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Stone Coalgebras

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Abstract

In this paper we argue that the category of Stone spaces forms an interesting base category for coalgebras, in particular, if one considers the Vietoris functor as an analogue to the power set functor on the category of sets.

We prove that the so-called descriptive general frames, which play a fundamental role in the semantics of modal logics, can be seen as Stone coalgebras in a natural way. This yields a duality between the category of modal algebras and that of coalgebras over the Vietoris functor.

Building on this idea, we introduce the notion of a Vietoris polynomial functor over the category of Stone spaces. For each such functor T we provide an adjunction between the category of T-sorted Boolean algebras with operators and the category of Stone coalgebras over T. Since the unit of this adjunction is an isomorphism, this shows that $\mathsf{Coalg}(T)^{\mathrm{op}}$ is a full reflective subcategory of BAO_T. Applications include a general theorem providing final coalgebras in the category of T-coalgebras.

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1 Introduction

Every coalgebra is based on a carrier which is an object in the so-called base category. Most of the literature on coalgebras either focuses on Set as the base category, or takes a very general perspective, allowing arbitrary base categories (possibly restricted by some constraints). The aim of this paper is to argue that, besides Set, the category Stone of Stone spaces is an interesting base category. We have a number of reasons for believing that *Stone coalgebras*, that is, coalgebras based on Stone, are of relevance.

To start with, in Section 3 we discuss interesting examples of Stone coalgebras, namely the ones that are associated with the *Vietoris functor* $\mathbb{V} : \mathsf{Stone} \to \mathsf{Stone}$. This $\mathbb{V}$ is the

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functorial extension of the Vietoris construction, which is a well-known topological analogue of the power set construction: the Vietoris topology of a topology τ is based on the collection of sets that are closed in τ [9]. This construction preserves a number of nice topological properties; in particular, it turns Stone spaces into Stone spaces [16]. As we will see further on, the category $\mathsf{Coalg}(\mathbb{V})$ of coalgebras over this Vietoris functor is of interest because it is isomorphic to the category DGF of descriptive general frames. This category in its turn is dual to that of modal algebras, and hence, unlike Kripke frames, descriptive general frames form a mathematically adequate semantics for modal logics [6].

The connection with modal logic thus forms a second reason as to why Stone coalgebras are of interest. Since coalgebras can be seen as a very general model of state-based dynamics, and modal logic as a logic for dynamic systems, the relation between modal logic and coalgebras is rather tight. Starting with the work of Moss [25], this has been an active research area [28, 15, 5, 27, 13, 7]. The relation between modal logic and coalgebras can be seen to dualize that between equational logic and algebra [22, 21], an important difference being that the relation with Set-based coalgebras seems to work smoothly only for modal languages that allow infinitary formulas. In the case of the Vietoris functor however, it follows from the duality between $\mathsf{Coalg}(\mathbb{V})$ and the category MA of modal algebras, that $\mathsf{Coalg}(\mathbb{V})$ provides a natural semantics for *finitary* modal logics.

As a digression into the field of modal logic and its connection with coalgebras, Section 4 briefly diverts from Stone spaces as a base category. We show that arbitrary, (that is, not necessarily descriptive) general frames, can also be seen as coalgebras if we take the category of represented Boolean algebras as our base category.

In the Sections 5 and 6 we further substantiate our case for Stone spaces as a coalgebraic base category, by introducing so-called Vietoris polynomial functors as the Stone-based analogs of Kripke polynomial functors over Set [28]. For each such functor T, we establish a translation between the category of T-sorted Boolean algebras with operators, as introduced by Jacobs [15], and the category of Stone coalgebras over T. Section 5 transfers the work of [15] from set-coalgebras to Stone-coalgebras. Section 6 shows that for Stone-coalgebras one obtains an adjunction between T-sorted Boolean algebras with operators and T-coalgebras. Although this adjunction is not a dual equivalence in general, we will see that each coalgebra can be represented as the translation of an algebra: $\mathsf{Coalg}(T)^{\mathrm{op}}$ is (isomorphic to) a full coreflective subcategory of BAO_T. As an application of this, we provide a final coalgebra in every category $\mathsf{Coalg}(T)$. One way to prove this uses another important result, namely that the initial algebra of BAO_T is *exact*, that is, belongs to the just mentioned coreflective subcategory of BAO_T.

Let us add two more observations on Stone-coalgebras. First, the duality of descriptive general frames and modal algebras shows that the (trivial) duality between the categories $\mathsf{Coalg}(T)$ and $\mathsf{Alg}(T^{\mathrm{op}})$ has non-trivial instances. Second, it might be interesting to note that Stone provides a meaningful example of a base category for coalgebras which is not finitely locally presentable.

Before we turn to the technical details of the paper, we want to emphasize that in our opinion the main value of this paper lies not so much in the technical contributions; in fact, many of the results that we list are known, or could be obtained by standard methods from known results. The interest of this work, we believe, rather lies in the fact that these results can be grouped together in a natural, coalgebraic light.

This is an extended version of [17].

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2 Preliminaries

We presuppose some familiarity with category theory, general topology, the (duality) theory of boolean algebras, and universal coalgebra. The main purpose of this section is to fix our notation and terminology.

Definition 2.1 (Coalgebras) Let C be a category and $T : \mathsf{C} \to \mathsf{C}$ an endofunctor. Then a T-coalgebra is a pair $(X, \xi : X \to TX)$ where X denotes an object of C and ξ a morphism of C. A T-coalgebra morphism $h : (X_1, \xi_1) \to (X_2, \xi_2)$ is a C-morphism $h : X_1 \to X_2$ satisfying $\xi_2 \circ h = Th \circ \xi_1$. The category $\mathsf{Coalg}(T)$ has T-coalgebras as its objects and T-coalgebra morphisms as arrows. Dually, we define a T-algebra to be a T^{op}-coalgebra and $\mathsf{Alg}(T) = (\mathsf{Coalg}(T^{\mathrm{op}}))^{\mathrm{op}}$. $\lhd$

Example 2.2 (Kripke frames) A *Kripke frame* is a structure $\mathbb{F} = (X, R)$ such that R is a binary relation on X. It is by now well-known that Kripke frames can be seen as coalgebras for the power set functor $\mathcal{P}$ over Set. The idea here is to replace the binary relation R of a frame $\mathbb{F} = (X, R)$ with the map $R[\_] : X \to \mathcal{P}(X)$ given by

$$R[s] := \{t \in X \mid Rst\}.$$

In fact, Kripke frames (and models) form some of the prime examples of coalgebras — many coalgebraic concepts have been developed as generalizations of notions applying to Kripke structures. This applies for instance to the notion of a *bounded morphism* between Kripke frames; we will use this terminology for $\mathcal{P}$-coalgebra morphisms.

Definition 2.3 (Stone spaces) A topological space $\mathbb{X} = (X, \tau)$ is called a Stone space if τ is a compact Hausdorff topology which is in addition zero-dimensional, that is, it has a basis of clopen sets. By $\mathsf{Clp}_{\mathbb{X}}$ we will denote the collection of clopen subsets of X. The category Stone of Stone spaces has as its objects Stone spaces and as its morphisms the continuous functions between them. $\lhd$

The following proposition states a basic fact about Stone spaces which is essential for obtaining the duality between Stone spaces and Boolean algebras and which we will also need later in the paper.

Proposition 2.4 *Let $\mathbb{X} = (X, \tau)$ be a Stone space. Then $\mathsf{Clp}_{\mathbb{X}}$ is the unique basis of τ that is closed under the boolean operations.*

Proof. $\mathbb{X}$ is a Stone space and therefore it is clear that $\mathsf{Clp}_{\mathbb{X}}$ is a basis of the topology. Suppose now we have a basis V of τ that is closed under the boolean operations. Then obviously $V \subseteq \mathsf{Clp}_{\mathbb{X}}$ because V has to be closed under taking complements. We want to show that also $\mathsf{Clp}_{\mathbb{X}} \subseteq V$. Consider an arbitrary clopen U. Then there are sets $B_{ij} \in V$ such that $U = \bigcup_{j\in J} \bigcap_{i=1}^{n_j} B_{ij}$. As $U \subseteq X$ is closed and $\mathbb{X}$ is Hausdorff we know that U is compact. Therefore we can find a $J' \subseteq_\omega J$ such that $U = \bigcup_{j\in J'} \bigcap_{i=1}^{n_j} B_{ij}$. As V is closed under boolean operations we get $U \in V$. QED

We now want to state the well-known Stone duality. In order to do that we first define two functors.

Definition 2.5 The category of Boolean algebras and homomorphisms between them is denoted as BA. The Stone space $(\mathsf{Sp}\,\mathbb{B}, \tau_{\mathbb{B}})$ corresponding to a Boolean algebra $\mathbb{B}$ is given by the collection $\mathsf{Sp}\,\mathbb{B}$ of ultrafilters of $\mathbb{B}$ and the topology $\tau_{\mathbb{B}}$ generated by basic opens of the form $\{u \in \mathsf{Sp}\,\mathbb{B} \mid b \in u\}$ for any b in $\mathbb{B}$. We let $\mathbb{S}\mathsf{p}$ denote the functor that associates with a Boolean algebra its corresponding Stone space, and with a Boolean homomorphism its inverse image function.

Conversely the functor mapping a Stone space $\mathbb{X}$ to the Boolean algebra $\mathsf{Clp}_{\mathbb{X}}$ of its clopens, and a continuous morphism to its inverse image function, is denoted as $\mathbb{C}\mathsf{lp}$. ◁

Definition 2.6 For any Boolean algebra $\mathbb{B}$ we define a map

$$\begin{array}{rcl} i_{\mathbb{B}} : \mathbb{B} & \to & \mathbb{C}\mathsf{lp}\,\mathbb{S}\mathsf{p}\,\mathbb{B} \\ b & \mapsto & \hat{b} := \{u \in \mathsf{Sp}\,\mathbb{B} \mid b \in u\} \end{array}$$

and for any Stone space $\mathbb{X}$ we define a map

$$\begin{array}{rcl} \epsilon_{\mathbb{X}} : \mathbb{X} & \to & \mathbb{S}\mathsf{p}\,\mathbb{C}\mathsf{lp}\mathbb{X} \\ x & \mapsto & \{U \in \mathsf{Clp}_{\mathbb{X}} \mid x \in U\} \end{array}$$

◁

Theorem 2.7 (Stone duality) *The families of morphisms* $(i_{\mathbb{B}})_{\mathbb{B}\in\mathsf{BA}}$ *and* $(\epsilon_{\mathbb{X}})_{\mathbb{X}\in\mathsf{Stone}}$ *are natural isomorphisms, hence the functors* $\mathbb{S}\mathsf{p} : \mathsf{BA} \to \mathsf{Stone}^{\mathrm{op}}$ *and* $\mathbb{C}\mathsf{lp} : \mathsf{Stone}^{\mathrm{op}} \to \mathsf{BA}$ *induce a dual equivalence between the categories* Stone *and* BA*:*

$$\mathsf{BA} \simeq \mathsf{Stone}^{\mathrm{op}}.$$

Definition 2.8 (Vietoris topology) Let $\mathbb{X} = (X, \tau)$ be a topological space. We let $K(\mathbb{X})$ denote the collection of all closed subsets of X. We define the operations $[\ni], \langle\ni\rangle : \mathcal{P}(X) \to \mathcal{P}(K(\mathbb{X}))$ by

$$\begin{array}{rcl} [\ni]U & := & \{F \in K(\mathbb{X}) \mid F \subseteq U\}, \\ \langle\ni\rangle U & := & \{F \in K(\mathbb{X}) \mid F \cap U \neq \varnothing\}. \end{array}$$

Given a subset $Q \subseteq \mathcal{P}(X)$, define

$$V_Q \;:=\; \{[\ni]U \mid U \in Q\} \cup \{\langle\ni\rangle U \mid U \in Q\}\,.$$

The Vietoris space $\mathbb{V}(\mathbb{X})$ associated with $\mathbb{X}$ is given by the topology $\upsilon_{\mathbb{X}}$ on $K(\mathbb{X})$ which is generated by the subbasis V_τ. $\lhd$

Modal logicians will recognize the above notation as indicating that $[\ni]$ and $\langle\ni\rangle$ are the 'box' and the 'diamond' associated with the converse membership relation $\ni \subseteq K(\mathbb{X}) \times X$.

In case the original topology is compact, then we might as well have generated the Vietoris topology in other ways. This has nice consequences for the case that the original topology is a Stone space.

Lemma 2.9 *Let $\mathbb{X} = (X, \tau)$ be a compact topological space and let $\mathcal{B}$ be a basis of τ that is closed under finite unions. Then the set*

$$V_{\mathcal{B}} := \{[\ni]U \mid U \in \mathcal{B}\} \cup \{\langle\ni\rangle U \mid U \in \mathcal{B}\}$$

forms a subbasis for $\upsilon_{\mathbb{X}}$.

Proof. We show that every element $O \in V$ is a union of elements of $V_{\mathcal{B}}$. Let $O \in V$.

First assume that $O = [\ni]U$ for some $U \in \tau$. Let $(B_i)_{i\in I} \subseteq \mathcal{B}$ be a collection of basic opens such that $U = \bigcup_{i\in I} B_i$. Then we get

$$\begin{aligned}
[\ni]U \quad &= \quad \left\{F \in K(\mathbb{X}) \mid F \subseteq \bigcup_{i\in I} B_i\right\} \\
&\overset{(\mathbb{X}\text{ compact})}{=} \quad \left\{F \mid F \subseteq \bigcup_{i'\in I'} B_{i'} \text{ for some } I' \subseteq_\omega I\right\} \\
&= \quad \bigcup_{I'\subseteq_\omega I} [\ni]\left(\bigcup_{i'\in I'} B_{i'}\right)
\end{aligned}$$

where the last equation proves our claim because of the fact that $\mathcal{B}$ is closed under taking finite unions.

Now assume that $O = \langle\ni\rangle U$ for some $U \in \tau$. Then again $U = \bigcup_{i\in I} B_i$, where $B_i \in \mathcal{B}$ for all $i \in I$, and therefore $\langle\ni\rangle U = \bigcup_{i\in I}\langle\ni\rangle B_i$ where $B_i \in \mathcal{B}$ for all $i \in I$. QED

Corollary 2.10 *Let $\mathbb{X} = (X, \tau)$ be a Stone space and let $\mathsf{Clp}_{\mathbb{X}}$ be the collection of the clopen subsets of $\mathbb{X}$. Then the set*

$$V_{\mathsf{Clp}_{\mathbb{X}}} := \{[\ni]U \mid U \in \mathsf{Clp}_{\mathbb{X}}\} \cup \{\langle\ni\rangle U \mid U \in \mathsf{Clp}_{\mathbb{X}}\}$$

forms a subbasis for the Vietoris topology $\upsilon_{\mathbb{X}}$.

Proof. The set $\mathsf{Clp}_{\mathbb{X}}$ fulfills the condition of Lemma 2.9. QED

The next two lemmas state that the Vietoris construction preserves various nice topological properties.

Lemma 2.11 *(cf. [24], Theorem 4.2, 4.9) Let $\mathbb{X} = (X, \tau)$ be a topological space.*

1. *If $\mathbb{X}$ is compact then $(K(\mathbb{X}), \upsilon_{\mathbb{X}})$ is compact.*
2. *If $\mathbb{X}$ is compact and Hausdorff, then $(K(\mathbb{X}), \upsilon_{\mathbb{X}})$ is compact and Hausdorff.*

Proof. (1) Let $V_{\mathcal{B}}$ be a subbasis of $\upsilon_{\mathbb{X}}$ defined as in Lemma 2.9. By the Alexander subbasis theorem it suffices to show that every covering of $K(\mathbb{X})$ by elements of $V_{\mathcal{B}}$ contains a finite subcovering. Suppose that

$$K(\mathbb{X}) \;\subseteq\; \bigcup_{i \in I} [\ni] U_i \cup \bigcup_{j \in J} \langle\ni\rangle V_j, \tag{1}$$

where the U_i's and the V_j's are elements of some basis $\mathcal{B}$ of τ.

We claim that

$$X \subseteq \bigcup_{i \in I} U_i \cup \bigcup_{j \in J} V_j. \tag{2}$$

Suppose for contradiction that there is an $x \in X$ such that $x \notin (\bigcup U_i) \cup (\bigcup V_j)$, and let $F_x := \bigcap \{F \in K(\mathbb{X}) \mid x \in F\}$. Since F_x is closed, it follows from (1) that there exists a $j \in J$ such that $F_x \in \langle\ni\rangle V_j$, that is, $F_x \cap V_j \neq \varnothing$, and $x \in -V_j \cap F_x$. As $-V_j \cap F_x$ is a closed set we get that $F_x \subseteq -V_j \cap F_x$ and this implies $F_x = -V_j \cap F_x$ which is clearly a contradiction to $V_j \cap F_x \neq \emptyset$. This proves (2).

Now consider the closed set $F^* = X \setminus \bigcup_{j \in J} V_j$. We know that $F^* \in \cup_{i \in I} [\ni] U_i$ which implies that there is an $i_0 \in I$ such that $F^* \subseteq U_{i_0}$. From this and the definition of F^* it is immediate that

$$X \subseteq U_{i_0} \cup \bigcup_{j \in J} V_j. \tag{3}$$

The compactness of $\mathbb{X}$ then provides us now with a set $J' \subseteq_\omega J$ such that

$$X \subseteq U_{i_0} \cup \bigcup_{j \in J'} V_j.$$

It is now easy to show that $K(\mathbb{X}) \subseteq [\ni] U_{i_0} \cup \bigcup_{j \in J'} \langle\ni\rangle V_j$. For, if an arbitrary $F \in K(\mathbb{X})$ is a subset of U_{i_0}, then $F \in [\ni] U_{i_0}$, while $F \not\subseteq U_{i_0}$ implies $F \cap \bigcup_{j \in J'} V_j \neq \varnothing$, whence $F \cap V_j \neq \emptyset$ for some $j \in J'$ and we get $F \in \langle\ni\rangle V_j$.

(2) To show that $(K(\mathbb{X}), \upsilon_{\mathbb{X}})$ is Hausdorff one has first to realize that $\mathbb{X}$ is a T_3-space, i.e. for every closed subset $F \subseteq X$ and $x \in X \setminus F$ there are $U_1, U_2 \in \tau$ such that $F \subseteq U_1$, $x \in U_2$ holds for any compact Hausdorff space (see e.g. [9], Thm. 3.1.9).

Now let $F_1, F_2 \in K(\mathbb{X})$ s.t. $F_1 \neq F_2$. Then we can without loss of generality assume that there is an $x \in F_2 \setminus F_1$. Because $\mathbb{X}$ is T_3 there exist $U_1, U_2 \in \tau$ such that $F_1 \subseteq U_1$, $x \in U_2$ and $U_1 \cap U_2 = \emptyset$. Therefore we get

$$F_1 \in [\ni] U_1,\; F_2 \in \langle\ni\rangle U_2 \text{ and } [\ni] U_1 \cap \langle\ni\rangle U_2 = \emptyset.$$

QED

Lemma 2.12 *(cf. [24], Theorem 4.9.6) Let $\mathbb{X} = (X, \tau)$ be a Stone space. Then $(K(\mathbb{X}), \tau_V)$ is also a Stone space.*

Proof. From Lemma 2.11 it follows that $(K(\mathbb{X}), \tau_V)$ is compact and Hausdorff. Furthermore we know that the collection $V_{\mathsf{Clp}_{\mathbb{X}}}$ of Corollary 2.10 forms a subbasis of τ_V. It is easy to see that for a clopen subset U of $\mathbb{X}$ the following equations hold:

$$\begin{array}{rcl} [\ni]U & = & -\langle\ni\rangle(-U) \\ \langle\ni\rangle U & = & -[\ni](-U) \end{array}$$

Therefore it is clear that $V_{\mathsf{Clp}_{\mathbb{X}}}$ is a clopen subbasis of τ_V. This implies that the clopen subsets of $K(\mathbb{X})$ form a basis of τ_V. QED

3 Descriptive general frames as Stone coalgebras

In this section we discuss what are probably the prime examples of Stone coalgebras, namely those for the Vietoris functor $\mathbb{V}$ (to be defined below). As we will see, the importance of these structures lies in the fact that the category $\mathsf{Coalg}(\mathbb{V})$ is isomorphic to the category of so-called *descriptive general frames*. We hasten to remark that when it comes down to the technicalities, this section contains little news; most of the results in this section can be obtained by exposing existing material from Esakia [10], Goldblatt [12], Johnstone [16], and Sambin and Vaccaro [31] in a new, coalgebraic framework.

General frames, and in particular, descriptive general frames, play a crucial role in the theory of modal logic. Together with their duals, the modal algebras, they provide an important class of structures interpreting modal languages. From a mathematical perspective they rank perhaps even higher than Kripke frames, since the Kripke semantics suffers from a fundamental incompleteness result: not every modal logic (in the technical sense of the word) is complete with respect to the class of Kripke frames on which it is valid (see e.g. [6], Chapter 4). Putting it differently, Kripke frames provide too poor a tool to make the required distinctions between modal logics. The algebraic semantics for modal logic does not suffer from this shortcoming: every modal logic is determined by the class of modal algebras on which it is valid.

Definition 3.1 (Modal algebras) Let $\mathbb{B}$ and $\mathbb{B}'$ be boolean algebras; an operation $g : B \to B'$ on their carriers is said to *preserve finite meets* if $g(\top) = \top'$ and $g(b_1 \wedge b_2) = g(b_1) \wedge' g(b_2)$. A *modal algebra* is a structure $\mathbb{A} = (A, \wedge, -, \bot, \top, g)$ such that the reduct $(A, \wedge, -, \bot, \top)$ of $\mathbb{A}$ is a Boolean algebra, and $g : A \to A$ preserves finite meets. The category of modal algebras (with homomorphisms) is denoted by MA. $\lhd$

The intended meaning of g is to provide an interpretation of the modal operator $\Box$. Thinking of $a \in A$ as the interpretation of a modal formula φ, $g(a)$ provides the interpretation of $\Box\varphi$.

Example 3.2 1. If (X, R) is a Kripke frame then $(\mathcal{P}X, \cap, -, \emptyset, X, [R])$ is a modal algebra where $[R](a) = \{x \in X \mid x\ R\ y \ \Rightarrow\ y \in a\}$.

2. Let Prop be a set of propositional variables and $\mathcal{L}(\mathsf{Prop})$ be the set of modal formulas over Prop quotiented by $\varphi \equiv \psi \iff \vdash_{\mathbf{K}} \varphi \leftrightarrow \psi$ where $\vdash_{\mathbf{K}}$ denotes derivability in the basic modal logic **K** (see eg [6]). Then $\mathcal{L}(\mathsf{Prop})$—equipped with the obvious operations—is a modal algebra. In fact, $\mathcal{L}(\mathsf{Prop})$ is the modal algebra free over Prop and is called the Lindenbaum-Tarski algebra (over Prop).

Remark 3.3 Although not needed in the following, we indicate how modal formulas are evaluated in modal algebras. Let φ be a modal formula taking propositional variables from Prop and let $\mathbb{A} = (A, \wedge, -, \bot, \top, g)$ be a modal algebra. Employing the freeness of the modal algebra $\mathcal{L}(\mathsf{Prop})$ we can identify valuations of variables $v : \mathsf{Prop} \to A$ with algebra morphisms $\mathcal{L}(\mathsf{Prop}) \to \mathbb{A}$ and define $\mathbb{A} \models \varphi$ if $v([\varphi]_{\equiv}) = \top$ for all morphisms $v : \mathcal{L}(\mathsf{Prop}) \to \mathbb{A}$.

However, modal algebras are fairly abstract in nature and many modal logicians prefer the intuitive, geometric appeal of Kripke frames. *General frames*, unifying the algebraic and the Kripke semantics in one structure, provide a nice compromise.

Definition 3.4 (General frames) Formally, a general frame is a structure $\mathbb{G} = (G, R, A)$ such that (G, R) is a Kripke frame and A is a collection of so-called *admissible* subsets of G that is closed under the boolean operations and under the operation $\langle R \rangle : \mathcal{P}(G) \to \mathcal{P}(G)$ given by:

$$\langle R \rangle X := \{y \in G \mid Ryx \text{ for some } x \in X\}.$$

A general frame $\mathbb{G} = (G, R, A)$ is called *differentiated* if for all distinct $s_1, s_2 \in G$ there is a 'witness' $a \in A$ such that $s_1 \in a$ while $s_2 \notin a$; *tight* if whenever t is not an R-successor of s, then there is a 'witness' $a \in A$ such that $t \in a$ while $s \notin \langle R \rangle a$; and *compact* if $\bigcap A_0 \neq \varnothing$ for every subset A_0 of A which has the finite intersection property. A general frame is *descriptive* if it is differentiated, tight and compact. $\lhd$

Example 3.5 1. Any Kripke frame (X, R) can be considered as a general frame $(X, R, \mathcal{P}X)$.

2. If $\mathbb{A} = (A, \wedge, -, \bot, \top, g)$ is a modal algebra then $(\mathsf{Sp}\,\mathbb{A}, R, \hat{A})$ where $R = \{(u, v) \mid a \in u \Rightarrow g(a) \in v\}$ and $\hat{A} = \{\{u \in \mathsf{Sp}\,\mathbb{A} \mid a \in u\} \mid a \in A\}$ is a descriptive general frame.

3. If $\mathbb{G} = (G, R, A)$ is a general frame then $(A, \cap, -, \emptyset, G, [R])$ is a modal algebra.

The following remark explains the terminology of 'admissible' subsets.

Remark 3.6 Let $\mathbb{G} = (X, R, A)$ be a general frame and consider a modal formula φ taking its propositional variables from the set Prop. Note that, given a function $v : X \to \prod_{\mathsf{Prop}} 2$, where $2 = \{0, 1\}$ is the set of truth values, (X, R, v) is a Kripke model. v is called a *valuation for* $\mathbb{G}$ if the extensions of all propositions are admissible, that is, if $\{x \in X \mid v(x)_p = 1\} \in A$ for all $p \in \mathsf{Prop}$. The validity of a modal formula in general frame is then defined as $\mathbb{G} \models \varphi$ if $(X, R, v) \models \varphi$ for all valuations v for $\mathbb{G}$.

Since Kripke frames (and models) form some of the prime examples of coalgebras, the question naturally arises whether (descriptive) general frames can be seen as coalgebras as well. In this and the following section we will answer this question in the positive.

Two crucial observations connect descriptive general frames with coalgebras. First, the admissible sets of a descriptive frame form a basis for a topology. This topology is compact, Hausdorff, and zero-dimensional because descriptive general frames are compact, differentiated and the admissible sets are closed under boolean operations. It follows that descriptive general frames give rise to a Stone space with the admissible sets appearing as the collection of clopens.

Second, the tightness condition of descriptive general frames can be reformulated as the requirement that the relation is *point-closed*; that is, the successor set of any point is closed in the Stone topology. This suggests that if we are looking for a coalgebraic counterpart of a descriptive general frame $\mathbb{G} = (G, R, A)$, it should be of the form

$$(G, \tau) \xrightarrow{R[\_]} (K(G), \tau_?)$$

where $K(G)$ is the collection of closed sets in the Stone topology τ on G and $\tau_?$ is some suitable topology on $K(G)$, which turns $K(G)$ again into a Stone space. A good candidate is the Vietoris topology: it is based on the closed sets of τ and it yields a Stone space if we started from one. Moreover, as we will see, choosing the Vietoris topology for $\tau_?$, continuity of the map $R[\_]$ corresponds to the admissible sets being closed under $\langle R\rangle$.

Turning these intuitions into a more precise statement, we will prove that the category of descriptive general frames and the category $\mathsf{Coalg}(\mathbb{V})$ of coalgebras for the Vietoris functor are in fact *isomorphic*. Before we can go into the details of this, there are two obvious tasks waiting: first, we have to define the morphisms that make the descriptive general frames into a category, and second, we have to show that the Vietoris construction, which until now has just been defined for objects, can be turned into a functor.

Definition 3.7 (General frame morphisms) A morphism $\theta : (G, R, A) \to (G', R', A')$ is a function from W to W' such that (i) $\theta : (W, R) \to (W', R')$ is a bounded morphism (see Example 2.2) and (ii) $\theta^{-1}(a') \in A$ for all $a' \in A'$.

We let GF (DGF) denote the category with general frames (descriptive general frames, respectively) as its objects, and the general frame morphisms as the morphisms. $\lhd$

In the future we will need the fact that there is a dual equivalence[1] between the categories of modal algebras and descriptive general frames:

$$\mathsf{MA} \simeq \mathsf{DGF}^{\mathrm{op}}.$$

We will now see how the Vietoris construction can be upgraded to a proper endofunctor on the category of Stone spaces. For that purpose, we need to show how continuous maps between Stone spaces can be lifted to continuous maps between their Vietoris spaces; as a first step, we need the fact that whenever $f : \mathbb{X} \to \mathbb{X}'$ is a continuous map between compact Hausdorff spaces, then the image map $f[\_]$ is of the right type, that is, sends closed sets to closed sets. Fortunately, this is standard topology.

[1]On objects the equivalence is given by Example 3.5, (2) and (3).

Lemma 3.8 *Let $f : \mathbb{X} \to \mathbb{X}'$ be a continuous map between compact Hausdorff spaces. Then the function $\mathbb{V}(f)$ given by*

$$\mathbb{V}(f)(F) \;:=\; f[F] \; (= \{f(x) \mid x \in X\})$$

maps closed sets in $\mathbb{X}$ to closed sets in $\mathbb{X}'$.

Proof. We show that $f[F]$ is compact. Then the claim follows from the fact that every compact subset of a Hausdorff space is closed (cf. [9] Thm. 3.1.8).
Suppose that

$$f[F] \subseteq \bigcup_{i \in I} U_i \text{ for } U_i \in \sigma.$$

Then $F \subseteq \bigcup_{i \in I} f^{-1}(U_i)$ and because of continuity of f and compactness of $\mathbb{X}$ we get

$$F \subseteq \bigcup_{i \in I'} f^{-1}(U_i)$$

for some $I' \subseteq_\omega I$. Hence

$$f[F] \subseteq \bigcup_{i \in I'} f[f^{-1}(U_i)] \subseteq \bigcup_{i \in I'} U_i$$

and we are done. QED

Moreover, $\mathbb{V}$ is functorial:

Lemma 3.9 *Let $f : \mathbb{X} \to \mathbb{X}'$ be a continuous map between compact Hausdorff spaces. Then the function $\mathbb{V}(f)$ is a continuous map from $\mathbb{V}(\mathbb{X})$ to $\mathbb{V}(\mathbb{X}')$, and satisfies the functorial laws: $\mathbb{V}(id_{\mathbb{X}}) = \mathbb{V}(id_{\mathbb{V}(\mathbb{X})})$, and $\mathbb{V}(f \circ g) = \mathbb{V}(f) \circ \mathbb{V}(g)$.*

Proof. Assume that f is a continuous map between the Stone spaces $\mathbb{X} = (X, \tau)$ and $\mathbb{X}' = (X', \tau')$. In order to show that $\mathbb{V}(f)$ is a continuous map from $\mathbb{V}(\mathbb{X})$ to $\mathbb{V}(\mathbb{X}')$, we show that the pre-images of subbasic elements of the Vietoris topology $\upsilon_{\mathbb{X}'}$ are open in the Vietoris topology $\upsilon_{\mathbb{X}}$.

Let U' be an arbitrary element of $V_{\mathbb{X}'}$; there are two cases to consider. To start with, if U' is of the form $[\ni]Q'$ for some $Q' \in \tau'$, then we see that $\mathbb{V}(f)^{-1}(U') = \{F \in K(\mathbb{X}) \mid \mathbb{V}(F) \in [\ni]Q'\} = \{F \in K(\mathbb{X}) \mid f[F] \subseteq Q'\} = \{F \in K(\mathbb{X}) \mid F \subseteq f^{-1}(Q')\} = [\ni]f^{-1}(Q')$. And second, if U' is of the form $\langle\ni\rangle Q'$ for some $Q' \in \tau'$, then we have $\mathbb{V}(f)^{-1}(U') = \{F \in K(\mathbb{X}) \mid \mathbb{V}(F) \in \langle\ni\rangle Q'\} = \{F \in K(\mathbb{X}) \mid f[F] \cap Q' \neq \varnothing\} = \{F \in K(\mathbb{X}) \mid F \cap f^{-1}(Q') \neq \varnothing\} = \langle\ni\rangle f^{-1}(Q')$. In both cases we find that $\mathbb{V}(f)^{-1}(U')$ is (basic) open, as required.

We leave it to the reader to verify that $\mathbb{V}$ satisfies the functorial laws. QED

Definition 3.10 (Vietoris functor) The *Vietoris functor* on the category of Stone spaces is given on objects as in Definition 2.8 and on morphisms as in Lemma 3.8, i.e., for $(X, \tau) \in \mathsf{Stone}$

$$\begin{aligned} (X, \tau) &\mapsto (K(\mathbb{X}), \tau_V) \\ (f : (X, \tau) \to (Y, \sigma)) &\mapsto \mathbb{V}(f) \end{aligned}$$

where $\mathbb{V}(f)[F] := f[F]$ for all closed $F \subseteq X$. $\lhd$

We now turn to the isomorphism between the categories DGF and $\mathsf{Coalg}(\mathbb{V})$. The following rather technical lemma allows us to define the required functors relating the two categories.

Lemma 3.11 *Let X, τ and A be such that τ is a Stone topology on X and A is the collection of clopens of τ, and likewise for X', τ' and A'. Furthermore, suppose that $R \subseteq X^2$ and $\gamma : X \to K(\mathbb{X})$ satisfy*

$$Rxy \text{ iff } y \in \gamma(x) \tag{4}$$

for all $x, y \in X$; and similarly for $R' \subseteq X'^2$ and $\gamma' : X' \to K(\mathbb{X}')$.

Then $\theta : X \to X'$ is $\mathbb{V}$-coalgebra homomorphism between $((X, \tau), \gamma)$ and $((X', \tau'), \gamma')$ if and only if it is a general frame morphism between (X, R, A) and (X', R', A').

Proof. Both directions of the proof are straightforward. We only show the direction from left to right, leaving the other direction to the reader. Suppose that θ is a coalgebra morphism. Then θ is a continuous map from (X, τ) to (X', τ'), so the θ-inverse of a clopen set in τ' is clopen in τ. This shows that $\theta^{-1}(a') \in A$ for all $a' \in A'$.

In order to show that θ is a bounded morphism, first let Rxy. This implies that $y \in \gamma(x)$. Because θ is a coalgebra morphism we have

$$\theta[\gamma(x)] = \gamma'(\theta(x)),$$

so we get $\theta(y) \in \gamma'(\theta(x))$, i.e. $R'\theta(x)\theta(y)$. Now suppose that $R'\theta(x)y'$. Then $y' \in \gamma'(\theta(x))$ so by the above equation $y' \in \theta[\gamma(x)]$; that is, there is a $y \in X$ such that Rxy and $\theta(y) = y'$. QED

Lemma 3.11, together with our earlier observation on the connection between the admissible sets of a descriptive general frame and the clopens of the Stone space induced by taking these admissible sets as a basis, ensures that the following definition is correct. That is, if the reader is willing to check for himself that the maps defined below are indeed functors.

Definition 3.12 We define the functor $\mathbb{C} : \mathsf{DGF} \to \mathsf{Coalg}(\mathbb{V})$ as follows:

$$(G, R, A) \quad \mapsto \quad (G, \sigma_A) \xrightarrow{R[\text{-}]} \mathbb{V}(G, \sigma_A)$$

Here σ_A denotes the Stone topology generated by taking A as a basis. Conversely, there is a functor $\mathbb{D} : \mathsf{Coalg}(\mathbb{V}) \to \mathsf{DGF}$ given by:

$$((X, \tau), \gamma) \quad \mapsto \quad (X, R_\gamma, \mathsf{Clp}_{(X,\tau)})$$

where R_γ is defined by $R_\gamma s_1 s_2$ iff $s_2 \in \gamma(s_1)$. On morphisms both functors act as the identity with respect to the underlying Set-functions. $\lhd$

Theorem 3.13 *The functors $\mathbb{C}$ and $\mathbb{D}$ form an isomorphism between the categories DGF and $\mathsf{Coalg}(\mathbb{V})$.*

Proof. The theorem can be easily proven by just spelling out the definitions. QED

Remark 3.14 For a set-coalgebra (X, ξ), a valuation of propositional variables $p \in \mathsf{Prop}$ is a function $X \to \prod_{\mathsf{Prop}} 2$ where 2 is the two-element set of truth-values. For a Stone-coalgebra $(\mathbb{X}, \xi)$, a valuation is a continuous map $v : \mathbb{X} \to \prod_{\mathsf{Prop}} 2$ where 2 is taken with the discrete topology. The continuity of v is equivalent to the statement that the propositional variables take their values in admissible sets. Indeed, writing $\pi_p : \prod_{\mathsf{Prop}} 2 \to 2$ $(p \in \mathsf{Prop})$ for the projections, continuity of v is equivalent to $v^{-1}(\pi_p^{-1}(\{1\}))$ clopen for all $p \in \mathsf{Prop}$. Observing that $v^{-1}(\pi_p^{-1}(\{1\})) = \{x \in X \mid v(x)_p = 1\}$ is the extension of p the claim now follows from the fact that the clopens coincide with the admissible sets.

Let us note two corollaries of Theorem 3.13. Using $\mathsf{MA} \simeq \mathsf{DGF}^{\mathrm{op}}$ and $(\mathsf{Coalg}(\mathbb{V}))^{\mathrm{op}} = \mathsf{Alg}(\mathbb{V}^{\mathrm{op}})$, it follows $\mathsf{MA} \simeq \mathsf{Alg}(\mathbb{V}^{\mathrm{op}})$. With $\mathsf{Stone}^{\mathrm{op}} \simeq \mathsf{BA}$ we obtain the following.

Corollary 3.15 *There is a functor* $H : \mathsf{BA} \to \mathsf{BA}$ *such that the category of modal algebras* MA *is equivalent to the category* $\mathsf{Alg}(H)$ *of algebras for the functor* H*.*

Proof. With the help of the contravariant functors $\mathbb{C}\mathsf{lp} : \mathsf{Stone} \to \mathsf{BA}$, $\mathbb{S}\mathsf{p} : \mathsf{BA} \to \mathsf{Stone}$, we let $H = \mathbb{C}\mathsf{lp}\mathbb{V}\mathbb{S}\mathsf{p}$. The claim now follows from the observation that $\mathsf{Alg}(H)$ is dual to $\mathsf{Coalg}(\mathbb{V})$: An algebra $HA \xrightarrow{\alpha} A$ corresponds to the coalgebra $\mathbb{S}\mathsf{p}\,A \xrightarrow{\mathbb{S}\mathsf{p}\,\alpha} \mathbb{S}\mathsf{p}\,HA \cong \mathbb{V}\,\mathbb{S}\mathsf{p}\,A$ and a coalgebra $\mathbb{X} \xrightarrow{\xi} \mathbb{V}\mathbb{X}$ corresponds to the algebra $H\mathbb{C}\mathsf{lp}\mathbb{X} \cong \mathbb{C}\mathsf{lp}\mathbb{V}\mathbb{X} \xrightarrow{\mathbb{C}\mathsf{lp}\xi} \mathbb{C}\mathsf{lp}\mathbb{X}$. QED

An explicit description of H not involving the Vietoris functor is given by the following proposition.

Proposition 3.16 *Let* $H : \mathsf{BA} \to \mathsf{BA}$ *be the functor that assigns to a Boolean algebra the free Boolean algebra over its underlying meet-semilattice. Then* $\mathsf{Alg}(H)$ *is isomorphic to the category of modal algebras* MA*.*

Proof. We use the well-known fact that MA is isomorphic to the category MPF which is defined as follows. An object of MPF is an endofunction $A \xrightarrow{m} A$ on a Boolean algebra A that preserves finite meets (i.e. binary meets and the top-element). A morphism $f : (A \xrightarrow{m} A) \longrightarrow (A' \xrightarrow{m'} A')$ is a Boolean algebra morphism $f : A \to A'$ such that $m' \circ f = f \circ m$. We also write $\mathsf{BA}_\wedge$ for the category with Boolean algebras as objects and finite meet preserving functions as morphisms.

To prove that $\mathsf{Alg}(H)$ and MPF are isomorphic categories, we first show that $\mathsf{BA}(HA, A) \cong \mathsf{BA}_\wedge(A, A)$,or slightly more general and precise, $\mathsf{BA}(HA, B) \cong \mathsf{BA}_\wedge(IA, IB)$ where $I : \mathsf{BA} \hookrightarrow \mathsf{BA}_\wedge$. (Here we denote, for a category C and objects A, B in C, the set of morphisms between A and B by $\mathsf{C}(A, B)$.) Indeed, consider the forgetful functors $U : \mathsf{BA} \to \mathsf{SL}$, $V : \mathsf{BA}_\wedge \to \mathsf{SL}$ to the category SL of meet-semilattices with top element and the left adjoint F of U. Using our assumption $H = FU$, we calculate $\mathsf{BA}(HA, B) = \mathsf{BA}(FUA, B) \cong \mathsf{SL}(UA, UB) \cong \mathsf{SL}(VIA, VIB) \cong \mathsf{BA}_\wedge(IA, IB)$. The isomorphisms $\varphi_A : \mathsf{BA}(HA, A) \to \mathsf{BA}_\wedge(A, A)$, $A \in \mathsf{BA}$, give us an isomorphism φ between the objects of $\mathsf{Alg}(H)$ and MPF. On morphisms, we define φ to be the identity. This is well-defined because the isomorphisms $\mathsf{BA}(HA, B) \cong \mathsf{BA}_\wedge(IA, IB)$ are natural in A and B. QED

As another corollary to the duality we obtain that $\mathsf{Coalg}(\mathbb{V})$ has cofree coalgebras.

Corollary 3.17 *The forgetful functor* $\mathsf{Coalg}(\mathbb{V}) \to \mathsf{Stone}$ *has a right adjoint.*

Proof. Consider the forgetful functors $R : \mathsf{MA} \to \mathsf{BA}$, $U : \mathsf{MA} \to \mathsf{Set}$, $V : \mathsf{BA} \to \mathsf{Set}$. Since U and V are monadic, R has a left adjoint. Hence, by duality, $\mathsf{Coalg}(\mathbb{V}) \to \mathsf{Stone}$ has a right adjoint. QED

4 General Frames as Coalgebras

In this section we will show how arbitrary general frames can be seen as coalgebras. Stone spaces provide an appropriate framework to study descriptive general frames because, due to the compactness, the admissible sets can be recovered from the topology: each Stone space has a (unique) basis of clopens that is closed under boolean operations. In the case of general frames, giving up compactness, we work directly with the admissible sets.

Definition 4.1 The category RBA (referential or represented Boolean algebras) has objects (X, A) where X a set and A a set of 'admissible' subsets of X closed under boolean operations. RBA has morphisms $f : (X, A) \to (Y, B)$ where f is a function $X \to Y$ such that $f^{-1}(b) \in A$ for all $b \in B$. $\lhd$

In the absence of tightness, the relation of the general frame will no longer be point-closed. Hence, its coalgebraic version has the full power set (and not only the closed subsets) as its codomain.

Definition 4.2 For $\mathbb{X} = (X, A) \in \mathsf{RBA}$ let $\mathbb{W}(\mathbb{X}) = (\mathcal{P}(X), v_{\mathbb{X}})$ where $v_{\mathbb{X}}$ is the Boolean algebra generated by $\{\{F \in \mathcal{P}X \mid F \cap a \neq \emptyset\} \mid a \in A\}$. On morphisms let $\mathbb{W}(f) = \mathcal{P}(f)$. $\lhd$

To see that $\mathbb{W}$ is a functor, we argue as in Lemma 3.9: For an RBA-morphism $f : (X, A) \to (Y, B)$ and $b \in B$, we calculate $\mathbb{W}(f)^{-1}(\{G \in \mathcal{P}Y \mid G \cap b \neq \emptyset\}) = \{F \in \mathcal{P}X \mid (\mathcal{P}f)(F) \cap b \neq \emptyset\} = \{F \in \mathcal{P}X \mid F \cap f^{-1}(b) \neq \emptyset\}$ which indeed belongs to $v_{(X,A)}$.

The following observation is the decisive one.

Lemma 4.3 *Let* $\mathbb{X} = (X, A) \in \mathsf{RBA}$ *and* R *a relation on* X. *Then* A *is closed under* $\langle R \rangle$ *iff* $R[\_] : X \to \mathcal{P}X$ *is an* RBA*-morphism* $\mathbb{X} \to \mathbb{W}(\mathbb{X})$.

Proof. Immediate from $\langle R \rangle a = (R[\_])^{-1}(\{F \in \mathcal{P}X \mid F \cap a \neq \emptyset\})$ and the fact that $(R[\_])^{-1}$ preserves all Boolean operations. QED

Theorem 4.4 *The categories* GF *and* $\mathsf{Coalg}(\mathbb{W})$ *are isomorphic.*

Proof. We define a mapping $\mathbb{C} : \mathsf{GF} \to \mathsf{Coalg}(\mathbb{W})$ on objects

$$(X, R, A) \quad \mapsto \quad (X, A) \xrightarrow{R[\_]} \mathbb{W}(X, A)$$

and on morphisms as the identity on the underlying set-maps. It is immediate from the respective definitions that a map $X \to Y$ is a general frame homomorphism $(X, R, A) \to (Y, S, B)$ iff it is a $\mathbb{W}$-coalgebra morphism $((X, A), R[\_]) \to ((Y, B), S[\_])$, hence $\mathbb{C}$ is a full and faithful functor. Moreover, $\mathbb{C}$ is a bijection on objects due to Lemma 4.3. QED

For a Kripke frame (X, R) or a set-coalgebra (X, ξ), a valuation of propositional variables $p \in \mathsf{Prop}$ is a function $X \to \prod_{\mathsf{Prop}} 2$ where 2 is the two-element set of truth-values. For general frames (X, R, A) a valuation is a function $v : X \to \prod_{\mathsf{Prop}} 2$ with the property that the extension of each proposition $p \in \mathsf{Prop}$ is admissible, ie, that $\{x \in X \mid v(x)_p = 1\} \in A$. We want to show that this can be expressed in terms of RBA-coalgebras by the requirement that a valuation is not only a map $X \to \prod_{\mathsf{Prop}} 2$ but an RBA-morphism $v : (X, A) \to \prod_{\mathsf{Prop}}(2, \mathcal{P}2)$ where $\prod_{\mathsf{Prop}}(2, \mathcal{P}2)$ here denotes the product in RBA.

Proposition 4.5 (Products in RBA) *Let (Y_i, B_i), $i = 1, 2$, be objects in* RBA*. Then the product $(Y_1, B_1) \times (Y_2, B_2)$ is given by $(Y_1 \times Y_2, B)$ where B is the Boolean algebra generated by all $\pi_i^{-1}(b_i)$, $b_i \in B_i$ (where $\pi_i : Y_1 \times Y_2 \to Y_i$ denote the projections).*

Proof. Consider $f_i : (Z, C) \to (Y_i, B_i)$. Since $Y_1 \times Y_2$ is a product in Set there is a unique map $h : Z \to Y_1 \times Y_2$ such that $f_i = \pi_i \circ h$. We have to show that h is indeed an RBA morphism. This follows from $h^{-1}(\pi_i^{-1}(b)) = f_i^{-1}(b) \in C$ for all $b \in B_i$. QED

Remark 4.6 The category RBA has more nice properties that resemble topological spaces. For example, an analogue of the above proposition holds for limits and colimits in general.

The next proposition shows that a valuation $v : X \to \prod_{\mathsf{Prop}} 2$ is an RBA-morphism iff the extension of each proposition is an admissible set.

Proposition 4.7 (Valuations for RBA-coalgebras) *Let (X, A) be an* RBA *and* Prop *a set. The map $v : X \to \prod_{\mathsf{Prop}} 2$ is an* RBA*-morphism $(X, A) \to \prod_{\mathsf{Prop}}(2, \mathcal{P}2)$ iff $\{x \in X \mid v(x)_p = 1\} \in A$ for all $p \in$* Prop.

5 Vietoris Polynomial Functors

In this section we introduce the notion of a Vietoris polynomial functor (short: VPF) as a natural analogue for the category Stone of what the so-called Kripke polynomial functors [28, 15] are for Set. This section can be therefore seen as a first application of the observation that coalgebras over Stone can be used as semantics for (coalgebraic) modal logics. Although we have kept most of this section self-contained, much of its content builds on the work by Jacobs in [15].

5.1 Polynomial functors

Definition 5.1 (Vietoris polynomial functors) The collection of *Vietoris polynomial functors*, in brief: VPFs, over Stone is inductively defined as follows:

$$T ::= \mathbb{I} \mid \mathbb{Q} \mid T_1 + T_2 \mid T_1 \times T_2 \mid T^D \mid \mathbb{V}T.$$

Here $\mathbb{I}$ is the identity functor on the category Stone; $\mathbb{Q}$ denotes a finite Stone space (that is, the functor $\mathbb{Q}$ is a constant functor); '+' and '×' denote disjoint union and binary product, respectively; and, for an arbitrary set D, T^D denotes the functor sending a Stone space $\mathbb{X}$ to the D-fold product[2] $(T(\mathbb{X})^D)$.

[2] We leave it as an exercise for the reader to verify that the class of Stone spaces is closed under taking topological products.

Associated with this we inductively define the notion of a *path*:

$$p \ ::= <> \mid \pi_1 \cdot p \mid \pi_2 \cdot p \mid \kappa_1 \cdot p \mid \kappa_2 \cdot p \mid [\mathrm{ev}(d)] \cdot p \mid \mathbb{V} \cdot p.$$

By induction on the complexity of paths we now define when two VPFs T_1 and T_2 are related by a path p, notation: $T_1 \overset{p}{\leadsto} T_2$:

$$\begin{array}{rclll}
T & \overset{<>}{\leadsto} & T & & \\
T_1 \times T_2 & \overset{\pi_i \cdot p}{\leadsto} & T' & \text{if} & T_i \overset{p}{\leadsto} T' \\
T_1 + T_2 & \overset{\kappa_i \cdot p}{\leadsto} & T' & \text{if} & T_i \overset{p}{\leadsto} T' \\
T^D & \overset{[\mathrm{ev}(d)] \cdot p}{\leadsto} & T' & \text{if} & T \overset{p}{\leadsto} T' \text{ and } d \in D \\
\mathbb{V}T & \overset{\mathbb{V} \cdot p}{\leadsto} & T' & \text{if} & T \overset{p}{\leadsto} T'.
\end{array}$$

Finally, for a VPF T we define $\mathsf{Ing}(T)$ to be the category with the set $\mathrm{Ing}(T) := \{S \mid \exists p.T \overset{p}{\leadsto} S\} \cup \{\mathbb{I}\}$ as the set of objects and the paths as morphisms between them. $\lhd$

Remark 5.2 There are at least two natural ways in which we could have generalized our definition of a Vietoris polynomial functor, while preserving all of the results in this and the next section. First, there is no reason why we should restrict our discussion to *finite* constants $\mathbb{Q}$. And second, dually to the Boolean product construction for Boolean algebras one can also define an infinite, 'Stone sum' of a collection of Stone spaces, provided that the index set is endowed with a Stone topology as well; we refer to Gehrke [11] for details on this construction.

Our reason to confine ourselves to the standard case, in which only finite constants and finite sums are allowed, is that we want to stay as close as possible to the work of Jacobs.

5.2 Algebras

It follows from the general definition of coalgebras, what the definition of a T-coalgebra is for an arbitrary VPF T. Dually, we will make good use of a kind of algebras for T; the definition of a so-called T-BAO may look slightly involved, but it is based on a simple generalization of the concept of a modal algebra. The generalization is that instead of dealing with one single Boolean algebra, we will be working with a *family* $(\Phi(S))_{S \in \mathrm{Ing}(T)}$ of Boolean algebras, linked by finite-meet preserving operations. As before, we let $\mathsf{BA}_\wedge$ denote the category with Boolean algebras as objects and finite-meet preserving operations as morphisms.

Definition 5.3 (T-BAO) Let T be a VPF. A *T-sorted Boolean algebra with operators*, T-BAO, consists of

- a functor $\Phi : \mathsf{Ing}(T)^{\mathrm{op}} \longrightarrow \mathsf{BA}_\wedge$, together with
- an additional map $\mathsf{next} : \Phi(T) \to \Phi(\mathbb{I})$ which preserves all Boolean operations.

This functor is supposed to satisfy the following conditions:

1. $\Phi(\mathbb{Q}) = \mathsf{Clp}_{\mathbb{Q}}$

2. the functions $\Phi(\pi_i)$ and $\Phi([\mathrm{ev}(d)])$ are Boolean homomorphisms

3. the functions $\Phi(\kappa_i)$ induced by the injection paths satisfy

$$\begin{array}{rcl} -\Phi(\kappa_1)(\bot) \vee -\Phi(\kappa_2)(\bot) & = & \top \\ -\Phi(\kappa_1)(\bot) \wedge -\Phi(\kappa_2)(\bot) & = & \bot \\ -\Phi(\kappa_i)(\bot) \wedge \Phi(\kappa_i)(-\alpha) & \leq & -\Phi(\kappa_i)(\alpha) \end{array}$$

$\lhd$

Example 5.4 Let $\mathbb{A} = (A, \wedge, -, \bot, \top, g)$ be a modal algebra, cf. Definition 3.1. This algebra gives rise to two different Boolean Algebras with Operators for the functor $\mathbb{V}\mathbb{I}$. Note that $\mathrm{Ing}(\mathbb{V}\mathbb{I}) = \{\mathbb{I}, \mathbb{V}\mathbb{I}\}$ and $\mathbb{V}\mathbb{I} \stackrel{\mathbb{V}}{\rightsquigarrow} \mathbb{I}$.

1. If we define $\Phi(\mathbb{I}) := \mathbb{A}$, $\Phi(\mathbb{V}\mathbb{I}) := \mathbb{A}$, $\Phi(\mathbb{V}) := g$, and take $\mathsf{next} : \Phi(\mathbb{V}\mathbb{I}) \to \Phi(\mathbb{I})$ to be the identity map, we get a $\mathbb{V}\mathbb{I}$-BAO (Φ, next) that corresponds to the original modal algebra.

2. Again $\Phi'(\mathbb{I}) := \mathbb{A}$. But let $\Phi'(\mathbb{V}\mathbb{I})$ be the free Boolean algebra over the meet-semilattice $\mathbb{A}$ (ie, in the notation from Proposition 3.16, $\Phi'(\mathbb{V}\mathbb{I}) = H\mathbb{A}$). Let $\Phi'(\mathbb{V}) : \Phi'(\mathbb{A}) \hookrightarrow \Phi'(\mathbb{V}\mathbb{I})$ be the (meet-preserving) inclusion of generators and next' the unique Boolean algebra morphism satisfying $\mathsf{next}' \circ \Phi'(\mathbb{V}) = g$. Then (Φ', next') is the $\mathbb{V}\mathbb{I}$-BAO obtained by considering the algebra (Φ, next) from the previous item as a $\mathbb{V}\mathbb{I}$-coalgebra and translating it back to an algebra. That is, in the notation of the next subsection, $(\Phi', \mathsf{next}') = \mathcal{AC}(\Phi, \mathsf{next})$.

We leave it for the reader to verify that the following is the natural generalization of the notion of a homomorphism between modal algebras.

Definition 5.5 (BAO_T) Let T be a Vietoris polynomial functor; a *morphism* from one T-BAO (Φ', next') to another (Φ, next) is a natural transformation $t : \Phi' \to \Phi$ such that for each ingredient S of T the component $t_S : \Phi'(S) \to \Phi(S)$ preserves the Boolean structure, such that $t_{\mathbb{I}}$ and t_T satisfy the following naturality condition with respect to next and next':

$$\mathsf{next} \circ t_T = t_{\mathbb{I}} \circ \mathsf{next}',$$

and such that $t_{\mathbb{Q}} = \mathsf{id}_{\mathsf{Clp}_{\mathbb{Q}}}$ for all constants $\mathbb{Q} \in \mathsf{Ing}(T)$. This yields the category BAO_T. $\lhd$

Example 5.6 With the notation from Example 5.4, $t : (\Phi', \mathsf{next}') \to (\Phi, \mathsf{next})$ defined by $t_{\mathbb{I}} = id$ and $t_{\mathbb{V}\mathbb{I}} = \mathsf{next}'$ is a BAO_T-morphism. It will be called $\alpha_{(\Phi,\mathsf{next})}$ in Section 6.2.

5.3 From coalgebras to algebras and back

It is not difficult to transform a T-coalgebra into a T-BAO; basically, we are dealing with a sorted version of Stone duality (see Definition 2.5 for terminology and notation), together with a path-indexed predicate lifting.

Lemma and Definition 5.7 *Let T be a VPF and let $\mathbb{X}$ be a Stone space. Then the following definition on the complexity of paths*

$$
\begin{array}{rcll}
\alpha^{<>} & := & \alpha & \\
\alpha^{\pi_1 \cdot p} & := & \pi_1^{-1}(\alpha^p) & \\
\alpha^{\pi_2 \cdot p} & := & \pi_2^{-1}(\alpha^p) & \\
\alpha^{\kappa_1 \cdot p} & := & \kappa_1(\alpha^p) \cup \kappa_2 S_2(X) & \textit{for } T_2 = S_1 + S_2 \\
\alpha^{\kappa_2 \cdot p} & := & \kappa_1 S_1(X) \cup \kappa_2(\alpha^p) & \textit{for } T_2 = S_1 + S_2 \\
\alpha^{[\mathrm{ev}(d)] \cdot p} & := & \pi_d^{-1}(\alpha^p) & \\
\alpha^{\mathbb{V} \cdot p} & := & \{\beta \mid \beta \subseteq \alpha^p \textit{ and } \beta \textit{ closed}\} & (= [\ni]\alpha)
\end{array}
$$

provides, for any path $T_1 \overset{p}{\leadsto} T_2$, a so-called predicate lifting

$$(\_)^p : \mathsf{Clp}_{T_2\mathbb{X}} \to \mathsf{Clp}_{T_1\mathbb{X}}.$$

Lemma 5.8 *For each Vietoris polynomial functor T, each T-coalgebra $(\mathbb{X}, \xi)$ gives rise to a T-BAO, namely, the 'complex algebra' functor $\mathcal{A}(\mathbb{X}, \xi) : \mathsf{Ing}(T)^{\mathrm{op}} \to \mathsf{BA}_\wedge$ given by*

$$
\begin{array}{rcl}
S & \mapsto & \mathbb{C}\mathsf{lp}S(\mathbb{X}) \\
\left(S_1 \overset{p}{\leadsto} S_2\right) & \mapsto & \left((\_)^p : \mathbb{C}\mathsf{lp}S_2(\mathbb{X}) \to \mathbb{C}\mathsf{lp}S_1(\mathbb{X})\right),
\end{array}
$$

accompanied by the map $\mathsf{next} : \mathbb{C}\mathsf{lp}(T\mathbb{X}) \to \mathbb{C}\mathsf{lp}(\mathbb{X})$ *given by* $\mathsf{next} := \xi^{-1}$.

Proof. To start with, we need to show that $\mathcal{A}(\mathbb{X}, \xi)$ is a functor from $\mathsf{Ing}(T)^{\mathrm{op}}$ to $\mathsf{BA}_\wedge$. To that aim, one has to prove that the predicate lifting $(\_)^p : \mathsf{Clp}_{T_1\mathbb{X}} \to \mathsf{Clp}_{T_2\mathbb{X}}$ constitutes a $\mathsf{BA}_\wedge$-morphism between $\mathbb{C}\mathsf{lp}T_1\mathbb{X}$ and $\mathbb{C}\mathsf{lp}T_2\mathbb{X}$; and that it satisfies the functorial laws.

Finally, we have to show that the functor $\mathcal{A}(\mathbb{X}, \xi) : \mathsf{Ing}(T)^{\mathrm{op}} \longrightarrow \mathsf{BA}_\wedge$, together with the map $\mathsf{next} := \xi^{-1}$, meets the requirements listed in Definition 5.3. All of these results can be proved in a fairly straightforward way. QED

Conversely, with each T-BAO Φ we may associate a T-coalgebra $\mathcal{C}(\Phi)$. Assume that T has the identity functor as an ingredient; given our results in the previous section, and the well-known Stone duality, it seems fairly obvious that we should take the dual Stone space $\mathbb{S}\mathsf{p}\,\Phi(\mathbb{I})$ as the carrier of this dual coalgebra. However, how to obtain T-coalgebra structure on this? Applying duality theory to the Boolean algebras obtained from Φ only seems to provide information on the spaces $\mathbb{S}\mathsf{p}\,\Phi(S)$, whereas we need to work with $S(\mathbb{S}\mathsf{p}\,\Phi(\mathbb{I}))$ in order to correctly define a T-coalgebra. Fortunately, in the next lemma and definition we show that there exists a map r which produces the S-structure. The definition of r is taken from [15]; what we have to show is that it works also in the topological setting.

Lemma and Definition 5.9 *(r_Φ) Let T be a VPF and let (Φ, next) be a T-BAO. Then the* *following definition by induction on the structure of ingredient functors of T*

$$
\begin{array}{rcl}
r_\Phi(\mathbb{I})(U) & := & U \\
r_\Phi(\mathbb{Q})(U) & := & a \quad \textit{where} \bigcap U = \{a\} \\
r_\Phi(S_1 \times S_2)(U) & := & \big\langle r_\Phi(S_1)(\Phi(\pi_1)^{-1}(U)), r_\Phi(S_2)(\Phi(\pi_2)^{-1}(U))\big\rangle \\
r_\Phi(S_1 + S_2)(U) & := & \begin{cases} \kappa_1 r_\Phi(S_1)(\Phi(\kappa_1)^{-1}(U)) \textit{ if } -\Phi(\kappa_1)(\bot) \in U \\ \kappa_2 r_\Phi(S_2)(\Phi(\kappa_2)^{-1}(U)) \textit{ if } -\Phi(\kappa_2)(\bot) \in U \end{cases} \\
r_\Phi(S^D)(U) & := & \lambda d \in D . r_\Phi(S)(\Phi(ev(d))^{-1}(U)) \\
r_\Phi(\mathbb{V}S)(U) & := & \big\{ r_\Phi(S)(V) \mid V \in \mathsf{Sp}\,\Phi(S) \textit{ and } \Phi(\mathbb{V})^{-1}(U) \subseteq V \big\}
\end{array}
$$

defines, for every $S \in \mathrm{Ing}(T)$ *a continuous map*

$$r_\Phi(S) : \mathbb{S}\mathsf{p}(\Phi(S)) \longrightarrow S(\mathbb{S}\mathsf{p}(\Phi(\mathbb{I}))).$$

Furthermore, the inverse image map next^{-1} *is a continuous map*

$$\mathsf{next}^{-1} : \mathbb{S}\mathsf{p}(\Phi(\mathbb{I})) \longrightarrow T(\mathbb{S}\mathsf{p}(\Phi(\mathbb{I}))).$$

Proof of 5.9. Let $S \in \mathsf{Ing}(T)$. Both claims (that is, the one concerning well-definedness and the one concerning the continuity of $r_\Phi(S)$) are proven simultaneously by induction on S. We only treat some crucial cases to prove our claim:

Case $S = \mathbb{I}$. Then $r_\Phi(\mathbb{I})$ is the identity function and therefore well-defined and continuous.

Case $S = \mathbb{Q}$. Then $\Phi(\mathbb{Q}) = \mathsf{Clp}_\mathbb{Q}$ and $r_\Phi(\mathbb{Q})$ is just the inverse of the isomorphism $\epsilon_\mathbb{Q}$, which is given by Stone duality (cf. Definition 2.6).

Case $S = S_1 \times S_2$: We only show that $r_\Phi(S)$ is continuous. Let $W_i \subseteq S_i(\mathsf{Sp}\,\Phi(\mathrm{Id}))$ be clopen sets for $i = 1, 2$. We show that $r_\Phi(S_1 \times S_2)^{-1}(W_1 \times W_2)$ is again clopen.

$$
\begin{array}{rcl}
r_\Phi(S_1 \times S_2)^{-1}(W_1 \times W_2) & = & \big\{ V \mid r_\Phi(S_i)\left(\Phi(\pi_i)^{-1}(V)\right) \in W_i \text{ for } i = 1,2 \big\} \\
& = & \big\{ V \mid \Phi(\pi_i)^{-1}(V) \in r_\Phi(S_i)^{-1}(W_1) \text{ for } i = 1,2 \big\}
\end{array}
$$

By I.H. we know that $r_\Phi(S_i)^{-1}(W_i)$ is a clopen set for $i = 1, 2$, say

$$r_\Phi(S_i)^{-1}(W_i) = \hat{a}_i \quad i = 1, 2.$$

Then

$$
\begin{array}{rcl}
r_\Phi(S_1 \times S_2)^{-1}(W_1 \times W_2) & = & \big\{ V \mid \Phi(\pi_i)^{-1}(V) \in \hat{a}_i \text{ for } i = 1,2 \big\} \\
& = & \{ V \mid \Phi(\pi_i)(a_i) \in V \text{ for } i = 1,2 \} \\
& = & \{ V \mid \Phi(\pi_1)(a_1) \wedge \Phi(\pi_2)(a_2) \in V \}
\end{array}
$$

Hence $r_\Phi(S_1 \times S_2)^{-1}(W_1 \times W_2)$ is of the form $\hat{b}$ for some $b \in \Phi(S_1 \times S_2)$, i.e. it is a clopen set in the Stone topology.

Case $S = \mathbb{V}S'$ Let $U \in \mathsf{Sp}\,\Phi(\mathbb{V}S')$ We first show that the set

$$F := \big\{V \mid V \in \mathsf{Sp}\,\Phi(S') \text{ and } \Phi(\mathbb{V})^{-1}(U) \subseteq V\big\}$$

is closed in $S'(\mathsf{Sp}\,\Phi(\mathbb{I}))$.

Let $V' \notin F$. Then there is an $a \in \Phi(\mathbb{V})^{-1}(U)$ such that $-a \in V'$. We therefore get $F \subseteq \hat{a}$ and $V' \in -\hat{a}$. This implies that we can find for any $V' \notin F$ an open set that contains V' and has an empty intersection with F and thus shows that F is closed. Because of Lemma 3.8 and the induction hypothesis it now follows that

$$r_\Phi(S)(U) = r_\Phi(S')[F]$$

is closed and therefore $r_\Phi(S)$ is well-defined. We now prove the continuity of $r_\Phi(S)$. Let $O \subseteq S'(\mathsf{Sp}\,\Phi(\mathbb{I}))$ be clopen. Then

$$\begin{aligned} r_\Phi(\mathbb{V}S')^{-1}([\ni](O)) &= \big\{U \in \mathsf{Sp}\,\Phi(\mathbb{V}S') \mid r_\Phi(\mathbb{V}S')(U) \in [\ni](O)\big\} \\ &= \big\{U \mid \big\{r_\Phi(S')(V) \mid \Phi(\mathbb{V})^{-1}(U) \subseteq V\big\} \subseteq O\big\} \\ &= \big\{U \mid \big\{V \mid \Phi(\mathbb{V})^{-1}(U) \subseteq V\big\} \subseteq r_\Phi(S')^{-1}(O)\big\} \end{aligned}$$

According to the induction hypothesis we have that $r_\Phi(S')^{-1}(O)$ is a clopen set, say with $b \in \Phi(S')$ s.t. $r_\Phi(S')^{-1}(O) = \hat{b}$. This leads us to

$$\begin{aligned} r_\Phi(\mathbb{V}S')^{-1}([\ni](O)) &= \Big\{U \mid \big\{V \mid \Phi(\mathbb{V})^{-1}(U) \subseteq V\big\} \subseteq \hat{b}\Big\} \\ &= \big\{U \mid \forall V \in \mathsf{Sp}\,\Phi(S')\,.\,\big(\Phi(\mathbb{V})^{-1}(U) \subseteq V \to b \in V\big)\big\} \\ &\overset{(!)}{=} \{U \mid \Phi(\mathbb{V})(b) \in U\} \end{aligned}$$

and we proved that $r_\Phi(\mathbb{V}S')^{-1}([\ni](O))$ is a clopen in the Stone topology. It remains to show that the equality (!) indeed holds:

$\supseteq$: trivial

$\subseteq$: Suppose $\Phi(\mathbb{V})(b) \notin U$. We will show that under this assumption there exists a $V' \in \mathsf{Sp}\,\Phi(S')$ such that $\Phi(\mathbb{V})^{-1}(U) \subseteq V'$ and $b \notin V'$. To that aim let $a_1, \ldots, a_n \in \Phi(\mathbb{V})^{-1}(U')$ such that

$$\bigwedge_{1 \le i \le n} a_i \wedge \neg b = \perp .$$

Then we have $\bigwedge_{1 \le i \le n} a_i \le b$ and therefore we get by monotonicity of $\Phi(\mathbb{V})$

$$U' \ni \Phi(\mathbb{V})\left(\bigwedge_{1 \le i \le n} a_i\right) \le \Phi(\mathbb{V})(b).$$

As U' is an ultrafilter we can conclude that $\Phi(\mathbb{V})(b) \in U'$, which contradicts our first assumption. This means that the set $\Phi(\mathbb{V})^{-1}(U') \cup \{\neg b\}$ has the finite intersection property and is contained in an ultrafilter V'. But this implies $U' \notin \{U \mid \Phi(\mathbb{V})^{-1}(U) \subseteq V \to b \in V\}$ and we proved our claim.

Let us now consider the sets of the form $r_\Phi(\mathbb{V}S')^{-1}(\langle\ni\rangle(O))$:

$$\begin{aligned} r_\Phi(\mathbb{V}S')^{-1}(\langle\ni\rangle(O)) &= r_\Phi(\mathbb{V}S')^{-1}(-([\ni](-O))) \\ &= -r_\Phi(\mathbb{V}S')^{-1}([\ni](-O)) \end{aligned}$$

Hence $r_\Phi(\mathbb{V}S')^{-1}(\langle\ni\rangle(O))$ is the complement of a clopen set and therefore a clopen set. The claim on the map next^{-1} is a simple consequence of Stone duality. QED

The above lemma allows us to define a T-coalgebra for a given T-BAO.

Definition 5.10 Let T be a VPF and let (Φ, next) be a T-BAO. We define the coalgebra $\mathcal{C}(\Phi, \mathsf{next})$ as the structure $(\mathbb{S}\mathsf{p}(\Phi(\mathbb{I})), r_\Phi(T) \circ \mathbb{S}\mathsf{p}(\mathsf{next}))$. ◁

5.4 Representing coalgebras

The maps $\mathcal{A}$ and $\mathcal{C}$ that allow us to move from a given T-BAO to a T-coalgebra and vice versa can be extended to functors. In Section 6 we will see that $\mathcal{C}$ is in fact right adjoint to $\mathcal{A}$; the main result in this subsection, Theorem 5.13 below, states that every coalgebra $(\mathbb{X}, \xi)$ is isomorphic to the 'ultrafilter coalgebra' obtained from the complex algebra of $(\mathbb{X}, \xi)$:

$$(\mathbb{X}, \xi) \cong \mathcal{C}(\mathcal{A}(\mathbb{X}, \xi)).$$

Now for the details. Fix a Vietoris polynomial functor T, and let $f : (\mathbb{X}, \xi) \to (\mathbb{X}', \xi')$ be a $\mathsf{Coalg}(T)$-morphism. Then we define $\mathcal{A}(f) : \mathcal{A}(\mathbb{X}', \xi') \to \mathcal{A}(\mathbb{X}, \xi)$ as follows. For each $S \in \mathrm{Ing}(T)$ let $\mathcal{A}(f)(S) := \mathbb{C}\mathsf{lp}(S(f))$. Naturality of $\mathcal{A}(f)$ can be proven by induction on paths and the additional condition in Definition 5.5 concerning the next functions is fulfilled because f is a T-coalgebra homomorphism.

Conversely, given a BAO_T-morphism $t : (\Phi, \mathsf{next}) \to (\Phi', \mathsf{next}')$, define the map $\mathcal{C}(t) : \mathsf{Sp}(\Phi'(\mathbb{I})) \to \mathsf{Sp}(\Phi(\mathbb{I}))$ to be the inverse image map of $t_{\mathbb{I}} : \Phi(\mathbb{I}) \to \Phi'(\mathbb{I})$. We leave it to the reader to verify that $\mathcal{C}(t)$ is in fact a $\mathsf{Coalg}(T)$ morphism between $\mathcal{C}(\Phi, \mathsf{next})$ and $\mathcal{C}(\Phi', \mathsf{next}')$ (cf. the proof of Proposition 5.3 in [15]).

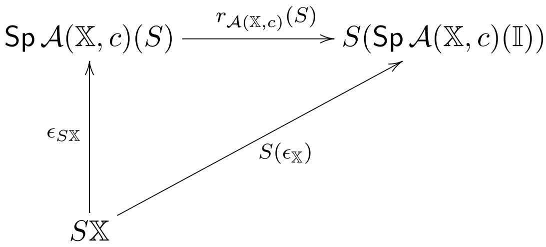
Lemma 5.11 *If we extend $\mathcal{A}$ and $\mathcal{C}$ as described above we obtain functors*

$$\mathcal{A} : \mathsf{Coalg}(T)^{\mathrm{op}} \to \mathsf{BAO}_T \quad \textit{and} \quad \mathcal{C} : \mathsf{BAO}_T \to \mathsf{Coalg}(T)^{\mathrm{op}}.$$

Proof. We already provided the arguments why $\mathcal{A}$ and $\mathcal{C}$ are well-defined. That they preserve the composition of morphisms and identities is obvious. QED

We now want to prove that any T-coalgebra has an "ultrafilter representation", i.e. every T-coalgebra is isomorphic to its double dual. To this end we first have to prove the following technical lemma.

Lemma 5.12 *Let $(\mathbb{X}, c)$ be a T-coalgebra. Then for each sort $S \in \mathsf{Ing}(T)$ the following diagram commutes:*



where $\epsilon_{\mathbb{X}} : \mathbb{X} \to \mathsf{Sp}(\mathsf{Clp}_{\mathbb{X}})$ *is defined as in Definition 2.6, i.e.* $\epsilon_{\mathbb{X}}(x) := \{\alpha \in \mathsf{Clp}_{\mathbb{X}} \mid x \in \alpha\}$.

Proof. The proof goes by induction on S. We will only treat the induction step where $S = \mathbb{V}S'$, since all other steps work exactly as in the proof of Lemma 5.6 in [15]. In order to prove the commutativity of the above diagram for $S = \mathbb{V}S'$, take an arbitrary $F \in \mathbb{V}S'(X)$. Then, by definition of r, we find

$$r_{\mathcal{A}(\mathbb{X},c)}(S)(\epsilon_{S\mathbb{X}}(F)) \;=\; r_{\mathcal{A}(\mathbb{X},c)}(S')\left[\{V \mid \mathcal{A}(\mathbb{X},c)(\mathbb{V})^{-1}(\epsilon_{S\mathbb{X}}(F)) \subseteq V\}\right].$$

Now observe that by definition of $\mathcal{A}$ and of $(\cdot)^{\mathbb{V}}$, respectively, we find that

$$(\mathcal{A}(\mathbb{X},c)(\mathbb{V}))^{-1}(\epsilon_{S\mathbb{X}}(F)) \;=\; \{\alpha \in \mathsf{Clp}_{S'\mathbb{X}} \mid (\alpha)^{\mathbb{V}} \in \epsilon_{S\mathbb{X}}(F)\} \;=\; \{\alpha \in \mathsf{Clp}_{S'\mathbb{X}} \mid F \subseteq \alpha\}. \tag{5}$$

It follows that

$$\begin{aligned} r_{\mathcal{A}(\mathbb{X},c)}(S)(\epsilon_{S\mathbb{X}}(F)) &\overset{(5)}{=} r_{\mathcal{A}(\mathbb{X},c)}(S')\,[\{V \mid \{\alpha \in \mathsf{Clp}_{S'\mathbb{X}} \mid F \subseteq \alpha\} \subseteq V\}] \\ &\overset{(!)}{=} \left\{S'(\epsilon_{\mathbb{X}})(u) \mid u \in F\right\} \\ &= S(\epsilon_{\mathbb{X}})(F) \end{aligned}$$

All that is left now is to prove (!). For the inclusion ($\supseteq$), let $u \in F$. Then we take $V := \epsilon_{S'\mathbb{X}}(u)$. By the inductive hypothesis we have

$$S'(\epsilon_{\mathbb{X}})(u) = r_{\mathcal{A}(\mathbb{X},c)}(S')(\epsilon_{S_1\mathbb{X}}(u))$$

and we have $\{\alpha \mid F \subseteq \alpha\} \subseteq V$. This gives us

$$S'(\epsilon_{\mathbb{X}})(u) \in r_{\mathcal{A}(c)}(S')\,[\{V \mid \{\alpha \in \mathsf{Clp}_{S'\mathbb{X}} \mid F \subseteq \alpha\} \subseteq V\}]\,.$$

For the converse inclusion ($\subseteq$), let $V \in \mathsf{Sp}\,\mathcal{A}(\mathbb{X},c)(S')$ be such that $\{\alpha \in \mathsf{Clp}_{S'\mathbb{X}} \mid F \subseteq \alpha\} \subseteq V$. By Stone duality we know that $\bigcap_{\alpha\in V} \alpha = \{u\}$ for exactly one $u \in S'\mathbb{X}$. This u must be an element of F, because

$$\bigcap_{\alpha\in V} \alpha \subseteq \bigcap\{\alpha \mid F \subseteq \alpha\} = F$$

and we get $\epsilon_{S'\mathbb{X}}(u) = V$. By the induction hypothesis this is the same as saying

$$r_{\mathcal{A}(\mathbb{X},c)}(S')(V) = S'(\epsilon_X)(u)$$

and this proves the inclusion. QED

The lemma gives us what we need to prove the following result, which we tend to see as a *representation theorem* stating that every T-coalgebra can be represented as the ultrafilter coalgebra of some T-BAO.

Theorem 5.13 *Let T be a Vietoris polynomial functor, and let $(\mathbb{X}, \xi)$ be a T-coalgebra. Then the map $\epsilon_{\mathbb{X}} : \mathbb{X} \to \mathsf{Sp}(\mathsf{Clp}_{\mathbb{X}})$ defined by $\epsilon_{\mathbb{X}}(x) := \{C \in \mathsf{Clp}_{\mathbb{X}} \mid x \in C\}$ is a $\mathsf{Coalg}(T)$-isomorphism witnessing that*

$$(\mathbb{X}, \xi) \cong \mathcal{C}(\mathcal{A}(\mathbb{X}, \xi)).$$

Proof. We calculate

$$\begin{array}{rclcl}
\mathcal{C}(\mathcal{A}(\mathbb{X}, c)) \circ \epsilon_{\mathbb{X}} & = & (r_{\mathcal{A}(\mathbb{X},c)}(T) \circ \mathbb{S}\mathsf{p}\,\mathbb{C}\mathsf{lp}(c)) \circ \epsilon_{\mathbb{X}} & = & r_{\mathcal{A}(\mathbb{X},c)}(T) \circ (\mathbb{S}\mathsf{p}\,\mathbb{C}\mathsf{lp}(c) \circ \epsilon_{\mathbb{X}}) \\
& \overset{\text{naturality of } \epsilon}{=} & r_{\mathcal{A}(\mathbb{X},c)}(T) \circ (\epsilon_{T\mathbb{X}} \circ c) & = & (r_{\mathcal{A}(\mathbb{X},c)}(T) \circ \epsilon_{T\mathbb{X}}) \circ c \\
& \overset{\text{Lemma 5.12}}{=} & T(\epsilon_{\mathbb{X}}) \circ c & &
\end{array}$$

i.e. $\epsilon_{\mathbb{X}}$ is a coalgebra homomorphism:

$$\begin{array}{ccc}
T\mathbb{X} & \xrightarrow{T(\epsilon_{\mathbb{X}})} & T\,\mathsf{Sp}\,\mathsf{Clp}_{\mathbb{X}} \\
{\scriptstyle c}\uparrow & & \uparrow{\scriptstyle \mathcal{C}(\mathcal{A}(\mathbb{X},c))} \\
\mathbb{X} & \xrightarrow[\epsilon_{\mathbb{X}}]{} & \mathsf{Sp}\,\mathsf{Clp}_{\mathbb{X}}
\end{array}$$

Because of Stone duality we know, that $\epsilon_{\mathbb{X}}$ is an isomorphism between Stone spaces and we can conclude that it is also an isomorphism between the two given coalgebras. QED

6 Representation and duality theorems

In the previous section, it has been shown that a T-coalgebra $(\mathbb{X}, c)$ can be represented as $\mathcal{C}(\Phi)$ where Φ is the T-BAO $\mathcal{A}(\mathbb{X}, c)$. More precisely, we have shown that there are functors

$$\mathcal{A} : \mathsf{Coalg}(T)^{\mathrm{op}} \to \mathsf{BAO}_T \qquad \text{and} \qquad \mathcal{C} : \mathsf{BAO}_T \to \mathsf{Coalg}(T)^{\mathrm{op}}$$

such that that for any coalgebra $(\mathbb{X}, c)$ there is an isomorphism $\epsilon_{(\mathbb{X},c)} : (\mathbb{X}, c) \to \mathcal{C}\mathcal{A}(\mathbb{X}, c)$.

To get a full representation theorem, in Section 6.2 we show that, similarly, for any T-BAO Φ there is a morphism

$$\alpha_\Phi : \mathcal{A}\mathcal{C}\Phi \to \Phi,$$

and, moreover, that $\mathcal{C}$ is right adjoint to $\mathcal{A}$ with α and the inverse γ of ϵ as unit and counit. Or, to put it differently, $\mathsf{Coalg}(T)^{\mathrm{op}}$ is (isomorphic to) a full coreflective subcategory of BAO_T. Section 6.1 provides the crucial technical lemma.

In contrast to the classical case of the duality $\mathsf{MA} \simeq \mathsf{DGF}^{\mathrm{op}}$, we do not obtain a dual equivalence between BAO_T and $\mathsf{Coalg}(T)$. The reader might have noticed already that this is due to the fact that the axiomatic definition of T-BAOs does not force a T-BAO Φ to respect T-structure. In section 6.3 we take a closer look at this, characterising the largest full subcategory of BAO_T on which the adjunction restricts to an equivalence. By showing that the initial algebra of BAO_T is *exact*, that is, belongs to this subcategory, we obtain the final T-coalgebra as its dual.

6.1 $\mathcal{C}$ is almost faithful

The functor $\mathcal{C}$ is not faithful in general; however, when it comes to morphisms having a complex algebra as their domain, we can prove the following.

Proposition 6.1 *Let $(\mathbb{X}, c)$ be a T-coalgebra and Φ be a T-BAO. Furthermore let $v, v' : \mathcal{A}(\mathbb{X}, c) \to \Phi$ be morphisms in BAO_T. Then $\mathcal{C}(v) = \mathcal{C}(v')$ implies $v = v'$.*

Proof. Let $(\mathbb{X}, c)$, Φ, v and v' be as in the statement of the Proposition, and assume that $\mathcal{C}(v) = \mathcal{C}(v')$. Then it is clear that we have $v_{\mathbb{I}} = v'_{\mathbb{I}}$. With the help of Lemma 6.2 below we therefore get $v = v'$. QED

The following lemma is the heart of the proof of Proposition 6.1. We state it separately because will need it later in Section 6.3.

Lemma 6.2 *Let $(\mathbb{X}, c)$ be a T-coalgebra and Φ a T-BAO. Furthermore let $v, v' : \mathcal{A}(\mathbb{X}, c) \to \Phi$ be natural transformations such that their components preserve all the Boolean structure, $v_{\mathbb{I}} = v'_{\mathbb{I}}$ and $v_{\mathbb{Q}} = v'_{\mathbb{Q}}$ for all constants $\mathbb{Q} \in \mathsf{Ing}(T)$. Then $v = v'$.*

Proof. Assume that we have two natural transformations $v, v' : \mathcal{A}(\mathbb{X}, c) \to \Phi$ as required in the lemma. In order to prove that $v = v'$, it suffices to show that

$$v_S = v'_S \text{ for all } S \in \mathsf{Ing}(T). \tag{6}$$

We will prove (6) by induction on S.

For the base case of the induction, there are two cases to consider: $S = \mathbb{I}$ and $S = \mathbb{Q}$ for some constant functor $\mathbb{Q}$. But in both cases it is in fact immediate that $v_S = v'_S$.

In the inductive case of the proof we will also make a case distinction. In each case, in order to show that $v_S(U) = v_{S'}(U)$ for every clopen U of $S\mathbb{X}$, we will first find a clopen subbasis $\mathcal{B}$ such that $v_S(W) = v'_S(W)$ for all subbasic W. This is sufficient, because of the following two facts. First, if $\mathcal{B}$ is a clopen subbasis of a Stone space, then every clopen set is a *finite* union of finite intersections of elements of $\mathcal{B}$ — this easily follows from Proposition 2.4. And second the components of v and v' preserve both finite unions and finite intersections of clopens.

Turning to the proper case distinction, we first consider the case where $S = S_1 \times S_2$ Then let $\mathcal{B} := (\_)^{\pi_1}[\mathsf{Clp}_{S_1\mathbb{X}}] \cup (\_)^{\pi_2}[\mathsf{Clp}_{S_2\mathbb{X}}]$, and consider an arbitrary $W \in \mathcal{B}$. Then

$$v_S(W) = v_S((V)^{\pi_i}) = \Phi(\pi_i) \circ v_{S_i}(V) = \Phi(\pi_i) \circ v'_{S_i}(V) = v'_S(W),$$

and the claim follows by the argument given above and the fact that $\mathcal{B}$ is a subbasis of the topology.

Now consider the case that $S = S_1 + S_2$. Then we define for $i = 1, 2$ the sets $\mathcal{B}_i := (\_)^{\kappa_i}[\mathsf{Clp}_{S_i\mathbb{X}}]$ and the set

$$\mathcal{B} := \{W \mid \exists W_1 \in \mathcal{B}_1.\exists W_2 \in \mathcal{B}_2.W = W_1 \cap W_2\}.$$

With $W \in \mathcal{B}$ we obtain:

$$\begin{array}{rcl} v_S(W) & = & v_S((V_1)^{\kappa_1} \cap (V_2)^{\kappa_2}) \quad = \quad v_S((V_1)^{\kappa_1}) \cap v_S((V_2)^{\kappa_2}) \\ & = & \Phi(\kappa_1)(v_{S_1}(V_1)) \cap \Phi(\kappa_2)(v_{S_2}(V_2)) \quad = \quad \Phi(\kappa_1)(v'_{S_1}(V_1)) \cap \Phi(\kappa_2)(v'_{S_2}(V_2)) \\ & \vdots & \\ & = & v'_S(W) \end{array}$$

It is obvious that $\mathcal{B}$ forms a clopen basis for the topology on $S\mathbb{X}$ and therefore we can prove $v_S = v'_S$.

Finally, we consider the case that $S = \mathbb{V}S'$. Let

$$\mathcal{B} := \left\{ W \mid W \in (\_)^{\mathbb{V}}[\mathsf{Clp}_{S'\mathbb{X}}] \right\} \cup \left\{ -W \mid W \in (\_)^{\mathbb{V}}[\mathsf{Clp}_{S'\mathbb{X}}] \right\},$$

and let $W \in \mathcal{B}$. Then again one can easily check that we have $v_S(W) = v'_S(W)$ for all $W \in \mathcal{B}$ and by the fact that $\mathcal{B}$ is a clopen subbasis of the Vietoris topology one can use the same arguments as in the other cases to show that $v_S = v'_S$. QED

6.2 Relating the categories

We show that the functors $\mathcal{C} : \mathsf{BAO}_T \to \mathsf{Coalg}(T)^{\mathrm{op}}$ and $\mathcal{A} : \mathsf{Coalg}(T)^{\mathrm{op}} \to \mathsf{BAO}_T$ form a so-called dual representation, i.e. $\mathcal{C}$ is right adjoint to $\mathcal{A}$ and the unit of the adjunction is an isomorphism. We first define the unit γ and the counit α of the adjunction. Recall that we proved in Theorem 5.13 that ϵ is an isomorphism; for r_Φ see Definition 5.9 and for $i_{\Phi(S)}$ Definition 2.6.

Definition 6.3 For a T-BAO (Φ, next) and a $S \in \mathsf{Ing}(T)$ we define

$$\alpha_\Phi : \mathcal{AC}(\Phi) \to \Phi$$

via $\alpha_\Phi(S) := j_{\Phi(S)} \circ \mathsf{Clp}(r_\Phi(S))$, where $j_{\Phi(S)}$ denotes the inverse of the isomorphism $i_{\Phi(S)} : \Phi(S) \to \mathbb{C}\mathsf{lp}\,\mathbb{S}\mathsf{p}\,\Phi(S)$.

For a T-coalgebra $(\mathbb{X}, c)$, we define

$$\gamma_{(\mathbb{X},c)} : (\mathbb{X}, c) \to \mathcal{CA}(\mathbb{X}, c) \quad \text{in } \mathsf{Coalg}(T)^{\mathrm{op}}$$

as the inverse $\gamma_{(\mathbb{X},c)} : \mathcal{CA}(\mathbb{X}, c) \to (\mathbb{X}, c)$ of the morphism $\epsilon_{(\mathbb{X},c)} : (\mathbb{X}, c) \to \mathcal{CA}(\mathbb{X}, c)$ in $\mathsf{Coalg}(T)$. $\lhd$

Now we can state

Theorem 6.4 *Let T be a VPF. Then $\mathcal{A} : \mathsf{Coalg}(T)^{\mathrm{op}} \to \mathsf{BAO}_T$ is a full embedding and has $\mathcal{C} : \mathsf{BAO}_T \to \mathsf{Coalg}(T)^{\mathrm{op}}$ as a right adjoint with γ and α as unit and counit. That is, $\mathsf{Coalg}(T)^{\mathrm{op}}$ is (isomorphic to) a full coreflective subcategory of BAO_T.*

The proof of this theorem is postponed to the end of this subsection. We first show that α is indeed a morphism of T-BAOs.

Lemma 6.5 *The family of maps $\alpha_\Phi(\_) : \mathcal{AC}\Phi \to \Phi$ is a morphism of T-BAOs.*

Proof. We have to show that $\alpha_\Phi(\_)$ is a natural transformation and that $\alpha_\Phi(\_)$ fulfills an additional naturality condition with respect to the next-operator.

Concerning the first claim we must prove that for all $S \overset{p}{\rightsquigarrow} S'$ in $\mathsf{Ing}(T)$ we have

$$\Phi(p) \circ \alpha_{\Phi(S')} = \alpha_{\Phi(S)} \circ (\_)^p.$$

It suffices to show, by a case distinction, that this equation holds for paths of length at most one. As all of these proofs boil down to a tedious but straightforward unravelling of definitions, we confine ourselves to the case that $p = \mathbb{V}$ and $S = \mathbb{V}S_1$. Take an arbitrary $U \in \mathsf{Clp}_{S_1\,\mathsf{Sp}\,\Phi(\mathbb{I})}$ and let $a \in \Phi(S_1)$ be such that $\mathbb{C}\mathsf{lp}(r_\Phi(S_1))(U) = \hat{a}$. Then

$$\begin{aligned}
\alpha_\Phi(S)((U)^{\mathbb{V}}) &= (j_{\Phi(S)} \circ \mathbb{C}\mathsf{lp}(r_{\Phi(S)}))((U)^{\mathbb{V}}) \\
&= (j_{\Phi(S)} \circ r^{-1}_{\Phi(S)})(\{\beta \subseteq U \mid \beta \subseteq S_1\,\mathsf{Sp}\,\Phi(\mathbb{I}) \text{ closed}\}) \\
&= j_{\Phi(S)}\left(\{u \in \mathsf{Sp}\,\Phi(S) \mid r_{\Phi(S)}(u) \subseteq U\}\right) \\
&= j_{\Phi(S)}\left(\{u \in \mathsf{Sp}\,\Phi(S) \mid \{r_{\Phi(S_1)}(v) \mid \Phi(\mathbb{V})^{-1}(u) \subseteq v\} \subseteq U\}\right) \\
&= j_{\Phi(S)}\left(\{u \in \mathsf{Sp}\,\Phi(S) \mid \{v \mid \Phi(\mathbb{V})^{-1}(u) \subseteq v\} \subseteq \mathbb{C}\mathsf{lp}(r_{\Phi(S_1)})(U)\}\right) \\
&= j_{\Phi(S)}\left(\{u \in \mathsf{Sp}\,\Phi(S) \mid \{v \mid \Phi(\mathbb{V})^{-1}(u) \subseteq v\} \subseteq \hat{a}\}\right) \\
&= j_{\Phi(S)}\left(\{u \in \mathsf{Sp}\,\Phi(S) \mid \Phi(\mathbb{V})^{-1}(u) \subseteq v \Rightarrow a \in v\}\right) \\
&= j_{\Phi(S)}\left(\{u \in \mathsf{Sp}\,\Phi(S) \mid \Phi(\mathbb{V})(a) \in u\}\right) \\
&= \Phi(\mathbb{V})(a) \\
&= \Phi(\mathbb{V})\left(j_{\Phi(S_1)} \circ \mathbb{C}\mathsf{lp}(r_{\Phi(S_1)})(U)\right) \\
&= \left(\Phi(\mathbb{V}) \circ \alpha_\Phi(S_1)\right)(U)
\end{aligned}$$

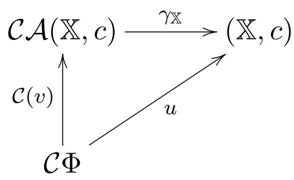
and we get $\alpha_\Phi(S) \circ (\_)^{\mathbb{V}} = \Phi(\mathbb{V}) \circ \alpha_\Phi(S_1)$, as required.

Now we turn to the second claim. The 'additional naturality condition with respect to the next-operator' is the following: $\mathsf{next} \circ \alpha_\Phi(T) = \alpha_\Phi(\mathbb{I}) \circ \mathbb{C}\mathsf{lp}(r_\Phi(T) \circ \mathbb{S}\mathsf{p}\,\mathsf{next})$. This is easily shown to hold:

$$\begin{aligned}
\alpha_\Phi(\mathbb{I}) \circ \mathbb{C}\mathsf{lp}(r_\Phi(T) \circ \mathbb{S}\mathsf{p}(\mathsf{next})) &= j_{\Phi(\mathbb{I})} \circ \mathbb{C}\mathsf{lp}(\mathbb{S}\mathsf{p}(\mathsf{next})) \circ \mathbb{C}\mathsf{lp}(r_{\Phi(T)}) \\
&= \mathsf{next} \circ j_{\Phi(T)} \circ \mathbb{C}\mathsf{lp}(r_\Phi(T)) \\
&= \mathsf{next} \circ \alpha_\Phi(T),
\end{aligned}$$

where the second identity is by the naturality of j. QED

Proof of Theorem 6.4. For the adjunction ([23], p. 81), we show that for all $(\mathbb{X}, c) \in \mathsf{Stone}$ and for all $u : \mathcal{C}(\Phi) \to (\mathbb{X}, c)$ there is a unique $v : \mathcal{A}(\mathbb{X}, c) \to \Phi$ such that the following diagram in $\mathsf{Coalg}(T)$ commutes:



Indeed, defining $v = \alpha_\Phi \circ \mathcal{A}(u)$, we calculate

$$\begin{aligned}
\gamma_{\mathbb{X}} \circ \mathcal{C}(\alpha_\Phi \circ \mathcal{A}(u)) &= \gamma_{\mathbb{X}} \circ \mathbb{S}\mathsf{p}\,(\alpha_\Phi(\mathbb{I}) \circ \mathcal{A}(u)(\mathbb{I})) \\
&= \gamma_{\mathbb{X}} \circ \mathbb{S}\mathsf{p}(j_{\Phi(\mathbb{I})} \circ r_\Phi(\mathbb{I}) \circ \mathbb{C}\mathsf{lp}(u)) \\
&= \gamma_{\mathbb{X}} \circ \mathbb{S}\mathsf{p}(\mathbb{C}\mathsf{lp}(u)) \circ \mathbb{S}\mathsf{p}(j_{\Phi(\mathbb{I})}) \\
&= u \circ \gamma_{\mathbb{S}\mathsf{p}(\Phi(\mathbb{I}))} \circ \mathbb{S}\mathsf{p}(j_{\Phi(\mathbb{I})}) \\
&= u
\end{aligned}$$

The last two steps use that $\mathbb{S}\mathsf{p}$ and $\mathbb{C}\mathsf{lp}$ are adjoint with (co)units j and γ, see Definitions 2.6 and 6.3. Uniqueness of v is Proposition 6.1.

To conclude the proof, recall that a left-adjoint is full and faithful iff the unit is iso ([23], p. 88). Hence $\mathcal{A}$ is full and faithful by Theorem 5.13. It remains to observe that $\mathcal{A}$ is injective on objects. QED

6.3 Exact T-BAOs and the final T-coalgebra

The aim of this section is to characterise the largest subcategory of BAO_T on which the adjunction from the previous subsection restricts to a dual equivalence. This dual equivalence is then used to obtain an alternative proof of the final coalgebra theorem.

The reader might have noticed already that our adjunction is not a dual equivalence since the definition of T-BAOs does not force a T-BAO Φ to respect T-structure. For example, if $S_1 \times S_2$ is an ingredient of T then it may well be that $\Phi(S_1 \times S_2) \neq \Phi(S_1) + \Phi(S_2)$.

Definition 6.6 Let S be a functor $\mathsf{Stone} \to \mathsf{Stone}$. Then

$$\hat{S} := \mathbb{C}\mathsf{lp} \circ S \circ \mathbb{S}\mathsf{p}\,.$$

defines a corresponding functor $\hat{S}$ on the category BA. $\lhd$

The following definition introduces *exact* T-BAOs, that is, those T-BAOs which do respect T-structure.

Definition 6.7 (exact T-BAO) A T-BAO Φ is called *exact* if there is a family of isomorphisms

$$\tau_S : \hat{S}(\Phi(\mathbb{I})) \to \Phi(S)$$

with the following properties:

- $\tau : \hat{(\_)}(\Phi(\mathbb{I})) \to \Phi$ is a natural transformation, where $\hat{(\_)}(\Phi(\mathbb{I})) : \mathsf{Ing}(T)^{\mathrm{op}} \to \mathsf{BA}_\wedge$ is defined on objects as in Definition 6.6 and on paths $p : S_1 \overset{p}{\leadsto} S_2$ as $(\_)^p$ in Definition 5.7 (with $\mathbb{X}$ being here $\Phi(\mathbb{I})$).

- $\tau_{\mathbb{I}} = j_{\Phi(\mathbb{I})}$, where again $j_{\Phi(\mathbb{I})}$ denotes the inverse of the isomorphism $i_{\Phi(\mathbb{I})} : \Phi(\mathbb{I}) \to \mathbb{C}\mathsf{lp}\,\mathbb{S}\mathsf{p}\,\Phi(\mathbb{I})$
- $\tau_{\mathbb{Q}} = \mathsf{id}_{\mathsf{Clp}_{\mathbb{Q}}}$ for every constant $\mathbb{Q} \in \mathsf{Ing}(T)$

$\lhd$

We will now see that exact T-BAOs are precisely those T-BAOs Φ for which the component α_Φ of the counit of the adjunction is an isomorphism.

Theorem 6.8 *Let T be a functor $T : \mathsf{Stone} \to \mathsf{Stone}$. Then the category BAO^e_T of exact T-BAOs is the largest full subcategory of BAO_T on which the above dual adjunction restricts to a dual equivalence to $\mathsf{Coalg}(T)$.*

Proof. Let $\mathcal{B}$ be the largest subcategory of BAO_T on which the adjunction $\mathcal{A} \dashv \mathcal{C}$ restricts to an equivalence. Then for any $\Phi \in \mathcal{B}$ the map $\alpha_\Phi : \mathcal{AC}\Phi \to \Phi$ consists of a family of isomorphisms going from $\mathcal{AC}\Phi(S) = \hat{S}(\Phi)(\mathbb{I})$ to $\Phi(S)$. Therefore we can define a family of isomorphisms $\tau_S : \hat{S}(\Phi)(\mathbb{I}) \to \Phi(S)$ by letting $\tau = \alpha_\Phi$. It is straightforward to check that this family satisfies the conditions in Def. 6.7. Hence $\Phi \in \mathsf{BAO}^e_T$.

Now let $\Phi \in \mathsf{BAO}^e_T$. We have to show that the counit α_Φ is an isomorphism. As $\Phi \in \mathsf{BAO}^e_T$ there is a family of isomorphisms

$$\tau_S : (\mathcal{AC}\Phi)(S) \to \Phi(S).$$

which is natural in S and for which we have $\tau_{\mathbb{I}} = j_{\Phi(\mathbb{I})} = \alpha_\Phi(\mathbb{I})$ and $\tau_{\mathbb{Q}} = \mathsf{id}_{\mathsf{Clp}_{\mathbb{Q}}} = \alpha_\Phi(\mathbb{Q})$ for all constants $\mathbb{Q} \in \mathsf{Ing}(T)$. Using Lemma 6.2 one can therefore show that $\tau_S = \alpha_S$ for all $S \in \mathsf{Ing}(T)$. But this means in particular that α_Φ is an isomorphism. QED

The duality between BAO^e_T and $\mathsf{Coalg}(T)$ can now be used to give an alternative proof of the final coalgebra theorem in the previous section. Let $\mathcal{L}_T$ be the Lindenbaum T-BAO of some VPF T. Then we obtain the final object of $\mathsf{Coalg}(T)$ by applying the functor $\mathcal{C}$ to $\mathcal{L}_T$:

Theorem 6.9 *Let T be a Vietoris polynomial functor. $\mathcal{CL}_T$ is the final object in $\mathsf{Coalg}(T)$.*

Proof. We prove the theorem by showing that $\alpha_{\mathcal{L}_T}$ is an isomorphism, i.e. $\mathcal{L}_T \in \mathsf{BAO}^e_T$. Finality of $\mathcal{CL}_T$ follows then immediately from the duality between $\mathsf{Coalg}(T)$ and BAO^e_T.

Since $\mathcal{L}_T$ is initial there is a morphism $m : \mathcal{L}_T \to \mathcal{ACL}_T$. Since $id_{\mathcal{L}_T}$ is the unique morphism $\mathcal{L}_T \to \mathcal{L}_T$ it follows that $\alpha_{\mathcal{L}_T} \circ m = id_{\mathcal{L}_T}$. We want to show that $m \circ \alpha_{\mathcal{L}_T} : \mathcal{ACL}_T \to \mathcal{ACL}_T$ is in fact the identity on $\mathcal{ACL}_T$. Since $\mathcal{A}$ is full (cf. Theorem 6.4) there is $f : \mathcal{CL}_T \to \mathcal{CL}_T$ in $\mathsf{Coalg}(T)$ such that $\mathcal{A}(f) = m \circ \alpha_{\mathcal{L}_T}$. We obtain $\alpha_{\mathcal{L}_T} \circ \mathcal{A}(f) = \alpha_{\mathcal{L}_T} \circ m \circ \alpha_{\mathcal{L}_T} = \alpha_{\mathcal{L}_T} = \alpha_{\mathcal{L}_T} \circ \mathcal{A}(id_{\mathcal{CL}_T})$ and the universal property of the coreflection tells us that $f = id_{\mathcal{CL}_T}$, hence, $id_{\mathcal{ACL}_T} = m \circ \alpha_{\mathcal{L}_T}$ and $\alpha_{\mathcal{L}_T}$ is iso. QED

Remark 6.10 In [15] Jacobs states a similar final coalgebra theorem for set-based Kripke polynomial functors. However, there is a defect in his proof.

The problem involves his functor $\mathcal{C} : \mathsf{BAO}_{T_J} \to \mathsf{Coalg}(T_J)^{\mathrm{op}}$. Note that Jacobs's functor T_J is the set-based analogue of our T. (To obtain Jacobs's T_J from ours, simply replace all occurrences of the Vietoris functor with the power set functor, and interpret all polynomial functors occurring in T in the standard way.) Thus Jacobs studies the relation between T_J-BAOs and *set-based* T_J-coalgebras. However, as mentioned already, on the algebraic side, we may identify T_J-BAOs with T-BAOs. Thus we may compare Jacobs's way of relating BAO_T with the Set-based $\mathsf{Coalg}(T_J)$ to our way of relating BAO_T to the Stone-based $\mathsf{Coalg}(T)$.

In [15] Jacobs assigns a modal logic to each Kripke polynomial functor, and he proves that the coalgebras for these functors form a sound and complete semantics for these logics. In order to obtain the final coalgebra for a so-called *finite* KPF T, that is, a KPF which may only contain the finite-power set functor, he maps the Lindenbaum-Tarski algebra $\mathcal{L}_T$ to its corresponding coalgebra $\mathcal{C}(\mathcal{L}_T)$, using the above-mentioned functor $\mathcal{C}$. His construction works, if the functor $\mathcal{C}$ maps a T-BAO for a finite KPF T to a T-coalgebra. This is however only the case for functors T not containing the finite power set functor.

This means that Jacobs's construction of final objects in $\mathsf{Coalg}(T)$ works only for Kripke polynomial functors that do not contain the power set functor or its finitary version. Moving from the category of sets to Stone enables us to repair this defect.

7 Conclusions

What we have done so far can be viewed from different perspectives. We summarise some of them, indicating possible future research directions.

Stone Coalgebras and Modal Logic Research on the relation between coalgebras and modal logic started with Moss [25] although earlier work, e.g. by Rutten [29] already showed that Kripke frames and models are instances of coalgebras. In [22, 21] it was shown that modal logic for coalgebras dualise equational logic for algebras, the idea being that equations describe quotients of free algebras and modal formulae describe subsets of final (or cofree) coalgebras.[3] But whereas, usually, any quotient of a free algebra can be defined by a set of ordinary equations, one needs *infinitary* modal formulae to define all subsets of a final coalgebra. As a consequence, while we have a satisfactory description of the coalgebraic semantics of infinitary modal logics, we do not completely understand the relationship between coalgebras and finitary modal logic. The results in this paper show that Stone coalgebras provide a natural and adequate semantics for finitary modal logics, but there is ample room for clarification here.

Another approach to a coalgebraic semantics for finitary modal logics was given in [19, 18]. There, the idea is to modify coalgebra morphisms in such a way that they capture not bisimulation but only bisimulation up to rank ω. Since finitary modal logics capture precisely bisimulation up to rank ω, the resulting category Beh_ω provides a convenient framework to

[3]Another account of the duality has been given in [20] where it was shown that modalities dualise algebraic operations. Related work on dualising equational logic include [14, 4, 2].

study the coalgebraic semantics of finitary modal logic. So an important next step is to understand the relation of both approaches.

Stone Coalgebras as Systems We investigated coalgebras over Stone spaces as models for modal logic. But what is the significance of Stone-coalgebras from the point of view of systems? Here, following [30], as systems we consider coalgebras over Set. Compared to these, the addition of (Stone) topological structure basically means two things. First, morphisms have to be continuous, i.e., the topologies allow for more specific notions of behaviour[4]. Second, the carriers have to be compact. This is quite a severe restriction and many interesting transition systems are not compact. So we would like to understand which set-coalgebras are Stone-coalgebras and how Stone-behavioural equivalence relates to Set-behavioural equivalence.

Generalising Stone Coalgebras Coalgebras over Stone spaces can be generalised in different ways. We have seen that replacing the topologies by represented Boolean algebras leads to general frames. But it will also be of interest to consider other topological spaces as base categories. Here are two examples.

First, can we find useful examples of coalgebras over topological spaces, if we drop the compactness condition? For instance, can the topologies be used to restrain the behaviour in order to guarantee fairness and liveness properties?

Second, there is a close relationship between Stone spaces and complete ultrametric spaces.[5] Now complete ultrametric spaces are used in the semantics of programming languages (see e.g. [8]), but they also form a base category for coalgebras in e.g. [33, 34]; this shows a clear need for further investigations. Moreover, using the results of [3] on how to partialise Stone spaces with a countable base using SFP-domains, it should be possible to establish a precise relation between modal logics for Stone-coalgebras and the logics for domains of [1].

Coalgebras and Duality Theory Whereas many, or most, common dualities are induced by a schizophrenic object (see [16], Section VI.4.1), the duality of modal algebras and descriptive general frames is not. For a contradiction, write $K : \mathsf{MA} \to \mathsf{DGF}$, $L : \mathsf{DGF} \to \mathsf{MA}$ for the contravariant functors witnessing the duality and suppose that there is a schizophrenic object S, that is, $\mathsf{MA}(\mathbb{A}, S) \cong UK(\mathbb{A})$ where U denotes the forgetful functor $\mathsf{DGF} \to \mathsf{Set}$. Then $\mathsf{Set}(1, U\mathbb{G}) \cong U\mathbb{G} \cong UKL\mathbb{G} \cong \mathsf{MA}(L\mathbb{G}, S) \cong \mathsf{DGF}(KS, KL\mathbb{G}) \cong \mathsf{DGF}(KS, \mathbb{G})$, showing that KS is a free object over one generator in DGF. But since DGF-morphisms are also bisimulations it is not hard to see that such an object cannot exists.

On the other hand, this duality is an instance of the duality $\mathsf{Alg}(T^{\mathrm{op}}) \cong \mathsf{Coalg}(T)^{\mathrm{op}}$ of algebras and coalgebras, with the Vietoris functor $\mathbb{V}$ as the functor T. It seems therefore of interest to explore which dualities are instances of the algebra/coalgebra duality. As a

[4]Recall that the notion of bisimulation or behavioural equivalence is defined in terms of the morphisms of the category. Requiring the morphisms to be continuous means that less states are identified under behavioural equivalence.

[5]A topological space is a Stone space with a countable base iff it is a complete totally bounded ultrametric space, see [32], Corollary 6.4.8.

first step in this direction, [26] shows that the duality between positive modal algebras and K^+-spaces can be described in a similar way as in Section 3 (although the technical details are substantially more complicated).

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