde{x}_0, \tilde{y}_0, t)$ if $(x,y) \neq (\tilde{x}_0, \tilde{y}_0)$, and $\widetilde{\Delta}(\tilde{x}_0, \tilde{y}_0, t)$ is positive definite. In addition, the minimum point $b_0(t)$ is unique as (x,y) varies in A_{B,B_0}.(cf also (7.4)).

(iii) $|E^{i\eta_1 X + i\eta_2 X^2}|^{v_1} \in L(R^2)$ for some $v_1 > 0$.

(iv) $\exists w_1 > 0, w_2 > 0$ such that both $|x|(F(x))^{w_1}$ and $y(1-F(y))^{w_2}$ are bounded when $x < 0$ and $y > 0$.

Then, we have

$$\widetilde{g}(t) = \widetilde{g}_{sp}(t)\{1 + m^{-1}\widetilde{R}_n(t)\},$$

where

$$\widetilde{g}_{sp}(t) = \sqrt{\frac{2\pi}{m-2}} \frac{J(t, \widetilde{b}_0)}{G^{1/2}(t, \widetilde{b}_0)\ |\widetilde{\Delta}(\widetilde{x}_0, \widetilde{y}_0, t)|^{1/2}} \exp\Big(-(m-2)\widetilde{\Lambda}(\widetilde{x}_0, \widetilde{y}_0, t)\Big),$$

where $\widetilde{x}_0(t)$, $\widetilde{y}_0(t)$, $\widetilde{d}_0(t)$, $\widetilde{u}_0(t)$ and $\widetilde{b}_0(t)$ are the solutions to equations (4.4).

Note that conditions $(i) - (iii)$ in Theorem 3 are similar to those in Theorem 1. The first three conditions guarantee that $f_{(\bar{Y},\bar{Z})}(y,z)$ has a uniform saddlepoint approximation as x and y vary in some compact set A_{B,B_0}. Since $(\bar{Y},\bar{Z}) \to (a(\bar{Y},\bar{Z}), b(\bar{Y},\bar{Z}))$ is a one-to-one and differentiable transformation, $f_{a(\bar{Y},\bar{Z})}(t)$ has a uniform saddlepoint approximation as x and y vary in A_{B,B_0}, i.e., $\widetilde{r}_m(x,y,t)$ is bounded as x and y vary. The fourth condition implies that the random variable X will have finite moments of arbitrarily small order. It is used in the proof of Lemma 13. We conjecture that it can be removed.

4.3 Saddlepoint approximation to the tail probability of the studentized trimmed mean

In this section, we shall derive a saddlepoint approximation to the tail probability of the studentized trimmed mean by integrating the density approximation obtained in Theorem 3. To simplify notations, let

$$\begin{cases} \widetilde{a}(t) & = \sqrt{\frac{2\pi}{m-2}} J(t, \widetilde{b}_0) G^{-1/2}(t, \widetilde{b}_0) |\widetilde{\Delta}(\widetilde{x}_0, \widetilde{y}_0, t)|^{-1/2} \\ \widetilde{h}(t) & = \widetilde{\Lambda}(\widetilde{x}_0, \widetilde{y}_0, t). \end{cases}$$

Then, we can rewrite $\widetilde{g}_{sp}(t)$ from Theorem 3 as

$$\widetilde{g}_{sp}(t) = \widetilde{a}(t) \exp\{-(m-2)\widetilde{h}(t)\}. \tag{4.5}$$

From the proof of Theorem 3, we see that $\widetilde{h}(t) = \widetilde{\Lambda}(\widetilde{x}_0, \widetilde{y}_0, t)$ achieves its minimum at $t = t_0$. Let

$$v = \sqrt{2(\widetilde{h}(t) - \widetilde{h}(t_0))}\ \ sgn(t - t_0). \tag{4.6}$$

$$\widetilde{\psi}(v) = \sqrt{\frac{2\pi}{m-2}} \widetilde{a}(t(v)) \exp\{-(m-2)\widetilde{h}(t_0)\} \left|\frac{dt}{dv}\right|. \tag{4.7}$$

Then, we have the following theorem whose proof is provided in the Appendix.

Theorem 4: *Under the conditions of Theorem 3, we have*

$$P(T \geq t) = 1 - \Phi(v\sqrt{m-2}) - \frac{\phi(v\sqrt{m-2})}{\sqrt{m-2}} \left(\frac{\widetilde{\psi}(0) - \widetilde{\psi}(v)}{v\widetilde{\psi}(0)} + O(m^{-1}) \right), \tag{4.8}$$

where v and $\widetilde{\psi}(v)$ are given in (4.6) and (4.7).

5. Numerical results.

In this section, we present some numerical evidence of the quality of our saddlepoint approximations. For simplicity, we shall do this only for ordinary trimmed means (cf. Theorems 1 and 2). Different distributions F and varying trimming percentages α and β are chosen in the simulations. The results are presented in Figures 1 and 4 below. In these figures, the left-hand panels give the right-tail probabilities $1-G(x)$, $1-G_{LR}(x)$ and $1-G_{ss}(x)$. On the other hand, the right-hand panel display absolute relative errors; i.e., we plot $\frac{|G_{ss}(x)-G(x)|}{1-G(x)}$ (dashed) and $\frac{|G_{LR}(x)-G(x)|}{1-G(x)}$ (dotted), where G denotes the exact d.f., computed by Monte Carlo using $N = 10^6$ samples from F,while G_{ss} is the integrated saddlepoint density (8), renormalized by dividing through its integral, which is computed by numerical integration; G_{LR} denotes the Lugannani-Rice type approximation given in Theorem 2. We note that in our simulations, $\psi(0)$ in Theorem 2 is calculated approximately by $\psi(a)$ for some small value a very close to zero.

Figure 1 deals with the case where F is standard normal, the trimming percentages α and β are both equal to .10, and the sample size $n = 20$. The results are very satisfactory. In Figure 2, we choose F to be a normal mixture, namely $F(x) = .9\Phi(x) + .1\Phi(x/5)$, the trimming percentages α and β are both equal to .25, and the sample size is again $n = 20$. The results are again very satisfactory, though not as good as in the first example. In this example, we find that, for $|x| > 5.1206$, the determinant appearing in (8) becomes negative; the exact probability that the trimmed mean takes values outside the interval (-5.1206,5.1206) is (estimated by Monte Carlo) 10^{-4}, so that the renormalization factor is in fact a little bit too small. In the first example (F is normal) these difficulties do not arise, as the determinant in (8) is positive for all values of x.

Figures 3 and 4 depict two cases of interest for which we find that the resulting saddlepoint approximations behave much less well as those described in Figures 1 and 2. Figure 3 deals with the normal mixture $F(x) = .7\Phi(x) + .3\Phi(x/5)$, trimming percentages $\alpha = \beta$=.1 and sample size $n = 20$, while in Figure 4 we present results for the case where F is Cauchy, trimming percentages $\alpha = \beta$=.25, and $n = 80$. The reason for taking a sample size as large as eighty in the Cauchy example is that, for smaller sample sizes, the determinant appearing in (8) is positive only for a rather small interval of x-values; the exact probability that the trimmed mean takes values outside this interval is (estimated by Monte-Carlo) $< 10^{-6}$. One way to improve upon this would be the use of higher order saddlepoint approximations to the trimmed mean.

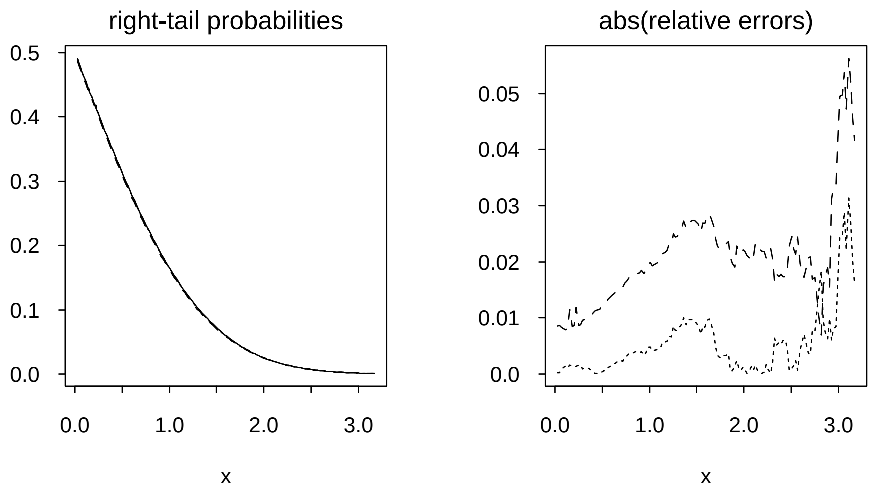

Figure 1: $\{Standard\ Normal,\alpha = \beta = .1, n = 20\}$;
$1 - G(x)$(solid,$N = 10^6$), Theorem2(dotted), $1 - G_{ss}(x)$(dashed);

relative errors w.r.t. exact (1-$G(x)$)

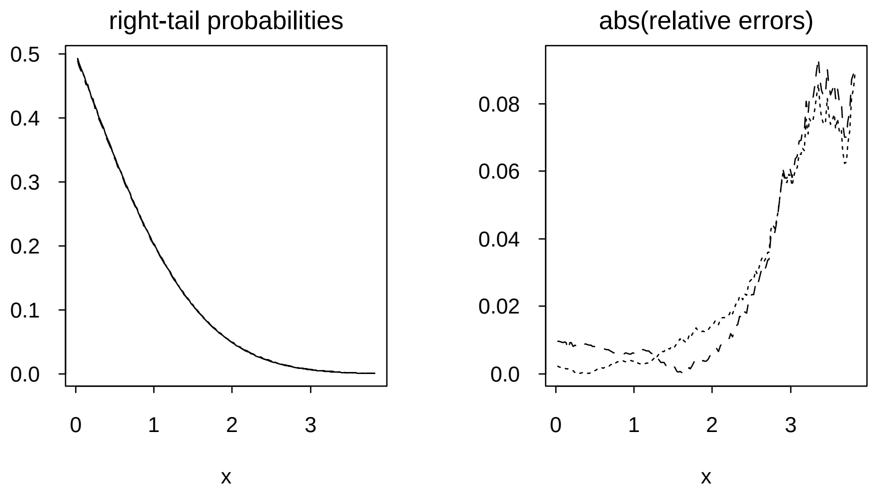

Figure 2: $\{$*Normal Mixture* $.9\Phi(x) + .1\Phi(x/5)$,$\alpha = \beta = .25, n = 20\}$; $1 - G(x)$(solid,$N = 10^6$), Theorem2(dotted), $1 - G_{ss}(x)$(dashed); relative errors w.r.t. exact (1-$G(x)$)

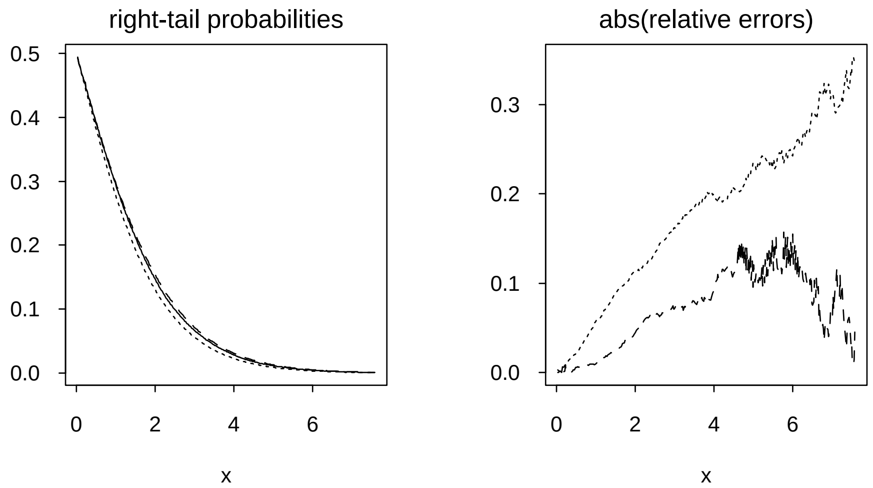

Figure 3: $\{$*Normal Mixture* $.7\Phi(x) + .3\Phi(x/5)$,$\alpha = \beta = .1, n = 20\}$; $1 - G(x)$(solid,$N = 10^6$), Theorem2(dotted), $1 - G_{ss}(x)$(dashed); relative errors w.r.t. exact (1-$G(x)$)

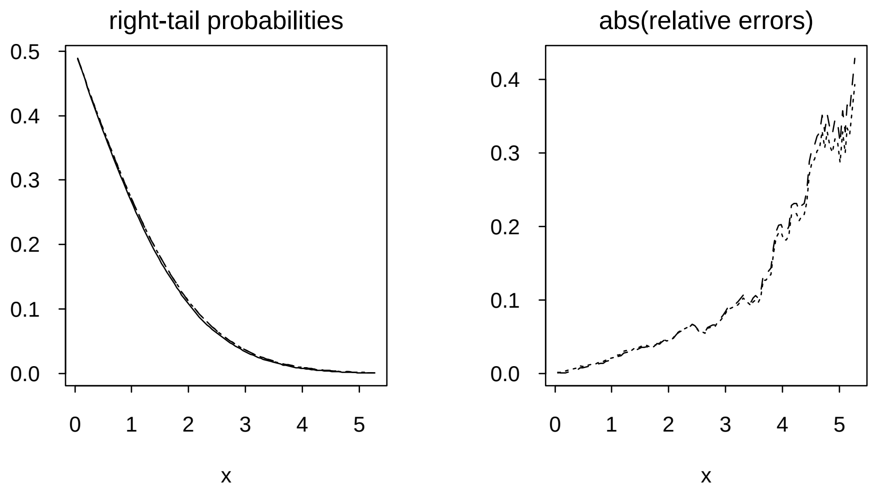

Figure 4: $\{Cauchy\ , \alpha = \beta = .25, n = 80\}$;
$1 - G(x)$(solid,$N = 10^6$), Theorem2(dotted), $1 - G_{ss}(x)$(dashed);
relative errors w.r.t. exact (1-$G(x)$)

6. Saddlepoint approximation to the bootstrap trimmed means

Suppose that we are interested in constructing a $(1-\delta)$ confidence interval for the population trimmed mean $\mu = (1-2\alpha)^{-1}\int_{F^{-1}(\alpha)}^{F^{-1}(1-\beta)} xdF(x)$. Let us assume that the underlying density function is symmetric, so that we can take $\alpha = \beta$. For simplicity, let us consider a one-sided confidence interval $[U, \infty)$, where $U = \overline{X}_{\alpha\alpha} - n^{-1/2}\hat{\tau}_{\alpha\alpha}x_{1-\delta}$ and $x_{1-\delta}$ satisfies $P(T_\alpha \le x_{1-\delta}) = 1-\delta$, where $T_\alpha = \overline{X}_{\alpha\alpha}/\hat{\tau}_{\alpha\alpha}$, our studentized trimmed mean. The coverage probability for this interval is precisely $1-\delta$, since $P(\mu \in [U,\infty)) = 1-\delta$. However, the underlying distribution F is unknown, so $x_{1-\delta}$ is unknown as well, and we can not use the "ideal" interval $[U,\infty)$. One way out of this problem is to employ bootstrap resampling to obtain an empirical approximation to the unknown distribution of T_α, using the data at hand. To be more specific, let $\{X_1^*, \cdots, X_n^*\}$ be a bootstrap resample from the observations $\{X_1, \cdots, X_n\}$. Let

$$T_\alpha^* = \frac{\overline{X^*}_{\alpha\alpha}}{\hat{\tau}_\alpha^*},$$

where $\overline{X^*}_{\alpha\alpha}$ is the trimmed mean of the bootstrap resample $\{X_1^*, \cdots, X_n^*\}$ and

$$\hat{\tau}_\alpha^{*2} = \frac{1}{n(1-2\alpha)^2}\left(\sum_{i=r}^{s}(X_{i:n}^* - \overline{X^*}_{\alpha\beta})^2 + (r-1)[(X_{r:n}^* - \overline{X^*}_{\alpha\beta})^2 + (X_{s:n}^* - \overline{X^*}_{\alpha\beta})^2]\right).$$

Therefore, $P(T_\alpha \le x)$ can be approximated by its bootstrap version $P^*(T_\alpha^* \le x)$, where P^* denotes the conditional distribution given the sample $\{X_1, \cdots, X_n\}$. Consequently, a bootstrap confidence interval for μ is given by $[U^*, \infty)$, where $U^* = \overline{X}^*_{\alpha\alpha} - n^{-1/2}\hat{\tau}_\alpha^* x_{1-\delta}^*$ and $x_{1-\delta}^*$ satisfies $P^*(T_\alpha^* \le x_{1-\delta}^*) = 1-\delta$.

The accuracy of the above bootstrap confidence intervals for μ of course depends very much on how close our bootstrap approximation $P^*(T_\alpha^* \le x)$ is to $P(T_\alpha \le x)$. This has been studied by Hall and Padmanabhan (1992). They show that bootstrap approximation performs better than the normal approximation in the sense that the error term is of smaller order than $n^{-1/2}$. They also derive an Edgeworth expansion to the distribution of the studentized trimmed mean in the non-bootstrap case, and show that it depends on the population density at the quantiles where trimming occurs. However, Hall and Padmanabhan (1992) remark that "the first term of the Edgeworth expansion is very complex and so it will not be written down explicitly". Recently, Gribkova and Helmers (2002) gave a simple explicit formula for the (empirical) Edgeworth expansion of the studentized trimmed mean. The empirical Edgeworth expansion can also be used to replace the bootstrap with absolute error of size $o(n^{-1/2})$.

In this section, we indicate briefly how to apply our saddlepoint approximations to the bootstrap in constructing confidence intervals for the population trimmed mean μ. The saddlepoint approximation intends to provide a fast and accurate approximation to the bootstrap distribution of the (studentized or otherwise) trimmed means in order to avoid the intensive Monte Carlo simulations, which would be needed to compute bootstrap confidence intervals like $[U^*, \infty)$. For related issues, see Davison and Hinkley (1988), Daniels and Young (1991) and Jing and Robinson (1994) among others. As our saddlepoint approximations for studentized trimmed means require a smoothness condition on the underlying distribution some smoothing is necessary in order to be applicable to the bootstrap.

To be more specific, let us define $\hat{f}_h(t)$ to be the kernel density estimator of $f(t)$, i.e.,

$$\hat{f}_h(t) = (nh_n)^{-1} \sum_{i=1}^{n} k\left(\frac{t - X_i}{h_n}\right),$$

where h_n is the bandwidth satisfying $h_n \to 0$ and $n\, h_n \to \infty$ as $n \to \infty$, and $k : \mathcal{R} \to \mathcal{R}$ is a kernel function which is assumed to be a density function. A kernel type estimator of the d.f. $F(t)$ is then given by

$$\hat{F}_h(t) = n^{-1} \sum_{i=1}^{n} K\left(\frac{t - x}{h_n}\right),$$

where $K(\cdot)$ is the distribution function of $k(\cdot)$. Note that the degree of smoothness of $\hat{F}_h(t)$ depends entirely on the smoothness of the kernel function $k(\cdot)$. For our purpose, it suffices for $k(\cdot)$ to be differentiable and its derivative to be continuous. Now instead of drawing resamples from the empirical distribution $\hat{F}_n$, we shall draw resamples from its smoothed version $\hat{F}_h(t)$. This can be achieved as follows. First draw a bootstrap sample $\{X_1^*, \cdots, X_n^*\}$ from $\hat{F}_n$ and then independently draw another sample $\{\epsilon_1, \cdots, \epsilon_n\}$ from the kernel distribution $K(\cdot)$. Then, a random sample from $\hat{F}_h(t)$ can be obtained by

$$X_i^{s*} = X_i{}^* + h_n \epsilon_i, \qquad i = 1, \cdots, n.$$

Therefore, the smoothed bootstrap approximation to $P(T_\alpha \leq x)$ is given by $P^*(T_\alpha^{\epsilon *} \leq x)$, where $T_\alpha^{\epsilon *}$ is similarly defined to $T_\alpha{}^*$, except that the bootstrap resample is now replaced by the above smoothed bootstrap resample. The probability $P^*(T_\alpha^{\epsilon *} \leq x)$, in turn, can be approximated by saddlepoint approximations obtained from Theorem 4 simply by replacing $F(\cdot)$, $f(\cdot)$, and $f'(\cdot)$ appearing (implicitly) in (4.8) by their kernel estimates $\hat{F}_h(t)$, $\hat{f}_h(t)$, and $\hat{f}_h'(t)$, respectively. The resulting saddlepoint approximation is referred to as the empirical (or bootstrap) saddlepoint approximation. See Feuerverger (1989), Jing, Feuerverger and Robinson (1994), Ronchetti and Welsh (1994) and Wang (1992).

7. Appendix

Throughout the appendix, we suppose that t is in the support of X, $x < t < y$. We will use (x, y) to denote the point or the open interval. They can be distinguished from the context.

We shall first present Lemmas 1-6, which will be used to prove Theorem 1.

Lemma 1: *Under Condition (iii) of Theorem 1, there exist some constant M and some even integer u such that*

$$\int_{-\infty}^{\infty} |F(y) - F(x)|^u |Ee^{i\eta Y_1}|^u d\eta \leq 2\pi M.$$

Proof: Let u be the smallest even integer which is greater than or equal to v. Since $Ee^{i\eta X} \in L^v(R)$ and $|Ee^{i\eta X}| \leq 1$, we have

$$Ee^{i\eta X} \in L^u(R). \tag{7.1}$$

Suppose $X_1, X_2, \cdots X_u$ are *i.i.d.* with the same distribution as X. So $|Ee^{i\eta X}|^u$ is the characteristic function of $(X_1+\cdots+X_{\frac{u}{2}})-(X_{\frac{u}{2}+1}+\cdots+X_u)$. (7.1) implies that $|Ee^{i\eta X}|^u \in L^1(R)$. Thus the density function $f_u(z)$ of $(X_1+\cdots+X_{\frac{u}{2}})-(X_{\frac{u}{2}+1}+\cdots+X_u)$ is bounded by some constant M. (See Feller (1971), Chapter XV, Section 3). Now Parseval inequality (See Feller(1971), Chapter XV, Section 3) gives

$$\begin{aligned} & \frac{1}{2\pi}\int_{-\infty}^{\infty} |E\exp i\eta[(Y_1+\cdots+Y_{\frac{u}{2}})-(Y_{\frac{u}{2}+1}+\cdots+Y_u)]|e^{-\frac{1}{2}a^2\eta^2}d\eta \qquad (7.2) \\ = & \frac{1}{\sqrt{2\pi}a}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2a^2}} f_{u(x,y)}(z)dz, \end{aligned}$$

where $Y_1, \cdots, Y_u$ are i.i.d. with the same distribution as Y_1, and $f_{u(x,y)}(z)$ is the density function of $(Y_1+\cdots+Y_{\frac{u}{2}})-(Y_{\frac{u}{2}+1}+\cdots+Y_u)$ and a is some positive constant. Noting the fact that $f_{u(x,y)}(z) \leq \frac{1}{[F(y)-F(x)]^u} f_u(z)$, we have, from (7.2),

$$\begin{aligned} & \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(y)-F(x)|^u |Ee^{i\eta Y_1}|^u e^{-\frac{1}{2}a^2\eta^2}d\eta \qquad (7.3) \\ & \leq \frac{1}{\sqrt{2\pi}a}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2a^2}} f_u(z)dz \\ & \leq M. \end{aligned}$$

In (7.3), letting $a \to 0$, we can complete the proof.

Denote the root of $K'_{Y_1}(\lambda) = t$ by $\tilde{\lambda}$. Let $Y(\tilde{\lambda})$ be the random variable with density function $f_{Y(\tilde{\lambda})}(z) = e^{\tilde{\lambda}z}f(z)I(x \leq z \leq y)/\int_x^y e^{\tilde{\lambda}z}f(z)dz$. For each pair of positive numbers B and B_0 such that $B > B_0, B - B_0 \geq |t|$, define

$$A_{B,B_0} := \{(x,y) : -B \leq x \leq t - B_0, \quad t + B_0 \leq y \leq B\}. \tag{7.4}$$

Lemma 2: *Under Condition (iii) of Theorem 1, we have*

$$\sup_{(x,y)\in A_{B,B_0}} \int_{-\infty}^{\infty} |Ee^{i\eta Y(\tilde{\lambda})}|^u d\eta < \infty, \tag{7.5}$$

where u is the smallest even integer greater than or equal to v.

Proof: Since $K'_{Y_1}(\tilde{\lambda}) = t$, we have

$$\int_x^y (z-t)e^{\tilde{\lambda}z}f(z)dz = 0.$$

Let $p(x,y,\lambda) = \int_x^y (z-t)e^{\lambda z}f(z)dz$. Since $p(x,y,\tilde{\lambda}) = 0$ and $\frac{\partial p(x,y,\tilde{\lambda})}{\partial \lambda} = \int_x^y (z-t)^2 e^{\tilde{\lambda}z}f(z)dz > 0$, it follows from the Implicit Function Theorem that there exists some $\epsilon > 0$ such that $\tilde{\lambda} = \tilde{\lambda}(x,y)$ is a continuous function on $A_\epsilon(x_1,y_1) = \{(x,y) : |x-x_1| \leq \epsilon, |y-y_1| \leq \epsilon\}$ for each point $(x_1,y_1) \in A_{B,B_0}$. Hence $\tilde{\lambda}$ is bounded on $A_\epsilon(x_1,y_1)$.

Define $\varphi(i\eta; x, y) := \frac{1}{F(y)-F(x)} \int_x^y e^{i\eta z} f(z)dz$. Lemma 1 shows that $\varphi(i\eta; x, y) \in L^u(R)$, where u is the smallest even integer greater than or equal to v. By changing the integration path, we have,

$$\begin{aligned}\int_{-\infty}^{\infty} \varphi^u(i\eta; x, y)d\eta &= \frac{1}{i}\int_{-i\infty}^{i\infty} \varphi^u(\eta; x, y)d\eta \\ &= \frac{1}{i}\int_{\tilde{\lambda}(x,y)-i\infty}^{\tilde{\lambda}(x,y)+i\infty} \varphi^u(\eta; x, y)d\eta \\ &= \int_{-\infty}^{\infty} \varphi^u(i\eta + \tilde{\lambda}(x, y); x, y)d\eta.\end{aligned}$$

By the definition of Lebesgue integrability, $\varphi^u(i\eta + \tilde{\lambda}(x, y)) \in L^1(R)$. Hence

$$Ee^{i\eta Y(\tilde{\lambda})} \in L^u(R). \tag{7.6}$$

Suppose $\tilde{Y}_1, \tilde{Y}_2, \cdots, \tilde{Y}_u$ are *i.i.d.* with the same distribution as $Y(\tilde{\lambda})$. So $|Ee^{i\eta Y(\tilde{\lambda})}|^u$ is the characteristic function of $(\tilde{Y}_1 + \cdots + \tilde{Y}_{\frac{u}{2}}) - (\tilde{Y}_{\frac{u}{2}+1} + \cdots + \tilde{Y}_u)$. (7.6) implies that $|Ee^{i\eta Y(\tilde{\lambda})}|^u \in L^1(R)$. Now Parseval inequality gives

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty} |E\exp i\eta[(\tilde{Y}_1 + \cdots + \tilde{Y}_{\frac{u}{2}}) - (\tilde{Y}_{\frac{u}{2}+1} + \cdots + \tilde{Y}_u)]|e^{-\frac{1}{2}a^2\eta^2}d\eta \qquad (7.7)\\ =\ &\frac{1}{\sqrt{2\pi}a}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2a^2}} f_{u(\tilde{\lambda};x,y)}(z)dz,\end{aligned}$$

where $f_{u(\tilde{\lambda};x,y)}(z)$ is the density function of $(\tilde{Y}_1 + \cdots + \tilde{Y}_{\frac{u}{2}}) - (\tilde{Y}_{\frac{u}{2}+1} + \cdots + \tilde{Y}_u)$. Note the following two facts:

1. $f_{u(\tilde{\lambda};x,y)}(z)$ is the convolution of $f_{\tilde{Y}_1}(z), \cdots, f_{\tilde{Y}_{\frac{u}{2}}}(z), f_{-\tilde{Y}_{\frac{u}{2}+1}}(z), \cdots, f_{-\tilde{Y}_u}(z)$, where $f_{\tilde{Y}_1}(z) = \cdots = f_{\tilde{Y}_{\frac{u}{2}}}(z) = f_{Y(\tilde{\lambda})}(z)$, and $f_{-\tilde{Y}_{\frac{u}{2}+1}}(z) = \cdots = f_{-\tilde{Y}_u}(z) = f_{Y(\tilde{\lambda})}(-z)$.

2. $f_u(z)$ is the convolution of $f_{X_1}(z), \cdots, f_{X_{\frac{u}{2}}}(z), f_{-X_{\frac{u}{2}+1}}(z), \cdots, f_{-X_u}(z)$, where $f_{X_1}(z) = \cdots = f_{X_{\frac{u}{2}}}(z) = f(z)$, and $f_{-X_{\frac{u}{2}+1}}(z) = \cdots = f_{-X_u}(z) = f(-z)$.

Since $f_{Y(\tilde{\lambda})}(z) = e^{\tilde{\lambda}z} f(z)I(x \le z \le y)/\int_x^y e^{\tilde{\lambda}z} f(z)dz \le Cf(z)$ for some absolute constant C as (x, y) varies in A_{B,B_0} by the boundedness of $\tilde{\lambda}$, we have $f_{u(\tilde{\lambda};x,y)}(z) \le C^u f_u(z)$. Hence by (7.7)

$$\begin{aligned}&\frac{1}{2\pi}\int_{-\infty}^{\infty} |Ee^{i\eta Y(\tilde{\lambda})}|^u e^{-\frac{1}{2}a^2\eta^2}d\eta \qquad (7.8)\\ \le\ &\frac{1}{\sqrt{2\pi}a}\int_{-\infty}^{\infty} e^{-\frac{z^2}{2a^2}} C^u f_u(z)dz \\ \le\ &C^u M,\end{aligned}$$

where M is the same as in Lemma 1. Letting $a \to 0$ in (7.8) shows $\int_{-\infty}^{\infty} |Ee^{i\eta Y(\tilde{\lambda})}|^u d\eta \le 2\pi C^u M$. So $\sup_{(x,y)\in A_\epsilon(x_1,y_1)} \int_{-\infty}^{\infty} |Ee^{i\eta Y(\tilde{\lambda})}|^u d\eta < \infty$. Since A_{B,B_0} can be covered by a finite number of $A_\epsilon(x_1, y_1)$, we can complete the proof.

Lemma 3: *Let $f(x) = F'(x)$. For arbitrary $\epsilon_1 > 0$, there exists $\delta > 0$ such that if $|\eta| \ge \delta$,*

$$\sup_{(x,y)\in A_{B,B_0}} |Ee^{i\eta Y_1 + \tilde{\lambda}Y_1}/Ee^{\tilde{\lambda}Y_1}| \le \epsilon_1.$$

Proof: Define $\hat{f}(\eta) = \int_x^y e^{i\eta z} e^{\tilde{\lambda} z} f(z)dz$. Then

$$\begin{aligned}
\hat{f}(\eta) &= -\int_x^y e^{i\eta(z+\frac{\pi}{\eta})} e^{\tilde{\lambda} z} f(z)dz \\
&= -\int_{x+\frac{\pi}{\eta}}^{y+\frac{\pi}{\eta}} e^{i\eta z} e^{\tilde{\lambda}(z-\frac{\pi}{\eta})} f(z-\frac{\pi}{\eta})dz.
\end{aligned}$$

So

$$\begin{aligned}
2\hat{f}(\eta) &= \int_{-\infty}^{\infty} [f(z)e^{\tilde{\lambda} z} I(x \le z \le y) - f(z-\frac{\pi}{\eta}) e^{\tilde{\lambda}(z-\frac{\pi}{\eta})} I(x \le z - \frac{\pi}{\eta} \le y)] e^{i\eta z} dz \\
&= \int_{-\infty}^{\infty} \left[f(z)e^{\tilde{\lambda} z} - f(z) e^{\tilde{\lambda}(z-\frac{\pi}{\eta})} I(x \le z \le y) \right] dz \\
&\quad + \int_{-\infty}^{\infty} \left[f(z) e^{\tilde{\lambda}(z-\frac{\pi}{\eta})} I(x \le z \le y) - f(z-\frac{\pi}{\eta}) e^{\tilde{\lambda}(z-\frac{\pi}{\eta})} I(x \le z - \frac{\pi}{\eta} \le y) \right] e^{i\eta z} dz \\
&= \int_x^y f(z) e^{\tilde{\lambda} z} (1 - e^{-\tilde{\lambda}\frac{\pi}{\eta}}) e^{i\eta z} dz \qquad (7.9) \\
&\quad + \int_x^{x+\frac{\pi}{\eta}} f(z) e^{\tilde{\lambda}(z-\frac{\pi}{\eta})} e^{i\eta z} dz - \int_y^{y+\frac{\pi}{\eta}} f(z-\frac{\pi}{\eta}) e^{\tilde{\lambda}(z-\frac{\pi}{\eta})} e^{i\eta z} dz \\
&\quad + \int_{x+\frac{\pi}{\eta}}^{y} \left[f(z) - f(z-\frac{\pi}{\eta}) \right] e^{\tilde{\lambda}(z-\frac{\pi}{\eta})} e^{i\eta z} dz.
\end{aligned}$$

From the proof of Lemma 2, we know that $\tilde{\lambda}$ is a continuous function on $A_\epsilon(x_1, y_1)$. Hence $\tilde{\lambda}$ is bounded on each $A_\epsilon(x_1, y_1)$. The compactness of A_{B,B_0} shows that $\tilde{\lambda}$ is also bounded on A_{B,B_0}. Thus $1 - e^{-\tilde{\lambda}\frac{\pi}{\eta}} \to 0$ uniformly on A_{B,B_0} as $|\eta| \to \infty$. This implies that

$$\int_x^y f(z) e^{\tilde{\lambda} z} (1 - e^{-\tilde{\lambda}\frac{\pi}{\eta}}) e^{i\eta z} dz \to 0 \qquad (7.10)$$

uniformly on A_{B,B_0} as $|\eta| \to \infty$. Since $F(x)$ is absolutely continuous with respect to the Lebesgue measure, it follows from Theorem 6.11 of Rudin (1987) that

$$\int_x^{x+\frac{\pi}{\eta}} f(z)dz \to 0$$

uniformly in x as $|\eta| \to \infty$. Hence

$$\begin{aligned}
&| \int_x^{x+\frac{\pi}{\eta}} f(z) e^{\tilde{\lambda}(z-\frac{\pi}{\eta})} e^{i\eta z} dz| \qquad (7.11) \\
&\le \int_x^{x+\eta} f(z)dz \sup_{(x,y)\in A_{B,B_0}, x \le z \le y} e^{\tilde{\lambda}(z-\frac{\pi}{\eta})} \\
&\to 0
\end{aligned}$$

uniformly on A_{B,B_0} as $|\eta| \to \infty$. Similarly,

$$\int_y^{y+\frac{\pi}{\eta}} f(z-\frac{\pi}{\eta}) e^{\tilde{\lambda}(z-\frac{\pi}{\eta})} e^{i\eta z} dz \to 0 \qquad (7.12)$$

uniformly on A_{B,B_0} as $|\eta| \to \infty$. Since

$$\int_{x+\frac{\pi}{\eta}}^{y} [f(z) - f(z-\frac{\pi}{\eta})] e^{\tilde{\lambda}(z-\frac{\pi}{\eta})} e^{i\eta z} dz \le \sup_{(x,y)\in A_{B,B_0}, x \le z \le y} e^{\tilde{\lambda}(z-\frac{\pi}{\eta})} \int_{-\infty}^{\infty} |f(z) - f(z-\frac{\pi}{\eta})| dz$$

and $\int_{-\infty}^{\infty}|f(z)-f(z-\frac{\pi}{\eta})|dz \to 0$ as $|\eta| \to \infty$ (Theorem 9.5 of Rudin(1987)), we have

$$\int_{x+\frac{\pi}{\eta}}^{y}[f(z)-f(z-\frac{\pi}{\eta})]e^{\tilde{\lambda}(z-\frac{\pi}{\eta})}e^{i\eta z}dz \to 0 \tag{7.13}$$

uniformly on A_{B,B_0} as $|\eta| \to \infty$.

Combining (7.9)-(7.13), we see that

$$\hat{f}(\eta) \to 0 \quad uniformly \quad on \quad A_{B,B_0} \quad as \quad |\eta| \to \infty.$$

Since $Ee^{\tilde{\lambda}Y_1}$ is bounded away from 0 as $(x,y) \in A_{B,B_0}$, we can complete the proof.

Lemma 4: *Suppose Conditions (i) and (iii) of Theorem 1 hold. Then $f_{\bar{Y}}(t)$ has a uniform saddlepoint approximation as (x,y) varies in A_{B,B_0}, i.e.*

$$f_{\bar{Y}}(t) = \sqrt{\frac{m}{2\pi K^{"}_{Y_1}(\tilde{\lambda})}}\exp\{-m[\tilde{\lambda}t - K_{Y_1}(\tilde{\lambda})]\}(1+m^{-1}r_m(x,y,t)), \tag{7.14}$$

where $|r_m(x,y,t)|$ is bounded by some absolute constant C_0.

Proof: Denote the mean and variance of $Y(\tilde{\lambda})$ by $\tilde{\mu}$ and $\tilde{\sigma}^2$, respectively.
Define $T(\tilde{\lambda}) = \frac{1}{\sqrt{m}\tilde{\sigma}}\sum_{j=1}^{m}(Y_j(\tilde{\lambda})-\tilde{\mu})$, where $Y_1(\tilde{\lambda}),\cdots,Y_m(\tilde{\lambda})$ are *i.i.d.* with the same distribution as $Y(\tilde{\lambda})$. In order to prove (7.14), it suffices to prove that the Edgeworth expansion of the density $f_{T(\tilde{\lambda})}(t)$ of $T(\tilde{\lambda})$ has a uniform error as (x,y) varies in A_{B,B_0}, *i.e.*

$$f_{T(\tilde{\lambda})}(t) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}t^2}[1+\frac{\tilde{\mu}_3}{6\tilde{\sigma}^3\sqrt{m}}(t^3-3t)] + m^{-1}r_m(t), \tag{7.15}$$

where $\tilde{\mu}_3 = E(Y(\lambda)-\tilde{\mu})^3$, $|r_m(t)|$ is bounded by some finite constant C_1 as (x,y) varies in A_{B,B_0}.

Indeed, Lemma 1 guarantees that

$$|m^{-1}r_m(t)| \le N_m := \frac{1}{2\pi}\int_{-\infty}^{\infty}|\varphi_1^m(\frac{i\eta}{\tilde{\sigma}\sqrt{m}}) - e^{-\frac{1}{2}\eta^2} - \frac{\tilde{\mu}_3}{6\tilde{\sigma}^3\sqrt{m}}(i\eta)^3e^{-\frac{1}{2}\eta^2}|d\eta, \tag{7.16}$$

where $\varphi_1(i\eta) = Ee^{i\eta(Y(\tilde{\lambda})-\tilde{\mu})}$.

By Lemma 3,$\forall\epsilon_1 < 1, \exists\delta > 0$ such that if $|\eta| \ge \delta, \sup_{(x,y)\in A_{B,B_0}}|\varphi_1(i\eta)| \le \epsilon_1$. Hence the contribution of the interval $(-\infty, -\delta\tilde{\sigma}\sqrt{m}) \cup (\delta\tilde{\sigma}\sqrt{m}, +\infty)$ to the integral in (7.16) is at most

$$\delta^{m-u}\int_{-\infty}^{\infty}|\varphi_1(\frac{i\eta}{\tilde{\sigma}\sqrt{m}})|^u d\eta + \int_{|\eta|>\delta\tilde{\sigma}\sqrt{m}} e^{-\frac{1}{2}\eta^2}(1+|\frac{\tilde{\mu}_3\eta^3}{\tilde{\sigma}^3}|)d\eta,$$

which decrease to 0 faster than any power of $\frac{1}{m}$ if we note Lemma 2 and the fact that $\tilde{\mu}_3$ is uniformly bounded and $\tilde{\sigma}^2$ is uniformly bounded away from 0 and ∞ as (x,y) varies in A_{B,B_0}.

Define $\psi(\eta) = \log\varphi_1(i\eta) + \frac{1}{2}\tilde{\sigma}^2\eta^2$. So we have

$$N_m = \frac{1}{2\pi}\int_{|\eta|\le\delta\tilde{\sigma}\sqrt{m}} e^{-\frac{1}{2}\eta^2}|\exp(m\psi(\frac{\eta}{\tilde{\sigma}\sqrt{m}})) - 1 - \frac{\tilde{\mu}_3}{6\tilde{\sigma}^3\sqrt{m}}(i\eta)^3|d\eta + o(\frac{1}{m}) \tag{7.17}$$

uniformly on A_{B,B_0} as $m \to \infty$.

The integrand can be estimated by the following well-known inequality (cf. Feller (1971))

$$\begin{aligned} |e^\alpha - 1 - \beta| &\le |e^\alpha - e^\beta + e^\beta - 1 - \beta| \\ &\le (|\alpha-\beta| + \frac{1}{2}\beta^2)e^\gamma, \end{aligned} \tag{7.18}$$

where $\gamma \geq \max(|\alpha|, |\beta|)$.

The function $\psi(\eta)$ is four times continuously differentiable, and $\psi(0) = \psi^{'}(0) = \psi^{''}(0) = 0$, $\psi^{'''}(0) = i^3\tilde{\mu}_3$. Since $\psi^{(4)}(\eta)$ is continuous, we can choose δ such that if $|\eta| < \delta$, $|\psi^{(4)}(\eta)|$ is uniformly bounded by some finite constant as $(x, y) \in A_{B,B_0}$. By the four term Taylor expansion we have

$$|\psi(\eta) - \frac{1}{6}\tilde{\mu}_3(i\eta)^3| \leq C_2\tilde{\sigma}^4|\eta|^4, \quad |\eta| \leq \delta, \tag{7.19}$$

for some finite constant C_2 as $(x, y) \in A_{B,B_0}$.

Next we shall choose sufficiently small δ so that

$$|\psi(\eta)| \leq \frac{1}{4}\tilde{\sigma}^2\eta, \quad |\frac{1}{6}\tilde{\mu}_3(i\eta)^3| \leq \frac{1}{4}\tilde{\sigma}^2\eta^2, \quad |\eta| \leq \delta, \tag{7.20}$$

uniformly as $(x, y) \in A_{B,B_0}$.

Thus if δ is so small that (7.19) and (7.20) hold, the integrand is at most

$$e^{-\frac{1}{4}\eta^2}(\frac{C_2}{m}\eta^4 + \frac{\tilde{\mu}_3^2}{72m}\eta^6).$$

This shows that (7.15) holds. So does (7.14).

Lemma 5: *If $F(x)$ is continuous at $x = t$, where t is in the support of X, then for any n satisfying $[n\alpha] \geq 2$, $[n\beta] \geq 2$ and $n - [n\alpha] - [n\beta] \geq 1$, $\Lambda(x, y, t)$ attains its minimum at some finite point (x_0, y_0).*

Proof: Suppose (x_n, y_n) is an arbitrary sequence in $\Omega(t)$. We will prove the following five assertions.

(I). If $x_n \to -\infty$, $y_n \to y_0$, where $t < y_0 \leq \infty$, then $\Lambda(x_n, y_n, t) \to \infty$.

(II). If $x_n \to x_0$, $y_n \to +\infty$, where $-\infty \leq x_0 < t$, then $\Lambda(x_n, y_n, t) \to \infty$.

(III). If $x_n \to t$, $y_n \to t$, then $\Lambda(x_n, y_n, t) \to \infty$.

(IV). If $x_n \to t$, $y_n \to y_0$, where $t < y_0 \leq \infty$, then $\Lambda(x_n, y_n, t) \to \infty$.

(v). If $x_n \to x_0$, $y_n \to t$, where $-\infty \leq x_0 < t$, then $\Lambda(x_n, y_n, t) \to \infty$.

Since $\Lambda(x, y, t) = \tilde{\lambda}t - \log\int_x^y e^{\tilde{\lambda}z}dF(z) - m^{-1}\log(C_{n\alpha\beta}[F(x)]^{r-2}[1 - F(y)]^{n-s-1})$, noting that

$$x_n \to -\infty \quad implies \quad F(x_n) \to 0,$$

$$y_n \to +\infty \quad implies \quad F(y_n) \to 1,$$

$$x_n \to t \quad and \quad y_n \to t \quad implies \quad \tilde{\lambda} \to t \quad and \quad \int_x^y e^{\tilde{\lambda}z}dF(z) \to 0,$$

we have the assertions of (I)-(III). Now we turn to the proof of (IV).

Since $K_{Y_1}^{'}(\tilde{\lambda}) = t$, we have

$$\int_x^y (t - z)e^{\tilde{\lambda}z}dF(z) = 0. \tag{7.21}$$

For each (x_n, y_n), we have a solution $\tilde{\lambda}_n$ to (7.21). Hence we have a sequence $\{\tilde{\lambda}_n, n \geq 1\}$. Now consider a convergent subsequence $\{\tilde{\lambda}_{n_k}, k = 1, 2, \cdots,\}$ of $\{\tilde{\lambda}_n, n \geq 1\}$. Hence we suppose $\tilde{\lambda}_{n_k} \to \lambda_0$. From (7.21), we have

$$\int_{x_{n_k}}^t (t - z)e^{\tilde{\lambda}_{n_k}z}dF(z) = \int_t^{y_{n_k}} (z - t)e^{\tilde{\lambda}_{n_k}z}dF(z). \tag{7.22}$$

If λ_0 is finite, the left-hand side of (7.22) goes to 0 but the right-hand side of (7.22) goes to some positive number as $x_{n_k} \to t$. If λ_0 is $+\infty$, we can consider the following formula

$$\int_{x_{n_k}}^t (t - z)e^{\tilde{\lambda}_{n_k}(z-t)}dF(z) = \int_t^{y_{n_k}} (z - t)e^{\tilde{\lambda}_{n_k}(z-t)}dF(z),$$

which is obtained from (7.22). The left side of the above formula goes to 0 but the right side goes to ∞. Therefore $\lambda_0 = -\infty$. And we can conclude that $\tilde{\lambda}_n \to -\infty$ as $n \to \infty$. Noting that $e^a \le 1 + ae^a$ if $a \ge 0$, we have for x_n sufficiently close to t,

$$\begin{aligned}\int_{x_n}^{t} e^{\tilde{\lambda}(z-t)} dF(z) &\le \int_{x_n}^{t} dF(z) + \int_{x_n}^{t} \tilde{\lambda}(z-t)e^{\tilde{\lambda}(z-t)} dF(z) \\ &= \int_{x_n}^{t} dF(z) - \int_{t}^{y_n} \tilde{\lambda}(z-t)e^{\tilde{\lambda}(z-t)} dF(z),\end{aligned} \tag{7.23}$$

where in the last equality we have used (7.22). Since $\tilde{\lambda}(z-t)e^{\tilde{\lambda}(z-t)}$ is bounded and goes to 0 for each $z > t$, we have $\int_t^{y_n} \tilde{\lambda}(z-t)e^{\tilde{\lambda}(z-t)} dF(z) \to 0$ by dominated convergence theorem. Hence $\int_{x_n}^{t} e^{\tilde{\lambda}(z-t)} dF(z) \to 0$ as $x \to t$. Therefore,

$$\int_{x_n}^{y_n} e^{\tilde{\lambda}(z-t)} dF(z) \to 0 \quad as \quad x \to t. \tag{7.24}$$

Observe that $\tilde{\lambda}t - \log \frac{\int_{x_n}^{y_n} e^{\tilde{\lambda}z} dF(z)}{F(y_n)-F(x_n)} = -\log \frac{\int_{x_n}^{y_n} e^{\tilde{\lambda}(z-t)} dF(z)}{F(y_n)-F(x_n)}$. Therefore we have proved (IV). The proof of (V) is the same as that of (IV).

Now (I)-(V) implies that $\Lambda(x, y, t)$ attains its minimum at some finite point (x_0, y_0) in $\Omega(t)$.

Remark A.1. Since $\Lambda(x, y, t)$ is differentiable in both x and y, Lemma 5 implies that (x_0, y_0) satisfies the equation (3.5). So Lemma 5 is just our Proposition 1.

Lemma 6: *Under the conditions of Theorem 1, for suitably chosen B and B_0 which are independent of n,*

$$\iint_{\Omega(t)/A_{B,B_0}} f_{\bar{Y}}(t) q_{r-1,s+1:n}(x, y) dxdy / \exp(-m\Lambda(x_0, y_0, t))$$

goes to 0 faster than any power of $\frac{1}{m}$.

Proof: From Lemma 1, $|Ee^{i\eta Y_1}|^u$ is integrable. So we can apply Fourier inversion theorem to get

$$\begin{aligned} f_{\bar{Y}}(t) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\eta t} E e^{i\eta \bar{Y}} d\eta \\ &= \frac{m}{2\pi} \int_{-\infty}^{\infty} e^{-i\eta m t} (E e^{i\eta Y_1})^m d\eta. \end{aligned}$$

Again, Lemma 1 shows that

$$|f_{\bar{Y}}(t)| \le \frac{m}{2\pi} \int_{-\infty}^{\infty} |Ee^{i\eta Y_1}|^m d\eta \le \frac{Mm}{|F(y) - F(x)|^u}.$$

Hence

$$\begin{aligned} &\iint_{\Omega(t)/A_{B,B_0}} f_{\bar{Y}}(t) q_{r-1,s+1:n}(x, y) dxdy \\ \le &\iint_{\Omega(t)/A_{B,B_0}} \frac{Mm}{[F(y) - F(x)]^u} q_{r-1,s+1:n}(x, y) dxdy \\ = &\iint_{\Omega(t)/A_{B,B_0}} Mm \exp\{-m\Lambda_3(x, y)\} f(x) f(y) dxdy, \end{aligned} \tag{7.25}$$

where $\Lambda_3(x,y) = -m^{-1}\log(C_{n\alpha\beta}[F(x)]^{r-2}[F(y)-F(x)]^{m-u}[1-F(y)]^{n-s-1})$.

Since $\Lambda(x,y,t) = \tilde{\lambda}t - \log\int_x^y e^{\tilde{\lambda}z}dF(z) - m^{-1}\log(C_{n\alpha\beta}[F(x)]^{r-2}[1-F(y)]^{n-s-1}) \leq \Lambda'(x,y,t) := \tilde{\lambda}t - \log\int_x^y e^{\tilde{\lambda}z}dF(z) - m^{-1}\log C_{n\alpha\beta} - \frac{2\alpha}{1-\alpha-\beta}\log F(x) - \frac{2\beta}{1-\alpha-\beta}\log(1-F(y))$ for sufficiently large m, we have $\Lambda(x_0,y_0,t) \leq \Lambda'(x_0',y_0',t) := \inf_{x<t<y}\Lambda'(x,y,t)$. The existence and finiteness of (x_0',y_0') can be proved as that of (x_0,y_0). From the expression of $\Lambda'(x,y,t)$, we see that x_0', y_0' are independent of n. Note the fact that $\lim_{x\to-\infty}F(x)=0$, $\lim_{y\to\infty}F(y)=1$, $\quad \lim_{x\to t, y\to t}(F(y)-F(x))=0$. $\forall\epsilon'>0$, we can choose positive numbers B and B_0 independent of n such that

$$\inf_{(x,y)\in\Omega(t)/A_{B,B_0}} \Lambda_3(x,y) > \Lambda'(x_0',y_0',t)+\epsilon' \geq \Lambda(x_0,y_0,t)+\epsilon'. \tag{7.26}$$

It follows from (7.26) that $\frac{1}{\exp\{-m\Lambda(x_0,y_0,t)\}}\iint_{\Omega(t)/A_{B,B_0}} Mm\exp\{-m\Lambda_3(x,y)\}f(x)f(y)dx\ dy$ goes to 0 faster than any power of m^{-1}. By (7.25), we can complete the proof.

Proof of Theorem 1: Lemma 6 assures us of the exponential smallness of

$$\iint_{\Omega(t)/A_{B,B_0}} f_{\bar{Y}}(t)q_{r-1,s+1:n}(x,y)dxdy/\exp(-m\Lambda(x_0,y_0,t)).$$

To complete the proof of Theorem 1, we need to consider the asymptotic expansion of

$$\iint_{A_{B,B_0}} f_{\bar{Y}}(t)\ q_{r-1,s+1:n}(x,y)\ dxdy.$$

Lemma 4 gives

$$\iint_{A_{B,B_0}} f_{\bar{Y}}(t)\ q_{r-1,s+1:n}(x,y) = \iint_{A_{B,B_0}}\sqrt{\frac{m}{2\pi K''_{Y_1}(\tilde{\lambda})}}f(x)f(y)\exp[-m\Lambda(x,y,t)] \times dxdy\{1+O(m^{-1})\}.$$

So we obtain a double integral of Laplace type. Conditions (i) and (ii) guarantee that we can use the formula (8.2.55) of Bleistein and Handelsman(1986) to get

$$\iint_{A_{B,B_0}}\sqrt{\frac{m}{2\pi K''_{Y_1}(\tilde{\lambda})}}f(x)f(y)\exp[-m\Lambda(x,y,t)]dxdy\{1+O(m^{-1})\}$$

$$= \sqrt{\frac{2\pi}{m}}\frac{\exp\{-m\Lambda(x_0(t),y_0(t),t)\}}{\sqrt{K''_{Y_1}(\tilde{\lambda}_0(t))|_{x=x_0(t),y=y_0(t)}|\Delta_0(t)|}}(1+O(\frac{1}{m})).$$

This completes the proof.

The following Lemmas 7-14 will be used to prove Theorem 3. Since the proofs of these lemmas are similar to those of Lemmas 1-4, we shall omit the details here.

Lemma 7: *Under Condition (iii) of Theorem 3, there exist some constant M_1 and some even integer u_1 such that*

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|F(y)-F(x)|^{u_1}|Ee^{i\eta_1 Y_1+i\eta_2 Y_1^2}|^{u_1}d\eta_1 d\eta_2 \leq (2\pi)^2 M_1.$$

Let d_0, u_0 be the solutions to the equations $\frac{\partial K(d_0,u_0)}{\partial d} = \bar{Y}(t,b)$, $\frac{\partial K(d_0,u_0)}{\partial u} = \bar{Z}(t,b)$.

Lemma 8: *Under Condition (iii) of Theorem 3, we have*

$$\sup_{(x,y)\in A_{B,B_0}} \iint_{R^2} |\exp(K(d_0+i\eta_1, u_0+i\eta_2) - K(d_0,u_0))|^{u_1} d\eta_1 d\eta_2 < \infty, \tag{7.27}$$

where u_1 is the smallest even integer greater than or equal to v_1.

Lemma 9: *Let $f(x) = F^{'}(x)$. For arbitrary $\epsilon_1' > 0$, there exists $\delta' > 0$ such that if $|\eta_1| + |\eta_2| \geq \delta'$, then*

$$\sup_{(x,y)\in A_{B,B_0}} |\exp(K(d_0+i\eta_1, u_0+i\eta_2) - K(d_0,u_0))| \leq \epsilon_1'. \tag{7.28}$$

Lemma 10: *Suppose Conditions (i) and (iii) of Theorem 3 hold. Then the density function $f_{(\bar{Y},\bar{Z})}(\bar{Y}(t,b),\bar{Z}(t,b))$ has a uniform saddlepoint approximation as (x,y) varies in A_{B,B_0}, i.e.*

$$\begin{aligned} f_{(\bar{Y},\bar{Z})}(\bar{Y}(t,b),\bar{Z}(t,b)) &= \frac{m-2}{2\pi}\Delta_s^{-\frac{1}{2}}(t,b)\exp[-(m-2)\Lambda_s(t,b)] \\ &\quad \times(1+m^{-1}\bar{r}_m(x,y,t)), \end{aligned} \tag{7.29}$$

where $|\bar{r}_m(x,y,t)|$ is bounded by some absolute constant C_0'.

Lemma 11:

$$\sup_{d,u}[d\bar{Y} + u\bar{Z} - K(d,u)] \to \infty \quad as \quad \bar{Y} \to x \quad from \quad the \quad right. \tag{7.30}$$

$$\sup_{d,u}[d\bar{Y} + u\bar{Z} - K(d,u)] \to \infty \quad as \quad \bar{Y} \to y \quad from \quad the \quad left. \tag{7.31}$$

Proof: We only prove (7.30). (7.31) is similar.

Since $\sup_{d,u}[d\bar{Y} + u\bar{Z} - K(d,u)] \geq \sup_d[d\bar{Y} - K(d,0)] = \tilde{d}\bar{Y} - K(\tilde{d},0)$, where $\tilde{d}$ satisfies the equation

$$\int_x^y \bar{Y}e^{\tilde{z}}dF(z) = \int_x^y ze^{\tilde{d}z}dF(z). \tag{7.32}$$

It suffices to prove $\tilde{d}\bar{Y} - K(\tilde{d},0) \to \infty$ as $\bar{Y} \to x$ from the right.

Let $h(d,\bar{Y}) = \int_x^y (z-\bar{Y})e^{dz}dF(z)$. Since $h(\tilde{d},\bar{Y}) = 0$, we have $\frac{\partial \tilde{d}}{\partial \bar{Y}} = \frac{\int_x^y e^{\tilde{d}z}dF(z)}{\int_x^y (z-\bar{Y})^2 e^{\tilde{d}z}dF(z)}$. Hence $\tilde{d}$ is a increasing function of $\bar{Y}$. Then if $\bar{Y} \to x$ from the right, we can suppose $\tilde{d} \to \tilde{d}_0$. From (7.32), we have

$$\int_x^{\bar{Y}} (\bar{Y}-z)e^{\tilde{d}z}dF(z) = \int_{\bar{Y}}^y (z-\bar{Y})e^{\tilde{d}z}dF(z). \tag{7.33}$$

If $\tilde{d}_0$ is finite, the left side of (7.33) goes to 0 but the right side goes to some positive number as $\bar{Y} \to x$.. Therefore $\tilde{d}_0 = -\infty$. Noting that $e^a \leq 1 + ae^a$ if $a \geq 0$, we have for $\bar{Y}$ sufficiently close to x,

$$\begin{aligned} \int_x^{\bar{Y}} e^{\tilde{d}(z-\bar{Y})}dF(z) &\leq \int_x^{\bar{Y}} dF(z) + \int_x^{\bar{Y}} \tilde{d}(z-\bar{Y})e^{\tilde{d}(z-\bar{Y})}dF(z) \\ &= \int_x^{\bar{Y}} dF(z) - \int_{\bar{Y}}^y \tilde{d}(z-\bar{Y})e^{\tilde{d}(z-\bar{Y})}dF(z), \end{aligned} \tag{7.34}$$

where in the last equality we have used (7.33). Since $\tilde{d}(z-\bar{Y})e^{\tilde{d}(z-\bar{Y})}$ is bounded and goes to 0 for each $z > \bar{Y}$, we have $\int_{\bar{Y}}^{y} \tilde{d}(z-\bar{Y})e^{\tilde{d}(z-\bar{Y})}dF(z) \to 0$ by dominated convergence theorem. Hence $\int_{x}^{\bar{Y}} e^{\tilde{d}(z-\bar{Y})}dF(z) \to 0$ as $\bar{Y} \to x$ from the right. Therefore, $\tilde{d}\bar{Y} - K(\tilde{d},0) = -\log\int_x^y \frac{e^{\tilde{d}(z-\bar{Y})}}{F(y)-F(x)}dF(z) \to \infty$ as $\bar{Y} \to x$ from the right.

Remark A.2. Since $b \equiv b(\overline{Y},\overline{Z}) = \left(\frac{m-2}{m}\overline{Y} + \frac{x+y}{m}\right)$, Lemma 11 implies that the equation $\frac{\partial\Lambda_s(t,b)}{\partial b}|_{d=d_0(t),u=u_0(t)} = 0$ has a solution $b = b_0(t) \in (x,y)$.

Lemma 12: *If $F(x)$ is contionuous at $x = t$, where t is in the support of X, then for any n satisfying $[n\alpha] \geq 1$ and $n - 2[n\alpha] \geq 3$, $\tilde{\Lambda}(x,y,t)$ attains its minimum at some finite point $(\tilde{x}_0,\tilde{y}_0)$.*

Proof: Suppose (x_n,y_n) is an arbitrary sequence in $\Omega(t)$. We will prove the following five assertions.

(I'). If $x_n \to -\infty$, $y_n \to y_0$, where $t < y_0 \leq \infty$, then $\tilde{\Lambda}(x_n,y_n,t) \to \infty$.

(II'). If $x_n \to x_0$, $y_n \to +\infty$, where $-\infty \leq x_0 < t$, then $\tilde{\Lambda}(x_n,y_n,t) \to \infty$.

(III'). If $x_n \to t$, $y_n \to t$, then $\tilde{\Lambda}(x_n,y_n,t) \to \infty$.

(IV'). If $x_n \to t$, $y_n \to y_0$, where $t < y_0 \leq \infty$, then $\tilde{\Lambda}(x_n,y_n,t) \to \infty$.

(V'). If $x_n \to x_0$, $y_n \to t$, where $-\infty \leq x_0 < t$, then $\tilde{\Lambda}(x_n,y_n,t) \to \infty$.

The proof of $(I') - (III')$ is similar to that of Lemma 5. Now we turn to the proof of (IV'). Since

$$\begin{aligned}
\tilde{\Lambda}_1(x,y,t) &= d_0(t)\bar{Y}(t,b_0(t)) + u_0(t)\bar{Z}(t,b_0(t)) - K(d_0(t),u_0(t)) \\
&= \sup_{d,u}[d\bar{Y}(t,b_0(t)) + u\bar{Z}(t,b_0(t)) - K(d,u)] \\
&\geq \sup_{d}[d\bar{Y}(t,b_0(t)) - K(d,0)],
\end{aligned}$$

it suffices to prove

$$\sup_d[d\bar{Y}(t,b_0(t)) - K(d,0)] \to \infty$$

as $x_n \to t$ and $y_n \to y_0$. This can be proved similarly as that of Lemma 5. It remains to prove (V'). Since $\tilde{\Lambda}_1(x,y,t) \geq \sup_u[u\bar{Z}(t,b_0(t)) - K(0,u)]$. This again follows similar lines of Lemma 5. $(I') - (V')$ give the assertion of Lemma 12.

Remark A.3. Lemma 12 implies that $(\tilde{x}_0,\tilde{y}_0)$ satisfies equation (4.4). So equation (4.4) has at least one solution $\tilde{x}_0, \tilde{y}_0, \tilde{d}_0, \tilde{u}_0, \tilde{b}_0$. Combining Lemmas 11 and 12 gives Proposition 2.

Lemma 13: *Under the conditions of Theorem 3, for suitably chosen B and B_0 which are independent of n,*

$$\int\int_{\Omega(t)/A_{B,B_0}} f_{a(\bar{Y},\bar{Z}}(t)q_{r,s:n}(x,y)dxdy/\exp(-(m-2)\tilde{\Lambda}(\tilde{x}_0,\tilde{y}_0,t))$$

goes to 0 faster than any power of $\frac{1}{m}$.

Proof: Lemma 7 shows $|Ee^{i\eta_1 Y_1 + i\eta_2 Z_1}|^{u_1}$ is integrable. So we can apply Fourier inversion theorem to get

$$\begin{aligned}
f_{(\bar{Y},\bar{Z})}(z_1,z_2) &= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i\eta_1 z_1 - i\eta_2 z_2}Ee^{i\eta_1\bar{Y}+i\eta_2\bar{Z}}d\eta_1 d\eta_2 \\
&= \frac{(m-2)^2}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(m-2)\eta_1 z_1 - i(m-2)\eta_2 z_2}(Ee^{i\eta_1 Y_1 + i\eta_2 Z_1})^{m-2}d\eta_1 d\eta_2.
\end{aligned}$$

Hence using Lemma 7, we have

$$f_{(\bar{Y},\bar{Z})}(z_1,z_2) \le \frac{(m-2)^2}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Ee^{i\eta_1 Y_1+i\eta_2\bar{Z}}|^{m-2}d\eta_1 d\eta_2 \le \frac{M_1(m-2)^2}{|F(y)-F(x)|^{u_1}}.$$

So

$$\begin{aligned} f_{a(\bar{Y},\bar{Z})}(t) &= \int_x^y f_{(\bar{Y},\bar{Z})}(\bar{Y}(t,b),\bar{Z}(t,b))|J|db \\ &\le \frac{M_1(m-2)^2}{|F(y)-F(x)|^{u_1}}\frac{2n(1-2\alpha)^2 m}{(m-2)^2t^3}\int_x^y b^2 db \\ &= \frac{M_1(m-2)^2}{|F(y)-F(x)|^{u_1}}\frac{2n(1-2\alpha)^2 m}{(m-2)^2t^3}\frac{y^3-x^3}{3}. \end{aligned}$$

Thus

$$\begin{aligned} &\iint_{\Omega(t)/A_{B,B_0}} f_{a(\bar{Y},\bar{Z})}(t)q_{r,s,:n}(x,y)dxdy \\ &\le \iint_{\Omega(t)/A_{B,B_0}} \frac{M_1(m-2)^2}{|F(y)-F(x)|^{u_1}}\frac{2n(1-2\alpha)^2 m}{(m-2)^2t^3}\frac{y^3-x^3}{3}q_{r,s:n}(x,y)dxdy \\ &\le \frac{2M_1(1-2\alpha)^2}{3t^3}\iint_{\Omega(t)/A_{B,B_0}} nm\exp\{-(m-2)\Lambda_3'(x,y)\} \\ &\quad (y^3-x^3)(F(x))^{3w_1}(1-F(y))^{3w_2}f(x)f(y)dxdy, \end{aligned} \tag{7.35}$$

where

$$\Lambda_3'(x,y) = -(m-2)^{-1}\log(D_{n\alpha\beta}[F(x)]^{r-1-3w_1}[F(y)-F(x)]^{m-2-u_1}[1-F(y)]^{n-s-3w_2}).$$

Condition (iv) of Theorem 3 implies that $(y^3-x^3)(F(x))^{3w_1}(1-F(y))^{3w_2}$ is bounded. Hence from (7.35)

$$\begin{aligned} &\iint_{\Omega(t)/A_{B,B_0}} f_{a(\bar{Y},\bar{Z})}(t)q_{r,s,:n}(x,y)dxdy \\ &\le M_2\iint_{\Omega(t)/A_{b,B_0}} nm\exp\{-(m-2)\Lambda_3'(x,y)\}f(x)x(y)dxdy, \end{aligned} \tag{7.36}$$

where M_2 is some absolute constant. As in the proof of Lemma 6, given $\epsilon_2>0$, we can select B and B_0 which are independent of n such that for n sufficiently large,

$$\inf_{\Omega(t)/A_{B,B_0}} \Lambda_3'(x,y) > \tilde{\Lambda}(\tilde{x}_0,\tilde{y}_0,t)+\epsilon_2. \tag{7.37}$$

Combining (7.35)-(7.37), we can complete the proof.

Lemma 14: *Given $t\neq 0$. Under Conditions $(C_1)-(C_2)$, (i) and (ii) of Theorem 3, $f_{a(\bar{Y},\bar{Z})}(t)$ has a uniform saddlepoint approximation as (x,y) varies in A_{B,B_0}, i.e.*

$$\begin{aligned} f_{a(\bar{Y},\bar{Z})}(t) &= \sqrt{\frac{m-2}{2\pi}}\ J(t,b_0(t))G^{-1/2}(t,b_0(t))\ \exp[-(m-2)\Lambda_s(t,b_0(t))] \\ &\quad \times\{1+m^{-1}\widetilde{r}_m(x,y,t)\}, \end{aligned}$$

where $|\widetilde{r}_m(x,y,t)|$ *is bounded as* (x,y) *varies in* A_{B,B_0}.

Proof: First, we will show that

$$\partial^2\Lambda_s(t,b_0(t))/\partial b^2 > 0 \tag{7.38}$$

as (x,y) varies in A_{B,B_0}. Lemma 11 and Remark A.2. imply $\partial\Lambda_s(t,b_0(t))/\partial b = 0$. Simple calculations show that

$$\begin{aligned}
& \partial\Lambda_s(t,b_0(t))/\partial b \\
= \;& d_0\frac{m}{m-2} + u_0(m-2)^{-1}(\frac{2n(1-2\alpha)^2}{t^2}b_0(t) + 2(m-2r+2)b_0(t) + 2(r+1)(x+y)), \\
& \partial^2\Lambda_s(t,b_0(t))/\partial b^2 \\
= \;& u_0(m-2)^{-1}(\frac{2n(1-2\alpha)^2}{t^2} + 2(m-2r+2)).
\end{aligned}$$

Since $(m-2)^{-1}(\frac{2n(1-2\alpha)^2}{t^2}+2(m-2r+2)) \to \frac{2(1-2\alpha)^2}{t^2(1-\alpha-\beta)}+2(m-2\alpha) > 0$, we see that $\partial^2\Lambda_s(t,b_0(t))/\partial b^2 > 0$ iff $u_0 > 0$ for sufficiently large n. Now we suppose $u_0 = 0$. Then $\partial\Lambda_s(t,b_0(t))/\partial b = 0$ gives $d_0 = 0$. Define x_0,y_0,b_0,t_0 by the following formula,

$$F(x_0) = \frac{r-1}{n-2}, \quad 1-F(y_0) = \frac{n-s}{n-2}. \tag{7.39}$$

$$b_0 = \left(\frac{(m-2)\int_{x_0}^{y_0} zdF(z)}{m(F(y_0)-F(x_0))} + \frac{x_0+y_0}{m}\right) \tag{7.40}$$

$$\begin{aligned}
t_0 = \;& \sqrt{n}(1-2\alpha)b_0\left(\frac{(m-2)\int_{x_0}^{y_0} z^2dF(z)}{F(y_0)-F(x_0)}\right. \\
& -(m-2r+2)\left(\frac{(m-2)\int_{x_0}^{y_0} zdF(z)}{m(F(y_0)-F(x_0))} + \frac{x_0+y_0}{m}\right)^2 + r(x_0+y_0)^2 \\
& \left.-2(r-1)(x_0+y_0)\left[\frac{(m-2)\int_{x_0}^{y_0} zdF(z)}{m(F(y_0)-F(x_0))} + \frac{x_0+y_0}{m}\right]\right)^{-1/2}.
\end{aligned} \tag{7.41}$$

Calculation shows that $\tilde{x}_0(t_0) = x_0$, $\tilde{y}_0(t_0) = y_0$, $\tilde{d}_0(t_0) = 0$, $\tilde{u}_0(t_0) = 0$ and $\tilde{b}_0(t_0) = b_0$ are the solutions to equations (4.4). Now from (7.39), we can easily see that $x_0-\xi_\alpha = O\left(n^{-1}\right)$ and $y_0-\xi_{1-\alpha} = O\left(n^{-1}\right)$. Furthermore, we have $\xi_\alpha + \xi_{1-\alpha} = 0$. Then, from these equations and the definition of t_0, we get $|t_0| = O\left(n^{-1}\right)$. Contradiction because t is a fixed non-zero number.

It is also impossible that $u_0 \to 0$ as $n \to \infty$. Otherwise $u_0 \to 0$ implies that $d_0 \to 0$. Equation (7.41) shows $t_0 \to 0$.

Hence we can suppose $\partial^2\Lambda_s(t,b_0(t))/\partial b^2$ is positive and bounded away from 0 as (x,y) varies in A_{B,B_0}.

Next we will show $\exists$ some fixed δ_f, such that for n sufficiently large,

$$\partial^2\Lambda_s(t,b)/\partial b^2 > 0$$

if $b \in (b_0(t)-\delta_f, b_0(t)+\delta_f)$ as (x,y) varies in A_{B,B_0}. Otherwise there exists a sequence $\{\delta_n\}$ such that $\delta_n \to 0$ as $n \to \infty$ and $\partial^2\Lambda_s(t,b_0(t)+\delta_n)/\partial b^2 \le 0$. Since A_{B,B_0} is compact, we can suppose $\lim_{n\to\infty}(b_0(t)+\delta_n) = b_0^*$. At the same time $\lim_{n\to\infty} b_0(t) = b_0^*$. Note the uniform convergence of $\partial^2\Lambda_s(t,b)/\partial b^2$ in any compact set as $n \to \infty$ when $\partial^2\Lambda_s(t,b)/\partial b^2$ is regarded as a function of x,y,b. We have $\partial^2\Lambda_s(t,b_0^*)/\partial b^2 \le 0$. But we have already shown that $\partial^2\Lambda_s(t,b_0(t))/\partial b^2$ is positive and bounded away from 0 as (x,y) varies in A_{B,B_0} for sufficiently large n. Contradiction.

So

$$
\begin{aligned}
& f_{a(\bar{Y},\bar{Z})}(t) \\
= & \int_x^y f_{(a(\bar{Y},\bar{Z}),b(\bar{Y},\bar{Z}))}(t,b)db \\
= & \frac{m-2}{2\pi}\int_x^y \Delta_s^{-\frac{1}{2}}(t,b)J(t,b)\exp[-(m-2)\Lambda_s(t,b)](1+\bar{r}_m(x,y,t)) \\
= & \frac{m-2}{2\pi}\left(\int_{|b-b_0(t)|\le\delta_f}+\int_x^{b_0(t)-\delta_f}+\int_{b_0(t)+\delta_f}^y\right) \\
& \Delta_s^{-\frac{1}{2}}(t,b)J(t,b)\exp[-(m-2)\Lambda_s(t,b)](1+\bar{r}_m(x,y,t))db.
\end{aligned}
$$

Laplace approximation gives the result. The uniform error comes from the compactness of A_{B,B_0}.

Proof of Theorem 3: Lemma 13 ensures the exponential smallness of

$$
\int\int_{\Omega(t)/A_{B,B_0}} f_{a(\bar{Y},\bar{Z})}(t)q_{r,s:n}(x,y)dxdy/\exp(-(m-2)\tilde{\Lambda}(\tilde{x}_0,\tilde{y}_0,t)).
$$

To complete the proof of Theorem 3, we need to consider the asymptotic expansion of

$$
\int\int_{A_{B,B_0}} f_{a(\bar{Y},\bar{Z})}(t)\ q_{r,s:n}(x,y)\ dxdy.
$$

Lemma 14 implies

$$
\begin{aligned}
& \int\int_{A_{B,B_0}} f_{a(\bar{Y},\bar{Z})}(t)q_{r,s:n}(x,y)dxdy \\
= & \int\int_{A_{B,B_0}} \sqrt{\frac{m-2}{2\pi}}\frac{J(t,b_0(t))}{G^{1/2}(t,b_0(t))}\exp[-(m-2)\tilde{\Lambda}(x,y,t)] \\
& \times f(x)f(y)\{1+m^{-1}\tilde{r}_m(x,y,t)\}dxdy,
\end{aligned}
$$

So we obtain a double integral of Laplace type. Condition $(i),(ii)$ guarantee that we can use the formula (8.2.55) of Bleistein and Handelsman(1986) to get

$$
\begin{aligned}
& \int\int_{A_{B,B_0}} \sqrt{\frac{m-2}{2\pi}}\frac{J(t,b_0(t))}{G^{1/2}(t,b_0(t))}\exp[-(m-2)\tilde{\Lambda}(x,y,t)] \\
& \times f(x)f(y)\{1+m^{-1}\tilde{r}_m(x,y,t)\}dxdy \\
= & \sqrt{\frac{2\pi}{m-2}}\frac{J(t,\widetilde{b}_0)}{G^{1/2}(t,\widetilde{b}_0)|\tilde{\Delta}(\tilde{x}_0,\tilde{y}_0,t)|^{1/2}}\exp[-(m-2)\tilde{\Lambda}(\tilde{x}_0,\tilde{y}_0,t)]\{1+m^{-1}\widetilde{R}_n(t)\}.
\end{aligned}
$$

This completes the proof.

Proof of Theorem 4. Setting $v=0$ in (4.7), we get

$$
\widetilde{\psi}(0)=\widetilde{a}(t_0)\ \exp\{-(m-2)\ \widetilde{h}(t_0)\}\ \left|\frac{dt}{dv}\right|_{v=0}.
$$

Note that $\widetilde{h}'(t) = \widetilde{\Lambda}'_x \widetilde{x}'_0(t) + \widetilde{\Lambda}'_y \widetilde{y}'_0(t) + \widetilde{\Lambda}'_t = \widetilde{\Lambda}'_t$. Using this and differentiating (4.6) result in

$$\begin{aligned}
\frac{dv}{dt} &= \frac{\widetilde{h}'(t)}{v} = \frac{\widetilde{\Lambda}'_t}{v} = \frac{\widetilde{\Lambda}'_{1t}}{v} \\
&= \frac{1}{v} \cdot \frac{d\Lambda_s(t, \widetilde{b}_0(t))}{dt} = v^{-1} \left(\Lambda'_{sa}(t, \widetilde{b}_0(t)) + \Lambda'_{sb}(t, \widetilde{b}_0(t)) b_0^{*'}(t) \right) \\
&= v^{-1} \; \Lambda'_{sa}(t, \widetilde{b}_0(t)) = \left(\frac{dv}{dt} \right)^{-1} \frac{d\Lambda'_{sa}(t, \widetilde{b}_0(t))}{dt} \\
&= \left(\frac{dv}{dt} \right)^{-1} \left(\Lambda''_{saa}(t, \widetilde{b}_0(t)) + \Lambda''_{sab}(t, \widetilde{b}_0(t)) b_0^{*'}(t) \right).
\end{aligned}$$

Differentiating $\Lambda'_{sb}(t, \widetilde{b}_0(t)) = 0$ with respect to t, we find

$$\frac{d\widetilde{b}_0(t)}{dt} = -\frac{\Lambda''_{sab}(t, \widetilde{b}_0(t))}{\Lambda''_{sbb}(t, \widetilde{b}_0(t))}.$$

Therefore,

$$\begin{aligned}
\frac{dv}{dt} &= \left(\Lambda''_{saa}(t, \widetilde{b}_0(t)) - \frac{(\Lambda''_{sab}(t, \widetilde{b}_0(t)))^2}{\Lambda''_{sbb}(t, \widetilde{b}_0(t))} \right)^{1/2} \\
\widetilde{\psi}(0) &= \sqrt{\frac{2\pi}{m-2}} \frac{J(t_0, b_0) \cdot (\frac{dt}{dv}) \mid_{v=0} \cdot \exp\{-(m-2)\widetilde{h}(t_0)\}}{|\Delta_s(t_0, b_0)|^{1/2} \cdot |\Lambda''_{sbb}(t_0, b_0)|^{1/2} \cdot |\widetilde{\Delta}(t_0)|^{-1/2}}.
\end{aligned}$$

From (4.5)–(4.7) and using an integration by parts similarly to Theorem 3.2.1 of Jensen (1995), we get

$$\begin{aligned}
\int_t^\infty \widetilde{g}_{sp}(t)dt &= \int_v^\infty \widetilde{\psi}(v) \exp\{-(m-2)v^2/2\} dv, \\
&= \left(1 - \Phi(v\sqrt{m-2})\right) \widetilde{\psi}(0) + \int_v^\infty \left(\widetilde{\psi}(v) - \widetilde{\psi}(0) \right) \exp\left\{-(m-2)v^2/2\right\} dv. \\
&= \left(1 - \Phi(v\sqrt{m-2})\right) \widetilde{\psi}(0) - \frac{\phi(v\sqrt{m-2})}{\sqrt{m-2}} \left(\frac{\widetilde{\psi}(0) - \widetilde{\psi}(v)}{v} + O(m^{-1}) \right).
\end{aligned}$$

From this, we get $\int_{-\infty}^\infty \widetilde{g}_{sp}(t)dt = \widetilde{\psi}(0)$. Finally we have

$$\begin{aligned}
P(T \geq t) &= \int_t^\infty \widetilde{g}_{sp}(t)\{1 + m^{-1}\widetilde{R}_n(t)\}dt \\
&= \int_t^\infty \widetilde{g}_{sp}(t)dt \Big/ \int_{-\infty}^\infty \widetilde{g}_{sp}(t)dt \\
&= 1 - \Phi(v\sqrt{m-2}) - \frac{\phi(v\sqrt{m-2})}{\sqrt{m-2}} \left(\frac{\widetilde{\psi}(0) - \widetilde{\psi}(v)}{v\widetilde{\psi}(0)} + O(m^{-1}) \right),
\end{aligned}$$

where, going through from the first line to the second one above, we have used the relation between the integration of the saddlepoint density approximations and renormalization outlined in Jing and Robinson (1994). This completes our proof.

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