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Realization theory for linear switched systems: Formal power series approach

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ABSTRACT

The paper deals with the realization theory of linear switched systems. Necessary and sufficient conditions are formulated for a family of input-output maps to be realizable by linear switched systems. Characterization of minimal realizations is presented. The paper treats two types of linear switched systems. The first one is when all switching sequences are allowed. The second one is when only a subset of switching sequences is admissible, but within this restricted set the switching times are arbitrary. The paper uses the theory of formal power series to derive the results on realization theory.

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Realization Theory For Linear Switched Systems: Formal Power Series Approach

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Abstract. The paper deals with the realization theory of linear switched systems. Necessary and sufficient conditions are formulated for a family of input-output maps to be realizable by linear switched systems. Characterization of minimal realizations is presented. The paper treats two types of linear switched systems. The first one is when all switching sequences are allowed. The second one is when only a subset of switching sequences is admissible, but within this restricted set the switching times are arbitrary. The paper uses the theory of formal power series to derive the results on realization theory.

1 Introduction

Linear switched systems are one of the best studied subclasses of hybrid systems. A vast literature is available on various issues concerning linear switched systems, for a comprehensive survey see [1]. Yet, to the author's knowledge, the only work available on the realization theory of linear switched systems is [2].

This paper extends the results of [2]. More specifically, the paper tries to solve the following problems.

1. *Reduction to a minimal realization*
   Consider a linear switched system Σ, and a subset of its input-output maps Φ. Find a minimal linear switched system which realizes Φ.
2. *Existence of a realization with arbitrary switching*
   Find necessary and sufficient condition for the existence of a linear switched system realizing a given set of input-output maps.
3. *Existence of a realization with constrained switching*
   Assume that a set of admissible switching sequences is defined. Assume that the switching times of the admissible switching sequences are arbitrary. Consider a set of input-output maps Φ defined only for the admissible sequences. Find sufficient and necessary conditions for the existence of a linear switched system realizing Φ. Give a characterization of the minimal realizations of Φ.

The motivation of the Problem 3 is the following. Assume that the switching is controlled by a finite automaton and the discrete modes are the states of this automaton. Assume that the automaton is driven by discrete input signals which trigger discrete-state transitions. Then the traces of this automaton combined with the switching times (which are arbitrary) give us the admissible switching sequences. If we can solve Problem 3 for such admissible switching sequences that the set of admissible sequences of discrete modes is a regular language, then we can solve the following problem. Construct a realization of a set of input-output maps by a linear switched system, such that switchings of that system are controlled by an automaton which is given in advance. Notice that the set of traces of an automaton is always a regular language.

The following results are proved in the paper.

- A switched system is a minimal realization of a set of input-output maps defined for all the switching sequences if and only if it is observable and semi-reachable from the set of states which induce the input-output maps of the given set. Minimal linear switched systems which realize a given set of input-output maps are unique up to similarity. Each linear switched system Σ can be transformed to a minimal realization of any set of input-output maps which are realized by Σ.
- A set of input/output maps is realizable by a linear switched system if and only if it has a *generalized kernel representation* and the rank of its Hankel-matrix is finite. There is a procedure to construct the realization from the columns of the Hankel-matrix, and this procedure yields a minimal realization.
- Consider a set of input-output maps Φ defined on some subset of switching sequences. Assume that the switching sequences of this subset have arbitrary switching times and that their discrete mode parts form a regular language L. Then Φ has a realization by a linear switched system if and only if Φ has a *generalized kernel representation with constraint* L and its Hankel-matrix is of finite rank. Again, there exists a procedure to construct a realization from the columns of the Hankel-matrix. The procedure yields an observable and semi-reachable realization of Φ. But this realization is not a realization with the smallest state-space dimension possible.

The problem addressed in this paper is more general than the one dealt with in [2]. There realization of one single input-output map was considered. Moreover the input-output map was supposed to be realized from the zero initial state and the input-output map was assumed to be defined on all the switching sequences. If only one input-output map is considered, which is defined for all switching sequences and zero for constant zero continuous inputs, then the results of the paper imply those of [2]. If the set of discrete modes contains only one element, then the results of the paper imply the classical ones for linear systems.

The main tool used in the paper is the theory of formal power series. The connection between realization theory and formal power series has been explored in several paper, for a summary see [3].

The outline of the paper is the following. Section 2 introduces the notation and concepts which are used in the rest of the paper. Section 3 presents certain properties of the input-output maps generated by switched linear systems. Section 4 contains the necessary results on formal power series. The material of Section 4 is an extension of the classical theory of rational formal power series, see [4]. The construction of the minimal linear switched system realizing a given set of input-output maps defined on all switching sequences is presented in Section 5. Section 6 presents realization theory for sets of input-output maps defined on the set of admissible switching sequences.

2 Switched Systems

This section contains the definition and elementary properties of linear switched system. The notation and notions described in this section are largely based on [2].

For sets A, B, denote by $PC(A, B)$ the class of piecewise-continuous maps from A to B. For a set Σ denote by Σ^* the set of finite strings of elements of Σ. For $w = a_1 a_2 \cdots a_k \in \Sigma^*$, $a_1, a_2, \ldots, a_k \in \Sigma$ the length of w is denoted by $|w|$, i.e. $|w| = k$. The empty sequence is denoted by ϵ. The length of ϵ is zero: $|\epsilon| = 0$. Let $\Sigma^+ = \Sigma^* \setminus \{\epsilon\}$. The concatenation of two strings $v = v_1 \cdots v_k, w = w_1 \cdots w_m \in \Sigma^*$ is the string $vw = v_1 \cdots v_k w_1 \cdots w_m$. We denote by w^k the string $\underbrace{w \cdots w}_{k-times}$. The word w^0 is just the empty word ϵ. Denote by T the set $[0, +\infty) \subseteq \mathbb{R}$. Denote by $\mathbb{N}$ the set of natural number including 0. Denote by $F(A, B)$ the set of all functions from the set A to the set B. By abuse of notation we will denote any constant function $f : T \to A$ by its value. That is, if $f(t) = a \in A$ for all $t \in T$, then f will be denoted by a. For any function f the range of f will be denoted by $\mathrm{Im} f$. If A, B are two sets, then the set $(A \times B)^*$ will be identified with the set $\{(u, w) \in A^* \times B^* \mid |u| = |w|\}$. For any two sets J, X an *indexed subset* of X with the *index set* J is simply a map $Z : J \to X$, denoted by $Z = \{a_j \in X \mid j \in J\}$, where $a_j = Z(j), j \in J$.

Let $f : A \times (B \times C)^+ \to D$. Then for each $a \in A$, $w \in B^+$ we define the function $f(a, w, .) : C^{|w|} \to D$ by $f(a, w, .)(v) = f(a, (w, v)), v \in C^{|w|}$. By abuse of notation we denote $f(a, w, .)(v)$ by $f(a, w, v)$. Denote by $\mathbb{N}^k$ the set of k tuples of non-negative integers. If $\alpha = (\alpha_1, \ldots, \alpha_k) \in \mathbb{N}^k$ and $\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{N}^m$, then $(\alpha, \beta) = (\alpha_1, \ldots, \alpha_k, \beta_1, \ldots, \beta_m) \in \mathbb{N}^{k+m}$. Let $\phi : \mathbb{R}^k \to \mathbb{R}^p$, and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k) \in \mathbb{N}^k$. We define $D^\alpha \phi$ by

$$D^\alpha \phi = \frac{d^{\alpha_1}}{dt_1^{\alpha_1}} \frac{d^{\alpha_2}}{dt_2^{\alpha_2}} \cdots \frac{d^{\alpha_k}}{dt_k^{\alpha_k}} \phi(t_1, t_2, \ldots, t_k)|_{t_1 = t_2 = \cdots = t_k = 0}.$$

Definition 1 (Linear switched systems, [2]) *A linear switched system (abbreviated as* LSS *) is a tuple* $\Sigma = (\mathcal{X}, \mathcal{U}, \mathcal{Y}, Q, \{(A_q, B_q, C_q) \mid q \in Q\})$ *where* Q *is a finite set,* $\mathcal{X} = \mathbb{R}^n$*,* $\mathcal{U} = \mathbb{R}^m$*,* $\mathcal{Y} = \mathbb{R}^p$ *for some* $n, p, m > 0$ *and* $A_q : \mathcal{X} \to \mathcal{X}$*,* $B_q : \mathcal{U} \to \mathcal{X}$*,* $C_q : \mathcal{X} \to \mathcal{Y}$ *are linear maps.*

The inputs of the switched system Σ are functions from $PC(T, \mathcal{U})$ and sequences from $(Q \times T)^+$. The elements of the set $(Q \times T)^+$ are called *switching sequences*. That is, the switching sequences are part of the input, they are specified externally and we allow any switching sequence to occur. Let $u \in PC(T, \mathcal{U})$ and

$w = (q_1,t_1)(q_2,t_2)\cdots(q_k,t_k) \in (Q\times T)^+$. The inputs u and w steer the system Σ from state x_0 to the state $x_\Sigma(x_0,u,w)$ given by

$$x_\Sigma(x_0,u,w) = \exp(A_{q_k}t_k)\exp(A_{q_{k-1}}t_{k-1})\cdots\exp(A_{q_1}t_1)x_0 +$$
$$\int_0^{t_k}\exp(A_{q_k}(t_k-s))B_{q_k}u(\sum_1^{k-1}t_i+s)ds +$$
$$\exp(A_{q_k}t_k)\int_0^{t_{k-1}}\exp(A_{q_{k-1}}(t_{k-1}-s))B_{q_{k-1}}u(\sum_1^{k-2}t_i+s)ds +$$
$$\cdots$$
$$\exp(A_{q_k}t_k)\exp(A_{q_{k-1}}t_{k-1})\cdots\exp(A_{q_2}t_2)\int_0^{t_1}\exp(A_{q_1}(t_1-s))B_{q_1}u(s)ds$$

Let $x(x_0,u,\epsilon) = x_0$. The *reachable set* of the system Σ from a set of initial states $\mathcal{X}_0$ is defined by $Reach(\Sigma,\mathcal{X}_0) = \{x_\Sigma(x_0,u,w) \in \mathcal{X} \mid u \in PC(T,\mathcal{U}), w \in (Q\times T)^*, x_0 \in \mathcal{X}_0\}$. Σ is said to be *reachable* from $\mathcal{X}_0$ if $Reach(\Sigma,\mathcal{X}_0) = \mathcal{X}$ holds. Σ is *semi-reachable* from $\mathcal{X}_0$ if $\mathcal{X}$ is the vector space of the smallest dimension containing $Reach(\Sigma,\mathcal{X}_0)$. Define the function $y_\Sigma : \mathcal{X}\times PC(T,\mathcal{U})\times(Q\times T)^+ \to \mathcal{Y}$ by $y_\Sigma(x,u,w) = C_{q_k}x_\Sigma(x,u,w)$ for all $x\in\mathcal{X}, u\in PC(T,\mathcal{U})$, $w = (q_1,t_1)(q_2,t_2)\cdots(q_k,t_k) \in (Q\times T)^+$. For each $x\in\mathcal{X}$ define the *input-output map of the system Σ induced by x* as the function $y_\Sigma(x,.,.) : PC(T,\mathcal{U})\times(Q\times T)^+ \to \mathcal{Y}$, given by $y_\Sigma(x,.,.)(u,w) = y_\Sigma(x,u,w)$. By abuse of notation we will use $y_\Sigma(x,u,w)$ for $y_\Sigma(x,.,.)(u,w)$.

Two states $x_1 \neq x_2 \in \mathcal{X}$ of the switched system Σ are *indistinguishable* if

$$\forall w\in(Q\times T)^+, u\in PC(T,\mathcal{U}):\ y_\Sigma(x_1,u,w) = y_\Sigma(x_2,u,w)$$

Σ is called *observable* if it has no pair of indistinguishable states. A set $\Phi \subseteq F(PC(T,\mathcal{U})\times(Q\times T)^+,\mathcal{Y})$ is said to be *realized* by a switched system $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{(A_q,B_q,C_q)\mid q\in Q\})$ if there exists $\mu:\Phi\to\mathcal{X}$ such that

$$\forall f\in\Phi:\ y_\Sigma(\mu(f),.,.) = f$$

Both Σ and (Σ,μ) are called a *realization* of Φ. Thus, Σ realizes Φ if and only if for each $f\in\Phi$ there exists a state $x\in\mathcal{X}$ such that $y_\Sigma(x,u,w) = f(u,w)$ for all $u\in PC(T,\mathcal{U}), w\in(Q\times T)^+$.

Define the set $\mathcal{X}_\Phi := \{x\in\mathcal{X}\mid y_\Sigma(x,.,.)\in\Phi\}$. Denote by $\dim\Sigma := \dim\mathcal{X}$ the dimension of the state space of the switched system Σ. A switched system Σ is a *minimal realization of* Φ if Σ is a realization of Φ and for each switched system Σ_1 such that Σ_1 is a realization of Φ it holds that $\dim\Sigma \le \dim\Sigma_1$. For any $L\subseteq Q^+$ define the subset of *admissible switching sequences* $TL \subseteq (Q\times T)^+$ by

$$TL := \{(w,\tau)\in(Q\times T)^+\mid w\in L\}$$

That is, TL is the set of all those switching sequences, for which the sequence of discrete modes belongs to L and the sequence of times is arbitrary. Notice that if $L = Q^+$ then $TL = (Q\times T)^+$.

Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$. The system $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{(A_q, B_q, C_q) \mid q \in Q\})$ *realizes Φ with constraint L* if there exists $\mu : \Phi \to \mathcal{X}$ such that

$$\forall f \in \Phi : \; y_{\Sigma}(\mu(f),.,.)|_{PC(T,\mathcal{U})\times TL} = f$$

Both Σ and (Σ,μ) will be called a *realization* of Φ. Notice that if $L = Q^+$ then Σ realizes Φ with constraint L if and only if Σ realizes Φ. Consider two LSS's $\Sigma_i = (\mathcal{X}_i,\mathcal{U},\mathcal{Y},Q,\{(A^i_q, B^i_q, C^i_q) \mid q \in Q\})$ $(i = 1, 2)$. The systems Σ_1 and Σ_2 are *algebraically similar* if there exists a vector space isomorphism $S : \mathcal{X}_1 \to \mathcal{X}_2$ such that the following holds

$$A^2_q = SA^1_qS^{-1},\, B^2_q = SB^1_q,\, C^2_q = C^1_qS^{-1} \;\forall q \in Q$$

3 Input-output maps of linear switched systems

This section deals with properties of input-output maps of linear switched systems. We define the notion of generalized kernel representation of a set of input-output maps, which turns out to be a notion of vital importance for the realization theory of switched systems. In fact, the realization problem is equivalent to finding a generalized kernel representation of a particular form for the specified set input-output maps. The section also contains a number of quite technical statements, which are used in the other parts of the paper.

Let $L \subseteq Q^+$. Define the languages $\mathrm{suffix}L = \{u \in Q^* \mid \exists w \in Q^* : wu \in L\}$ and $\widetilde{L} = \{u_1^{i_1}u_2^{i_2}\cdots u_k^{i_k} \in Q^* \mid u_1u_2\cdots u_k \in \mathrm{suffix}L, u_j \in Q, i_j \geq 0, j = 1,2,\dots k, i_1, i_k > 0, k > 0\}$.

Definition 2 (Generalized kernel-reprsentation with constraint L) *A set $\Phi \subseteq F(PC(T,\mathcal{U})\times TL, \mathcal{Y})$ is said to have* generalized kernel representation with constraint L *if there exist functions*

$$K^{f,\Phi}_w : \mathbb{R}^k \to \mathbb{R}^{p\times 1} \quad \text{and } G^{\Phi}_w : \mathbb{R}^k \to \mathbb{R}^{p\times m},\; f \in \Phi, w \in \widetilde{L}, |w| = k$$

such that the following holds.

1. $\forall w \in \widetilde{L}, \forall f \in \Phi$: *$K^f_w$ is analytic and G^{Φ}_w is analytic*
2. *For each $f \in \Phi$ and $w, v \in Q^*$ such that $wqqv, wqv \in \widetilde{L}$, it holds that*

$$K^{f,\Phi}_{wqqv}(t_1,\dots,t_k,t,t',t_{k+1},\dots t_{k+l}) = K^{f,\Phi}_{wqv}(t_1,\dots t_k,t+t',t_{k+1}\dots t_{k+l})$$
$$G^{\Phi}_{wqqv}(t_1,\dots,t_k,t,t',t_{k+1},\dots t_{k+l}) = G^{\Phi}_{wqv}(t_1,\dots t_k,t+t',t_{k+1}\dots t_{k+l})$$

where $k := |w|$ and $l := |v|$.

3. $\forall vw \in \widetilde{L}, \forall f \in \Phi :$

$$K^{f,\Phi}_{vqw}(t_1,\dots,t_l,0,t_{l+1},\dots,t_{k+l}) = K^{f,\Phi}_{vw}(t_1,t_2,\dots,t_{k+l}) \quad \text{if } |w| > 0$$
$$G^{\Phi}_{vqw}(t_1,\dots,t_l,0,t_{l+1},\dots,t_{k+l}) = G^{\Phi}_{vw}(t_1,\dots,t_{l+k}) \quad \text{if } |v| > 0, |w| > 0$$

where $k = |w|$ and $l = |v|$.

4. *For each* $f \in \Phi$, $w = w_1 w_2 \cdots w_k \in L$, $w_1, \ldots, w_k \in Q$, $\underline{t} = (t_1, \ldots, t_k) \in T^k$*:*

$$f(u, w, \underline{t}) = K_w^{f,\Phi}(t_1, \ldots, t_k) + \int_0^{t_k} G_{w_k}^{\Phi}(t_k - s)\sigma_k u(s) ds +$$
$$\int_0^{t_{k-1}} G_{w_{k-1}w_k}^{\Phi}(t_{k-1} - s, t_k)\sigma_{k-1} u(s) ds + \cdots + \int_0^{t_1} G_w^{\Phi}(t_1 - s, t_2, \ldots, t_k) u(s) ds$$

where $\sigma_j u(s) = u(s + \sum_1^{j-1} t_i)$*.*

We say that Φ has a *generalized kernel representation* if it has a generalized kernel representation with the constraint $L = Q^+$. Using the notation above, define the function $y_0^{\Phi} : PC(T, \mathcal{U}) \times TL \to \mathcal{Y}$ by

$$y_0^{\Phi}(u, w, \underline{t}) := \int_0^{t_k} G_{w_k}^{\Phi}(t_k - s)\sigma_k u(s) ds + \int_0^{t_{k-1}} G_{w_{k-1}w_k}^{\Phi}(t_{k-1} - s, t_k) \times$$
$$\times \sigma_{k-1} u(s) ds + \cdots + \int_0^{t_1} G_{w_1 w_2 \cdots w_k}^{\Phi}(t_1 - s, t_2, \ldots, t_k) u(s) ds$$

It follows from the fact that Φ has a generalized kernel representation that y_0^{Φ} can be expressed by $\forall f \in \Phi : y_0^{\Phi}(u, w, \tau) = f(u, w, \tau) - f(0, w, \tau)$

Assume that $\widetilde{L} \ni w = z_1^{\alpha_1} \cdots z_k^{\alpha_k}$ such that $z_1, \ldots, z_k \in Q, \alpha \in \mathbb{N}^k$, $\alpha_k > 0$ and $z_1 \cdots z_k \in \widetilde{L}$. Then by using Part 2 and Part 3 of Definition 2 one gets

$$\begin{aligned} K_w^{f,\Phi}(t_1, \ldots, t_{|w|}) &= K_{z_l \cdots z_k}^{f,\Phi}(\textstyle\sum_{j=1}^{\alpha_l} t_j, \ldots, \sum_{j=1+\alpha_l+\cdots+\alpha_{k-1}}^{\alpha_l+\cdots+\alpha_k} t_j) \\ G_w^{\Phi}(t_1, \ldots, t_{|w|}) &= G_{z_l \cdots z_k}^{\Phi}(\textstyle\sum_{j=1}^{\alpha_l} t_j, \ldots, \sum_{j=1+\alpha_l+\cdots+\alpha_{k-1}}^{\alpha_l+\cdots+\alpha_k} t_j) \end{aligned} \tag{1}$$

where $f \in \Phi$, $l = \min\{z \mid \alpha_z > 0\}$ and $\sum_{j=a}^{b} t_j$ is taken to be 0 if $a > b$. Using the formula above, the chain rule, and induction it is straightforward to show that

$$D^{\beta} K_w^{f,\Phi} = D^{\gamma} K_{z_l \cdots z_k}^{f,\Phi} \text{ and } D^{\beta} G_w^{\Phi} = D^{\gamma} G_{z_l \cdots z_k}^{\Phi}, \, w = z_1^{\alpha_1} \cdots z_k^{\alpha_k} \tag{2}$$

where $\beta \in \mathbb{N}^{|w|}, l = \min\{z \mid \alpha_z > 0\}, \gamma \in \mathbb{N}^{k-l+1}$ and $\gamma_i = \sum_{j=1+\alpha_l+\cdots+\alpha_{l+i-2}}^{\alpha_l+\cdots+\alpha_{l+i-1}} \beta_j$ for each $i = 1, \ldots, k-l+1$. Formula (1) implies that the functions $\{K_w^{f,\Phi}, G_w^{\Phi} \mid f \in \Phi, w \in \text{suffix} L\}$ completely determine the functions $\{K_w^{f,\Phi}, G_w^{\Phi} \mid f \in \Phi, w \in \widetilde{L}\}$. Indeed, for any $w \in \widetilde{L}$ there exist $d_1, \ldots, d_r \in Q$ and $\xi \in \mathbb{N}^r$ such that $d_1 \cdots d_r \in \text{suffix} L$, $w = d_1^{\xi_1} \cdots d_r^{\xi_r}$ and $\xi_r > 0, \xi_1 > 0$. Applying (1) yields that $K_w^{\Phi,f}$ and G_w^{Φ} are uniquely determined by $K_{d_1 \cdots d_r}^{\Phi,f}$ and $G_{d_1 \cdots d_r}^{\Phi}$ respectively.

If Φ has a realization by a linear switched system, then Φ has a generalized kernel representation. In fact, (Σ, μ) is a realization of Φ with constraint L if and only if Φ has a generalized kernel representation defined by

$$G_{w_1 w_2 \cdots w_k}^{\Phi}(t_1, t_2, \ldots, t_k) = C_{w_k} \exp(A_{w_k} t_k) \exp(A_{w_{k-1}} t_{k-1}) \cdots \exp(A_{w_1} t_1) B_{w_1}$$
$$K_{w_1 w_2 \cdots w_k}^{f,\Phi}(t_1, t_2, \ldots, t_k) = C_{w_k} \exp(A_{w_k} t_k) \exp(A_{w_{k-1}} t_{k-1}) \cdots \exp(A_{w_1} t_1) \mu(f).$$

for each $w_1 w_2 \cdots w_k \in \widetilde{L}, w_1, \ldots, w_k \in Q$. Moreover, if (Σ, μ) is a realization of Φ, then $y_0^{\Phi} = y_{\Sigma}(0, ., .)|_{PC(T,\mathcal{U}) \times TL}$.

If the set Φ has a generalized kernel representation with constraint L, then the collection of analytic functions $\{K_w^{f,\Phi}, G_w^{\Phi} \mid w \in \text{suffix}L, f \in \Phi\}$ determines Φ. Since $K_w^{f,\Phi}$ is analytic, we get that $\{D^{\alpha} K_w^{f,\Phi}, D^{\alpha} G_w^{\Phi} \mid \alpha \in \mathbb{N}^{|w|}\}$ determines $K_w^{f,\Phi}$ locally. By applying the formula $\frac{d}{dt}\int_0^t f(t,\tau)d\tau = f(t,t) + \int_0^t \frac{d}{dt} f(t,\tau)d\tau$ and Part 4 of Definition 2 one gets

$$D^{\alpha} K_w^{f,\Phi} = D^{\alpha} f(0, w, .) \ \text{ and } D^{\alpha} G_{w_l w_{l+1} \cdots w_k}^{\Phi} e_z = D^{\beta} y_0^{\Phi}(e_z, w, .) \tag{3}$$

where $w = w_1 \cdots w_k,\ w_1, \ldots, w_k \in Q,\ l \leq k,\ \mathbb{N}^k \ni \beta = (\underbrace{0, 0, \ldots, 0}_{k-l-times}, \alpha_1 + 1, \alpha_2, \ldots, \alpha_l)$ and e_z is the zth unit vector of $\mathbb{R}^m$, i.e $e_z^T e_j = \delta_{zj}$. Formula (3) implies that all the high-order derivatives of the functions $K_w^{f,\Phi}, G_w^{\Phi}$ ($f \in \Phi$, $w \in \text{suffix}L$) at zero can be computed from high-order derivatives with respect to the switching times of the functions from Φ. With the notation above, using the principle of analytic continuation and formula (3), one gets the following

Proposition 1 *Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$. The pair (Σ, μ) is a realization of Φ with constraint L if and only if Φ has a generalized kernel representation with constraint L and for each $w \in L$, $f \in \Phi$, $j = 1, 2, \ldots, m$ and $\alpha \in \mathbb{N}^{|w|}$ the following holds*

$$\begin{aligned} D^{\alpha} y_0^{\Phi}(e_j, w, .) = D^{\beta} G_{w_l \cdots w_k}^{\Phi} e_j &= C_{w_1} A_{w_k}^{\alpha_k} A_{w_{k-1}}^{\alpha_{k-1}} \cdots A_{w_l}^{\alpha_l - 1} B_{w_l} e_j \\ D^{\alpha} f(0, w, .) = \quad D^{\alpha} K_w^{f,\Phi} &= C_{w_k} A_{w_k}^{\alpha_k} A_{w_{k-1}}^{\alpha_{k-1}} \cdots A_{w_l}^{\alpha_l} \mu(f) \end{aligned} \tag{4}$$

where $l = \min\{h \mid \alpha_h > 0\}$, e_z is the zth unit vector of $\mathcal{U}$, $\beta = (\alpha_l - 1, \ldots, \alpha_{|w|})$ and $w = w_l \cdots w_k$, $w_j \in Q$.

The following reformulation of Proposition 1 will be used in Section 6. Let $S = \{(\alpha, w) \mid \alpha \in \mathbb{N}^{|w|}, w \in Q^*\}$. For each $w \in Q^*$, $q_1, q_2 \in Q$ define $F_{q_1,q_2}(w) = \{(v, (\alpha, z)) \in Q^* \times S \mid vz \in L, q_2 w q_1 = z_1 z_1^{\alpha_1} \cdots z_k^{\alpha_k} z_k, z = z_1 \cdots z_k, z_1, \ldots, z_k \in Q\}$, $F_{q_1}(w) = \{(v, (\alpha, z)) \in Q^* \times S \mid vz \in L, w q_1 = z_1^{\alpha_1} \cdots z_k^{\alpha_k} z_k, \alpha_1 > 0, z = z_1 \cdots z_k, z_1, \ldots, z_k \in Q\}$. Define $\widetilde{L}_{q_1,q_2} = \{w \in Q^* \mid F_{q_1,q_2}(w) \neq \emptyset\}$, $\widetilde{L}_q = \{w \in Q^* \mid F_q(w) \neq \emptyset\}$. Let $\mathbb{O}_l = (0, 0, \ldots, 0) \in \mathbb{N}^l$. For any $\alpha \in \mathbb{N}^k$ let $\beta^+ = (\beta_1 + 1, \beta_2, \ldots, \beta_k) \in \mathbb{N}^k$. With the notation above, formula (4) holds for any $w \in L$, $j = 1, 2, \ldots, m$, $\alpha \in \mathbb{N}^{|w|}$ and $f \in \Phi$ if and only if

$$\begin{aligned} &\forall (v, (\beta, z)) \in F_{q_1,q_2}(w) : \\ &\quad D^{(\mathbb{O}_{|v|}, \beta^+)} y_0^{\Phi}(e_j, vz, .) = D^{(0,\beta,0)} G_{q_2 z q_1}^{\Phi} e_j = C_{q_1} A_{z_k}^{\beta_k} \cdots A_{z_1}^{\beta_1} B_{q_2} e_j \\ &\quad \forall (v, (\beta, z)) \in F_q(w) : \\ &\quad\quad D^{(\mathbb{O}_{|v|}, \beta)} f(0, vz, .) = \quad D^{(\beta,0)} K_{zq}^{f,\Phi} \quad = \ C_q A_{z_k}^{\beta_k} \cdots A_{z_1}^{\beta_1} \mu(f) \end{aligned} \tag{5}$$

holds for any $f \in \Phi$, $w \in \widetilde{L}$, $q, q_1, q_2 \in Q$ and $j = 1, 2, \ldots, m$.

4 Formal Power Series

The section presents results on formal power series. The material of this section is based on the classical theory of formal power series, see [4]. However, a number of concepts and results are extensions of the standard ones. In particular, the definition of the rationality is more general than the one occurring in the literature. Consequently, the theorems characterizing minimality are extensions of the well-known results.

Let X be a finite alphabet, let K be a field. Denote by $K^{p\times m}$ the set of p by m matrices with elements from K. We will identify the sets K^p and $K^{p\times 1}$. A *formal power series* S with coefficients in $K^{p\times m}$ is a map $S : X^* \to K^{p\times m}$. We denote by $K^{p\times m} \ll X^* \gg$ the set of all formal power series with coefficients in $K^{p\times m}$. Let $S \in K^{p\times m} \ll X^* \gg$. For each $i = 1,\dots,p$, $j = 1,\dots,m$ define the formal power series $S_{i,j} \in K \ll X^* \gg$ and $S_{.j} \in K^p \ll X^* \gg$ by the following equations $S_{i,j}(w) = (S(w))_{i,j}$, $S_{.j}(w) = \begin{bmatrix} S_{1,j}(w) \; S_{1,j}(w) \cdots S_{p,j}(w)\end{bmatrix}^T$ An indexed set of formal power series $\Psi = \{S_j \in K^{p\times 1} \ll X^* \gg \mid j \in J\}$ is called *rational* if there exists a vector space $\mathcal{X}$ over K, $\dim \mathcal{X} < +\infty$, linear maps $C : \mathcal{X} \to K^p$, $A_\sigma \in \mathcal{X} \to \mathcal{X}$, $\sigma \in X$ and an indexed set $B = \{B_j \in \mathcal{X} \mid j \in J\}$ of elements of $\mathcal{X}$ such that

$$S_j(\sigma_1\sigma_2\cdots\sigma_k) = CA_{\sigma_k}A_{\sigma_{k-1}}\cdots A_{\sigma_1}B_j.$$

The 4-tuple $R = (\mathcal{X}, \{A_x\}_{x\in X}, B, C)$ is called a *representation* of S. The number $\dim \mathcal{X}$ is called the *dimension* of the representation R and it is denoted by $\dim R$. In the sequel the following short-hand notation will be used $A_w := A_{w_k}A_{w_{k-1}}\cdots A_{w_1}$ for $w = w_1\cdots w_k$. A_ϵ is the identity map. A power series $S \in K^{p\times m} \ll X^* \gg$ is called *rational* if the set $\{S_{.j} \in K^{p\times 1} \ll X^* \gg \mid j = 1,2,\dots m\}$ is rational. A representation R_{min} of Ψ is called *minimal* if for each representation R of Ψ it holds that $\dim R_{min} \leq \dim R$.

Let $L \subseteq X^*$. If L is a regular language then the power series $\bar{L} \in K \ll X^* \gg$, $\bar{L}(w) = \begin{cases} 1 & \text{if } w \in L \\ 0 & \text{otherwise}\end{cases}$ is a rational power series. Consider two power series $S, T \in K^{p\times m} \ll X^* \gg$. Define the *Hadamard product* $S \odot T \in K^{p\times m} \ll X^* \gg$ by $(S \odot T)_{i,j}(w) = S_{i,j}(w)T_{i,j}(w)$. Let $w \in X^*$ and define $w \circ S \in K^{p\times m} \ll X^* \gg$ – the *left shift of S by w* by $\forall v \in X^* : w \circ S(v) = S(wv)$ The following statements are generalizations of the results on rational power series from [4]. Let $\Psi = \{S_j \in K^p \ll X^* \gg \mid j \in J\}$. Define $W_\Psi = \mathrm{Span}\{w \circ S_j \in K^{p\times 1} \ll X^* \gg \mid j \in J, w \in X^*\}$. Define the Hankel-matrix H_Ψ of Ψ as $H_\Psi \in K^{(X^*\times I)\times(X^*\times J)}$, $I = \{1,2,\dots,p\}$ and $(H_\Psi)_{(u,i)(v,j)} = (S_j)_i(vu)$. Notice that $\dim W_\Psi = \mathrm{rank}\ H_\Psi$.

Theorem 1 *Let $\Psi = \{S_j \in K^p \ll X^* \gg \mid j \in J\}$. The following are equivalent. (i) Ψ is rational. (ii)* $\dim W_\Psi = \mathrm{rank}\ H_\Psi < +\infty$, *(iii) The tuple $R_\Psi = (W_\Psi, \{A_\sigma\}_{\sigma\in X}, B, C)$, where $A_\sigma : W_\Psi \to W_\Psi$, $A_\sigma(T) = \sigma \circ T$, $B = \{B_j \in W_\Psi \mid j \in J\}$, $B_j = S_j$ for each $j \in J$, $C : W_\Psi \to K^p$, $C(T) = T(\epsilon)$, defines a representation of Ψ.*

The representation R_Ψ is called *free*. Since the linear space spanned by the column vectors of H_Ψ and the space W_Ψ are isomorphic, one can construct a representation of Ψ over the space of column vectors of H_{Ψ} in a way similar to the construction of R_{Ψ}. Theorem 1 implies the following lemma.

Theorem 1 *Let* $\Psi = \{S_j \in K^p \ll X^* \gg \mid j \in J\}$ *and* $\Theta = \{T_j \in K^p \ll X^* \gg \mid j \in J\}$ *be rational indexed sets. Then* $\Psi \odot \Theta := \{S_j \odot T_j \mid j \in J\}$ *is a rational set. Moreover,* $\text{rank } H_{\Psi \odot \Theta} \leq \text{rank } H_{\Psi} \cdot \text{rank } H_{\Theta}$.

Let $R = (\mathcal{X}, \{A_{\sigma}\}_{\sigma \in X}, B, C)$ be a representation of $\Psi \subseteq K^p \ll X^* \gg$. Define the subspaces W_R and O_R of $\mathcal{X}$ by

$$W_R = \text{Span}\{A_w B_j \mid w \in X^*, j \in J\} \ , \ \ O_R = \bigcap_{w \in X^*} \ker C A_w \tag{6}$$

Theorem 2 (Minimal representation) *Let* $\Psi = \{S_j \in K^p \ll X^* \gg \mid j \in J\}$*. The following are equivalent. (i)* $R_{min} = (\mathcal{X}, \{A_{\sigma}^{min}\}_{\sigma \in X}, B^{min}, C^{min})$ *is a minimal representation of* Ψ*, (ii)* $W_{R_{min}} = \mathcal{X}$ *and* $O_{R_{min}} = \{0\}$*, (iii)* $\text{rank } H_{\Psi} = \dim W_{\Psi} = \dim R_{min}$*, (iv) If* $R = (\mathcal{X}_R, \{A_x\}_{x \in X}, B, C)$ *is a representation of* Ψ*, then there exists a surjective vector space morphism* $T : W_R \to \mathcal{X}$ *such that*

$$TA_x|_{W_R} = A_x^{min} T, TB_j = B_j^{min}, (j \in J), C|_{W_R} = C^{min} T$$

for all $x \in X$ *and* $j \in J$*. In particular, if* R *is a minimal representation, then* $T : \mathcal{X}_R = W_R \to \mathcal{X}$ *is a vector space isomorphism and*

$$A_x^{min} = TA_x T^{-1} \ \ x \in X, B_j^{min} = TB_j, C^{min} = CT^{-1}$$

Using the theorem above it is easy to check that the free representation R_{Ψ} is minimal. One can also give a procedure, similar to reachability and observability reduction for linear systems, such that the procedure transforms any representation of Ψ to a minimal representation of Ψ. If $R = (\mathcal{X}, \{A_{\sigma}\}_{\sigma \in \Sigma}, B, C)$ is a representation of Ψ, then for any vector space isomorphism $T : \mathcal{X} \to \mathbb{R}^n$, $n = \dim R$, the tuple $R^{'} = (\mathbb{R}^n, \{TA_{\sigma}T^{-1}\}_{\sigma \in \Sigma}, TB, CT^{-1})$ is also a representation of Ψ. It is easy to see that R is minimal if and only if $R^{'}$ is minimal. From now on, we will silently assume that $\mathcal{X} = \mathbb{R}^n$ holds for any representation considered.

5 Realization of input-output maps by linear switched systems with arbitrary switching

In this section the solution to the realization problem will be presented. That is, given a set of input-output maps we will formulate necessary and sufficient conditions for the existence of a linear switched system realizing that set. In addition, characterization of minimal systems realizing the given set of input-output maps will be given. In this section we assume that there are no restrictions on switching sequences. That is, in this section we study realization with the trivial constraint $L = Q^+$.

The main tool of this section is the theory of rational formal power series. The main idea of the solution is the following. We associate a set of formal power series Ψ_Φ with the set of input-output maps Φ . Any representation of Ψ_Φ yields a realization of Φ and any realization of Φ yields a representation of Ψ_Φ. Moreover, minimal representations give rise to minimal realizations and vice versa. Then we can apply the theory of rational formal power series to characterize minimal realizations.

Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times (Q \times T)^+, \mathcal{Y})$. The fact that all switching sequences are allowed and formula (2) yield the following reformulation of Proposition 1. The LSS $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{(A_q,B_q,C_q) \mid q \in Q\})$ is a realization of Φ if and only if Φ has a generalized kernel representation and there exists $\mu : \Phi \to \mathcal{X}$ such that for each $q_1, q_2 \in Q, f \in \Phi, j = 1,2,\ldots,m$ it holds that

$$\begin{array}{lclcl} D^{(1,\mathbb{I}_k,0)} y_0^{\Phi}(e_z, q_2 w q_1, .) &=& D^{(0,\mathbb{I}_k,0)} G^{\Phi}_{q_2 w q_1} e_z &=& C_{q_1} A_{w_k} \cdots A_{w_1} B_{q_2} e_z \\ D^{(\mathbb{I}_k,0)} f(0, wq_1, .) &=& D^{(\mathbb{I}_k,0)} K^{f,\Phi}_{wq_1} &=& C_{q_1} A_{w_k} \cdots A_{w_1} \mu(f) \end{array} \quad (7)$$

where $\mathbb{I}_k = (1,1,\ldots,1) \in \mathbb{N}^k$.

The statement above allows us to reformulate the realization problem in terms of rationality of certain power series. Define the formal power series $S_{q_1,q_2,z}, S_{f,q_1} \in \mathbb{R}^p \ll Q^* \gg$, ($q_1, q_2 \in Q, f \in \Phi,\ z \in \{1,2,\ldots,m\}$) by

$$S_{q_1,q_2,z}(w) = D^{(1,\mathbb{I}_{|w|},0)} y_0^{\Phi}(e_z, q_2 w q_1, .)\ ,\ S_{f,q_1}(w) = D^{(\mathbb{I}_{|w|},0)} f(0, wq_1, .)$$

for each $w \in Q^*$. Notice that the functions $G^{\Phi}_w, K^{f,\Phi}_w$ are not involved in the definition of the series of $S_{q_1,q_2,z}$ and S_{f,q_1}. On the other hand, if Φ has a generalized kernel representation, then $S_{q_1,q_2,z}(w) = D^{(0,\mathbb{I}_{|w|},0)} G^{\Phi}_{q_2 w q_1} e_z$ and $S_{f,q_1}(w) = D^{(\mathbb{I}_{|w|},0)} K^{f,\Phi}_{wq_1}$. For each $q \in Q, z = 1,2,\ldots,m,\ f \in \Phi$ define the formal power series $S_{q,z}, S_f \in \mathbb{R}^{p|Q|} \ll Q^* \gg$ by

$$S_{q,z} := \begin{bmatrix} S^T_{q_1,q,z} & S^T_{q_2,q,z} & \cdots & S^T_{q_N,q,z} \end{bmatrix}^T \ S_f = \begin{bmatrix} S^T_{f,q_1} & S^T_{f,q_2} & \cdots & S^T_{f,q_N} \end{bmatrix}^T$$

where $Q = \{q_1, q_2, \ldots, q_N\}$.

Define the set $J_\Phi = \Phi \cup \{(q,z) \mid q \in Q, z = 1,2,\ldots,m\}$. Define the *indexed set of formal power series associated with* Φ by

$$\Psi_\Phi = \{S_j \mid j \in J_\Phi\} \quad (8)$$

Define the *Hankel-matrix of* Φ, H_Φ as the Hankel-matrix of the associated set of formal power series, i.e. $H_\Phi := H_{\Psi_\Phi}$.

Let $\Sigma = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{(A_q,B_q,C_q) \mid q \in Q\})$ be a LSS, and assume that (Σ,μ) is a realization of Φ. Define the *representation associated with* (Σ,μ) by

$$R_{\Sigma,\mu} = (\mathcal{X}, \{A_q\}_{q\in Q}, \widetilde{B}, \widetilde{C})$$

where $\widetilde{C} : \mathcal{X} \to \mathbb{R}^{p|Q|}$, $\widetilde{C} = \begin{bmatrix} C^T_{q_1} & C^T_{q_2} & \cdots & C^T_{q_N} \end{bmatrix}^T$, and $\widetilde{B} = \{\widetilde{B}_j \in \mathcal{X} \mid j \in J_\Phi\}$ is defined by $\widetilde{B}_f = \mu(f), f \in \Phi$ and $\widetilde{B}_{q,l} = B_q e_l, l = 1,2,\ldots,m, q \in Q$, $e_l \in \mathcal{U}, (e_l)_z = \delta_{lz}$, i.e. e_z is the zth standard base vector. Conversely, consider a representation $R = (\mathcal{X}, \{A_q\}_{q\in Q}, \widetilde{B}, \widetilde{C})$ of Ψ_Φ. Then define (Σ_R, μ_R) *the realization associated with* R by

$$\Sigma_R = (\mathcal{X},\mathcal{U},\mathcal{Y},Q,\{(A_q,B_q,C_q) \mid q \in Q\})\ ,\ \mu(f) = \widetilde{B}_f$$

where $\widetilde{C} = \begin{bmatrix} C_{q_1}^T \; C_{q_2}^T \; \cdots \; C_{q_N}^T \end{bmatrix}^T$ and $B_q e_l = \widetilde{B}_{q,l}$. It is easy to see that $\Sigma_{R_{\Sigma,\mu}} = \Sigma$, $\mu_{R_{\Sigma,\mu}} = \mu$ and $R_{\Sigma_R,\mu_R} = R$. In fact, the following theorems hold.

Theorem 3 *Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times (Q \times T)^+, \mathcal{Y})$. If (Σ, μ) is a realization of Φ then $R_{\Sigma,\mu}$ is a representation of Ψ_Φ. Conversely, if Φ has a generalized kernel representation and $R = (\mathcal{X}, \{A_q\}_{q\in Q}, \widetilde{B}, \widetilde{C})$ is a representation of Ψ_Φ then (Σ_R, μ_R) is a realization of Φ .*

The theorem above and the discussion after Theorem 1 imply that a realization of Φ can be constructed on the space of the column vectors of H_Φ. In fact, the following is a straightforward consequence of Theorem 1 and the above theorem.

Theorem 4 (Realization of input/output map) *Let $\Phi \subseteq F(PC(T,\mathcal{U}) \times (Q \times T)^+, \mathcal{Y})$. The following are equivalent. (i) Φ has a realization by a linear switched system, (ii) Φ has a generalized kernel representation and Ψ_Φ is rational, (iii) Φ has a generalized kernel representation and* $\mathrm{rank}\ H_\Phi < +\infty$.

The theory of rational power series allows us to formulate necessary and sufficient conditions for a linear switched system to be minimal. Before formulating a characterization of minimal realizations, some additional work has to be done. Let R be a representation and recall from Section 4 the sets W_R and O_R. Let (Σ, μ) be a realization of Φ. Let $R = R_{\Sigma,\mu}$ be the representation associated with (Σ, μ). Using the results of [5] and (6) the following can be shown. Σ is observable if and only if $O_R = \{0\}$, and the set W_R is the smallest vector space containing $Reach(\Sigma, \mathrm{Im}\mu)$. Consequently, Σ is semi-reachable from $\mathrm{Im}\mu$ if and only if $W_{R_{\Sigma,\mu}} = \mathcal{X}$. From the discussion after Theorem 2 we get that any realization of Φ can be transformed to an observable and semi-reachable realization of Φ. It can also be shown that if (Σ, μ) is a realization of Φ and Σ is observable, then $\mathcal{X}_\Phi = \mathrm{Im}\mu$. Moreover, Theorem 3 implies the following. If (Σ, μ) is a minimal realization of Φ, then $R_{\Sigma,\mu}$ is a minimal representation of Ψ_Φ. Conversely, if R is a minimal representation of Ψ_Φ, then (Σ_R, μ_R) is a minimal realization of Φ. This observation and the discussion above together with Theorem 2 imply the following theorem.

Theorem 5 (Minimal realization) *If (Σ, μ) is a realization of Φ, then the following are equivalent. (i) (Σ, μ) is minimal, (ii) Σ is semi-reachable from $\mathcal{X}_\Phi$ and it is observable, (iii) $\dim \Sigma = \mathrm{rank}\ \ H_\Phi$, (iv) For each linear switched system $\Sigma^{'}$ realizing Φ the inequality $\dim \Sigma \leq \dim \Sigma^{'}$ holds, (v) Let $\Sigma^{'} = (\mathcal{X}_1, \mathcal{U}, \mathcal{Y}, Q, \{(A_q^1, B_q^1, C_q^1) \mid q \in Q\})$ be a linear switched system such that $(\Sigma^{'}, \mu^{'})$ realizes Φ and $\Sigma^{'}$ is semi-reachable from $\mathrm{Im}\mu^{'}$. Then there exists a surjective linear map $T : \mathcal{X}_1 \to \mathcal{X}$, such that*

$$A_q T = T A_q^1,\;\; B_q = T B_q^1,\;\; C_q T = C_q^1,\;\; T\mu^{'} = \mu$$

In particular, all minimal linear switched systems realizing Φ are algebraically similar.

The theorems above contain the results of [2] as a special case. Indeed, if $\Phi = \{f\}$ and $\mu(f) = 0$, then the set $Reach(\Sigma, \{0\})$ is a vector space by [5]. That is,

Σ is semi-reachable from $\{0\}$ if and only if Σ is reachable from 0. Now it is straightforward to see that Theorem 5 contains the results of [2].

6 Realization of input-output maps with constraints on the switching

In this section the solution of the realization problem with constraints will be presented. That is, given $L \subseteq Q^+$ and $\Phi \subseteq F(PC(T,\mathcal{U}) \times TL, \mathcal{Y})$ we will study linear switched systems realizing Φ with constraint L. As in the previous section, the theory of formal power series will be our main tool for solving the realization problem. The solution of the realization problem for Φ goes as follows. As in the previous section, we associate a set of formal power series Ψ_Φ with the set of maps Φ. We will show that any representation of Ψ_Φ gives rise to a realization of Φ with constraint L. If L is regular, then any realization of Φ with constraint L gives rise to a representation of Ψ_Φ. Unfortunately minimal representations of Ψ_Φ do not yield minimal realizations of Φ. However, any minimal representation of Ψ_Φ yields an observable and semi-reachable realization of Φ. Notice that *if L is finite, then L is regular.*

Recall from Section 3 the definition of the languages $\widetilde{L}$, $\widetilde{L}_{q_1,q_2}$, $\widetilde{L}_q$ and the sets $F_{q_1,q_2}(w)$, $F_q(w)$. Let $E = (1,1,\ldots,1) \in \mathbb{R}^{1\times p}$. Define the power series $C_{q_1,q_2} \in \mathbb{R}^p \ll Q^* \gg$ by $C_{q_1,q_2}(w) = \begin{cases} E & \text{if } w \in \widetilde{L}_{q_1,q_2} \\ 0 & \text{otherwise} \end{cases}$. Define the power series $C_q, C \in \mathbb{R}^{p|Q|\times 1} \ll Q^* \gg$ by

$$C_q = \left[C_{q,q_1}, C_{q,q_2}, \cdots, C_{q,q_N}\right]^T \ , \ C = \left[Z_{q_1}, Z_{q_2}, \ldots, Z_{q_N}\right]^T$$

where $Z_q(w) = \begin{cases} E & \text{if } w \in \widetilde{L}_q \\ 0 & \text{otherwise} \end{cases}$ and $Q = \{q_1, \ldots, q_N\}$. It is a straightforward exercise in automaton theory to show that if L is regular, then the languages $\widetilde{L}_q$ and $\widetilde{L}_{q_1,q_2}$ are regular. Thus, we get that if L regular then the power series C_q, C are rational. Recall that for any $\alpha \in \mathbb{N}^k$, α^+ denotes $\alpha^+ = (\alpha_1 + 1, \alpha_2, \ldots, \alpha_k)$. We define the formal power series $S_{q_1,q_2,j}, S_{q,f} \in \mathbb{R}^p \ll Q^* \gg$, $q_1, q_2, q \in Q$, $j = 1, 2, \ldots, m$, $f \in \Phi$.

$$S_{q_1,q_2,j}(w) = \begin{cases} D^{(\mathbb{O}_{|v|},\alpha^+)} y_0^\Phi(e_j, vz, .) & \text{if } w \in \widetilde{L}_{q_1,q_2} \text{ and } (v,(\alpha,z)) \in F_{q_1,q_2}(w) \\ 0 & \text{otherwise} \end{cases}$$

$$S_{q,f}(w) = \begin{cases} D^{(\mathbb{O}_{|v|},\alpha)} f(0, vz, .) & \text{if } w \in \widetilde{L}_q \text{ and } (v,(\alpha,z)) \in F_q(w) \\ 0 & \text{otherwise} \end{cases}$$

We will argue that if Φ has a generalized kernel representation with constraint L, then the series $S_{q_1,q_2,z}$ and $S_{q,f}$ are well-defined. From Part 3 of Definition 2 and formulas (3) and (2) it follows that $D^{(\mathbb{O}_{|v|},\alpha^+)} y_0^\Phi(e_j, vz, .) = D^\alpha G_z^\Phi e_j = D^{(0,\alpha,0)} G_{q_2 z q_1} e_j = D^{(0,\mathbb{I}_{|w|},0)} G^\Phi_{q_2 w q_1} e_j$ and $D^{(\mathbb{O}_{|v|},\alpha)} f(0, vz, .) = D^{(\mathbb{O}_{|v|},\alpha)} K^{\Phi,f}_{vz} = D^\alpha K^{\Phi,f}_z = D^{(\mathbb{I}_w,0)} K^{\Phi,f}_{wq}$, where $\mathbb{I}_{|w|} = (1,1,\ldots,1) \in \mathbb{N}^{|w|}$. That is, $S_{q_1,q_2,j}(w)$

and $S_{q,f}(w)$ do not depend on the choice of $(v,(\alpha,z)) \in F_{q_1,q_2}(w)$ or $(v,(\alpha,z)) \in F_q(w)$ respectively. Define formal power series $S_{q,j}, S_f \in \mathbb{R}^{p|Q|\times 1}$ for each $j \in \{1,2,\ldots,m\}$, $q \in Q$ and $f \in \Phi$ by

$$S_{q,j} = \left[S^T_{q_1,j}\ S^T_{q_2,j}\ \ldots\ S^T_{q_N,j}\right]^T,\ S_f = \left[S^T_{q_1,f}\ S^T_{q_2,f}\ \ldots\ S^T_{q_N,f}\right]^T$$

where $Q = \{q_1,\ldots,q_N\}$. Define the set of formal power series *associated with* Φ by

$$\Psi_\Phi = \{S_z \mid z \in \Phi \cup (Q \times \{1,2,\ldots,m\})\}$$

Define the *Hankel-matrix of* Φ H_Φ as the Hankel-matrix of Ψ_Φ, i.e. $H_\Phi = H_{\Psi_\Phi}$.

Let (Σ,μ) be a realization. Define $\Theta = \{y_\Sigma(\mu(f),.,.) \in F(PC(T,\mathcal{U}) \times (Q \times T)^+,\mathcal{Y}) \mid f \in \Phi\}$. Recall the definition of the set of formal power series Ψ_Θ associated with Θ as defined in (8), Section 5. Denote by $T_{q,z}$ the element of Ψ_Θ indexed by $(q,z) \in (Q \times \{1,2,\ldots,m\})$ and denote by $T_{y_\Sigma}(\mu f,.,.)$ the element of Ψ_Θ indexed by $y_\Sigma(\mu(f),.,.) \in \Theta$. With the notation above, combining Proposition 1, formula (5) one gets the following theorems.

Theorem 6 *(Σ,μ) is a realization of Φ with constraint L if and only if Φ has a general kernel representation with constraint L and*

$$S_f = T_{y_\Sigma(\mu(f),.,.)} \odot C \text{ and } S_{q,z} = T_{q,z} \odot C_q,\ f \in \Phi, q \in Q, z = 1,2,\ldots,m$$

If Φ has a generalized kernel representation with constraint L and R is a representation of Ψ_Φ, then (Σ_R,μ_R) realizes Φ with constraint L.

Define the language $comp(L) = \{wq \in Q^* \mid w \in Q^*, q \in Q, \widetilde{L}_q = \emptyset\}$. Intuitively, the language $comp(L)$ contains those sequences which can never be observed if the switching system is run with constraint L. Using Theorem 6 and Lemma 1 from Section 4 one gets the following.

Theorem 7 *Consider a language $L \subseteq Q^+$ and a set $\Phi \subseteq F(PC(T,\mathcal{U}) \times TL,\mathcal{Y})$ of input-output maps. Assume that L is regular. Then the following are equivalent. (i) Φ has a realization by a linear switched system with constraint L, (ii) Φ has a generalized kernel representation with constraint L and Ψ_Φ is rational, (iii) Φ has a generalized kernel representation with constraint L and* rank $H_\Phi < +\infty$, *(iv) There exists a realization (Σ,μ) of Φ with constraint L such that (Σ,μ) is a minimal realization of $\Phi' = \{y_\Sigma(\mu(f),.,.) \in F(PC(T,\mathcal{U}) \times (Q \times T)^+,\mathcal{Y}) \mid f \in \Phi\}$ and*

$$y_\Sigma(\mu(f),.,.)|_{PC(T,\mathcal{U})\times T(comp(L))} = 0\ ,\ \forall f \in \Phi$$

Moreover, if $(\widetilde{\Sigma},\widetilde{\mu})$ is an arbitrary linear switched system realizing Φ with constraint L, then $\dim \Sigma \leq M \dim \widetilde{\Sigma}$, *for some $M > 0$. The constant M can be determined from L in the following way. Let $\Omega = \{C_q \in \mathbb{R}^p \ll Q^* \gg \mid q \in Q \cup \{\epsilon\}\}$, where $C_\epsilon = C$. Then $M =$* rank H_Ω.

In fact, the result of part (iv) of Theorem 7 is sharp in the following sense. One can construct an input-output map y, a language L, and realizations Σ_1 and

Σ_2 such that the following holds. Both Σ_1 and Σ_2 realize y from the initial state zero with constraint L, they are both reachable from zero and observable, but $\dim \Sigma_1 = 1$ and $\dim \Sigma_2 = 2$. We will give the construction of such Σ_1 and Σ_2 below. Let $Q = \{1, 2\}$, $L = \{q_1^k q_2 \mid k > 0\}$, $\mathcal{Y} = \mathcal{U} = \mathbb{R}$. Define $y : PC(T, \mathcal{U}) \times TL \to \mathcal{Y}$ by

$$y(u(.), w) = \int_0^{t_{m+1}} e^{2(t_{m+1}-s)} u(s + T_m) ds + \int_0^{T_m} e^{2t_{m+1}+T_m-s} u(s) ds$$

where $w = (\underbrace{q_1 \cdots q_1}_{m-times} q_2, t_1 \cdots t_m t_{m+1}) \in TL$, $T_m = \sum_1^m t_i$. Define the system $\Sigma_1 = (\mathbb{R}, \mathbb{R}, \mathbb{R}, Q, \{(A_{1,q}, B_{1,q} C_{1,q}) \mid q \in \{q_1, q_2\}\})$ by $A_{1,q_1} = 1, B_{1,q_1} = 1, C_{1,q_1} = 1$ and $A_{1,q_2} = 2, B_{1,q_2} = 1, C_{1,q_2} = 1$. Define the system $\Sigma_2 = (\mathbb{R}^2, \mathbb{R}, \mathbb{R}, Q, \{(A_{2,q}, B_{2,q}, C_{2,q}) \mid q \in Q\})$ by $A_{2,q_1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B_{2,q_1} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, C_{2,q_1} = \begin{bmatrix} 0 & 0 \end{bmatrix}$ and $A_{2,q_2} = \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix}, B_{2,q_2} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C_{2,q_2} = \begin{bmatrix} 1 & 1 \end{bmatrix}$. Both Σ_1 and Σ_2 are reachable and observable, therefore they are the minimal realizations of $y_{\Sigma_1}(0, ., .)$ and $y_{\Sigma_2}(0, ., .)$. Moreover, it is easy to see that $y_{\Sigma_1}(0, ., .)|_{PC(T,\mathcal{U}) \times TL} = y = y_{\Sigma_2}(0, ., .)|_{PC(T,\mathcal{U}) \times TL}$ In fact, Σ_2 can be obtained by constructing the minimal representation of $\Psi_{\{y\}}$, i.e., Σ_2 is a minimal realization of y satisfying part (iv) of Theorem 7.

7 Conclusions

Solution to the realization problem for linear switched systems has been presented. The realization problem considered is to find a realization of a family of input-output maps. Moreover, it is allowed to restrict the input-output maps to some subsets of switching sequences. Thus, the realization problem covers the case of linear switched systems where the switching is controlled by an automaton and the automaton is known in advance. The results of the paper extend those of [2], where a much more restricted realization problem was studied. The paper offers a new technique, the theory of formal power series, to deal with realization problem for switched systems. Topics of further research include realization theory for piecewise-affine systems, switched systems with switching controlled by an automaton or a timed automaton and non-linear switched systems.

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