Centrum voor Wiskunde en Informatica

REPORT*RAPPORT*

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Probability, Networks and Algorithms

Large deviations of infinite intersections of events in Gaussian processes

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REPORT PNA-E0409 JUNE 2004

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ISSN 1386-3711

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ABSTRACT
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2000 Mathematics Subject Classification: 60G15, 60K25, 60F10
Keywords and Phrases: Sample-path large deviations; Schilder's theorem; busy period; reproducing kernel Hilbert space; optimization

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Abstract The large deviations principle for Gaussian measures in Banach space is given by the generalized Schilder's theorem. After assigning a norm $||f||$ to paths f in the reproducing kernel Hilbert space of the underlying Gaussian process, the probability of an event A can be studied by minimizing the norm over all paths in A. The minimizing path f^*, if it exists, is called the *most probable path* and it determines the corresponding exponential decay rate. The main objective of our paper is to identify f^* for the class of sets A that are such that the minimization is over a closed convex set in an infinite-dimensional Hilbert space. The 'smoothness' (i.e., mean-square differentiability) of the Gaussian process involved has a crucial impact on the structure of the solution. Notably, as an example of a non-smooth process, we analyze the special case of fractional Brownian motion, and the set A consisting of paths f such that $f(t) \geq t$ for $t \in [0,1]$. For $H > \frac{1}{2}$, we prove that there is an $s^* \in (0, \frac{1}{2})$ such that the optimum path is *at* the diagonal for $t \in [0,s^*] \cup \{1\}$, whereas it is *strictly above* the diagonal for $t \in (s^*,1)$; for $H < \frac{1}{2}$ an analogous result is derived. For smooth processes, such as integrated Ornstein-Uhlenbeck, f^* has an essentially different nature, and is found by imposing conditions also on the derivatives of the path.

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1. Introduction

The large deviation principle (LDP) for Gaussian measures in Banach space, usually known as the (generalized) Schilder's theorem, has been established more than two decades ago by Bahadur and Zabell [3], see also [2,4]. In this LDP, a central role is played by the norm $||f||$ of paths f in the reproducing kernel Hilbert space of the underlying Gaussian process. More precisely, the probability of the Gaussian process being in some closed set A has exponential decay rate $\frac{1}{2}||f^*||^2$, where f^* is the path in A with minimum norm, i.e., $\operatorname{argmin}_{f\in A}||f||$. The path f^* has the interpretation of the *most probable path* (MPP) in A: if the Gaussian process happens to fall in A, with overwhelming probability it will be close to f^*.

For various specific sets A the MPP has been found. Addie *et al.* [1] consider a queueing system fed by a Gaussian process with stationary increments, and succeed in finding the MPP leading to overflow. This problem is relatively easy as the overflow event can be written as an infinite *union* of events $A = \cup_{t>0} A_t$, such that the decomposition

$$\inf_{f\in A} ||f|| = \inf_{t>0} \inf_{f\in A_t} ||f||$$

applies. Here A_t corresponds to the event of overflow at time t, and due to the fact that the infimum over A_t turns out to be straightforward, the problem can be solved. In this paper we look at the intrinsically more involved situation that A is an *intersection*, rather than a union, of events: $A = \cap_t A_t$; decay rates, and the corresponding MPPs, of these intersections are then usually considerably harder to determine. In our setting the norm has to be minimized over a convex set in an infinite-dimensional Hilbert space.

Few results are known on MPPs of these infinite intersections of events. In Norros [11] it was shown that the event of a queue with fractional Brownian motion (fBm) input having a busy period longer than, say, 1,

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corresponds to an infinite intersection of events; the set A consists of all f such that $f(t) \geq t$ for all $t \in [0,1]$. However, the shape of the MPP in A remained an open problem in [11]. Interestingly, it was shown that the straight line, i.e., the path $f(t) = t$, is *not* optimal, unlike in the case of Markovian input, see [14, Thm. 11.24]. In [8,9] buffer overflow in tandem, priority, and generalized processor sharing queues was analyzed: first it was shown that in these queues overflow relates to an infinite intersection of events, and then explicit lower bounds on the minimizing norm (corresponding to upper bounds on the overflow probability) were given. Conditions were given under which this lower bound is tight – in that case obviously the path corresponding to the lower bound is also the MPP.

An important element in our analysis is the 'smoothness' of the Gaussian process involved. Here we rely on results from Tutubalin and Freidlin [15] and Piterbarg [13], showing that the infinitesimal space of a Gaussian process Z (at time t) is essentially a finite-dimensional space generated by the value Z_t of the process itself, but in addition also its derivatives at t, say, Z_t', Z_t'',...,$Z_t^{(k)}$. The implication of this result is that in our study, processes without derivatives (such as fBm) had to be treated in another way than smooth processes (such as the so-called integrated Ornstein-Uhlenbeck process).

This paper is organized as follows. Section 2 presents preliminaries on Gaussian processes and a number of other prerequisites. In Section 3 we focus on the most probable path in the set of paths f such that $f(t) \geq \zeta(t)$, for a function ζ and t in some compact set $S \subset \mathbb{R}$. Our general result characterizes the MPP in this infinite intersection of events. In case the Gaussian process does not have derivatives, the MPP can be expressed as a conditional mean. Section 4 gives explicit results for the case $\zeta(t) = t$ and $S = [0,1]$, i.e., the busy-period problem. We illustrate the impact of the smoothness by focusing on examples of both a process without (fBm) and with (integrated Ornstein-Uhlenbeck) derivatives. In the case of fBm, we prove that for $H > \frac{1}{2}$ the MPP is *at* the diagonal in some interval $[0,s^*]$, and evidently also at the end of the busy period, but *strictly above* the diagonal in between (corresponding to a positive queue length); for $H < \frac{1}{2}$ an analogous result is derived. In the case of integrated Ornstein-Uhlenbeck, we show how the MPP is derived by imposing conditions at two points, namely the derivative at $t = 0$ and the value of the function at $t = 1$.

2. Preliminaries

This section describes some prerequisites, e.g., some fundamental results on Gaussian processes.

2.1. Gaussian process, path space, and reproducing kernel Hilbert space

The following framework will be used throughout the paper. Let $Z = (Z_t)_{t\in\mathbb{R}}$ be a centered Gaussian process with stationary increments, completely characterized by its variance function $v(t) \doteq \mathrm{Var}(Z_t)$. The covariance function of Z can be written as

$$\Gamma(t,s) \doteq \mathrm{Cov}(Z_t, Z_s) = \frac{1}{2}(v(s) + v(t) - v(s-t)).$$

For a finite subset S of $\mathbb{R}$, denote by $\Gamma(S,t)$ the column vector $\{\Gamma(s,t) : s \in S\}$, by $\Gamma(t,S)$ the corresponding row vector, and by $\Gamma(S)$ the matrix

$$\Gamma(S) \doteq \{\Gamma(s,t) : s \in S,\ t \in S\}.$$

In addition to the basic requirement that $v(t)$ results in a positive semi-definite covariance function, we impose the following assumptions on $v(t)$:

(i) $v(t)$ is continuous, and $\Gamma(S)$ is non-singular for any finite subset S of $\mathbb{R}$ with distinct elements;
(ii) there is a number $\alpha_0 \in (0,2]$ such that $v(h)/h^{\alpha_0}$ is bounded for $h \in (0,1)$;
(iii) $\lim_{t\to\infty} v(t) = \infty$, and $\lim_{t\to\infty} v(t)/t^{\alpha_\infty} = 0$ for some $\alpha_\infty \in (0,2)$.

The assumption (ii) guarantees the existence of a version with continuous paths, by virtue of Kolmogorov's lemma. Denote by Ω the function space

$$\Omega \doteq \left\{ \omega :\ \omega \text{ continuous } \mathbb{R} \to \mathbb{R},\ \omega(0) = 0,\ \lim_{t\to\infty} \frac{\omega(t)}{1+|t|} = \lim_{t\to-\infty} \frac{\omega(t)}{1+|t|} = 0 \right\}.$$

Equipped with the norm

$$\|\omega\|_\Omega \doteq \sup\left\{\frac{|\omega(t)|}{1+|t|} : t \in \mathbb{R}\right\},$$

Ω is a separable Banach space. We choose Ω as our basic probability space by letting $\mathbb{P}$ be the unique probability measure on the Borel sets of Ω such that the random variables $Z_t(\omega) = \omega(t)$ form a realization of Z.

The *reproducing kernel Hilbert space* R related to Z is defined by starting from the functions $\Gamma(t,\cdot)$ and defining an inner product by $\langle \Gamma(s,\cdot), \Gamma(t,\cdot)\rangle = \Gamma(s,t)$. The space is then closed with linear combinations, and completed with respect to the norm $\|\cdot\|^2 = \langle\cdot,\cdot\rangle$. Thus, the mapping

$$Z_t \mapsto \Gamma(t,\cdot) \tag{1}$$

is extended to an isometry between the Gaussian space G of Z, i.e., the smallest closed linear subspace of L^2 containing the random variables Z_t, and the function space R. The inner product definition generalizes to the *reproducing kernel property*:

$$\langle f, \Gamma(t,\cdot)\rangle = f(t), \quad f \in R. \tag{2}$$

The topology of R is finer than that corresponding to a weighted supremum distance between the paths: by Cauchy-Schwarz and (2),

$$\sup_{t\in\mathbb{R}} \frac{|f(t)|}{1+|t|} \le \|f\| \cdot \sup_{t\in\mathbb{R}} \frac{\|\Gamma(t,\cdot)\|}{1+|t|}, \tag{3}$$

where the supremum on the right hand side is finite by (iii). We see that all elements of R are continuous functions, R is a subset of Ω, and the topology of R is finer than that of Ω.

2.2. Large deviations: generalized Schilder's theorem

The generalization of Schilder's theorem on large deviations of Brownian motion to Gaussian measures in a Banach space is originally due to Bahadur and Zabell [3] (see also [2,4]). Here is a formulation appropriate to our case; for the definition of *good* rate function, see, e.g., [4, Section 2.1].

Theorem 1. *The function* $I : \Omega \to [0,\infty]$,

$$I(\omega) \doteq \begin{cases} \frac{1}{2}\|\omega\|_R^2, & \text{if } \omega \in R, \\ \infty, & \text{otherwise,} \end{cases} \tag{4}$$

is a good rate function for the centered Gaussian measure $\mathbb{P}$*, and* $\mathbb{P}$ *satisfies the large deviations principle:*

$$\text{for } F \text{ closed in } \Omega: \quad \limsup_{n\to\infty} \frac{1}{n}\log\mathbb{P}\left(\frac{Z}{\sqrt{n}} \in F\right) \le -\inf_{\omega\in F} I(\omega);$$

$$\text{for } G \text{ open in } \Omega: \quad \liminf_{n\to\infty} \frac{1}{n}\log\mathbb{P}\left(\frac{Z}{\sqrt{n}} \in G\right) \ge -\inf_{\omega\in G} I(\omega).$$

We call a function $f \in A$ such that $I(f) = \inf_{\omega\in A} I(\omega) < \infty$ a *most probable path* of A. A most probable path can be intuitively understood as a point of maximum likelihood, although there is no counterpart to the Lebesgue measure on Ω. If A is convex and closed and has a non-empty intersection with R, then the most probable path exists and is unique.

2.3. Notes on optimization

The following standard fact from optimization theory is crucial in our analysis, see, e.g., Exercise 3.13.23 in [7].

Proposition 1. *Let* H *be a Hilbert space. Consider a set* $A = \{x \in H : \langle x, y_i\rangle \ge a_i,\ i \in I\}$*, where* I *is a finite index set and* $y_i \in H$*. Assume that* $x^* = \operatorname{argmin}\{\|x\| : x \in A\}$ *and denote* $I^* = \{i \in I : \langle x^*, y_i\rangle = a_i\}$*. Then* $x^* \in \operatorname{Span}\{y_i : i \in I^*\}$.

The intuitive content of Proposition 1 is that conditions which are not tightly met (i.e., satisfied with equality) at the optimal point do not appear in the solution. If the finite set of linear conditions is replaced by an infinite one, the result does not hold without further assumptions. One particular generalization will be considered in Section 3.

We also need the following basic infinite-dimensional result.

Proposition 2. *Let H be a Hilbert space, and let $y_i \in H$, $a_i \in \mathbb{R}$, $i = 1,2,\ldots$, and denote*

$$A_n = \{x \in H : \langle x, y_i\rangle \geq a_i,\ i = 1,\ldots,n\},$$
$$A_\infty = \{x \in H : \langle x, y_i\rangle \geq a_i,\ i = 1,2,\ldots\}.$$

Assume that the convex set A_∞ is non-empty and let

$$\alpha_n = \operatorname{argmin}_{x\in A_n}\|x\|,\ n = 1,2,\ldots,\infty.$$

Then $\lim_{n\to\infty}\alpha_n = \alpha_\infty$.

Proof. Obviously $\|\alpha_n\| \leq \|\alpha_\infty\|$. We show first that $\|\alpha_n\| \to \|\alpha_\infty\|$. The closed ball $B(0,\|\alpha_\infty\|)$ is weakly compact. Let α_0 be a weak accumulation point of the sequence α_n. By definition of the weak topology, for each n, there is a subsequence m_j such that

$$\langle \alpha_0, y_n\rangle = \lim_{j\to\infty}\langle \alpha_{m_j}, y_n\rangle \geq a_n.$$

Thus, $\alpha_0 \in A_n$ for every n. It follows that $\alpha_0 \in A_\infty$ and, since the sequence $\|\alpha_n\|$ is non-decreasing, that $\|\alpha_n\| \nearrow \|\alpha_\infty\|$.

Now, by a basic characterization of minimum norm elements in closed convex sets, we have $\langle \alpha_n, \alpha_\infty - \alpha_n\rangle \geq 0$, since $\alpha_\infty \in A_\infty \subseteq A_n$ and α_n is the minimum norm element of A_n. But then

$$\|\alpha_n - \alpha_\infty\|^2 = \|\alpha_\infty\|^2 - \|\alpha_n\|^2 - 2\langle \alpha_n, \alpha_\infty - \alpha_n\rangle \leq \|\alpha_\infty\|^2 - \|\alpha_n\|^2 \to 0.$$

□

2.4. Derivatives and the infinitesimal space

We call the Gaussian process Z *smooth* at t, if it has a mean-square derivative at t, that is, there exists a random variable $Z'_t \in G$ such that

$$\lim_{h\to 0}\mathbb{E}\left\{\left(\frac{Z_{t+h} - Z_t}{h} - Z'_t\right)^2\right\} = 0.$$

It follows from the stationarity of increments that if Z is smooth at 0, then it is smooth at all $t \in \mathbb{R}$. On the other hand, applying the above definition at $t = 0$, we see that process Z is non-differentiable if $\lim_{h\to 0} v(h)/h^2 = \infty$.

Here are some more properties of a smooth Gaussian process with stationary increments. The proofs are straightforward and left as an exercise.

Proposition 3. *Assume that Z is smooth. Then*

(i) $\Gamma(s,t)$ has partial derivatives, and the isometry counterpart of Z'_t in R is the function

$$\Gamma'(s,t) \doteq \frac{\mathrm{d}}{\mathrm{d}s}\Gamma(s,t);$$

(ii) all functions $f \in R$ are differentiable at every point, and evaluation of a derivative at t can be obtained by taking an inner product with $\Gamma'(t,\cdot)$:

$$f'(t) = \langle f, \Gamma'(t,\cdot)\rangle, \quad f \in R,\ t \in \mathbb{R};$$

(iii) the variance function v is twice differentiable everywhere, and

$$\operatorname{Var}\left(Z'_0\right) = \frac{1}{2}v''(0);$$

(iv) for any $s,t \in \mathbb{R}$,

$$\langle \Gamma'(s,\cdot), \Gamma'(t,\cdot)\rangle = \frac{\nu''(t-s)}{2}.$$

For any subset A of a Banach space X, denote by $\operatorname{Span} A$ the smallest closed linear subspace of X containing the set A. For any set $V \subseteq \mathbb{R}$, denote

$$G_V \doteq \operatorname{Span}\{Z_t : t \in V\}, \quad R_V \doteq \operatorname{Span}\{\Gamma(t,\cdot) : t \in V\}.$$

The *infinitesimal space* of the Gaussian process Z at timepoint t is defined as

$$G_{t\pm} \doteq \bigcap_{u>0} G_{[t-u,t+u]}.$$

By the stationarity of increments, the structure of $G_{t\pm} - Z_t$ is the same for all t. In R, we denote by $R_{t\pm}$ the isometry counterpart of $G_{t\pm}$.

A subspace G_V (resp. R_V) augmented with the infinitesimal spaces at all points in V is denoted by G_V^o (resp. R_V^o):

$$G_V^o \doteq \operatorname{Span} \bigcup_{t\in V} G_{t\pm} \quad R_V^o \doteq \operatorname{Span} \bigcup_{t\in V} R_{t\pm}. \tag{5}$$

The infinitesimal space of a stationary Gaussian process X was characterized by Tutubalin and Freidlin [15]. Under a mild spectral condition, $G_{t\pm}$ is a finite-dimensional space generated by the random variable X_t and the derivatives of the process at t, say X_t', X_t'',...,$X_t^{(k)}$. Moreover, the corresponding 'infinitesimal σ-algebra' is also generated by these random variables, and some sets of measure zero. Note also that, by this result, the infinitesimal σ-algebra is the same for one- and two-sided neighborhoods in the definition.

The generalization to non-stationary Gaussian processes is by Piterbarg [13]. Denote by $\mathscr{D}$ the Schwarz space (i.e., the space of $C^\infty(R)$ functions $f(x)$, such that the k-th derivative $f^{(k)}(x)$ vanishes faster than any inverse power, for $x \to \infty$ and any $k \in \{0,1,\ldots\}$). Let H be the closure of functions in $\mathscr{D}$ with respect to the inner product $\langle \phi_1, \phi_2\rangle = \int_{\mathbb{R}^2} \Gamma(s,t)\phi_1(s)\phi_2(t)\,\mathrm{d}s\mathrm{d}t$. The following result is due to Piterbarg [13, Th. 1].

Theorem 2. *Suppose that*

(i) $\mathscr{D} \subset R$ and the embedding is continuous and dense;
(ii) The space H is closed under local shifts; see for the precise definition [13, Thm. 1]*;*
(iii) In the region $\{(s,t) : s,t \in \mathbb{R},\ s \neq t\}$, the function $\Gamma(s,t)$ has mixed partial derivatives of any order.

Then $G_{t\pm}$ equals the closed linear hull of all existing mean-square derivatives of Z at t.

Note that if Z has continuously differentiable paths and the spectral density of Z', denoted by $f(\lambda)$, satisfies $f(\lambda) \geq 1/\lambda^p$ for some $p > 0$, then the characterization of $G_{t\pm}$ is immediately obtained from [15].

When Z is a Brownian motion, it follows easily from the independence of increments that the infinitesimal space is trivial, i.e., $G_{t\pm} = \{Z_t\}$. This implies the same property for fractional Brownian motions with self-similarity parameter $H \in (0,1)$. Indeed, the transformed process

$$M_t = \int_0^t s^{\frac{1}{2}-H}(t-s)^{\frac{1}{2}-H}\,\mathrm{d}Z_t, \quad t \geq 0,$$

is a process with independent increments and $\operatorname{Span}\{M_s : s \in [0,t]\} = G_{[0,t]}$, see [10, 12].

2.5. A note on conditional expectations

For a finite-dimensional Gaussian vector X, the conditional distribution with respect to any linear condition $AX = a$ is again Gaussian. Moreover, the mean of this distribution is linear in a, whereas its variance is independent of a. It is less obvious how conditional distributions and expectations with respect to linear conditions should be defined in the infinite-dimensional case. In this subsection we show how certain conditional expectations with respect to infinite-dimensional linear conditions can be defined in an elementary way.

Let $S \subset \mathbb{R}$ be a non-empty finite set of timepoints. For any $u \in \mathbb{R}$, the conditional expectation of Z_u given the vector Z_S has the expression

$$\mathbb{E}[Z_u \,|\, Z_S] = \Gamma(u,S)\Gamma(S)^{-1}Z_S.$$

Thus, we have for any particular vector $\boldsymbol{x}$ a natural expression for a particular condition (although evidently the probability of the condition is zero):

$$\mathbb{E}[Z_u \,|\, Z_S = \boldsymbol{x}] = \Gamma(u,S)\Gamma(S)^{-1}\boldsymbol{x}.$$

Note that the expression is linear in $\boldsymbol{x}$. We give another point of view to the above formula by defining for each $\boldsymbol{x}$ a random variable

$$Y_{\boldsymbol{x}} = \boldsymbol{x}^T\Gamma(S)^{-1}Z_S. \tag{6}$$

We obtain, for the one particular condition $\{Z_S = \boldsymbol{x}\}$, the conditional expectations of all Z_u's as covariances with one and the same random variable $Y_{\boldsymbol{x}}$:

$$\mathbb{E}\{Y_{\boldsymbol{x}}Z_u\} = \mathbb{E}[Z_u \,|\, Z_S = \boldsymbol{x}] \quad \text{for all } u \in \mathbb{R}. \tag{7}$$

Further, the isometry counterpart of $Y_{\boldsymbol{x}}$ in R is the element f that satisfies

$$\langle f, \Gamma(u,\cdot)\rangle = \mathbb{E}\{Y_{\boldsymbol{x}}Z_u\} \quad \text{for all } u \in \mathbb{R}.$$

By the reproducing kernel property, this element is the function

$$u \mapsto \mathbb{E}[Z_u \,|\, Z_S = \boldsymbol{x}].$$

From this, we deduce the following characterization of the most probable path going through a finite number of specified points.

Proposition 4. *For any finite $S \subset \mathbb{R}$ and any $\boldsymbol{x} \in \mathbb{R}^{|S|}$, the conditional expectation given the values on S and the most probable path satisfying $f(S) = \boldsymbol{x}$ are equal, i.e.,*

$$f^*(u) = \mathbb{E}[Z_u \,|\, Z_S = \boldsymbol{x}] \quad \textit{for all } u \in \mathbb{R}.$$

Proof. As shown above, the random variable $Y_{\boldsymbol{x}}$ defined in (6) is the random variable with smallest variance that satisfies $\mathbb{E}\{Y_{\boldsymbol{x}}Z_s\} = \mathbb{E}[Z_s \,|\, Z_S = \boldsymbol{x}]$ for all $s \in S$. By this minimum variance characterization, its isometry counterpart in R is the most probable path f^*. The claim follows now from (7). □

Remark 1. In the case that Z is smooth, Proposition 4 still holds if there appear as conditions also values of Z' at some points, or those of higher derivatives if they exist. The generalization to those cases is straightforward and we skip the details.

It is not clear for us how far Proposition 4 can be generalized to an infinite-dimensional setting. We now show how this can be done when the conditions are in R.

Proposition 5. *Let S be a closed subset of $\mathbb{R}$ and let $\zeta \in R$. Let f^* be the most probable path satisfying $f(s) = \zeta(s)$ for every $s \in S$. Then, for every increasing sequence of finite subsets of S such that $\overline{\bigcup_n S_n} = S$, and for every $u \in \mathbb{R}$,*

$$f^*(u) = \lim_{n\to\infty} \mathbb{E}[Z_u \,|\, Z_s = \zeta(s)\ \forall s \in S_n].$$

Proof. Take any sequence S_n and denote $A_n = \{f \in R :\ f(s) = \zeta(s)\ \forall s \in S_n\}$,

$$f_n = \operatorname{argmin}_{f\in A_n} \|f\|, \quad n = 1,2,\ldots,$$

and $A = \bigcap_n A_n$. Since an equality can be obtained as a pair of non-strict inequalities in opposite directions, and since $f^* \in A_\infty$, we can apply Proposition 2 to see that $f_n \to f^*$ as $n \to \infty$. The expression of $f_n(u)$ is obtained from Proposition 4. □

Consequently, it is unambiguous to define, for any closed set $S \subset \mathbb{R}$ and any $\zeta \in R$,

$$\mathbb{E}[Z_u \,|\, Z_s = \zeta(s)\ \forall s \in S] \doteq \lim_{n\to\infty} \mathbb{E}[Z_u \,|\, Z_s = \zeta(s)\ \forall s \in T_n], \tag{8}$$

where T_n is any sequence of finite sets that approaches S from within.

2.6. The Gaussian queue

Our motivation for doing this study came from queues with Gaussian input, where we encountered the problem of identifying the most probable paths in sets of the type $\{Z_t \geq \zeta(t), \forall t \in S\}$. We here present two prominent examples of this.

Busy period The first example relates to the busy period in a queue fed by Gaussian input. The queue length process with input Z and service rate 1 is commonly defined as

$$Q_t = \sup_{s \leq t}(Z_t - Z_s - (t-s)).$$

Following [11], let K_T be the set of paths that are such that the ongoing busy period at time 0 is longer than $T > 0$:

$$K_T \doteq \{A < 0 < B : B - A > T\},$$

with the random interval $[A,B]$ defined as

$$[A,B] \doteq [\sup\{t \leq 0 :\ Q_t = 0\}, \inf\{t \geq 0 :\ Q_t = 0\}].$$

When interested in the decay rate of the probability of a *long* busy period, Norros [11] showed that for fBm, with $v(t) = t^{2H}$, without losing generality, attention can be restricted to the set

$$B = \{f \in R :\ f(s) \geq s,\ \forall s \in [0,1]\}$$

of paths in R that create non-proper busy periods starting at 0 and straddling the interval $[0,1]$; this is due to

$$\lim_{T\to\infty} \frac{1}{T^{2-2H}} \log \mathbb{P}(Z \in K_T) = -\inf_{f \in B} \frac{1}{2}\|f\|^2.$$

The problem is to determine the MPP in B, i.e., $\beta^* \doteq \operatorname{argmin}_{f\in B}\|f\|$. Since B is convex and closed, β^* is uniquely determined, but [11] does not succeed in finding an explicit characterization. Both Kozachenko *et al.* [6] and Dieker [5] consider the extension of this setup to a regularly varying (rather than purely polynomial) variance function: $v(t) = L(t)t^{2H}$ for a slowly varying $L(\cdot)$, and show that, under specific conditions,

$$\lim_{T\to\infty} \frac{L(T)}{T^{2-2H}} \log \mathbb{P}(Z \in K_T) = -\inf_{f \in B} \frac{1}{2}\|f\|^2;$$

hence in this case the same minimization problem needs to be solved.

Tandem The second example corresponds to overflow in the second queue of a tandem queueing network. Assume that the first queue is emptied at a constant rate c_1, whereas the second has link rate c_2 (with $c_1 > c_2$). Clearly, the steady-state queue length of the first queue can be represented as

$$Q_1 = \sup_{s \geq 0}(Z_{-s} - c_1 s).$$

Also, the total queue length behaves as a queue with link rate c_2:

$$Q_1 + Q_2 = \sup_{t \geq 0}(Z_{-t} - c_2 t).$$

Therefore, expressing the occupancy of the second queue as the difference of the total buffer content and the content of the first queue,

$$\{Q_2 \geq b\} = \{\exists t \geq 0 : \forall s \geq 0 : Z_{-t} - Z_{-s} - c_2 t + c_1 s \geq b\};$$

it is easily seen that we can restrict ourselves to $s \in [0,t]$, and $t \geq t_b \doteq b/(c_1 - c_2)$. By a straightforward time-shift, we conclude that the decay rate of our interest equals $-\inf_{f\in U} \frac{1}{2}\|f\|^2$, where

$$U \doteq \bigcup_{t \geq t_b} U_t, \text{ with } U_t \doteq \{f \in R : \forall s \in [0,t] : f(s) \geq b + c_2 t - c_1(t-s)\}.$$

This decay rate obviously reads $-\inf_{t\geq t_b} \inf_{f\in U_t} \frac{1}{2}\|f\|^2$. Mandjes and van Uitert [8] partly solve the problem of finding the MPP in U_t: for large values of c_1 (above some explicit threshold value c_1^F) the MPP is known, and for small c_1 the MPP is known under some additional condition (that is *not* fulfilled in the case of fBm).

3. The most probable path in $\{Z \geq \zeta \text{ on } S\}$

The central problem in this paper is of the following form: given a function ζ and a set of timepoints S, what is the most probable path in the event $\{Z \geq \zeta \text{ on } S\}$? In the rest of the paper, we assume that Gaussian process Z satisfies the conditions of Theorem 2 so that the infinitesimal spaces are generated simply by $Z_t, \ldots, Z_t^{(k)}$, where k is the number of derivatives.

In order to keep the presentation simpler, we only consider sets $\{Z \geq \zeta \text{ on } S\}$, with $\zeta \in R$. There are two immediate generalizations, which may be included without too much extra effort. The requirement that $\zeta \in R$ is certainly quite restrictive; point-wise and certain discontinuous conditions can also be handled along the same lines. On the other hand, instead of considering $\{Z \geq \zeta \text{ on } S\}$, one could also study sets $\{Z\text{sign}(\zeta) \geq \zeta\text{sign}(\zeta) \text{ on } S\}$.

Our first general result is a generalization of Proposition 1.

Theorem 3. *Let $\zeta \in R$ and let $S \subseteq \mathbb{R}$ be compact. Denote*

$$B_S \doteq \{ f \in R : \ f(s) \geq \zeta(s) \ \forall s \in S \}.$$

There exists a function $\beta^* \in B_S$ with minimal norm, i.e.,

$$\beta^* \doteq \operatorname{argmin}_{f \in B_S} \| f \|.$$

Moreover,

$$\beta^* \in R^o_{S^*},$$

where

$$S^* = \{ t \in S : \ \beta^*(t) = \zeta(t) \}.$$

If the infinitesimal space of the process Z is trivial, i.e., $G_{t\pm}$ is generated by random variable Z_t, then $\beta^* \in R_{S^*}$, and

$$\beta^*(t) = \mathbb{E}\left[Z_t \,|\, Z_s = \zeta(s) \ \forall s \in S^* \right].$$

Remark: The notation $R^o_{S^*}$ is explained in (5) in Section 2.4, and the meaning of the conditional expectation in (8) in Section 2.5.

Proof. Since B_S contains ζ and it is convex and closed, it has a unique element with minimum norm. Let S_n be a non-decreasing sequence of finite subsets of S such that $S_\infty \doteq \bigcup S_n$ is dense in S. Denote

$$B_n = \{ f \in R : \ f(s) \geq \zeta(s) \ \forall s \in S_n \}, \quad n = 1, 2, \ldots,$$

and let β_n be the element in B_n with smallest norm. By Proposition 2, the sequence β_n converges, and since the functions in R are continuous, the limit is β^*.

Let U be a bounded open interval such that $S \subset U$. For $m = 1, 2, \ldots$ denote

$$U_m = \left\{ t \in U : \ \beta^*(t) > \zeta(t) + \frac{1}{m} \right\}.$$

Since

$$|\beta_n(t) - \beta^*(t)| = |\langle \beta_n - \beta^*, \Gamma(t, \cdot) \rangle| \leq \|\beta_n - \beta^*\| \sup_{u \in U} \sqrt{\Gamma(u,u)}$$

for all $t \in U$, there is a number n_m such that $\beta_{n_m}(t) > t + 1/(2m)$ for all $t \in U_m$.

By Proposition 1,

$$\beta_{n_m} \in \operatorname{Span}\{ \Gamma(s, \cdot) : s \in S_{n_m} \cap U_m^c \} \subseteq R_{S \setminus U_m}.$$

Since the sequence of closed subspaces $R_{S \setminus U_m}$ is decreasing in m and $\beta_{n_m} \to \beta^*$, it follows that

$$\beta^* \in \bigcap_{m=1}^{\infty} R_{U_m^c} = R^o_{S^*}.$$

The last assertion follows directly from Proposition 5. □

Remark 2. The set S^* in Theorem 3 need not be the smallest set fulfilling the assertions. For example, if ζ is the minimum-norm function with condition $\zeta(1) = 1$, and $1 \in S$, then the theorem would give the set S itself as S^*, although the singleton $\{1\}$ would suffice.

Remark 3. In the case of trivial infinitesimal space, Theorem 3 has a clear intuitive content: the 'cheapest' way to push the process above ζ is to push it exactly to the curve $t \mapsto \zeta(t)$ in the subset S^*; the points in $S \setminus S^*$ then come 'for free'.

The information provided by Theorem 3 is still insufficient for characterizing the MPP in any concrete case. Such a characterization can often be obtained by studying 'least likely' finite-dimensional approximations of β^*, defined in such a way that their norm is always less or equal to $\|\beta^*\|$. This idea is borrowed from [8,9].

For any set $V \subseteq S$, denote

$$B_V \doteq \{f \in R : \ f(t) \geq \zeta(t) \ \forall t \in V\}, \quad L_V \doteq \{f \in R : \ f(t) = \zeta(t) \ \forall t \in V\}.$$

Let the unique element with smallest norm in B_V and L_V be, respectively,

$$\varphi^V \doteq \operatorname{argmin}_{\varphi \in B_V} \|\varphi\|, \quad \overline{\varphi}^V \doteq \operatorname{argmin}_{\varphi \in L_V} \|\varphi\|.$$

In this context we identify a vector $\boldsymbol{t} \in \mathbb{R}^n$ with the set of its distinct components. Note that for any $V \subseteq S$, $\|\varphi^V\|$ is a lower bound of $\|\beta^*\|$, but it is possible that $\|\overline{\varphi}^V\| > \|\beta^*\|$.

Next, we state a proposition showing that the coefficients of the $\Gamma(v,\cdot)$, $v \in V$ in the representation of $\overline{\varphi}^V$ are strictly positive if every v is needed to make function φ^V feasible.

Proposition 6. *Assume a finite V. If for each $v \in V$ it holds that $\overline{\varphi}^{V\setminus\{v\}}(v) < \zeta(v)$, then the coefficients θ_v in the representation*

$$\overline{\varphi}^V = \sum_{v \in V} \theta_v \Gamma(v,\cdot)$$

are all strictly positive.

Proof. Take $v \in V$ and denote

$$\overline{\varphi}^{V\setminus\{v\}} = \sum_{t \in V\setminus\{v\}} \tilde{\theta}_t \Gamma(t,\cdot).$$

The assumption that $\overline{\varphi}^{V\setminus\{v\}}(v) < \zeta(v)$ implies that $\|\overline{\varphi}^V\| > \|\overline{\varphi}^{V\setminus\{v\}}\|$. Thus

$$\begin{aligned} 0 &< \|\overline{\varphi}^V - \overline{\varphi}^{V\setminus\{v\}}\|^2 \\ &= \langle \overline{\varphi}^V - \overline{\varphi}^{V\setminus\{v\}}, \sum_{t \in V\setminus\{v\}} (\theta_t - \tilde{\theta}_t)\Gamma(t,\cdot) + \theta_v \Gamma(v,\cdot) \rangle \\ &= \theta_v(\zeta(v) - \overline{\varphi}^{V\setminus\{v\}}(v)). \end{aligned}$$

□

The nature of the MPP in S depends crucially on the smoothness of Z. Section 3.1 is on the non-smooth case, and Section 3.2 on the smooth case.

3.1. The case of non-smooth Z

Theorem 4 describes the MPP for non-smooth Z. Proposition 7 is crucial in the proof of Theorem 4.

Proposition 7. *Assume that Gaussian process Z satisfies the assumptions of Theorem 2. Then the mappings $T \mapsto \overline{\varphi}^T$ and $T \mapsto \varphi^T$ from $\{J \subset \mathbb{R} : \ |J| < \infty\}$ to R are continuous for every fixed $\zeta \in R$, if and only if $G_{0\pm}$, the infinitesimal space of Z, is trivial.*

Proof. First we show the continuity of $\overline{\varphi}^T$ and φ^T under the triviality assumption, i.e., $G_{0\pm} = \{0\}$. Consider the map $T \mapsto \overline{\varphi}^T$.

1. Let T_n and T be finite subsets of $\mathbb{R}$ such that $T_n \to T$. (Notice that in principle T can have a lower cardinality than the T_n.) For every $\varepsilon > 0$, let n_ε be the smallest number such that $T_n \subset T + [-\varepsilon, \varepsilon]$ for all $n \geq n_\varepsilon$.
2. For a closed subspace Y of R, denote by P_Y the orthogonal projection on Y. For closed sets $V \subset \mathbb{R}$ we also use the shorthand notation $P_V \doteq P_{R_V}$. Note that evidently $\overline{\varphi}^{T_n} = P_{T_n}\zeta$, and $\overline{\varphi}^{T} = P_T\zeta$.
3. Further, for any $t \in \mathbb{R}$, denote by

$$R^c_{t,\varepsilon} \doteq R_{[t-\varepsilon,t+\varepsilon]} \ominus R_t = \{ f \in R_{[t-\varepsilon,t+\varepsilon]} \,:\, \langle f, \Gamma(t,\cdot)\rangle = 0\},$$

i.e., $R^c_{t,\varepsilon}$ is the orthogonal complement of R_t with respect to $R_{[t-\varepsilon,t+\varepsilon]}$. The orthogonal complement of R_T with respect to $R_{T+[-\varepsilon,\varepsilon]}$ satisfies

$$R^c_{T,\varepsilon} \doteq R_{T+[-\varepsilon,\varepsilon]} \ominus R_T \subseteq \mathrm{Span}\left\{ R^c_{t,\varepsilon} : t \in T \right\}. \tag{9}$$

4. Now, for $n \geq n_\varepsilon$ (which is needed in the second equality),

$$\overline{\varphi}^{T_n} = P_{T_n}\zeta = P_{T_n}P_{T+[-\varepsilon,\varepsilon]}\zeta = P_{T_n}P_T\zeta + P_{T_n}P_{R^c_{T,\varepsilon}}\zeta.$$

As $n \to \infty$, the first term converges to $P_T\zeta$ (due to the assumed convergence $T_n \to T$; note that $P_T\zeta$ is a *finite* combination of $\Gamma(t,\cdot)$'s, $t \in T$). On the other hand, $P_{T_n}P_{R^c_{T,\varepsilon}}\zeta \to 0$ (as $n \to \infty$), because the triviality of the infinitesimal spaces, in conjunction with (9), implies $\lim_{\varepsilon\to 0} P_{R^c_{T,\varepsilon}} f = 0$ for any fixed $f \in R$.

Then consider the map $T \mapsto \varphi^T$.

1. For any finite T, denote

$$\overline{T} \doteq \left\{ t \in T : \varphi^T(t) = \zeta(t) \right\},$$

and note that $\varphi^T = \overline{\varphi}^{\overline{T}}$. Choose $\varepsilon > 0$ such that for all $t_i, t_j \in T$ it holds that $|t_i - t_j| > 2\varepsilon$. Denote also $\tilde{T}_n \doteq T_n \cap (\overline{T} + [-\varepsilon,\varepsilon])$. Then $\tilde{T}_n \to \overline{T}$ as $n \to \infty$, and by the first part of the proposition we have

$$\overline{\varphi}^{\tilde{T}_n} \to \overline{\varphi}^{\overline{T}} = \varphi^T. \tag{10}$$

2. Let then T' be any accumulation point of the sequence $\overline{T}_n$, and let (n_k) be a subsequence such that $\overline{T}_{n_k} \to T'$. By the continuity of $\overline{\varphi}^T$,

$$\varphi^{T_{n_k}} = \overline{\varphi}^{\overline{T}_{n_k}} \to \overline{\varphi}^{T'}. \tag{11}$$

3. For any $t \in T$, take $t_k \in T_{n_k}$ such that $t_k \to t$. Because convergence in R implies uniform convergence on compacts by (3),

$$\overline{\varphi}^{T'}(t) = \lim_{k\to\infty} \overline{\varphi}^{T'}(t_k) = \lim_{k\to\infty} \varphi^{T_{n_k}}(t_k) \geq \lim_{k\to\infty} \zeta(t_k) = \zeta(t),$$

where the first equality is due to $\overline{\varphi}^{T'}$ being continuous, the second by virtue of (11), the inequality because $t_k \in T_{n_k}$, and the last equality due to ζ being continuous. Thus, $\overline{\varphi}^{T'} \in B_T$. As φ^T is the element of B_T with minimal norm, we conclude that $\|\overline{\varphi}^{T'}\| \geq \|\varphi^T\|$.
4. Now we prove that $\overline{\varphi}^{\tilde{T}_{n_k}} \in B_{T_{n_k}}$ for large k. For any $t \in \tilde{T}_{n_k}$ evidently $\overline{\varphi}^{\tilde{T}_{n_k}}(t) = \zeta(t)$. Now pick $t \in T_{n_k} \setminus \tilde{T}_{n_k}$. By (10) and continuity of $\overline{\varphi}^{\tilde{T}_{n_k}}$ and ζ, we see that $\overline{\varphi}^{\tilde{T}_{n_k}}(t) > \zeta(t)$ for k large enough.
5. The fact that $\overline{\varphi}^{\tilde{T}_{n_k}} \in B_{T_{n_k}}$ for large k, in conjunction with the property that $\varphi^{T_{n_k}}$ is the element of $B_{T_{n_k}}$ with minimum norm, implies the inequality $\|\varphi^{T_{n_k}}\| \leq \|\overline{\varphi}^{\tilde{T}_{n_k}}\|$ for large k. Thus, we have obtained the chain

$$\|\varphi^T\| \leq \|\overline{\varphi}^{T'}\| = \lim_{k\to\infty} \|\varphi^{T_{n_k}}\| \leq \lim_{k\to\infty} \|\overline{\varphi}^{\tilde{T}_{n_k}}\| = \|\varphi^T\|$$

and see that equality must hold everywhere. By the uniqueness of the minimum norm element, we deduce that $\overline{\varphi}^{T'} = \varphi^T$. Finally, because the limit is independent of the accumulation point T', we get the desired convergence $\varphi^{T_n} \to \varphi^T$.

Finally, let us show that the existence of Z_0' implies that the mappings φ^T and $\overline{\varphi}^T$ cannot be continuous. We first verify this statement for $\overline{\varphi}^T$. Suppose the mean-square derivative Z_0' exists. Take $T_n = \{1/n\}$ and let ζ be any element in R such that $\zeta'(0) > 0$. Then $\lim T_n = \{0\}$ and $\overline{\varphi}^{\{0\}} = 0$, but

$$\overline{\varphi}^{T_n} = \frac{\zeta(\frac{1}{n})}{\Gamma(\frac{1}{n},\frac{1}{n})}\Gamma\left(\frac{1}{n},\cdot\right) \to \frac{\zeta'(0)}{2\nu''(0)}\Gamma'(0,\cdot).$$

Since $\varphi^{\{s\}} = \overline{\varphi}^{\{s\}}$ whenever $\zeta(s) \geq 0$, we obtain a counterexample for φ^T as well. □

We now consider sets V of at most n timepoints such that the norm of φ^V is as *large* as possible: let

$$b_n = \sup\left\{\|\varphi^V\| : V \subseteq S,\ |V| \leq n\right\}.$$

By Proposition 2, $b_n \uparrow \|\beta^*\|$ (cf. the proof of Theorem 3). The following theorem shows that for each n, the value b_n is attained at some set S_n, and provides detailed information on this set. This theorem is the key element in our method for identifying most probable paths satisfying an infinite number of conditions. We shall see later that the theorem does not hold in the smooth case.

Theorem 4. *Assume that Gaussian process Z satisfies the conditions of Theorem 2 and that the infinitesimal spaces are trivial. Let b_n be as above, and denote by n^* the possibly infinite number $n^* = \inf\{n \in \mathbb{N} :\ b_n = b_{n+1}\}$. Then*

(i) For each n, there exists a (generally non-unique) set $S_n \subseteq S$ with at most n elements such that $\|\varphi^{S_n}\| = b_n$;
(ii) If $\|\varphi^{S_n}\| = \|\varphi^{S_{n+1}}\|$ for some n, then $\beta^* = \varphi^{S_{n^*}}$;
(iii) If $n \leq n^*$, then $\varphi^{S_n} = \overline{\varphi}^{S_n}$;
(iv) $\lim_{n\to\infty} \varphi^{S_n} = \beta^*$;
(v) Assume that $n^* = \infty$. Then

$$\bigcap_{m=1}^{\infty} \overline{\bigcup_{n=m}^{\infty} S_n} \subseteq S^*,$$

where S^* is the set defined in Theorem 3.

Proof. (i): Take any n if $n^* = \infty$, otherwise any $n \leq n^*$. For $m = 1, 2, \ldots$, choose an n-element set $T_m \subseteq S$ such that

$$\|\varphi^{T_m}\| > b_{n-1} + \left(1 - \frac{1}{m}\right)(b_n - b_{n-1}).$$

If there were a point $t \in T_m$ such that $\varphi^{T_m}(t) > \zeta(t)$, we could, by Proposition 1, remove it from the optimization without changing the optimal point, i.e., we would have $\varphi^{T_m\setminus\{t\}} = \varphi^{T_m}$. This is not possible however, because we required $\|\varphi^{T_m}\| > b_{n-1}$. Thus we have $\varphi^{T_m} = \overline{\varphi}^{T_m}$.

Let us identify the sets T_m with elements in

$$D_S^n \doteq \{\boldsymbol{t} \in \mathbb{R}^n : t_1 \leq \cdots \leq t_n,\ t_i \in S\ \forall i\}.$$

Since D_S^n is compact, the sequence T_m has a subsequence T_{m_k} converging to some element $S_n \in D_S^n$, that might have less than n distinct elements. In any case, Proposition 7 yields that

$$\|\varphi^{S_n}\| = \lim_{k\to\infty} \|\varphi^{T_{m_k}}\| = b_n. \tag{12}$$

Finally, the proof of the next claim shows that in the case $n^* < \infty$ we can just take $S_n = S_{n^*}$ for $n > n^*$.

(ii): If $\|\varphi^{S_n}\| = \|\varphi^{S_{n+1}}\|$ but $\varphi^{S_n} \neq \beta^*$, then $\varphi^{S_n} \notin B_S$. Then some of the hyperplanes $L_{\{t\}}$ strictly separates φ^{S_n} from B_S, that is, $\varphi^{S_n}(t) < \zeta(t)$. It follows that

$$\varphi^{S_n\cup\{t\}} \neq \varphi^{S_n},$$

which by the uniqueness minimum norm elements implies that $\|\varphi^{S_n\cup\{t\}}\| > \|\varphi^{S_n}\|$.

(iii): This was shown already in the proof of claim (i).

(iv): Take an arbitrary sequence of sets $\{D_n\}$ satisfying $D_n \subset D_{n+1} \subseteq S$ and having a dense limit set $D_\infty \doteq \lim_{n\to\infty} D_n$ in S. Then by the continuity of Γ, R_{D_∞} is dense in R_S, which implies that $\varphi^{D_n} \to \beta^*$. Since $\|\varphi^{D_n}\| \leq \|\varphi^{S_n}\|$ for any n, $\|\varphi^{S_n}\| \to \|\beta^*\|$.

It suffices to show that

$$\|\varphi^{S_n} - \beta^*\|^2 \leq \|\beta^*\|^2 - \|\varphi^{S_n}\|^2.$$

But this is easily seen to be equivalent to the condition $\langle \varphi^{S_n}, \beta^* - \varphi^{S_n} \rangle \geq 0$, which is true since β^* is on the same side of the hyperplane $\{ f : \langle \varphi^{S_n}, f \rangle = \|\varphi^{S_n}\|^2 \}$ as the set B_S.

(v): By Cauchy-Schwarz,

$$\|\beta^* - \varphi^{S_n}\| \geq \frac{\beta^*(s) - \varphi^{S_n}(s)}{\|\Gamma(s,\cdot)\|} = \frac{\beta^*(s) - \zeta(s)}{\|\Gamma(s,\cdot)\|}$$

for any n and any $s \in S_n$. Denote

$$\tilde{U}_\varepsilon = \left\{ t \in S : \frac{\beta^*(t) - \zeta(t)}{\|\Gamma(t,\cdot)\|} > \varepsilon \right\}.$$

If $\|\beta^* - \varphi^{S_n}\| \leq \varepsilon$, then $S_n \subseteq \tilde{U}_\varepsilon^c$. On the other hand,

$$\bigcap_{\varepsilon>0} \tilde{U}_\varepsilon^c = S^*.$$

□

The claim (iii) of the previous proposition is crucial, because it makes it possible to compute the paths φ^{S_n} when the set S_n is known. Our example with fractional Brownian motion in Section 4.1 indicates that the explicit identification of the S_n's is usually impossible in practice, but general properties can often be deduced.

Here are some other useful properties of the paths φ^{S_n}:

Proposition 8. *Assume that Gaussian process Z satisfies the conditions of Theorem 2 and that the infinitesimal spaces are trivial. Let $n \leq n^*$.*

(i) For each $s \in S_n$,

$$\overline{\varphi}^{S_n\setminus\{s\}}(s) < \zeta(s).$$

(ii) The coefficients θ_s in the unique representation

$$\varphi^{S_n} = \sum_{s\in S_n} \theta_s \Gamma(s,\cdot) \tag{13}$$

are all strictly positive.

Proof. (i): By claim (ii) of Theorem 4, all points in S_n are relevant. It follows that we cannot have $\overline{\varphi}^{S_n\setminus\{s\}}(s) = s$, because otherwise we would have $\overline{\varphi}^{S_n\setminus\{s\}} = \overline{\varphi}^{S_n} = \varphi^{S_n}$. Assume that

$$\overline{\varphi}^{S_n\setminus\{s\}}(s) > \zeta(s).$$

Then

$$\overline{\varphi}^{S_n\setminus\{s\}} \in B_{S^n}.$$

Since $\overline{\varphi}^{S_n\setminus\{s\}} \neq \overline{\varphi}^{S_n}$ and $\overline{\varphi}^{S_n} \in L_{S_n\setminus\{s\}}$, we obtain the contradictory chain of inequalities

$$\|\overline{\varphi}^{S_n\setminus\{s\}}\| < \|\overline{\varphi}^{S_n}\| = \|\varphi^{S_n}\| < \|\overline{\varphi}^{S_n\setminus\{s\}}\|.$$

Thus, $\overline{\varphi}^{S_n\setminus\{s\}}(s) < \zeta(s)$.

(ii): Follows from Proposition 6. □

So far we have made rather few assumptions on the variance function. In the last general proposition in the non-smooth case, we make the additional assumption that $v(t) = \Gamma(t,t)$ be everywhere differentiable, including the origin (necessarily then $v'(0) = 0$). We show that φ^{S_n} then touches ζ smoothly at the points of S_n that are interior points of S.

Proposition 9. *Assume that Gaussian process Z satisfies the conditions of Theorem 2 and that the infinitesimal spaces are trivial. Consider a connected closed set S. Assume v be differentiable on the whole $\mathbb{R}$. Let $n \le n^*$ and denote $S_n = \{s_i\}_{i=1}^n$, where* $\min\{s \in S\} \le s_1 < s_2 < \cdots < s_n \le \max\{s \in S\}$.

(i) For $i = 2, \dots, n-1$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi^{S_n}(t)\big|_{t=s_i} = \zeta'(s_i),$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\varphi^{S_n}(t)\big|_{t=s_1} \ge \zeta'(s_1), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\varphi^{S_n}(t)\big|_{t=s_n} \le \zeta'(s_n),$$

where an inequality can be replaced by an equality, if point s_1 or s_n is an inner point of S.

(ii) Assume additionally that $v(t)$ be twice differentiable outside the origin, and $v''(0) = \infty$. Then the curve $\varphi^{S_n}(t)$ touches the line $\zeta(t)$ from below at the points $s_1, \dots, s_{n-1}$.

Proof. (i): Denote $\mathbf{t} = (t_1, \dots, t_n)$, $\zeta(\mathbf{t}) = (\zeta(t_1), \dots, \zeta(t_n))^T$ and

$$f(\cdot) = \zeta(\mathbf{t})^T \Gamma(\mathbf{t})^{-1} \begin{pmatrix} \Gamma(t_1, \cdot) \\ \vdots \\ \Gamma(t_n, \cdot) \end{pmatrix} = \theta(\mathbf{t})\Gamma(\mathbf{t}, \cdot),$$

where $\theta(\mathbf{t}) = \zeta(\mathbf{t})^T \Gamma(\mathbf{t})^{-1}$. Thus $f(t_i) = \zeta(t_i)$ for $i = 1, \dots, n$. Taking the derivative of f at points t_k, $k = 1, \dots, n$, gives

$$f'(t_k) = \sum_{i \ne k} \theta_i(\mathbf{t}) \frac{\partial}{\partial t_k} \Gamma(t_i, t_k) + \frac{1}{2}\theta_k(\mathbf{t})\, v'(t_k) \tag{14}$$

(note that here we need that $v'(0) = 0$).

Since the s_i maximize the norm,

$$\frac{\partial}{\partial t_k} \|f\|^2 \Big|_{\mathbf{t}=\mathbf{s}} = 0 \quad \text{for } k = 2, \dots, n-1. \tag{15}$$

Observing that $\|f\|^2 = \langle f, \theta(\mathbf{t})\Gamma(\mathbf{t}, \cdot)\rangle = \theta(\mathbf{t})\zeta(\mathbf{t})$, this condition can be written as

$$(\partial_k \theta(\mathbf{s}))\, \zeta(\mathbf{s}) = -\zeta'(s_k)\theta_k(\mathbf{s}), \quad k = 2, \dots, n-1, \tag{16}$$

where $\partial_k \theta(\mathbf{t}) = \frac{\partial}{\partial t_k}\theta(\mathbf{t})$.

On the other hand, we can write $\|f\|^2 = \theta(\mathbf{t})\Gamma(\mathbf{s})\theta(\mathbf{t})^T$ and obtain the expressions

$$\begin{aligned}
\frac{\partial}{\partial t_k}\|f\|^2 &= \frac{\partial}{\partial t_k} \sum\sum \theta_i(\mathbf{t})\Gamma(t_i, t_j)\theta_j(\mathbf{t}) \\
&= \sum\sum 2\theta_i(\mathbf{t})\Gamma(t_i, t_j)\, (\partial_k \theta_j(\mathbf{t})) + \sum_{i \ne k} 2\theta_k(\mathbf{t})\theta_i(\mathbf{t}) \left(\frac{\partial}{\partial t_k}\Gamma(t_i, t_k) \right) + \theta_k(\mathbf{t})^2 v'(t_k) \\
&= 2\,(\partial_k \theta(\mathbf{t}))\, \Gamma(\mathbf{t})\theta(\mathbf{t})^T + 2\theta_k(\mathbf{t})\, f'(t_k), \quad k = 1, \dots, n-1,
\end{aligned}$$

where the last line follows from (14). Finally, notice that $\Gamma(\mathbf{t})\theta(\mathbf{t})^T = \zeta(\mathbf{t})$, replace $\mathbf{t}$ by $\mathbf{s}$, and use (16) to get

$$f'(s_k) = -\frac{1}{\theta_k(\mathbf{s})}\, (\partial_k \theta(\mathbf{s}))\, \zeta(\mathbf{s}) = \zeta'(s_k).$$

For points s_1 and s_n, the equality in (15) is replaced by an inequality. Otherwise, the proof is similar.

(ii): By claim (i), it is enough to show that

$$\frac{\mathrm{d}^2}{\mathrm{d}^2 t}\varphi^{S_n}(t) < 0$$

at the points $s_1, \dots, s_{n-1}$. A direct computation yields

$$\frac{\mathrm{d}^2}{\mathrm{d}^2 t}\varphi^{S_n}(t) = \frac{1}{2}\sum_i \theta_i(\mathbf{s})(v''(t) - v''(t - s_i)).$$

By claim (ii) of Proposition 8 and the assumption $v''(0) = \infty$, this expression equals $-\infty$ at all the points s_i. □

3.2. The case of smooth Z

When process Z has derivatives up to the order $k \in \{1,2,\ldots\}$, the analysis gets more involved since the mappings $T \mapsto \varphi^T$ and $T \mapsto \overline{\varphi}^T$ are not continuous anymore. Fortunately, in the case of smooth processes, only a small number of points is often enough to determine the most probable path. For example, the most probable path for the busy period of a queue fed by integrated Ornstein-Uhlenbeck process is solved using just two points (Section 4.2) whereas infinitely many points are needed in the case of fractional Brownian input (Section 4.1). The general approach is left for future studies. In this paper, as a starting point, we present in Section 4.2 the solution of the special case of busy periods of the integrated Ornstein-Uhlenbeck inputs (which are once differentiable).

Instead of imposing conditions on the values of Z at some points, in the smooth case we could equivalently also put requirements on the infinitesimal neighborhoods of those points. More precisely, we can require that the *projections* $P_{t\pm}$ to the infinitesimal spaces $R_{t\pm}$ satisfy the original condition in some ε-neighborhood, i.e., for V again a finite subset of S,

$$B_V = \{ f \in R \,:\, P_{t\pm} f(s) \geq P_{t\pm}\zeta(s),\ \forall t \in V,\ \forall s \in [t-\varepsilon, t+\varepsilon] \cap S, \text{ for some } \varepsilon > 0 \}. \tag{17}$$

For any $f \in B_V$ we have naturally $f(t) \geq \zeta(t)$ for all $t \in V$. Moreover, if ζ is nicely behaving, it is also possible that $f(s) \geq \zeta(s)$ in the neighborhood of $t \in V$, even if $f(t) = \zeta(t)$. There is no easy way to write a generalization to L_V, since $R_{t\pm}$ is spanned by $\Gamma(t,\cdot), \ldots, \Gamma^{(k)}(t,\cdot)$, and often only some subset of these derivatives results in a sharp condition.

As an example, let us consider a connected closed set S and the case of $k = 1$, i.e., processes which are once differentiable. Proposition 3 implies that for any $t \in V$ the condition in (17) can be written as

$$\begin{aligned} 0 &\leq \big(f(t)-\zeta(t),\ f'(t)-\zeta'(t)\big) \begin{pmatrix} v(t) & \frac{1}{2}v'(t) \\ \frac{1}{2}v'(t) & \frac{1}{2}v''(0) \end{pmatrix}^{-1} \begin{pmatrix} \Gamma(t,s) \\ \Gamma'(t,s) \end{pmatrix} \\ &= (f(t)-\zeta(t))g_1(s) + (f'(t)-\zeta'(t))g_2(s), \end{aligned}$$

for s in some ε-environment of t, and the $g_i(s)$ functions defined appropriately; notice that $\mathrm{Var}(Z_t') = \frac{1}{2}v''(0)$ and $\mathrm{Cov}(Z_t, Z_t') = \frac{1}{2}(v'(t) - v'(0)) = \frac{1}{2}v'(t)$ for smooth Z. One can show that $g_1(s)$ is positive for s in the neighborhood of t, whereas $g_2(s)$ changes its sign at t. Denote $S^i := \{s \in S : |s-y| > 0\ \forall y \in \mathbb{R} \setminus S\}$, $S^l := \min\{s \in S\}$ and $S^r := \max\{s \in S\}$, i.e., the inner, left boundary and right boundary points of S. Then B_V can written as the intersection $B_V = B_V^{(i)} \cap B_V^{(l)} \cap B_V^{(r)}$, where

$$\begin{aligned} B_V^{(i)} &= \big\{ f \in R : \forall t \in V \cap S^i : \{f(t) > \zeta(t)\} \text{ or } \{f(t) = \zeta(t) \text{ and } f'(t) = \zeta'(t)\} \big\}, \\ B_V^{(l)} &= \big\{ f \in R : \forall t \in V \cap S^l : \{f(t) > \zeta(t)\} \text{ or } \{f(t) = \zeta(t) \text{ and } f'(t) \geq \zeta'(t)\} \big\}, \\ B_V^{(r)} &= \big\{ f \in R : \forall t \in V \cap S^r : \{f(t) > \zeta(t)\} \text{ or } \{f(t) = \zeta(t) \text{ and } f'(t) \leq \zeta'(t)\} \big\}. \end{aligned}$$

4. Example: busy periods of Gaussian queues

As an application of the results derived in Section 3, we consider the problem of busy periods of a queue with Gaussian input, introduced in Section 2.6. We consider both an example of non-smooth input (fBm, Section 4.1) and smooth input (integrated Ornstein-Uhlenbeck, Section 4.2).

4.1. Fractional Brownian motion

Our results enable an explicit characterization of β^* in the case that Z is a fractional Brownian motion (fBm), $S = [0,1]$, and $\zeta(t) = t$ for $t \in S$. As discussed in Section 2.6, this gives the logarithmic asymptotics of the probability of long busy periods in a queue with fBm input.

Assume that the fBm Z has self-similarity parameter $H \in (0,1)$, such that

$$\Gamma(s,t) = \frac{1}{2}(s^{2H} + t^{2H} - |s-t|^{2H}).$$

Let us first state some properties of the derivative of φ^{S_n} for fixed $n \leq n^*$.

Proposition 10. *Let $H > 1/2$, and let $n \le n^*$. Denote $\psi(t) = \frac{\mathrm{d}}{\mathrm{d}t}\varphi^{S_n}(t)$ and $S_n = \{s_i\}_{i=1}^n$, where $0 < s_1 < s_2 < \cdots < s_n \le 1$. Then*

(i) $s_n = 1$;
(ii) $\psi(s_i) = 1$ *and* $\psi'(s_i) = -\infty$ *for* $i = 1, \dots, n-1$;
(iii) $\psi(0) < 1$, *and* $\psi(t) = 1$ *for only one point in* $(0, s_1)$;
(iv) For each $i = 1, \dots, n-2$, $\psi(t) = 1$ *for only one point in* (s_i, s_{i+1});
(v) $\psi(1) < 1$, *and* $\psi(t) = 1$ *for two points in* $(s_{n-1}, 1)$.

Proof. (i) Denote $\mathbf{s} = (s_1, \dots, s_n)^T$. The self-similarity of fBm gives

$$\Gamma(s_i, s_j) = s_n^{2H} \Gamma\left(\frac{s_i}{s_n}, \frac{s_j}{s_n}\right).$$

Thus,

$$\|\overline{\varphi}^{S_n}\|^2 = \mathbf{s}^T \Gamma(\mathbf{s})\mathbf{s} = s_n^{2-2H} \tilde{\mathbf{s}}^T \Gamma(\tilde{\mathbf{s}})\tilde{\mathbf{s}},$$

where $\tilde{\mathbf{s}} = \left(\frac{s_1}{s_n}, \dots, \frac{s_{n-1}}{s_n}, 1\right) = (\tilde{s}_1, \dots, \tilde{s}_{n-1}, 1)$. Since $\varphi^{S_n} = \overline{\varphi}^{S_n}$ for $n \le n^*$, and by recalling that S_n maximizes the norm, we conclude $s_n = 1$.

(ii) This follows from Proposition 9; note that $v''(0) = \infty$.

(iii) Write $\psi(t)$ in the form

$$\psi(t) = C\left[t^\alpha + \sum_{s \in S_n,\ s>t} \rho_s (s-t)^\alpha - \sum_{s \in S_n,\ s<t} \rho_s (t-s)^\alpha\right], \tag{18}$$

where

$$\alpha \doteq 2H - 1 \in (0,1), \quad C \doteq H \sum_{s \in S_n} \theta_s, \quad \rho_s \doteq \frac{\theta_s}{\sum_{r \in S_n} \theta_r} \in (0,1).$$

Note that in the right hand side of (18), the first term is increasing and concave, the second is decreasing and concave, and the third (negative) is decreasing and convex. Hence ψ is strictly concave between 0 and s_1. Due to this property, in conjunction with $\psi(s_1) = 1$, ψ can obtain the value 1 at most once in $(0, s_1)$. On the other hand, this does happen at least once by the mean value theorem, since $\varphi^{S_n}(s_1) = \int_0^{s_1} \psi(\tau)\mathrm{d}\tau = s_1$.

(iv) Since $\psi'(s_i) < 0$, $i = 1, \dots, n-1$, it is enough to show that within (s_i, s_{i+1}), ψ' can change its sign at most twice. Write

$$\psi'(t) = C\alpha\left[t^\beta - \sum_{s \in S_n,\ s>t} \rho_s (s-t)^\beta - \sum_{s \in S_n,\ s<t} \rho_s (t-s)^\beta\right],$$

where $\beta \doteq \alpha - 1 \in (-1, 0)$. With $t \in (s_i, s_{i+1})$, make the change of variable

$$\begin{aligned} x &= s_i^\beta - t^\beta \\ t &= \left(s_i^\beta - x\right)^{1/\beta}, \quad x \in \left(0, s_i^\beta - s_{i+1}^\beta\right). \end{aligned}$$

This transforms the first term t^β into a linear function. The powers in the first sum read, in terms of x:

$$g_j(x) \doteq \left(s_j - \left(s_i^\beta - x\right)^{1/\beta}\right)^\beta, \quad j > i.$$

A straightforward calculation shows that $g_j''(x) > 0$, thus g_j is convex. An essentially identical calculation shows the convexity of the functions

$$h_j(x) \doteq \left(\left(s_i^\beta - x\right)^{1/\beta} - s_j\right)^\beta, \quad j \le i,$$

appearing in the second sum. Now the stated follows by observing that the convex function

$$\sum_{j=i+1}^{n} \rho_{s_j} g_j(x) + \sum_{j=1}^{i} \rho_{s_j} h_j(x)$$

can cross the linear function $s_i^\beta - x$ at most twice.

(v) The sign-change argument of the previous item also works on the interval $(s_{n-1},1)$. It remains to note that

$$\psi(1) = \sum_{s\in S_n} \theta_s \frac{\mathrm{d}}{\mathrm{d}t} \Gamma(s,t)\big|_{t=1} < \sum_{s\in S_n} \theta_s \Gamma(s,1) = 1,$$

as a consequence of the fact

$$\frac{\mathrm{d}}{\mathrm{d}t}\Gamma(s,t) < \frac{\Gamma(s,t)}{t}, \quad 0 < s \le t.$$

Thus since $\psi(1) < 1$, there are two points on $(s_{n-1},1)$ such that $\psi(t) = 1$. □

Applying the previous proposition together with results of Section 3, we get the following qualitative characterizations of the paths φ^{S_n}.

Proposition 11. *Let $H > 1/2$ and $S_n = \{s_i\}_{i=1}^n$, where $0 < s_1 < s_2 < \cdots < s_n = 1$.*

(i) The function $\varphi^{S_n}(t)$ is concave for $t \ge 1/2$;
(ii) For $n \ge 2$, $s_{n-1} \le 1/2$;
(iii) There exists a time point $u_n \in (s_{n-1},1)$ such that

$$\varphi^{S_n}(t) \le t, \quad t \in [0,u_n],$$
$$\varphi^{S_n}(t) \ge t, \quad t \in [u_n,1];$$

(iv) $\varphi^{S_n}(t) < t$ on $[0,u_n]$ unless $t \in S_n \cup \{0,u_n\}$, and $\varphi^{S_n}(t) > t$ on $(u_n,1)$;
(v) The number n^* is infinite.

Proof. (i) Since for any $t > 0$ the second derivative of $\Gamma(t,\cdot)$ is negative after the point $t/2$ (i.e., $\frac{\mathrm{d}^2}{\mathrm{d}s^2}\Gamma(t,s) \le 0$ for all $s \ge t/2$), and the coefficients θ_s in the representation (13) are positive by claim (ii) of Proposition 8, the second derivative of φ^{S_n} is negative after the time point $1/2$. This proves the claim on concavity for $t \ge 1/2$.

(ii) By Propositions 9 and 10, $\frac{\mathrm{d}}{\mathrm{d}t}\varphi^{S_n}(t)$ must be increasing somewhere after s_{n-1}, i.e., there is a subinterval of $(s_{n-1},1)$ where $\varphi^{S_n}(t)$ is convex. However by (i), $\varphi^{S_n}(t)$ is concave in $[1/2,1]$.

(iii) and (iv) Follows directly from Proposition 10.

(v) The infiniteness of n^* follows from the fact that the above characterization of the S_n's was shown to hold for any n. (If n^* were finite, we would have $\varphi^{S_{n^*}}(t) \ge t$ for all $t \in [0,1]$.) □

Proposition 12. *Let $H < 1/2$. The number n^* is infinite. Let $S_n = \{s_i\}_{i=1}^n$, where $0 < s_1 < s_2 < \cdots < s_n \le 1$. The number s_n is 1 for all n. The function $\varphi^{S_n}(t)$ is concave for $t \le 1/2$. There exists a time point $u_n \in (0,s_1)$ such that*

$$\varphi^{S_n}(t) \ge t, \quad t \in [0,u_n],$$
$$\varphi^{S_n}(t) \le t, \quad t \in [u_n,1].$$

Moreover, $\varphi^{S_n}(t) < t$ on $[u_n,1]$ unless $t \in S_n \cup \{u_n\}$, and $\varphi^{S_n}(t) > t$ on $(0,u_n)$.

Proof. The proof is a simpler variant of the case $H > 1/2$, since φ^{S_n} turns out to be convex inside each interval (s_j,s_{j+1}). This is seen by the applying the change of variable used in item (iv) in the proof of Proposition 10, applied directly to the path itself instead of the second derivative. As regards the form of φ^{S_n} in $(0,s_1)$, we only need to note that the derivative of φ^{S_n} is convex in this interval. □

Examples of the shapes of the paths φ^{S_n} are shown in Figure 1. We can now prove our main result on fBm:

Theorem 5. *For an fBm with $H > 1/2$, the set S^* has the form*

$$S^* = [0,s^*] \cup \{1\},$$

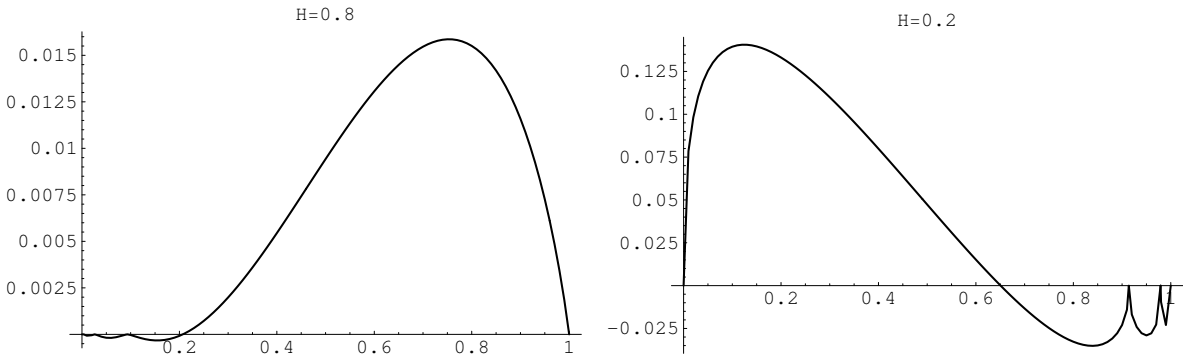

Figure 1. The shapes of $\varphi^{S_3}(t)-t$ for fBm with $H=0.8$ (left; in this case s_1 is too close to 0 to be seen in the figure) and $H=0.2$ (right).

where $s^*\in(0,1)$. *The function* β^* *has the expression*

$$\begin{aligned}\beta^*(t) &= \mathbb{E}\,[\,Z_t\,|\,Z_s=s,\,\forall s\in[0,s^*],Z_1=1]\\ &= \chi_{[0,s^*]}(t)+\frac{\mathrm{Cov}\,[\,Z_t,Z_1\,|\,\mathscr{F}]}{\mathrm{Var}\,[\,Z_t\,|\,\mathscr{F}]}(1-\chi_{[0,s^*]}(1)),\end{aligned}$$

where $\mathscr{F}=\sigma(Z_s:\ s\in[0,s^*])$, *and*

$$\|\beta^*\|^2=\|\chi_{[0,s^*]}\|^2+\frac{(1-\chi_{[0,s^*]}(1))^2}{\mathrm{Var}\,(Z_1-\mathbb{E}\,[\,Z_1\,|\,\mathscr{F}_s,\ s\in[0,s^*]])},$$

where $\chi_{[0,t]}$ *is the most probable path in R satisfying* $\chi_{[0,t]}(s)=s$ *for all* $s\in[0,t]$.

For an fBm with $H=1/2$ *(i.e., the Brownian motion), we have*

$$S^*=[0,1].$$

For an fBm with $H<1/2$, *we have*

$$S^*=[s^*,1],$$

where $s^*\in(0,1)$,

$$\beta^*(t)=\mathbb{E}\,[\,Z_t\,|\,Z_s=s,\,\forall s\in[s^*,1]]=\chi_{[s^*,1]}\quad\textit{and}\quad\|\beta^*\|^2=\|\chi_{[s^*,1]}\|^2,$$

where $\chi_{[t,1]}$ *is the most probable path in R satisfying* $\chi_{[t,1]}(s)=s$ *for all* $s\in[t,1]$.

Remark 4. For the case $H=1/2$, S^* is not the minimal set, the singleton $\{1\}$ would suffice.

Proof. $H>1/2$:

1^{o} Set S^* cannot be the whole interval since the case $\beta^*(t)=t$ for all $t\in[0,1]$ is ruled out because we know from [11] that $\chi_{[0,1]}$ is *not* the optimal busy period path. On the other hand, $S^*\neq\{1\}$, since $\Gamma(1,\cdot)$ is not in B.

By claim (iv) of Theorem 4, β^* is a limit of the functions φ^{S_n}. By Proposition 11, $\varphi^{S_n}(t)$ is at or below the diagonal on $[0,u_n]$ and strictly above it on $(u_n,1)$. On the other hand, Proposition 10 shows that on each interval (s_i,s_{i+1}) (for $i=0,\dots,n-1$; $s_0=0$ and $s_n=1$) φ^{S_n} is first concave then convex and finally concave again. Thus, on interval $[u_n,1]$, φ^{S_n} is either concave or first convex and then concave; this behavior is qualitatively illustrated by the φ^{S_3} shown in Figure 1. Combine this with the properties mentioned in the first paragraph to get

$$\lim_{n\to\infty}\varphi^{S_n}(t)=t,\ \forall t\in[0,s^*]\cup\{1\}\quad\text{and}\quad\lim_{n\to\infty}\varphi^{S_n}(t)>t,\ \forall t\in(s^*,1)\cup\{1\}$$

for some $s^*\in(0,1)$.

$2^{\rm o}$ For any function $f \in R$, define

$$\varphi_f(t) = \mathbb{E}\,[\,Z_t \,|\, Z_s = f(s)\ \forall s \in [0,s^*]\,],$$
$$\psi_f(t) = \mathbb{E}\,[\,Z_t \,|\, Z_s = f(s)\ \forall s \in [0,s^*];\ Z_1 = 1\,].$$

The conditional distribution of the pair (Z_t, Z_1) w.r.t. $\mathscr{F}$ is a two-dimensional Gaussian distribution with (random) mean $\mathbb{E}\,[\,(Z_t,Z_1)\,|\,\mathscr{F}]$. Thus, the further conditioning on $\{Z_1 = 1\}$ can be computed according to the formula of conditional expectation in a bivariate Gaussian distribution:

$$\psi_f(t) = \varphi_f(t) + \frac{\mathrm{Cov}\,[\,Z_t, Z_1 \,|\, \mathscr{F}]}{\mathrm{Var}\,[\,Z_1 \,|\, \mathscr{F}]}\,(1-\varphi_f(1)) = \varphi_f(t) + c(t)((1-\varphi_f(1)),$$

where $c(t) = \mathrm{Cov}\,[\,Z_t, Z_1 \,|\, \mathscr{F}]\,/\mathrm{Var}\,[\,Z_1 \,|\, \mathscr{F}]$ does not depend on f. Applying this to the function $f(t) \equiv 0$ yields

$$c(t) = \psi_0(t).$$

Since $\langle \psi_0, \Gamma(u,\cdot)\rangle = 0$ for $u \in [0,s^*]$, ψ_0 minimizes the R-norm in the set

$$R^{\perp}_{[0,s^*]} \cap \{f:\ f(1) = 1\}.$$

Denote by P the orthogonal projection on the subspace $R_{[0,s^*]}$. For $g \in R^{\perp}_{[0,s^*]}$, we have

$$g(1) = \langle g, \Gamma(1,\cdot)\rangle = \langle g, (I-P)\Gamma(1,\cdot)\rangle,$$

and it follows that the element g in $R^{\perp}_{[0,s^*]} \cap \{f:\ f(1) = 1\}$ with minimal norm must be a multiple of $(I - P)\Gamma(1,\cdot)$. Thus,

$$\psi_0 = \frac{1}{\|(I-P)\Gamma(1,\cdot)\|^2}\,(I-P)\Gamma(1,\cdot).$$

The counterpart of $P\Gamma(1,\cdot)$ in the isometry (1) is $\mathbb{E}\,[\,Z_1 \,|\, \mathscr{F}]$, and it follows that the counterpart of ψ_0 is the random variable

$$\frac{Z_1 - \mathbb{E}\,[\,Z_1 \,|\, \mathscr{F}]}{\mathrm{Var}\,(Z_1 - \mathbb{E}\,[\,Z_1 \,|\, \mathscr{F}])}.$$

Thus,

$$\|\psi_0\|^2 = \mathrm{Var}\,(Z_1 - \mathbb{E}\,[\,Z_1 \,|\, \mathscr{F}])^{-1}.$$

Now, note that

$$\beta^*(t) = \mathbb{E}\,[\,Z_t \,|\, Z_s = s, \forall s \in [0,s^*], Z_1 = 1] = \psi_{\chi_{[0,s^*]}},$$

$\varphi_{\chi_{[0,s^*]}} = \chi_{[0,s^*]}$, and ψ_0 is orthogonal to $\chi_{[0,s^*]}$. Thus,

$$\|\beta^*\|^2 = \|\chi_{[0,s^*]}\|^2 + \frac{(1-\chi_{[0,s^*]}(1))^2}{\mathrm{Var}\,(Z_1 - \mathbb{E}\,[\,Z_1 \,|\, \mathscr{F}])}.$$

$H = 1/2$: A well known result.

$H < 1/2$: Using the similar type of argument as for $H > 1/2$, it is seen that the shapes of the φ^{S_n} (see Figure 1) are such that the limiting path must be of the form $\beta^*(t) > t$ if $t \in (0,s^*)$ and $\beta^*(t) = t$ if $t \in \{0\} \cup [s^*,1]$ for some $s^* \in (0,1)$. □

The quantities in the expression of β^* can be computed. The function $\chi_{[0,s^*]}$ is the counterpart of the random variable M_{s^*} in [12] in the isometry (1), see also [11]. Let us focus on the case $H > 1/2$. Note first that for a multivariate Gaussian distribution the conditional variances and covariances, given a subset of the variables, are constants, and this carries over to Gaussian processes as well. Then apply the general relation,

$$\mathrm{Cov}[Z_s, Z_t | Z_u,\, u \in [0,1]] = \mathbb{E}Z_sZ_t - \mathrm{Cov}(\mathbb{E}[Z_s|\mathscr{F}], \mathbb{E}[Z_t|\mathscr{F}]),$$

recall the prediction formula of Thm. 5.3 in [12]

$$\mathbb{E}[Z_t | Z_u,\, u \in [0,s^*]] = \int_0^{s^*} \Psi_t(s^*,u)\mathrm{d}Z_u,$$

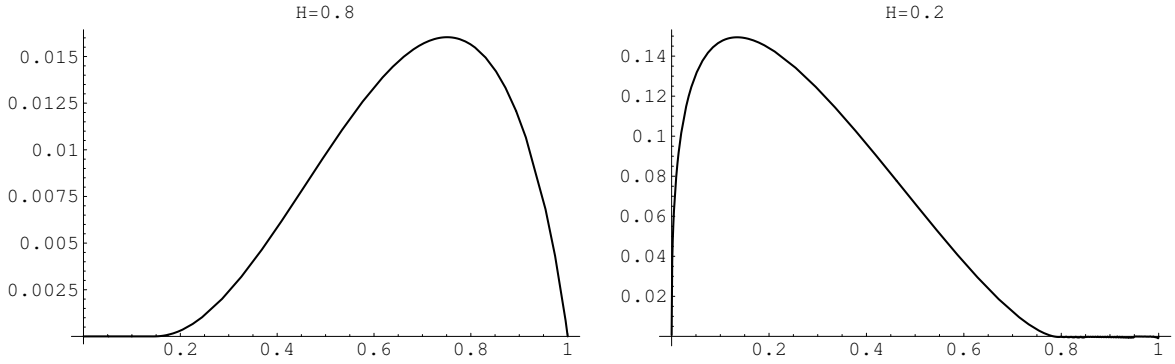

Figure 2. The difference $\beta^*(t)-t$ for fBm with $H=0.8$ (left) and $H=0.2$ (right).

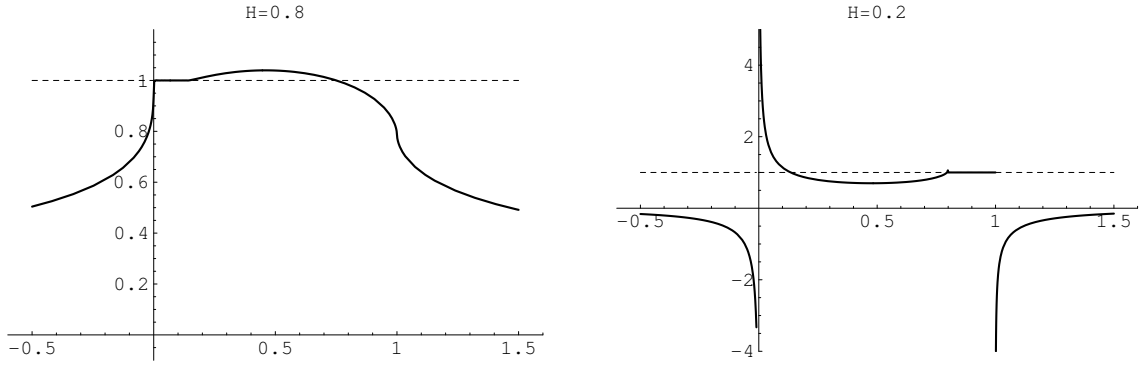

Figure 3. The derivative of $\beta^*(t)$ for fBm with $H=0.8$ and $H=0.2$. The dashed lines correspond to the server rate 1.

and use the covariance formula

$$\begin{aligned}&\mathrm{Cov}\left(\int_0^{s^*}\Psi_s(s^*,u)\mathrm{d}Z_u,\int_0^{s^*}\Psi_t(s^*,v)\mathrm{d}Z_v\right)\\&=H(2H-1)\int_0^{s^*}\int_0^{s^*}\Psi_s(s^*,u)\Psi_t(s^*,v)|u-v|^{2H-2}\mathrm{d}u\mathrm{d}v.\end{aligned}$$

The expression of $\Psi_s(s^*,u)$ contains an integral, and numerical computation of β^* from an expression containing multiple integrals may be hard. As regards the number s^*, we have not found how to obtain any explicit expression for it.

However, by knowing the structure of S^*, or even by just knowing from Theorem 3 that the MPP is determined by a set where it touches the diagonal, it is easy to obtain discrete approximations of the MPPs using some graphical mathematical tool. Figures 2 and 3 show the shapes of the paths β^* in two fBm cases.

4.2. Integrated Ornstein-Uhlenbeck process

Consider a Gaussian process Z_t with stationary increments and variance $v(t)=t-1+e^{-t}$. This is an integrated Ornstein-Uhlenbeck model, which can be interpreted as the Gaussian counterpart of the Anick-Mitra-Sondhi model [1]. Since the rate process is defined by the stochastic differential equation

$$\mathrm{d}X_t=-\beta X_t\mathrm{d}t+\sigma\mathrm{d}W_t,$$

where W denotes the standard Brownian motion, Z is exactly once differentiable and the infinitesimal space $G_{t\pm}$ is generated by Z_t and Z_t'; in the above differential equation both β and σ should be equated to 1 to get the desired variance function. The differentiability property can also be deduced by observing the spectral density of Z_t', which is $1/(4\pi(1+\lambda^2))$.

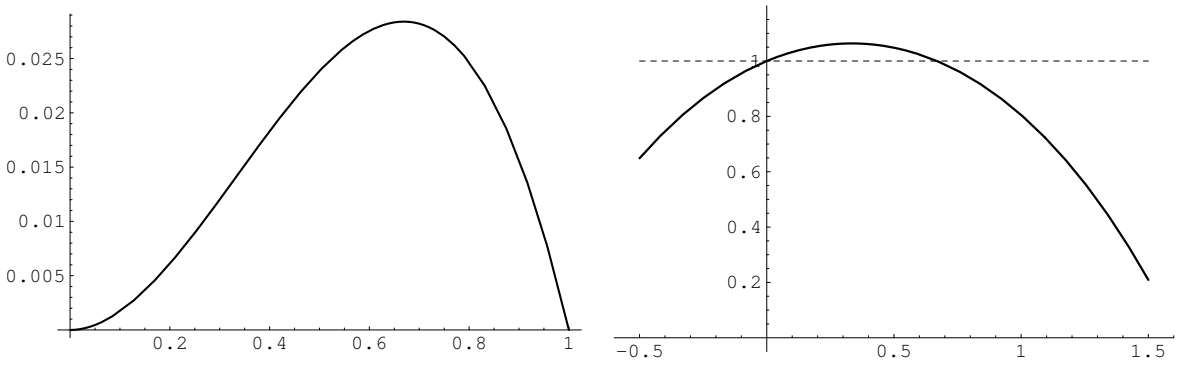

Figure 4. Integrated Ornstein-Uhlenbeck model with $v(t) = t - 1 + e^{-t}$. On the left, the difference $\beta^*(t) - t$. On the rigth, the derivative of $\beta^*(t)$ (solid line) and the server rate (dashed line).

Input paths in B, i.e., path resulting in a busy period starting at $t = 0$ and lasting at least till $t = 1$, necessarily belong to the set

$$F = \{ f \in R : f'(0) \geq 1,\ f(1) \geq 1 \}.$$

The next theorem shows that the most probable path in F is also the most probable path in B, despite $B \subseteq F$. The resulting path is shown in Figure 4.

Theorem 6. *Assume that $v(t) = t - 1 + e^{-t}$. Then the most probable path in $B = \{ f \in R :\ f(s) \geq s,\ \forall s \in [0,1] \}$ is given by*

$$\beta^*(t) = t + \frac{(e-1)^2(t-1+e^{-t}) - (e^t-1)^2 e^{-t}}{4e - 1 - e^2}. \tag{19}$$

Proof. Application of Proposition 3 gives that the mimimizing path in F is

$$f^* = \operatorname{argmin} \{ \| f \| : f \in R,\ \langle f, \Gamma'(0,\cdot) \rangle \geq 1,\ \langle f, \Gamma(1,\cdot) \rangle \geq 1 \}.$$

It is easy to see that both conditions $\langle f, \Gamma'(0,\cdot) \rangle \geq 1$ and $\langle f, \Gamma(1,\cdot) \rangle \geq 1$ are needed, and by Proposition 1, $f^* \in \operatorname{Span}\{ \Gamma'(0,\cdot), \Gamma(1,\cdot) \}$. Thus,

$$f^* = (1,\, 1) \begin{pmatrix} \frac{1}{2}v''(0) & \frac{1}{2}v'(1) \\ \frac{1}{2}v'(1) & v(1) \end{pmatrix}^{-1} \begin{pmatrix} \Gamma'(0,\cdot) \\ \Gamma(1,\cdot) \end{pmatrix}.$$

Inserting $v(t) = t - 1 + e^{-t}$ and doing some simple manipulations gives that $f^*(t)$ equals the formula in the right hand side of (19). One can show that $f^*(t) \geq t$ for all $t \in [0,1]$, for example, using the Taylor series representation. Thus the optimum path f^* in the 'larger set' F is also in the 'smaller set' B. Conclude that $\beta^* = f^*$. □

Acknowledgment. The authors express their thanks to Prof. Mihail Lifshits for pointing out a serious error in an earlier version of this paper, and for several other valuable comments.

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