Supplementary Material I: Analysis of the zero-temperature Ising MDP

In this section we provide the details of our analysis of the zero-temperature Ising MDP and prove our main result, i.e., Theorem 4. Section S.I.1 introduces some additional notation and terminology required for the complete analysis of the zero-temperature Ising MDP. In Section S.I.2.1, we formalize the construction of the auxiliary MDP and show that it is indeed an accurate approximation of the original MDP. Finally, Section S.I.2.2 reports an outline of the proof of Theorem 4. Some details of the proofs are deferred to Supplementary Material III.

S.I.1 Additional notation and terminology

In order to formalize the analysis outlined in Section 3, we introduce some additional notation and terminology.

Given a site $i \in \Lambda$, let $N_h(i) \subseteq \Lambda$ and $N_v(i) \subseteq \Lambda$ denote the sets of its horizontal neighbors and its vertical neighbors. We define the following distance measure $\delta : \Lambda \times \Lambda \to \mathbb{N}_0$ on the lattice:

$$\delta((x,y),(x',y')) = \min\{|x'-x|, N-|x'-x|\} + \min\{|y'-y|, N-|y'-y|\},$$

where $(x,y),(x',y') \in \Lambda \times \Lambda$, taking into account a torus edge correction. Furthermore, letting $P(\Lambda)$ denote the power set of Λ, we define a distance measure $\tilde{\delta} : \Lambda \times P(\Lambda) \to \mathbb{N}_0$ as

$$\tilde{\delta}((x,y),W) = \min_{(x',y')\in W} \delta((x,y),(x',y')), \quad (x,y) \in \Lambda, \quad W \subseteq \Lambda.$$

Let $\mu((x,y),W)$ denote the set of spins in W that are closest to (x,y) with respect to this distance measure, i.e.,

$$\mu((x,y),W) = \operatorname*{arg\,min}_{(x',y')\in W} \delta((x,y),(x',y')), \quad (x,y) \in \Lambda, \quad W \subseteq \Lambda.$$

In addition, we define a horizontal distance measure $\delta_h : \Lambda \times \Lambda \to \mathbb{N}_0^\infty$ as

$$\delta_h((x,y),(x',y')) = \begin{cases} \min\{|x'-x|, N-|x'-x|\}, & \text{if } y = y,' \\ \infty, & \text{otherwise.} \end{cases}$$

Accordingly, let $\tilde{\delta}_h : \Lambda \times P(\Lambda) \to \mathbb{N}_0^\infty$ be defined as

$$\tilde{\delta}_h((x,y),W) = \min_{(x',y')\in \Lambda} \delta_h((x,y),(x',y')), \quad (x,y) \in \Lambda, \quad W \subseteq \Lambda,$$

and $\mu_h((x,y),W)$, $(x,y) \in \Lambda$, $W \subseteq \Lambda$ as

$$\mu_h((x,y),W) = \begin{cases} \emptyset, & \text{if } \delta_h((x,y),(x',y')) = \infty, \\ & \text{for all } (x',y') \in W, \\ \arg\min_{(x',y')\in W} \delta_h((x,y),(x',y')), & \text{otherwise.} \end{cases}$$

The vertical counterparts δ_v, $\tilde{\delta}_v$ and μ_v are defined in an analogous way.

Given a configuration $\sigma \in S$, let $\Lambda_+(\sigma) \subseteq \Lambda$ denote the set of sites with spin $+1$ in σ, i.e., $\Lambda_+(\sigma) = \{i \in \Lambda | \sigma(i) = +1\}$. Furthermore, let $R(\sigma)$ denote all sites in the smallest rectangle that circumscribes the set of sites with spin $+1$. We define the set of *corner sites*, the sets of *horizontal and vertical boundary sites* and the set of *interior sites* of σ as

$$C(\sigma) = \left\{ i \in \Lambda \;\middle|\; \sigma(i) = +1, \sum_{j\in N_h(i)} \sigma(j) = 0 \text{ and } \sum_{j\in N_v(i)} \sigma(j) = 0 \right\},$$

$$B_h(\sigma) = \left\{ i \in \Lambda \;\middle|\; \sigma(i) = +1, \sum_{j\in N_h(i)} \sigma(j) = 2 \text{ and } \sum_{j\in N_v(i)} \sigma(j) = 0 \right\},$$

$$B_v(\sigma) = \left\{ i \in \Lambda \;\middle|\; \sigma(i) = +1, \sum_{j\in N_v(i)} \sigma(j) = 2 \text{ and } \sum_{j\in N_h(j)} \sigma(j) = 0 \right\},$$

$$I(\sigma) = \{ i \in \Lambda \mid \sigma(i) = \sigma(j) = +1 \text{ for all } j \in N(i) \}.$$

Let $\hat{\sigma} \in S$ denote the configuration obtained from σ by setting the spins of all sites in $R(\sigma)$ to $+1$, i.e.,

$$\hat{\sigma}(i) = \begin{cases} +1, & \text{if } i \in R(\sigma), \\ \sigma(i), & \text{otherwise,} \end{cases} \quad i \in \Lambda.$$

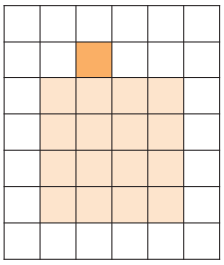


Figure S1: Configuration drawn on the dual lattice. Each orange square corresponds to a site with spin $+1$ and each white square represents a site with spin -1. The dark orange site, which has spin $+1$, is susceptible. After all, it has three neighbors with spin -1, so flipping the spin at this site would lead to a lower energy configuration.

We proceed to introduce some notation and terminology specific to the zero-temperature behaviour of the Ising MDP. We call a configuration $\sigma' \in S$ a *downhill configuration* of a configuration $\sigma \in S$ if there exists a sequence $\omega = (\sigma_0, \sigma_1, \dots, \sigma_\ell)$ for some $\ell \in \mathbb{N}$ that satisfies $\sigma_0 = \sigma$, $\sigma_\ell = \sigma'$, $H(\sigma_{t+1}) \leq H(\sigma_t)$ and $\sum_{i\in\Lambda} |\sigma_{t+1}(i) - \sigma_t(i)| \leq 1$ for all $t = 0, \dots, \ell - 1$. Let the probability of following this sequence under the zero-temperature Metropolis dynamics be denoted by $p_\infty(\omega)$, i.e.,

$$p_\infty(\omega) = \prod_{t=0}^{\ell-1} p_\infty(\sigma_t, \sigma_{t+1}).$$

Note that, using this formulation, a configuration $\sigma' \in S$ is a downhill configuration of $\sigma \in S$ if and only if there exists a sequence $\omega = (\sigma_0, \sigma_1, \dots, \sigma_\ell)$ for some $\ell \in \mathbb{N}$ such that $\sigma_0 = \sigma$, $\sigma_\ell = \sigma'$ and $p_\infty(\omega) > 0$. Similarly, we define $\sigma' \in S$ to be an *ℓ-step downhill configuration* of $\sigma \in S$ if there exists a sequence $\omega = (\sigma_0, \sigma_1, \dots, \sigma_\ell)$ of specified length $\ell \in \mathbb{N}$ that satisfies $\sigma_0 = \sigma$, $\sigma_\ell = \sigma'$ and $H(\sigma_{t+1}) \leq H(\sigma_t)$ and $\sum_{i\in\Lambda} |\sigma_{t+1}(i) - \sigma_t(i)| \leq 1$ for all $t = 0, \dots, \ell - 1$. Let $\Gamma(\sigma) \subseteq S$ denote the set of all downhill configurations of $\sigma \in S$. Similarly, let $\Gamma_\ell(\sigma) \subseteq S$ denote the set of all ℓ-step downhill configurations of $\sigma \in S$, where $\ell \in \mathbb{N}$.

We call a sequence of configurations $\omega = (\sigma_0, \sigma_1, \dots, \sigma_\ell)$, $\sigma_0, \dots, \sigma_\ell \in S$, a *downhill path* if $H(\sigma_{t+1}) \leq H(\sigma_t)$ and $\sum_{i\in\Lambda} |\sigma_{t+1}(i) - \sigma_t(i)| = 1$ for all $t = 0, \dots, \ell - 1$. For any $\sigma, \eta \in S$, let $\Omega(\sigma, \eta)$ denote the collection of all downhill paths leading from configuration σ to η. Furthermore, for any $\sigma \in S$, let $\Omega(\sigma)$ denote the set of all downhill paths starting at configuration σ. Finally, let Ω denote the set of all downhill paths. Hence, $\Omega = \cup_{\sigma\in S}\Omega(\sigma)$ and $\Omega(\sigma) = \cup_{\eta\in S}\Omega(\sigma, \eta)$, $\sigma \in S$. For a set $W \subseteq \Lambda$, let W^* denote the set of sequences of any length with elements in W. Given a downhill path $\omega = (\sigma_0, \sigma_1, \dots, \sigma_\ell)$, let $\zeta(\omega) = (x_1, x_2, \dots, x_\ell) \in \Lambda^*$ denote the corresponding sequence of sites at which spins are flipped along this path, i.e., x_k is the site whose spin was flipped to obtain configuration σ_k from configuration σ_{k-1}, $k = 1, \dots, \ell$. We now define $\Theta(\sigma, \eta)$, $\Theta(\sigma)$, $\Theta \subseteq \Lambda^*$, $\sigma, \eta \in S$, as the sets of *downhill sequences* corresponding to the paths in $\Omega(\sigma, \eta)$, $\Omega(\sigma)$ and Ω. Note that $\zeta : \Omega \to \Theta$ is a one-to-one mapping from the set of downhill paths to the set of sequences.

We call a site $i \in \Lambda$ *susceptible* in a configuration $\sigma \in S$ if $H(\sigma^i) \leq H(\sigma)$. Note that a site i is susceptible in σ if and only if $p_\infty(\sigma, \sigma^i) > 0$. Recall that in our setting, we imposed a small external magnetic field $h \in (0, 1)$. This implies that a site with spin $+1$ is susceptible if and only if it has at least three neighboring sites with spin -1. A site with spin -1, on the other hand, is susceptible if and only if it has at least two neighboring sites with spin +1. Figure S1 shows an example of a susceptible site with spin $+1$. The set of susceptible sites in a configuration $\sigma \in S$ is denoted by $\Delta(\sigma)$.

We call a configuration $\sigma \in S$ *fragile* if at least one site is susceptible in σ, that is, if $\Delta(\sigma) \neq \emptyset$. A configuration that is not fragile, we call *robust*. Some examples of fragile and robust configurations are depicted in Figure S2. Let $U \subseteq S$ denote the set of robust configurations. We define $U^{(1)}$ to be the set of robust configurations that are not the all-minus configuration, denoted by σ^-, and in which the sites with spin $+1$ form a single droplet. Theorem S2 characterizes these configurations geometrically as configurations in which the sites with spin $+1$ form a rectangle.

Let $U(\sigma) \subseteq U$ denote the set of robust configurations that are downhill configurations of $\sigma \in S$, i.e., $U(\sigma) = U \cap \Gamma(\sigma)$. Accordingly, let $U_\ell(\sigma) = U \cap \Gamma_\ell(\sigma)$ denote the set of robust configurations that are ℓ-step downhill configurations of σ for some $\ell \in \mathbb{N}$. The sets $U^{(1)}(\sigma)$ and $U^{(1)}_\ell(\sigma)$ for $\sigma \in S$ and $\ell \in \mathbb{N}$ are defined in an analogous way, replacing U with $U^{(1)}$.

We call a downhill path $\omega = (\sigma_0, \sigma_1, \dots, \sigma_\ell)$, $\sigma_0, \dots, \sigma_\ell \in S$, a *closed downhill path* if it ends in a robust configuration, i.e., if $\sigma_\ell \in U$. We refer to the corresponding sequence of spins $\zeta(\omega)$ as a *closed downhill sequence*. Let $\tilde{\Omega}(\sigma, \eta)$, $\tilde{\Omega}(\sigma)$

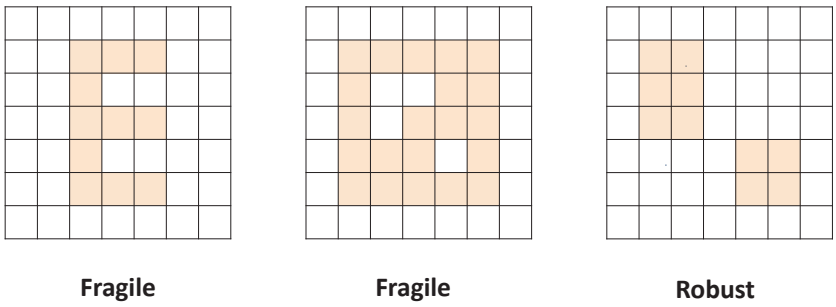


Figure S2: Examples of fragile and robust configurations.

and $\tilde{\Omega}$ denote the sets of closed downhill paths, defined analogously to $\Omega(\sigma, \eta)$, $\Omega(\sigma)$ and Ω, and let $\tilde{\Theta}(\sigma, \eta)$, $\tilde{\Theta}(\sigma)$ and $\tilde{\Theta}$ denote the corresponding sets of closed downhill sequences.

Given a sequence $\mathbf{x} \in \Lambda^*$ and a configuration $\sigma \in S$, let $\sigma^{\mathbf{x}}$ denote the configuration obtained after consecutively flipping the spins at sites in $\mathbf{x}$ from σ. For a set $W \subseteq \Lambda$ and a sequence $\mathbf{x} \in \Lambda^*$, let $\mathbf{x}^W$ denote the subsequence of $\mathbf{x}$ that consists only of those elements of $\mathbf{x}$ that are part of W. We refer to $\mathbf{x}^W$ as the *restriction of* $\mathbf{x}$ to W. We now call a set $W \subseteq \Lambda$ *essential* for a configuration $\sigma \in S$ if it satisfies the following two requirements:

- All closed downhill sequences $\mathbf{x} \in \tilde{\Theta}(\sigma)$ that have the same restriction to W lead to the same robust configuration, i.e., if $\mathbf{x}^W = \mathbf{y}^W$, for $\mathbf{x}, \mathbf{y} \in \tilde{\Theta}(\sigma)$, then $\sigma^{\mathbf{x}} = \sigma^{\mathbf{y}}$.
- The susceptibility of the sites in W only depends on the evolution of the configuration on W, that is, for any downhill sequence $\mathbf{x} \in \Theta(\sigma)$, we have $\Delta(\sigma^{\mathbf{x}}) \cap W = \Delta(\sigma^{\mathbf{x}^W}) \cap W$.

The above properties imply that the robust configuration at the end of a closed downhill path is fully determined by the evolution of the spins at sites in the essential set. Furthermore, the dynamics governing the spins at sites in the essential set, starting from the configuration σ, is also fully specified by the evolution of the spins at sites in the set. As a consequence, when computing transition probabilities to the next robust configuration from a configuration σ, it suffices to focus on the essential set for σ, which greatly simplifies computations.

S.I.2 Analysis of the zero-temperature Ising MDP

Using the concepts defined in Section S.I.1, we now proceed to formalize the analysis of the low-temperature Ising MDP. First, observe that for any $\ell \in \mathbb{N}$, we have $P_{\infty,\ell}(\sigma'|\sigma, d(\sigma)) = 0$ for all $\sigma' \notin \Gamma_\ell(\sigma^{d(\sigma)})$. Recalling in addition that the value function is finite, due to the fact that the reward function is bounded, the equations in expression (8) at zero temperature reduce to

$$v^\pi_{\lambda,\infty,k}(\sigma) = r(\sigma, d(\sigma)) + \lambda \sum_{\sigma' \in \Gamma_k(\sigma^{d(\sigma)})} P_{\infty,k}(\sigma'|\sigma, d(\sigma)) v^\pi_{\lambda,\infty,k}(\sigma'), \quad \sigma \in S. \tag{S1}$$

We show that dropping the contributions of fragile configurations in the summation in expression (S1) in fact leads to a valid approximation of the zero-temperature value function. To this end, we define a transition probability kernel $Q_{\infty,k} : S \times A \times S \to [0, 1]$ by conditioning on the event that the system transitions to a configuration $\sigma' \in U$, given a starting configuration $\sigma \in S$ and action $a \in A$, i.e.,

$$Q_{\infty,k}(\sigma'|\sigma, a) := \begin{cases} \dfrac{P_{\infty,k}(\sigma'|\sigma, a)}{\sum\limits_{\sigma'' \in U} P_{\infty,k}(\sigma''|\sigma, a)}, & \text{if } \sigma' \in U, \\ 0, & \text{otherwise,} \end{cases} \tag{S2}$$

for $\sigma, \sigma' \in S$, $a \in A$. In the result that follows, it is convenient to write equation (S1) in vector notation, i.e., as

$$\mathbf{v}^\pi_{\lambda,\infty,k} = \mathbf{r}^\pi + \lambda \mathbf{P}^\pi_{\infty,k} \mathbf{v}^\pi_{\lambda,\infty,k},$$

where $\mathbf{v}^\pi_{\lambda,\infty,k} \in \mathbb{R}^{|S|}$, $\mathbf{r}^\pi \in \mathbb{R}^{|S|}$ and $\mathbf{P}^\pi_{\infty,k} \in \mathbb{R}^{|S| \times |S|}$ denote the vector of values, the vector of rewards and the zero-temperature transition probability matrix specified by policy π with discount factor $\lambda \in (0, 1)$ and adjustment

time $k \in \mathbb{N}$. As established in Section 2.2, the vector of expected total discounted rewards $\mathbf{v}^{\pi}_{\lambda,\infty,k}$ of a policy $\pi \in \Pi$ is the unique fixed point of the operator $\mathcal{F}^{\pi}_{\lambda,k} : \mathbb{R}^{|S|} \to \mathbb{R}^{|S|}$ defined as

$$\mathcal{F}^{\pi}_{\lambda,\infty,k}\mathbf{x} := \mathbf{r}^{\pi} + \lambda \mathbf{P}^{\pi}_{\infty,k}\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^{|S|}. \tag{S3}$$

In analogy to expression (S3), we define the operator $\mathcal{G}^{\pi}_{\lambda,\infty,k} : \mathbb{R}^{|S|} \to \mathbb{R}^{|S|}$ as

$$\mathcal{G}^{\pi}_{\lambda,\infty,k}\mathbf{x} := \mathbf{r}^{\pi} + \lambda \mathbf{Q}^{\pi}_{\infty,k}\mathbf{x}, \quad \mathbf{x} \in \mathbb{R}^{|S|}. \tag{S4}$$

Theorem S1 asserts that the fixed point of the operator $\mathcal{G}^{\pi}_{\lambda,\infty,k}$ can approach the zero-temperature value function $v^{\pi}_{\lambda,\infty,k}$ arbitrarily closely and that the accuracy of this approximation can be controlled by selecting a sufficiently long adjustment time k.

Theorem S1. *Consider the operators $\mathcal{F}^{\pi}_{\lambda,\infty,k}$ and $\mathcal{G}^{\pi}_{\lambda,\infty,k}$ defined in expressions (S3) and (S4) respectively. For all $\varepsilon > 0$, there exists a positive integer K such that for all $k \geq K$, we have*

$$||\mathcal{F}^{\pi}_{\lambda,\infty,k}\mathbf{x} - \mathcal{G}^{\pi}_{\lambda,\infty,k}\mathbf{x}||_{\infty} < \varepsilon ||\mathbf{x}||_{\infty} \quad \textit{for all } \mathbf{x} \in \mathbb{R}^{|S|} \textit{ and } \pi \in \Pi.$$

Proof. Let $\pi = d^{\infty} \in \Pi$, $\mathbf{x} \in \mathbb{R}^{|S|}$ and $\varepsilon > 0$. We start by showing that there exists a positive integer K such that for all $k \geq K$, we have

$$\sum_{\sigma' \in U} P_{\infty,k}(\sigma'|\sigma, a) \geq 1 - \frac{\varepsilon}{2} \quad \text{for all } \sigma \in S, a \in A. \tag{S5}$$

First, observe that $0 < h < 1$ implies that flipping at a susceptible site strictly decreases the energy of a configuration. Since the energy cannot drop below its global minimum $H_{\min} = -N^2(2+h)$, each closed downhill path has finite length. Let L denote the maximum length of such a path. Note that the probability of transitioning from a fragile configuration $\sigma \in S$ to a different configuration at zero temperature is given by

$$\sum_{\substack{\sigma' \in S \\ \sigma' \neq \sigma}} p_{\infty}(\sigma, \sigma') = \frac{|\Delta(\sigma)|}{N^2}.$$

Since each fragile configuration has at least 1 susceptible site, the probability of taking a step along a downhill path is at least $1/N^2$. The minimum number of steps necessary to reach a robust configuration is at most L. Hence, we can lower bound the expression $\sum_{\sigma' \in U} P_{\infty,k}(\sigma'|\sigma, a)$ by the probability of obtaining at least L successes in k Bernoulli trials with success probability $1/N^2$. That is,

$$\sum_{\sigma' \in U} P_{\infty,k}(\sigma'|\sigma, a) \geq \mathbb{P}(X \geq L), \quad \text{for all } \sigma \in S, \quad a \in A,$$

where $X \sim \text{Bin}\left(k, 1/N^2\right)$ and $k \geq L$. Hence,

$$\sum_{\sigma' \in U} P_{\infty,k}(\sigma'|\sigma, a) \geq 1 - \mathbb{P}(X < L) = 1 - \sum_{i=0}^{L-1} \binom{k}{i} \left(\frac{1}{N^2}\right)^i \left(1 - \frac{1}{N^2}\right)^{k-i}.$$

For a given $i = 0, \ldots, L-1$, we have

$$0 \leq \lim_{k\to\infty} \binom{k}{i} \left(\frac{1}{N^2}\right)^i \left(1 - \frac{1}{N^2}\right)^{k-i} \leq \lim_{k\to\infty} \frac{k^i}{i!} \left(\frac{1}{N^2}\right)^i e^{(k-i)\log(1-1/N^2)} = 0.$$

Hence,

$$\lim_{k\to\infty} \left[1 - \sum_{i=0}^{L-1} \binom{k}{i} \left(\frac{1}{N^2}\right)^i \left(1 - \frac{1}{N^2}\right)^{k-i}\right] = 1.$$

This implies that there exists a positive integer $K \geq L$ such that expression (S5) is satisfied for all $k \geq K$.

We have

$$||\mathbf{P}^{\pi}_{\infty,k} - \mathbf{Q}^{\pi}_{\infty,k}||_{\infty} = \max_{\sigma \in S} \sum_{\sigma' \in S} \left| P^{\pi}_{\infty,k}(\sigma'|\sigma, d(\sigma)) - Q^{\pi}_{\infty,k}(\sigma'|\sigma, d(\sigma)) \right|.$$

Since, $P^{\pi}_{\infty,k}(\sigma'|\sigma, d(\sigma)) = Q^{\pi}_{\infty,k}(\sigma'|\sigma, d(\sigma)) = 0$ for all $\sigma \in S$, $\sigma' \in S \setminus \Gamma_k(\sigma^{d(\sigma)})$ and $Q^{\pi}_{\infty,k}(\sigma'|\sigma, d(\sigma)) = 0$ for all $\sigma \in S$ and $\sigma' \in \Gamma_k(\sigma^{d(\sigma)}) \setminus U_k(\sigma^{d(\sigma)})$, we obtain

$$||\mathbf{P}^{\pi}_{\infty,k} - \mathbf{Q}^{\pi}_{\infty,k}||_{\infty} = \max_{\sigma \in S} \sum_{\sigma' \in \Gamma_k(\sigma^{d(\sigma)})} \left| P^{\pi}_{\infty,k}(\sigma'|\sigma, d(\sigma)) - Q^{\pi}_{\infty,k}(\sigma'|\sigma, d(\sigma)) \right| \tag{S6}$$

$$= \max_{\sigma \in S} \Big\{ \sum_{\sigma' \in U_k(\sigma^{d(\sigma)})} \left| P^{\pi}_{\infty,k}(\sigma'|\sigma, d(\sigma)) - Q^{\pi}_{\infty,k}(\sigma'|\sigma, d(\sigma)) \right| + \sum_{\sigma' \in \Gamma_k(\sigma^{d(\sigma)}) \setminus U_k(\sigma^{d(\sigma)})} P^{\pi}_{\infty,k}(\sigma'|\sigma, d(\sigma)) \Big\}.$$

For the first summation, using expression (S2) yields

$$\begin{aligned}
&\sum_{\sigma' \in U_k(\sigma^{d(\sigma)})} \left| P^{\pi}_{\infty,k}(\sigma'|\sigma, d(\sigma)) - Q^{\pi}_{\infty,k}(\sigma'|\sigma, d(\sigma)) \right| \\
&= \sum_{\sigma' \in U_k(\sigma^{d(\sigma)})} \left| \sum_{\sigma'' \in U} P_{\infty,k}(\sigma''|\sigma, d(\sigma)) Q^{\pi}_{\infty,k}(\sigma'|\sigma, d(\sigma)) - Q^{\pi}_{\infty,k}(\sigma'|\sigma, d(\sigma)) \right| \\
&= 1 - \sum_{\sigma' \in U} P_{\infty,k}(\sigma'|\sigma, d(\sigma)).
\end{aligned}$$

Choosing $k \geq K$, we now obtain

$$\sum_{\sigma' \in U_k(\sigma^{d(\sigma)})} \left| P^{\pi}_{\infty,k}(\sigma'|\sigma, d(\sigma)) - Q^{\pi}_{\infty,k}(\sigma'|\sigma, d(\sigma)) \right| \leq \frac{\varepsilon}{2}.$$

As for the second summation in expression (S6), we have, again for $k \geq K$,

$$\sum_{\sigma' \in \Gamma_k(\sigma^{d(\sigma)}) \setminus U_k(\sigma^{d(\sigma)})} P^{\pi}_{\infty,k}(\sigma'|\sigma, d(\sigma)) = 1 - \sum_{\sigma' \in U} P^{\pi}_{\infty,k}(\sigma'|\sigma, d(\sigma)) \leq \frac{\varepsilon}{2}.$$

Hence, we obtain

$$||\tilde{\mathbf{P}}^{\pi}_{\infty,k} - \mathbf{Q}^{\pi}_{\infty,k}||_{\infty} \leq \varepsilon.$$

This now implies

$$||\mathcal{F}^{\pi}_{\lambda,\infty,k}\mathbf{x} - \mathcal{G}^{\pi}_{\lambda,\infty,k}\mathbf{x}||_{\infty} = ||\lambda^k(\tilde{\mathbf{P}}^{\pi}_{\infty,k} - \mathbf{Q}^{\pi}_{\infty,k})\mathbf{x}||_{\infty} \leq \lambda^k ||\tilde{\mathbf{P}}^{\pi}_{\infty,k} - \mathbf{Q}^{\pi}_{\infty,k}||_{\infty} \cdot ||\mathbf{x}||_{\infty} < \varepsilon ||\mathbf{x}||_{\infty},$$

for all $k \geq K$. □

S.I.2.1 The auxiliary MDP

Theorem S1 suggests that the fixed point of the operator $\mathcal{G}^{\pi}_{\lambda,\infty,k}$ for some $\pi \in \Pi$ provides an accurate approximation of the zero-temperature value function $v^{\pi}_{\lambda,\infty,k}$ for a sufficiently long adjustment time k. Armed with this result, we construct an auxiliary MDP that serves as a caricature version of the original Ising MDP and ranges only over the all-minus configuration, denoted by σ^- together with the set $U^{(1)}$, i.e., the set of robust configurations in which the sites with spin $+1$ form a single droplet. The following result provides a geometrical characterization of such robust configurations.

Theorem S2. *A configuration $\sigma \in S$ belongs to $U^{(1)}$ if and only if $\sigma \neq \sigma^-$ and the sites with spin $+1$ form a rectangle of size $i \times j$, for some $i, j \in \{2, 3, \ldots, N-3, N-2, N\}$.*

Proof. A configuration $\sigma \neq \sigma^-$ is robust if and only if each site with spin -1 has at most one neighbor with spin $+1$ and each site with spin $+1$ has at least two neighbors with spin $+1$. A configuration in which the sites with spin $+1$ form one droplet satisfies these requirements if and only if these sites form a rectangle of size $i \times j$, where $i, j \in \{2, 3, \ldots, N-3, N-2, N\}$. □

We now formalize the construction of the auxiliary MDP $(\hat{S}, \hat{A}, \hat{P}, \hat{r})$ described in Section 3.1. Recall that the state space $\hat{S}$ consist of 2-dimensional vectors representing the size of a rectangle in the Ising MDP, i.e.,

$$\hat{S} = \{(i,j) | i, j = 2, 3, \ldots, N-3, N-2, N\} \cup \{(0,0)\}.$$

Here, the vector $(0,0)$ corresponds to the all-minus configuration. To formally relate the original zero-temperature Ising MDP to the auxiliary MDP, we define a mapping $\mathcal{I} : U^{(1)} \cup \{\sigma^-\} \to \hat{S}$ as $\mathcal{I}(\sigma) = (i,j)$ if $\sigma \in U^{(1)}$ and the sites with spin $+1$ in σ form a rectangle of size $i \times j$ and $\mathcal{I}(\sigma^-) = (0,0)$. Given $(i,j) \in \hat{S}$, we now let $\sigma_{(i,j)} \in \mathcal{I}^{-1}((i,j))$,

without loss of generality, denote the configuration in $\mathcal{I}^{-1}((i,j))$ in which the lower left corner of the rectangle is located at the origin of the lattice. The action space $\hat{A}_{(i,j)}$ corresponding to a state $(i,j) \in \hat{S}$ is defined as

$$\hat{A}(i,j) = \begin{cases} \{0\}, & \text{if } i=j=0 \text{ or } i=j=N, \\ \{a'_{11}, a'_{12}, a'_{21}, a'_{22}, a_0, \tilde{a}, 0\}, & \text{if } i=j=2, \\ \{a'_{11}, a'_{12}, a_{21}, a_{22}, a'_{21}, a'_{22}, a_0, \tilde{a}, 0\}, & \text{if } i=2,\ 3 \le j \le N-3, \\ \{a'_{11}, a_{21}, a_{22}, a'_{21}, a'_{22}, a_0, \tilde{a}, 0\}, & \text{if } i=2,\ j=N-2, \\ \{a_{11}, a_{12}, a'_{11}, a'_{12}, a'_{21}, a'_{22}, a_0, \tilde{a}, 0\}, & \text{if } 3 \le i \le N-3,\ j=2, \\ \{a_{11}, a_{12}, a'_{11}, a'_{12}, a'_{21}, a_0, \tilde{a}, 0\}, & \text{if } i=N-2,\ j=2, \\ \{a_{11}, a_{12}, a'_{11}, a'_{12}, a_{21}, a_{22}, a'_{21}, a'_{22}, a_0, \tilde{a}, 0\}, & \text{if } 3 \le i,j \le N-3, \\ \{a_{11}, a_{12}, a'_{11}, a'_{12}, a_{21}, a'_{21}, a_0, \tilde{a}, 0\}, & \text{if } i=N-2,\ 3 \le j \le N-3, \\ \{a_{11}, a'_{11}, a_{21}, a_{22}, a'_{21}, a'_{22}, a_0, \tilde{a}, 0\}, & \text{if } 3 \le i \le N-3,\ j=N-2, \\ \{a_{11}, a'_{11}, a_{21}, a'_{21}, a_0, \tilde{a}, 0\}, & \text{if } i=j=N-2, \\ \{a_{21}, 0\}, & \text{if } i=N-2,\ j=N, \\ \{a_{11}, a_{12}, 0\}, & \text{if } i=N,\ 2 \le j \le N-3, \\ \{a_{11}, 0\}, & \text{if } i=N,\ j=N-2, \\ \{a_{21}, a_{22}, 0\}, & \text{if } 2 \le i \le N-3,\ j=N. \end{cases}$$

To clarify the connection to the original MDP, we define a mapping $\mathcal{J}_\sigma : A \to \hat{A}_{\mathcal{I}(\sigma)}$ for each $\sigma \in U^{(1)} \cup \{\sigma^-\}$ as

$$\mathcal{J}_\sigma(a) = \begin{cases} a_{1\ell}, & \text{if } \tilde{\delta}_v(a, V_+(\sigma)) = \ell \text{ and } \mu_v(a, V_+(\sigma)) \notin C(\sigma), \\ a'_{1\ell}, & \text{if } \tilde{\delta}_v(a, V_+(\sigma)) = \ell \text{ and } \mu_v(a, V_+(\sigma)) \in C(\sigma), \\ a_{2\ell}, & \text{if } \tilde{\delta}_h(a, V_+(\sigma)) = \ell \text{ and } \mu_h(a, V_+(\sigma)) \notin C(\sigma), \\ a'_{2\ell}, & \text{if } \tilde{\delta}_h(a, V_+(\sigma)) = \ell \text{ and } \mu_h(a, V_+) \in C(\sigma), \\ a_0, & \text{if } \tilde{\delta}(a, V_+(\sigma)) = 2 \text{ and } \tilde{\delta}_h(a, V_+(\sigma)) = \tilde{\delta}_v(a, V_+(\sigma)) = \infty, \\ \tilde{a}, & \text{if } a \in C(\sigma), \\ 0, & \text{otherwise,} \end{cases}$$

for $\ell = 1, 2$, $a \in A$. Hence, $a_{1\ell}$ and $a_{2\ell}$, $\ell = 1, 2$, correspond to the actions of flipping the spin at a site at distance ℓ from the horizontal and the vertical side of the rectangle, for which the closest site belonging to the rectangle is not a corner site of the rectangle. Actions $a'_{1\ell}$, $a'_{2\ell}$, $\ell = 1, 2$, correspond to the actions of flipping the spin at a site at distance ℓ from the horizontal and the vertical side of the rectangle, where the closest site belonging to the rectangle is a corner site. The action a_0 represents the action of flipping a spin at a site that is diagonally adjacent to the rectangle. The action $\tilde{a}$ corresponds to the action of flipping the spin at a corner site of the rectangle. The action 0 represents flipping the spin at any other site or doing nothing. We refer to Figure 2 for a visualization of the action space.

Recall that the reward function of the auxiliary process is given by

$$\hat{r}(s,a) = \mathbb{1}\{s = (N,N)\}$$

for all $s \in \hat{S}$, $a \in \hat{A}$.

Finally, we define the transition probability kernel $\hat{P} : \hat{S} \times \hat{A} \times \hat{S} \to [0,1]$, which is based on the the transition probability kernel $Q_{\infty,k} : S \times A \times S \to [0,1]$ given by expression (S2). To this end, we first provide some preliminary results. The following statement asserts that for any configuration $\sigma \in U^{(1)}$, action $a \in A$ and state $(i',j') \in \hat{S}$, there is at most one robust downhill configuration of σ^a that corresponds to state (i',j').

Lemma S3. *For any configuration $\sigma \in U^{(1)}$, action $a \in A$ and state $(i',j') \in \hat{S}$, the set $U^{(1)}(\sigma^a) \cap \mathcal{I}^{-1}((i',j'))$ is either empty or a singleton.*

Proof. Let $\sigma \in U^{(1)}$ and suppose that $\mathcal{I}(\sigma) = (i,j)$. Consider the case that $\mathcal{J}_\sigma(a) = 0$. This implies that $a \in \{0\} \cup [V_+(\sigma) \setminus C(\sigma)] \cup \{i \in \Lambda | \tilde{\delta}(a, V_+(\sigma)) > 2\}$. Suppose first that $a = 0$. In this case, we have $U^{(1)}(\sigma^a) \cap \mathcal{I}^{-1}((i',j')) = \{\sigma\}$ if $(i',j') = (i,j)$ and $U^{(1)}(\sigma^a) \cap \mathcal{I}^{-1}((i',j')) = \emptyset$ otherwise. Suppose now that either $a \in V_+(\sigma) \setminus C(\sigma)$ or a satisfies $\tilde{\delta}(a, V_+(\sigma)) > 2$. In both cases, the only downhill path starting at configuration σ^a is (σ^a, σ), so we again obtain $U^{(1)}(\sigma^a) \cap \mathcal{I}^{-1}((i',j')) = \{\sigma\}$ if $(i',j') = (i,j)$ and $U^{(1)}(\sigma^a) \cap \mathcal{I}^{-1}((i',j')) = \emptyset$ otherwise.

We proceed to consider the case $\mathcal{J}_\sigma(a) = \tilde{a}$. If $i,j \ge 3$, then the only downhill path starting at configuration σ^a is (σ^a, σ), which again yields $U^{(1)}(\sigma^a) \cap \mathcal{I}^{-1}((i',j')) = \{\sigma\}$ if $(i',j') = (i,j)$ and $U^{(1)}(\sigma^a) \cap \mathcal{I}^{-1}((i',j')) = \emptyset$

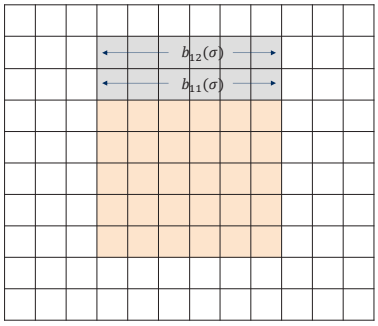


Figure S3: Sets $b_{11}(\sigma)$ and $b_{12}(\sigma)$.

otherwise. Suppose now that $a \in C(\sigma)$ and either $i = 2$, $j > 2$ or $i > 2$, $j = 2$. Without loss of generality, we consider the case $i = 2$ and $j > 2$. In this situation, σ^a has two susceptible sites, namely a itself and the unique horizontal neighbor a' of a that has spin $+1$. Flipping the spin at site a leads back to configuration σ, whereas flipping the spin at site a' leads to the robust configuration $\sigma^{\{a,a'\}}$, which satisfies $\mathcal{I}(\sigma^{\{a,a'\}}) = (i, j-1)$. Hence, we obtain

$$U^{(1)}(\sigma^a) \cap \mathcal{I}^{-1}((i',j')) = \begin{cases} \{\sigma\}, & \text{if } (i',j') = (i,j), \\ \{\sigma^{\{a,a'\}}\}, & \text{if } (i',j') = (i,j-1), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Suppose that $i = j = 2$. In this case, σ^a has three susceptible sites, namely a, the unique horizontal neighbor a' of a that has spin $+1$ and the unique vertical neighbor a'' of a that has spin $+1$. If either the spin at site a' or the spin at site a'' flips first from σ^a, any closed downhill path will end in the all-minus configuration σ^-, which corresponds to state $(0,0)$. On the other hand, flipping the spin at site a first leads back to configuration σ. Thus, we have

$$U^{(1)}(\sigma^a) \cap \mathcal{I}^{-1}((i',j')) = \begin{cases} \{\sigma\}, & \text{if } (i',j') = (i,j), \\ \{\sigma^-\}, & \text{if } (i',j') = (0,0), \\ \emptyset, & \text{otherwise.} \end{cases}$$

Finally, consider the situation $\mathcal{J}_\sigma(a) \neq 0$. In this case, the only sites whose spins can flip on a downhill path starting at configuration σ^a are those in the set $R(\sigma^a) \setminus V_+(\sigma)$, i.e., the sites in the circumscribed rectangle of the set of sites with spin $+1$ in σ^a which are not part of the rectangle in configuration σ. We consider $\mathcal{J}_\sigma(a) = a_{12}$. The remaining cases can be dealt with in a similar way. Let $b_{11}(\sigma)$ and $b_{12}(\sigma)$ denote the horizontal sets of sites lying at distance 1 and 2 from the rectangle in σ respectively, as depicted in Figure S3.

We have $R(\sigma^a) \setminus V_+(\sigma) = b_{11}(\sigma) \cup b_{12}(\sigma)$, so the only sites whose spins can flip on a downhill path starting at configuration σ^a are those in the set $b_{11}(\sigma) \cup b_{12}(\sigma)$. This implies that the only robust endpoints of downhill paths starting at σ^a are the configurations σ itself, $\sigma^{b_{11}(\sigma)}$ and $\sigma^{b_{11}(\sigma) \cup b_{12}(\sigma)}$. Thus, we obtain $U^{(1)}(\sigma^a) \cap \mathcal{I}^{-1}((i',j')) = \{\sigma\}$ if $(i',j') = (i,j)$, $U^{(1)}(\sigma^a) \cap \mathcal{I}^{-1}((i',j')) = \{\sigma^{b_{11}(\sigma)}\}$ if $(i',j') = (i,j+1)$, $U^{(1)}(\sigma^a) \cap \mathcal{I}^{-1}((i',j')) = \{\sigma^{b_{11}(\sigma) \cup b_{12}(\sigma)}\}$ if $(i',j') = (i,j+2)$ and $U^{(1)}(\sigma^a) \cap \mathcal{I}^{-1}((i',j')) = \emptyset$ otherwise. Analogous results for $\mathcal{J}_\sigma(a) \in \{a_{11}, a_{21}, a_{22}, a'_{11}, a'_{12}, a'_{21}, a'_{22}, a_0\}$ can be obtained in a similar way. It follows that the set $U^{(1)}(\sigma^a) \cap \mathcal{I}^{-1}((i',j'))$ is either empty or a singleton for any $\sigma \in U^{(1)}$ and $a \in A$. □

If the set $U^{(1)}(\sigma^a_{(i,j)}) \cap \mathcal{I}^{-1}((i',j'))$, for $(i,j), (i',j') \in \hat{S}$ and $a \in A$, is nonempty, let the unique configuration in this set be denoted by $\eta_{\{(i,j),a,(i',j')\}}$.

Lemma S4. *Let $(i,j), (i',j') \in \hat{S}$ and $a, a' \in A$. If $\mathcal{J}_{\sigma_{(i,j)}}(a) = \mathcal{J}_{\sigma_{(i,j)}}(a')$, then*

$$\lim_{k\to\infty} Q_{\infty,k}(\eta_{\{(i,j),a,(i',j')\}} | \sigma_{(i,j)}, a) = \lim_{k\to\infty} Q_{\infty,k}(\eta_{\{(i,j),a',(i',j')\}} | \sigma_{(i,j)}, a').$$

Proof. By symmetry, the statement is obvious for

$$\mathcal{J}_{\sigma_{(i,j)}}(a) = \mathcal{J}_{\sigma_{(i,j)}}(a') \in \{a'_{11}, a'_{12}, a'_{21}, a'_{22}, \tilde{a}, a_0\}.$$

We consider $\mathcal{J}_{\sigma_{(i,j)}}(a) = \mathcal{J}_{\sigma_{(i,j)}}(a') = a_{12}$. The result for the remaining cases can be obtained in a very similar way. Let the site a be labeled as 1, let its vertical neighbor that is adjacent to the rectangle be labeled as 2 and let the

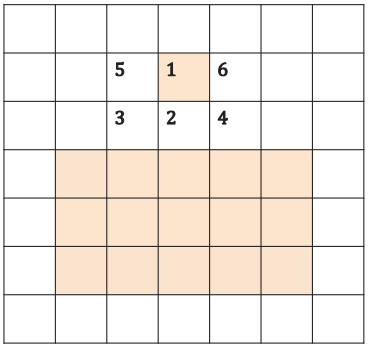


Figure S4: Post-decision configuration $\sigma^a_{(i,j)}$.

horizontal neighbors of sites 1 and 2 be labeled as 5, 6 and 3, 4 respectively, as illustrated in Figure S4. Let $b_{11}(\sigma_{(i,j)})$ and $b_{12}(\sigma_{(i,j)})$ again denote the horizontal sets lying at distance 1 and 2 from the rectangle in $\sigma_{(i,j)}$ respectively, as depicted in Figure S3. Observe first of all that once two adjacent sites in one of these sets reach spin $+1$, the spins at these sites can no longer flip to -1 on any downhill path. Furthermore, note that all remaining sites in this set will inevitably have spin $+1$ in any robust configuration at the end of such a downhill path. The only way to gain two $+$-spins in set $b_{11}(\sigma_{(i,j)})$ on a downhill path from configuration $\sigma^a_{(i,j)}$ is if the spin at site 2 flips first, followed by either the spin at site 3 or the spin at site 4. The only way for set $b_{12}(\sigma_{(i,j)})$ to gain two $+$-spins on a downhill path from configuration $\sigma^a_{(i,j)}$ is through flipping either the spin at site 5 or at site 6, which is only possible if the spin at site 3 or 4 respectively has flipped to $+1$ and the spin at site 1 has not flipped back in the meantime. It follows from these arguments that the robust configuration at the end of a closed downhill path is fully determined by the evolution of the spins at sites in the set $W = \{1, 2, 3, 4, 5, 6\}$. That is, for any two closed downhill sequences $\mathbf{x}, \mathbf{y} \in \tilde{\Theta}(\sigma^a_{(i,j)})$ that satisfy $\mathbf{x}^W = \mathbf{y}^W$, we have $(\sigma^a_{(i,j)})^{\mathbf{x}} = (\sigma^a_{(i,j)})^{\mathbf{y}}$. In addition, the robust configuration we reach after taking action a only depends on the rectangle through the sites that are adjacent to sites 2, 3 and 4. Since the configuration of the set of sites labeled 2, 3 and 4 is the same up to translation for each $a \in A$ that satisfies $\mathcal{J}_{\sigma_{(i,j)}}(a) = a_{12}$, we can conclude that

$$\lim_{k\to\infty} Q_{\infty,k}(\eta_{\{(i,j),a,(i',j')\}}|\sigma_{(i,j)}, a) = \lim_{k\to\infty} Q_{\infty,k}(\eta_{\{(i,j),a',(i',j')\}}|\sigma_{(i,j)}, a'),$$

if $\mathcal{J}_{\sigma_{(i,j)}}(a) = \mathcal{J}_{\sigma_{(i,j)}}(a') = a_{12}$. □

We now define the transition probability kernel $\hat{P} : \hat{S} \times \hat{A} \times \hat{S} \to [0, 1]$ as

$$\hat{P}((i',j')|(i,j),\hat{a}) = \begin{cases} \lim_{k\to\infty} Q_{\infty,k}(\eta_{\{(i,j),a,(i',j')\}}|\sigma_{(i,j)}, a), & \text{if } U^{(1)}(\sigma^a_{(i,j)}) \cap \mathcal{I}^{-1}((i',j')) \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases} \tag{S7}$$

for $(i,j), (i',j') \in \hat{S}$, $\hat{a} \in \hat{A}_{(i,j)}$, where $a \in \mathcal{J}^{-1}_{\sigma_{(i,j)}}(\hat{a})$. By Lemma S4, each choice of $a \in \mathcal{J}^{-1}_{\sigma_{(i,j)}}(\hat{a})$ yields the same result and therefore expression (S7) is well-defined.

The following result provides a way to compute the probabilities $\hat{P}((i',j')|(i,j),\hat{a})$ explicitly.

Lemma S5. *Consider $(i,j), (i',j') \in \hat{S}$ and $a \in A$ such that $U^{(1)}(\sigma^a_{(i,j)}) \cap \mathcal{I}^{-1}((i',j')) \neq \emptyset$. Then,*

$$\lim_{k\to\infty} Q_{\infty,k}(\eta_{\{(i,j),a,(i',j')\}}|\sigma_{(i,j)}, a) = \begin{cases} 1, & \text{if } \sigma^a_{(i,j)} = \eta_{\{(i,j),a,(i',j')\}}, \\ \sum_{\omega\in\Omega(\sigma^a_{(i,j)},\eta_{\{(i,j),a,(i',j')\}})} \prod_{\ell=0}^{|\omega|-1} \frac{1}{|\Delta(\omega_\ell)|}, & \text{otherwise.} \end{cases} \tag{S8}$$

Proof. Note that a configuration $\sigma \in S$ is an absorbing state in the process that evolves according to this dynamics if and only if it is robust. We define

$$\hat{p}(\sigma,\sigma') = \lim_{k\to\infty} p^{(k)}_\infty(\sigma,\sigma'), \quad \sigma,\sigma' \in S.$$

It follows that $\hat{p}(\sigma,\sigma') > 0$ if and only if the configuration σ' is robust. We first show that for any $\sigma,\sigma' \in S$ that satisfy $\sigma' \in U(\sigma)$ and $\sigma \neq \sigma'$, we have

$$\hat{p}(\sigma,\sigma') = \sum_{\omega\in\Omega(\sigma,\sigma')} \prod_{\ell=0}^{|\omega|-1} \frac{1}{|\Delta(\omega_\ell)|}. \tag{S9}$$

Recall that $p_\infty(\sigma,\sigma^i) = 0$ for all $i \notin \Delta(\sigma)$ and $p_\infty(\sigma,\sigma^i) = 1/N^2$ for all $i \in \Delta(\sigma)$. Conditioning on the first time a susceptible site, if any, is selected by the Metropolis dynamics during the adjustment period yields

$$p_\infty^{(k)}(\sigma,\sigma') = \sum_{t=1}^{k} \left(1 - \frac{|\Delta(\sigma)|}{N^2}\right)^{t-1} \frac{|\Delta(\sigma)|}{N^2} \sum_{i\in\Delta(\sigma)} \frac{1}{|\Delta(\sigma)|} p_\infty^{(k-t)}(\sigma^i,\sigma').$$

Taking the limit $k\to\infty$, we now obtain

$$\hat{p}(\sigma,\sigma') = \sum_{t=1}^{\infty} \left(1 - \frac{|\Delta(\sigma)|}{N^2}\right)^{t-1} \frac{|\Delta(\sigma)|}{N^2} \sum_{i\in\Delta(\sigma)} \frac{1}{|\Delta(\sigma)|} \hat{p}(\sigma^i,\sigma') = \sum_{i\in\Delta(\sigma)} \frac{1}{|\Delta(\sigma)|} \hat{p}(\sigma^i,\sigma').$$

We show that this implies the validity of expression (S9) by induction over the length $L(\sigma,\sigma')$ of the longest downhill path that leads from σ to σ', i.e., $L(\sigma,\sigma') = \max_{\omega\in\Omega(\sigma,\sigma')} |\omega| - 1$. Note that $\Omega(\sigma,\sigma') \neq \emptyset$ and $L(\sigma,\sigma') \geq 1$, since $\sigma' \in U(\sigma)$ and $\sigma \neq \sigma'$. For $L = 1$, we obtain

$$\hat{p}(\sigma,\sigma') = \sum_{i\in\Delta(\sigma)} \frac{1}{|\Delta(\sigma)|} \hat{p}(\sigma^i,\sigma') = \sum_{\substack{i\in\Delta(\sigma)\\ \sigma^i=\sigma'}} \frac{1}{|\Delta(\sigma)|} = \sum_{\omega\in\Omega(\sigma,\sigma')} \prod_{\ell=0}^{|\omega|-1} \frac{1}{|\Delta(\omega_\ell)|}.$$

Now, assume that expression (S9) holds if $L \leq n$. For $L = n+1$, we then obtain

$$\hat{p}(\sigma,\sigma') = \sum_{i\in\Delta(\sigma)} \frac{1}{|\Delta(\sigma)|} \hat{p}(\sigma^i,\sigma').$$

If $\sigma^i = \sigma'$, for $i \in \Delta(\sigma)$, then $\hat{p}(\sigma^i,\sigma') = 1$. On the other hand, if $\sigma^i \neq \sigma'$ for $i \in \Delta(\sigma)$, then either $\sigma' \notin U(\sigma^i)$ or $\sigma' \in U(\sigma^i)$ and $L(\sigma^i,\sigma') \leq n$. In the first case, we have $\hat{p}(\sigma^i,\sigma') = 0$. In the second case, the induction hypothesis implies that

$$\hat{p}(\sigma^i,\sigma') = \sum_{\omega\in\Omega(\sigma^i,\sigma')} \prod_{\ell=0}^{|\omega|-1} \frac{1}{|\Delta(\omega_\ell)|}.$$

Thus, we obtain

$$\hat{p}(\sigma,\sigma') = \sum_{\substack{i\in\Delta(\sigma),\\ \sigma^i=\sigma'}} \frac{1}{|\Delta(\sigma)|} + \sum_{\substack{i\in\Delta(\sigma)\\ \sigma^i\neq\sigma'\\ \sigma'\in U(\sigma^i)}} \frac{1}{|\Delta(\sigma)|} \sum_{\omega\in\Omega(\sigma^i,\sigma')} \prod_{\ell=0}^{|\omega|-1} \frac{1}{|\Delta(\omega_\ell)|} = \sum_{\omega\in\Omega(\sigma,\sigma')} \prod_{\ell=0}^{|\omega|-1} \frac{1}{|\Delta(\omega_\ell)|}.$$

Now, consider $(i,j),(i',j') \in \hat{S}$ and $a \in A$ such that $U^{(1)}(\sigma^a_{(i,j)}) \cap \mathcal{I}^{-1}((i',j')) \neq \emptyset$. By definition of the transition probability kernel $Q_{\infty,k}$, we have

$$\lim_{k\to\infty} Q_{\infty,k}(\eta_{\{(i,j),a,(i',j')\}}|\sigma_{(i,j)},a) = \hat{p}(\sigma^a_{(i,j)},\eta_{\{(i,j),a,(i',j')\}}). \tag{S10}$$

First, suppose that $\eta_{\{(i,j),a,(i',j')\}} = \sigma^a_{(i,j)}$. Since $\eta_{\{(i,j),a,(i',j')\}}$ is a robust configuration, it immediately follows that

$$\lim_{k\to\infty} Q_{\infty,k}(\eta_{\{(i,j),a,(i',j')\}}|\sigma_{(i,j)},a) = 1.$$

Secondly, suppose that $\eta_{\{(i,j),a,(i',j')\}} \neq \sigma^a_{(i,j)}$. It now follows from expression (S9) that

$$\lim_{k\to\infty} Q_{\infty,k}(\eta_{\{(i,j),a,(i',j')\}}|\sigma_{(i,j)},a) = \sum_{\omega\in\Omega(\sigma^a_{(i,j)},\eta_{\{(i,j),a,(i',j')\}})} \prod_{\ell=0}^{|\omega|-1} \frac{1}{|\Delta(\omega_\ell)|},$$

establishing expression (S8). □

Lemma S6 now provides the explicit expressions for the transition probabilities $\hat{P}((i',j')|(i,j),\hat{a})$ for each $(i,j),(i',j') \in \hat{S}$, $\hat{a} \in \hat{A}_{(i,j)}$.

Lemma S6. *The transition probability kernel $\hat{P} : \hat{S} \times \hat{A} \times \hat{S} \to [0,1]$ is given by*

$$\hat{P}((i',j')|(i,j),a_{11}) = \begin{cases} 1/3, & \text{if } i'=i, j'=j, \\ 2/3, & \text{if } i'=i, j'=j+1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } \begin{matrix} i=3,\dots,N-2,N, \\ j=2,\dots,N-3, \end{matrix} \tag{S11}$$

$$\hat{P}((i',j')|(i,j),a_{21}) = \begin{cases} 1/3, & \text{if } i'=i, j'=j, \\ 2/3, & \text{if } i'=i+1, j'=j, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } \begin{matrix} i=2,\dots,N-3, \\ j=3,\dots,N-2,N, \end{matrix} \tag{S12}$$

$$\hat{P}((i',j')|(i,j),a_{12}) = \begin{cases} 5/9, & \text{if } i'=i, j'=j, \\ 7/27, & \text{if } i'=i, j'=j+1, \\ 5/27, & \text{if } i'=i, j'=j+2, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } \begin{matrix} i=3,\dots,N-2,N, \\ j=2,\dots,N-4, \end{matrix} \tag{S13}$$

$$\hat{P}((i',j')|(i,j),a_{22}) = \begin{cases} 5/9, & \text{if } i'=i, j'=j, \\ 7/27, & \text{if } i'=i+1, j'=j, \\ 5/27, & \text{if } i'=i+2, j'=j, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } \begin{matrix} i=2,\dots,N-4, \\ j=3,\dots,N-2,N, \end{matrix} \tag{S14}$$

$$\hat{P}((i',j')|(i,j),a_{0}) = \begin{cases} 4/9, & \text{if } i'=i, j'=j, \\ 1/9, & \text{if } i'=i+1, j'=j \\ & \text{or } i'=i, j'=j+1, \\ 1/3, & \text{if } i'=i+1, j'=j+1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } \begin{matrix} i=2,\dots,N-3, \\ j=2,\dots,N-3, \end{matrix} \tag{S15}$$

$$\hat{P}((i',j')|(i,j),a'_{11}) = \begin{cases} 1/2, & \text{if } i'=i, j'=j, \\ 1/2, & \text{if } i'=i, j'=j+1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } \begin{matrix} i=2,\dots,N-2, \\ j=2,\dots,N-3, \end{matrix} \tag{S16}$$

$$\hat{P}((i',j')|(i,j),a'_{21}) = \begin{cases} 1/2, & \text{if } i'=i, j'=j, \\ 1/2, & \text{if } i'=i+1, j'=j, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } \begin{matrix} i=2,\dots,N-3, \\ j=2,\dots,N-2, \end{matrix} \tag{S17}$$

$$\hat{P}((i',j')|(i,j),a'_{12}) = \begin{cases} 5/8, & \text{if } i'=i, j'=j, \\ 1/4, & \text{if } i'=i, j'=j+1, \\ 1/8, & \text{if } i'=i, j'=j+2, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } \begin{matrix} i=2,\dots,N-2, \\ j=2,\dots,N-4, \end{matrix} \tag{S18}$$

$$\hat{P}((i',j')|(i,j),a'_{22}) = \begin{cases} 5/8, & \text{if } i'=i, j'=j, \\ 1/4, & \text{if } i'=i+1, j'=j, \\ 1/8, & \text{if } i'=i+2, j'=j, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } \begin{matrix} i=2,\dots,N-4, \\ j=2,\dots,N-2, \end{matrix} \tag{S19}$$

$$\hat{P}((i',j')|(i,N-2),a_{11}) = \begin{cases} 1/4, & \text{if } i'=i, j'=N-2, \\ 3/4, & \text{if } i'=i, j'=N, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } i=3,\dots,N-2,N, \tag{S20}$$

$$\hat{P}((i',j')|(N-2,j),a_{21}) = \begin{cases} 1/4, & \text{if } i'=N-2, j'=j, \\ 3/4, & \text{if } i'=N, j'=j, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } j=3,\dots,N-2,N, \tag{S21}$$

$$\hat{P}((i',j')|(i,N-3),a_{12}) = \begin{cases} 7/18, & \text{if } i'=i, j'=N-3, \\ 31/144, & \text{if } i'=i, j'=N-2 \\ 19/48, & \text{if } i'=i, j'=N, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } i=3,\dots,N-2,N, \tag{S22}$$

$$\hat{P}((i',j')|(N-3,j),a_{22}) = \begin{cases} 7/18, & \text{if } i'=N-3, j'=j, \\ 31/144, & \text{if } i'=N-2, j'=j, \\ 19/48, & \text{if } i'=N, j'=j, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } j=3,\dots,N-2,N, \tag{S23}$$

$$\hat{P}((i',j')|(i,N-2),a_{0}) = \begin{cases} 5/12, & \text{if } i'=i, j'=j, \\ 1/8, & \text{if } i'=i, j'=N, \\ 1/9, & \text{if } i'=i+1, j'=j, \\ 25/72, & \text{if } i'=i+1, j'=N, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } i=2,\dots,N-3, \tag{S24}$$

$$\hat{P}((i',j')|(N-2,j),a_{0}) = \begin{cases} 5/12, & \text{if } i'=i, j'=j, \\ 1/8, & \text{if } i'=N, j'=j, \\ 1/9, & \text{if } i'=i, j'=j+1, \\ 25/72, & \text{if } i'=N, j'=j+1, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } j=2,\dots,N-3, \tag{S25}$$

$$\hat{P}((i',j')|(N-2,N-2),a_{0}) = \begin{cases} 7/18, & \text{if } i'=j'=N-2, \\ 1/8, & \text{if } i'=N, j'=N-2 \\ & \text{or } i'=N-2, j'=N, \\ 13/36, & \text{if } i'=j'=N, \\ 0, & \text{otherwise,} \end{cases} \tag{S26}$$

$$\hat{P}((i',j')|(i,N-2),a'_{11}) = \begin{cases} 1/3, & \text{if } i'=i, j'=N-2, \\ 2/3, & \text{if } i'=i, j'=N, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } i=2,\dots,N-2, \tag{S27}$$

$$\hat{P}((i',j')|(N-2,j),a'_{21}) = \begin{cases} 1/3, & \text{if } i'=N-2, j'=j, \\ 2/3, & \text{if } i'=N, j'=j, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } j=2,\dots,N-2, \tag{S28}$$

$$\hat{P}((i',j')|(i,N-3),a'_{12}) = \begin{cases} 4/9, & \text{if } i'=i, j'=N-3, \\ 5/27, & \text{if } i'=i, j'=N-2, \\ 10/27, & \text{if } i'=i, j'=N, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } i=2,\dots,N-2, \tag{S29}$$

$$\hat{P}((i',j')|(N-3,j),a'_{22}) = \begin{cases} 4/9, & \text{if } i'=N-3, j'=j, \\ 5/27, & \text{if } i'=N-2, j'=j, \\ 10/27, & \text{if } i'=N, j'=j, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } j=2,\dots,N-2, \tag{S30}$$

$$\hat{P}((i',j')|(i,j),\tilde{a}) = \begin{cases} 1, & \text{if } (i',j')=(i,j), \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } i,j=3,\dots,N-2, \tag{S31}$$

$$\hat{P}((i',j')|(2,j),\tilde{a}) = \begin{cases} 1/2, & \text{if } (i',j')=(2,j), \\ 1/2, & \text{if } (i',j')=(2,j-1), \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } j=3,\dots,N-2, \tag{S32}$$

$$\hat{P}((i',j')|(i,2),\tilde{a}) = \begin{cases} 1/2, & \text{if } (i',j')=(i,2), \\ 1/2, & \text{if } (i',j')=(i-1,2), \\ 0, & \text{otherwise,} \end{cases} \quad \text{for } i=3,\dots,N-2, \tag{S33}$$

$$\hat{P}((i',j')|(2,2),\tilde{a}) = \begin{cases} 1/3, & \text{if } (i',j')=(2,2), \\ 2/3, & \text{if } (i',j')=(0,0), \\ 0, & \text{otherwise,} \end{cases} \quad \text{and} \tag{S34}$$

$$\hat{P}((i,j)|(i,j),0) = 1, \quad \text{for all } (i,j)\in\hat{S}. \tag{S35}$$

Proof. We give the proof of expression (S13). Let $(i,j)\in\hat{S}$, $i=3,\dots,N$, $j=2,\dots,N-4$ and let $a\in\mathcal{J}^{-1}_{\sigma_{(i,j)}}(a_{12})$. Consider again Figure S4. First, the arguments laid out in the proof of Lemma S4 imply that $U^{(1)}(\sigma^a_{(i,j)})\cap\mathcal{I}^{-1}((i',j'))\neq\emptyset$ if and only if $(i',j')=(i,j)$, $(i',j')=(i,j+1)$ or $(i',j')=(i,j+2)$. Hence, $\hat{P}((i',j')|(i,j),a_{12})=0$ if $(i',j')\notin\{(i,j),(i,j+1),(i,j+2)\}$.

Now, suppose that $(i',j')\in\{(i,j),(i,j+1),(i,j+2)\}$. The proof of Lemma S4 implies that the transition probability $\hat{P}((i',j')|(i,j),a_{12})$ depends only on the evolution of the site whose spin was flipped, denoted by a, its vertical neighbor that is adjacent to the rectangle, the horizontal neighbors of the latter site and the horizontal neighbors of a itself, labeled 1, 2, 3, 4, 5 and 6 respectively, as illustrated in Figure S4. Letting $W=\{1,2,3,4,5,6\}$, we showed that for any two closed downhill sequences $\mathbf{x},\mathbf{y}\in\tilde{\Theta}(\sigma^a_{(i,j)})$ that satisfy $\mathbf{x}^W=\mathbf{y}^W$, we have $(\sigma^a_{(i,j)})^{\mathbf{x}}=(\sigma^a_{(i,j)})^{\mathbf{y}}$. We now proceed to show that the susceptibility of the sites in the set $W=\{1,2,3,4,5,6\}$ is not affected by the evolution of the configuration on the remainder of the lattice, i.e., that for any downhill sequence $\mathbf{x}\in\Theta(\sigma)$, we have $\Delta((\sigma^a_{(i,j)})^{\mathbf{x}})\cap W=\Delta((\sigma^a_{(i,j)})^{\mathbf{x}^W})\cap W$. Consider the sequence $\mathbf{y}=(2)$ and let $\mathbf{x}\in\Theta(\sigma)$ be any sequence that satisfies $\mathbf{x}^W=\mathbf{y}$. In configuration $\sigma^a_{(i,j)}$, the only susceptible sites are 1 and 2. This implies that the first element of $\mathbf{x}$ is site 2. After flipping the spin at site 2, the set of susceptible sites is $\{1,3,4\}$. Therefore, we have $\mathbf{x}=\mathbf{y}=(2)$ and thus $\Delta((\sigma^a_{(i,j)})^{\mathbf{x}})\cap W=\Delta((\sigma^a_{(i,j)})^{\mathbf{x}^W})\cap W$. Now consider the sequence $\mathbf{y}=(2,4)$ and let $\mathbf{x}\in\Theta(\sigma^a_{(i,j)})$ again be any sequence that satisfies $\mathbf{x}^W=\mathbf{y}$. The same argument as before implies that the first two entries of $\mathbf{x}$ are the sites 2 and 4. At this point, the only sites outside of W whose spins can flip on any downhill path are the sites in the set $b_{11}(\sigma_{(i,j)})\setminus W$, where $b_{11}(\sigma_{(i,j)})$ is the horizontal set lying at distance 1 from the rectangle in $\sigma_{(i,j)}$, as depicted in Figure S3. Observe, however, that flipping the spin at any site in this set leaves the set of susceptible sites that are part of W, namely $\{1,6\}$, unaffected. We thus again obtain $\Delta((\sigma^a_{(i,j)})^{\mathbf{x}})\cap W=\Delta((\sigma^a_{(i,j)})^{\mathbf{x}^W})\cap W$. This argument can easily be extended to the sequence $\mathbf{y}=(2,4,1)$. Since after flipping the spins at the sites in this sequence, the spins

Table S1: Derivation of expression (S13), corresponding to Figure S4. Similar derivation leads to expression (S14).

Sequences of susceptible sites	Probability of selecting sequence	Next state	Transition probability
(1)	$1/2$	(i,j)	$5/9$
$(2,1,2)$	$1/18$		
$(2,1,3)$	$1/18$	$(i,j+1)$	$7/27$
$(2,1,4)$	$1/18$		
$(2,3,1)$	$1/18$		
$(2,3,4,1)$	$1/54$		
$(2,4,1)$	$1/18$		
$(2,4,3,1)$	$1/54$		
$(2,3,4,5)$	$1/54$	$(i,j+2)$	$5/27$
$(2,3,4,6)$	$1/54$		
$(2,4,3,5)$	$1/54$		
$(2,4,3,6)$	$1/54$		
$(2,3,5)$	$1/18$		
$(2,4,6)$	$1/18$		

at sites 2 and 4 can no longer flip back on any downhill path and the only sites that will become susceptible on any downhill path are those in the set $b_{11}(\sigma_{(i,j)})$, it follows that any closed downhill sequence $\mathbf{x} \in \tilde{\Theta}(\sigma^a_{(i,j)})$ that satisfies $\mathbf{x}^W = \mathbf{y}$ leads to the robust configuration $\sigma^{b_{11}(\sigma_{(i,j)})}_{(i,j)}$. In Table S1 we listed the restrictions to W of all sequences in the set $\tilde{\Theta}(\sigma^a_{(i,j)})$. The same argument as above can be applied to any of the sequences in this table. We can thus conclude that for any downhill sequence $\mathbf{x} \in \Theta(\sigma)$, we have $\Delta((\sigma^a_{(i,j)})^{\mathbf{x}}) \cap\ W = \Delta((\sigma^a_{(i,j)})^{\mathbf{x}^W}) \cap W$. It follows that this set is essential for the configuration $\sigma^a_{(i,j)}$. Thus, to compute $\hat{P}((i',j')|(i,j),a_{12})$ it suffices to focus on the restrictions of the closed downhill sequences to W that lead to the robust configuration $\eta_{\{(i,j),a,(i',j')\}}$. Let the set of these sequences be denoted by $\tilde{\Theta}^W(\sigma^a_{(i,j)}, \eta_{\{(i,j),a,(i',j')\}})$. To keep notation simple in the computation that follows, we abbreviate $\sigma^a_{(i,j)}$ and $\eta_{\{(i,j),a,(i',j')\}}$ as σ^a and η. By the properties of the essential set, it follows that

$$\begin{aligned}\sum_{\omega\in\tilde{\Omega}(\sigma^a,\eta)} \prod_{\ell=0}^{|\omega|-1} \frac{1}{|\Delta(\omega_\ell)|} &= \sum_{\mathbf{x}\in\tilde{\Theta}(\sigma^a,\eta)} \frac{1}{|\Delta(\sigma^a)|} \prod_{\ell=1}^{|\mathbf{x}|-1} \frac{1}{|\Delta((\sigma^a)^{(x_1,\ldots,x_\ell)})|} \\ &= \sum_{\mathbf{y}\in\tilde{\Theta}^W(\sigma^a,\eta)} \frac{1}{|\Delta(\sigma^a)\cap W|} \prod_{\ell=1}^{|\mathbf{y}|-1} \frac{1}{|\Delta((\sigma^a)^{(y_1,\ldots,y_\ell)})\cap W|}.\end{aligned} \tag{S36}$$

For each sequence in $\mathbf{y} \in \tilde{\Theta}^W(\sigma^a, \eta)$, Table S1 gives the probability that this sequence is selected, i.e., the quantity

$$\frac{1}{|\Delta(\sigma^a \cap W)|} \prod_{\ell=1}^{|\mathbf{y}|-1} \frac{1}{|\Delta((\sigma^a)^{(y_1,\ldots,y_\ell)}) \cap W|},$$

and the state that corresponds to the robust configuration that the sequence leads to. Using Table S1 to evaluate expression (S36), we obtain expression (S13).

The remaining transition probabilities can be computed in a similar way. Supporting figures depicting the essential sets and tables listing the relevant sequences of susceptible sites are provided in Supplementary Material II. □

S.I.2.2 Proof of Theorem 4

A convenient feature of the auxiliary MDP is the fact that it is not possible for a rectangle of size at least 3x3 to shrink, as by the definition of the transition probability kernel, the probability that a state variable that is greater than 3 decreases in the auxiliary MDP is zero. This fact allows us to solve the Bellman equations recursively, starting from state (N, N), which corresponds to the all-plus configuration. Before doing so, we first show the suboptimality of the actions 0 and $\tilde{a}$, such that we can leave these actions out of consideration when solving the Bellman equations.

Lemma S7. *Any stationary deterministic optimal policy* $\pi^* = (d^*)^\infty$ *satisfies* $d^*(i,j) \neq 0$ *for all* $(i,j) \in \hat{S}$, $(i,j) \notin \{(0,0),(N,N)\}$.

Proof. Suppose that $\pi^* = (d^*)^\infty$ is an optimal policy, with $d^*(i,j) = 0$ for some $(i,j) \in \hat{S}$, $(i,j) \neq (N,N)$. By definition of the transition probabilities and the reward function, this implies $v^*(i,j) = 0$. Now, let $\pi = d^\infty$ denote another policy that satisfies $d(i,j) \neq 0$ for all $(i,j) \neq (0,0)$. Note that the definition of the transition probabilities imply that for any state $(i,j) \neq (0,0)$, there exists a sequence of states $(s_1, s_2, \ldots, s_k)$ for some $k \in \mathbb{N}$ that satisfies $s_1 = (i,j)$, $s_k = (N,N)$ and $\hat{P}(s_{\ell+1}|s_\ell, d(s_\ell)) > 0)$ for all $\ell = 1, \ldots, k-1$. From the definition of the reward function, it now follows that $v^\pi(i,j) > 0$ and thus $v^\pi(i,j) > v^*(i,j)$, which contradicts the optimality of policy π^*. This completes the proof. □

Lemma S8. *Any stationary deterministic optimal policy* $\pi^* = (d^*)^\infty$ *satisfies* $d^*(i,j) \neq \tilde{a}$ *for all* $(i,j) \in \hat{S}$, $i,j \notin \{0,N\}$.

Proof. First consider $s = (i,j) \in \hat{S}$, where either $i,j = 3, \ldots, N-2$ or $(i,j) = (2,2)$. Suppose that $\pi^* = (d^*)^\infty$ is an optimal policy that satisfies $d^*(s) = \tilde{a}$. By expressions (S31), (S34) and (5), this implies $v^*(s) = 0$. As shown in the proof of Lemma S7, there exists a policy $\pi = d^\infty$ that satisfies $v^\pi(i,j) > 0 = v^*(i,j)$, which contradicts the optimality of π^*. Hence $d^*(s) \neq \tilde{a}$.

We proceed to prove the statement for states $(i,j) \in \hat{S}$, where either $i = 2$ and $j = 3, \ldots, N-2$ or $i = 3, \ldots, N-2$ and $j = 2$. Suppose that $\pi^* = (d^*)^\infty$ is an optimal policy that satisfies $d^*(2,j) = \tilde{a}$ for some $j = 3, \ldots, N-2$. Let j^* denote the largest such j for which this holds. We assume that $3 < j^* < N-2$. The argument can easily be extended to $j^* = 3$ and $j^* = N-2$. Note that rectangles of size (i,j), where $i,j = 3, \ldots, N-2, N$ or $i = 2$, $j = j^*+1, \ldots, N-2, N$, cannot shrink under policy π^*. In addition, by the definition of the reward function, the optimality of π^* and the result of Lemma S7, we obtain $v^*(i,j) < v^*(i+1,j)$ and $v^*(i,j) < v^*(i,j+1)$ for $i,j = 3, \ldots, N-2, N$ or $i = 2$, $j = j^*+1, \ldots, N-2, N$. We proceed to show that at least one of the following must hold:

1. $d^*(2, j^*-1) \in \{a'_{12}, a_{21}, a'_{21}, a_{22}, a'_{22}, a_0\}$,
2. $d^*(2,j) \in \{a_{21}, a'_{21}, a_{22}, a'_{22}, a_0\}$, for some $j = 3, \ldots, j^*-2$,
3. $d^*(2,2) = a_0$.

If none of these statements holds, we have $d^*(2, j^*-1) \in \{a'_{11}, \tilde{a}\}$, $d^*(2,j) \in \{a'_{11}, a'_{12}, \tilde{a}\}$, $j = 3, \ldots, j^*-2$, and $d^*(2,2) \in \{a'_{11}, a'_{12}, a'_{21}, a'_{22}\}$. It now follows from the definition of the transition probabilities and from symmetry that the only rectangles that can be reached from state $(2, j^*)$ are those in the set $\{(2,j),(j,2)|j = 2, \ldots, j^*\}$. By the definition of the reward function, this implies $v^*(2,j^*) = 0$. Since there exists a policy $\pi = d^\infty$ that satisfies $v^\pi(s) > 0$ for all $s \in S \setminus \{(0,0)\}$, this contradicts the optimality of π^*. Hence, one of the statements $1-3$ must hold. Let j' denote the largest j for which one of these statements is attained. We assume that $j' \in \{3, \ldots, j^*-2\}$ and $d^*(2,j') = a_{21}$. All the remaining scenarios can be handled in a similar way. We construct a policy $\pi = d^\infty$, where $d : \hat{S} \to \hat{A}$ is given by

$$d(i,j) = \begin{cases} a_{21}, & \text{if } i = 2, \quad j = j^*, \\ d^*(i,j), & \text{otherwise.} \end{cases}$$

That is, we replace the action in state $(2,j^*)$ by the action prescribed in state $(2,j')$. Now, let τ denote the first hitting time to the state $(2,j')$ in the process induced by policy π^*, started from state $(2,j^*)$. Note that the state s_t at time t, for $t = 1, 2, \ldots$, satisfies $s_t \in \{(2,j)|j = j'+1, \ldots, j^*\}$ for all $t < \tau$ and that $\tau \geq j^* - j'$ by definition of π^* and the transition probabilities. This implies that $v^*(2,j^*) \leq \lambda^{j^*-j'} v^*(2,j')$. Also, by the fact that $d(i,j) = d^*(i,j)$ for $i,j = 3, \ldots, N-2, N$, and for $i = 2$, $j = j^*+1, \ldots, N-2, N$ and the fact that these rectangles cannot shrink, we have $v^\pi(i,j) = v^*(i,j)$ for all $i,j = 3, \ldots, N-2, N$ and $i = 2$, $j = j^*+1, \ldots, N-2, N$. We now obtain, using expression (S12),

$$v^*(2,j^*) \leq \lambda^{j^*-j'} v^*(2,j') = \lambda^{j^*-j'} \frac{2}{3-\lambda} v^*(3,j'),$$

and

$$v^\pi(2,j^*) = \frac{2}{3-\lambda} v^\pi(3,j^*) = \frac{2}{3-\lambda} v^*(3,j^*).$$

Since $v^*(3,j') < v^*(3,j^*)$, this yields $v^*(2,j^*) < v^\pi(2,j^*)$, contradicting the optimality of π^*. Thus, $d^*(i,j) \neq \tilde{a}$ for all $(i,j) \in \hat{S}$, $i,j \notin \{0,N\}$. □

It follows from Lemmas S7 and S8 that we can leave actions 0 and $\tilde{a}$ out of consideration. The following result implies the suboptimality of actions a'_{11}, a'_{12}, a'_{21} and a'_{22}. For rectangles with side lengths greater than 2, it shows that these actions are always inferior to their counterparts a_{11}, a_{12}, a_{21} and a_{22} respectively. Hence, in finding the optimal policy, when considering a rectangle that has a side of length greater than 2, we leave the corresponding actions in the set $\{a'_{11}, a'_{12}, a'_{21}, a'_{22}\}$ out of consideration.

Lemma S9. *Let $v^* : \hat{S} \to \mathbb{R}$ denote the optimal value function of the auxiliary MDP. For any state $s = (x, y) \in \hat{S}$ with $3 \leq x \leq N-2$ and $y \leq N-2$, we have*

$$\hat{r}(s, a'_{11}) + \lambda \sum_{s' \in \hat{S}} \hat{P}(s'|s, a'_{11}) v^*(s') < \hat{r}(s, a_{11}) + \lambda \sum_{s' \in \hat{S}} \hat{P}(s'|s, a_{11}) v^*(s'), \tag{S37}$$

and for any state $s = (x, y) \in \hat{S}$ with $3 \leq x \leq N-2$ and $y \leq N-3$, we have

$$\hat{r}(s, a'_{12}) + \lambda \sum_{s' \in \hat{S}} \hat{P}(s'|s, a'_{12}) v^*(s') < \hat{r}(s, a_{12}) + \lambda \sum_{s' \in \hat{S}} \hat{P}(s'|s, a_{12}) v^*(s'). \tag{S38}$$

Similarly, for any state $s = (x, y) \in \hat{S}$ with $3 \leq y \leq N-2$ and $x \leq N-2$, we have

$$\hat{r}(s, a'_{21}) + \lambda \sum_{s' \in \hat{S}} \hat{P}(s'|s, a'_{21}) v^*(s') < \hat{r}(s, a_{21}) + \lambda \sum_{s' \in \hat{S}} \hat{P}(s'|s, a_{21}) v^*(s'), \tag{S39}$$

and for any state $s = (x, y) \in \hat{S}$ with $3 \leq y \leq N-2$ and $x \leq N-3$, we have

$$\hat{r}(s, a'_{22}) + \lambda \sum_{s' \in \hat{S}} \hat{P}(s'|s, a'_{22}) v^*(s') < \hat{r}(s, a_{22}) + \lambda \sum_{s' \in \hat{S}} \hat{P}(s'|s, a_{22}) v^*(s'). \tag{S40}$$

Proof. We prove the validity of expressions (S37) and (S38). Expressions (S39) and (S40) follow immediately by symmetry. We first consider expression (S37) for $y = N-2$. The definition of the reward function and expressions (S20) and (S27) yield

$$\hat{r}(s, a'_{11}) + \lambda \sum_{s' \in \hat{S}} \hat{P}(s'|s, a'_{11}) v^*(s') = \frac{\lambda}{3} v^*(x, N-2) + \frac{2\lambda}{3} v^*(x, N),$$
$$\hat{r}(s, a_{11}) + \lambda \sum_{s' \in \hat{S}} \hat{P}(s'|s, a_{11}) v^*(s') = \frac{\lambda}{4} v^*(x, N-2) + \frac{3\lambda}{4} v^*(x, N).$$

The facts that $\hat{r}(x, y) = 1$ if $(x, y) = (N, N)$ and $\hat{r}(x, y) = 0$ otherwise, together with the fact that for each $s \in \hat{S}$, $s \neq (N, N)$, there exists $s' \in \hat{S}$, $s' \neq s$ such that $\hat{P}(s'|s, d(s)) > 0$ if $d(s) \neq 0$ and the fact that rectangles cannot shrink, imply that $v^*(x, y) < v^*(x, y+1)$ for all $3 \leq y \leq N-1$. Therefore,

$$\begin{aligned}\frac{\lambda}{3} v^*(x, N-2) + \frac{2\lambda}{3} v^*(x, N) &= \left(\frac{\lambda}{3} - \frac{\lambda}{12}\right) v^*(x, N-2) + \frac{\lambda}{12} v^*(x, N-2) + \frac{2\lambda}{3} v^*(x, N) \\ &< \frac{\lambda}{4} v^*(x, N-2) + \frac{3\lambda}{4} v^*(x, N).\end{aligned}$$

Now suppose that $y < N-2$. Using the definition of the reward function and expressions (S11) and (S16), we obtain

$$\hat{r}(s, a'_{11}) + \lambda \sum_{s' \in \hat{S}} \hat{P}(s'|s, a'_{11}) v^*(s') = \frac{\lambda}{2} v^*(x, y) + \frac{\lambda}{2} v^*(x, y+1),$$
$$\hat{r}(s, a_{11}) + \lambda \sum_{s' \in \hat{S}} \hat{P}(s'|s, a_{11}) v^*(s') = \frac{\lambda}{3} v^*(x, y) + \frac{2\lambda}{3} v^*(x, y+1).$$

Again invoking the fact that $v^*(x, y) < v^*(x, y+1)$ for all $3 \leq y \leq N-1$, we have

$$\begin{aligned}\frac{\lambda}{2} v^*(x, y) + \frac{\lambda}{2} v^*(x, y+1) &= \left(\frac{\lambda}{2} - \frac{\lambda}{6}\right) v^*(x, y) + \frac{\lambda}{6} v^*(x, y) + \frac{\lambda}{2} v^*(x, y+1) \\ &< \frac{\lambda}{3} v^*(x, y) + \frac{2\lambda}{3} v^*(x, y+1),\end{aligned}$$

which proves expression (S37).

The validity of expression (S38) can be established in a similar way. We first consider the case $y = N - 3$. From the definition of the reward function and expressions (S22) and (S29), it follows that

$$\hat{r}(s, a'_{12}) + \lambda \sum_{s' \in \hat{S}} \hat{P}(s'|s, a'_{12}) v^*(s') = \frac{4\lambda}{9} v^*(x, N-3) + \frac{5\lambda}{27} v^*(x, N-2) + \frac{10\lambda}{27} v^*(x, N)$$

and

$$\hat{r}(s, a_{12}) + \lambda \sum_{s' \in \hat{S}} \hat{P}(s'|s, a_{12}) v^*(s') = \frac{7\lambda}{18} v^*(x, N-3) + \frac{31\lambda}{144} v^*(x, N-2) + \frac{19\lambda}{48} v^*(x, N).$$

Again using the fact that $v^*(x, y) < v^*(x, y+1)$ for all $3 \leq y \leq N-1$, we obtain

$$\begin{aligned}
&\frac{4\lambda}{9} v^*(x, N-3) + \frac{5\lambda}{27} v^*(x, N-2) + \frac{10\lambda}{27} v^*(x, N) \\
&= \frac{4\lambda}{9} v^*(x, N-3) + \left(\frac{5\lambda}{27} - \frac{11\lambda}{432} \right) v^*(x, N-2) + \frac{11\lambda}{432} v^*(x, N-2) + \frac{10\lambda}{27} v^*(x, N) \\
&< \frac{4\lambda}{9} v^*(x, N-3) + \frac{23\lambda}{144} v^*(x, N-2) + \frac{19\lambda}{48} v^*(x, N) \\
&= \left(\frac{4\lambda}{9} - \frac{\lambda}{18} \right) v^*(x, N-3) + \frac{\lambda}{18} v^*(x, N-3) + \frac{23\lambda}{144} v^*(x, N-2) + \frac{19\lambda}{48} v^*(x, N) \\
&< \frac{7\lambda}{18} v^*(x, N-3) + \frac{31\lambda}{144} v^*(x, N-2) + \frac{19\lambda}{48} v^*(x, N).
\end{aligned}$$

Finally, for $y < N - 3$, the definition of the reward function and expressions (S13) and (S18) yield

$$\hat{r}(s, a'_{12}) + \lambda \sum_{s' \in \hat{S}} \hat{P}(s'|s, a'_{12}) v^*(s') = \frac{5\lambda}{8} v^*(x, y) + \frac{\lambda}{4} v^*(x, y+1) + \frac{\lambda}{8} v^*(x, y+2)$$

and

$$\hat{r}(s, a_{12}) + \lambda \sum_{s' \in \hat{S}} \hat{P}(s'|s, a_{12}) v^*(s') = \frac{5\lambda}{9} v^*(x, y) + \frac{7\lambda}{27} v^*(x, y+1) + \frac{5\lambda}{27} v^*(x, y+2).$$

Invoking once more the fact that $v^*(x, y) < v^*(x, y+1)$ for all $3 \leq y \leq N-1$, we conclude

$$\begin{aligned}
&\frac{5\lambda}{8} v^*(x, y) + \frac{\lambda}{4} v^*(x, y+1) + \frac{\lambda}{8} v^*(x, y+2) \\
&= \frac{5\lambda}{8} v^*(x, y) + \left(\frac{\lambda}{4} - \frac{13\lambda}{216} \right) v^*(x, y+1) + \frac{13\lambda}{216} v^*(x, y+1) + \frac{\lambda}{8} v^*(x, y+2) \\
&< \frac{5\lambda}{8} v^*(x, y) + \frac{41\lambda}{216} v^*(x, y+1) + \frac{5\lambda}{27} v^*(x, y+2) \\
&= \left(\frac{5\lambda}{8} - \frac{5\lambda}{72} \right) v^*(x, y) + \frac{5\lambda}{72} v^*(x, y) + \frac{41\lambda}{216} v^*(x, y+1) + \frac{5\lambda}{27} v^*(x, y+2) \\
&< \frac{5\lambda}{9} v^*(x, y) + \frac{7\lambda}{27} v^*(x, y+1) + \frac{5\lambda}{27} v^*(x, y+2).
\end{aligned}$$

□

Lemma S9 implies that we can leave actions $a'_{11}, a'_{12}, a'_{21}$ and a'_{22} out of consideration whenever the corresponding side of the rectangle has length greater than 2. In Lemma S10, we provide a condition under which these actions are suboptimal when the side length equals 2 as well. First, we let $(\hat{S}, \hat{A}', \hat{P}', \hat{r}')$ denote an MDP that is identical to the auxiliary MDP, apart from the fact that the action spaces of rectangles that have a side of length 2 are extended with the corresponding actions in the set $\{a_{11}, a_{12}, a_{21}, a_{22}\}$. That is,

$$\hat{A}'(i,j) = \begin{cases}
\hat{A}(i,j), & \text{if } i, j \neq 2, \\
\hat{A}(i,j) \cup \{a_{11}, a_{12}\}, & \text{if } i = 2, \quad j = 3, ..., N-3, \\
\hat{A}(i,j) \cup \{a_{11}\}, & \text{if } i = 2, \quad j = N-2, \\
\hat{A}(i,j) \cup \{a_{21}, a_{22}\}, & \text{if } i = 3, ..., N-3, \quad j = 2, \\
\hat{A}(i,j) \cup \{a_{21}\}, & \text{if } i = N-2, \quad j = 2, \\
\hat{A}(i,j) \cup \{a_{11}, a_{12}, a_{21}, a_{22}\}, & \text{if } i = j = 2.
\end{cases}$$

Let corresponding transition probabilities be defined analogously to expressions (S11)–(S14), (S20)–(S23), i.e.,

$$\hat{P}'((i',j')|(2,j),a_{11}) = \begin{cases} 1/3, & \text{if } i'=2, j'=j, \\ 2/3, & \text{if } i'=i, j'=j+1, \\ 0, & \text{otherwise,} \end{cases}$$

for $j = 2, \ldots, N-3$ in accordance with expression (S11) and

$$\hat{P}'((i',j')|(2,N-2),a_{11}) = \begin{cases} 1/4, & \text{if } i'=2, j'=N-2, \\ 3/4, & \text{if } i'=2, j'=N, \\ 0, & \text{otherwise,} \end{cases}$$

in accordance with (S20). The remaining cases follow from symmetry. Let the reward function be defined analogously to expression (5). Lemma S10 shows that this artificially extended MDP can provide an easy way to discard actions $a'_{11}, a'_{12}, a'_{21}$ and a'_{22} for rectangles with side length 2.

Lemma S10. *Let $\pi^* = (d^*)^\infty$ denote an optimal policy in the MDP $(\hat{S}, \hat{A}, \hat{P}, \hat{r})$. Suppose that, for each optimal policy $\hat{\pi}^* = (\hat{d}^*)^\infty$ in the MDP $(\hat{S}, \hat{A}', \hat{P}', \hat{r}')$, all of the following conditions hold:*

1. $\hat{d}^*(2,j) \notin \{a_{11}, a_{12}\}$, $j = 3, \ldots, N-3$,
2. $\hat{d}^*(2, N-2) \neq a_{11}$,
3. $\hat{d}^*(i,2) \notin \{a_{21}, a_{22}\}$, $i = 3, \ldots, N-3$,
4. $\hat{d}^*(N-2, 2) \neq a_{21}$,
5. $\hat{d}^*(2,2) \notin \{a_{11}, a_{12}, a_{21}, a_{22}\}$.

This implies

1. $d^*(2,j) \notin \{a'_{11}, a'_{12}\}$, $j = 3, \ldots, N-3$,
2. $d^*(2, N-2) \neq a'_{11}$,
3. $d^*(i,2) \notin \{a'_{21}, a'_{22}\}$, $i = 3, \ldots, N-3$,
4. $d^*(N-2, 2) \neq a'_{21}$,
5. $d^*(2,2) \notin \{a'_{11}, a'_{12}, a'_{21}, a'_{22}\}$.

Proof. From the fact that none of the actions that were added to $\hat{A}$ to obtain the action space $\hat{A}'$ are optimal in the MDP $(\hat{S}, \hat{A}', \hat{P}', \hat{r})$, it follows that the optimal value function of the latter MDP is equivalent to that of the MDP $(\hat{S}, \hat{A}, \hat{P}, \hat{r})$. We show that this implies $d^*(s) \neq a'_{11}$ for $s = (2,j)$, $j = 3, \ldots, N-3$. The remaining cases follow from a similar argument. Since a_{11} is not optimal in state s in the MDP $(\hat{S}, \hat{A}', \hat{P}', \hat{r})$, there is an action $a^* \in \hat{A}'(s) \setminus \{a_{11}\}$ such that

$$\hat{r}'(s, a_{11}) + \lambda \sum_{s' \in \hat{S}} \hat{P}'(s'|s, a_{11}) v^*(s') < \hat{r}'(s, a^*) + \lambda \sum_{s' \in \hat{S}} \hat{P}'(s'|s, a^*) v^*(s'), \tag{S41}$$

where $v^* : S \to \mathbb{R}$ denotes the optimal value function of both the MDPs. The same arguments that led to Lemma S9 can be applied to show that

$$\hat{r}'(s, a'_{11}) + \lambda \sum_{s' \in \hat{S}} \hat{P}'(s'|s, a'_{11}) v^*(s') < \hat{r}'(s, a_{11}) + \lambda \sum_{s' \in \hat{S}} \hat{P}'(s'|s, a_{11}) v^*(s'), \tag{S42}$$

It now follows from expressions (S41) and (S42) that

$$\hat{r}'(s, a'_{11}) + \lambda \sum_{s' \in \hat{S}} \hat{P}'(s'|s, a'_{11}) v^*(s') < \hat{r}'(s, a^*) + \lambda \sum_{s' \in \hat{S}} \hat{P}'(s'|s, a^*) v^*(s').$$

Since this statement does not depend on the presence of action a_{11} in the action space $\hat{A}'(s)$, we can conclude that for any stationary deterministic optimal policy $\pi^* = (d^*)^\infty$ in the MDP $(\hat{S}, \hat{A}, \hat{P}, \hat{r})$, we have $d^*(2,j) \neq a'_{11}$, $j = 3, \ldots, N-3$. □

Now that actions 0, $\tilde{a}$, a'_{11}, a'_{21}, a'_{12} and a'_{22} are dealt with, we unravel the structure of the optimal policy in the auxiliary MDP, which is specified in Theorem 4. Figure 5 provides a visualization of this result.

Proof of Theorem 4

Since the state and action spaces are both finite, it suffices to show that the value function $v_\lambda^{\pi^*} : \hat{S} \to \mathbb{R}$ of a stationary, deterministic policy $\pi^* = (d^*)^\infty$ satisfies the Bellman optimality equations (11) if and only if it has the specified form. We consider states of the form (i, j), $i \geq j$. The analogous results for states of the form (i, j), $i < j$ follow directly from symmetry. To obtain the desired result for states $(i, 2)$, $i = 2, \ldots, N-2$ (respectively $(2, j)$, $j = 2, \ldots, N-2$), we prove the statement for the MDP $(\hat{S}, \hat{A}', \hat{P}', \hat{r}')$. The result for the original auxiliary MDP $(\hat{S}, \hat{A}, \hat{P}, \hat{r})$ then follows immediately from Lemma S10.

Let Π_1 denote the space of stationary, deterministic policies $\pi_1 = d_1^\infty$ that satisfy $d_1(i, j) \in A_1^*(i, j)$ for all $(i, j) \in \hat{S}$. Similarly, let Π_2 denote the space of stationary, deterministic policies $\pi_2 = d_2^\infty$ that satisfy $d_2(i, j) \in A_2^*(i, j)$ for all $(i, j) \in \hat{S}$. We proceed to show that $v_\lambda^{\pi^*} : \hat{S} \to \mathbb{R}$ for $\lambda \in (\lambda_c, 1)$ satisfies the Bellman optimality equations if and only if $\pi^* \in \Pi_1$, that $v_\lambda^{\pi^*} : \hat{S} \to \mathbb{R}$ for $\lambda \in (0, \lambda_c)$ satisfies the Bellman optimality equations if and only if $\pi^* \in \Pi_2$ and that $v_{\lambda_c}^{\pi^*} : \hat{S} \to \mathbb{R}$ satisfies the Bellman optimality equations if and only if $\pi^* \in \Pi_1 \cup \Pi_2$.

To this end, let $\pi_1 = d_1^\infty$ and $\pi_2 = d_2^\infty$ be two policies in Π_1 and Π_2 respectively. Note that $A_k^*(i, j)$, $k = 1, 2$, is a singleton for all $(i, j) \in \hat{S}$, $(i, j) \neq (N-3, N-3)$. Since actions a_{12} and a_{22} have the same effect in state $(N-3, N-3)$, we let $d_1(N-3, N-3) = d_2(N-3, N-3) = a_{22}$ without loss of generality. Using the transition probabilities (S11–S30), we obtain the following recursive expressions for $v_\lambda^{\pi_1}$ and $v_\lambda^{\pi_2}$:

$$v_\lambda^{\pi_k}(N, N) = \frac{1}{1-\lambda}, \quad k = 1, 2, \tag{S43}$$

$$v_\lambda^{\pi_k}(N, N-2) = \frac{3\lambda}{4-\lambda} v_\lambda^{\pi_k}(N, N), \quad k = 1, 2, \tag{S44}$$

$$v_\lambda^{\pi_k}(N, N-3) = \frac{31\lambda}{8(18-7\lambda)} v_\lambda^{\pi_k}(N, N-2) + \frac{57\lambda}{8(18-7\lambda)} v_\lambda^{\pi_k}(N, N), \quad k = 1, 2, \tag{S45}$$

$$v_\lambda^{\pi_1}(N, j) = \frac{2\lambda}{3-\lambda} v_\lambda^{\pi_1}(N, j+1), \quad j = 2, \ldots, N-4, \tag{S46}$$

$$v_\lambda^{\pi_2}(N, j) = \begin{cases} \dfrac{2\lambda}{3-\lambda} v_\lambda^{\pi_2}(N, j+1), & \text{if } j = N-4, \\ \dfrac{7\lambda}{3(9-5\lambda)} v_\lambda^{\pi_2}(N, j+1) + \dfrac{5\lambda}{3(9-5\lambda)} v_\lambda^{\pi_2}(N, j+2), & \text{if } j = 2, \ldots, N-5, \end{cases} \tag{S47}$$

$$v_\lambda^{\pi_k}(N-2, N-2) = \frac{9\lambda}{2(18-7\lambda)} v_\lambda^{\pi_k}(N, N-2) + \frac{13\lambda}{2(18-7\lambda)} v_\lambda^{\pi_k}(N, N), \quad k = 1, 2, \tag{S48}$$

$$v_\lambda^{\pi_k}(N-2, N-3) = \frac{31\lambda}{8(18-7\lambda)} v_\lambda^{\pi_k}(N-2, N-2) + \frac{57\lambda}{8(18-7\lambda)} v_\lambda^{\pi_k}(N-2, N), \quad k = 1, 2, \tag{S49}$$

$$v_\lambda^{\pi_k}(N-2, j) = \frac{12}{12-5\lambda} \left(\frac{\lambda}{8} v_\lambda^{\pi_k}(N, j) + \frac{25\lambda}{72} v_\lambda^{\pi_k}(N, j+1) + \frac{\lambda}{9} v_\lambda^{\pi_k}(N-2, j+1) \right), \tag{S50}$$
$$j = 2, \ldots, N-4, \quad k = 1, 2,$$

$$v_\lambda^{\pi_k}(N-3, N-3) = \frac{31\lambda}{8(18-7\lambda)} v_\lambda^{\pi_k}(N-2, N-3) + \frac{57\lambda}{8(18-7\lambda)} v_\lambda^{\pi_k}(N, N-3), \quad k = 1, 2, \tag{S51}$$

$$v_\lambda^{\pi_k}(i, j) = \frac{\lambda}{9-4\lambda} \Big(v_\lambda^{\pi_k}(i, j+1) + v_\lambda^{\pi_k}(i+1, j) + 3 v_\lambda^{\pi_k}(i+1, j+1) \Big), \tag{S52}$$
$$i = 2, \ldots, N-3, \quad j = 2, \ldots, N-4, \quad k = 1, 2.$$

Analogous expressions for states of the form (i, j), $i < j$ follow directly from symmetry. Note that expressions (S43–S52) together with Corollary 2 allow us to compute the quantity $\mathbb{E}\left[\lambda^{\tau_{(N,N)}^{(i,j),\pi_k}}\right]$ for $k = 1, 2$, $(i, j) \in \hat{S}$. It suffices to show that for any state $s = (i, j) \in \hat{S}$, we have

$$\hat{r}(s, a) + \lambda \sum_{s' \in \hat{S}} \hat{P}(s'|s, a) v_\lambda^{\pi_1}(s') < \hat{r}(s, d_1(s)) + \lambda \sum_{s' \in \hat{S}} \hat{P}(s'|s, d_1(s)) v_\lambda^{\pi_1}(s'), \tag{S53}$$

for all $a \in \hat{A}(i, j), a \notin A_1^*(s)$, $\lambda > \lambda_c$,

$$\hat{r}(s, a) + \lambda \sum_{s' \in \hat{S}} \hat{P}(s'|s, a) v_\lambda^{\pi_2}(s') < \hat{r}(s, d_2(s)) + \lambda \sum_{s' \in \hat{S}} \hat{P}(s'|s, d_2(s)) v_\lambda^{\pi_2}(s'), \tag{S54}$$

for all $a \in \hat{A}(i,j), a \notin A_2^*(s)$, $\lambda < \lambda_c$ and

$$\begin{aligned} v_{\lambda_c}^{\pi_1}(s) = v_{\lambda_c}^{\pi_2}(s) &:= v_{\lambda_c}(s), \\ \hat{r}(s,d_1(s)) + \lambda_c \sum_{s' \in \hat{S}} \hat{P}(s'|s,d_1(s)) v_{\lambda_c}(s') &= \hat{r}(s,d_2(s)) + \lambda \sum_{s' \in \hat{S}} \hat{P}(s'|s,d_2(s)) v_{\lambda_c}(s') \\ &> \hat{r}(s,a) + \lambda_c \sum_{s' \in \hat{S}} \hat{P}(s'|s,a) v_{\lambda_c}(s'), \end{aligned} \tag{S55}$$

for all $a \in \hat{A}(i,j), a \notin A_1^*(s) \cup A_2^*(s)$. Note that the validity of these inequalities is trivial for states (N,N) and $(N,N-2)$.

Assuming that $v_{\lambda^c}^{\pi_1}(N,j) = v_{\lambda^c}^{\pi_2}(N,j) := v_{\lambda^c}(N,j)$, which will be proved at a later point, the inequalities for states (N,j), $j = 2,\dots,N-3$, reduce to

$$\lambda \sum_{(i',j') \in \hat{S}} \hat{P}((i',j')|(N,N-3),a_{12}) v_\lambda^{\pi_k}(i',j') > \lambda \sum_{(i',j') \in \hat{S}} \hat{P}((i',j')|(N,N-3),a_{11}) v_\lambda^{\pi_k}(i',j'),$$
for $k=1, \lambda \in [\lambda_c, 1)$ and $k=2, \lambda \in (0, \lambda_c]$,

$$\lambda \sum_{(i',j') \in \hat{S}} \hat{P}((i',j')|(N,N-4),a_{11}) v_\lambda^{\pi_k}(i',j') > \lambda \sum_{(i',j') \in \hat{S}} \hat{P}((i',j')|(N,N-4),a_{12}) v_\lambda^{\pi_k}(i',j'),$$
for $k=1, \lambda \in [\lambda_c, 1)$ and $k=2, \lambda \in (0, \lambda_c]$,

$$\lambda \sum_{(i',j') \in \hat{S}} \hat{P}((i',j')|(N,j),a_{11}) v_\lambda^{\pi_1}(i',j') > \lambda \sum_{(i',j') \in \hat{S}} \hat{P}((i',j')|(N,j),a_{12}) v_\lambda^{\pi_1}(i',j'),$$
for $\lambda \in (\lambda^c, 1), \quad j = 2,\dots,N-5$,

$$\lambda \sum_{(i',j') \in \hat{S}} \hat{P}((i',j')|(i,j),a_{12}) v_\lambda^{\pi_2}(i',j') > \lambda \sum_{(i',j') \in \hat{S}} \hat{P}((i',j')|(i,j),a_{11}) v_\lambda^{\pi_2}(i',j'),$$
for $\lambda \in (0, \lambda^c)$ and $j = 2,\dots,N-5$, and

$$\lambda_c \sum_{(i',j') \in \hat{S}} \hat{P}((i',j')|(N,j),a_{11}) v_{\lambda_c}(i',j') = \lambda_c \sum_{(i',j') \in \hat{S}} \hat{P}((i',j')|(N,j),a_{12}) v_{\lambda_c}(i',j'),$$
for $j = 2,\dots,N-5$.

Inserting expressions (S11, S13, S22), we obtain the following set of inequalities:

$$8v_\lambda^{\pi_k}(N,N-3) - 65v_\lambda^{\pi_k}(N,N-2) + 57v_\lambda^{\pi_k}(N,N) > 0, \quad k=1, \lambda \in [\lambda_c,1) \text{ and } k=2, \lambda \in (0,\lambda_c], \tag{S56}$$
$$-6v_\lambda^{\pi_k}(N,N-4) + 11v_\lambda^{\pi_k}(N,N-3) - 5v_\lambda^{\pi_k}(N,N-2) > 0, \tag{S57}$$
$$k=1, \lambda \in [\lambda_c,1) \text{ and } k=2, \lambda \in (0,\lambda_c], \tag{S58}$$
$$-6v_\lambda^{\pi_1}(N,j) + 11v_\lambda^{\pi_1}(N,j+1) - 5v_\lambda^{\pi_1}(N,j+2) > 0, \quad \lambda \in (\lambda_c,1), \quad j = 2,\dots,N-5, \tag{S59}$$
$$6v_\lambda^{\pi_2}(N,j) - 11v_\lambda^{\pi_2}(N,j+1) + 5v_\lambda^{\pi_2}(N,j+2) > 0, \quad \lambda \in (0,\lambda_c), \quad j = 2,\dots,N-5, \text{ and} \tag{S60}$$
$$-6v_{\lambda_c}(N,j) + 11v_{\lambda_c}(N,j+1) - 5v_{\lambda_c}(N,j+2) = 0, \quad j = 2,\dots,N-5. \tag{S61}$$

In a similar way, expressions (S53–S55) for states $(N-2,j)$, $j = 2,\dots,N-3$, using expressions (S11, S13, S21, S22, S25, S26), the fact that actions a_{11} and a_{21} have the same effect in state $(N-2,N-2)$ by symmetry and the fact that $v_\lambda^{\pi_k}(N-2,N) = v_\lambda^{\pi_k}(N,N-2)$ for $k=1,2$, $\lambda \in (0,1)$, boil down to

$$5v_\lambda^{\pi_k}(N-2,N-2) - 18v_\lambda^{\pi_k}(N,N-2) + 13v_\lambda^{\pi_k}(N,N) > 0, \tag{S62}$$
$$8v_\lambda^{\pi_k}(N-2,N-3) - 65v_\lambda^{\pi_k}(N-2,N-2) + 57v_\lambda^{\pi_k}(N-2,N) > 0, \tag{S63}$$
$$20v_\lambda^{\pi_k}(N-2,N-3) + 31v_\lambda^{\pi_k}(N-2,N-2) + 57v_\lambda^{\pi_k}(N-2,N) - 108v_\lambda^{\pi_k}(N,N-3) > 0, \tag{S64}$$
$$-4v_\lambda^{\pi_k}(N-2,N-3) + 15v_\lambda^{\pi_k}(N-2,N-2) - 18v_\lambda^{\pi_k}(N,N-3) + 7v_\lambda^{\pi_k}(N,N-2) > 0, \tag{S65}$$
$$6v_\lambda^{\pi_k}(N-2,j) - 40v_\lambda^{\pi_k}(N-2,j+1) + 9v_\lambda^{\pi_k}(N,j) + 25v_\lambda^{\pi_k}(N,j+1) > 0, \quad j = 2,\dots,N-4, \tag{S66}$$
$$-30v_\lambda^{\pi_k}(N-2,j) - 32v_\lambda^{\pi_k}(N-2,j+1) - 40v_\lambda^{\pi_k}(N-2,j+2) + 27v_\lambda^{\pi_k}(N,j) + 75v_\lambda^{\pi_k}(N,j+1) > 0, \tag{S67}$$
$$j = 2,\dots,N-4, \tag{S68}$$
$$12v_\lambda^{\pi_k}(N-2,j) + 8v_\lambda^{\pi_k}(N-2,j+1) - 45v_\lambda^{\pi_k}(N,j) + 25v_\lambda^{\pi_k}(N,j+1) > 0, \quad j = 2,\dots,N-4, \tag{S69}$$

for $k = 1, \lambda \in [\lambda_c, 1)$ and $k = 2, \lambda \in (0, \lambda_c]$.

For states $(N-3, j)$, $j = 2, \ldots, N-3$, by inserting expressions (S11, S12, S13, S15, S23) and using the facts that $v_\lambda^{\pi_k}(N-2, N-3) = v_\lambda^{\pi_k}(N-3, N-2)$ for $k = 1, 2$, $\lambda \in (0, 1)$ and actions a_{21} and a_{11} have an equivalent effect in state $(N-3, N-3)$, inequalities (S53–S55) reduce to

$$8v_\lambda^{\pi_k}(N-3,N-3) - 65v_\lambda^{\pi_k}(N-2,N-3) + 57v_\lambda^{\pi_k}(N,N-3) > 0, \tag{S70}$$
$$-8v_\lambda^{\pi_k}(N-3,N-3) - v_\lambda^{\pi_k}(N-2,N-3) - 48v_\lambda^{\pi_k}(N-2,N-2) + 57v_\lambda^{\pi_k}(N,N-3) > 0, \tag{S71}$$
$$v_\lambda^{\pi_k}(N-3,j) - 5v_\lambda^{\pi_k}(N-3,j+1) + v_\lambda^{\pi_k}(N-2,j) + 3v_\lambda^{\pi_k}(N-2,j+1) > 0, \quad j = 2, \ldots, N-4, \tag{S72}$$
$$-3v_\lambda^{\pi_k}(N-3,j) - 4v_\lambda^{\pi_k}(N-3,j+1) - 5v_\lambda^{\pi_k}(N-3,j+2) + 3v_\lambda^{\pi_k}(N-2,j) \tag{S73}$$
$$+ 9v_\lambda^{\pi_k}(N-2,j+1) > 0, \quad j = 2, \ldots, N-4,$$
$$v_\lambda^{\pi_k}(N-3,j) + v_\lambda^{\pi_k}(N-3,j+1) - 5v_\lambda^{\pi_k}(N-2,j) + 3v_\lambda^{\pi_k}(N-2,j+1) > 0, \quad j = 2, \ldots, N-4, \tag{S74}$$
$$8v_\lambda^{\pi_k}(N-3,j) + 16v_\lambda^{\pi_k}(N-3,j+1) - 15v_\lambda^{\pi_k}(N-2,j) + 48v_\lambda^{\pi_k}(N-2,j+1) - 57v_\lambda^{\pi_k}(N,j) > 0, \tag{S75}$$
$$j = 2, \ldots, N-4, \tag{S76}$$

for $k = 1, \lambda \in [\lambda_c, 1)$ and $k = 2, \lambda \in (0, \lambda_c]$.

Finally, inequalities (S53–S55) for states (i, j), $i, j = 2, \ldots, N-4$, using expressions (S11–S15), reduce to

$$v_\lambda^{\pi_k}(i,j) + v_\lambda^{\pi_k}(i+1,j) - 5v_\lambda^{\pi_k}(i,j+1) + 3v_\lambda^{\pi_k}(i+1,j+1) > 0, \tag{S77}$$
$$v_\lambda^{\pi_k}(i,j) - 5v_\lambda^{\pi_k}(i+1,j) + v_\lambda^{\pi_k}(i,j+1) + 3v_\lambda^{\pi_k}(i+1,j+1) > 0, \tag{S78}$$
$$-3v_\lambda^{\pi_k}(i,j) + 3v_\lambda^{\pi_k}(i+1,j) - 4v_\lambda^{\pi_k}(i,j+1) + 9v_\lambda^{\pi_k}(i+1,j+1) - 5v_\lambda^{\pi_k}(i,j+2) > 0, \tag{S79}$$
$$-3v_\lambda^{\pi_k}(i,j) - 4v_\lambda^{\pi_k}(i+1,j) + 3v_\lambda^{\pi_k}(i,j+1) + 9v_\lambda^{\pi_k}(i+1,j+1) - 5v_\lambda^{\pi_k}(i+2,j) > 0, \tag{S80}$$

for $k = 1, \lambda \in [\lambda_c, 1)$ and $k = 2, \lambda \in (0, \lambda_c]$.

Thus, in order to obtain the desired result, it suffices to prove that the value functions of policies π_1 and π_2 satisfy expressions (S56–S80), in addition to the equations $v_{\lambda_c}^{\pi_1}(N,j) = v_{\lambda_c}^{\pi_2}(N,j)$ for all $j = 2, \ldots, N-2, N$. To prove the set of (in)equalities, we make use of the recursive expressions (S43–S52) for $v_\lambda^{\pi_1}$ and $v_\lambda^{\pi_2}$. The proof crucially relies on the fact that rectangles of size at least 3×3 cannot shrink under the dynamics of the auxiliary MDP, which allows us to show the validity of the expressions by means of backward induction over the size of the rectangle. For states of the form (N, j), $(N-2, j)$ and $(N-3, j)$, $j = 2, \ldots, N$, we use backward induction over the length of the vertical side of the rectangle. For states of the form (i, j), $i, j = 2, \ldots, N-4$, i.e., expressions (S77–S80), we invoke a more involved induction argument, which can be outlined as follows:

- *Induction base:* First, we show that the expressions are satisfied for states of the form $(i, N-4)$ and $(N-4, j)$, $i, j = 2, \ldots, N-4$, using an embedded induction argument over the length of the shortest side of the rectangle.
- *Induction hypothesis*: We assume that the expressions are valid for states of the form $(i, n+1)$ and $(n+1, j)$, $i, j = 2, \ldots, n+1$ for some $n < N-4$.
- *Induction step:* First, we show that the induction hypothesis implies the correctness of the expressions for state (n, n). Using this result, together with the induction hypothesis, we invoke another embedded induction argument over the length of the shortest side of the rectangle to show that the expressions hold for states of the form (i, n) and (n, j), $i, j = 2, \ldots, n-1$.

The details of the induction proofs are provided in Supplementary Material III. □

Supplementary Material II: supporting tables and figures for computation of transition probabilities

The derivations of expressions (S11, S12) and (S15)–(S26) are based on similar arguments as in the proof of Lemma S6 and are clarified in Figures S5–S12 and the corresponding Tables S2–S9.

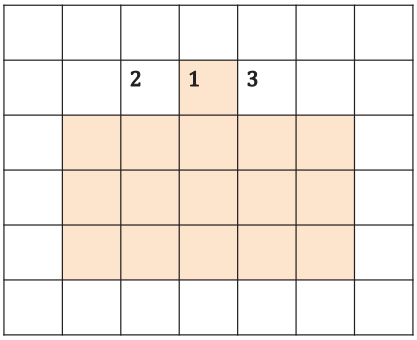


Figure S5: Post-decision configuration after flipping a spin at distance 1 from the horizontal side of the rectangle.

Table S2: Derivation of expression (S11), corresponding to Figure S5. Similar derivation leads to expression (S12).

Sequences of susceptible spins	Probability of selecting sequence	Next state	Transition probability
(1)	$1/3$	(i,j)	$1/3$
(2) (3)	$1/3$ $1/3$	$(i,j+1)$	$2/3$

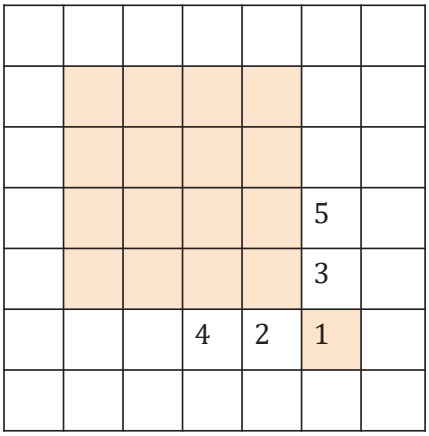


Figure S6: Post-decision configuration corresponding to expression (S15).

Table S3: Derivation of expression (S15), corresponding to Figure S6.

Sequences of susceptible spins	Probability of selecting sequence	Next state	Transition probability
(1)	$1/3$	(i,j)	$4/9$
$(2,1,2)$	$1/18$		
$(3,1,3)$	$1/18$		
$(2,1,4)$	$1/18$	$(i+1,j)$	$1/9$
$(2,4,1)$	$1/18$		
$(3,1,5)$	$1/18$	$(i,j+1)$	$1/9$
$(3,5,1)$	$1/18$		
$(2,3)$	$1/9$	$(i+1,j+1)$	$1/3$
$(3,2)$	$1/9$		
$(2,4,3)$	$1/18$		
$(3,5,2)$	$1/18$		

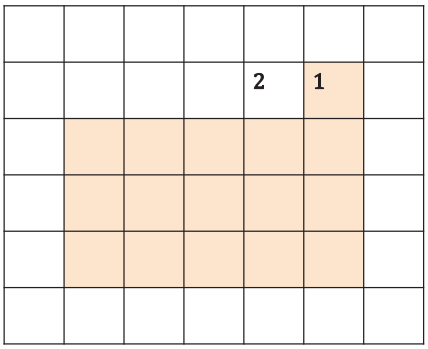


Figure S7: Post-decision configuration corresponding to expression (S16).

Table S4: Derivation of expression (S16), corresponding to Figure S7. Similar derivation leads to expression (S17).

Sequences of susceptible spins	Probability of selecting sequence	Next state	Transition probability
(1)	$1/2$	(i,j)	$1/2$
(2)	$1/2$	$(i,j+1)$	$1/2$

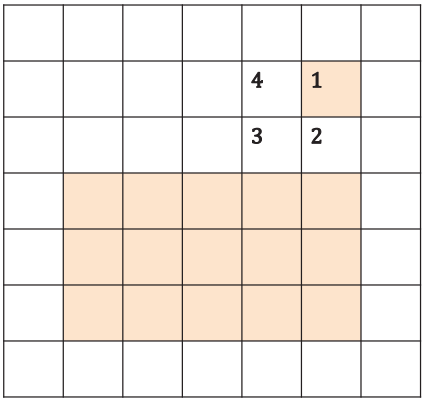


Figure S8: Post-decision configuration corresponding to expression (S18).

Table S5: Derivation of expression (S18), corresponding to Figure S8. Similar derivation leads to expression (S19).

Sequences of susceptible spins	Probability of selecting sequence	Next state	Transition probability
(1) $(2,1,2)$	$1/2$ $1/8$	(i,j)	$5/8$
$(2,1,3)$ $(2,3,1)$	$1/8$ $1/8$	$(i,j+1)$	$1/4$
$(2,3,4)$	$1/8$	$(i,j+2)$	$1/8$

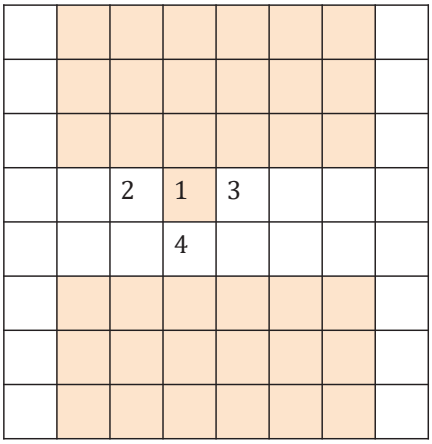


Figure S9: Post-decision configuration corresponding to expression (S20).

Table S6: Derivation of expression (S20), corresponding to Figure S9. Similar derivation leads to expression (S21).

Sequences of susceptible spins	Probability of selecting sequence	Next state	Transition probability
(1)	$1/4$	$(i,N-2)$	$1/4$
(2) (3) (4)	$1/4$ $1/4$ $1/4$	(i,N)	$3/4$

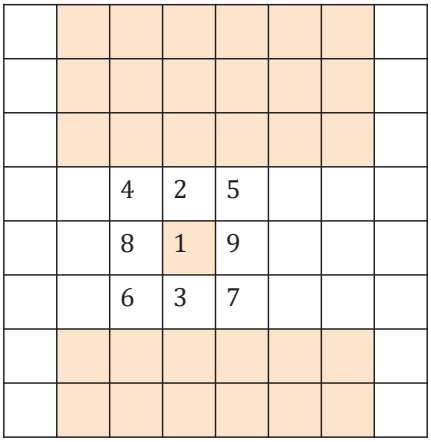


Figure S10: Post-decision configuration corresponding to expression (S22).

Table S7: Derivation of expression (S22), corresponding to Figure S10. Similar derivation leads to expression (S23).

Sequences of susceptible spins	Probability of selecting sequence	Next state	Transition probability
(1)	$1/3$		
$(2,1,2)$	$1/36$	$(i, N-3)$	$7/18$
$(3,1,3)$	$1/36$		
$(2,1,4)$	$1/36$		
$(2,1,5)$	$1/36$		
$(3,1,6)$	$1/36$		
$(3,1,7)$	$1/36$		
$(2,4,1)$	$1/48$		
$(2,5,1)$	$1/48$		
$(3,6,1)$	$1/48$	$(i, N-2)$	$31/144$
$(3,7,1)$	$1/48$		
$(2,4,5,1)$	$1/192$		
$(2,5,4,1)$	$1/192$		
$(3,6,7,1)$	$1/192$		
$(3,7,6,1)$	$1/192$		
$(2,3)$	$1/12$		
$(3,2)$	$1/12$		
$(2,4,3)$	$1/48$		
$(2,5,3)$	$1/48$		
$(3,6,2)$	$1/48$		
$(3,7,2)$	$1/48$		
$(2,4,5,3)$	$1/192$		
$(2,5,4,3)$	$1/192$		
$(3,6,7,2)$	$1/192$		
$(3,7,6,2)$	$1/192$		
$(2,4,5,8)$	$1/192$		
$(3,6,7,8)$	$1/192$	(i, N)	$19/48$
$(3,7,6,8)$	$1/192$		
$(2,5,4,8)$	$1/192$		
$(2,4,5,9)$	$1/192$		
$(2,5,4,9)$	$1/192$		
$(3,6,7,9)$	$1/192$		
$(3,7,6,9)$	$1/192$		
$(2,4,8)$	$1/48$		
$(2,5,9)$	$1/48$		
$(3,6,8)$	$1/48$		
$(3,7,9)$	$1/48$		

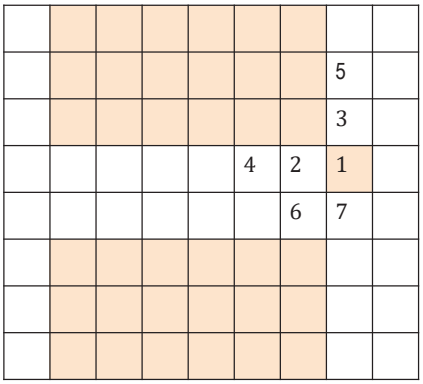


Figure S11: Post-decision configuration corresponding to expression (S24).

Table S8: Derivation of expression (S24), corresponding to Figure S11. Similar derivation leads to expression (S25).

Sequences of susceptible spins	Probability of selecting sequence	Next state	Transition probability
(1)	$1/3$	$(i, N-2)$	$5/12$
$(2,1,2)$	$1/36$		
$(3,1,3)$	$1/18$		
$(2,1,4)$	$1/36$	(i, N)	$1/8$
$(2,1,6)$	$1/36$		
$(2,4,1)$	$1/24$		
$(2,6,1)$	$1/36$		
$(3,1,5)$	$1/18$	$(i+1, N-2)$	$1/9$
$(3,5,1)$	$1/18$		
$(2,3)$	$1/12$	$(i+1, N)$	$25/72$
$(2,4,3)$	$1/24$		
$(2,6,3)$	$1/36$		
$(2,6,7)$	$1/36$		
$(3,2)$	$1/9$		
$(3,5,2)$	$1/18$		

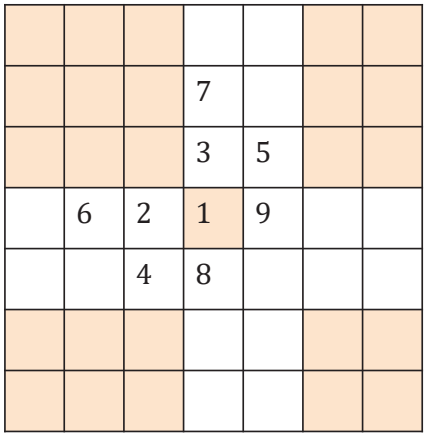


Figure S12: Post-decision configuration corresponding to expression (S26).

Table S9: Derivation of expression (S26), corresponding to Figure S12.

Sequences of susceptible spins	Probability of selecting sequence	Next state	Transition probability
(1)	$1/3$	$(N-2, N-2)$	$7/18$
$(2,1,2)$	$1/36$		
$(3,1,3)$	$1/36$		
$(2,1,4)$	$1/36$	$(N-2, N)$	$1/8$
$(2,1,6)$	$1/36$		
$(2,4,1)$	$1/36$		
$(2,6,1)$	$1/24$		
$(3,1,5)$	$1/36$	$(N, N-2)$	$1/8$
$(3,1,7)$	$1/36$		
$(3,5,1)$	$1/36$		
$(3,7,1)$	$1/24$		
$(2,3)$	$1/12$	(N, N)	$13/36$
$(3,2)$	$1/12$		
$(2,4,3)$	$1/36$		
$(3,5,2)$	$1/36$		
$(2,4,8)$	$1/36$		
$(3,5,9)$	$1/36$		
$(2,6,3)$	$1/24$		
$(3,7,2)$	$1/24$		

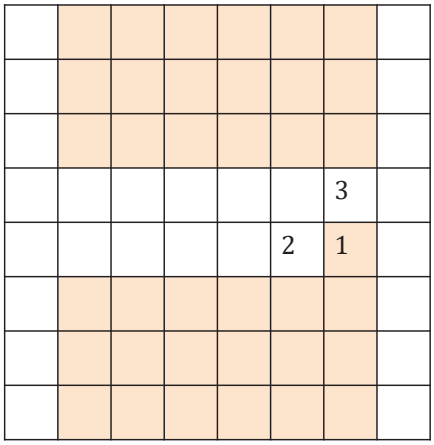


Figure S13: Post-decision configuration corresponding to expression (S27).

Table S10: Derivation of expression (S27), corresponding to Figure S13. Similar derivation leads to expression (S28)

Sequences of susceptible spins	Probability of selecting sequence	Next state	Transition probability
(1)	$1/3$	$(i, N-2)$	$1/3$
(2)	$1/3$	(i, N)	$1/3$
(3)	$1/3$		

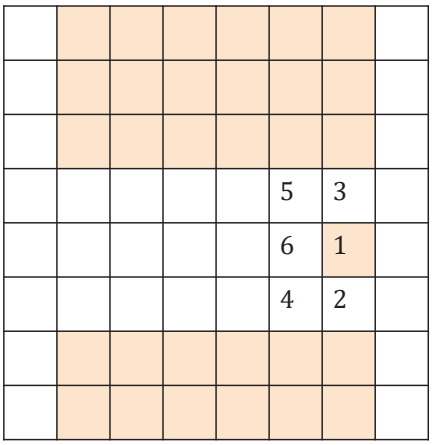


Figure S14: Post-decision configuration corresponding to expression (S29).

Table S11: Derivation of expression (S29), corresponding to Figure S14. Similar derivation leads to expression (S30).

Sequences of susceptible spins	Probability of selecting sequence	Next state	Transition probability
(1)	$1/3$	$(i, N-3)$	$4/9$
$(2,1,2)$	$1/18$		
$(3,1,3)$	$1/18$		
$(2,1,4)$	$1/18$	$(i, N-2)$	$5/27$
$(2,4,1)$	$1/27$		
$(3,1,5)$	$1/18$		
$(3,5,1)$	$1/27$		
$(2,3)$	$1/9$	(i, N)	$10/27$
$(2,4,3)$	$1/27$		
$(2,4,6)$	$1/27$		
$(3,2)$	$1/9$		
$(3,5,2)$	$1/27$		
$(3,5,6)$	$1/27$		

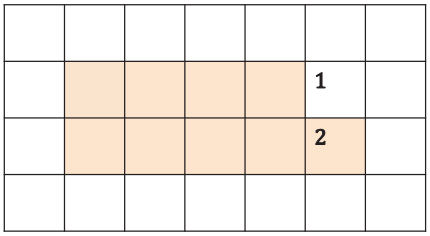


Figure S15: Post-decision configuration corresponding to expression (S33).

Table S12: Derivation of expression (S33), corresponding to Figure S15. Similar derivation leads to expression (S32).

Sequences of susceptible spins	Probability of selecting sequence	Next state	Transition probability
(1)	$1/2$	$(i, 2)$	$1/2$
(2)	$1/2$	$(i-1, 2)$	$1/2$

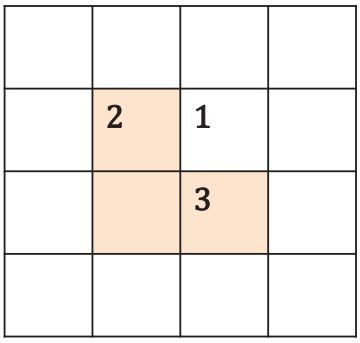


Figure S16: Post-decision configuration corresponding to expression (S34).

Table S13: Derivation of expression (S34), corresponding to Figure S16.

Sequences of susceptible spins	Probability of selecting sequence	Next state	Transition probability
(1)	$1/3$	$(2,2)$	$1/3$
(2) (3)	$1/3$ $1/3$	$(0,0)$	$2/3$

Supplementary Material III: Details of the proof of Theorem 4

Let policies $\pi_1 = d_1^\infty$ and $\pi_2 = d_2^\infty$ denote two arbitrary policies in Π_1 and Π_2 respectively. Recall that the value functions $v_\lambda^{\pi_1} : \hat{S} \to \mathbb{R}$ and $v_\lambda^{\pi_2} : \hat{S} \to \mathbb{R}$ satisfy the following recursive expressions:

$$v_\lambda^{\pi_k}(N,N) = \frac{1}{1-\lambda}, \quad k=1,2, \tag{S81}$$

$$v_\lambda^{\pi_k}(N,N-2) = \frac{3\lambda}{4-\lambda} v_\lambda^{\pi_k}(N,N), \quad k=1,2, \tag{S82}$$

$$v_\lambda^{\pi_k}(N,N-3) = \frac{31\lambda}{8(18-7\lambda)} v_\lambda^{\pi_k}(N,N-2) + \frac{57\lambda}{8(18-7\lambda)} v_\lambda^{\pi_k}(N,N), \quad k=1,2, \tag{S83}$$

$$v_\lambda^{\pi_1}(N,j) = \frac{2\lambda}{3-\lambda} v_\lambda^{\pi_1}(N,j+1), \quad j=2,\dots,N-4, \tag{S84}$$

$$v_\lambda^{\pi_2}(N,j) = \begin{cases} \dfrac{2\lambda}{3-\lambda} v_\lambda^{\pi_2}(N,j+1), & \text{if } j = N-4, \\ \dfrac{7\lambda}{3(9-5\lambda)} v_\lambda^{\pi_2}(N,j+1) + \dfrac{5\lambda}{3(9-5\lambda)} v_\lambda^{\pi_2}(N,j+2), & \text{if } j = 2,\dots,N-5, \end{cases} \tag{S85}$$

$$v_\lambda^{\pi_k}(N-2,N-2) = \frac{9\lambda}{2(18-7\lambda)} v_\lambda^{\pi_k}(N,N-2) + \frac{13\lambda}{2(18-7\lambda)} v_\lambda^{\pi_k}(N,N), \quad k=1,2, \tag{S86}$$

$$v_\lambda^{\pi_k}(N-2,N-3) = \frac{31\lambda}{8(18-7\lambda)} v_\lambda^{\pi_k}(N-2,N-2) + \frac{57\lambda}{8(18-7\lambda)} v_\lambda^{\pi_k}(N-2,N), \quad k=1,2, \tag{S87}$$

$$v_\lambda^{\pi_k}(N-2,j) = \frac{12}{12-5\lambda}\Big(\frac{\lambda}{8} v_\lambda^{\pi_k}(N,j) + \frac{25\lambda}{72} v_\lambda^{\pi_k}(N,j+1) + \frac{\lambda}{9} v_\lambda^{\pi_k}(N-2,j+1)\Big), \tag{S88}$$
$$j=2,\dots,N-4, \quad k=1,2,$$

$$v_\lambda^{\pi_k}(N-3,N-3) = \frac{31\lambda}{8(18-7\lambda)} v_\lambda^{\pi_k}(N-2,N-3) + \frac{57\lambda}{8(18-7\lambda)} v_\lambda^{\pi_k}(N,N-3), \quad k=1,2, \tag{S89}$$

$$v_\lambda^{\pi_k}(i,j) = \frac{\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(i,j+1) + v_\lambda^{\pi_k}(i+1,j) + 3v_\lambda^{\pi_k}(i+1,j+1)\Big), \tag{S90}$$
$$i=2,\dots,N-3, \quad j=2,\dots,N-4, \quad k=1,2.$$

In Table S14, explicit expressions for these value functions are recorded for several states, which will be required at a later point in the proof. Using expressions (S81-S90), we show that $v_\lambda^{\pi_1}$ and $v_\lambda^{\pi_2}$ satisfy $v_{\lambda^c}^{\pi_1}(N,j) = v_{\lambda^c}^{\pi_2}(N,j) := v_{\lambda^c}(N,j)$ for all $j = 2,\dots,N$ and expressions (S56-S61, S62-S69, S70-S75, S77-S80), which are recollected below:

$$8v_\lambda^{\pi_k}(N,N-3) - 65v_\lambda^{\pi_k}(N,N-2) + 57v_\lambda^{\pi_k}(N,N) > 0, \quad \text{for } k=1, \lambda \in [\lambda_c, 1) \text{ and } k=2, \lambda \in (0,\lambda_c], \tag{S91}$$

$$-6v_\lambda^{\pi_k}(N,N-4) + 11v_\lambda^{\pi_k}(N,N-3) - 5v_\lambda^{\pi_k}(N,N-2) > 0, \quad \text{for } k=1, \lambda \in [\lambda_c, 1) \tag{S92}$$
$$\text{and } k=2, \lambda \in (0,\lambda_c],$$

$$-6v_\lambda^{\pi_1}(N,j) + 11v_\lambda^{\pi_1}(N,j+1) - 5v_\lambda^{\pi_1}(N,j+2) > 0, \quad \lambda \in (\lambda^c, 1), \quad j=2,\dots,N-5, \tag{S93}$$

$$6v_\lambda^{\pi_2}(N,j) - 11v_\lambda^{\pi_2}(N,j+1) + 5v_\lambda^{\pi_2}(N,j+2) > 0, \quad \lambda \in (0,\lambda^c), \quad j=2,\dots,N-5, \tag{S94}$$

$$-6v_{\lambda_c}(N,j) + 11v_{\lambda^c}(N,j+1) - 5v_{\lambda_c}(N,j+2) = 0, \quad j=2,\dots,N-5, \tag{S95}$$

$$5v_\lambda^{\pi_k}(N-2,N-2) - 18v_\lambda^{\pi_k}(N,N-2) + 13v_\lambda^{\pi_k}(N,N) > 0, \quad \text{for } k=1, \lambda \in [\lambda_c, 1) \tag{S96}$$
$$\text{and } k=2, \lambda \in (0,\lambda_c],$$

$$8v_\lambda^{\pi_k}(N-2,N-3) - 65v_\lambda^{\pi_k}(N-2,N-2) + 57v_\lambda^{\pi_k}(N-2,N) > 0, \quad \text{for } k=1, \lambda \in [\lambda_c, 1) \tag{S97}$$
$$\text{and } k=2, \lambda \in (0,\lambda_c],$$

$$20v_\lambda^{\pi_k}(N-2,N-3) + 31v_\lambda^{\pi_k}(N-2,N-2) + 57v_\lambda^{\pi_k}(N-2,N) - 108v_\lambda^{\pi_k}(N,N-3) > 0, \tag{S98}$$
$$\text{for } k=1, \lambda \in [\lambda_c, 1) \text{ and } k=2, \lambda \in (0,\lambda_c],$$

$$-4v_\lambda^{\pi_k}(N-2,N-3) + 15v_\lambda^{\pi_k}(N-2,N-2) - 18v_\lambda^{\pi_k}(N,N-3) + 7v_\lambda^{\pi_k}(N,N-2) > 0, \tag{S99}$$
$$\text{for } k=1, \lambda \in [\lambda_c, 1) \text{ and } k=2, \lambda \in (0,\lambda_c],$$

$$6v_\lambda^{\pi_k}(N-2,j) - 40v_\lambda^{\pi_k}(N-2,j+1) + 9v_\lambda^{\pi_k}(N,j) + 25v_\lambda^{\pi_k}(N,j+1) > 0, \quad j=2,\dots,N-4, \tag{S100}$$
$$\text{for } k=1, \lambda \in [\lambda_c, 1) \text{ and } k=2, \lambda \in (0,\lambda_c],$$

$$-30v_\lambda^{\pi_k}(N-2,j)-32v_\lambda^{\pi_k}(N-2,j+1)-40v_\lambda^{\pi_k}(N-2,j+2)+27v_\lambda^{\pi_k}(N,j)+75v_\lambda^{\pi_k}(N,j+1)>0, \tag{S101}$$
$j=2,\dots,N-4,$ for $k=1,\lambda\in[\lambda_c,1)$ and $k=2,\lambda\in(0,\lambda_c]$,

$$12v_\lambda^{\pi_k}(N-2,j)+8v_\lambda^{\pi_k}(N-2,j+1)-45v_\lambda^{\pi_k}(N,j)+25v_\lambda^{\pi_k}(N,j+1)>0,\quad j=2,\dots,N-4, \tag{S102}$$
for $k=1,\lambda\in[\lambda_c,1)$ and $k=2,\lambda\in(0,\lambda_c]$

$$8v_\lambda^{\pi_k}(N-3,N-3)-65v_\lambda^{\pi_k}(N-2,N-3)+57v_\lambda^{\pi_k}(N,N-3)>0,\quad \text{for } k=1,\lambda\in[\lambda_c,1) \tag{S103}$$
and $k=2,\lambda\in(0,\lambda_c]$,

$$-8v_\lambda^{\pi_k}(N-3,N-3)-v_\lambda^{\pi_k}(N-2,N-3)-48v_\lambda^{\pi_k}(N-2,N-2)+57v_\lambda^{\pi_k}(N,N-3)>0, \tag{S104}$$
for $k=1,\lambda\in[\lambda_c,1)$ and $k=2,\lambda\in(0,\lambda_c]$,

$$v_\lambda^{\pi_k}(N-3,j)-5v_\lambda^{\pi_k}(N-3,j+1)+v_\lambda^{\pi_k}(N-2,j)+3v_\lambda^{\pi_k}(N-2,j+1)>0, j=2,\dots,N-4, \tag{S105}$$
for $k=1,\lambda\in[\lambda_c,1)$ and $k=2,\lambda\in(0,\lambda_c]$,

$$-3v_\lambda^{\pi_k}(N-3,j)-4v_\lambda^{\pi_k}(N-3,j+1)-5v_\lambda^{\pi_k}(N-3,j+2)+3v_\lambda^{\pi_k}(N-2,j) \tag{S106}$$
$+9v_\lambda^{\pi_k}(N-2,j+1)>0,\quad j=2,\dots,N-4,\quad$ for $k=1,\lambda\in[\lambda_c,1)\quad$ and $k=2,\lambda\in(0,\lambda_c]$,

$$v_\lambda^{\pi_k}(N-3,j)+v_\lambda^{\pi_k}(N-3,j+1)-5v_\lambda^{\pi_k}(N-2,j)+3v_\lambda^{\pi_k}(N-2,j+1)>0, \tag{S107}$$
$j=2,\dots,N-4,$ for $k=1,\lambda\in[\lambda_c,1)$ and $k=2,\lambda\in(0,\lambda_c]$

$$8v_\lambda^{\pi_k}(N-3,j)+16v_\lambda^{\pi_k}(N-3,j+1)-15v_\lambda^{\pi_k}(N-2,j)+48v_\lambda^{\pi_k}(N-2,j+1)-57v_\lambda^{\pi_k}(N,j)>0, \tag{S108}$$
$j=2,\dots,N-4,$ for $k=1,\lambda\in[\lambda_c,1)\quad$ and $k=2,\lambda\in(0,\lambda_c]$,

$$v_\lambda^{\pi_k}(i,j)+v_\lambda^{\pi_k}(i+1,j)-5v_\lambda^{\pi_k}(i,j+1)+3v_\lambda^{\pi_k}(i+1,j+1)>0,\quad \lambda\in(0,1),\quad i,j=2,\dots,N-4, \tag{S109}$$
for $k=1,\lambda\in[\lambda_c,1)$ and $k=2,\lambda\in(0,\lambda_c]$,

$$v_\lambda^{\pi_k}(i,j)-5v_\lambda^{\pi_k}(i+1,j)+v_\lambda^{\pi_k}(i,j+1)+3v_\lambda^{\pi_k}(i+1,j+1)>0,\quad \lambda\in(0,1),\quad i,j=2,\dots,N-4, \tag{S110}$$
for $k=1,\lambda\in[\lambda_c,1)$ and $k=2,\lambda\in(0,\lambda_c]$,

$$-3v_\lambda^{\pi_k}(i,j)+3v_\lambda^{\pi_k}(i+1,j)-4v_\lambda^{\pi_k}(i,j+1)+9v_\lambda^{\pi_k}(i+1,j+1)-5v_\lambda^{\pi_k}(i,j+2)>0, \tag{S111}$$
$i,j=2,\dots,N-4$ for $k=1,\lambda\in[\lambda_c,1)\quad$ and $k=2,\lambda\in(0,\lambda_c]$

$$-3v_\lambda^{\pi_k}(i,j)-4v_\lambda^{\pi_k}(i+1,j)+3v_\lambda^{\pi_k}(i,j+1)+9v_\lambda^{\pi_k}(i+1,j+1)-5v_\lambda^{\pi_k}(i+2,j)>0, \tag{S112}$$
$i,j=2,\dots,N-4,$ for $k=1,\lambda\in[\lambda_c,1)\quad$ and $k=2,\lambda\in(0,\lambda_c]$.

We start by showing that $v_{\lambda_c}^{\pi_1}(N,j)=v_{\lambda_c}^{\pi_2}(N,j):=v_{\lambda_c}(N,j)$, for all $j=2,\dots,N-2$. This statement is trivial for $j=N-4,N-3,N-2,N$, by the fact that $d^1(N,j)=d^2(N,j)$ for these values of j and the fact that rectangles cannot shrink. Let $v_{\lambda_c}(N,j):=v_{\lambda_c}^{\pi_1}(N,j)=v_{\lambda_c}^{\pi_2}(N,j)$ for $j=N-4,N-3,N-2,N$. Using expression (S84) and recalling that $\lambda_c=15/17$, we compute $v_{\lambda_c}(N,N-4)=19550/3551$ and $v_{\lambda_c}(NN-3)=23460/3551$. We now use a backward induction argument to show that $v_{\lambda_c}^{\pi_1}(N,j)=v_{\lambda_c}^{\pi_2}(N,j)$ holds for $j=2,\dots,N-5$. First of all, we verify its validity for $j=N-5$. By expression (S84), we have

$$v_{\lambda_c}^{\pi_1}(N,N-5)=\frac{2\lambda_c}{3-\lambda_c}v_{\lambda_c}^{\pi_1}(N,N-4)=\frac{48875}{10653}.$$

By expression (S85), we obtain

$$v_{\lambda_c}^{\pi_2}(N,N-5)=\frac{7\lambda_c}{3(9-5\lambda_c)}v_{\lambda_c}^{\pi_2}(N,N-4)+\frac{5\lambda_c}{3(9-5\lambda_c)}v_{\lambda_c}^{\pi_2}(N,N-3)=\frac{48875}{10653}.$$

Thus, we have

$$v_{\lambda^c}^{\pi_1}(N,N-5)=v_{\lambda^c}^{\pi_2}(N,N-5)=\frac{48875}{10653}:=v_{\lambda^c}(N,N-5).$$

Similarly, we find

$$v_{\lambda_c}^{\pi_1}(N,N-6)=v_{\lambda^c}^{\pi_2}(N,N-6)=\frac{244375}{63918}:=v_{\lambda_c}(N,N-6).$$

Now, suppose that $v_{\lambda_c}^{\pi_1}(N,j)=v_{\lambda_c}^{\pi_2}(N,j):=v_{\lambda_c}(N,j)$ for all $j\geq n+1$, $n=2,\dots,N-7$. We proceed to show that this implies $v_{\lambda_c}^{\pi_1}(N,n)=v_{\lambda_c}^{\pi_2}(N,n):=v_{\lambda_c}(N,n)$. Using expression (S84) and the induction hypothesis, we obtain

$$\begin{aligned}v_{\lambda_c}^{\pi_1}(N,n)&=\frac{2\lambda_c}{3-\lambda_c}v_{\lambda_c}(N,n+1)\\&=\frac{7\lambda_c}{3(9-5\lambda_c)}v_{\lambda_c}(N,n+1)+\left(\frac{2\lambda_c}{3-\lambda_c}-\frac{7\lambda_c}{3(9-5\lambda_c)}\right)\frac{2\lambda_c}{3-\lambda_c}v_{\lambda_c}(N,n+2)\end{aligned}$$

$$= \frac{7\lambda_c}{3(9-5\lambda^c)} v_{\lambda_c}(N, n+1) + \frac{2\lambda_c^2(33-23\lambda_c)}{3(3-\lambda_c)^2(9-5\lambda_c)} v_{\lambda_c}(N, n+2)$$
$$= \frac{7\lambda_c}{3(9-5\lambda_c)} v_{\lambda_c}(N, n+1) + \frac{5\lambda_c}{3(9-5\lambda_c)} v_{\lambda_c}(N, n+2),$$

where we used the fact that

$$\frac{2\lambda_c^2(33-23\lambda_c)}{3(3-\lambda_c)^2(9-5\lambda_c)} = \frac{25}{78} = \frac{5\lambda_c}{3(9-5\lambda_c)}.$$

Thus, by expression (S85) and the induction hypothesis, we have $v_{\lambda^c}^{\pi_1}(N, n) = v_{\lambda^c}^{\pi_2}(N, n) := v_{\lambda_c}(N, n)$. It follows that $v_{\lambda^c}^{\pi_1}(N, j) = v_{\lambda^c}^{\pi_2}(N, j) := v_{\lambda_c}(N, j)$ for all $j = 2, \dots, N-2$. Note that this immediately implies the validity of equation (S95) above.

We now turn our attention to the remaining inequalities listed above. Inequalities (S91), (S92), (S96), (S97), (S98), (S99), (S103) and (S104) can be verified easily using the explicit expressions for the value functions collected in Table S14. We now proceed to prove the remaining inequalities by means of induction over the size of the rectangle.

Inequality (S93) First, we verify the validity of inequality (S93) for $j = N-5$ using the explicit expressions for $v_\lambda^{\pi_1}(N, N-5)$, $v_\lambda^{\pi_1}(N, N-4)$ and $v_\lambda^{\pi}(N, N-3)$, $\lambda \in (\lambda_c, 1)$ recorded in Table S14. Now, assume that it holds for $j = n+1$ for some $n = 2, \dots, N-6$. Using expression (S84), we obtain for $j = n$:

$$-6v_\lambda^{\pi_1}(N, n) + 11v_\lambda^{\pi_1}(N, n+1) - 5v_\lambda^{\pi_1}(N, n+2)$$
$$= \frac{2\lambda}{3-\lambda}\Big(-6v_\lambda^{\pi_1}(N, n+1) + 11v_\lambda^{\pi_1}(N, n+2) - 5v_\lambda^{\pi_1}(N, n+3)\Big).$$

From the induction hypothesis, it now follows that

$$-6v_\lambda^{\pi_1}(N, n) + 11v_\lambda^{\pi_1}(N, n+1) - 5v_\lambda^{\pi_1}(N, n+2) > 0.$$

Thus, inequality (S93) holds for all $j = 2, \dots, N-5$.

Inequality (S94) We first verify the correctness of inequality (S94) for $j = N-5$ and $j = N-6$ using the explicit expressions collected in Table S14. Suppose now that it is valid for all $j \in \{n+1, \dots, N-5\}$ for some $n = 2, \dots, N-7$. Using expression (S85), we obtain

$$\begin{aligned}
&6v_\lambda^{\pi_2}(N, n) - 11v_\lambda^{\pi_2}(N, n+1) + 5v_\lambda^{\pi_2}(N, n+2)\\
&= 6\left(\frac{7\lambda}{3(9-5\lambda)} v_\lambda^{\pi_2}(N, n+1) + \frac{5\lambda}{3(9-5\lambda)} v_\lambda^{\pi_2}(N, n+2)\right)\\
&- 11\left(\frac{7\lambda}{3(9-5\lambda)} v_\lambda^{\pi_2}(N, n+2) + \frac{5\lambda}{3(9-5\lambda)} v_\lambda^{\pi_2}(N, n+3)\right)\\
&+ 5\left(\frac{7\lambda}{3(9-5\lambda)} v_\lambda^{\pi_2}(N, n+3) + \frac{5\lambda}{3(9-5\lambda)} v_\lambda^{\pi_2}(N, n+4)\right)\\
&= \frac{7\lambda}{3(9-5\lambda)}\Big(6v_\lambda^{\pi_2}(N, n+1) - 11v_\lambda^{\pi_2}(N, n+2) + 5v_\lambda^{\pi_2}(N, n+3)\Big)\\
&+ \frac{5\lambda}{3(9-5\lambda)}\Big(6v_\lambda^{\pi_2}(N, n+2) - 11v_\lambda^{\pi_2}(N, n+3) + 5v_\lambda^{\pi_2}(N, n+4)\Big).
\end{aligned}$$

Invoking the induction hypothesis now yields

$$6v_\lambda^{\pi_2}(N, n) - 11v_\lambda^{\pi_2}(N, n+1) + 5v_\lambda^{\pi_2}(N, n+2) > 0.$$

It follows that inequality (S94) holds for all $j = 2, \dots, N-5$.

Inequality (S100) We start by verifying the validity of inequality (S100) for $j = N-4$ using the explicit expressions for the value functions in Table S14. Then, we assume that it holds for $j = n+1$ for some $n = 2, \dots, N-5$ and show that its validity carries over to $j = n$. Expression (S88) implies

$$\begin{aligned}
&6v_\lambda^{\pi_k}(N-2, n) - 40v_\lambda^{\pi_k}(N-2, n+1) + 9v_\lambda^{\pi_k}(N, n) + 25v_\lambda^{\pi_k}(N, n+1)\\
&= \frac{72}{12-5\lambda}\left(\frac{\lambda}{8} v_\lambda^{\pi_k}(N, n) + \frac{25\lambda}{72} v_\lambda^{\pi_k}(N, n+1) + \frac{\lambda}{9} v_\lambda^{\pi_k}(N-2, n+1)\right)
\end{aligned}$$

$$
\begin{aligned}
&- \frac{480}{12-5\lambda}\left(\frac{\lambda}{8}v_\lambda^{\pi_k}(N,n+1)+\frac{25\lambda}{72}v_\lambda^{\pi_k}(N,n+2)+\frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,n+2)\right)+9v_\lambda^{\pi_k}(N,n)+25v_\lambda^{\pi_k}(N,n+1)\\
&=\frac{4\lambda}{3(12-5\lambda)}\Big(6v_\lambda^{\pi_k}(N-2,n+1)-40v_\lambda^{\pi_k}(N-2,n+2)+9v_\lambda^{\pi_k}(N,n+1)+25v_\lambda^{\pi_k}(N,n+2)\Big)\\
&+\frac{36(3-\lambda)}{12-5\lambda}v_\lambda^{\pi_k}(N,n)+\frac{4(75-43\lambda)}{12-5\lambda}v_\lambda^{\pi_k}(N,n+1)-\frac{200\lambda}{12-5\lambda}v_\lambda^{\pi_k}(N,n+2),
\end{aligned}
$$

for $k=1,2$, $\lambda\in(0,1)$. From the induction hypothesis, it follows that it suffices to show that

$$
\frac{36(3-\lambda)}{12-5\lambda}v_\lambda^{\pi_k}(N,\ell)+\frac{4(75-43\lambda)}{12-5\lambda}v_\lambda^{\pi_k}(N,\ell+1)-\frac{200\lambda}{12-5\lambda}v_\lambda^{\pi_k}(N,\ell+2)\geq 0, \tag{S113}
$$

for all $\ell=2,\dots,N-5$. We prove this statement by means of another induction argument, distinguishing between $k=1$ and $k=2$. For $k=1$, we first verify the validity of expression (S113) for $\ell=N-5$, using the explicit expressions for $v_\lambda^{\pi_1}(N,N-5)$, $v_\lambda^{\pi_1}(N,N-4)$ and $v_\lambda^{\pi_1}(N,N-3)$ recorded in Table S14. Now, assume that it holds for $\ell=\tilde{\ell}+1$ for some $\tilde{\ell}=2,\dots,N-6$. By expression (S84) and the induction hypothesis, we obtain

$$
\begin{aligned}
&\frac{36(3-\lambda)}{12-5\lambda}v_\lambda^{\pi_1}(N,\tilde{\ell})+\frac{4(75-43\lambda)}{12-5\lambda}v_\lambda^{\pi_1}(N,\tilde{\ell}+1)-\frac{200\lambda}{12-5\lambda}v_\lambda^{\pi_1}(N,\tilde{\ell}+2)\\
&=\frac{2\lambda}{3-\lambda}\Big(\frac{36(3-\lambda)}{12-5\lambda}v_\lambda^{\pi_1}(N,\tilde{\ell}+1)+\frac{4(75-43\lambda)}{12-5\lambda}v_\lambda^{\pi_1}(N,\tilde{\ell}+2)-\frac{200\lambda}{12-5\lambda}v_\lambda^{\pi_1}(N,\tilde{\ell}+3)\Big)>0.
\end{aligned}
$$

This establishes the validity of expression (S113) for all $\ell=2,\dots,N-5$ and $k=1$.
We now turn our attention to $k=2$. In this case, we verify the validity of expression (S113) for $\ell=N-5$ and $\ell=N-6$, using Table S14, and make the slightly stronger assumption that it holds for all $\ell\geq\tilde{\ell}+1$ for some $\tilde{\ell}=2,\dots,N-7$. By expression (S85), we obtain

$$
\begin{aligned}
&\frac{36(3-\lambda)}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell})+\frac{4(75-43\lambda)}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell}+1)-\frac{200\lambda}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell}+2)\\
&=\frac{36(3-\lambda)}{12-5\lambda}\left(\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+1)+\frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+2)\right)\\
&+\frac{4(75-43\lambda)}{12-5\lambda}\left(\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+2)+\frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+3)\right)\\
&-\frac{200\lambda}{12-5\lambda}\left(\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+3)+\frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+4)\right)\\
&=\frac{7\lambda}{3(9-5\lambda)}\Big(\frac{36(3-\lambda)}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell}+1)+\frac{4(75-43\lambda)}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell}+2)-\frac{200\lambda}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell}+3)\Big)\\
&+\frac{5\lambda}{3(9-5\lambda)}\Big(\frac{36(3-\lambda)}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell}+2)+\frac{4(75-43\lambda)}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell}+3)-\frac{200\lambda}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell}+4)\Big).
\end{aligned}
$$

The induction hypothesis now implies

$$
\frac{36(3-\lambda)}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell})+\frac{4(75-43\lambda)}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell}+1)-\frac{200\lambda}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell}+2)\geq 0,
$$

which proves the correctness of inequality (S113) for all $\ell=2,\dots,N-5$, $k=2$ and $\lambda\in(0,\lambda^c]$. It now follows that

$$
6v_\lambda^{\pi_k}(N-2,n)-40v_\lambda^{\pi_k}(N-2,n+1)+9v_\lambda^{\pi_k}(N,n)+25v_\lambda^{\pi_k}(N,n+1)>0,
$$

for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. Thus, inequality (S100) is valid for all $j=2,\dots,N-4$, for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$.

Inequality (S101) First, we verify the correctness of inequality (S101) for $j=N-4$ and $j=N-5$, for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$, using Table S14. Now, assume that it holds for all $j=n+1,\dots N-4$, for some $n=2,\dots,N-6$, for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. By expression (S88) we obtain

$$
\begin{aligned}
&-30v_\lambda^{\pi_k}(N-2,n)-32v_\lambda^{\pi_k}(N-2,n+1)-40v_\lambda^{\pi_k}(N-2,n+2)+27v_\lambda^{\pi_k}(N,n)+75v_\lambda^{\pi_k}(N,n+1)\\
&=-\frac{360}{12-5\lambda}\left(\frac{\lambda}{8}v_\lambda^{\pi_k}(N,n)+\frac{25\lambda}{72}v_\lambda^{\pi_k}(N,n+1)+\frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,n+1)\right)
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{384}{12-5\lambda}\left(\frac{\lambda}{8}v_\lambda^{\pi_k}(N,n+1)+\frac{25\lambda}{72}v_\lambda^{\pi_k}(N,n+2)+\frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,n+2)\right)\\
&- \frac{480}{12-5\lambda}\left(\frac{\lambda}{8}v_\lambda^{\pi_k}(N,n+2)+\frac{25\lambda}{72}v_\lambda^{\pi_k}(N,n+3)+\frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,n+3)\right)+27v_\lambda^{\pi_k}(N,n)+75v_\lambda^{\pi_k}(N,n+1)\\
&= \frac{4\lambda}{3(12-5\lambda)}\Big(-30v_\lambda^{\pi_k}(N-2,n+1)-32v_\lambda^{\pi_k}(N-2,n+2)-40v_\lambda^{\pi_k}(N-2,n+3)+27v_\lambda^{\pi_k}(N,n+1)\\
&+75v_\lambda^{\pi_k}(N,n+2)\Big)+\frac{36(9-5\lambda)}{12-5\lambda}v_\lambda^{\pi_k}(N,n)+\frac{900-584\lambda}{12-5\lambda}v_\lambda^{\pi_k}(N,n+1)-\frac{880\lambda}{3(12-5\lambda)}v_\lambda^{\pi_k}(N,n+2)\\
&- \frac{500\lambda}{3(12-5\lambda)}v_\lambda^{\pi_k}(N,n+3),
\end{aligned}
$$

for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. From the induction hypothesis, it follows that it suffices to show that

$$
\frac{36(9-5\lambda)}{12-5\lambda}v_\lambda^{\pi_k}(N,\ell)+\frac{900-584\lambda}{12-5\lambda}v_\lambda^{\pi_k}(N,\ell+1)-\frac{880\lambda}{3(12-5\lambda)}v_\lambda^{\pi_k}(N,\ell+2)-\frac{500\lambda}{3(12-5\lambda)}v_\lambda^{\pi_k}(N,\ell+3)\geq 0, \tag{S114}
$$

for all $\ell=2,\ldots,N-6$, for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. We first consider the case $k=1$, $\lambda\in[\lambda_c,1)$. After verifying the correctness of expression (S114) for $\ell=N-6$, using Table S14, we assume that it holds for $\ell=\tilde{\ell}+1$, for some $\tilde{\ell}=2,\ldots,N-7$. Expression (S84) and this induction hypothesis imply

$$
\begin{aligned}
&\frac{36(9-5\lambda)}{12-5\lambda}v_\lambda^{\pi_1}(N,\tilde{\ell})+\frac{900-584\lambda}{12-5\lambda}v_\lambda^{\pi_1}(N,\tilde{\ell}+1)-\frac{880\lambda}{3(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{\ell}+2)-\frac{500\lambda}{3(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{\ell}+3)\\
&=\frac{2\lambda}{3-\lambda}\left(\frac{36(9-5\lambda)}{12-5\lambda}v_\lambda^{\pi_1}(N,\tilde{\ell}+1)+\frac{900-584\lambda}{12-5\lambda}v_\lambda^{\pi_1}(N,\tilde{\ell}+2)-\frac{880\lambda}{3(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{\ell}+3)\right.\\
&\left.-\frac{500\lambda}{3(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{\ell}+4)\right)\geq 0.
\end{aligned}
$$

Hence, expression (S114) holds for all $\ell=2,\ldots,N-6$, for $k=1$, $\lambda\in[\lambda_c,1)$.

We proceed to consider the case $k=2$, $\lambda\in(0,\lambda_c]$. First of all, we again use Table S14 to verify the validity of expression (S114) for $\ell=N-6$ and $\ell=N-7$. Then, we make the slightly stronger assumption that it is satisfied for all $\ell=\tilde{\ell}+1,\ldots N-6$, for some $\tilde{\ell}=2,\ldots,N-8$. Using expression (S85) we obtain

$$
\begin{aligned}
&\frac{36(9-5\lambda)}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell})+\frac{900-584\lambda}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell}+1)-\frac{880\lambda}{3(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+2)-\frac{500\lambda}{3(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+3)\\
&=\frac{36(9-5\lambda)}{12-5\lambda}\left(\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+1)+\frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+2)\right)+\frac{900-584\lambda}{12-5\lambda}\left(\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+2)\right.\\
&\left.+\frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+3)\right)-\frac{880\lambda}{3(12-5\lambda)}\left(\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+3)+\frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+4)\right)\\
&-\frac{500\lambda}{3(12-5\lambda)}\left(\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+4)+\frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+5)\right)\\
&=\frac{7\lambda}{3(9-5\lambda)}\left(\frac{36(9-5\lambda)}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell}+1)+\frac{900-584\lambda}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell}+2)-\frac{880\lambda}{3(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+3)\right.\\
&\left.-\frac{500\lambda}{3(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+4)\right)+\frac{5\lambda}{3(9-5\lambda)}\left(\frac{36(9-5\lambda)}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell}+2)+\frac{900-584\lambda}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell}+3)\right.\\
&\left.-\frac{880\lambda}{3(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+4)-\frac{500\lambda}{3(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+5)\right).
\end{aligned}
$$

The induction hypothesis now implies that

$$
\frac{36(9-5\lambda)}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell})+\frac{900-584\lambda}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell}+1)-\frac{880\lambda}{3(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+2)-\frac{500\lambda}{3(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+3)\geq 0.
$$

Thus, expression (S114) holds for all $\ell = 2, \ldots, N-6$ for $k = 2$, $\lambda \in (0, \lambda_c]$ as well. From this, we can conclude that

$$-30v_\lambda^{\pi_k}(N-2,n) - 32v_\lambda^{\pi_k}(N-2,n+1) - 40v_\lambda^{\pi_k}(N-2,n+2) + 27v_\lambda^{\pi_k}(N,n) + 75v_\lambda^{\pi_k}(N,n+1) > 0,$$

for $k = 1$, $\lambda \in [\lambda_c, 1)$ and $k = 2$, $\lambda \in (0, \lambda_c]$. Thus, we have established that inequality (S101) holds for all $j = 2, \ldots, N-4$ for $k = 1$, $\lambda \in [\lambda_c, 1)$ and $k = 2$, $\lambda \in (0, \lambda_c]$.

Inequality (S102) We again start by verifying the validity of inequality (S102) for $j = N-4$, for $k = 1$, $\lambda \in [\lambda_c, 1)$ and $k = 2$, $\lambda \in (0, \lambda_c]$, using Table S14. Assume that it is satisfied for $j = n+1$, for some $n = 2, \ldots, N-5$, for $k = 1$, $\lambda \in [\lambda_c, 1)$ and $k = 2$, $\lambda \in (0, \lambda_c]$. Expression (S88) yields

$$\begin{aligned}
&12v_\lambda^{\pi_k}(N-2,n) + 8v_\lambda^{\pi_k}(N-2,n+1) - 45v_\lambda^{\pi_k}(N,n) + 25v_\lambda^{\pi_k}(N,n+1)\\
&= \frac{144}{12-5\lambda}\left(\frac{\lambda}{8}v_\lambda^{\pi_k}(N,n) + \frac{25\lambda}{72}v_\lambda^{\pi_k}(N,n+1) + \frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,n+1)\right)\\
&+ \frac{96}{12-5\lambda}\left(\frac{\lambda}{8}v_\lambda^{\pi_k}(N,n+1) + \frac{25\lambda}{72}v_\lambda^{\pi_k}(N,n+2) + \frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,n+2)\right) - 45v_\lambda^{\pi_k}(N,n) + 25v_\lambda^{\pi_k}(N,n+1)\\
&= \frac{4\lambda}{3(12-5\lambda)}\Big(12v_\lambda^{\pi_k}(N-2,n+1) + 8v_\lambda^{\pi_k}(N-2,n+2) - 45v_\lambda^{\pi_k}(N,n+1) + 25v_\lambda^{\pi_k}(N,n+2)\Big)\\
&- \frac{27(20-9\lambda)}{12-5\lambda}v_\lambda^{\pi_k}(N,n) + \frac{3(100-\lambda)}{12-5\lambda}v_\lambda^{\pi_k}(N,n+1),
\end{aligned}$$

for $k = 1, 2$, $\lambda \in (0, 1)$. The induction hypothesis implies that a sufficient condition for inequality (S102) to hold for $j = n$ is given by

$$-\frac{27(20-9\lambda)}{12-5\lambda}v_\lambda^{\pi_k}(N,\ell) + \frac{3(100-\lambda)}{12-5\lambda}v_\lambda^{\pi_k}(N,\ell+1) \geq 0, \tag{S115}$$

for all $\ell = 2, \ldots, N-5$, for $k = 1$, $\lambda \in [\lambda_c, 1)$ and $k = 2$, $\lambda \in (0, \lambda^c]$. First, we consider the case $k = 1$, $\lambda \in [\lambda_c, 1)$. After verifying the validity of expression (S115) for $\ell = N-5$, using Table S14, we assume that it holds for $\ell = \tilde{\ell} + 1$ for some $\tilde{\ell} = 2, \ldots, N-6$. It now follows from expression (S84) and this induction hypothesis that

$$\begin{aligned}
&-\frac{27(20-9\lambda)}{12-5\lambda}v_\lambda^{\pi_1}(N,\tilde{\ell}) + \frac{3(100-\lambda)}{12-5\lambda}v_\lambda^{\pi_1}(N,\tilde{\ell}+1)\\
&= \frac{2\lambda}{3-\lambda}\left(-\frac{27(20-9\lambda)}{12-5\lambda}v_\lambda^{\pi_1}(N,\tilde{\ell}+1) + \frac{3(100-\lambda)}{12-5\lambda}v_\lambda^{\pi_1}(N,\tilde{\ell}+2)\right) \geq 0,
\end{aligned}$$

for $\lambda \in [\lambda^c, 1)$. This implies that expression (S115) is satisfied for all $\ell = 2, \ldots, N-5$, for $k = 1$, $\lambda \in [\lambda^c, 1)$. We proceed to consider the case $k = 2$, $\lambda \in (0, \lambda^c]$. Again, we use Table S14 to verify the correctness of expression (S115) for $\ell = N-5$ and $\ell = N-6$. Assume that it is satisfied for $\ell = \tilde{\ell}+1, \ldots, N-5$, where $\tilde{\ell} = 2, \ldots, N-7$. Using expression (S85), we obtain

$$\begin{aligned}
&-\frac{27(20-9\lambda)}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell}) + \frac{3(100-\lambda)}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell}+1)\\
&= -\frac{27(20-9\lambda)}{12-5\lambda}\left(\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+1) + \frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+2)\right)\\
&+ \frac{3(100-\lambda)}{12-5\lambda}\left(\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+2) + \frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{\ell}+3)\right)\\
&= \frac{7\lambda}{3(9-5\lambda)}\left(-\frac{27(20-9\lambda)}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell}+1) + \frac{3(100-\lambda)}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell}+2)\right)\\
&+ \frac{5\lambda}{3(9-5\lambda)}\left(-\frac{27(20-9\lambda)}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell}+2) + \frac{3(100-\lambda)}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell}+3)\right).
\end{aligned}$$

By the induction hypothesis, it now follows that

$$-\frac{27(20-9\lambda)}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell}) + \frac{300-3\lambda}{12-5\lambda}v_\lambda^{\pi_2}(N,\tilde{\ell}+1) \geq 0,$$

for all $\lambda \in (0, \lambda_c]$. This implies that expression (S115) holds for all $\ell = 2, \ldots, N-5$, for $k = 2$, $\lambda \in (0, \lambda_c]$ as well. Thus, we can conclude that

$$12v_\lambda^{\pi_k}(N-2,n) + 8v_\lambda^{\pi_k}(N-2,n+1) - 45v_\lambda^{\pi_k}(N,n) + 25v_\lambda^{\pi_k}(N,n+1) > 0,$$

for $k = 1$, $\lambda \in [\lambda_c, 1)$ and $k = 2$, $\lambda \in (0, \lambda_c]$. This establishes the validity of inequality (S102) for all $j = 2, \ldots, N-4$, for $k = 1$, $\lambda \in [\lambda_c, 1)$ and $k = 2$, $\lambda \in (0, \lambda_c]$.

Inequality (S105) First of all, we verify the correctness of inequality (S105) for $j = N-4$, for $k = 1$, $\lambda \in [\lambda_c, 1)$ and $k = 2$, $\lambda \in (0, \lambda_c]$ using Table S14. We proceed to assume that it holds for $j = n+1$ for some $n = 2, \ldots, N-5$, for $k = 1$, $\lambda \in [\lambda_c, 1)$ and $k = 2$, $\lambda \in (0, \lambda_c]$. Using expression (S90), we obtain

$$
\begin{aligned}
&v_\lambda^{\pi_k}(N-3,n) - 5v_\lambda^{\pi_k}(N-3,n+1) + v_\lambda^{\pi_k}(N-2,n) + 3v_\lambda^{\pi_k}(N-2,n+1) \\
&= \frac{\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(N-2,n) + v_\lambda^{\pi_k}(N-3,n+1) + 3v_\lambda^{\pi_k}(N-2,n+1)\Big) \\
&- \frac{5\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(N-2,n+1) + v_\lambda^{\pi_k}(N-3,n+2) + 3v_\lambda^{\pi_k}(N-2,n+2)\Big) + v_\lambda^{\pi_k}(N-2,n) + 3v_\lambda^{\pi_k}(N-2,n+1) \\
&= \frac{\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(N-3,n+1) - 5v_\lambda^{\pi_k}(N-3,n+2) + v_\lambda^{\pi_k}(N-2,n+1) + 3v_\lambda^{\pi_k}(N-2,n+2)\Big) \\
&+ \frac{3(3-\lambda)}{9-4\lambda}v_\lambda^{\pi_k}(N-2,n) + \frac{3(9-5\lambda)}{9-4\lambda}v_\lambda^{\pi_k}(N-2,n+1) - \frac{18\lambda}{9-4\lambda}v_\lambda^{\pi_k}(N-2,n+2),
\end{aligned}
$$

for $k = 1, 2$, $\lambda \in (0, 1)$. From the induction hypothesis, it follows that it suffices to show that

$$
\frac{3(3-\lambda)}{9-4\lambda}v_\lambda^{\pi_k}(N-2,\ell) + \frac{3(9-5\lambda)}{9-4\lambda}v_\lambda^{\pi_k}(N-2,\ell+1) - \frac{18\lambda}{9-4\lambda}v_\lambda^{\pi_k}(N-2,\ell+2) \geq 0, \tag{S116}
$$

holds for all $\ell = 2, \ldots, N-5$, for $k = 1$, $\lambda \in [\lambda_c, 1)$ and for $k = 2$, $\lambda \in (0, \lambda_c]$. We prove this statement by means of another induction argument. First of all, we verify its validity for $\ell = N-5$ using Table S14. Now suppose that it is true for $\ell = \tilde{\ell} + 1$, for some $\tilde{\ell} = 2, \ldots, N-6$, for $k = 1$, $\lambda \in [\lambda_c, 1)$ and for $k = 2$, $\lambda \in (0, \lambda_c]$. Using expression (S88), we obtain

$$
\begin{aligned}
&\frac{3(3-\lambda)}{9-4\lambda}v_\lambda^{\pi_k}(N-2,\tilde{\ell}) + \frac{3(9-5\lambda)}{9-4\lambda}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+1) - \frac{18\lambda}{9-4\lambda}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+2) \\
&= \frac{3(3-\lambda)}{9-4\lambda}\frac{12}{12-5\lambda}\left(\frac{\lambda}{8}v_\lambda^{\pi_k}(N,\tilde{\ell}) + \frac{25\lambda}{72}v_\lambda^{\pi_k}(N,\tilde{\ell}+1) + \frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+1)\right) \\
&+ \frac{3(9-5\lambda)}{9-4\lambda}\frac{12}{12-5\lambda}\left(\frac{\lambda}{8}v_\lambda^{\pi_k}(N,\tilde{\ell}+1) + \frac{25\lambda}{72}v_\lambda^{\pi_k}(N,\tilde{\ell}+2) + \frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+2)\right) \\
&- \frac{18\lambda}{9-4\lambda}\frac{12}{12-5\lambda}\left(\frac{\lambda}{8}v_\lambda^{\pi_k}(N,\tilde{\ell}+2) + \frac{25\lambda}{72}v_\lambda^{\pi_k}(N,\tilde{\ell}+3) + \frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+3)\right) \\
&= \frac{4\lambda}{3(12-5\lambda)}\Big(\frac{3(3-\lambda)}{9-4\lambda}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+1) + \frac{3(9-5\lambda)}{9-4\lambda}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+2) - \frac{18\lambda}{9-4\lambda}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+3)\Big) \\
&+ \frac{9\lambda(3-\lambda)}{2(12-5\lambda)(9-4\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}) + \frac{\lambda(78-35\lambda)}{(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}+1) + \frac{\lambda(225-179\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}+2) \\
&- \frac{75\lambda^2}{(12-5\lambda)(9-4\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}+3),
\end{aligned}
$$

for $k = 1, 2$, $\lambda \in (0, 1)$. By the induction hypothesis, it follows that it suffices to show that

$$
\begin{aligned}
&\frac{9\lambda(3-\lambda)}{2(12-5\lambda)(9-4\lambda)}v_\lambda^{\pi_k}(N,m) + \frac{\lambda(78-35\lambda)}{(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,m+1) + \frac{\lambda(225-179\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,m+2) \\
&- \frac{75\lambda^2}{(12-5\lambda)(9-4\lambda)}v_\lambda^{\pi_k}(N,m+3) \geq 0,
\end{aligned} \tag{S117}
$$

for all $m = 2, \ldots, N-6$, for $k = 1$, $\lambda \in [\lambda_c, 1)$ and $k = 2$, $\lambda \in (0, \lambda_c]$. We prove this statement by yet another induction argument. Consider the case $k = 1$, $\lambda \in [\lambda_c, 1)$. Table S14 yields that expression (S117) holds with equality for $m = N-6$. Now, suppose that it holds with equality for $m = \tilde{m} + 1$, for some $\tilde{m} = 2, \ldots, N-7$. By expression (S84) and this induction hypothesis, we obtain

$$
\begin{aligned}
&\frac{9\lambda(3-\lambda)}{2(12-5\lambda)(9-4\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}) + \frac{\lambda(78-35\lambda)}{(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+1) + \frac{\lambda(225-179\lambda)}{2(12-5\lambda)(9-4\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+2) \\
&- \frac{75\lambda^2}{(12-5\lambda)(9-4\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+3)
\end{aligned}
$$

$$= \frac{2\lambda}{3-\lambda}\Bigg[\frac{9\lambda(3-\lambda)}{2(12-5\lambda)(9-4\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+1) + \frac{\lambda(78-35\lambda)}{(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+2)$$
$$+ \frac{\lambda(225-179\lambda)}{2(12-5\lambda)(9-4\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+3) - \frac{75\lambda^2}{(12-5\lambda)(9-4\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+4)\Bigg] = 0,$$

for all $\lambda \in [\lambda_c, 1)$. Hence, expression (S117) is satisfied with equality for all $m = 2, \ldots, N-6$, for $k = 1$, $\lambda \in [\lambda_c, 1)$. Now, consider the case $k = 2$, $\lambda \in (0, \lambda_c]$. Again using Table S14, we verify the correctness of expression (S117) for $m = N-6$ and $m = N-7$. Assume that it holds for all $m \geq \tilde{m}+1$, for some $\tilde{m} = 2, \ldots, N-8$. Expression (S85) yields

$$\frac{9\lambda(3-\lambda)}{2(12-5\lambda)(9-4\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}) + \frac{\lambda(78-35\lambda)}{(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+1) + \frac{\lambda(225-179\lambda)}{2(12-5\lambda)(9-4\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+2)$$
$$- \frac{75\lambda^2}{(12-5\lambda)(9-4\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+3)$$
$$= \frac{9\lambda(3-\lambda)}{2(12-5\lambda)(9-4\lambda)}\left(\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+1) + \frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+2)\right)$$
$$+ \frac{\lambda(78-35\lambda)}{(9-4\lambda)(12-5\lambda)}\left(\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+2) + \frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+3)\right)$$
$$+ \frac{\lambda(225-179\lambda)}{2(12-5\lambda)(9-4\lambda)}\left(\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+3) + \frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+4)\right)$$
$$- \frac{75\lambda^2}{(12-5\lambda)(9-4\lambda)}\left(\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+4) + \frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+5)\right)$$
$$= \frac{7\lambda}{3(9-5\lambda)}\Bigg[\frac{9\lambda(3-\lambda)}{2(12-5\lambda)(9-4\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+1) + \frac{\lambda(78-35\lambda)}{(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+2)$$
$$+ \frac{\lambda(225-179\lambda)}{2(12-5\lambda)(9-4\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+3) - \frac{75\lambda^2}{(12-5\lambda)(9-4\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+4)\Bigg]$$
$$+ \frac{5\lambda}{3(9-5\lambda)}\Bigg[\frac{9\lambda(3-\lambda)}{2(12-5\lambda)(9-4\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+2) + \frac{\lambda(78-35\lambda)}{(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+3)$$
$$+ \frac{\lambda(225-179\lambda)}{2(12-5\lambda)(9-4\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+4) - \frac{75\lambda^2}{(12-5\lambda)(9-4\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+5)\Bigg].$$

By the induction hypothesis, we now obtain

$$\frac{9\lambda(3-\lambda)}{2(12-5\lambda)(9-4\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}) + \frac{\lambda(78-35\lambda)}{(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+1) + \frac{\lambda(225-179\lambda)}{2(12-5\lambda)(9-4\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+2)$$
$$- \frac{75\lambda^2}{(12-5\lambda)(9-4\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+3) \geq 0,$$

for all $\lambda \in (0, \lambda_c]$. This implies that expression (S117) holds for all $m = 2, \ldots, N-6$ for $k = 2$, $\lambda \in (0, \lambda_c]$ as well. It follows that expression (S116) is valid for all $\ell = 2, \ldots, N-5$, for $k = 1$, $\lambda \in [\lambda_c, 1)$, $k = 2$, $\lambda \in (0, \lambda_c]$. This in turn implies the validity of inequality (S105) for all $j = 2, \ldots, N-4$, for $k = 1$, $\lambda \in [\lambda_c, 1)$, $k = 2$, $\lambda \in (0, \lambda_c]$.

Inequality (S106) We again start by verifying inequality (S106) for $j = N-4$ and $j = N-5$, for $k = 1$, $\lambda \in [\lambda_c, 1)$ and $k = 2$, $\lambda \in (0, \lambda_c]$, using Table S14. Suppose that it is satisfied for $j = n+1$ for some $n = 2, \ldots, N-6$, for $k = 1$, $\lambda \in [\lambda_c, 1)$ and $k = 2$, $\lambda \in (0, \lambda_c]$. Expressions (S88) and (S90) yield

$$-3v_\lambda^{\pi_k}(N-3,n) - 4v_\lambda^{\pi_k}(N-3,n+1) - 5v_\lambda^{\pi_k}(N-3,n+2) + 3v_\lambda^{\pi_k}(N-2,n) + 9v_\lambda^{\pi_k}(N-2,n+1)$$
$$= -\frac{3\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(N-2,n) + v_\lambda^{\pi_k}(N-3,n+1) + 3v_\lambda^{\pi_k}(N-2,n+1)\Big)$$
$$- \frac{4\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(N-2,n+1) + v_\lambda^{\pi_k}(N-3,n+2) + 3v_\lambda^{\pi_k}(N-2,n+2)\Big)$$

$$
\begin{aligned}
&- \frac{5\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(N-2,n+2) + v_\lambda^{\pi_k}(N-3,n+3) + 3v_\lambda^{\pi_k}(N-2,n+3)\Big) \\
&+ \frac{36}{12-5\lambda}\left(\frac{\lambda}{8}v_\lambda^{\pi_k}(N,n) + \frac{25\lambda}{72}v_\lambda^{\pi_k}(N,n+1) + \frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,n+1)\right) \\
&+ \frac{108}{12-5\lambda}\left(\frac{\lambda}{8}v_\lambda^{\pi_k}(N,n+1) + \frac{25\lambda}{72}v_\lambda^{\pi_k}(N,n+2) + \frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,n+2)\right) \\
&= \frac{\lambda}{9-4\lambda}\Big(-3v_\lambda^{\pi_k}(N-3,n+1) - 4v_\lambda^{\pi_k}(N-3,n+2) - 5v_\lambda^{\pi_k}(N-3,n+3) + 3v_\lambda^{\pi_k}(N-2,n+1) \\
&+ 9v_\lambda^{\pi_k}(N-2,n+2)\Big) - \frac{3\lambda}{9-4\lambda}v_\lambda^{\pi_k}(N-2,n) - \left(\frac{16\lambda}{9-4\lambda} - \frac{4\lambda}{12-5\lambda}\right)v_\lambda^{\pi_k}(N-2,n+1) \\
&- \left(\frac{26\lambda}{9-4\lambda} - \frac{12\lambda}{12-5\lambda}\right)v^{\pi_k}(N-2,n+2) - \frac{15\lambda}{9-4\lambda}v_\lambda^{\pi_k}(N-2,n+3) + \frac{9\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,n) \\
&+ \frac{26\lambda}{12-5\lambda}v_\lambda^{\pi_k}(N,n+1) + \frac{75\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,n+2),
\end{aligned}
$$

for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. The induction hypothesis now implies that it suffices to show that

$$
\begin{aligned}
&- \frac{3\lambda}{9-4\lambda}v_\lambda^{\pi_k}(N-2,\ell) - \left(\frac{16\lambda}{9-4\lambda} - \frac{4\lambda}{12-5\lambda}\right)v_\lambda^{\pi_k}(N-2,\ell+1) \\
&- \left(\frac{26\lambda}{9-4\lambda} - \frac{12\lambda}{12-5\lambda}\right)v^{\pi_k}(N-2,\ell+2) - \frac{15\lambda}{9-4\lambda}v_\lambda^{\pi_k}(N-2,\ell+3) + \frac{9\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,\ell) \\
&+ \frac{26\lambda}{12-5\lambda}v_\lambda^{\pi_k}(N,\ell+1) + \frac{75\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,\ell+2) \geq 0,
\end{aligned}
\tag{S118}
$$

for all $\ell=2,\dots,N-6$, for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. We prove this statement by means of an embedded induction argument. First of all, we use Table S14 to verify its validity for $\ell=N-6$. Now, we assume that it holds for $\ell=\tilde{\ell}+1$, for some $\tilde{\ell}=2,\dots,N-7$, for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda^c]$. Using expression (S88), we obtain

$$
\begin{aligned}
&- \frac{3\lambda}{9-4\lambda}v_\lambda^{\pi_k}(N-2,\tilde{\ell}) - \left(\frac{16\lambda}{9-4\lambda} - \frac{4\lambda}{12-5\lambda}\right)v_\lambda^{\pi_k}(N-2,\tilde{\ell}+1) - \left(\frac{26\lambda}{9-4\lambda} - \frac{12\lambda}{12-5\lambda}\right)v^{\pi_k}(N-2,\tilde{\ell}+2) \\
&- \frac{15\lambda}{9-4\lambda}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+3) + \frac{9\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}) + \frac{26\lambda}{12-5\lambda}v_\lambda^{\pi_k}(N,\tilde{\ell}+1) + \frac{75\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}+2) \\
&= -\frac{3\lambda}{9-4\lambda}\frac{12}{12-5\lambda}\left(\frac{\lambda}{8}v_\lambda^{\pi_k}(N,\tilde{\ell}) + \frac{25\lambda}{72}v_\lambda^{\pi_k}(N,\tilde{\ell}+1) + \frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+1)\right) \\
&- \left(\frac{16\lambda}{9-4\lambda} - \frac{4\lambda}{12-5\lambda}\right)\frac{12}{12-5\lambda}\left(\frac{\lambda}{8}v_\lambda^{\pi_k}(N,\tilde{\ell}+1) + \frac{25\lambda}{72}v_\lambda^{\pi_k}(N,\tilde{\ell}+2) + \frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+2)\right) \\
&- \left(\frac{26\lambda}{9-4\lambda} - \frac{12\lambda}{12-5\lambda}\right)\frac{12}{12-5\lambda}\left(\frac{\lambda}{8}v_\lambda^{\pi_k}(N,\tilde{\ell}+2) + \frac{25\lambda}{72}v_\lambda^{\pi_k}(N,\tilde{\ell}+3) + \frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+3)\right) \\
&- \frac{15\lambda}{9-4\lambda}\frac{12}{12-5\lambda}\left(\frac{\lambda}{8}v_\lambda^{\pi_k}(N,\tilde{\ell}+3) + \frac{25\lambda}{72}v_\lambda^{\pi_k}(N,\tilde{\ell}+4) + \frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+4)\right) + \frac{9\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}) \\
&+ \frac{26\lambda}{12-5\lambda}v_\lambda^{\pi_k}(N,\tilde{\ell}+1) + \frac{75\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}+2) \\
&= \frac{4\lambda}{3(12-5\lambda)}\Bigg[-\frac{3\lambda}{9-4\lambda}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+1) - \left(\frac{16\lambda}{9-4\lambda} - \frac{4\lambda}{12-5\lambda}\right)v_\lambda^{\pi_k}(N-2,\tilde{\ell}+2) \\
&- \left(\frac{26\lambda}{9-4\lambda} - \frac{12\lambda}{12-5\lambda}\right)v^{\pi_k}(N-2,\tilde{\ell}+3) - \frac{15\lambda}{9-4\lambda}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+4) + \frac{9\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}+1) \\
&+ \frac{26\lambda}{12-5\lambda}v_\lambda^{\pi_k}(N,\tilde{\ell}+2) + \frac{75\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}+3)\Bigg] + \frac{9\lambda(9-5\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell})
\end{aligned}
$$

$$+\frac{\lambda(468-281\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}+1)+\frac{\lambda(2025-1534\lambda)}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}+2)-\frac{785\lambda^2}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}+3)$$
$$-\frac{125\lambda^2}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}+4),$$

for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. The induction hypothesis implies that in order to prove expression (S118) for $\ell=\tilde{\ell}$, it suffices to show that

$$\begin{aligned}&\frac{9\lambda(9-5\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,m)+\frac{\lambda(468-281\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,m+1)\\&+\frac{\lambda(2025-1534\lambda)}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,m+2)-\frac{785\lambda^2}{6(9-4\lambda)(12-5\lambda))}v_\lambda^{\pi_k}(N,m+3)\\&-\frac{125\lambda^2}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,m+4)\geq 0,\end{aligned}\tag{S119}$$

for all $m=2,\dots,N-7$, for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. We prove this statement by means of another induction argument. First, we consider the case $k=1$, $\lambda\in[\lambda_c,1)$. After verifying expression (S119) for $m=N-7$, using Table S14, we assume that it holds for $m=\tilde{m}+1$, for some $\tilde{m}=2,\dots,N-8$. This induction hypothesis, together with expression (S84) now implies

$$\begin{aligned}&\frac{9\lambda(9-5\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{m})+\frac{\lambda(468-281\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+1)+\frac{\lambda(2025-1534\lambda)}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+2)\\&-\frac{785\lambda^2}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+3)-\frac{125\lambda^2}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+4)\\&=\frac{2\lambda}{3-\lambda}\left[\frac{9\lambda(9-5\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+1)+\frac{\lambda(468-281\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+2)\right.\\&+\frac{\lambda(2025-1534\lambda)}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+3)-\frac{785\lambda^2}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+4)\\&\left.-\frac{125\lambda^2}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+5)\right]\geq 0.\end{aligned}$$

Thus, expression (S119) holds for all $m=2,\dots,N-7$, for $k=1$, $\lambda\in[\lambda_c,1)$. We now direct our attention to the case $k=2$, $\lambda\in(0,\lambda^c]$. From Table S14, it follows that expression (S119) holds with equality for $m=N-7$ and $m=N-8$. We proceed to assume that it holds with equality for all $m=\tilde{m}+1,\dots,N-7$, for some $\tilde{m}=2,\dots,N-9$. By expression (S85), we have

$$\begin{aligned}&\frac{9\lambda(9-5\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m})+\frac{\lambda(468-281\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+1)+\frac{\lambda(2025-1534\lambda)}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+2)\\&-\frac{785\lambda^2}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+3)-\frac{125\lambda^2}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+4)\\&=\frac{9\lambda(9-5\lambda)}{2(9-4\lambda)(12-5\lambda)}\left[\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+1)+\frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+2)\right]\\&+\frac{\lambda(468-281\lambda)}{2(9-4\lambda)(12-5\lambda)}\left[\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+2)+\frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+3)\right]\\&+\frac{\lambda(2025-1534\lambda)}{6(9-4\lambda)(12-5\lambda)}\left[\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+3)+\frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+4)\right]\\&-\frac{785\lambda^2}{6(9-4\lambda)(12-5\lambda)}\left[\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+4)+\frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+5)\right]\\&-\frac{125\lambda^2}{2(9-4\lambda)(12-5\lambda)}\left[\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+5)+\frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+6)\right]\\&=\frac{7\lambda}{3(9-5\lambda)}\left[\frac{9\lambda(9-5\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{m}+1)+\frac{\lambda(468-281\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{m}+2)\right.\end{aligned}$$

$$+\frac{\lambda(2025-1534\lambda)}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{m}+3)-\frac{785\lambda^2}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{m}+4)$$
$$-\frac{125\lambda^2}{2(12-5\lambda)(9-4\lambda)}v_\lambda^{\pi_k}(N,\tilde{m}+5)\Bigg]+\frac{5\lambda}{3(9-5\lambda)}\Bigg[\frac{9\lambda(9-5\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{m}+2)$$
$$+\frac{\lambda(468-281\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{m}+3)+\frac{\lambda(2025-1534\lambda)}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{m}+4)$$
$$-\frac{785\lambda^2}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{m}+5)-\frac{125\lambda^2}{2(12-5\lambda)(9-4\lambda)}v_\lambda^{\pi_k}(N,\tilde{m}+6)\Bigg].$$

From the induction hypothesis, it now follows that

$$\frac{9\lambda(9-5\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m})+\frac{\lambda(468-281\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+1)+\frac{\lambda(2025-1534\lambda)}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+2)$$
$$-\frac{785\lambda^2}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+3)-\frac{125\lambda^2}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+4)=0,$$

for all $\lambda\in(0,\lambda_c]$. Hence, expression (S119) is satisfied for all $m=2,\ldots,N-7$, for $k=2$, $\lambda\in(0,\lambda_c]$ as well. It follows that expression (S118) holds for all $\ell=2,\ldots,N-6$, for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. This in turn implies the validity of expression (S106) for all $j=2,\ldots,N-4$, for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$.

Inequality (S107) First of all, we use Table S14 to verify inequality (S107) for $j=N-4$, for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. We now assume that it holds for $j=n+1$, for some $n=2,\ldots,N-5$, for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. By expressions (S88) and (S90), we obtain

$$\begin{aligned}
&v_\lambda^{\pi_k}(N-3,n)+v_\lambda^{\pi_k}(N-3,n+1)-5v_\lambda^{\pi_k}(N-2,n)+3v_\lambda^{\pi_k}(N-2,n+1)\\
&=\frac{\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(N-2,n)+v_\lambda^{\pi_k}(N-3,n+1)+3v_\lambda^{\pi_k}(N-2,n+1)\Big)\\
&+\frac{\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(N-2,n+1)+v_\lambda^{\pi_k}(N-3,n+2)+3v_\lambda^{\pi_k}(N-2,n+2)\Big)\\
&-\frac{60}{12-5\lambda}\left[\frac{\lambda}{8}v_\lambda^{\pi_k}(N,n)+\frac{25\lambda}{72}v_\lambda^{\pi_k}(N,n+1)+\frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,n+1)\right]\\
&+\frac{36}{12-5\lambda}\left[\frac{\lambda}{8}v_\lambda^{\pi_k}(N,n+1)+\frac{25\lambda}{72}v_\lambda^{\pi_k}(N,n+2)+\frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,n+2)\right]\\
&=\frac{\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(N-3,n+1)+v_\lambda^{\pi_k}(N-3,n+2)-5v_\lambda^{\pi_k}(N-2,n+1)+3v_\lambda^{\pi_k}(N-2,n+2)\Big)\\
&+\frac{\lambda}{9-4\lambda}v_\lambda^{\pi_k}(N-2,n)-\left(\frac{20\lambda}{3(12-5\lambda)}-\frac{9\lambda}{9-4\lambda}\right)v_\lambda^{\pi_k}(N-2,n+1)+\frac{4\lambda}{12-5\lambda}v_\lambda^{\pi_k}(N-2,n+2)\\
&-\frac{15\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,n)-\frac{49\lambda}{3(12-5\lambda)}v_\lambda^{\pi_k}(N,n+1)+\frac{25\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,n+2),
\end{aligned}$$

for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. By the induction hypothesis, it suffices to show that

$$\begin{aligned}
&\frac{\lambda}{9-4\lambda}v_\lambda^{\pi_k}(N-2,\ell)-\left(\frac{20\lambda}{3(12-5\lambda)}-\frac{9\lambda}{9-4\lambda}\right)v_\lambda^{\pi_k}(N-2,\ell+1)+\frac{4\lambda}{12-5\lambda}v_\lambda^{\pi_k}(N-2,\ell+2)\\
&-\frac{15\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,\ell)-\frac{49\lambda}{3(12-5\lambda)}v_\lambda^{\pi_k}(N,\ell+1)+\frac{25\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,\ell+2)\geq 0,
\end{aligned}\tag{S120}$$

for all $\ell=2,\ldots,N-5$, for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. We prove this statement by another induction argument. First, we verify its validity for $\ell=N-5$ using Table S14. Now, suppose that it holds for $\ell=\tilde{\ell}+1$, for some $\tilde{\ell}=2,\ldots,N-6$, for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. Using expression (S88), we obtain

$$\frac{\lambda}{9-4\lambda}v_\lambda^{\pi_k}(N-2,\tilde{\ell})-\left(\frac{20\lambda}{3(12-5\lambda)}-\frac{9\lambda}{9-4\lambda}\right)v_\lambda^{\pi_k}(N-2,\tilde{\ell}+1)+\frac{4\lambda}{12-5\lambda}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+2)$$
$$-\frac{15\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell})-\frac{49\lambda}{3(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}+1)+\frac{25\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}+2)$$

$$
\begin{aligned}
&= \frac{\lambda}{9-4\lambda}\frac{12}{12-5\lambda}\left[\frac{\lambda}{8}v_\lambda^{\pi_k}(N,\tilde{\ell}) + \frac{25\lambda}{72}v_\lambda^{\pi_k}(N,\tilde{\ell}+1) + \frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+1)\right] \\
&- \left(\frac{20\lambda}{3(12-5\lambda)} - \frac{9\lambda}{9-4\lambda}\right)\frac{12}{12-5\lambda}\left[\frac{\lambda}{8}v_\lambda^{\pi_k}(N,\tilde{\ell}+1) + \frac{25\lambda}{72}v_\lambda^{\pi_k}(N,\tilde{\ell}+2) + \frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+2)\right] \\
&+ \frac{4\lambda}{12-5\lambda}\frac{12}{12-5\lambda}\left[\frac{\lambda}{8}v_\lambda^{\pi_k}(N,\tilde{\ell}+2) + \frac{25\lambda}{72}v_\lambda^{\pi_k}(N,\tilde{\ell}+3) + \frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+3)\right] - \frac{15\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}) \\
&- \frac{49\lambda}{3(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}+1) + \frac{25\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}+2) \\
&= \frac{4\lambda}{3(12-5\lambda)}\left[\frac{\lambda}{9-4\lambda}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+1) - \left(\frac{20\lambda}{3(12-5\lambda)} - \frac{9\lambda}{9-4\lambda}\right)v_\lambda^{\pi_k}(N-2,\tilde{\ell}+2)\right. \\
&+ \frac{4\lambda}{12-5\lambda}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+3) - \frac{15\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}+1) - \frac{49\lambda}{3(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}+2) \\
&\left.+ \frac{25\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}+3)\right] - \frac{9\lambda(15-7\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}) - \frac{\lambda(147-83\lambda)}{(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}+1) \\
&+ \frac{25\lambda(9-\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}+2),
\end{aligned}
$$

for $k = 1, 2$ and $\lambda \in (0, 1)$. From the induction hypothesis, it follows that a sufficient condition for expression (S120) to hold for $\ell = \tilde{\ell}$ is

$$
-\frac{9\lambda(15-7\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,m) - \frac{\lambda(147-83\lambda)}{(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,m+1) + \frac{25\lambda(9-\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,m+2) \geq 0, \tag{S121}
$$

for all $m = 2, \ldots, N-6$, for $k = 1$, $\lambda \in [\lambda_c, 1)$ and $k = 2$, $\lambda \in (0, \lambda_c]$. We prove this statement by another embedded induction argument. Consider the case $k = 1$, $\lambda \in [\lambda_c, 1)$. We start by verifying the validity of expression (S121) for $m = N-6$ using Table S14. Now, assume that it holds for $m = \tilde{m}+1$, for some $\tilde{m} = 2, \ldots, N-7$. Expression (S84) and this induction hypothesis now yield

$$
\begin{aligned}
&-\frac{9\lambda(15-7\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}) - \frac{\lambda(147-83\lambda)}{(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+1) + \frac{25\lambda(9-\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+2) \\
&= \frac{2\lambda}{3-\lambda}\left[-\frac{9\lambda(15-7\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+1) - \frac{\lambda(147-83\lambda)}{(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+2)\right. \\
&\left.+ \frac{25\lambda(9-\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+3)\right] \geq 0,
\end{aligned}
$$

for all $\lambda \in [\lambda_c, 1)$. Thus, expression (S121) is satisfied for all $m = 2, \ldots, N-6$, for $k = 1$, $\lambda \in [\lambda_c, 1)$.
We proceed to consider the case $k = 2$, $\lambda \in (0, \lambda_c]$. Again using Table S14, we verify the correctness of expression (S121) for $m = N-6$ and $m = N-7$. Suppose now that it is satisfied for all $m \geq \tilde{m}+1$, for some $\tilde{m} = 2, \ldots, N-8$. Using expression (S85), we obtain

$$
\begin{aligned}
&-\frac{9\lambda(15-7\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}) - \frac{\lambda(147-83\lambda)}{(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+1) + \frac{25\lambda(9-\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+2) \\
&= -\frac{9\lambda(15-7\lambda)}{2(9-4\lambda)(12-5\lambda)}\left[\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+1) + \frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+2)\right] \\
&- \frac{\lambda(147-83\lambda)}{(9-4\lambda)(12-5\lambda)}\left[\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+2) + \frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+3)\right] \\
&+ \frac{25\lambda(9-\lambda)}{2(9-4\lambda)(12-5\lambda)}\left[\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+3) + \frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+4)\right] \\
&= \frac{7\lambda}{3(9-5\lambda)}\left[-\frac{9\lambda(15-7\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+1) - \frac{\lambda(147-83\lambda)}{(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+2)\right.
\end{aligned}
$$

$$+\frac{25\lambda(9-\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+3)\Bigg]+\frac{5\lambda}{3(9-5\lambda)}\Bigg[-\frac{9\lambda(15-7\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+2)$$
$$-\frac{\lambda(147-83\lambda)}{(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+3)+\frac{25\lambda(9-\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+4)\Bigg].$$

It now follows from the induction hypothesis that

$$-\frac{9\lambda(15-7\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m})-\frac{\lambda(147-83\lambda)}{(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+1)+\frac{25\lambda(9-\lambda)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+2)\geq 0.$$

Hence, expression (S121) holds for all $m=2,\ldots,N-6$, for $k=2$, $\lambda\in(0,\lambda_c]$ as well. This implies that expression (S120) is valid for all $\ell=2,\ldots,N-5$ for $k=1$, $\lambda\in[\lambda_c,1)$ and for $k=2$, $\lambda\in(0,\lambda_c]$. This in turn establishes the correctness of expression (S107) for all $j=2,\ldots,N-4$, for $k=1$, $\lambda\in[\lambda_c,1)$ and for $k=2$, $\lambda\in(0,\lambda_c]$.

Inequality (S108) Again, we start by verifying inequality (S108) for $j=N-4$ using Table S14, for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. Now, suppose that it is satisfied for $j=n+1$, for some $n=2,\ldots,N-5$, for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. Using expressions (S88) and (S90), we obtain

$$\begin{aligned}
&8v_\lambda^{\pi_k}(N-3,n)+16v_\lambda^{\pi_k}(N-3,n+1)-15v_\lambda^{\pi_k}(N-2,n)+48v_\lambda^{\pi_k}(N-2,n+1)-57v_\lambda^{\pi_k}(N,n)\\
&=\frac{8\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(N-2,n)+v_\lambda^{\pi_k}(N-3,n+1)+3v_\lambda^{\pi_k}(N-2,n+1)\Big)\\
&+\frac{16\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(N-2,n+1)+v_\lambda^{\pi_k}(N-3,n+2)+3v_\lambda^{\pi_k}(N-2,n+2)\Big)\\
&-\frac{180}{12-5\lambda}\left(\frac{\lambda}{8}v_\lambda^{\pi_k}(N,n)+\frac{25\lambda}{72}v_\lambda^{\pi_k}(N,n+1)+\frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,n+1)\right)\\
&+\frac{576}{12-5\lambda}\left(\frac{\lambda}{8}v_\lambda^{\pi_k}(N,n+1)+\frac{25\lambda}{72}v_\lambda^{\pi_k}(N,n+2)+\frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,n+2)\right)-57v_\lambda^{\pi_k}(N,n)\\
&=\frac{\lambda}{9-4\lambda}\Big(8v_\lambda^{\pi_k}(N-3,n+1)+16v_\lambda^{\pi_k}(N-3,n+2)-15v_\lambda^{\pi_k}(N-2,n+1)+48v_\lambda^{\pi_k}(N-2,n+2)\\
&-57v_\lambda^{\pi_k}(N,n+1)\Big)+\frac{8\lambda}{9-4\lambda}v_\lambda^{\pi_k}(N-2,n)+\left[\frac{55\lambda}{9-4\lambda}-\frac{20\lambda}{12-5\lambda}\right]v_\lambda^{\pi_k}(N-2,n+1)\\
&+\frac{64\lambda}{12-5\lambda}v_\lambda^{\pi_k}(N-2,n+2)-\frac{1368-525\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,n)+\left[\frac{19\lambda}{2(12-5\lambda)}+\frac{57\lambda}{9-4\lambda}\right]v_\lambda^{\pi_k}(N,n+1)\\
&+\frac{200}{12-5\lambda}v_\lambda^{\pi_k}(N,n+2),
\end{aligned}$$

for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. By the induction hypothesis, it follows that it suffices to show that

$$\begin{aligned}
&\frac{8\lambda}{9-4\lambda}v_\lambda^{\pi_k}(N-2,\ell)+\left[\frac{55\lambda}{9-4\lambda}-\frac{20\lambda}{12-5\lambda}\right]v_\lambda^{\pi_k}(N-2,\ell+1)+\frac{64\lambda}{12-5\lambda}v_\lambda^{\pi_k}(N-2,\ell+2)\\
&-\frac{1368-525\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,\ell)+\left[\frac{19\lambda}{2(12-5\lambda)}+\frac{57\lambda}{9-4\lambda}\right]v_\lambda^{\pi_k}(N,\ell+1)+\frac{200}{12-5\lambda}v_\lambda^{\pi_k}(N,\ell+2)\geq 0,
\end{aligned}\tag{S122}$$

for all $\ell=2,\ldots,N-5$, for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. We prove this statement by an embedded induction argument. First of all, we use Table S14 to verify its validity for $\ell=N-5$. Now, we assume that it holds for $\ell=\tilde{\ell}+1$ for some $\tilde{\ell}=2,\ldots,N-6$, for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. Expression (S88) yields

$$\begin{aligned}
&\frac{8\lambda}{9-4\lambda}v_\lambda^{\pi_k}(N-2,\tilde{\ell})+\left[\frac{55\lambda}{9-4\lambda}-\frac{20\lambda}{12-5\lambda}\right]v_\lambda^{\pi_k}(N-2,\tilde{\ell}+1)+\frac{64\lambda}{12-5\lambda}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+2)\\
&-\frac{1368-525\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell})+\left[\frac{19\lambda}{2(12-5\lambda)}+\frac{57\lambda}{9-4\lambda}\right]v_\lambda^{\pi_k}(N,\tilde{\ell}+1)+\frac{200}{12-5\lambda}v_\lambda^{\pi_k}(N,\tilde{\ell}+2)\\
&=\frac{8\lambda}{9-4\lambda}\frac{12}{12-5\lambda}\left[\frac{\lambda}{8}v_\lambda^{\pi_k}(N,\tilde{\ell})+\frac{25\lambda}{72}v_\lambda^{\pi_k}(N,\tilde{\ell}+1)+\frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+1)\right]\\
&+\left[\frac{55\lambda}{9-4\lambda}-\frac{20\lambda}{12-5\lambda}\right]\frac{12}{12-5\lambda}\left[\frac{\lambda}{8}v_\lambda^{\pi_k}(N,\tilde{\ell}+1)+\frac{25\lambda}{72}v_\lambda^{\pi_k}(N,\tilde{\ell}+2)+\frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+2)\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{64\lambda}{12-5\lambda}\frac{12}{12-5\lambda}\left[\frac{\lambda}{8}v_\lambda^{\pi_k}(N,\tilde{\ell}+2)+\frac{25\lambda}{72}v_\lambda^{\pi_k}(N,\tilde{\ell}+3)+\frac{\lambda}{9}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+3)\right]\\
&-\frac{1368-525\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell})+\left[\frac{19\lambda}{2(12-5\lambda)}+\frac{57\lambda}{9-4\lambda}\right]v_\lambda^{\pi_k}(N,\tilde{\ell}+1)+\frac{200}{12-5\lambda}v_\lambda^{\pi_k}(N,\tilde{\ell}+2)\\
&=\frac{4\lambda}{3(12-5\lambda)}\left[\frac{8\lambda}{9-4\lambda}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+1)+\left[\frac{55\lambda}{9-4\lambda}-\frac{20\lambda}{12-5\lambda}\right]v_\lambda^{\pi_k}(N-2,\tilde{\ell}+2)\right.\\
&+\frac{64\lambda}{12-5\lambda}v_\lambda^{\pi_k}(N-2,\tilde{\ell}+3)-\frac{1368-525\lambda}{2(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}+1)\\
&\left.+\left[\frac{19\lambda}{2(12-5\lambda)}+\frac{57\lambda}{9-4\lambda}\right]v_\lambda^{\pi_k}(N,\tilde{\ell}+2)+\frac{200}{12-5\lambda}v_\lambda^{\pi_k}(N,\tilde{\ell}+3)\right]\\
&-\frac{3(4104-3399\lambda+692\lambda^2)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell})+\frac{\lambda(8721-3067\lambda)}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}+1)\\
&+\frac{10800-4800\lambda+919\lambda^2}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,\tilde{\ell}+2)-\frac{800\lambda(1-\lambda)}{3(12-5\lambda)^2}v_\lambda^{\pi_k}(N,\tilde{\ell}+3),
\end{aligned}
$$

for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. The induction hypothesis now implies that it suffices to show that

$$
\begin{aligned}
&-\frac{3(4104-3399\lambda+692\lambda^2)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,m)+\frac{\lambda(8721-3067\lambda)}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,m+1)\\
&+\frac{10800-4800\lambda+919\lambda^2}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_k}(N,m+2)-\frac{800\lambda(1-\lambda)}{3(12-5\lambda)^2}v_\lambda^{\pi_k}(N,m+3)\geq 0,
\end{aligned}
\tag{S123}
$$

for all $m=2,\ldots,N-6$, for $k=1$, $\lambda\in[\lambda_c,1)$ and for $k=2$, $\lambda\in(0,\lambda_c]$. We prove this statement by yet another induction argument. First of all, consider the case $k=1$, $\lambda\in[\lambda_c,1)$. We start by verifying the correctness of expression (S123) for $m=N-6$. Now, assume that it is satisfied for $m=\tilde{m}+1$ for some $\tilde{m}=2,\ldots,N-7$. Using expression (S84) and this induction hypothesis, we now obtain

$$
\begin{aligned}
&-\frac{3(4104-3399\lambda+692\lambda^2)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{m})+\frac{\lambda(8721-3067\lambda)}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+1)\\
&+\frac{10800-4800\lambda+919\lambda^2}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+2)-\frac{800\lambda(1-\lambda)}{3(12-5\lambda)^2}v_\lambda^{\pi_1}(N,\tilde{m}+3)\\
&=\frac{2\lambda}{3-\lambda}\left[-\frac{3(4104-3399\lambda+692\lambda^2)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+1)+\frac{\lambda(8721-3067\lambda)}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+2)\right.\\
&\left.+\frac{10800-4800\lambda+919\lambda^2}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_1}(N,\tilde{m}+3)-\frac{800\lambda(1-\lambda)}{3(12-5\lambda)^2}v_\lambda^{\pi_1}(N,\tilde{m}+4)\right]\geq 0.
\end{aligned}
$$

Hence, expression (S123) holds for all $m=2,\ldots,N-6$, for $k=1$, $\lambda\in[\lambda_c,1)$.
We proceed to consider the case $k=2$, $\lambda\in(0,\lambda_c]$. Again using Table S14, we verify the validity of expression (S123) for $m=N-6$ and $m=N-7$. Now, we suppose that it is true for all $m=\tilde{m}+1,\ldots,N-6$ for some $\tilde{m}=2,\ldots,N-8$. By expression (S85), we have

$$
\begin{aligned}
&-\frac{3(4104-3399\lambda+692\lambda^2)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m})+\frac{\lambda(8721-3067\lambda)}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+1)\\
&+\frac{10800-4800\lambda+919\lambda^2}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+2)-\frac{800\lambda(1-\lambda)}{3(12-5\lambda)^2}v_\lambda^{\pi_2}(N,\tilde{m}+3)\\
&=-\frac{3(4104-3399\lambda+692\lambda^2)}{2(9-4\lambda)(12-5\lambda)}\left[\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+1)+\frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+2)\right]\\
&+\frac{\lambda(8721-3067\lambda)}{6(9-4\lambda)(12-5\lambda)}\left[\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+2)+\frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+3)\right]\\
&+\frac{10800-4800\lambda+919\lambda^2}{6(9-4\lambda)(12-5\lambda)}\left[\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+3)+\frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+4)\right]
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{800\lambda(1-\lambda)}{3(12-5\lambda)^2}\left[\frac{7\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+4)+\frac{5\lambda}{3(9-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+5)\right]\\
&=\frac{7\lambda}{3(9-5\lambda)}\left[-\frac{3(4104-3399\lambda+692\lambda^2)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+1)+\frac{\lambda(8721-3067\lambda)}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+2)\right.\\
&\left.+\frac{10800-4800\lambda+919\lambda^2}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+3)-\frac{800\lambda(1-\lambda)}{3(12-5\lambda)^2}v_\lambda^{\pi_2}(N,\tilde{m}+4)\right]\\
&+\frac{5\lambda}{3(9-5\lambda)}\left[-\frac{3(4104-3399\lambda+692\lambda^2)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+2)+\frac{\lambda(8721-3067\lambda)}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+3)\right.\\
&\left.+\frac{10800-4800\lambda+919\lambda^2}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+4)-\frac{800\lambda(1-\lambda)}{3(12-5\lambda)^2}v_\lambda^{\pi_2}(N,\tilde{m}+5)\right],
\end{aligned}
$$

for all $\lambda\in(0,\lambda_c]$. Invoking the induction hypothesis, we can now conclude that

$$
\begin{aligned}
&-\frac{3(4104-3399\lambda+692\lambda^2)}{2(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m})+\frac{\lambda(8721-3067\lambda)}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+1)\\
&+\frac{10800-4800\lambda+919\lambda^2}{6(9-4\lambda)(12-5\lambda)}v_\lambda^{\pi_2}(N,\tilde{m}+2)-\frac{800\lambda(1-\lambda)}{3(12-5\lambda)^2}v_\lambda^{\pi_2}(N,\tilde{m}+3)\geq 0,
\end{aligned}
$$

for all $\lambda\in(0,\lambda_c]$. Thus, expression (S123) holds for all $m=2,\ldots,N-6$ for $k=2$, $\lambda\in[\lambda_c,1)$ as well. Hence, expression (S122) is satisfied for all $\ell=2,\ldots,N-5$, for $k=1$, $\lambda\in[\lambda_c,1)$ and for $k=2$, $\lambda\in(0,\lambda_c]$, which in turn implies the validity of inequality (S108) for all $j=2,\ldots,N-4$, for $k=1$, $\lambda\in[\lambda_c,1)$ and for $k=2$, $\lambda\in(0,\lambda_c]$.

We finalize the proof by showing the validity of inequalities (S109) and (S111). The correctness of inequalities (S110) and (S112) then follow immediately from symmetry. In the proofs of inequalities (S109) and (S111), we make use of the more involved induction argument outlined in the main part of the proof of Theorem 4.

Inequality (S109)

- *Induction base:* We show that inequality (S109) holds for states $(i,N-4)$, $(N-4,j)$, $i,j=2,\ldots,N-4$, for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. To this end, we use an embedded induction argument over the length of the shortest side of the rectangle. By symmetry, it suffices to handle the case $(N-4,j)$, $j=2,\ldots,N-4$. First of all, we verify the validity of the inequality for state $(N-4,N-4)$ using Table S14. Now, suppose that it is true for state $(N-4,n+1)$ for some $n=2,\ldots,N-5$, for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. By expression (S90), we have

$$
\begin{aligned}
&v_\lambda^{\pi_k}(N-4,n)+v_\lambda^{\pi_k}(N-3,n)-5v_\lambda^{\pi_k}(N-4,n+1)+3v_\lambda^{\pi_k}(N-3,n+1)\\
&=\frac{\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(N-3,n)+v_\lambda^{\pi_k}(N-4,n+1)+3v_\lambda^{\pi_k}(N-3,n+1)\Big)\\
&+\frac{\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(N-2,n)+v_\lambda^{\pi_k}(N-3,n+1)+3v_\lambda^{\pi_k}(N-2,n+1)\Big)\\
&-\frac{5\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(N-3,n+1)+v_\lambda^{\pi_k}(N-4,n+2)+3v_\lambda^{\pi_k}(N-3,n+2)\Big)\\
&+\frac{3\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(N-2,n+1)+v_\lambda^{\pi_k}(N-3,n+2)+3v_\lambda^{\pi_k}(N-2,n+2)\Big),
\end{aligned}
$$

  for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. Through rearranging, we obtain

$$
\begin{aligned}
&v_\lambda^{\pi_k}(N-4,n)+v_\lambda^{\pi_k}(N-3,n)-5v_\lambda^{\pi_k}(N-4,n+1)+3v_\lambda^{\pi_k}(N-3,n+1)\\
&=\frac{\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(N-3,n)+v_\lambda^{\pi_k}(N-2,n)-5v_\lambda^{\pi_k}(N-3,n+1)+3v_\lambda^{\pi_k}(N-2,n+1)\Big)\\
&+\frac{\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(N-4,n+1)+v_\lambda^{\pi_k}(N-3,n+1)-5v_\lambda^{\pi_k}(N-4,n+2)+3v_\lambda^{\pi_k}(N-3,n+2)\Big)\\
&+\frac{3\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(N-3,n+1)+v_\lambda^{\pi_k}(N-2,n+1)-5v_\lambda^{\pi_k}(N-3,n+2)+3v_\lambda^{\pi_k}(N-2,n+2)\Big),
\end{aligned}
$$

for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. The induction hypothesis and inequality (S105) now yield,

$$v_\lambda^{\pi_k}(N-4,n)+v_\lambda^{\pi_k}(N-3,n)-5v_\lambda^{\pi_k}(N-4,n+1)+3v_\lambda^{\pi_k}(N-3,n+1)>0,$$

for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$. Hence, inequality (S109) holds for all states $(N-4,j)$, $j=2,\ldots,N-4$, for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$.

- *Induction hypothesis*: Suppose that inequality (S109) holds for states $(i,n+1)$, $(n+1,i)$ for all $i=2,\ldots,n+1$ for some $n=2,\ldots,N-5$. We refer to this induction hypothesis as the *global induction hypothesis*.
- *Induction step*: First of all, we show that the induction hypothesis implies the validity of inequality (S109) for state (n,n). Using expression (S90), we obtain

$$\begin{aligned}
&v_\lambda^{\pi_k}(n,n)+v_\lambda^{\pi_k}(n+1,n)-5v_\lambda^{\pi_k}(n,n+1)+3v_\lambda^{\pi_k}(n+1,n+1)\\
&=\frac{\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(n+1,n)+v_\lambda^{\pi_k}(n,n+1)+3v_\lambda^{\pi_k}(n+1,n+1)\Big)\\
&+\frac{\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(n+2,n)+v_\lambda^{\pi_k}(n+1,n+1)+3v_\lambda^{\pi_k}(n+2,n+1)\Big)\\
&-\frac{5\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(n+1,n+1)+v_\lambda^{\pi_k}(n,n+2)+3v_\lambda^{\pi_k}(n+1,n+2)\Big)\\
&+\frac{3\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(n+2,n+1)+v_\lambda^{\pi_k}(n+1,n+2)+3v_\lambda^{\pi_k}(n+2,n+2)\Big)\\
&=\frac{\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(n+1,n)+v_\lambda^{\pi_k}(n+2,n)-5v_\lambda^{\pi_k}(n+1,n+1)+3v_\lambda^{\pi_k}(n+2,n+1)\Big)\\
&+\frac{\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(n,n+1)+v_\lambda^{\pi_k}(n+1,n+1)-5v_\lambda^{\pi_k}(n,n+2)+3v_\lambda^{\pi_k}(n+1,n+2)\Big)\\
&+\frac{3\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(n+1,n+1)+v_\lambda^{\pi_k}(n+2,n+1)-5v_\lambda^{\pi_k}(n+1,n+2)+3v_\lambda^{\pi_k}(n+2,n+2)\Big)>0,
\end{aligned}$$

for $k=1$, $\lambda\in[\lambda_c,1)$ and $\lambda\in(0,\lambda_c]$ by the induction hypothesis, establishing the correctness of inequality (S109) for state (n,n).

Now, in addition to the global induction hypothesis, we make the assumption that inequality (S109) holds for some state $(n,\ell+1)$, where $\ell=2,\ldots,n-1$. We show that, under the global and additional induction hypotheses, the validity of inequality (S109) for state $(n,\ell+1)$ carries over to state (n,ℓ). This implies the correctness of inequality (S109) for all states (i,n), (n,i), $i=2,\ldots,n$ under the global induction hypothesis. Using expression (S90), we obtain

$$\begin{aligned}
&v_\lambda^{\pi_k}(n,\ell)+v_\lambda^{\pi_k}(n+1,\ell)-5v_\lambda^{\pi_k}(n,\ell+1)+3v_\lambda^{\pi_k}(n+1,\ell+1)\\
&=\frac{\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(n+1,\ell)+v_\lambda^{\pi_k}(n,\ell+1)+3v_\lambda^{\pi_k}(n+1,\ell+1)\Big)\\
&+\frac{\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(n+2,\ell)+v_\lambda^{\pi_k}(n+1,\ell+1)+3v_\lambda^{\pi_k}(n+2,\ell+1)\Big)\\
&-\frac{5\lambda}{3(9-4\lambda)}\Big(v_\lambda^{\pi_k}(n+1,\ell+1)+v_\lambda^{\pi_k}(n,\ell+2)+3v_\lambda^{\pi_k}(n+1,\ell+2)\Big)\\
&+\frac{3\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(n+2,\ell+1)+v_\lambda^{\pi_k}(n+1,\ell+2)+3v_\lambda^{\pi_k}(n+2,\ell+2)\Big)\\
&=\frac{\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(n+1,\ell)+v_\lambda^{\pi_k}(n+2,\ell)-5v_\lambda^{\pi_k}(n+1,\ell+1)+3v_\lambda^{\pi_k}(n+2,\ell+1)\Big)\\
&+\frac{\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(n,\ell+1)+v_\lambda^{\pi_k}(n+1,\ell+1)-5v_\lambda^{\pi_k}(n,\ell+2)+3v_\lambda^{\pi_k}(n+1,\ell+2)\Big)\\
&+\frac{3\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(n+1,\ell+1)+v_\lambda^{\pi_k}(n+2,\ell+1)-5v_\lambda^{\pi_k}(n+1,\ell+2)+3v_\lambda^{\pi_k}(n+2,\ell+2)\Big)>0,
\end{aligned}$$

by the global and additional induction hypotheses. It follows that inequality (S109) is satisfied for states (n,i), (i,n) for all $i=2,\ldots,n$ under the global induction hypothesis. From the full induction argument, we can now conclude that inequality (S109) holds for all states (i,j), $i,j=2,\ldots,N-4$, for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$.

Inequality (S111)

- *Induction base:* We prove that inequality (S111) is valid for states $(i, N-4)$, $(N-4, j)$, $i, j = 2, \ldots, N-4$, for $k = 1$, $\lambda \in [\lambda_c, 1)$ and $k = 2$, $\lambda \in (0, \lambda_c]$, again by means of an embedded induction argument over the length of the shortest side of the rectangle. By symmetry, it is sufficient to prove the statement for states $(N-4, j)$, $j = 2, \ldots, N-4$. First, we use Table S14 to verify the correctness of inequality (S111) for state $(N-4, N-4)$. Now, assume that it holds for state $(N-4, n+1)$ for some $n = 2, \ldots, N-5$, for $k = 1$, $\lambda \in [\lambda_c, 1)$ and $k = 2$, $\lambda \in (0, \lambda_c]$. Expression (S90) yields

$$
\begin{aligned}
&- 3v_\lambda^{\pi_k}(N-4,n) + 3v_\lambda^{\pi_k}(N-3,n) - 4v_\lambda^{\pi_k}(N-4,n+1) + 9v_\lambda^{\pi_k}(N-3,n+1) - 5v_\lambda^{\pi_k}(N-4,n+2)\\
&= -\frac{3\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(N-3,n) + v_\lambda^{\pi_k}(N-4,n+1) + 3v_\lambda^{\pi_k}(N-3,n+1)\Big)\\
&+ \frac{3\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(N-2,n) + v_\lambda^{\pi_k}(N-3,n+1) + 3v_\lambda^{\pi_k}(N-2,n+1)\Big)\\
&- \frac{4\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(N-3,n+1) + v_\lambda^{\pi_k}(N-4,n+2) + 3v_\lambda^{\pi_k}(N-3,n+2)\Big)\\
&+ \frac{9\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(N-2,n+1) + v_\lambda^{\pi_k}(N-3,n+2) + 3v_\lambda^{\pi_k}(N-2,n+2)\Big)\\
&- \frac{5\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(N-3,n+2) + v_\lambda^{\pi_k}(N-4,n+3) + 3v_\lambda^{\pi_k}(N-3,n+3)\Big).
\end{aligned}
$$

  Through rearranging, we obtain

$$
\begin{aligned}
&- 3v_\lambda^{\pi_k}(N-4,n) + 3v_\lambda^{\pi_k}(N-3,n) - 4v_\lambda^{\pi_k}(N-4,n+1) + 9v_\lambda^{\pi_k}(N-3,n+1) - 5v_\lambda^{\pi_k}(N-4,n+2)\\
&= \frac{\lambda}{9-4\lambda}\Big(-3v_\lambda^{\pi_k}(N-3,n) + 3v_\lambda^{\pi_k}(N-2,n) - 4v_\lambda^{\pi_k}(N-3,n+1) + 9v_\lambda^{\pi_k}(N-2,n+1)\\
&- 5v_\lambda^{\pi_k}(N-3,n+2)\Big) + \frac{\lambda}{9-4\lambda}\Big(-3v_\lambda^{\pi_k}(N-4,n+1) + 3v_\lambda^{\pi_k}(N-3,n+1) - 4v_\lambda^{\pi_k}(N-4,n+2)\\
&+ 9v_\lambda^{\pi_k}(N-3,n+2) - 5v_\lambda^{\pi_k}(N-4,n+3)\Big) + \frac{3\lambda}{9-4\lambda}\Big(-3v_\lambda^{\pi_k}(N-3,n+1) + 3v_\lambda^{\pi_k}(N-2,n+1)\\
&- 4v_\lambda^{\pi_k}(N-3,n+2) + 9v_\lambda^{\pi_k}(N-2,n+2) - 5v_\lambda^{\pi_k}(N-3,n+3)\Big).
\end{aligned}
$$

  From the induction hypothesis and inequality (S106), it now follows that

$$
\begin{aligned}
&- 3v_\lambda^{\pi_k}(N-4,n) + 3v_\lambda^{\pi_k}(N-3,n) - 4v_\lambda^{\pi_k}(N-4,n+1) + 9v_\lambda^{\pi_k}(N-3,n+1)\\
&- 5v_\lambda^{\pi_k}(N-4,n+2) > 0.
\end{aligned}
$$

  Thus, inequality (S111) holds for all states $(N-4, j)$, $j = 2, \ldots, N-4$ for $k = 1$, $\lambda \in [\lambda_c, 1)$ and $k = 2$, $\lambda \in (0, \lambda_c]$.

- *Induction hypothesis:* Suppose that inequality (S111) is valid for states $(i, n+1)$, $(n+1, j)$ for all $i, j = 2, \ldots, n+1$ for some $n = 2, \ldots, N-5$. We refer to this induction hypothesis as the *global induction hypothesis.*

- *Induction step:* First of all, we show that the induction hypothesis implies the correctness of inequality (S111) for state (n, n). By expression (S90), we have

$$
\begin{aligned}
&- 3v_\lambda^{\pi_k}(n,n) + 3v_\lambda^{\pi_k}(n+1,n) - 4v_\lambda^{\pi_k}(n,n+1) + 9v_\lambda^{\pi_k}(n+1,n+1) - 5v_\lambda^{\pi_k}(n,n+2)\\
&= -\frac{3\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(n+1,n) + v_\lambda^{\pi_k}(n,n+1) + 3v_\lambda^{\pi_k}(n+1,n+1)\Big)\\
&+ \frac{3\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(n+2,n) + v_\lambda^{\pi_k}(n+1,n+1) + 3v_\lambda^{\pi_k}(n+2,n+1)\Big)\\
&- \frac{4\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(n+1,n+1) + v_\lambda^{\pi_k}(n,n+2) + 3v_\lambda^{\pi_k}(n+1,n+2)\Big)\\
&+ \frac{9\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(n+2,n+1) + v_\lambda^{\pi_k}(n+1,n+2) + 3v_\lambda^{\pi_k}(n+2,n+2)\Big)\\
&- \frac{5\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(n+1,n+2) + v_\lambda^{\pi_k}(n,n+3) + 3v_\lambda^{\pi_k}(n+1,n+3)\Big).
\end{aligned}
$$

Rearranging yields

$$
\begin{aligned}
&-3v_\lambda^{\pi_k}(n,n)+3v_\lambda^{\pi_k}(n+1,n)-4v_\lambda^{\pi_k}(n,n+1)+9v_\lambda^{\pi_k}(n+1,n+1)-5v_\lambda^{\pi_k}(n,n+2)\\
&=\frac{\lambda}{9-4\lambda}\Big(-3v_\lambda^{\pi_k}(n+1,n)+3v_\lambda^{\pi_k}(n+2,n)-4v_\lambda^{\pi_k}(n+1,n+1)+9v_\lambda^{\pi_k}(n+2,n+1)\\
&-5v_\lambda^{\pi_k}(n+1,n+2)\Big)+\frac{\lambda}{9-4\lambda}\Big(-3v_\lambda^{\pi_k}(n,n+1)+3v_\lambda^{\pi_k}(n+1,n+1)\\
&-4v_\lambda^{\pi_k}(n,n+2)+9v_\lambda^{\pi_k}(n+1,n+2)-5v_\lambda^{\pi_k}(n,n+3)\Big)+\frac{3\lambda}{9-4\lambda}\Big(-3v_\lambda^{\pi_k}(n+1,n+1)\\
&+3v_\lambda^{\pi_k}(n+2,n+1)-4v_\lambda^{\pi_k}(n+1,n+2)+9v_\lambda^{\pi_k}(n+2,n+2)-5v_\lambda^{\pi_k}(n+1,n+3)\Big).
\end{aligned}
$$

Invoking the induction hypothesis three times, we now obtain

$$-3v_\lambda^{\pi_k}(n,n)+3v_\lambda^{\pi_k}(n+1,n)-4v_\lambda^{\pi_k}(n,n+1)+9v_\lambda^{\pi_k}(n+1,n+1)-5v_\lambda^{\pi_k}(n,n+2)>0,$$

for $k=1$, $\lambda\in[\lambda_c,1)$ and $\lambda\in(0,\lambda_c]$. Hence, inequality (S111) is satisfied for state (n,n).
We proceed to make the assumption that inequality (S111) holds for some state $(n,\ell+1)$, where $\ell=2,\dots,n-1$, in addition to the global induction hypothesis. We show that the global and additional induction hypotheses imply the validity of inequality (S111) for state (n,ℓ). Invoking expression (S90), we obtain

$$
\begin{aligned}
&-3v_\lambda^{\pi_k}(n,\ell)+3v_\lambda^{\pi_k}(n+1,\ell)-4v_\lambda^{\pi_k}(n,\ell+1)+9v_\lambda^{\pi_k}(n+1,\ell+1)-5v_\lambda^{\pi_k}(n,\ell+2)\\
&=-\frac{3\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(n+1,\ell)+v_\lambda^{\pi_k}(n,\ell+1)+3v_\lambda^{\pi_k}(n+1,\ell+1)\Big)\\
&+\frac{3\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(n+2,\ell)+v_\lambda^{\pi_k}(n+1,\ell+1)+3v_\lambda^{\pi_k}(n+2,\ell+1)\Big)\\
&-\frac{4\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(n+1,\ell+1)+v_\lambda^{\pi_k}(n,\ell+2)+3v_\lambda^{\pi_k}(n+1,\ell+2)\Big)\\
&+\frac{9\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(n+2,\ell+1)+v_\lambda^{\pi_k}(n+1,\ell+2)+3v_\lambda^{\pi_k}(n+2,\ell+2)\Big)\\
&-\frac{5\lambda}{9-4\lambda}\Big(v_\lambda^{\pi_k}(n+1,\ell+2)+v_\lambda^{\pi_k}(n,\ell+3)+3v_\lambda^{\pi_k}(n+1,\ell+3)\Big)\\
&=\frac{\lambda}{9-4\lambda}\Big(-3v_\lambda^{\pi_k}(n+1,\ell)+3v_\lambda^{\pi_k}(n+2,\ell)-4v_\lambda^{\pi_k}(n+1,\ell+1)+9v_\lambda^{\pi_k}(n+2,\ell+1)\\
&-5v_\lambda^{\pi_k}(n+1,\ell+2)\Big)+\frac{\lambda}{9-4\lambda}\Big(-3v_\lambda^{\pi_k}(n,\ell+1)+3v_\lambda^{\pi_k}(n+1,\ell+1)-4v_\lambda^{\pi_k}(n,\ell+2)\\
&+9v_\lambda^{\pi_k}(n+1,\ell+2)-5v_\lambda^{\pi_k}(n,\ell+3)\Big)+\frac{3\lambda}{9-4\lambda}\Big(-3v_\lambda^{\pi_k}(n+1,\ell+1)\\
&+3v_\lambda^{\pi_k}(n+2,\ell+1)-4v_\lambda^{\pi_k}(n+1,\ell+2)+9v_\lambda^{\pi_k}(n+2,\ell+2)-5v_\lambda^{\pi_k}(n+1,\ell+3)\Big).
\end{aligned}
$$

It now follows from the global and additional induction hypotheses that

$$-3v_\lambda^{\pi_k}(n,\ell)+3v_\lambda^{\pi_k}(n+1,\ell)-4v_\lambda^{\pi_k}(n,\ell+1)+9v_\lambda^{\pi_k}(n+1,\ell+1)-5v_\lambda^{\pi_k}(n,\ell+2)>0.$$

Thus, inequality (S111) holds for all states (n,i), (i,n), $i=2,\dots,n$ under the global induction hypothesis. The full induction argument now implies that inequality (S111) is valid for all states (i,j), $i,j=2,\dots,N-4$ for $k=1$, $\lambda\in[\lambda_c,1)$ and $k=2$, $\lambda\in(0,\lambda_c]$.

Some explicit expressions for the optimal value function

(i,j)	k,λ	$v_\lambda^{\pi_k}(i,j)$
(N,N)	$k=1,2,\ \lambda\in(0,1)$	$\dfrac{1}{1-\lambda}$
$(N,N-2)$	$k=1,2,\ \lambda\in(0,1)$	$\dfrac{3\lambda}{(1-\lambda)(4-\lambda)}$
$(N,N-3)$	$k=1,2,\ \lambda\in(0,1)$	$\dfrac{3\lambda(19+3\lambda)}{2(4-\lambda)(1-\lambda)(18-7\lambda)}$
$(N,N-4)$	$k=1,2,\ \lambda\in(0,1)$	$\dfrac{3\lambda^2(19+3\lambda)}{(18-7\lambda)(1-\lambda)(3-\lambda)(4-\lambda)}$
$(N,N-5)$	$k=1,\ [\lambda_c,1)$	$\dfrac{6\lambda^3(19+3\lambda)}{(18-7\lambda)(1-\lambda)(3-\lambda)^2(4-\lambda)}$
	$k=2,\ (0,\lambda_c]$	$\dfrac{3\lambda^2(5+3\lambda)(19+3\lambda)}{2(4-\lambda)(3-\lambda)(1-\lambda)(9-5\lambda)(18-7\lambda)}$
$(N,N-6)$	$k=1,\ [\lambda_c,1)$	$\dfrac{12\lambda^4(19+3\lambda)}{(18-7\lambda)(1-\lambda)(3-\lambda)^3(4-\lambda)}$
	$k=2,\ (0,\lambda_c]$	$\dfrac{\lambda^3(19+3\lambda)(125-29\lambda)}{2(9-5\lambda)^2(4-\lambda)(3-\lambda)(1-\lambda)(18-7\lambda)}$
$(N,N-7)$	$k=1,\ [\lambda_c,1)$	$\dfrac{24\lambda^5(19+3\lambda)}{(18-7\lambda)(1-\lambda)(3-\lambda)^4(4-\lambda)}$
	$k=2,\ (0,\lambda_c]$	$\dfrac{\lambda^3(19+3\lambda)(675+905\lambda-428\lambda^2)}{6(4-\lambda)(3-\lambda)(1-\lambda)(9-5\lambda)^3(18-7\lambda)}$
$(N,N-8)$	$k=1,\ [\lambda_c,1)$	$\dfrac{48\lambda^6(19+3\lambda)}{(18-7\lambda)(1-\lambda)(3-\lambda)^5(4-\lambda)}$
	$k=2,\ (0,\lambda_c]$	$\dfrac{\lambda^4(19+3\lambda)(21600-6955\lambda-821\lambda^2)}{18(9-5\lambda)^4(4-\lambda)(3-\lambda)(1-\lambda)(18-7\lambda)}$
$(N-2,N-2)$	$k=1,2,\ \lambda\in(0,1)$	$\dfrac{\lambda(7\lambda+26)}{(4-\lambda)(1-\lambda)(18-7\lambda)}$
$(N-2,N-3)$	$k=1,2,\ \lambda\in(0,1)$	$\dfrac{\lambda^2(971-245\lambda)}{2(18-7\lambda)^2(1-\lambda)(4-\lambda)}$
$(N-2,N-4)$	$k=1,2,\ \lambda\in(0,1)$	$\dfrac{\lambda^2(2401\lambda^3-16714\lambda^2-1653\lambda+76950)}{12(18-7\lambda)^2(4-\lambda)(3-\lambda)(1-\lambda)(12-5\lambda)}$
$(N-2,N-5)$	$k=1,\ [\lambda_c,1)$	$\dfrac{\lambda^3(3231900-2216520\lambda+167001\lambda^2+109700\lambda^3-11417\lambda^4)}{18(18-7\lambda)^2(12-5\lambda)^2(4-\lambda)(3-\lambda)^2(1-\lambda)}$

	$k=2$, $(0,\lambda_c]$	$\frac{\lambda^3(21053520-21676626\lambda+5403027\lambda^2+601490\lambda^3-258755\lambda^4)}{36(18-7\lambda)^2(12-5\lambda)^2(4-\lambda)(3-\lambda)(1-\lambda)(9-5\lambda)}$
$(N-2,N-6)$	$k=1$, $[\lambda_c,1)$	$\frac{\lambda^4}{27(18-7\lambda)^2(4-\lambda)(3-\lambda)^3(1-\lambda)(12-5\lambda)^3}\Big(119118600$ $-135213192\lambda+45728820\lambda^2-1408311\lambda^3-1454032\lambda^4+122059\lambda^5\Big)$
	$k=2$, $\lambda\in(0,\lambda_c]$	$\frac{\lambda^3}{108(18-7\lambda)^2(9-5\lambda)^2(4-\lambda)(3-\lambda)(1-\lambda)(12-5\lambda)^3}$ $\cdot\Big(149508000-269088480\lambda-1961414838\lambda^2+1470626901\lambda^3$ $-283284744\lambda^4-34167685\lambda^5+11723950\lambda^6\Big)$
$(N-3,N-3)$	$k=1,2$, $\lambda\in(0,1)$	$\frac{\lambda^2(29241+8296\lambda-5593\lambda^2)}{8(4-\lambda)(1-\lambda)(18-7\lambda)^3}$
$(N-3,N-4)$	$k=1,2$, $\lambda\in(0,1)$	$\frac{\lambda^3}{24(4-\lambda)(3-\lambda)(1-\lambda)(9-4\lambda)(12-5\lambda)(18-7\lambda)^3}$ $\cdot\Big(28579716-34122249\lambda+14845917\lambda^2-2786335\lambda^3+191191\lambda^4\Big)$
$(N-3,N-5)$	$k=1$, $[\lambda_c,1)$	$\frac{\lambda^3}{72(12-5\lambda)^2(9-4\lambda)^2(4-\lambda)(3-\lambda)^2(1-\lambda)(18-7\lambda)^3}$ $\cdot\Big(8077903200-7782658200\lambda-3094709976\lambda^2+7348557735\lambda^3$ $-4088759886\lambda^4+1083552000\lambda^5-142794154\lambda^6+7639681\lambda^7\Big)$
	$k=2$, $(0,\lambda_c]$	$\frac{\lambda^3}{72(12-5\lambda)^2(9-4\lambda)^2(4-\lambda)(3-\lambda)(1-\lambda)(9-5\lambda)(18-7\lambda)^3}$ $\cdot\Big(24233709600-28194716520\lambda-7657684272\lambda^2+27365238693\lambda^3$ $-17216824914\lambda^4+4906807638\lambda^5-649171070\lambda^6+30101645\lambda^7\Big)$
$(N-4,N-4)$	$k=1,2$, $\lambda\in(0,1)$	$\frac{\lambda^3}{24(9-4\lambda)^2(4-\lambda)(3-\lambda)(1-\lambda)(12-5\lambda)(18-7\lambda)^3}\Big(85266756$ $-20495835\lambda-73183797\lambda^2+55332465\lambda^3-14767511\lambda^4+1389122\lambda^5\Big)$

Table S14: Some explicit expressions for the optimal value function.