Centrum voor Wiskunde en Informatica

REPORTRAPPORT

PNA

Probability, Networks and Algorithms

Probability, Networks and Algorithms

A goodness of fit statistic for the geometric distribution

J.A. Ferreira

REPORT PNA-R0309 JULY 31, 2003

CWI is the National Research Institute for Mathematics and Computer Science. It is sponsored by the Netherlands Organization for Scientific Research (NWO).
CWI is a founding member of ERCIM, the European Research Consortium for Informatics and Mathematics.

CWI's research has a theme-oriented structure and is grouped into four clusters. Listed below are the names of the clusters and in parentheses their acronyms.

Probability, Networks and Algorithms (PNA)

Software Engineering (SEN)

Modelling, Analysis and Simulation (MAS)

Information Systems (INS)

P.O. Box 94079, 1090 GB Amsterdam (NL)
Kruislaan 413, 1098 SJ Amsterdam (NL)
Telephone +31 20 592 9333
Telefax +31 20 592 4199

ISSN 1386-3711

A Goodness of fit Statistic for the Geometric Distribution

J.A. Ferreira
CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands
ferreira@bio.vu.nl

ABSTRACT
We propose a goodness of fit statistic for the geometric distribution and compare it in terms of power, via simulation, with the chi-square statistic. The statistic is based on the Lau-Rao theorem and can be seen as a discrete analogue of the total time on test statistic. The results suggest that the test based on the new statistic is generally superior to the chi-square test.

2000 Mathematics Subject Classification: 60G51, 60G17.
Keywords and Phrases: Geometric distribution, goodness of fit, characterization, empirical process, total time on test statistic, chi-square test, power study, simulation.

1 Introduction

Let $X_1, \ldots, X_n$ be independent random variables taking values on $\mathbb{N}$, with common distribution function (d.f.) F, and denote by F_n the associated empirical d.f. Let $\mathcal{G}$ be the family of geometric distributions on $\mathbb{N}$ with generic element $F_0(x;\alpha) = 1-(1-\alpha)^{[x]}$, $x \geq 0$, $0 < \alpha < 1$ ($[x]$ denoting as usual the integer part of x). For testing the hypothesis that $F \in \mathcal{G}$ on the basis of the sample $X_1, \ldots, X_n$, we consider the *integral statistic*

$$\begin{aligned} I_n &= n^{1/2} \sum_{i=1}^{\infty} \bar{F}_n(i)\,[F_n(i) - F_n(i-1)] - \\ &\quad \alpha_n \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} (j-i-1)\,[F_n(j) - F_n(j-1)]\,[F_n(i) - F_n(i-1)]\,, \end{aligned} \tag{1.1}$$

where $\alpha_n := \bar{X}_n^{-1}$, $\bar{X}_n$ is the sample average, and $\bar{F}_n := 1 - F_n$.

Put

$$\sigma^2(\alpha) = \frac{\alpha^3 (1-\alpha)^2 \{1 + (1-\alpha)^2\}}{\{1-(1-\alpha)^2\}\{1-(1-\alpha)^3\}\{1-(1-\alpha)^4\}}, \quad 0 < \alpha < 1. \tag{1.2}$$

As explained below, under the hypothesis that $F \in \mathcal{G}$, the asymptotic distribution of $I_n^* := I_n \Big/ \sqrt{\sigma^2(\alpha_n)}$ is standard normal. Thus a test of approximate size γ based on I_n^* consists of rejecting the null hypothesis if $|\,I_n^*\,| > z_{1-\gamma/2}$, where z_p is the quantile of probability p of the standard normal distribution.

The purpose of this note is to advocate the use of I_n as a goodness of fit statistic for the geometric model. The statistic is easy to calculate, and its scaled version I_n^* has a convenient asymptotic null distribution, being applicable in most practical situations.

Besides the classical chi-square statistic and the statistic proposed by Vit (1974), there are by now several goodness of fit statistics for the geometric distribution; see the paper of Bracquemond et al. (2002) and the references therein. However, given the relevance of this model in applied science, it seems still of some interest to consider competing statistics.

The rationale behind the definition of I_n is explained in Section 2. We show how the statistic is obtained as an integral of a certain empirical process, whose definition is based on a characterization result known as the Lau-Rao theorem, and that it can be regarded as a discrete version of the well-known 'total time on test statistic', which is widely used for testing exponentiality (e.g. Hollander and Proschan (1975)). We also point out that the integral statistic may not be consistent against all alternatives.

In Section 3 we present a small simulation study comparing the power of the integral and chi-square statistics for testing the geometric distribution against some negative binomial, shifted-Poisson and logarithmic alternatives. Our results suggest that the test based on I_n is generally superior to the chi-square test. We also include some recommendations on the use of the integral statistic.

In practice, the chi-square statistic is still the most popular goodness of fit statistic for the geometric distribution (and most discrete distributions), and that is why we have chosen it here as the standard for comparison. As to the alternatives chosen, we have focused on models which are quite close to the geometric distribution; 'distant models' do not seem interesting to us because they are easy to reject with the sample sizes used here, and because in practical situations researchers do not usually want to test a model with data that manifestly violates it.

Before proceeding, let us note that the test procedure outlined above needs to be slightly rectified. If the event $[\alpha_n = 1] = [X_i = 1,\, i = 1, \ldots, n] = [I_n = 0]$ occurs, then (see (1.2)) so does $[\sigma^2(\alpha_n) = 0]$, and then I_n^* is not defined. In this event, we shall *reject* the hypothesis that F is geometric.—This makes sense because a sample whose elements are all 1 provides evidence of a distribution degenerate at 1, a model excluded from our definition of $F_0(\,\cdot\,; \alpha)$.

2 Definition of the Statistic

Let μ be a σ-finite measure on $[0, +\infty)$ such that $\mu\{0\} < 1$, and f a non-negative, Borel-measurable, locally integrable (with respect to Lebesgue measure) function not identically equal to zero and satisfying the functional equation

$$f(x) = \int_{[0,\infty)} f(x+y)\, \mu(dy) \quad \text{for almost all } x \geq 0. \tag{2.1}$$

Then, according to the Lau-Rao theorem (see Rao and Shanbhag (1994), p. 29), f must be essentially proportional to the exponential or geometric functions. To be precise, *either* μ is arithmetic with some span $\lambda > 0$ (i.e., μ is concentrated on the semi-lattice $\{\lambda, 2\lambda, \ldots\}$) and $f(x+n\lambda) = f(x)b^n$ for almost all $x \geq 0$ and $n = 0, 1, 2, \ldots$, *or* μ is non-arithmetic and $f(x) \propto e^{\eta x}$

for almost all $x \geq 0$, where the constants b and η are determined from $\sum_{n=0}^{\infty} b^n \mu(\{n\lambda\}) = 1$ and $\int_{[0,\infty)} f(y)\, \mu(dy) = 1$, respectively.

Now let $f = \bar{F} := 1 - F$ for some probability distribution function F concentrated on $[0, +\infty)$. If we impose certain restrictions on the supports of μ and F (namely that F be assumed arithmetic with the same span λ as μ whenever μ is arithmetic, or non-arithmetic whenever μ is non-arithmetic), then we conclude that f satisfies (2.1) if and only if F is the geometric distribution on the semi-lattice $\{\lambda, 2\lambda, \dots\}$, or the exponential distribution, or the mixture of one of these and the degenerate distribution at the origin.

We have recently introduced an empirical process associated with (2.1), defined in terms of the sample $X_1, \dots, X_n$ by

$$Z_n(x) = \bar{F}_n(x) - \int \bar{F}_n(x + y)\mu_n(dy), \quad x \geq 0,$$

where μ_n is a certain sample analogue of μ. The properties of F_n, together with the characterization theorem just stated, suggest that when F is *essentially* exponential or geometric the process Z_n should behave in a symmetric fashion around zero, and that such pattern should occur only when F is one of those distributions. Using empirical process theory, we have proved a precise form of this statement, and also the weak convergence of a normalized version of Z_n to a Gaussian process. But what matters to us here is the idea, implied by the previous argument, that integral statistics such as $\int Z_n(x) dF_n(x)$ and $\int Z_n^2(x) dF_n(x)$ should be generally sensitive to departures from the exponential or geometric distributions.

To give a motivating example suppose F is continuous and $\mu_n(dy) = \alpha_n dy$ on $[0, +\infty)$ (the sample analogue of $\mu(dy) = \alpha dy$) with α_n as in Section 1. Then it can be checked that (i) Z_n is related to the 'total time on test process' studied by Csörgő et al. (1986), and consequently that (ii) $n^{1/2} Z_n$ is, under the null hypothesis of exponentiality, asymptotically Gaussian with mean zero and covariance function r defined by $r(s,t) = F(s)(1 - F(t))$, $s \leq t$, and finally that (iii) the integral $\int Z_n(x) dF_n(x)$ is a linear function of the total time on test statistic (e.g. Hollander and Proschan (1975), p. 590), and hence its asymptotic distribution is normal.

Several power studies (e.g. Stephens (1986)) have shown that both $\int Z_n(x) dF_n(x)$ and $\int Z_n^2(x) dF_n(x)$ with $\mu_n(dy) = \alpha_n dy$ are powerful statistics for testing the exponential distribution; moreover, this choice of μ_n seems to be one of the most sensible for omnibus statistics among the many possible measures μ_n one can take in the definition of Z_n. These considerations suggest that a good choice of μ_n for testing the geometric distribution should be $\mu_n(dy) = \alpha_n[1 + dy]$ on $[0, +\infty)$ (the measure concentrated on $\{0, 1, 2, \dots\}$ having mass α_n at each point, which is the sample analogue of $\mu(dy) = \alpha[1 + dy]$).

Thus, assume F is concentrated on $\mathbb{N}$ and $\mu_n(dy) = \alpha_n[1 + dy]$ on $[0, +\infty)$. Then the integral $n^{1/2} \int Z_n(x) dF_n(x)$ is precisely I_n, which (in analogy with the previous example) can be seen as the discrete analogue of the total time on test statistic, and the following are facts of interest: (i) $n^{1/2} Z_n$ is, under the null hypothesis that F is geometric, asymptotically Gaussian with zero means and covariance function r given by $r(s,t) = F(s)(1 - F(t + 1))$, $s \leq t$; (ii) I_n is, under the null hypothesis that F is geometric, asymptotically normal with mean zero and variance $\sigma^2(\alpha)$, where α is the (unknown) parameter of the geometric model and σ^2 has been defined in (1.2).

These statements follow from the following proposition, which is a special case of Theorem 5.1 and Proposition 6.1 of Ferreira (2003):

Proposition *Let F be concentrated on $\mathbb{N}$, have mean $1/\theta$ for some $\theta \in (0, \infty)$, and satisfy $\int \bar{F}(x)^{1/2} dx < \infty$. Put $Z(x) = \bar{F}(x) - \int \bar{F}(x+y)\theta[dy+1]$, $x \geq 0$, and $Z_n^* = n^{1/2}(Z_n - Z)$, where $Z_n(x) = \bar{F}_n(x) - \int \bar{F}_n(x+y)\theta_n[dy+1]$, $x \geq 0$, $\theta_n := \bar{X}_n^{-1}$ and F_n is the empirical distribution function based on a random sample $X_1, \ldots, X_n$ with distribution function F. Then as $n \to \infty$*

$$Z_n^* \to^d W \quad \text{and} \quad I_n(F) := n^{1/2}\left[\int Z_n dF_n - \int Z\, dF\right] \to^d \int W\, dF - \int Y\, dZ,$$

where $Y := B \circ F$, B is a brownian bridge and W a Gaussian process defined in terms of Y by $W(x) = \int Y(x+t)\theta[dt+1] - Y(x) - \theta^2 \int \bar{F}(x+t)[dt+1] \int Y(t)dt$, $x \geq 0$.

We note that this result is tailored to test $\int ZdF = 0$ against $\int ZdF \neq 0$. One can prove that the tests for the exponential and geometric models based on $\int Z_n dF_n$ with a general μ_n are consistent only against alternatives F satisfying $\int ZdF \neq 0$. As pointed out by Spurrier (1984), p. 1645 (see also Example 6.1 in Ferreira (2003)), there exist F other than the exponential that satisfy $\int ZdF = 0$ with $\mu(dy) = \alpha\, dy$, and the total time on test statistic is not consistent against such alternatives. Similarly, there should be alternatives F different from the geometric distribution satisfying $\int ZdF = 0$ with $\mu(dy) = \alpha[1 + dy]$, and against these I_n^* will not be consistent.

Making the analogy with the total time on test statistic, known to be a powerful and versatile statistic, this shortcoming of the integral statistic does not seem very serious. It can always be overcome by calculating the chi-square statistic for those sets of data that look suspiciously non-geometric (e.g. with too large a variance) but are not rejected by I_n^*. In any case, the seemingly greater sensitivity of I_n^* over the chi-square statistic is enough to recommend it *at least* as a supplementary tool. [The use of the quadratic integral $\int Z_n^2(x)dF_n(x)$ would avoid consistency problems, but this can be used only for testing the exponential distribution because its asymptotic distribution depends on the unknown parameter α unless the data are continuous and μ_n is suitably chosen.]

3 Comparison with the Chi-square Statistic via Simulation

We shall compare the performances of the test based on I_n and the chi-square test against several alternatives that are relatively similar to the geometric distribution. The tests will have a nominal size of 0.10.

Since the application of the chi-square test is not completely definite, namely because there are different methods of choosing classes and critical points, we need to decide exactly on how the test is to be performed. We shall use a common approach suitable for discrete distributions on $\mathbb{N}$: Partition the sample space into the C classes $\{1\}$, $\{2\}$,...,$\{C-1\}$ and $\{C, C+1, \ldots\}$, where C is the smallest integer satisfying $P_{\alpha_n}\{C\} < 5/n$, n is the sample size, P_α is the family of probability measures postulated by the null hypothesis and defined in terms of an unknown parameter α, and α_n is an appropriate estimate of α. In this way, each of the classes $\{1\}$, $\{2\}$,...,$\{C-1\}$ contains an 'estimated' expected number of sample observations ≥ 5, while the same number for the class $\{C, C+1, \ldots\}$ will not be, in principle, much less than 5. (This well-known rule aims at improving the approximation to the chi-square distribution.)

Writing $e_i = nP_{\alpha_n}\{i\}$, $i = 1, \ldots, C-1$, $e_C = nP_{\alpha_n}\{C, C+1, \ldots\}$, and denoting by o_i the number of sample points in the i-th class, the chi-square statistic is calculated as $\mathrm{X}_n^2 = \sum_{i=1}^{C}(o_i - e_i)^2/e_i$. It is known that, under the null hypothesis, X_n^2 is asymptotically a random variable bounded below and above by a chi-square random variable with $C-2$ degrees of freedom and a chi-square random variable with $C-1$ degrees of freedom, respectively. The chi-square test of (approximate or nominal) size 0.10 consists of rejecting the null hypothesis if X_n^2 exceeds the quantile of probability 0.90 of one (usually the first) of these two random variables.

We find that the chi-square distribution with $C-2$ degrees of freedom provides a good approximation to the distribution of X_n^2 when the model is geometric. Therefore, in what follows we use the quantile of probability 0.90 of the chi-square distribution with $C-2$ degrees of freedom, which we designate as usual by $\chi^2_{C-2,0.90}$. (See Table 1 for an assessment of this procedure.)

In the case of the geometric distribution, the number of classes C is the smallest integer i satisfying $i > 1 - \log(n\alpha_n/5) / \log(1-\alpha_n)$. For small sample sizes and certain values of α_n one finds sometimes that $C = 2$, which precludes the use of $\chi^2_{C-2,0.90}$. Whenever this happens, we take the number of classes as 3.

Another particularity of our case is that we can frequently observe $\alpha_n = 1$ if α is near 1 and the sample is small. In such occurrences the test based on I_n consists of rejecting the null hypothesis; we shall also convention that the chi-square test rejects the null hypothesis if $\alpha_n = 1$. The possible effect of these last two dispositions on the results is very small, and for sample sizes of $n \geq 50$ it can even be ignored.

We can check the appropriateness of the chi-square test now described and the use of the normal 90% quantiles for testing with I_n by comparing the actual significance levels with the nominal 0.10 significance level. Table 1 shows estimates of the actual sizes of the tests based on X_n^2 and I_n when the model is geometric with parameter $\alpha = 0.15, 0.25, 0.50, 0.75, 0.85$. The estimates were obtained by simulation and consist of proportions of rejections of the null hypothesis (that the model is geometric) out of 10,000 trials. Standard errors are omitted for lack of space, but they can be readily calculated using the information from the table and the size of the simulated samples. [We have used the pseudo-random number generator of Wichmann and Hill (1982). The results for X_n^2 and I_n were obtained with the same sequence of pseudo-random numbers. Similarly for Table 2.]

For X_n^2, the closeness between nominal and actual sizes seems satisfactory except when $\alpha = 0.15, 0.25$ and the sample size is ≤ 50. For I_n, the approximation seems satisfactory for samples of size $n \geq 50$ if $\alpha = 0.15, 0.25$, and for samples of size $n \geq 100$ if $\alpha \geq 0.50$. While X_n^2 overestimates the size of the test, I_n underestimates it, and this asymmetry has to be accounted for when interpreting the power results. In any case, it is clear that for $n \geq 100$ the two tests will be compared on equal footing.

In applications one will generally have an idea about the true value of α (e.g. through a confidence interval), so the information in Table 1 may serve as a guide to the actual size of a 10% test based on the integral statistic. Note that for sample sizes as small as 20 and a wide range of α values we have the guarantee that the test based on I_n has an actual size of 5% to 10%, a fact with obvious practical relevance.

Table 1: *Estimates of actual significance levels of the tests with nominal size 0.10 based on* X_n^2 *and* I_n, $n = 20, 50, 100, 200, 350$, *for geometric distributions with parameter* $\alpha = 0.15, 0.25, 0.50, 0.75, 0.85$.

		α				
n	Statistic	0.15	0.25	0.50	0.75	0.85
20	X_n^2	0.2028	0.1714	0.1223	0.1162	0.1028
	I_n	0.0856	0.0815	0.0762	0.0531	0.0808
50	X_n^2	0.1618	0.1208	0.1065	0.1090	0.0910
	I_n	0.0902	0.0916	0.0864	0.0830	0.0603
100	X_n^2	0.1074	0.1091	0.1027	0.1011	0.1026
	I_n	0.0981	0.1020	0.0957	0.0929	0.0915
200	X_n^2	0.1031	0.1012	0.1000	0.096	0.0927
	I_n	0.0978	0.0995	0.0968	0.0992	0.0935
350	X_n^2	0.1024	0.1012	0.1027	0.1021	0.0921
	I_n	0.0987	0.0999	0.0972	0.0968	0.0958

The alternatives to be considered include six examples of the negative binomial, shifted-Poisson and logarithmic distributions on $\mathbb{N}$. See Figure 1 for a comparison of their probability functions with 'neighbouring' geometric probability functions. We aimed at choosing models that are plausible alternatives to the geometric distribution (i.e., having an overall shape relatively similar to it), while at the same time possessing some interesting tail feature. Thus the Poisson distribution has a lighter tail than the geometric model, the logarithmic distribution a heavier tail, and the negative binomial distribution more or less the same tail as the geometric distribution.

The negative binomial distribution with parameters r and p, to which we refer as NB(r,p), has probability function $\Gamma(r+x-1)\,\Gamma(x)^{-1}\,\Gamma(r)^{-1}\,p^r\,(1-p)^{x-1}$ for $x = 1, 2, \ldots$, $r > 0$, $0 < p < 1$ (Γ is the gamma function). We consider three alternatives of this family: NB(3/2,1/3), NB(1/2,1/3) and NB(2,6/7). The means and variances of these distributions are, respectively, 4 and 9, 2 and 3, $\approx 4/3$ and ≈ 0.389. Each of the models can be compared with a 'neighbouring' geometric model, defined as the geometric distribution with (approximately) the same mean. Thus, writing Geo(α) for the geometric distribution with parameter α, the neighbouring geometric models of NB(3/2,1/3), NB(1/2,1/3) and NB(2,6/7) are, respectively, Geo(1/4), Geo(1/2) and Geo(3/4), whose means and variances are 4 and 12, 2 and 2, 4/3 and ≈ 0.444. Three of the plots in Figure 1 represent the negative binomial alternatives and their neighbouring geometric probability functions.

The Poisson distribution on $\mathbb{N}$ with parameter $\lambda > 0$, which we denote by Poi+(λ), corresponds to a Poisson random variable shifted one unit to the right; its probability function is therefore $e^{-\lambda}\lambda^{x-1}/(x-1)!$, $x = 1, 2, \ldots$ We consider a Poi+(1/3) alternative; this has mean 4/3 and variance 1/3, and corresponds to a neighbouring Geo(3/4) model, whose mean and variance, as just said, are $\approx 4/3$ and ≈ 0.444, respectively.

The logarithmic distribution on $\mathbb{N}$ with parameter $p \in (0, 1)$, denoted by Lo(p), has probability function $x^{-1}(-p^x/\log(1-p))$, $x = 1, 2, \ldots$ As alternatives, we consider Lo(0.715), whose mean and variance are 2 and ≈ 3.026, and Lo(0.423), with mean $\approx 4/3$ and variance ≈ 0.534. The corresponding neighbouring geometric models are Geo(1/2) (mean 2, variance 2) and Geo(3/4) (mean $\approx 4/3$, variance ≈ 0.444).

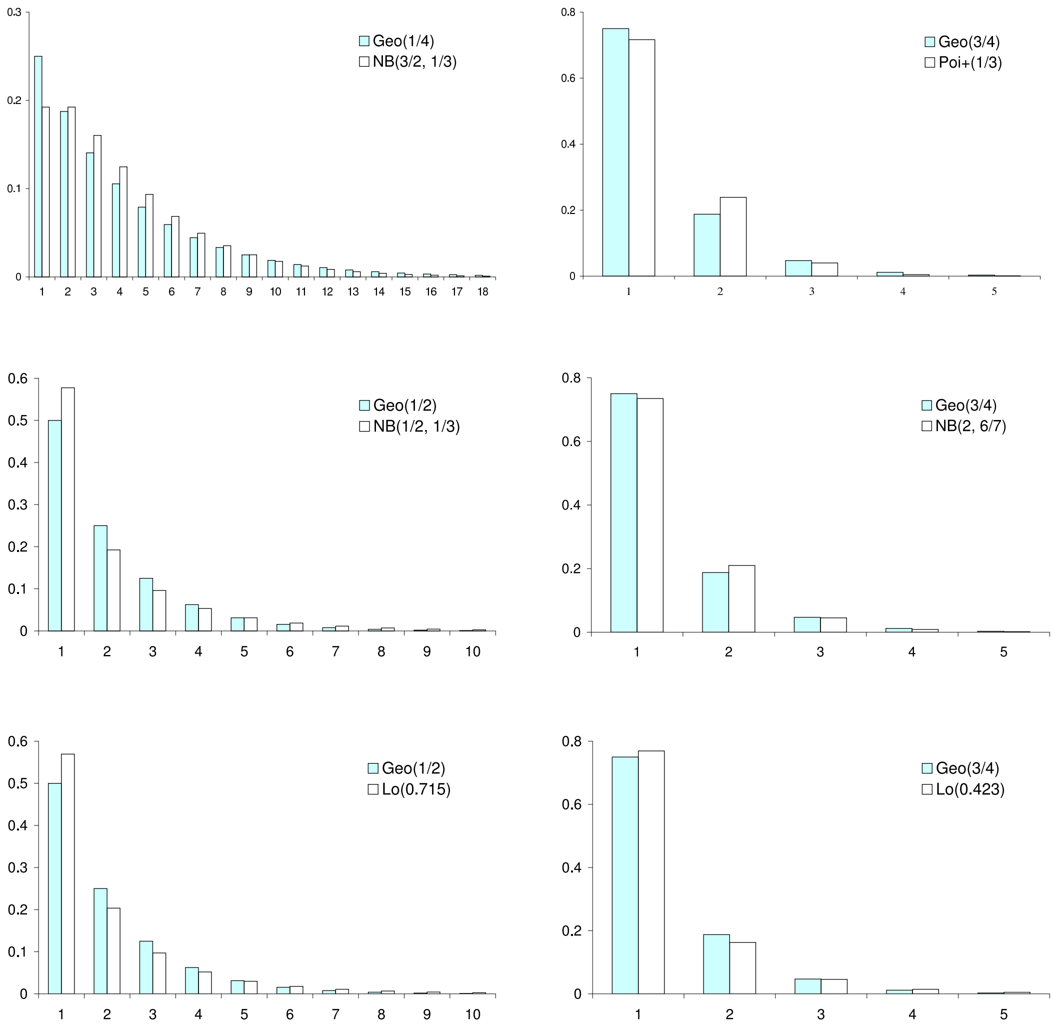

Figure 1: *Comparison between the negative binomial, Poisson and logarithmic alternative probability functions and their 'neighbouring' geometric probability functions.*

Estimates of the power of the tests based on X_n^2 and I_n with sample sizes of $n = 20, 50, 100, 200$ and 350 are given in Table 2. The results consist of proportions of rejections of the null hypothesis out of 1,000 simulation runs.

For a sample size of $n = 20$, I_n is slightly biased against the Poi+(1/3) and NB(2,6/7) alternatives, but note that for these alternatives the actual significance levels should be between 5% and 7%. The test is clearly sensitive for all other sample sizes, and we may infer that it is consistent against the alternatives considered.

Table 2: *Power estimates of the 10% tests based on* X_n^2 *and* I_n, $n = 20, 50, 100, 200, 350$, *against* NB(3/2,1/3), NB(1/2,1/3), Lo(0.715), Poi+(1/3), NB(2,6/7) *and* Lo(0.423) *alternatives.*

		Alternatives					
n	Statistic	NB $\left(\frac{3}{2},\frac{1}{3}\right)$	NB $\left(\frac{1}{2},\frac{1}{3}\right)$	Lo(0.715)	Poi+$\left(\frac{1}{3}\right)$	NB $\left(2,\frac{6}{7}\right)$	Lo(0.423)
20	X_n^2	0.234	0.164	0.151	0.236	0.143	0.107
	I_n	0.149	0.242	0.226	0.027	0.028	0.116
50	X_n^2	0.242	0.230	0.185	0.322	0.173	0.125
	I_n	0.332	0.435	0.391	0.230	0.101	0.154
100	X_n^2	0.333	0.437	0.339	0.418	0.161	0.151
	I_n	0.520	0.724	0.644	0.435	0.158	0.234
200	X_n^2	0.538	0.739	0.634	0.608	0.194	0.252
	I_n	0.784	0.926	0.891	0.723	0.252	0.356
350	X_n^2	0.716	0.938	0.884	0.839	0.262	0.383
	I_n	0.962	0.993	0.990	0.920	0.360	0.522

Overall, the performances of I_n and X_n^2 are qualitatively similar, but it is clear that the integral statistic performs generally better against all alternatives. The main exceptions to this rule occur for $n = 20$, and may be due to the underestimation, in the case of the integral statistic, and overestimation, in the case of the chi-square statistic, of the nominal 0.10 significance level. The superiority of I_n over X_n^2 is particularly evident for larger sample sizes against those alternatives that are more difficult to detect, namely NB(3/2,1/3), NB(2,6/7) and Lo(0.423).

Acknowledgements

This work is based on part of my Ph.D. thesis, written at the University of Sheffield, U.K. I am very grateful to D.N. Shanbhag, my supervisor, for all his comments and suggestions, and to Fundação para a Ciência e Tecnologia for funding my research through the Sub-programa Ciência e Tecnologia do 2° quadro comunitário de apoio.

References

Bracquemond, C., Crétois, E., and Gaudoin, O. (2002). *A comparative study of goodness-of-fit tests for the geometric distribution and application to discrete time reliability* (Tech. Rep.). LMC-IMAG, 51 rue des Mathématiques, B. P. 53, 38041 Grenoble Cedex 9, France: Équipe de Statistique et Modélisation Stochastique, Laboratoire de Modélisation et Calcul.

Csörgő, M., Csörgő, S., and Horváth, L. (1986). *An asymptotic theory for reliability and concentration processes.* Springer-Verlag. (Lecture Notes in Statistics 30)

Ferreira, J. (2003). An empirical process with applications to testing the exponential and geometric models. In C. Rao and D. Shanbhag (Eds.), *Handbook of Statistics: Stochastic processes, Modelling and Simulation* (Vol. 21, p. 167-225). Elsevier Science B.V.

Hollander, M., and Proschan, F. (1975). Tests for the mean residual life. *Biometrika*, *62*(3), 585–593.

Rao, C., and Shanbhag, D. (1994). *Choquet-Deny type functional equations with applications to stochastic models.* Chichester: Wiley.

Spurrier, J. (1984). An overview of tests for exponentiality. *Communications in Statistics–Theory and Methods*, *13*(13), 1635-1654.

Stephens, M. (1986). Tests based on EDF statistics. In R. D'Agostino and M. Stephens (Eds.), *Goodness of fit techniques* (p. 95-193). Marcel Dekker.

Vit, P. (1974). Testing for homogeneity; the geometric distribution. *Biometrika*, *61*, 565-568.

Wichmann, B., and Hill, I. (1982). Algorithm as183: An efficient and portable pseudo-random number generator. *Applied Statistics*, *2*, 188–190.