Centrum voor Wiskunde en Informatica

REPORT RAPPORT

MAS

Modelling, Analysis and Simulation

Modelling, Analysis and Simulation

MAS

Realization theory for linear switched systems

M. Petreczky

REPORT MAS-R0402 DECEMBER 2004

P.O. Box 94079, 1090 GB Amsterdam (NL)
Kruislaan 413, 1098 SJ Amsterdam (NL)
Telephone +31 20 592 9333
Telefax +31 20 592 4199

ISSN 1386-3703

Realization theory for linear switched systems

ABSTRACT
The paper deals with the realization theory of linear switched systems. First, it presents a procedure for constructing a minimal realization from a given linear switched system. Second, it gives necessary and sufficient conditions for an input-output map to be realizable by a linear switched system. The proof of the sufficiency also yields a procedure for constructing a minimal representation of the input-output map.

2000 Mathematics Subject Classification: 93B15, 93B20, 93B25, 93C99
Keywords and Phrases: Hybrid systems, linear switched systems, realization theory, minimal realization, Kalman-decomposition, reachability, observability

Realization Theory for Linear Switched Systems

Mihály Petreczky
Centrum voor Wiskunde en Informatica
P.O.Box 94079, 1090 GB Amsterdam, The Netherlands
Email: M.Petreczky@cwi.nl

Abstract

The paper deals with the realization theory of linear switched systems. First, it presents a procedure for constructing a minimal realization from a given linear switched system. Second, it gives necessary and sufficient conditions for an input-output map to be realizable by a linear switched system. The proof of the sufficiency also yields a procedure for constructing a minimal representation of the input-output map.

1 Introduction

Linear switched systems are one of the best studied subclasses of hybrid systems. A vast literature is available on various issues concerning linear switched systems, for a comprehensive survey see [5]. Yet, to the author's knowledge, no literature exists on the realization theory of linear switched systems.

This paper tries to fill the gap by presenting results on the realization theory of linear switched systems. More specifically, the paper tries to answer the following two questions.

- Does there exist an algorithm, which, given a linear switched system Σ, constructs a minimal linear switched system $\Sigma^{'}$ such that Σ and $\Sigma^{'}$ are input-output equivalent.
- Given an input-output map y, what are the necessary and sufficient conditions for the existence of a linear switched system realizing the map y. Does there exist a procedure to construct a minimal linear switched system which realizes y.

The paper presents a procedure for constructing a *minimal* (with the state-space of the smallest possible dimension, observable and controllable) linear switched system from a given linear switched system. The minimal linear switched system constructed by the procedure is equivalent as a realization to the original system. The procedure also gives a Kalman-like decomposition of the matrices of the original system. It is also proven that all minimal systems are *algebraically* *similar*, meaning that they are defined on vector spaces of the same dimension and their matrices can be transformed to each other by a basis transformation.

The paper also deals with the inverse problem i.e., consider an input-output function and formulate necessary and sufficient conditions for the existence of a linear switched system which is a realization of the given input-output map. The paper presents a set of conditions which are necessary and sufficient for the existence of such a realization. The proof of the sufficiency of these conditions also gives a procedure for constructing a minimal realization of the given input-output map. The necessary and sufficient conditions include a finite-rank condition which is reminiscent of the Hankel-matrix rank condition for linear systems. In fact, the classical conditions for the realizability of an input-output map by a linear system and the classical construction of the minimal linear system realizing the given input-output map are a special case of the results presented in the paper.

In order to develop realization theory for linear switched systems, abstract realization theory for initialized systems (see [7]) has been used. In fact, even the definition of minimality for linear switched systems isn't that obvious. The approach taken in this paper is to treat switched systems as a subclass of abstract initialized systems and use the concepts developed for abstract initialized systems.

Although the results on the realization theory of linear switched systems bear a certain resemblance to those of finite-dimensional linear systems, the former is by no means a straightforward extension of the latter. As the results of this and other papers demonstrate, the approach "apply the well-known linear system theory to each continuous system and combine the results in a smart way" doesn't always work. Reachability, observability and the realization theory of linear switched systems belong to the class of problems, for which classical linear system theory can't be applied. This also shows up on the results. For example, if a linear switched system is reachable, it doesn't mean that any of the linear systems constituting the switched system has to be reachable, nor does it imply that any point of the continuous state space can be reached by some continuous component. The same holds for the observability (in sense of indistinguishability) of linear switched systems. The reader who wishes to verify these statements is encouraged to consult [8]. In the light of these remarks it is not that surprising that a minimal linear switched system may have non-minimal continuous components. That is, if a linear switched system is minimal, it does not imply that any of its continuous components is minimal. On the other hand, the approach to the realization theory taken in the paper bears a certain resemblance with the works on realization theory for nonlinear systems presented in [3, 4, 1]. In some sense linear switched systems have more in common with non-linear than with linear systems.

The outline of the paper is the following. The first section, Section 2 sets up some notation which will be used throughout the paper. Section 3 describes some properties and concepts related to linear switched systems which are used in the rest of the paper. Section 4 presents the minimization procedure and the Kalman-decomposition for linear switched systems. The construction of the minimal linear switched system realizing a given input-output map can be found in Section 5

2 Preliminaries

The section sets up the notation and some terminology which will be used in the paper. Denote by $\mathbb{R}_+$ the set $[0,+\infty) \subseteq \mathbb{R}$. Denote by $\mathbb{N}$ the set of natural numbers $\{0,1,2,\ldots\}$ For $A=[a,b]$, $a,b \in \mathbb{R}\cup\{+\infty,-\infty\}$ and $B \subseteq \mathbb{R}^p$ denote by $PC(A,B)$ the class of piecewise-continuous mappings from A to B. That is, $f \in PC(A,B)$ if and only if on each compact interval f has finitely many points of discontinuity and at each point of discontinuity f has finite left and right limits.

For a set A denote by A^+ the set of finite strings of elements of A, excluding the empty string. Denote by A^* the set of strings over A including the empty string, i.e. $A^* = A^+ \cup \{\epsilon\}$, where ϵ denotes the empty string. For $w = a_1a_2\cdots a_k \in A^+$ the length of w is denoted by $|w|$, i.e. $|w| = k > 0$. Let $|\epsilon| = 0$. Note that in our setting $|w| > 0$ for all $w \in A^+$.

The set of all partial mappings from set A to set B will be denoted by B^A. Let $f : [T_0,T] \to B$. Then the function $\mathrm{Shift}_\sigma(f) : [\sigma+T_0, T+\sigma] \to B$ is defined by $\mathrm{Shift}_\sigma(f)(t) = f(t-\sigma)$ for $\sigma + T_0 \le t \le T + \sigma$. Let A,B be sets. Then the projection $\pi_A : A \times B \to A$ is defined by $\pi_A(a,b) = a$ for $(a,b) \in A \times B$.

Let $S \subseteq \mathbb{R}^n$ i.e. S is an arbitrary subset of $\mathbb{R}^n$ and $f : S \to \mathbb{R}^k$. The function f is said to be *analytic* if there exists an open set $U \subseteq \mathbb{R}^n$ and a function $g : U \to \mathbb{R}^k$ such that $S \subseteq U$, g is analytic in the usual sense and $g|_S = f$.

Let Q be a set, T be a subset of $\mathbb{R}_+$. For each $w = (q_1,t_1)(q_2,t_2)\cdots(q_k,t_k) \in (Q\times T)^*$ define the function $\widetilde{w} \in Q^T$ in the following way. Let $\mathrm{dom}(\widetilde{w}) = [0, \sum_1^k t_i]$. Let $\forall t \in [\sum_1^j t_i, \sum_1^{j+1} t_i) : \widetilde{w}(t) = q_{j+1}$, $j = 0,\ldots,k-1$ and $\widetilde{w}(\sum_1^k t_i) = q_k$. That is, the function $\widetilde{w}$ is a piecewise-constant function, such that its "i-th constant piece" has value q_i and the "duration" of the "i-th constant piece" is t_i. The reason for introducing this function is the following. Consider the relation $\sim$ on $(Q\times T)^+$ defined by

$$(q_1,t_1)(q_2,t_2)\cdots(q_{i-1},t_{i-1})(q,t)(q,t^{'})(q_i,t_i)\cdots(q_k,t_k) \quad \sim$$
$$(q_1,t_1)(q_2,t_2)\cdots\cdots(q_{i-1},t_{i-1})(q,t+t^{'})(q_i,t_i)\cdots(q_k,t_k)$$

and $w(q,0)v \sim wv$. Denote the reflexive transitive closure of $\sim$ by $\sim^*$. Then $w \sim^* u$ if and only if $\widetilde{u} = \widetilde{w}$ for each $u,w \in (Q\times T)^+$.

Let A, B be two finite sets. The set $\{(u,v) \in A^+ \times B^+ \mid |u| = |v|\}$ will be identified with the set $(A\times B)^+$. No distinction will be made between these two sets. For example, (aa,bb) and $(a,b)(a,b)$ will be considered to be the same.

3 Linear switched systems: basic definition and properties

The section is divided into several subsections. Subsection 3.1 contains the definition of switched systems along with the reformulation of some important system theoretic concepts for switched systems. This subsection also describes some basic properties of the input-output behavior induced by switched systems. Subsection 3.2 deals with the definition and basic properties of minimal switched systems. Subsection 3.3 introduces linear switched systems and gives a brief overview of those properties of linear switched systems which are relevant for the realization theory.

3.1 Switched systems

The notion of switched system considered in the paper is the standard one ([5]). That is, a switched system has a continuous state space, but its input space contains both continuous and discrete components. In other words, the sequence of discrete components is determined externally, the evolution of the system does not influence which discrete component will be chosen at a certain point of time. More precisely, the state evolution is described by a finite collection of differential equations. The collection of differential equations is indexed by the discrete component of the input space. The right hand-side of each differential equation also depends on the continuous input component. The differential equations are assumed to have solution on the whole time-axis. The sequence of application of the differential equations is determined externally, the evolution of the system does not influence which differential equation will be chosen at a certain point of time. Therefore the sequences of discrete components, which are indices of the differential equations, will be regarded as inputs. The allowed continuous input functions are assumed to be bounded on any bounded interval. The allowed discrete input is assumed to be piecewise-constant. Notice that switched systems can also be viewed as systems with a state-space given by direct product of a discrete and continuous component. The input space is continuous in this case. The resulting system is a non-deterministic one. In this paper we want to avoid this case, exactly because realization theory of non-deterministic systems is full of complications even in the most simple setting. For example, even for systems on sets, reachability and observability doesn't guarantee minimality nor uniqueness up to isomorphism. The interested reader is referred to [2, 6, 7] for more information on realization theory of abstract control systems. Notice that the class of switched system defined in this paper is a subclass of nonlinear systems.

Definition 3.1 (Switched systems). *A switched (control) system is a tuple*

$$\Sigma = (T, \mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{f_\sigma \mid \sigma \in Q, u \in \mathcal{U}\}, \{h_\sigma | \sigma \in Q\}, x_0)$$

where

- $T = [0, K] \subseteq \mathbb{R}_+$ *is the time index,* $K > 0$
- $\mathcal{X} = \mathbb{R}^n$ *is the state-space*
- $\mathcal{Y} = \mathbb{R}^p$ *is the output-space*
- $\mathcal{U} = \mathbb{R}^m$ *is the input-space*
- Q *is the finite set of discrete modes*
- *for each* $\sigma \in Q$ *the map* $f_\sigma : \mathcal{X} \times \mathcal{U} \to \mathcal{X}$ *is such that for each* $u(.) \in PC(T,\mathcal{U})$ *the differential equation* $\dot{x}\,(t) = f_\sigma(x(t), u(t))$ *with initial condition* $x(t_0) = x_0$ *has a unique solution on the whole* T
- $h_\sigma : \mathcal{X} \to \mathcal{Y}$ *is smooth map for each* $\sigma \in Q$
- $x_0 \in \mathcal{X}$ *is the initial state*

In the sequel we will always assume that $T = \mathbb{R}_+$. Using the notation above, for a given switched system Σ define the mapping $x_\Sigma : \mathcal{X} \times PC(T,\mathcal{U}) \times (Q \times T)^+ \to \mathcal{X}^T$ in the following way. For each $x_{init} \in \mathcal{X}$, $u(.) \in PC(T,\mathcal{U})$ and $w = (q_1, t_1), ..., (q_k, t_k) \in (Q \times T)^+$ let $\mathrm{dom}(x_\Sigma(x_{init}, u(.), w)) = \mathrm{dom}(\widetilde{w})$. By the assumption of the Definition 3.1 for each $q \in Q$ and $u(.) \in PC(T,\mathcal{U})$ the differential equation $\frac{d}{dt}x(\tau) = f_q(x(\tau), u(\tau)), x(\tau_0) = x_0$ has a unique solution on T. For $t \in [0, t_1]$ define $x_\Sigma(x_{init}, u(.), w)(t)$ by $x_\Sigma(x_{init}, u(.), w)(t) = x(t)$, where $x : T \to \mathcal{X}$ is the unique solution of the differential equation

$$\frac{d}{dt}x(t) = f_{q_1}(x(t), u(t)),\, x(0) = x_{init}$$

For $i = 1, \dots, k-1$ and for $t \in (\sum_1^i t_j, \sum_1^{i+1} t_j]$ let $x_\Sigma(x_{init}, u(.), w)(t) = x(t)$ where $x : T \to \mathcal{X}$ is the unique solution of the differential equation

$$\frac{d}{dt}x(t) = f_{q_{i+1}}(x(t), u(t)),\, x(\sum_1^i t_j) = x_\Sigma(x_{init}, u(.), w)(\sum_1^i t_j)$$

The definition of $x_\Sigma(x_{init}, u(.), w)$ can be given a concise form by requiring $x_\Sigma(x_{init}, u(.), w)$ to be continuous and satisfy the following equations

$$\begin{aligned} &\forall t \in \left(\textstyle\sum_1^i t_j, \sum_1^{i+1} t_j\right) : \\ &\quad \frac{d}{dt}x_\Sigma(x_{init}, u(.), w)(t) = f_{q_i}(x_\Sigma(x_{init}, u(.), w)(t), u(t)) \end{aligned} \tag{1}$$

and $x_\Sigma(x_{init}, u(.), w)(0) = x_{init}$. Formula (1) implies that $x_\Sigma(x_{init}, u(.), w)$ in fact depends on the piecewise-constant function $\widetilde{w} : [0, \sum_1^k t_i] \to Q$, i.e. $\widetilde{w_1} = \widetilde{w_2} \implies x_\Sigma(x_{init}, u(.), w_1) = x_\Sigma(x_{init}, u(.), w_2)$ holds. Define the mapping $y_\Sigma : \mathcal{X} \times PC(T,\mathcal{U}) \times (Q \times T)^+ \to \mathcal{Y}^T$ in the following way. For each $x_{init} \in \mathcal{X}$, $u(.) \in$

$PC(T,\mathcal{U})$ and $w = (q_1, t_1), ..., (q_k, t_k) \in (Q \times T)^+$ let $\mathrm{dom}(y(x_{init}, w, u(.)) = \mathrm{dom}(\widetilde{w})$ and

$$\forall t \in \left[\sum_1^i t_j, \sum_1^{i+1} t_j\right) : y_\Sigma(x_{init}, u(.), w)(t) = h_{q_i}(x_\Sigma(x_{init}, u(.), w)(t)) \quad (2)$$

Define $y_\Sigma(x_{init}, u(.), w)(\sum_1^k t_i)$ being equal to $h_{q_k}(x_\Sigma(x_{init}, u(.), w)(\sum_1^k t_i))$. From the definition of the map y_Σ it follows that $\widetilde{w_1} = \widetilde{w_2} \implies y_\Sigma(x_{init}, u(.), w_1) = y_\Sigma(x_{init}, u(.), w_2)$. That is, $x_\Sigma(x_{init}, u(.), w)$ and $y_\Sigma(x_{init}, u(.), w)$ depend on $\widetilde{w}$ rather than on w.

Recall the notion of *initialized system* from [7]. In the sequel, we will identify switched systems with initialized systems. More precisely, with a given switched system $\Sigma = (T, \mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{f_q \mid q \in Q, u \in \mathcal{U}\}, \{h_q \mid q \in Q\}, x_0)$ we associate the initialized system $\Sigma_{init} = (T, \mathcal{X}, \mathcal{Y}, \mathcal{U} \times Q, \phi, h, x_0)$ where ϕ and h are defined in the following way. The domain D_ϕ of the state-transition map is defined as the set of tuples $(\tau, \sigma, x, \omega) \in T \times T \times \mathcal{X} \times (\mathcal{U} \times Q)^{[\sigma,\tau)}$ such that $\pi_Q \circ \omega$ is piecewise constant. The mapping $\phi : D_\phi \to \mathcal{X}$ is defined as $\phi(\tau, \sigma, x_i, \omega) = x_\Sigma(x_i, \mathrm{Shift}_{-\sigma}(\pi_{\mathcal{U}} \circ \omega), w)(\tau - \sigma)$ where $w = (q_1, t_1)(q_2, t_2) \cdots (q_k, t_k) \in (Q \times T)^+$ is any sequence such that $\widetilde{w} = \pi_Q \circ \omega$ holds. Since $x_\Sigma(x_0, u(.), w)$ depends on $\widetilde{w}$ rather than on w, the mapping ϕ above is well defined. The readout map $h : \mathcal{U} \times Q \times T \times X \to \mathcal{Y}$ is defined as $h(u, q, t, x) = h_q(x)$. It is easy to see that the initialized system corresponding to a switched system is time-invariant and complete. In the sequel whenever the term "initialized system" is used, we will mean time-invariant complete initialized system.

Note that in the definition of *initialized systems* in [7] the readout map depends on the time and state only. However it is easy to see that the whole theory also holds if one allows readout maps which depend on the input. For more on this see Chapter 2, Section 2.12 of [7].

The identification of switched systems with the initialized systems allows us to use the terminology and results of [7]. In particular, notions such as input-output behavior, system morphism, response (input-output) map of a system from a state, the reachable set, reachability, observability (indistinguishability), canonical systems, system equivalence, minimal system, minimal representation, of an input-output map are well defined for initialized systems. Since switched systems form a subclass of initialized systems, these definitions can be directly applied to switched systems. However, for the sake of completeness these relevant notions will be repeated specifically for switched systems.

Let $\Sigma = (T, \mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{f_q \mid q \in Q, u \in \mathcal{U}\}, \{h_q \mid q \in Q\}, x_0)$ be a switched system. The map

$$y_\Sigma : PC(T,\mathcal{U}) \times (Q \times T)^+ \to \mathcal{Y}^T$$

defined by $y_\Sigma(u(.), w) = y_\Sigma(x_0, u(.), w)$ $(u(.) \in PC(T,\mathcal{U}),\ w \in (Q \times T)^+)$ is called the *input-output map* (or the *input-output behavior*) induced by Σ. The switched system Σ is said to be a *realization* of an input-output map $\psi : PC(T,\mathcal{U}) \times (Q \times T)^+ \to \mathcal{Y}^T$ if $y_\Sigma = \psi$, i.e. the input-output behavior induced by Σ is identical to ψ. A *system morphism* $\phi : \Sigma_1 \to \Sigma_2$ between switched systems

$$\Sigma_1 = (T, \mathcal{X}_1, \mathcal{U}, \mathcal{Y}, Q, \{f_q^1 \mid q \in Q, u \in \mathcal{U}\}, \{h_q^1 \mid q \in Q\}, x_0^1)$$

and

$$\Sigma_2 = (T, \mathcal{X}_2, \mathcal{U}, \mathcal{Y}, Q, \{f_q^2 \mid q \in Q, u \in \mathcal{U}\}, \{h_q^2 \mid q \in Q\}, x_0^2)$$

is a mapping $\phi : \mathcal{X}_1 \to \mathcal{X}_2$ such that

- $\phi(x_0^1) = x_0^2$
- for each $x \in \mathcal{X}_1$, $u(.) \in PC(T,\mathcal{U})$, $w \in (Q \times T)^+$ and $t \in \mathrm{dom}(\widetilde{w})$ it holds that $\phi(x_{\Sigma_1}(x, u(.), w)(t)) = x_{\Sigma_2}(\phi(x), u(.), w)(t)$
- for each $q \in Q$ and $x \in X_1$ it holds that $h_q^1(x) = h_q^2(\phi(x))$

An immediate consequence of the characterization above is that whenever $\phi : \Sigma_1 \to \Sigma_2$ is a system morphism then it holds that $y_{\Sigma_1}(x, u(.), w) = = y_{\Sigma_2}(\phi(x), u(.), w)$ for each $x \in \mathcal{X}_1$, $u(.) \in PC(T,\mathcal{U})$ and $w \in (Q \times T)^+$. Thus the switched systems Σ_1 and Σ_2 above induce the same input-output behavior. Two switched systems

$$\Sigma_1 = (T, \mathcal{X}_1, \mathcal{U}, \mathcal{Y}, Q, \{f_q^1 \mid q \in Q, u \in \mathcal{U}\}, \{h_q^1 \mid q \in Q\}, x_0^1)$$

and

$$\Sigma_2 = (T, \mathcal{X}_2, \mathcal{U}, \mathcal{Y}, Q, \{f_q^2 \mid q \in Q, u \in \mathcal{U}\}, \{h_q^2 \mid q \in Q\}, x_0^2)$$

are called *(input-output) equivalent* if they induce the same input-output behavior, i.e. $y_{\Sigma_1} = y_{\Sigma_2}$ holds.

Consequently, if two switched systems are related by a system morphism, then they are input-output equivalent. A system morphism is called *isomorphism* whenever it is bijective as a mapping between the state spaces. Two systems are called an *isomorphic* if there exists an isomorphism between them.

A switched system $\Sigma = (T, \mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{f_q \mid q \in Q, u \in \mathcal{U}\}, \{h_q \mid q \in Q\}, x_0)$ is *reachable* if

$$\begin{aligned} Reach(\Sigma) = \{x_\Sigma(x_0, u(.), w)(t) \mid u(.) \in PC(T,\mathcal{U}), \\ w \in (Q \times T)^+, t \in \mathrm{dom}(\widetilde{w})\} = \mathcal{X} \end{aligned}$$

A switched system $\Sigma = (T, \mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{f_q \mid q \in Q, u \in \mathcal{U}\}, \{h_q \mid q \in Q\}, x_0)$ is called *observable* if for each $x_1, x_2 \in \mathcal{X}$ the equality $\forall w \in (Q \times T)^+, u(.) \in PC(T,\mathcal{U}) : y_\Sigma(x_1, u(.), w) = y_\Sigma(x_2, u(.), w)$ implies $x_1 = x_2$. A reachable and observable switched system is called *canonical*.

Consider a switched system $\Sigma = (T, \mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{f_q \mid q \in Q, u \in \mathcal{U}\}, \{h_q \mid q \in Q\}, x_0)$. The input-output behavior induced by Σ is a map $y : PC(T,\mathcal{U}) \times (Q \times T)^+ \to \mathcal{Y}^T$. For each map $y : PC(T,\mathcal{U}) \times (Q \times T)^+ \to \mathcal{Y}^T$ we shall define a map $\widetilde{y} : (\mathcal{U} \times Q \times T)^+ \to \mathcal{Y}$ such that Σ is a realization of y if and only if Σ is a realization of $\widetilde{y}$ in the sense defined below. Denote by $PC_{const}(T,\mathcal{U})$ the set of piecewise-constant input functions. It is well-known that for each $u(.) \in PC(T,\mathcal{U})$ there exists a sequence $u_n(.) \in PC_{const}(T,\mathcal{U}), n \in \mathbb{N}$ such that $\lim_{n\to+\infty} u_n(.) = u(.)$. Given a switched system Σ, by the continuity of the solutions of differential equations we get that $\lim_{n\to+\infty} x_\Sigma(x, u_n(.), w)(t) = x_\Sigma(x, u(.), w)(t)$ and $\lim_{n\to+\infty} y_\Sigma(x, u_n(.), w)(t) = y_\Sigma(x, u(.), w)(t)$. It is also easy to see that for any $u(.) \in PC_{const}(T,\mathcal{U})$ and for any $w \in (Q \times T)^+$ there exists a sequence $z = (q_1, t_1)(q_2, t_2)\cdots(q_k, t_k) \in (Q \times T)^+$ such that $\widetilde{w} = \widetilde{z}$ and $u|_{[\sum_1^i t_i, \sum_1^{i+1} t_i)}$ is constant for $i = 0, \ldots, k-1$. This, of course, implies that $x_\Sigma(x, u(.), w) = x_\Sigma(x, u(.), z)$ and $y_\Sigma(x, u(.), w) = y_\Sigma(x, u(.), z)$. This simple fact lies in the heart of the proof of Proposition 3.1.

Let $\phi : PC(T,\mathcal{U}) \times (Q \times T)^+ \to \mathcal{Y}^T$. Define $\widetilde{\phi} : (\mathcal{U} \times Q \times T)^+ \to \mathcal{Y}$ as

$$\widetilde{\phi}((u_1, q_1, t_1)(u_2, q_2, t_2)\cdots(u_k, q_k, t_k)) = \phi(\widetilde{v}, (q_1, t_1)(q_2, t_2)\cdots(q_k, t_k), \sum_1^k t_i)$$

where $v = (u_1, t_1)(u_2, t_2)\cdots(u_k, t_k) \in (\mathcal{U} \times T)^+$. Define the realization of a map $\psi : (\mathcal{U} \times Q \times T)^+ \to \mathcal{Y}$ in the following way

Definition 3.2. *Consider a function $\psi : (\mathcal{U} \times Q \times T)^+ \to \mathcal{Y}$ and a switched system*

$$\Sigma = (T, \mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{f_q \mid q \in Q, u \in \mathcal{U}\}, \{h_q \mid q \in Q\}, x_0)$$

The switched system Σ is a realization of ψ if $\widetilde{y}_\Sigma = \psi$.

The following proposition, proof of which is straightforward, gives the justification of the concept introduced in Definition 3.2

Proposition 3.1. *Consider a function $y : PC(T,\mathcal{U}) \times (Q \times T)^+ \to \mathcal{Y}^T$. If the input-output map y has a realization by a switched system then the following conditions hold*

1. *For each $w, z \in (Q \times T)^+$, $u \in PC(T,\mathcal{U})$ it holds that $\mathrm{dom}(y(u(.), w)) = \mathrm{dom}(\widetilde{w})$ and $\widetilde{z} = \widetilde{w} \implies y(u(.), w) = y(u(.), z)$.*

2. *For each $w \in (Q \times T)^+$ and $u_n, u(.) \in PC(T,\mathcal{U})$:*

$$\lim_{n\to\infty} u_n(.) = u(.) \implies \lim_{n\to\infty} y(u_n(.), w)(t) = y(u(.), w)(t), (\forall t \in \mathrm{dom}(\widetilde{w})).$$

If y is an arbitrary map which satisfies conditions 1 and 2, then a switched system Σ is a realization of y if and only if it is a realization of $\widetilde{y}$ in the sense of Definition 3.2

3.2 Definition of minimal switched systems

For linear systems the definition of minimality is clear, but for more general systems there is no standard definition of minimality. The definition of minimality used in this paper is analogous to that of abstract system theory, see [6, 2]. We first define minimality for initialized systems. In the sequel we will use the terminology of [7]. Let Θ be any subclass of initialized systems. An initialized system $\Sigma \in \Theta$ is called Θ*–minimal*, if for each reachable initialized system $\Sigma^{'} \in \Theta$ such that $\Sigma^{'}$ and Σ induce the same input-output behavior, there exists a unique surjective system morphism $\phi : \Sigma^{'} \to \Sigma$. It is an easy consequence of the definition that all Θ–minimal systems realizing the same input-output behavior are isomorphic. Denote by Ω the whole class of initial systems. It follows from Section 6.8, Theorem 30 of [7] that each canonical initialized system is Ω–minimal. It also follows from Section 6.8 of [7] that for each input-output map realizable by initialized systems there exists a canonical realization of that input-output map. Thus we get that for each input-output map realizable by initialized systems there exist a Ω–minimal initialized system realizing it. Since all minimal systems are isomorphic and reachability and observability are preserved by isomorphisms, we get that an initial system is Ω–minimal if and only if it is canonical, i.e. reachable and observable. Notice that existence of a minimal system realizing an input-output map is a property of the input-output map. Moreover, if an input-output map has a realization by an initialized system belonging to a certain class Θ (for example it has a realization by a switched system), then the input-output map need not have a Θ–minimal realization. It is easy to see that if $\Theta^{'} \subseteq \Theta$ then each Θ–minimal system belonging to $\Theta^{'}$ is $\Theta^{'}$–minimal. In particular, each canonical system $\Sigma \in \Theta$ is Θ–minimal.

Let Ω_{sw} be the class of switched systems, let $\Omega^{'} \subseteq \Omega_{sw}$ be a subclass of switched systems. The subclass $\Omega^{'}$ can be considered as a subclass of initialized systems. A switched system $\Sigma \in \Omega^{'}$ is called *minimal* if Σ is $\Omega^{'}$–minimal when considered as an initialized system. As a consequence any canonical switched system $\Sigma \in \Omega^{'}$ is $\Omega^{'}$–minimal. Later we will show that for linear switched systems (to be defined later) each minimal linear switched system has a state space of the smallest dimension among all linear switched systems realizing the same behavior.

3.3 Linear switched systems

In this paper we will be concerned with linear switched systems. Consider a switched system $\Sigma = (T, \mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{f_q \mid q \in Q, u \in \mathcal{U}\}, \{h_q \mid q \in Q\}, x_0)$. The switched system Σ is called *linear switched system* if

- $x_0 = 0$
- For each $q \in Q$ there exist linear mappings

$$A_q : \mathcal{X} \to \mathcal{X} \quad B_q : \mathcal{U} \to \mathcal{X} \quad C_q : \mathcal{X} \to \mathcal{Y}$$

  such that

$$f_q(x,u) = A_q x + B_q u \quad \text{and} \quad h_q(x) = C_q x.$$

To make the notation simpler, linear switched system will be denoted by $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$ Notice that for linear switched systems the initial state is taken to be 0, so there is no need to indicate the initial state in the shorthand notation. Consider two linear switched systems

$$\Sigma_1 = (\mathcal{X}_1, \mathcal{U}, \mathcal{Y}, Q, \{(A_{q,1}, B_{q,1}, C_{q,1}) \mid q \in Q\})$$

and

$$\Sigma_2 = (\mathcal{X}_2, \mathcal{U}, \mathcal{Y}, Q, \{(A_{q,2}, B_{q,2}, C_{q,2}) \mid q \in Q\})$$

Systems Σ_1 and Σ_2 are said to be *algebraically similar* if there exists a bijective linear map $S : \mathcal{X}_1 \to \mathcal{X}_2$ such that for all $q \in Q$ it holds that $A_{q,2} = SA_{q,1}S^{-1}$, $B_{q,2} = SB_{q,1}$ and $C_{q,2} = C_{q,1}S^{-1}$.

Notice that the mapping S doesn't depend on $q \in Q$. In fact, it is easy to see that S defines a system isomorphism. In our model system morphisms do not depend on $q \in Q$. This choice is implied by our perception of discrete modes as inputs. Since in our model the discrete modes are regarded as inputs, dependence of system morphisms on discrete modes would be equivalent to the dependence of system morphisms on input. If the mapping S was allowed to depend on q, the mapping S would not only cease to be an isomorphism of system, but it would also be possible to have algebraically similar systems with different input-output behavior. Indeed, consider the following example.

Example Consider the following two linear switched systems Σ_1 and Σ_2, with two discrete modes q_1 and q_2 each. The continuous state space is $\mathbb{R}^2$, the continuous input space is $\mathbb{R}$, the output space is $\mathbb{R}$. The system $\Sigma_1 = (\mathbb{R}^2, \mathbb{R}, \mathbb{R}, \{q_1, q_2\}, \{(A^1_q, B^1_q, C^1_q) \mid q \in \{q_1, q_2\}\})$ is of the form.

$$A^1_{q_1} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B^1_{q_1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C^1_{q_1} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$A^1_{q_2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B^1_{q_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C^1_{q_2} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

The switched system $\Sigma_2 = (\mathbb{R}^2, \mathbb{R}, \mathbb{R}, \{q_1, q_2\}, \{(A^2_q, B^2_q, C^2_q) \mid q \in \{q_1, q_2\}\})$ is of the form

$$A^2_{q_1} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B^2_{q_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad C^2_{q_1} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

$$A^2_{q_2} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B^2_{q_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C^2_{q_2} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Now, consider the following mappings

$$f_{q_1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad , f_{q_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Now, $A^2_q = f_q A^1_q f_q^{-1}$, $B^2_q = f_q B^1_q$ and $C^2_q = C^1_q f_q^{-1}$ for $q = q_1, q_2$. So, if in the definition of algebraic similarity we allowed the linear transformations depend on q, then Σ_1 and Σ_2 would be algebraically similar. But Σ_1 and Σ_2 are not input-output equivalent. To see this, compute $y_{\Sigma_1}(u(.),(q_1,t_1)(q_2,t_2))(t_1+t_2)$ and $y_{\Sigma_2}(u(.),(q_1,t_1)(q_2,t_2))(t_1+t_2)$ for

$$u(t) = \begin{cases} v \in \mathbb{R} & t \in [0,t_1) \\ 0 & t \in [t_1,t_2] \end{cases}$$

Then one can see that

$$y_{\Sigma_1}(u(.),(q_1,t_1)(q_2,t_2))(t_1+t_2) = 0.5t_1^2 v + t_2 t_1 v + t_1 v$$

and

$$y_{\Sigma_2}(u(.),(q_1,t_1)(q_2,t_2))(t_1+t_2) = t_1 v + 0.5t_1^2 t_2 v + 0.5t_1^2 v$$

So, we get that

$$y_{\Sigma_1}(u(.),(q_1,t_1)(q_2,t_2))(t_1+t_2) \neq y_{\Sigma_2}(u(.),(q_2,t_2)(q_1,t_1))(t_1+t_2)$$

which contradicts to the assumption that Σ_1 and Σ_2 are input-output equivalent.

The example above also demonstrates that for linear switched systems the Markov parameters of the continuous components don't determine the input-output behavior of the whole switched system. In the above example the Markov parameters of the continuous components of both systems are $1,1,0,0,\ldots$. That is, the Markov parameters are the same, but the input-output behaviors of the two systems are different. In Section 5 a generalization of Markov-parameters will be presented, so that the input-output behavior of linear switched system is uniquely determined by those parameters.

The following result on reachability and observability of linear switched systems is an easy reformulation of the results in [8] [1] .

Proposition 3.2. *Consider a linear switched system*

$$\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{(A_q,B_q,C_q) \mid q \in Q\})$$

(1) For each $w=(q_1,t_1)(q_2,t_2)\cdots(q_k,t_k)$, $u \in PC(T,\mathcal{U})$ the following holds

$$\begin{aligned} x_\Sigma(x_0,u,w) = {} & \exp(A_{q_k}t_k)\exp(A_{q_{k-1}}t_{k-1})\cdots\exp(A_{q_1}t_1)x_0 + \\ & + \int_0^{t_k} \exp(A_{q_k}(t_k-s))B_{q_k}u(s+\sum_1^{k-1} t_i)ds + \\ & + \exp(A_{q_k}t_k)\int_0^{t_{k-1}} \exp(A_{q_{k-1}}(t_{k-1}-s))B_{q_{k-1}}u(s+\sum_1^{k-2} t_i)ds \\ & \cdots \\ & + \exp(A_{q_k}t_k)\exp(A_{q_{k-1}}t_{k-1})\cdots \\ & \cdots \exp(A_{q_2}t_2)\int_0^{t_1} \exp(A_{q_1}(t_1-s))B_{q_1}u(s)ds \end{aligned}$$

[1]The results on reachability and observability from [8] can be proven in a rather different way than the one used in [8] As an alternative the author used geometric theory of nonlinear systems. These results however won't be discussed here.

$$
\begin{aligned}
y_\Sigma(x_0,u,w) = & C_{q_k}\exp(A_{q_k}t_k)\exp(A_{q_{k-1}}t_{k-1})\cdots\exp(A_{q_1}t_1)x_0+ \\
& + \int_0^{t_k} C_{q_k}\exp(A_{q_k}(t_k-s))B_{q_k}u(s+\sum_1^{k-1}t_i)ds + \\
& + C_{q_k}\exp(A_{q_k}t_k)\int_0^{t_{k-1}}\exp(A_{q_{k-1}}(t_{k-1}-s))B_{q_{k-1}}u(s+\sum_1^{k-2}t_i)ds \\
& \dots \\
& + C_{q_k}\exp(A_{q_k}t_k)\exp(A_{q_{k-1}}t_{k-1})\cdots \\
& \cdots\exp(A_{q_2}t_2)\int_0^{t_1}\exp(A_{q_1}(t_1-s))B_{q_1}u(s)ds
\end{aligned}
$$

(2) The structure of the reachable set is the following

$$
\begin{aligned}
Reach(\Sigma) = \mathrm{Span}\{ & A_{q_1}^{j_1}A_{q_2}^{j_2}\cdots A_{q_k}^{j_k}B_z u \mid \\
& q_1,q_2,\dots,q_k,z\in Q, j_1,j_2,\dots,j_k\geq 0, u\in\mathcal{U}\}
\end{aligned}
$$

Moreover, there exists $w=(q_1,t_1)(q_2,t_2),\dots,(q_k,t_k)$ such that

$$
Reach(\Sigma) = \{x_\Sigma(0,w,u(.),t_k) \mid u(.)\in PC_{const}(T,\mathcal{U})\}
$$

(3) For each $x_1,x_2\in\mathcal{X}$ the states x_1 and x_2 are indistinguishable if and only if

$$
x_1-x_2 \in \bigcap_{q_1,q_2,\dots,q_k,z\in Q,j_1,j_2,\dots j_k\geq 0} \ker C_z A_{q_1}^{j_1}A_{q_2}^{j_2}\cdots A_{q_k}^{j_k}
$$

Remark Notice that if a linear switched system is reachable, the linear systems making up the switched systems need not be reachable . Moreover, the reachable set of the switched system may be bigger than the union of the reachable sets of the linear components. Indeed, consider the following switched system $\Sigma=(\mathbb{R}^3,\mathbb{R},\mathbb{R},\{q_1,q_2\},\{(A_q,B_q,C_q)\mid q=q_1,q_2\})$

$$
A_{q_1}=\begin{bmatrix}0&1&0\\0&0&0\\0&0&0\end{bmatrix},\quad B_{q_1}=\begin{bmatrix}0\\1\\0\end{bmatrix},\quad C_{q_1}=\begin{bmatrix}1&1&1\end{bmatrix}
$$

$$
A_{q_2}=\begin{bmatrix}0&0&0\\0&0&0\\0&1&0\end{bmatrix},\quad B_{q_2}=\begin{bmatrix}0\\0\\0\end{bmatrix},\quad C_{q_2}=\begin{bmatrix}1&1&1\end{bmatrix}
$$

Since $A_{q_1}B_{q_1}=[1,0,0]^T$, $A_{q_2}B_{q_1}=[0,0,1]^T$, we get that

$$
\mathbb{R}^3 = \mathrm{Span}\{B_{q_1},A_{q_1}B_{q_1},A_{q_2}B_{q_1}\}\subseteq Reach(\Sigma)
$$

So $Reach(\Sigma)=\mathbb{R}^3$, i.e. the system is reachable. Yet, neither (A_{q_1},B_{q_1}) nor (A_{q_2},B_{q_2}) are reachable, moreover $Reach(A_{q_1},B_{q_1})=\mathbb{R}^2$, $Reach(A_{q_2},B_{q_2})=0$, so $Reach(A_{q_1},B_{q_1})\oplus Reach(A_{q_2},B_{q_2})\neq Reach(\Sigma)$.

4 Minimization of linear switched systems

This section gives a procedure to construct a minimal linear switched system equivalent to a given linear switched system. Also a Kalman-like decomposition for linear switched systems will be presented. It will also be shown that two equivalent minimal linear switched systems are algebraically similar, and that a minimal linear switched system has a state space of smaller dimension than any other linear switched system realizing the same input-output map.

For a given linear switched system we will construct an equivalent canonical system. The steps of the construction are similar to the construction of the canonical initialized system equivalent to a given one. In its full generality the procedure is described in Section 6.8 of [7]. The challenge is to show that at each step of the general procedure we get a linear switched system. This will be done below.

Theorem 4.1. *Let $\widetilde{\Sigma}$ be an arbitrary linear switched system. Then there exists a canonical linear switched system $\widetilde{\Sigma}_{can}$ equivalent to $\widetilde{\Sigma}$.*

Proof. First, given a linear switched system $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) | q \in Q\})$, we take the restriction of Σ to its reachable set by defining the system

$$\Sigma_r = (Reach(\Sigma), \mathcal{U}, \mathcal{Y}, Q, \{(A^r_q, B^r_q, C^r_q) \mid q \in Q\})$$

where for each $q \in Q$ the map $A^r_q = A_q|_{Reach(\Sigma)} : Reach(\Sigma) \to Reach(\Sigma)$ is the restriction of A_q to $Reach(\Sigma)$, $B^r_q = B_q : \mathcal{U} \to Reach(\Sigma)$ and $C^r_q = C_q|_{Reach(\Sigma)} : Reach(\Sigma) \to \mathcal{Y}$ is the restriction of C_q to $Reach(\Sigma)$. It is easy to see that Σ^r is a well-defined linear switched system, it is reachable and it is equivalent to Σ. Indeed, by Proposition 3.2 for each $q \in Q$ it holds that $\mathrm{Im}(B_q) \subseteq Reach(\Sigma)$. So B^r_q is well defined for each $q \in Q$. Again from Proposition 3.2 it follows that to see that A^r_q is well defined it is enough to show that $A^r_q(A^{j_1}_{q_1} A^{j_2}_{q_2} \cdots A^{j_k}_{q_k} B_z u) \in Reach(\Sigma)$ for all $q_1, q_2, \dots q_k, z \in Q, u \in \mathcal{U}, j_1, j_2, \dots, j_k \geq 0$. But $A^r_q x = A_q x$ for all $x \in Reach(\Sigma)$, so we get

$$A^r_q(A^{j_1}_{q_1} A^{j_2}_{q_2} \cdots A^{j_k}_{q_k} B_z u) = A_q A^{j_1}_{q_1} A^{j_2}_{q_2} \cdots A^{j_k}_{q_k} B_z u \in Reach(\Sigma)$$

So, for each $q \in Q$ the map A^r_q is well defined. The map C^r_q is trivially well defined. Notice that the construction of Σ_r goes along the same lines as the construction of the reachable initialized system equivalent to a given one, as it is described in [7].

The next step is to construct an observable linear switched system from a reachable linear switched system in such a way that the new reachable and observable system is equivalent to the original one.

Let $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, , Q, \{(A_q, B_q, C_q) \mid q \in Q\})$ be a linear switched system. Define $O_\Sigma = \bigcap_{q_1, q_2, \dots, q_k, z \in Q, j_1, j_2, \dots, j_k \geq 0} \ker C_z A^{j_1}_{q_1} A^{j_2}_{q_2} \cdots A^{j_k}_{q_k}$. Let $W = O^\perp_\Sigma$ be the orthogonal complement of O_Σ. Assume that Σ is reachable. Consider the system $\Sigma^o = (W, \mathcal{U}, \mathcal{Y}, Q, \{(A^o_q, B^o_q, C^o_q) \mid q \in Q\})$ where $A^o_q = \tilde{A}_q|_W : W \to W$, and $\tilde{A}_q$ is defined by $z = \tilde{A}_q x \iff A_q x = z + z^{'}, z \in W, z^{'} \in O_\Sigma$.

$C_q^o = C_q|_W : W \to \mathcal{Y}$, and $B_q^o : \mathcal{U} \to W$ is given by the rule $B_q^o u = z \Leftrightarrow B_q u = z + z^{'}$ such that $z \in W, z^{'} \in O_{\Sigma}$. Then the system Σ^o is well-defined, it is reachable and observable (i.e. canonical) and equivalent to Σ. The construction of Σ^o is a slight modification of the construction of the canonical initialized system presented in Section 6.8 of [7]. Note that W is isomorphic to $\mathcal{X}/O_{\Sigma}$. In fact, a linear switched system can be defined on $\mathcal{X}/O_{\Sigma}$ in such a way, that it will be isomorphic to Σ^o. This linear switched system defined on $\mathcal{X}/O_{\Sigma}$ corresponds to the canonical initialized system described in Section 6.8 of [7].

Using the notation above define $\widetilde{\Sigma}_{can}$ to be $(\widetilde{\Sigma}_r)^o$. Then $\widetilde{\Sigma}_{can}$ is indeed canonical and equivalent to $\widetilde{\Sigma}$. □

Denote by Ω_{lin} the class of linear switched systems considered as a subclass of initialized systems. From Subsection 3.2 it follows that any canonical linear switched system is Ω_{lin}-minimal. We will show that any linear switched system Σ which is Ω_{lin}–minimal has state-space of the smallest dimension among all linear switched systems equivalent to it.

Lemma 4.1. *Consider two linear switched systems*

$$\Sigma_1 = (\mathcal{X}_1, \mathcal{U}, \mathcal{Y}, Q, \{(A_q^1, B_q^1, C_q^1) \mid q \in Q\})$$

$$\Sigma_2 = (\mathcal{X}_2, \mathcal{U}, \mathcal{Y}, Q, \{(A_q^2, B_q^2, C_q^2) \mid q \in Q\})$$

Assume that Σ_1 is reachable. Then for any system morphism $\phi : \Sigma_1 \to \Sigma_2$ the corresponding map $\phi : \mathcal{X}_1 \to \mathcal{X}_2$ is linear.

Proof. The fact that ϕ is a system morphism means that the following holds. $\forall u \in PC(T, \mathcal{U}), \forall w \in (Q \times T)^*, \forall t \in \mathrm{dom}(\widetilde{w}), \forall x \in X_1 : \phi(x_{\Sigma_1}(x, u(.), w)(t)) = x_{\Sigma_2}(\phi(x), u(.), w)(t)$, $\phi(0) = 0$ and $C_q^1 x = C_q^2 \phi(x)$. Now, we shall prove that ϕ is a linear map. Notice that by [8] there exists a $w = (q_1, t_1)(q_2, t_2) \cdots (q_k, t_k) \in (Q \times T)^+$ such that $R_w = \{x_{\Sigma_1}(0, u(.), w)(t_k) \mid u(.) \in PC(T, \mathcal{U})\} = Reach(\Sigma_1) = \mathcal{X}_1$. Then for each $x_1, x_2 \in \mathcal{X}_1$ we have that

$$\phi(\alpha x_1 + \beta x_2) = \phi(x_{\Sigma_1}(0, \alpha u_1(.) + \beta u_2(.), w)(t_k)) = x_{\Sigma_2}(0, \alpha u_1(.) + \beta u_2(.), w)(t_k) = \alpha x_{\Sigma_2}(0, u_1(.), w)(t_k) + \beta x_{\Sigma_2}(0, u_2(.), w)(t_k)$$

So, ϕ is indeed a linear map. □

An important consequence of this lemma is the following theorem

Theorem 4.2. *Let $\Sigma_{min} = (\mathcal{X}_{min}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q^{min}, B_q^{min}, C_q^{min}) \mid q \in Q\})$ be a linear switched system. Then Σ_{min} is a minimal linear switched system if and only if for any linear switched system $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$ such that Σ is equivalent to Σ_{min} the following holds*

$$\dim \mathcal{X}_{min} \leq \dim \mathcal{X} \tag{3}$$

Proof. `"only if" part`
Consider the linear switched system Σ_r, i.e. the restriction of Σ to $Reach(\Sigma)$. Clearly $\dim Reach(\Sigma) \leq \dim \mathcal{X}$. The system Σ_r is reachable and equivalent to Σ, hence it is equivalent to Σ_{min}. By definition of Ω_{lin}–minimality there exists a surjective system morphism $\phi : \Sigma_r \to \Sigma_{min}$. By Lemma 4.1 the map $\phi : Reach(\Sigma) \to \mathcal{X}_{min}$ is linear, and by the surjectivity of the system morphism it is surjective. That is,

$$\dim \mathcal{X}_{min} = \dim \mathrm{Im}(\phi) \leq \dim Reach(\Sigma) \leq \dim \mathcal{X}$$

`"if" part`
Assume Σ_{min} has the property (3). Then Σ_{min} must be reachable. Assume the opposite. The restriction of Σ_{min} to its reachable set would give a system equivalent to Σ_{min} with state space $Reach(\Sigma_{min})$. But $\dim Reach(\Sigma_{min}) < \dim \mathcal{X}_{min}$, which contradicts to (3). Let $\Sigma_{can} = (\mathcal{X}_{can}, \mathcal{U}, \mathcal{Y}, Q, \{(A^{can}_q, B^{can}_q, C^{can}_q) \mid q \in Q\})$ be a canonical linear switched system equivalent to Σ_{min}. Such a system always exists by Theorem 4.1. The system Σ_{can} is minimal, so there exists a surjective system morphism $\phi : \Sigma_{min} \to \Sigma_{can}$. Then ϕ is a surjective linear map, so we get that $\dim \mathcal{X}_{can} \leq \dim \mathcal{X}_{min}$. But by (3) we have that $\dim \mathcal{X}_{can} \geq \dim \mathcal{X}_{min}$. It implies that $\dim \mathcal{X}_{can} = \dim \mathcal{X}_{min}$, that is, ϕ is an isomorphism. Since Σ_{can} is minimal and Σ_{min} is isomorphic to it, we get that Σ_{min} is minimal too. □

For reachable linear switched systems, isomorphism of systems is equivalent to algebraic similarity.

Theorem 4.3. *Two reachable linear switched systems*

$$\Sigma_1 = (\mathcal{X}_1, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$$

$$\Sigma_2 = (\mathcal{X}_2, \mathcal{U}, \mathcal{Y}, Q, \{(A^{'}_q, B^{'}_q, C^{'}_q) \mid q \in Q\})$$

are isomorphic if and only if they are algebraically similar

Proof. It is clear that if Σ_1 and Σ_2 are algebraically similar then Σ_1 and Σ_2 are isomorphic. Assume that $\phi : \Sigma_1 \to \Sigma_2$ is an isomorphism of systems. From Lemma 4.1 it follows that $\phi : \mathcal{X}_1 \to \mathcal{X}_2$ is a linear map. Since ϕ is isomorphism, we have that the linear map $\phi : \mathcal{X}_1 \to \mathcal{X}_2$ is bijective. We get that ϕ^{-1} is a linear bijective map too.

What we need to show is that for each $q \in Q$ the following holds.

$$A^{'}_q = \phi A_q \phi^{-1}, \quad B^{'}_q = \phi B_q \quad , C^{'}_q = C_q \phi^{-1}$$

It follows immediately from the fact that ϕ is a bijective system morphism that $C^{'}_q \phi = C_q$, which implies $C^{'}_q = C_q \phi^{-1}$.

We show that $A^{'}_q = \phi A_q \phi^{-1}$ for all $q \in Q$. For each $q \in Q$, $x_{\Sigma_1}(x, 0, (q,t))(t) = \exp(A_q t)x$ and $x_{\Sigma_2}(\phi(x), 0, (q,t))(t) = \exp(A^{'}_q t)\phi(x)$. So we get that $\phi(\exp(A_q t)x) = \exp(A^{'}_q t)\phi(x)$ for all $t > 0$. Taking the derivative of

t at 0 we get that for all $x \in X_1$ it holds that $\phi(A_q x) = A^{'}_q \phi(x)$, which implies $A^{'}_q = \phi A_q \phi^{-1}$ for all $q \in Q$.

It is left to show that $B^{'}_q = \phi B_q$. Denote the constant function taking the value $u \in \mathcal{U}$ by $const_u$. Then $\phi(x_{\Sigma_1}(0, const_u, (q,t)))(t) = \phi(\int_0^t \exp(A_q(t-s))B_q u\, ds) = x_{\Sigma_2}(0, const_u, (q,t))(t) = \int_0^t \exp(A^{'}_q(t-s))B^{'}_q u\, ds$ for all $t > 0, u \in \mathcal{U}$. Again, after taking derivatives by t at $t = 0$ we get $\phi B_q u = B^{'}_q u$. That is, we get $B^{'}_q = \phi B_q$. So, Σ_1 and Σ_2 are indeed algebraically similar. □

Since all equivalent minimal linear switched systems are isomorphic, one gets the following result.

Corollary 4.1. *All minimal equivalent linear switched systems are algebraically similar.*

The following theorem sums up the results of the discussion above.

Theorem 4.4 (Existence and uniqueness of minimal realization). *For linear switched systems the following statements hold.*

1. *Given a linear switched system $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$ there exists a system $\Sigma_{min} = (\mathcal{Z}, \mathcal{U}, \mathcal{Y}, \{(A^{min}_q, B^{min}_q, C^{min}_q) \mid q \in Q\})$ such that Σ^{min} is minimal and equivalent to Σ. Such a minimal system is unique up to algebraic similarity.*

2. *A linear switched system is minimal if and only if it is canonical.*

3. *A linear switched system Σ_{min} is minimal if and only if for each equivalent linear switched system Σ the dimension of the state-space of Σ is not smaller than the dimension of the state-space of Σ_{min}*

Proof. The statement of part 1 follows from Theorem 4.1, the fact that each canonical linear switched system is minimal (see Subsection 3.2) and Corollary 4.1.

Let Σ be a minimal linear switched system. By Theorem 4.1 there exists a canonical system Σ_{can} equivalent to Σ. But by Section 3.2 Σ_{can} is minimal, therefore Σ_{can} and Σ are isomorphic. Since any isomorphism preserves reachability and observability we get that Σ_{min} is reachable and observable, hence canonical. So the statement of part 2 is proven.

The statement of part 3 follows directly from Theorem 4.2. □

The construction of the minimal representation described above yields the following Kalman-decomposition of a linear switched system.

Theorem 4.5. *Given a linear switched system $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$ there exists a basis transformation on $\mathcal{X}$ compatible with decomposition $X = W_{or} \oplus W_{rno} \oplus W_{onr} \oplus W_{nonr}$ where $W_{or} \oplus W_{rno} = Reach(\Sigma)$, $W_{onr} \oplus$*

$W_{nonr} = O_\Sigma$ *such that in the new basis the matrix representation of maps* A_q, B_q, C_q *has the following form*

$$A_q = \begin{bmatrix} A_q^1 & 0 & A_q^2 & 0 \\ A_q^3 & A_q^4 & A_q^5 & A_q^6 \\ 0 & 0 & A_q^7 & 0 \\ 0 & 0 & A_q^8 & A_q^9 \end{bmatrix}, \quad B_q = \begin{bmatrix} B_q^1 \\ B_q^2 \\ 0 \\ 0 \end{bmatrix}, \quad C_q = \begin{bmatrix} C_q^1 & 0 & C_q^2 & 0 \end{bmatrix}$$

where

- $\Sigma_{or} = (W_{or}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q^1, B_q^1, C_q^1) \mid q \in Q\})$ *is minimal and equivalent to* Σ.
- $\Sigma_{rno} = (Reach(\Sigma), \mathcal{U}, \mathcal{Y}, Q, \{(\begin{bmatrix} A_q^1 & 0 \\ A_q^3 & A_q^4 \end{bmatrix}, \begin{bmatrix} B_q^1 \\ B_q^2 \end{bmatrix}, \begin{bmatrix} C_q^1 & 0 \end{bmatrix}) \mid q \in Q\})$ *is a reachable system equivalent to* Σ.
- $\Sigma_{rno} = (O_\Sigma^\perp, \mathcal{U}, \mathcal{Y}, Q, \{(\begin{bmatrix} A_q^1 & A_q^2 \\ 0 & A_q^7 \end{bmatrix}, \begin{bmatrix} B_q^1 \\ 0 \end{bmatrix}, \begin{bmatrix} C_q^1 & C_q^2 \end{bmatrix}) \mid q \in Q\})$ *is an observable system equivalent to* Σ.

5 Constructing a minimal representation for input-output maps

Below necessary and sufficient conditions for the existence of realization by a linear switched system will be presented. Also a procedure will be described to construct a minimal representation for a realizable input-output map. The well-known condition for existence of realization by a linear system is a special case of the condition given here. The construction of a minimal linear representation of an input-output map is also a particular case of the procedure presented below. By Proposition 3.1 it is enough to determine conditions for realizability of input-output maps of the form $y : (\mathcal{U} \times Q \times T)^+ \to \mathcal{Y}$.

Below conditions on $y : (\mathcal{U} \times Q \times T)^+ \to \mathcal{Y}$ will be given, which will be proven necessary and sufficient for realizability of y in the sense of Definition 3.2. Before proceeding further some notation has to be introduced. Let $u_1 = u_{11}u_{12}\cdots u_{1k}, u_2 = u_{21}u_{22}\cdots u_{2k} \in \mathcal{U}^+$, then $\alpha u_1 + \beta u_2 = (\alpha u_{11} + \beta u_{21})(\alpha u_{12} + \beta u_{22})\cdots(\alpha u_{1k} + \beta u_{2k}) \in \mathcal{U}^+$ for $\alpha, \beta \in \mathbb{R}$. Let $u = u_1 u_2 \cdots u_k \in \mathcal{U}^+, w = w_1 w_2 \cdots w_k \in Q^+, \tau = \tau_1 \tau_2 \cdots t_k \in T^+$, then $y(u, w, \tau)$ is defined as

$$y(u, w, \tau) = y((u_1, w_1, \tau_1)(u_2, w_2, \tau_2)\cdots(u_k, w_k, \tau_k))$$

Let $\phi : \mathbb{R}^{k+r} \to \mathbb{R}^p$. Whenever we want to refer to the arguments of ϕ explicitly we will use the notation $\phi(t_1, t_2, \ldots, t_k, s_1, s_2, \ldots, s_r)$, or in vector notation $\phi(\underline{t}, \underline{s})$, where $\underline{t} = (t_1, t_2, \ldots, t_k)$ and $\underline{s} = (s_1, s_2, \ldots, s_r)$ are formal k and r-tuples respectively. If $\underline{a} \in \mathbb{R}^k$ then we use the notation $\phi(\underline{t}, \underline{s})|_{\underline{t} = \underline{a}}$ for the function $\mathbb{R}^r \ni \underline{b} \mapsto \phi(\underline{a}, \underline{b})$. For any $\alpha = (\alpha_k, \alpha_{k-1}, \cdots, \alpha_1) \in \mathbb{N}^k$ denote by $\frac{d^\alpha}{dt^\alpha}\phi$ the partial derivative

$$\frac{d^\alpha}{dt^\alpha}\phi = \frac{d}{dt_k^{\alpha_k} dt_{k-1}^{\alpha_{k-1}} \cdots dt_1^{\alpha_1}} \phi(t_k, t_{k-1}, \dots, t_1, s_r, s_{r-1}, \dots, s_1) : \mathbb{R}^{k+r} \to \mathbb{R}^p$$

If we want to refer to the components of $\alpha \in \mathbb{N}^k$ explicitly, we will use the notation $\frac{d^{(\alpha_k, \alpha_{k-1}, \cdots, \alpha_1)}}{dt^{(\alpha_k, \alpha_{k-1}, \cdots, \alpha_1)}}\phi = \frac{d^\alpha}{dt^\alpha}\phi$. If $\underline{t} = (t_1, t_2, \dots, t_k)$ then denote by $\underline{t}^l$ the tuple $(t_l, t_{l+1}, \dots, t_k)$ and by ${}^l\underline{t}$ the tuple $(t_1, t_2, \dots, t_l)$ for $l < k$.

For any $u \in \mathcal{U}^+, w \in Q^+$ the function $y(u, w, \tau) : T^+ \to \mathcal{Y}$ will be identified with the function $T^{|w|} \ni (t_1, t_2, \dots, t_k) \mapsto y(u, w, t_1 t_2 \cdots t_k)$

Consider the matrices $A_{q_1}, A_{q_2}, \cdots A_{q_k} \in \mathbb{R}^{n \times n}$ and define the function $\exp_{q_1 q_2 \cdots q_k} : T^k \to \mathbb{R}^{n \times n}$ by

$$\exp_{q_k q_{k-1} \cdots q_1}(t_1, t_2, \dots, t_k) = \exp(A_{q_k} t_k) \exp(A_{q_{k-1}} t_{k-1}) \cdots \exp(A_{q_1} t_1)$$

Definition 5.1 (Realizability conditions). *Consider a map $y : (\mathcal{U} \times Q \times T)^+ \to \mathcal{Y}$. The map y is said to satisfy the* realizability conditions *if the following properties hold*

1. *Linearity of the input-output function*
   For all $u_1, u_2 \in \mathcal{U}^+$, $w \in Q^+$, $\tau \in T^+$ such that $|u_1| = |u_2| = |w| = |\tau|$ and for all $\alpha, \beta \in \mathbb{R}$ it holds that
   $$y(\alpha u_1 + \beta u_2, w, \tau) = \alpha y(u_1, w, \tau) + \beta y(u_2, w, \tau)$$
2. *Zero-time behavior*
   $$y(u, w, \underbrace{00 \cdots 0}_{|w|-times}) = 0$$
3. *Analyticity in switching times*
   For all $w \in Q^+, u \in \mathcal{U}^+$ such that $|w| = |u|$ the function $y(u, w, .) : T^{|w|} \to \mathcal{Y}$ defined by $(t_1, t_2, \dots, t_{|w|}) \mapsto y(u, w, t_1 t_2 \cdots t_k)$ is analytic.
4. *Repetition of the same input*
   For all $w_1, w_2 \in Q^+, u_1, u_2 \in \mathcal{U}^+, \tau_1, \tau_2 \in T^*$ such that $|w_i| = |u_i| = |\tau_i|, (i = 1, 2)$ and for all $q \in Q, u \in \mathcal{U}, t_1, t_2 \in T$ it holds that
   $$y(u_1 u u u_2, w_1 q q w_2, \tau_1 t_1 t_2 \tau_2) = y(u_1 u u_2, w_1 q w_2, \tau_1 (t_1 + t_2) \tau_2)$$
   The condition is equivalent to stating that for each $z, l \in (\mathcal{U} \times Q \times T)^+$
   $$\widetilde{z} = \widetilde{l} \implies y(z) = y(l)$$
5. *Decomposition of concatenation of inputs*
   For each $w_1, w_2 \in Q^+, u_1, u_2 \in \mathcal{U}^+, \tau_1, \tau_2 \in T^+$ such that $|w_i| = |u_i| = |\tau_i|$, $(i = 1, 2)$ it holds that
   $$y(u_1 u_2, w_1 w_2, \tau_1 \tau_2) = y(u_2, w_2, \tau_2) + y(u_1 \underbrace{00 \cdots 0}_{|u_2|-times}, w_1 w_2, \tau_1 \tau_2)$$

6. ***Elimination of zero duration***
For all $w_1, w_2, v \in Q^+, \tau_1, \tau_2 \in T^+, u_1, u_2, u \in \mathcal{U}^+$ such that $|u_i| = |w_i| = |\tau_i|$ and $|v| = |u|$ it holds that

$$y(u_1 u u_2, w_1 v w_2, \tau_1 \underbrace{00\cdots 0}_{|u|-times} \tau_2) = y(u_1 u_2, w_1 w_2, \tau_1 \tau_2)$$

Proposition 5.1. *If a map $y : (\mathcal{U} \times Q \times T)^+ \to \mathcal{Y}$ is realizable by a linear switched system, then it satisfies the realizability conditions.*

Analyticity of the input-output maps allows to rephrase the property that a linear switched system realizes an input-output map in terms of the high-order derivatives of the input-output map.

Let A_q, B_q, C_q, $(q \in Q)$ be linear maps over suitable spaces and let $j_1, j_2, \ldots, j_k \geq 0$. If $l = \inf\{z \in \mathbb{N} | j_z > 0\} = -\infty$, i.e. $j_1 = j_2 = \cdots = j_k = 0$, then by $C_{q_k} A_{q_k}^{j_k} A_{q_{k-1}}^{j_{k-1}} \cdots A_{q_l}^{j_l - 1} B_{q_l}$ we mean simply the identically zero map.

Proposition 5.2. *Consider the linear switched system*

$$\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) | q \in Q\})$$

Then for each $w = q_1 q_2 \cdots q_k \in Q^+$, $u = u_1 u_2 \cdots u_k \in \mathcal{U}$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$ the following holds

$$\frac{d^\alpha}{dt^\alpha} \widetilde{y}_\Sigma(u, w, \underline{t})|_{\underline{t}=0} = C_{q_k} A_{q_k}^{\alpha_k} A_{q_{k-1}}^{\alpha_{k-1}} \cdots A_{q_l}^{\alpha_l - 1} B_{q_l} u_l$$

where $l = \min\{z | \alpha_z > 0\}$.

Proof. Define the function $\widetilde{x}_\Sigma : (\mathcal{U} \times Q \times T)^+ \to \mathcal{X}$ in the following way. For $w = w_1 w_2 \cdots w_k \in Q^+$, $\tau = t_1 t_2 \cdots t_k \in T^+$ and $u = u_1 u_2 \cdots u_k \in \mathcal{U}^+$ define $\widetilde{x}_\Sigma(u, w, \tau)$ by $\widetilde{x}_\Sigma(u, w, \tau) = x_\Sigma(0, \widetilde{v}, z)(\sum_1^k t_i)$ where $v = (u_1, t_1)(u_2, t_2) \cdots (u_k, t_k)$ and $z = (w_1, t_1)(w_2, t_2) \cdots (w_k, t_k)$. It is easy to see that $\widetilde{x}_\Sigma$ satisfies the realizability conditions. We shall use this, the fact that $\widetilde{y}_\Sigma$ satisfies the realizability properties and the following basic property of linear switched systems (see [8])

$$\begin{aligned}
&\widetilde{y}_\Sigma(u_1 u_2 \cdots u_l \underbrace{0\cdots 000}_{k-l-times}, q_1 q_2 \cdots q_k, t_1 t_2 \cdots t_k) = C_{q_k} \exp(A_{q_k} t_k) \times \\
&\quad \exp(A_{q_{k-1}} t_{k-1}) \cdots \exp(A_{q_{l+1}} t_{l+1}) \widetilde{x}_\Sigma(u_1 u_2 \cdots u_l, q_1 q_2 \cdots q_l, t_1 t_2 \cdots t_l) \\
&\quad = C_{q_k} \exp_{q_k q_{k-1} \cdots q_{l+1}}(t_k, t_{k-1}, \ldots, , t_{l+1}) \widetilde{x}_\Sigma(u_1 u_2 \cdots u_l, q_1 q_2 \cdots q_l, t_1 t_2 \cdots t_l)
\end{aligned}$$

From condition 5 of the realizability conditions one gets

$$\begin{aligned}
&\widetilde{y}_\Sigma(u_1 u_2 \cdots u_l u_{l+1} \cdots u_k, q_1 q_2 \cdots q_l q_{l+1} \cdots q_k, t_1 t_2 \cdots t_l t_{l+1} \cdots t_k) = \\
&\quad \widetilde{y}_\Sigma(u_{l+1} \cdots u_k, q_{l+1} \cdots q_k, t_{l+1} \cdots t_k) + \widetilde{y}_\Sigma(u_1 \cdots u_l 00 \cdots 0, w, t_1 t_2 \cdots t_k)
\end{aligned}$$

where $w = q_1 q_2 \cdots q_k$. Combining the two expressions above one gets

$$\begin{aligned}
&\frac{d^\alpha}{dt^\alpha}\widetilde{y}_\Sigma(u,w,\underline{t})|_{\underline{t}=0} = \frac{d^\alpha}{dt^\alpha}\widetilde{y}_\Sigma(u_1u_2\cdots u_l 00\cdots 0, w, \underline{t})|_{\underline{t}=0} \\
&= \frac{d^\alpha}{dt^\alpha}(C_{q_k}\exp_{q_kq_{k-1}\cdots q_{l+1}}(\underline{t}^{l+1})\widetilde{x}_\Sigma(u_1u_2\cdots u_l, q_lq_2\cdots q_1, {}^l\underline{t}))|_{\underline{t}=0} \\
&= \frac{d^\alpha}{dt^\alpha}C_{q_k}\exp_{q_kq_{k-1}\cdots q_{l+1}}(\underline{t}^{l+1}) \times \\
&\quad (\widetilde{x}_\Sigma(u_l,q_l,t_l) + \widetilde{x}_\Sigma(u_1u_2\cdots u_{l-1}0, q_1q_2\cdots q_{l-1}q_l, {}^l\underline{t}))|_{\underline{t}=0} \\
&= \frac{d^{(\alpha_k,\alpha_{k-1},\cdots,\alpha_l)}}{dt^{(\alpha_k,\alpha_{k-1},\cdots,\alpha_l)}}C_{q_k}\exp_{q_k,q_{k-1},\cdots q_{l+1}}(\underline{t}^{l+1}) \times \\
&\quad (\widetilde{x}_\Sigma(u_l,q_l,t_l) + \exp(A_{q_l}t_l)\widetilde{x}_\Sigma(u_1u_2\cdots u_{l-1}, q_1q_2\cdots q_{l-1}, {}^l\underline{t}))|_{\underline{t}=0}
\end{aligned}$$

where $l = \min\{z \mid \alpha_z > 0\}$. In the derivation above the condition 5 of the realizability conditions was applied to $\widetilde{x}_\Sigma$. Since $\widetilde{x}_\Sigma(u_1u_2\cdots u_{l-1}, q_1q_2\cdots q_{l-1}, 00\cdots 0) = 0$ we get that

$$\begin{aligned}
&\frac{d^\alpha}{dt^\alpha}\widetilde{y}_\Sigma(u,w,\underline{t})_{\underline{t}=0} = \frac{d^{(\alpha_k,\alpha_{k-1},\ldots,\alpha_l)}}{dt^{(\alpha_k,\alpha_{k-1},\ldots,\alpha_l)}}(C_{q_k}\exp_{q_k,q_{k-1},\ldots,q_{l+1}}(\underline{t}^{l+1})\widetilde{x}_\Sigma(u_l,q_l,t_l)|_{\underline{t}=0} \\
&= \frac{d^{(\alpha_k,\alpha_{k-1},\ldots,\alpha_l)}}{dt^{(\alpha_k,\alpha_{k-1},\ldots,\alpha_l)}}(C_{q_k}\exp_{q_k,q_{k-1},\ldots,q_{l+1}}(\underline{t}^{l+1}) \\
&\quad \int_0^{t_l}\exp(A_{q_l}(t_l-s))B_{q_l}u_l\,ds)|_{\underline{t}=0} \\
&= (\frac{d^{(\alpha_k,\alpha_{k-1},\ldots,\alpha_{l+1})}}{dt^{(\alpha_k,\alpha_{k-1},\ldots,\alpha_{l+1})}}C_{q_k}\exp_{q_k,q_{k-1},\ldots,q_{l+1}}(\underline{t}^{l+1}) \times \\
&\quad (\frac{d}{dt_l^{\alpha_l-1}}(\exp(A_{q_l}t_l)B_{q_l}u_l) + \int_0^{t_l}\frac{d}{dt_l^{\alpha_l}}\exp(A_{q_l}(t_l-s))B_{q_l}u_l\,ds)|_{\underline{t}=0} \\
&= \frac{d^{(\alpha_k,\alpha_{k-1},\ldots,\alpha_{l+1})}}{dt^{(\alpha_k,\alpha_{k-1},\ldots,\alpha_{l+1})}}(C_{q_k}\exp(A_{q_k}t_k) \times \\
&\quad \times\exp(A_{q_{k-1}}t_{k-1})\cdots\exp(A_{q_{l+1}}t_{l+1})A_{q_l}^{\alpha_l-1}B_{q_l}u_l)|_{\underline{t}=0} \\
&= C_{q_k}A_{q_k}^{\alpha_k}A_{q_{k-1}}^{\alpha_{k-1}}\cdots A_{q_l}^{\alpha_l-1}B_{q_l}u_l.
\end{aligned}$$

In the last equation the fact was used that $\frac{d}{dt^j}Z\exp(At)L|_{t=0} = ZA^jL$ holds for any A, L, Z matrices of compatible dimensions. □

Proposition 5.2 , and the fact that $\widetilde{y}_\Sigma(u,w,,t_1t_2\cdots t_l)$ is analytic in $(t_1,t_2,\cdots,t_l)$ implies the following corollary.

Corollary 5.1. *Let $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{(A_q,B_q,C_q) \mid q\in Q\})$ be a linear switched system. Consider a map $y : (\mathcal{U}\times Q\times T)^+ \to \mathcal{Y}$ and assume that for each $w\in Q^+$, $u\in\mathcal{U}^+$, $|u| = |w|$ the map $(t_1,t_2,\ldots,t_{|w|}) \mapsto y(u,w,t_1t_2\cdots t_{|w|})$ is analytic. Then Σ is a realization of y if and only if*

$$\begin{aligned}
&\forall u = u_1u_2\cdots u_k\in\mathcal{U}^+, \forall w = q_1q_2\cdots q_k\in Q^+, \forall\alpha\in\mathbb{N}^k \\
&\quad\frac{d^\alpha}{dt^\alpha}y(u,w,\underline{t})|_{\underline{t}=0} = C_{q_k}A_{q_k}^{\alpha_k}A_{q_{k-1}}^{\alpha_{k-1}}\cdots A_{q_l}^{\alpha_l-1}B_{q_l}u_l
\end{aligned} \tag{4}$$

where $l = \min\{z|\alpha_z > 0\}$

The corollary above says that the matrices of the form $C_{q_k} A_{q_k}^{\alpha_k} A_{q_{k-1}}^{\alpha_{k-1}} \cdots A_{q_1}^{\alpha_1} B_z$ $(q_1, q_2, \ldots, q_k, z \in Q,\ \alpha \in \mathbb{N}^k)$ determine the input-output behavior of linear switched systems. In fact, for the case of one discrete mode these matrices are the Markov-parameters of the system. The matrices (4) can be viewed as a generalization of the concept of Markov parameters.

Now we shall introduce a few concepts, which are needed to formulate the generalization of the Hankel-matrix for linear switched systems. Let $\mathcal{Y} = \mathbb{R}^p$, $T = \mathbb{R}_+$ and Q be an arbitrary finite set. Define the following set

$$Z = \{\phi : Q^+ \to Y^{T^+} \mid \forall w \in Q^+ : \mathrm{dom}(\phi(w)) = T^{|w|}$$
$$\text{and } \phi(w) : T^{|w|} \to Y \text{ is analytic } \}$$

Then Z is a vector space with respect to point-wise addition and multiplication by scalar, i.e. $\forall \phi_1, \phi_2 \in Z, \forall w \in Q^+, t \in T^{|w|}$:

$$(\alpha\phi_1 + \beta\phi_2)(w, t) := \alpha\phi_1(w, t) + \beta\phi_2(w, t)\ ,\ \alpha, \beta \in \mathbb{R}$$

Define the set D as follows

$$D = \{f : (Q \times \mathbb{N})^+ \to \mathcal{Y}\}$$

It is easy to see that D is a vector space with respect to point-wise addition and multiplication by real numbers, i.e.

$$\forall f_1, f_2 \in D, \forall w \in (Q \times \mathbb{N})^+ : (\alpha f_1 + \beta f_2)(w) := \alpha f_1(w) + \beta f_2(w)\ ,\ \alpha, \beta \in \mathbb{R}$$

Define the mapping $F : Z \to D$ in the following way

$$F(\phi)((q_1, \alpha_1)(q_2, \alpha_2) \cdots (q_k, \alpha_k)) = \frac{d^\alpha}{dt^\alpha} \phi(q_1 q_2 \cdots q_k)(\underline{t})|_{\underline{t}=0} \qquad (5)$$

That is, the function F stores the germs of functions from Z in sequences of the form $(Q \times \mathbb{N})^+ \to \mathcal{Y}$.

For each $f \in Z$ and for each sequence $w \in Q^+$ the value of $F(f)$ at $(w, \alpha_1 \alpha_2 \cdots \alpha_{|w|})$ equals the partial derivative $\frac{d^\alpha}{dt^\alpha}$ at $(0, 0, \ldots, 0) \in T^{|w|}$ of the analytic function $f(w) : T^{|w|} \to \mathcal{Y}$. Thus, the proof of the following theorem is straightforward.

Proposition 5.3. *The mapping $F : Z \to D$ defined above is an injective vector space homomorphism.*

Now we are ready to define the generalized Hankel-matrix. Consider a mapping $y : (\mathcal{U} \times Q \times T)^+ \to \mathcal{Y}$ and assume that it satisfies the realizability conditions. For each $(w, u) = (w_1, u_1)(w_2, u_2) \cdots (w_k, u_k) \in (Q \times \mathcal{U})^+$ and $\alpha \in \mathbb{N}^k$ define the mapping $\frac{d^\alpha}{dt^\alpha} y_{(w,u)} : Q^+ \to Y^{T^+}$ in the following way. For all $v \in Q^+$ let $\mathrm{dom}(\frac{d^\alpha}{dt^\alpha} y_{(w,u)}(v)) = T^{|v|}$. For each fixed $\tau \in T^{|v|}$

$$\frac{d^\alpha}{dt^\alpha} y_{(w,u)}(v)(\tau) = \frac{d^\alpha}{dt^\alpha} y(u \underbrace{00 \cdots 0}_{|v|-times}, wv, \underline{t}\tau)|_{\underline{t}=0}$$

Then by analyticity of $y(u00\cdots0, wv, .)$ the mapping $\frac{d^\alpha}{dt^\alpha}y_{(w,u)}$ belongs to Z. Consider the following subspace of Z

$$\mathcal{X}_y = \mathrm{Span}\{\frac{d^\alpha}{dt^\alpha}y_{(w,u)} \mid (w,u) \in (Q\times\mathcal{U})^+, \alpha \in \mathbb{N}^{|w|}\} \tag{6}$$

The Hankel-matrix of y can be defined in the following way

Definition 5.2 (Hankel-matrix). *Consider a mapping $y : (\mathcal{U}\times Q\times T)^+ \to \mathcal{Y}$ such that y satisfies the realizability condition. Using the notation above define the map $H_y = F|_{\mathcal{X}_y} : \mathcal{X}_y \to D$. The map H_y will be called the Hankel-map (or Hankel-matrix) of the mapping y.*

It is easy to see that H_y is a linear mapping, therefore it makes sense to speak about its rank, $\mathrm{rank}H_y := \dim \mathrm{Im}H_y \in \mathbb{N}\cup\{\infty\}$.

Lemma 5.1. *Consider the mapping $y : (\mathcal{U}\times Q\times T)^+ \to \mathcal{Y}$ and assume that y has a realization by a linear switched system. Then y satisfies the realizability conditions and* $\mathrm{rank}H_y < +\infty$.

Proof. Assume that the linear switched system
$\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{(A_q,B_q,C_q)|q\in Q\})$ is a realization of y. Then by Corollary 5.1

$$\begin{aligned}
H_y(\frac{d^\alpha}{dt^\alpha}y_{(w,u)})((q_1,\beta_1)(q_2,\beta_2)\cdots(q_l,\beta_l)) = \\
= \frac{d^\beta}{d\tau^\beta}\frac{d^\alpha}{dt^\alpha}y(u00\cdots0, wq_1q_2\cdots q_l, \underline{t}\underline{\tau})|_{\underline{t}=0,\underline{\tau}=0} \\
= C_{q_l}A_{q_l}^{\beta_l}A_{q_{l-1}}^{\beta_{l-1}}\cdots A_{q_1}^{\beta_1}A_{w_k}^{\alpha_k}\cdots A_{w_b}^{\alpha_b-1}B_{w_b}u_b
\end{aligned}$$

where $b = \min\{z|\alpha_z > 0\}$.

Let $r = \dim \mathit{Reach}(\Sigma) < +\infty$. Choose a basis $e_1, e_2, \dots, e_r$ of $\mathit{Reach}(\Sigma)$. Assume that $e_i = A_{q(i)_{k(i)}}^{\alpha(i,k(i))}A_{q(i)_{k(i)-1}}^{\alpha(i,k(i)-1)}\cdots A_{q(i)_1}^{\alpha(i,1)-1}B_{q(i)_1}u(i)$. For each $i = 1,2,\dots,r$ define

$$f_i = \frac{d^{(\alpha(i,k(i)),\alpha(i,k(i)-1),\dots\alpha(i,1))}}{dt^{(\alpha(i,k(i)),\alpha(i,k(i)-1),\dots\alpha(i,1))}}y_{(q(i)_1q(i)_2\cdots q(i)_{k(i)},u(i)\underbrace{00\cdots0}_{k(i)-1-times})}$$

Then we claim that $H_y(f_i)$ generates $\mathrm{Im}H_y$. Indeed, take an arbitrary $f = \frac{d^\alpha}{dt^\alpha}y_{(w,u)}$ Define $\widetilde{f} = A_{w_k}^{\alpha_k}A_{w_{k-1}}^{\alpha_{k-1}}\cdots A_{w_l}^{\alpha_l-1}B_{w_l}u_l$ where $l = \min\{z|\alpha_z > 0\}$. Then there exist scalars $\gamma_i \in \mathbb{R}$ such that $\widetilde{f} = \sum_{z=1}^r \gamma_i e_i$. But for each $x = (q_1,d_1)(q_2,d_2)\cdots(q_e,d_e) \in (Q\times\mathbb{N})^+$ it holds that $H_y(f)(x) = C_{q_e}A_{q_e}^{d_e}\cdots A_{q_1}^{d_1}\widetilde{f}$. Then $H_y(f_i)(x) = C_{q_e}A_{q_e}^{d_e}\cdots A_{q_1}^{d_1}e_i$, so we get that

$$(\sum_{j=1}^r \gamma_j H_y(f_j))(x) = \sum_{j=1}^r \gamma_j C_{q_e}A_{q_e}^{d_e}\cdots A_{q_1}^{d_1}e_j = C_{q_e}A_{q_e}^{d_e}\cdots A_{q_1}^{d_1}\widetilde{f} = H_y(f)(x)$$

so that we get that

$$H_y(f) = \sum_{j=1}^{r} \gamma_j H_y(f_j)$$

That is, the set $\{H_y(f_i) \mid i = 1, 2, \dots, r\}$ is a finite generator of $\mathrm{Im}H_y$. $\square$

Now we are ready to state the main theorem of the section.

Theorem 5.1. *Consider a map $y : (\mathcal{U} \times Q \times T)^+ \to \mathcal{Y}$. The map y is realizable by a linear switched system if and only if it satisfies the realizability conditions and its Hankel-map is of finite rank, i.e. $n = \mathrm{rank}H_y < +\infty$. If y is realizable, and $\mathrm{rank}H_y < +\infty$ then there exists a minimal linear switched system*

$$\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$$

which realizes it and $\dim \mathcal{X} = n = \mathrm{rank}H_y$. This minimal representation is unique up to algebraic similarity.

Proof. Lemma 5.1 and Proposition 5.1 imply the necessity of the condition. The last statement of the theorem follows from Corollary 4.1 In order to prove sufficiency, a minimal linear switched system will be constructed that realizes y. The proof will be divided into several steps.

(1) Consider $H = \mathrm{Im}H_y$. For each $q \in Q$ define the following linear maps $\mathcal{A}_q : H \to H$, $\mathcal{C}_q : H \to \mathcal{Y}$ and $\mathcal{B}_q : \mathcal{U} \to H$ as follows

$$\begin{aligned} &\forall (q_1, j_1)(q_2, j_2) \cdots (q_k, j_k) : \\ &\quad (\mathcal{A}_q \phi)((q_1, j_1)(q_2, j_2) \cdots (q_k, j_k)) := \phi((q, 1)(q_1, j_1)(q_2, j_2) \cdots (q_k, j_k)) \end{aligned}$$

$$\mathcal{B}_q u := H_y(\frac{d}{dt} y_{(q,u)}), \quad \mathcal{C}_q \phi := \phi((q, 0))$$

It is clear that $\mathcal{B}_q$ and $\mathcal{C}_q$ are well defined linear mappings. It is left to show that $\mathcal{A}_q$ is well defined. It is clear that $\mathcal{A}_q : H \to D$ is linear. We need to show that $\mathcal{A}_q(H) \subseteq H$. In fact, the following is true: for all $f = \frac{d^\alpha}{dt^\alpha} y_{(w,u)} \in \mathcal{X}_y$ it holds that

$$\mathcal{A}_q(H_y(f)) = H_y(\frac{d^{(1,\alpha)}}{dt^{(1,\alpha)}} y_{(wq,u0)}) \tag{7}$$

Indeed, denote by ϕ the right-hand side of (7). Then

$$\begin{aligned} &\phi((q_1, \beta_1)(q_2, \beta_2) \cdots (q_z, \beta_z)) = \\ &\quad = \frac{d^\beta}{d\tau^\beta} \frac{d^{(1,\alpha)}}{dt^{(1,\alpha)}} y(u0\underbrace{00\cdots0}_{z-times}, wqq_1q_2\cdots q_z, t_1t_2\cdots t_kt_{k+1}\tau_1\tau_2\cdots\tau_z)|_{\underline{t}=0,\underline{\tau}=0} \\ &\quad = \frac{d^{(\beta,1)}}{d\tau^{(\beta,1)}} \frac{d^\alpha}{dt^\alpha} y(u000\cdots0, wqq_1q_2\cdots q_z, t_1t_2\cdots t_k\tau_1\tau_2\tau_2\cdots\tau_{z+1})|_{\underline{t}=0,\underline{\tau}=0} \\ &\quad = H_y(f)((q, 1)(q_1, \beta_1)(q_2, \beta_2) \cdots (q_z, \beta_z)) \end{aligned}$$

(2) For each $q_1q_2\cdots q_k, z \in Q^+$, $\alpha \in \mathbb{N}^k$ and $u \in \mathcal{U}$ the following holds

$$\mathcal{A}_{q_k}^{\alpha_k}\cdots\mathcal{A}_{q_1}^{\alpha_1}\mathcal{B}_z u = H_y\Big(\frac{d^{(\alpha,1)}}{dt^{(\alpha,1)}}y_{(zq_1q_2\cdots q_k,u\underbrace{00\cdots 0}_{k-times})}\Big) \tag{8}$$

It is easy to see that $\frac{d^{(1,\alpha)}}{dt^{(1,\alpha)}}y_{(wqq,vu0)} = \frac{d^{(\alpha_m+1,\alpha_{m-1},\dots,\alpha_1)}}{dt^{(\alpha_m+1,\alpha_{m-1},\dots,\alpha_1)}}y_{(wq,vu)}$, $m=|wq|$. The correctness of (8) follows now from the repeated application of (7). We also get the following equalities.

$$\mathcal{A}_{q_k}^{\alpha_k}\mathcal{A}_{q_{k-1}}^{\alpha_{k-1}}\cdots\mathcal{A}_{q_1}^{\alpha_1-1}\mathcal{B}_{q_1}u_1 = H_y\Big(\frac{d^{\alpha}}{dt^{\alpha}}y_{(q_1q_2\cdots q_k,u_1\underbrace{00\cdots 0}_{k-1-times})}\Big) \tag{9}$$

$$\mathcal{C}_q\mathcal{A}_{q_k}^{\alpha_k}\mathcal{A}_{q_{k-1}}^{\alpha_{k-1}}\cdots\mathcal{A}_{q_1}^{\alpha_1-1}\mathcal{B}_{q_1}u_1 = \frac{d^{\alpha}}{dt^{\alpha}}y(q_1q_2\cdots q_kq,u_1\underbrace{00\cdots 0}_{k-times},\underline{t}s)|_{\underline{t}=0,s=0} \tag{10}$$

where $\alpha_1 > 0$.

(3) Using condition 5 of realizability conditions one gets for any $k \geq l \in \mathbb{N}$

$$\begin{aligned}
&\frac{d^{\alpha}}{dt^{\alpha}}y_{(q_1q_2\cdots q_k,u_1u_2\cdots u_k)}(v)(\tau) = \\
&\quad = \frac{d^{\alpha}}{dt^{\alpha}}y(q_1q_2\cdots q_kv,u_1u_2\cdots u_k\underbrace{0\cdots 00}_{|v|-times},\underline{t}\tau)|_{\underline{t}=0} \\
&\quad = \frac{d^{\alpha}}{dt^{\alpha}}(y(q_{l+1}q_{l+2}\cdots q_kv,u_{l+1}\cdots u_k\underbrace{0\cdots 00}_{|v|-times},\underline{t}^{l+1}\tau)+ \\
&\qquad +y(q_lq_{l+1}\cdots q_kv,u_l\underbrace{00\cdots 0}_{|v|+k-l-times},\underline{t}^{l}\tau)) \\
&\qquad +y(q_1q_2\cdots q_kv,u_1u_2\cdots u_{l-1}0\underbrace{0\cdots 00}_{|v|+k-l-times},\underline{t}\tau))|_{\underline{t}=0} \\
&\quad = \frac{d^{\alpha}}{dt^{\alpha}}y(q_{l+1}q_{l+2}\cdots q_kv,u_{l+1}\cdots u_k\underbrace{0\cdots 00}_{|v|-times},\underline{t}^{l+1}\tau) \\
&\qquad +\frac{d^{(\alpha_k,\alpha_{k-1},\dots,\alpha_l)}}{dt^{(\alpha_k,\alpha_{k-1},\dots,\alpha_l)}}y_{(q_lq_{l+1}\cdots q_k,u_l\underbrace{00\cdots 0}_{k-l-times})}(v)(\tau) \\
&\qquad +\frac{d^{\alpha}}{dt^{\alpha}}y_{(q_1q_2\cdots q_k,u_1u_2\cdots u_{l-1}0\underbrace{00\cdots 0}_{k-l-times})}(v)(\tau)
\end{aligned}$$

Assume that $l=\min\{z|\alpha_z>0\}$. Now, since the function

$$y(q_{l+1}q_{l+2}\cdots q_kv,u_{l+1}u_{l+2}\cdots u_k\underbrace{0\cdots 00}_{|v|-times},t_{l+1}t_{l+2}\cdots t_k\tau)$$

doesn't depend on t_l, we get that

$$\frac{d^{\alpha}}{dt^{\alpha}}(y(q_{l+1}q_{l+2}\cdots q_kv,u_{l+1}\cdots u_k\underbrace{0\cdots 00}_{|v|-times},\underline{t}\tau)|_{\underline{t}=0}=0$$

For the third term of the sum

$$\forall w = w_1 w_2 \cdots w_z \in Q^+, \tau = \tau_1\tau_2\cdots\tau_z \in T^z :$$

$$\frac{d^\alpha}{dt^\alpha} y_{(q_1q_2\cdots q_k, u_1u_2\cdots u_{l-1}0\underbrace{00\cdots0}_{k-l-times})}(w)(\tau)$$

$$= \frac{d^\alpha}{dt^\alpha} y(u_1u_2\cdots u_{l-1}0\underbrace{00\cdots0}_{k-l-times}\underbrace{00\cdots0}_{z-times}, q_1q_2\cdots q_k w_1w_2\cdots w_z, \underline{t}\tau)|_{\underline{t}=0}$$

$$= \frac{d^{(\alpha_k,\alpha_{k-1},\dots,\alpha_l)}}{dt^{(\alpha_k,\alpha_{k-1},\dots,\alpha_l)}} y(0\underbrace{00\cdots0}_{k-l-times}\underbrace{00\cdots0}_{z-times}, q_l\cdots q_k w_1w_2\cdots w_z, \underline{t}\tau)|_{\underline{t}=0} = 0$$

In the last two steps the condition 6 of the realizability conditions and the equality $y(00\cdots0, w, \tau) = 0$ were applied. So, we get that the following holds:

$$\frac{d^\alpha}{dt^\alpha} y_{(q_1q_2\cdots q_k, u_1u_2\cdots u_k)} = \frac{d^{(\alpha_k,\alpha_{k-1},\dots,\alpha_l)}}{dt^{(\alpha_k,\alpha_{k-1},\dots,\alpha_l)}} y_{(q_lq_{l+1}\cdots q_k, u_l\underbrace{00\cdots0}_{k-l-times})}$$

Taking into account equalities (9) and (10) one immediately gets

$$H_y(\frac{d^\alpha}{dt^\alpha} y_{(q_1q_2\cdots q_k, u_1u_2\cdots u_k)}) = \mathcal{A}^{\alpha_k}_{q_k}\mathcal{A}^{\alpha_{k-1}}_{q_{k-1}}\cdots\mathcal{A}^{\alpha_l-1}_{q_l}\mathcal{B}_{q_l}u_l \tag{11}$$

and

$$\frac{d^\alpha}{dt^\alpha} y(q_1q_2\cdots q_k, u_1u_2\cdots u_k, \underline{t})|_{\underline{t}=0} = \mathcal{C}_{q_k}\mathcal{A}^{\alpha_k}_{q_k}\mathcal{A}^{\alpha_{k-1}}_{q_{k-1}}\cdots\mathcal{A}^{\alpha_l-1}_{q_l}\mathcal{B}_{q_l}u_l \tag{12}$$

(4) Consider vector spaces

$$W = \mathrm{Span}\{\mathcal{A}^{\alpha_k}_{q_k}\mathcal{A}^{\alpha_{k-1}}_{q_{k-1}}\cdots\mathcal{A}^{\alpha_1}_{q_1}\mathcal{B}_z u \mid u \in \mathcal{U}, q_1, q_2, \dots q_k, z \in Q, \alpha \in \mathbb{N}^k\}$$

and

$$O = \bigcap_{q_1,q_2,\dots,q_k,z\in Q,\alpha\in\mathbb{N}^k} \ker \mathcal{C}_z\mathcal{A}^{\alpha_k}_{q_k}\mathcal{A}^{\alpha_{k-1}}_{q_{k-1}}\cdots\mathcal{A}^{\alpha_1}_{q_1}$$

From (8) and (11) it follows that $H = H_y(\mathcal{X}_y) = W$. We will show that $O = \{0\}$. Let $f = \frac{d^\alpha}{dt^\alpha} y_{(x,v)} \in \mathcal{X}_y$. Then

$$\begin{aligned}\mathcal{C}_{w_z}\mathcal{A}^{\beta_z}_{w_z}\mathcal{A}^{\beta_{z-1}}_{w_{z-1}}\cdots\mathcal{A}^{\beta_1}_{w_1}H_y f &= \mathcal{C}_{w_z}H_y(\frac{d^\beta}{d\tau^\beta}\frac{d^\alpha}{dt^\alpha} y_{(xw,v\underbrace{0\cdots0}_{z-times})}) \\ &= H_y(f)((w_1,\beta_1)(w_2,\beta_2)\cdots(w_z,\beta_z))\end{aligned}$$

For each $z \in O$ there exist $f_1, f_2, \dots f_r$ and $\alpha_i \in \mathbb{R}, i = 1, 2, \dots r$ such that $f_i = \frac{d^{(\alpha(i,k(i)),\alpha(i,k(i)-1),\dots,\alpha(i,1))}}{dt^{(\alpha(i,k(i)),\alpha(i,k(i)-1),\dots,\alpha(i,1))}} y_{(w_i,u_i)}$ and $z = \sum_{i=1}^r \gamma_i H_y(f_i)$. For each $(w, \beta) = (w_1, \beta_1)(w_2, \beta_2)\cdots(w_k, \beta_k) \in (Q \times \mathbb{N})^+$ it holds that

$\mathcal{C}_{w_k}\mathcal{A}_{w_k}^{\beta_k}\mathcal{A}_{w_{k-1}}^{\beta_{k-1}}\cdots\mathcal{A}_{z_1}^{\beta_1}z = 0$. But

$$\begin{aligned}\mathcal{C}_{w_k}\mathcal{A}_{w_k}^{\beta_k}\mathcal{A}_{w_{k-1}}^{\beta_{k-1}}\cdots\mathcal{A}_{z_1}^{\beta_1}\sum_{i=1}^{r}\gamma_i H_y f_i &= \sum_{i=1}^{r}\gamma_i\mathcal{C}_{w_k}\mathcal{A}_{w_k}^{\beta_k}\mathcal{A}_{w_{k-1}}^{\beta_{k-1}}\cdots\mathcal{A}_{z_1}^{\beta_1}H_y(f_i) \\ &= \sum_{i=1}^{r}\gamma_i H_y(f_i)((w,\beta)) = z(w,h)\end{aligned}$$

So for each $(w,\beta)\in(Q\times\mathbb{N})^+$ we get that $z((w,\beta))=0$, that is, $z=0$.

(5) Since $n=\dim H$ there is a $T: H\rightarrow\mathbb{R}^n$ vector space isomorphism. Define on $\mathbb{R}^n$ the following linear switched system $\Sigma=(\mathbb{R}^n,\mathcal{U},\mathcal{Y},Q,\{(A_q,B_q,C_q)|q\in Q\})$ where

$$A_q = T\mathcal{A}_qT^{-1},\quad B_q = T\mathcal{B}_q,\quad C_q=\mathcal{C}_qT^{-1}$$

Then for each $q_1,q_2,\dots q_k\in Q$, $u\in\mathcal{U}$, $\alpha\in\mathbb{N}^k$ we get that

$$C_{q_k}A_{q_k}^{\alpha_k}\cdots A_{q_1}^{\alpha_1-1}B_{q_1}u = \mathcal{C}_{q_k}\mathcal{A}_{q_k}^{\alpha_k}\cdots\mathcal{A}_{q_1}^{\alpha_1-1}\mathcal{B}_{q_1}u$$

This and (12) together with Corollary 5.1 imply that Σ is indeed a realization of y. Also, we get that $Reach(\Sigma)=TW=TH=\mathbb{R}^n$, so Σ is reachable. Again, $TO=O_\Sigma=\{0\}$, so Σ is observable. That is, Σ is a minimal linear switched system that realizes y and its state space is of dimension n. □

As a consequence of the theorem we get the following corollary

Corollary 5.2. *Let $\Sigma=(\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{(A_q,B_q,C_q)|q\in Q\})$ be a linear switched system. Let $y:=\widetilde{y}_\Sigma$. Then $\mathrm{rank}H_y\le\dim\mathcal{X}$. The system Σ is minimal if and only it holds that $\mathrm{rank}H_y=n=\dim\mathcal{X}$.*

6 Conclusions

Procedures for minimization of linear switched systems and construction of a minimal linear switched system representation of an input/output map were described in the paper. Future research is directed towards extension of the results for the case when not all switching sequences are admissible. Another task of future research is to make the connection between the nonlinear realization theory presented in [3, 4, 1] and the approach of the current paper more transparent.

Acknowledgment The author would like to express his gratitude to Jan H. van Schuppen, who suggested the topic of the paper and whose suggestions and remarks helped the further development of the paper. The author is also grateful to Pieter Collins for the fruitful discussions, which helped to conceive and develop the results of the paper. The author wants to thank the anonymous reviewers for the useful suggestions and comments.

References

[1] F. Celle and J.P. Gauthier. Realizations of nonlinear analytic input-output maps. *Math. Systems Theory*, 19:227 – 237, 1987.

[2] H. Ehrig. *Universal Theory of Automata*. B.G.Taubner, Stuttgart, 1974.

[3] Bronisław Jakubczyk. Existence and uniqueness of realizations of nonlinear systems. *SIAM J. Control and Optimization*, 18(4):455 – 471, 1980.

[4] Bronisław Jakubczyk. Construction of formal and analytic realizations of nonlinear systems. In *Feedback Control of Linear and Nonlinear Systems*, volume 39 of *Lecture Notes in Control and Information Sciences*, pages 147 – 156, 1982.

[5] Daniel Liberzon. *Switching in Systems and Control*. Birkhäuser, Boston, 2003.

[6] M.D. Mesarovic and Y. Takahara. In *Abstract Systems Theory*, volume 116 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, 1989.

[7] E.D. Sontag. *Mathematical Control Theory*. Spinger-Verlag, 1990.

[8] Zhendong Sun, S.S. Ge, and T.H. Lee. Controllability and reachability criteria for switched linear systems. *Automatica*, 38:115 – 786, 2002.