

On Zeros and Algorithms for Disordered Systems: Mean-Field Spin Glasses

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Abstract

Spin glasses are fundamental probability distributions at the core of statistical physics, the theory of average-case computational complexity, and modern high-dimensional statistical inference. In the mean-field setting, we design deterministic quasipolynomial-time algorithms for estimating the partition function to arbitrarily high accuracy for all inverse temperatures in the second moment regime. In particular, for the Sherrington–Kirkpatrick model, our algorithms succeed for the entire replica-symmetric phase. To achieve this, we study the locations of the zeros of the partition function. Notably, our methods are conceptually simple, and apply equally well to the spherical case and the case of Ising spins.

CCS Concepts

• **Mathematics of computing** → **Stochastic processes**; • **Theory of computation** → **Generating random combinatorial structures**; **Approximation algorithms analysis**.

Keywords

spin glasses, partition functions, approximate counting, zero-freeness, Barvinok interpolation

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1 Introduction

We study *spin glasses*, which are probabilistic models of inhomogeneous materials (e.g. disordered magnetic alloys) in statistical mechanics. These and related models have also formed the bedrock for the study of average-case algorithm design, as well as the computational complexity of high-dimensional statistical inference problems; see e.g. [22, 52] and references therein. For concreteness, we

consider *mixed p -spin models (with Gaussian interactions)*, which are specified by the random Hamiltonian

$$\mathcal{H}_G(\sigma) \stackrel{\text{def}}{=} \sum_{p=2}^{p_{\max}} \frac{\gamma_p}{n^{\frac{p-1}{2}}} \sum_{i_1, \dots, i_p=1}^n G_{i_1, \dots, i_p} \prod_{j=1}^p \sigma_{i_j}, \quad \forall \sigma \in \mathbb{R}^n, \quad (1)$$

where $\{\gamma_p\}_{p=2}^{p_{\max}}$ are deterministic nonnegative real coefficients, $p_{\max} \geq 2$ is an arbitrary integer, and $G_{i_1, \dots, i_p} \sim \mathcal{N}(0, 1)$ are independent across all $2 \leq p \leq p_{\max}$ and $i_1, \dots, i_p \in [n]$. These models are *mean-field* in the sense that each spin σ_i interacts directly with every other spin σ_j in $\mathcal{H}_G(\sigma)$. For a background probability measure ϱ on $\mathbb{R}^n$ and an *inverse temperature* $\beta \geq 0$, define the corresponding (random) *Gibbs measure* on $\mathbb{R}^n$ by

$$d\mu_{G, \beta}(\sigma) \propto \exp(\beta \cdot \mathcal{H}_G(\sigma)) d\varrho(\sigma),$$

where the normalization constant is given by the *partition function*

$$Z_G(\beta) \stackrel{\text{def}}{=} \mathbb{E}_{\sigma \sim \varrho} [\exp(\beta \cdot \mathcal{H}_G(\sigma))].$$

In this paper, we will consider the cases where ϱ is the uniform measure over either the discrete Boolean hypercube $C_n \stackrel{\text{def}}{=} \{\pm 1\}^n$ or the (rescaled) Euclidean sphere $S_n \stackrel{\text{def}}{=} \{\sigma \in \mathbb{R}^n : \|\sigma\|_2^2 = n\}$. Note that by setting $\varrho = \text{Unif}(C_n)$, $p_{\max} = 2$ and $\gamma_2 = 1/\sqrt{2}$, the above recovers perhaps the most intensely studied mean-field spin glass: the *Sherrington–Kirkpatrick model* [66].

Given these data, two natural and intimately related computational tasks are as follows:

Counting: For an error tolerance $\eta > 0$ and a failure probability $0 < \delta < 1$, compute an *estimate* $\widehat{Z}$ satisfying $e^{-\eta} \cdot Z_G(\beta) \leq \widehat{Z} \leq e^{\eta} \cdot Z_G(\beta)$ with probability at least $1 - \delta$.

Sampling: For $\eta > 0$, sample a random configuration $\widehat{\sigma}$ such that $\mathcal{D}(\text{Law}(\widehat{\sigma}), \mu_{G, \beta}) \leq \eta$ for some prescribed (pseudo-)metric $\mathcal{D}(\cdot, \cdot)$ on the space of probability measures.

These are fundamental and important algorithmic primitives in high-dimensional statistics and computational physics. Over the last few decades, a deep and beautiful connection has emerged between the computational complexity of these tasks for a multitude of statistical physics models, and the presence/absence of structural phase transitions in these models; see e.g. [12, 61] and references therein. In this vein, our contributions are two-fold:

- (1) Inspired by the seminal Lee–Yang theory of phase transitions [48] (see Section 1.3 for further background), we control the



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locations of the zeros of $Z_G(\beta)$ in the complex plane. To the best of our knowledge, this perspective on mean-field spin glasses is novel.

- (2) Leveraging our understanding of the zeros, we design deterministic quasipolynomial-time algorithms which *compute* the partition function $Z_G(\beta)$ to high accuracy for nearly all β below a certain natural threshold β_{2nd} (see [Definition 1](#)). For the Sherrington–Kirkpatrick model, β_{2nd} is known to coincide with the *replica-symmetry breaking threshold* β_{RS} , a well-known phase transition also believed to separate the computationally tractable and intractable regimes of β for counting and sampling. As we discuss in [Sections 1.2](#) and [1.3](#), β_{2nd} far exceeds the current range of β in which efficient counting/sampling algorithms are known. At a high level, our algorithms are based on Barvinok’s interpolation method, which additively approximates $\log Z_G(\beta)$ using a truncation of its Taylor series.

One advantage of our approach is its conceptual simplicity, as it relies only on the second moment method and elementary complex analysis. Moreover, our techniques can simultaneously treat the cubical and spherical cases in a completely unified manner.

1.1 Main Results

We begin by stating the range of inverse temperatures β in which we obtain efficient algorithms for estimating $Z_G(\beta)$. For this, as is customary in the theory of mean-field spin glasses, we first define the *mixture function* associated with the parameters $\{\gamma_p\}_{p=2}^{p_{\max}}$ by

$$\xi(s) \stackrel{\text{def}}{=} \sum_{p=2}^{p_{\max}} \gamma_p^2 \cdot s^p.$$

This is relevant because under the choice of normalization in [Eq. \(1\)](#), the collection of random variables $\{\mathcal{H}_G(\sigma)\}_{\sigma \in \mathbb{R}^n}$ appearing in the Gibbs measure and the partition function is a Gaussian process with covariance

$$\mathbb{E}_G [\mathcal{H}_G(\tau) \cdot \mathcal{H}_G(\sigma)] = n \cdot \xi \left(\frac{\langle \tau, \sigma \rangle}{n} \right), \quad (2)$$

which depends only on the *overlap* $\frac{\langle \tau, \sigma \rangle}{n}$.

Definition 1 (Second Moment Regime). *Define the following (context-dependent) entropy functional on $[-1, 1]$:*

$$h(m) \stackrel{\text{def}}{=} -\frac{1}{2} \cdot ((1-m) \log(1-m) + (1+m) \log(1+m))$$

when $\varrho = \text{Unif}(C_n)$, and

$$h(m) \stackrel{\text{def}}{=} \frac{1}{2} \log(1-m^2)$$

when $\varrho = \text{Unif}(S_n)$. For a mixture function ξ , we say $\beta \geq 0$ satisfies the second moment condition (with respect to ϱ, ξ) if the function $\varphi : [-1, 1] \rightarrow \mathbb{R}$ defined by

$$\varphi(m) \stackrel{\text{def}}{=} \beta^2 \cdot \xi(m) + h(m)$$

has 0 as its unique global maximizer, and that $\varphi''(0) < 0$. We define the corresponding second moment threshold by

$$\beta_{2nd}(\varrho, \xi) \stackrel{\text{def}}{=} \sup \{ \beta \geq 0 : \beta \text{ satisfies the second moment condition} \}$$

We suppress the dependence on ϱ, ξ when they are clear from context.

This is the supremal (positive real) β where the second moment method locates the in-probability limiting free energy $\frac{1}{n} \log Z_G$. In [Section 2](#) we explain how the second moment method is related to our techniques. In [Section 1.2](#), we compare the second moment threshold with other notable phase transition thresholds appearing in the literature.

Theorem 1.1 (Algorithms). *For any $0 < \varepsilon, \eta < 1$, there exists $C = \Theta(1/\varepsilon)$ and $\delta = \delta(\varepsilon, n) > 0$ tending to 0 as $n \rightarrow \infty$ such that*

$$\mathbb{P}_G \left[e^{-\eta} \cdot \exp(\widehat{P}(\beta)) \leq Z_G(\beta) \leq e^{\eta} \cdot \exp(\widehat{P}(\beta)) \right] \geq 1 - \delta$$

for all $\beta \in \mathbb{D}(0, (1-\varepsilon) \cdot \beta_{2nd})$, where $\widehat{P}(\beta)$ denotes the Taylor polynomial of $\beta \mapsto \log Z_G(\beta)$ of degree $C \log(n/\eta)$. In particular, there is a deterministic algorithm running in time $n^{C \log(n/\eta)}$ which, on input $G, \{\gamma_p\}_{p=2}^{p_{\max}}, \beta$, outputs an $e^{\pm\eta}$ -multiplicative approximation to $Z_G(\beta)$ with probability at least $1 - o_n(1)$, for all $\beta \in \mathbb{D}(0, (1-\varepsilon) \cdot \beta_{2nd})$.

Remark 1 (Certifiable Approximation). A natural additional feature one would like from our algorithms is *certifiability*: There is an *efficiently testable* property $\mathcal{P}$ of the input instance G that holds with probability $1 - o_n(1)$ and moreover, the algorithm *must* output a $(1 \pm \varepsilon)$ -approximation to $Z_G(\beta)$ on the event $\mathcal{P}$. A natural candidate for $\mathcal{P}$ is zero-freeness of $Z_G(\beta)$ as described in [Theorem 1.2](#). However, we are not aware of an efficient algorithm to test this property. We leave the question of designing a certifiable approximation algorithm to future work.

We highlight that as a special case, we obtain deterministic quasipolynomial-time algorithms for estimating the partition function of the Sherrington–Kirkpatrick model, where $\varrho = \text{Unif}(C_n)$ and $\xi(s) = \frac{1}{2}s^2$, for all inverse temperatures up to the *replica-symmetry breaking threshold* $\beta_{RS} = \beta_{2nd} = 1$. This threshold is known to separate the high-temperature and low-temperature regimes in these models; see [Section 1.2](#) for a more in-depth discussion.

Following Barvinok’s interpolation method [[11–14](#)], our algorithms are based on truncating the Taylor series of $\log Z_G(\beta)$ to logarithmically many terms. The low-order coefficients are then computed exactly via brute force. The crucial property which permits this truncation is *stability* (or *zero-freeness*) of the partition function $Z_G(\beta)$. This is our main technical contribution.

Theorem 1.2 (Zeros). *For any $0 < \varepsilon < 1$, there exists $\delta = \delta(\varepsilon, n) > 0$ tending to 0 as $n \rightarrow \infty$ such that with probability at least $1 - \delta$ over the randomness of G , the partition function $Z_G(\beta)$ is nonzero on the disk $\mathbb{D}(0, (1-\varepsilon) \cdot \beta_{2nd})$.*

Taking inspiration from [[35](#)], we prove these using *Jensen’s Formula*, a widely used tool in the analysis of random polynomials. Our approach continues a long line of research studying the connections between the stability of partition functions, phase transitions in large-scale systems in statistical physics, and computational complexity of counting and sampling. In our setting, since we are viewing $Z_G(\beta)$ as a function of the interaction strength β , [Theorem 1.2](#) says that with high probability there are no *Fisher zeros* [[36](#)] of the model inside the centered disk of radius equal to the second moment threshold. Constraining the locations of the *Lee–Yang zeros* [[48](#)], i.e. the zeros of the partition function where the input

variables are given by the *external fields*, remains an interesting open problem. This is especially relevant for *sampling* from the associated Gibbs measure, since one needs to be able to handle the presence of external fields in order to invoke standard reductions between counting and sampling [46]. We emphasize that due to the lack of self-reducibility, our work does not immediately yield an efficient algorithm which samples from the Gibbs distribution. Even for the Sherrington–Kirkpatrick model, this remains an outstanding open problem as of this writing, although for general mixed p -spin models, efficient sampling is expected to not always be possible up to β_{2nd} ; see Section 1.2.

1.2 Comparison with Other Thresholds

Using the powerful but nonrigorous “replica method”, physicists identified a natural phase transition threshold β_{RS} such that in the *replica symmetric (RS) phase* $0 \leq \beta < \beta_{RS}$, the model exhibits “high-temperature behavior” (see e.g. [27, 66, 73]). For the models we study in this paper, an equivalent description of this phase is that the “quenched free energy” $\frac{1}{n} \mathbb{E}_G [\log Z_G(\beta)]$ and the “annealed free energy” $\frac{1}{n} \log \mathbb{E}_G [Z_G(\beta)]$ agree in the large n limit. Note that the former quantity is of greater interest from the physics perspective, while the latter quantity admits a simple and explicit formula. This has since been vindicated in a number of works in the mathematical physics literature (see e.g. [3, 24]). The RS phase coincides with a natural range of β for which one can hope to obtain high-accuracy estimates of the partition function $Z_G(\beta)$ in polynomial time. This is the regime in which we operate. In the specific setting of the Sherrington–Kirkpatrick model, where $\xi(s) = \frac{1}{2}s^2$ and $\varrho = \text{Unif}(C_n)$, it is well-known that $\beta_{RS} = \beta_{2nd} = 1$ [3, 24]. Thus, for this model, our algorithmic result is essentially sharp in terms of the range of β .

For the remainder of this discussion, we restrict our attention to the spherical case; a similar picture is predicted to hold for the discrete hypercube. In this setting,

$$\beta_{RS} \stackrel{\text{def}}{=} \sup \left\{ \beta \geq 0 : \beta^2 \cdot \xi(m) + m + \log(1-m) < 0, \forall m \in (0, 1) \right\}. \quad (3)$$

As described in [43, 44], the replica-symmetric phase can be further broken up into several, typically distinct, thresholds

$$0 \leq \beta_{\text{unique}} \leq \beta_{SL} \leq \beta_{\text{shatter}} \leq \beta_{RS},$$

which are particularly relevant for the related task of *sampling* from the Gibbs measure $\mu_{G,\beta}$:

Uniqueness Threshold: This is the threshold β_{unique} below which local Markov chains (e.g. *Glauber dynamics* and *Langevin diffusion*) for sampling from $\mu_{G,\beta}$ are believed to satisfy a *Poincaré Inequality* with high probability (and hence, converge to $\mu_{G,\beta}$ in polynomial time even under a worst-case initialization [25]). Above this threshold, it is known that worst-case mixing is exponentially slow [9]. For a precise definition of this threshold in the spherical pure p -spin case, see [10].

Stochastic Localization Threshold: This is the threshold β_{SL} below which *algorithmic stochastic localization* [5, 44]

successfully samples from $\mu_{G,\beta}$, and above which algorithmic stochastic localization fails [44]. This is also the threshold below which *simulated annealing* [43] is proven to sample from $\mu_{G,\beta}$ (though simulated annealing may also succeed beyond β_{SL}).

Shattering Threshold: Below this threshold β_{shatter} and above β_{unique} , *metastable states* are present in the Gibbs measure [9], which obstruct local Markov chains from mixing rapidly from a worst-case initial distribution. Nonetheless, because these metastable states in aggregate only constitute a vanishingly small proportion of the Gibbs mass, it has been predicted in the physics literature that local Markov chains still mix rapidly from a *random initialization*; see e.g. [4, 25, 26]. Mixing from a suitable *warm start* has been proved for β up to β_{SL} [43]. Rigorous analysis of any algorithm that succeeds up to β_{shatter} (which can be Glauber or Langevin dynamics from random initialization, or another algorithm entirely) remains an interesting open problem.

Above this threshold β_{shatter} , $\mu_{G,\beta}$ *shatters* into exponentially many well-separated clusters, each of which contains only an exponentially small fraction of the Gibbs mass [25, 32, 37]. There is preliminary evidence suggesting that efficient sampling from $\mu_{G,\beta}$ is impossible for all $\beta > \beta_{\text{shatter}}$ [5, 32].

We refer interested readers to [43, 44] for precise characterizations of β_{SL} , β_{shatter} , analogous to Definition 1 and Eq. (3).

Because sampling from $\mu_{G,\beta}$ is intimately related to computing the partition function $Z_G(\beta)$ [46], it is interesting to compare these thresholds with β_{2nd} . For this, let us focus on the *pure p -spin* case, where $\xi(s) = s^p$ for some positive integer $p \geq 2$. In this case, it is known (see [43, 44]) that

$$\beta_{\text{unique}} \asymp \frac{1}{\sqrt{\log p}}, \quad \beta_{SL}, \beta_{\text{shatter}} \asymp 1, \quad \beta_{RS} \asymp \sqrt{\log p},$$

$$\frac{\beta_{SL}}{\beta_{\text{shatter}}} \approx \frac{\sqrt{e}}{2}.$$

Now recall that $h(m) = \frac{1}{2} \log(1-m^2)$ in the spherical case. Since the following numerical inequalities hold

$$2 \cdot h(m) \leq m + \log(1-m) \leq h(m), \quad \forall m \in [0, 1],$$

it is clear that $\beta_{2nd} \asymp \beta_{RS} \asymp \sqrt{\log p}$ as well. Thus, Theorem 1.1 covers a significantly larger range than $0 \leq \beta \leq \beta_{\text{shatter}}$, which is the limit of what is believed to be achievable in the realm of efficient sampling [5, 32]. Note, however, that $\beta_{2nd} < \beta_{RS}$ in general. We leave extending Theorem 1.1 to the full replica-symmetric phase to future work.

1.3 Prior Work

Free Energy Formulas. There is an enormous literature devoted to deriving and proving explicit formulas for the limiting “quenched free energy”:

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E}_G [\log Z_G(\beta)].$$

For the Sherrington–Kirkpatrick model, landmark results include the *replica-symmetric formula* [66], the celebrated *Parisi formula* [57–59], and subsequent breakthroughs which made the latter mathematically rigorous [23, 38, 39, 54, 56, 69, 70]; we refer interested

readers to [22, 55, 71, 72] and references therein for more details. Note that such formulas entail only an $o(n)$ -additive approximation to $\mathbb{E}_G [\log Z_G(\beta)]$, which is of order $O(n)$. In contrast, following the theoretical computer science tradition, our goal is to design algorithms which estimate $\log Z_G$ up to η -additive error for *any* $\eta > 0$ prescribed as input. Moreover, our algorithms must work with high probability over the realization of G , as opposed to in expectation.

The Plefka and Aizenman–Lebowitz–Ruelle Expansions. In [63], an alternative but nonrigorous derivation of the Thouless–Anderson–Palmer (TAP) Equations [73] for the Sherrington–Kirkpatrick model was proposed, based on truncating the Taylor expansion of $\beta \mapsto \log Z_G(\beta)$ to second-order:

$$\begin{aligned} \log Z_G(\beta) \approx & \log Z_G(0) + \mathbb{E}_{\sigma \sim \varrho} \left[\frac{1}{\sqrt{2n}} \sigma^\top G \sigma \right] \cdot \beta \\ & + \text{Var}_{\sigma \sim \varrho} \left[\frac{1}{\sqrt{2n}} \sigma^\top G \sigma \right] \cdot \frac{\beta^2}{2}. \end{aligned}$$

This is known in the literature as the *Plefka expansion*. However, we are unaware of any mathematically rigorous result prior to our work which validates the above approximation. [Theorem 1.2](#) vindicates the $O(\log n)$ -order Plefka expansion with probability $1 - o_n(1)$ for all β with $|\beta| < \beta_{2\text{nd}}$.

Aizenman, Lebowitz and Ruelle [3] introduced a cluster expansion approximation of the Sherrington–Kirkpatrick partition function and rigorously showed that it $o_n(1)$ -additively approximates $\log Z_G(\beta)$ for all (positive real) $\beta < \beta_{2\text{nd}} = 1$. An analogue of this expansion for the mixed p -spin models will play an important role in our proof; see [Section 2.2](#). However, we emphasize that the cluster expansion of [3] only achieves an $o_n(1)$ error for *some* $o_n(1)$, whereas [Theorem 1.1](#) achieves any *arbitrarily small* prescribed error.

Localization and Markov Chains. For the Sherrington–Kirkpatrick model, [34] (see also [17]) showed that Glauber dynamics for sampling from $\mu_{G,\beta}$ mixes in $O(n^2)$ -time for $\beta < \frac{1}{4}$. Subsequently, the mixing time was improved to $O(n \log n)$ in [6], and the range of β was extended to $\beta < 0.295$ [8]. In [5] (see also [31] for the mixed p -spin setting), it was shown that an algorithmic version of Eldan’s *stochastic localization* [33] samples from the Gibbs measure up to $o(n)$ Wasserstein distance for all $\beta < 1 = \beta_{\text{RS}}$, which is the tight range of β ; it is believed that Glauber dynamics efficiently samples up to arbitrary total variation error in this regime as well. For more general mixed p -spin models on the hypercube, $O(n^2)$ -mixing for Glauber dynamics was established in [2] under a more restrictive condition than the replica-symmetric phase. This was improved to $O(n \log n)$ in [7]. Besides [5, 31], all these sampling results can be converted into efficient algorithms for computing $(1 \pm \eta)$ -multiplicative approximations to $Z_G(\beta)$ via standard counting-to-sampling reductions [46].

In the spherical setting, [44] used algorithmic stochastic localization to sample from the Gibbs measure with $o_n(1)$ total variation error for $\beta < \beta_{\text{SL}}$. [43] then showed that annealed Langevin dynamics samples from the Gibbs measure up to subexponentially small total variation error in the same regime. While the lack of

self-reducibility obstructs standard sampling-to-counting reductions, we note that sampling up to arbitrarily small total variation error does yield an algorithm for estimating the partition function $Z_G(\beta)$ in a black-box manner. Indeed, we have

$$(\log Z_G)'(\beta) = \mathbb{E}_{\sigma \sim \mu_{G,\beta}} [\mathcal{H}_G(\sigma)],$$

which can be approximated using the Monte Carlo method given sampling access to the Gibbs measure $\mu_{G,\beta}$. This can then be integrated over β .

Approximate Counting from Zero-Freeness. Since the introduction of the #P complexity class in theoretical computer science [74, 75], there has been significant interest in designing FPTAS for *approximate counting* tasks, that is, efficient deterministic algorithms which estimate the partition function up to arbitrarily small relative error. Towards this, Barvinok proposed to truncate the Taylor series of the logarithm of the partition function. This is often referred to as Barvinok’s *interpolation method*, and it has since been used to design quasipolynomial-time approximation algorithms for a wide variety of counting problems, including the permanent [13, 14], graph homomorphisms [11, 15, 16], etc. We highlight [35], which applies the interpolation method to compute permanents of *random* complex matrices. Our work draws heavily upon theirs. For some of these counting problems these algorithms have been improved to run in polynomial time [60], thereby obtaining a genuine FPTAS.

Now, the radius of convergence is governed by the location of the nearest zero of the partition function. Building on the seminal Lee–Yang theory [48], it was gradually realized that there is a deep connection between the absence of phase transitions and the computational complexity of estimating the partition function. This has been formalized in a number of works; see e.g. [19, 21, 28, 62] for the hardcore gas model, [20, 41, 49–51] for the Ising and Potts models, [18, 40] for Holant-type problems, [42] for quantum many-body systems, etc. We refer interested readers to [12, 61] for a more comprehensive discussion.

Zeros of the Random Energy Model. After posting the initial version of our paper, we discovered that Fisher zeros of the *random energy model* (and its generalized version) had previously been studied. This is a simplified spin glass model, originally proposed by Derrida [29], where $\mathcal{H}(\sigma)$ is an independently drawn Gaussian for all $\sigma \in \{\pm 1\}^n$. Its asymptotic free energy admits closed-form formulas, from which nonrigorous physics predictions were made about partition function zeros; see e.g. [30, 67]. Rigorous mathematical results in this direction were obtained more recently [47]. For this model, a short calculation reveals that our techniques can easily establish zero-freeness in a disk of radius equal to the second moment threshold, with high probability over the disorder. This then yields a quasipolynomial-time algorithm to estimate the partition function in the same regime.

2 Techniques and Overview of the Paper

2.1 Counting Zeros via Jensen’s Formula

[Theorem 1.1](#) will follow easily from the zero-freeness established in [Theorem 1.2](#). Indeed, following Barvinok (see e.g. [12]), the algorithm reduces to computing the Taylor coefficients of $\widehat{P}(\beta)$ and

truncating the resulting series. In the rest of this overview, we discuss the proof of [Theorem 1.2](#), which occupies most of this paper. We will set out to prove

$$\mathbb{E}_G [\#\{\omega \in \mathbb{D}(0, r) : Z_G(\omega) = 0\}] = o_n(1), \quad (4)$$

from which [Theorem 1.2](#) follows by Markov's inequality. At the heart of our methodology is the following tool from complex analysis and the study of zeros of random polynomials, which was used in a similar context in [\[35\]](#).

Theorem 2.1 (Jensen's Formula). *Let $\Omega \subset \mathbb{C}$ be an open set that contains the disk $\overline{\mathbb{D}(0, R)}$ for some $R > 0$. Let $f : \Omega \rightarrow \mathbb{C}$ be an analytic function such that $f(0) \neq 0$. Then*

$$\sum_{\omega \in \mathbb{D}(0, R) : f(\omega) = 0} \log \left(\frac{R}{|\omega|} \right) = \mathbb{E}_\theta \left[\log \left| \frac{f(Re^{i\theta})}{f(0)} \right| \right],$$

where the expectation is taken with respect to $\theta \sim \text{Unif}[-\pi, \pi]$.

Throughout this paper we set

$$R = \left(1 - \frac{\varepsilon}{2}\right) \beta_{2\text{nd}} \quad \text{and} \quad r = (1 - \varepsilon) \beta_{2\text{nd}}.$$

The starting point of our proof is the following estimate on the expected number of zeros of a random analytic function on $\mathbb{D}(0, r)$, in terms of the function's second moment. This is proved in [Section 2.3](#).

Lemma 2.2. *Let Ω be as in [Theorem 2.1](#), and $f_G : \Omega \rightarrow \mathbb{C}$ be a random G -measurable function, which is analytic with $f_G(0) \neq 0$ deterministically. Then,*

$$\mathbb{E}_G [\#\{\omega \in \mathbb{D}(0, r) : f_G(\omega) = 0\}] \leq \frac{1}{2\varepsilon} \mathbb{E}_\theta \log \mathbb{E}_G \left[\left| \frac{f_G(Re^{i\theta})}{f_G(0)} \right|^2 \right]$$

The most natural choice of $f_G(\beta)$ to use in [Lemma 2.2](#) is $f_G(\beta) = Z_G(\beta)$. With this choice, we will see that [Lemma 2.2](#) reduces the mixed p -spin glass to a significantly simpler mean-field model where the interactions are deterministic, the *mixed p -spin Curie-Weiss model*. The partition function of this model is

$$Z_{\text{CW}}(\beta) \stackrel{\text{def}}{=} \mathbb{E}_{\sigma \sim \varrho} \left[\exp \left(\beta \cdot n \cdot \xi \left(\frac{\langle \sigma, 1 \rangle}{n} \right) \right) \right], \quad (5)$$

where again, we suppress the dependence on ξ . In the corresponding Gibbs distribution, higher probability mass is placed on configurations σ which align with 1 or -1 , i.e. all interactions are ferromagnetic in nature. In the case where $\varrho = \text{Unif}(C_n)$ and $\xi(s) = \frac{1}{2}s^2$, this is the partition function of the standard Curie-Weiss model in statistical physics, which may be viewed as the ferromagnetic Ising model on the complete graph.

Formally, the following lemma, proved in [Section 2.3](#), evaluates the right-hand side of [Lemma 2.2](#) for $f_G(\beta) = Z_G(\beta)$.

Lemma 2.3. *We have that*

$$\mathbb{E}_\theta \log \mathbb{E}_G \left[\left| \frac{Z_G(Re^{i\theta})}{Z_G(0)} \right|^2 \right] = \log Z_{\text{CW}}(R^2).$$

Unfortunately, this is not enough to prove [Theorem 1.2](#). In the second moment regime we study, Z_{CW} is bounded by a constant, but is usually not $1 + o_n(1)$. Indeed, for the Sherrington-Kirkpatrick model (i.e. the mixed p -spin glass with $\varrho = \text{Unif}(C_n)$ and $\xi(s) = \frac{1}{2}s^2$), the corresponding Z_{CW} is the standard Curie-Weiss partition function. It is well-known that Z_{CW} has a phase transition at $\beta_{2\text{nd}} = 1$, and $Z_{\text{CW}}(\beta^2) = (1 - \beta^2)^{-1/2} + o_n(1)$ for positive real $\beta < 1$; see e.g. [Lemma 2.1](#) in [\[68\]](#). For general mixture functions, an analogous bound is

$$Z_{\text{CW}}(|\beta|^2) = \frac{1}{\sqrt{1 - |\beta|^2 \xi''(0)}} + o_n(1) = \frac{1}{\sqrt{1 - 2|\beta|^2 \gamma_2^2}} + o_n(1) \quad (6)$$

for all $|\beta| < \beta_{2\text{nd}}$. This generalizes earlier results for the pure p -spin case [\[53\]](#). Combining [Eq. \(6\)](#) with [Lemmas 2.2](#) and [2.3](#), where we choose $f_G(\beta) = Z_G(\beta)$, shows

$$\mathbb{E}_G [\#\{\omega \in \mathbb{D}(0, r) : Z_G(\omega) = 0\}] \leq O_\varepsilon(1),$$

where $O_\varepsilon(1)$ denotes an ε -dependent constant. This falls short of the $o_n(1)$ desired in [Eq. \(4\)](#).

Remark 2. An exception to the above discussion is the case where quadratic interactions are absent, i.e. $\gamma_2 = 0$. In this case [Eq. \(6\)](#) shows $Z_{\text{CW}}(|\beta|^2) = 1 + o_n(1)$. Then, [Lemmas 2.2](#) and [2.3](#) directly imply [Eq. \(4\)](#), which yields [Theorem 1.2](#).

We will be able to obtain the $o_n(1)$ bound in [Eq. \(4\)](#) by applying [Lemma 2.2](#) with $f_G(\beta) = X_G(\beta)$, where $X_G(\beta)$ is a suitably reweighted version of $Z_G(\beta)$. We explain this in the next subsection.

2.2 Reweighting the Partition Function

The fact that the bound in [Lemma 2.2](#) with $f_G(\beta) = Z_G(\beta)$ loses a constant factor is related to a common issue faced by the *second moment method*. We describe this first in the simpler setting of a positive real-valued random variable Z . The second moment method is a strategy for proving that $Z > 0$ with high probability, in which we aim to show that

$$\mathbb{E}[Z^2] = (1 + o_n(1)) \cdot \mathbb{E}[Z]^2. \quad (7)$$

If this holds, the Cauchy-Schwarz inequality implies $\mathbb{P}(Z > 0) = 1 - o_n(1)$. However, in many applications [Eq. \(7\)](#) does not hold, and instead we only have the weaker estimate

$$\mathbb{E}[Z^2] = O(1) \cdot \mathbb{E}[Z]^2. \quad (8)$$

Typically, this is because the random variable Z has $O(1)$ -multiplicative fluctuations. From [Eq. \(8\)](#) we can only conclude $\mathbb{P}(Z > 0) = \Omega(1)$. In such settings, one way to recover a high-probability bound is to identify a random variable $A > 0$ such that

$$X \stackrel{\text{def}}{=} A \cdot Z = 1 + o_n(1) \quad \text{with high probability,}$$

and then aim to show [Eq. \(7\)](#) with X in place of Z . That is, A is a reweighting factor that “explains the fluctuations” of Z . This method is well-known in the combinatorics literature as the *small subgraph conditioning method* [\[64, 65\]](#).

In our setting, the random variable $Z_G(\beta)$ likewise has $O(1)$ -multiplicative fluctuations, which cause a direct application of

Lemma 2.2 to lose a constant factor. To remedy this, we will guess a reweighting factor $A_G(\beta)$ such that we (heuristically) expect

$$X_G(\beta) \stackrel{\text{def}}{=} A_G(\beta) \cdot Z_G(\beta) = 1 + o_n(1), \quad \forall \beta \in \overline{\mathbb{D}(0, R)} \quad (9)$$

with high probability. Our choice of $A_G(\beta)$ takes the form of a cluster expansion in the entries of the quadratic part of G , and is motivated by the cluster expansion of [3]. This follows the work [1], which applies a similar cluster expansion-reweighted second moment to the (positive real-valued) partition function of the symmetric binary perceptron.

We will choose $A_G(\beta)$ to be analytic in β on an open neighborhood $\Omega \supseteq \overline{\mathbb{D}(0, R)}$, so that X_G is also analytic in Ω and its zeros are a superset of those of Z_G . We then show that for $f_G(\beta) = X_G(\beta)$, the upper bound obtained in **Lemma 2.2** is $o_n(1)$. (In order to make this calculation tractable, it will also be important that $A_G(\beta)$ is complex analytic in G .) The main point of the analysis is that the reweighting factor $A_G(\beta)$ contributes a factor of $\sqrt{1 - |\beta|^2 \xi''(0)}$ to the second moment. This cancels the Curie–Weiss partition function **Eq. (6)**, leaving

$$\mathbb{E}_G \left[\left| \frac{X_G(\beta)}{X_G(0)} \right|^2 \right] = 1 + o_n(1), \quad \forall \beta \in \overline{\mathbb{D}(0, R)}.$$

Then **Theorem 1.2** readily follows from **Lemma 2.2**. Finally, we note that while our choice of $A_G(\beta)$ is motivated by spin glass considerations, for the formal proof it is enough to control the upper bound from **Lemma 2.2**, and this is purely a second moment calculation. In particular, we do not need to prove that $A_G(\beta)$ satisfies **Eq. (9)**.

2.3 Reduction from Zero-Counting to Second Moment

In this subsection, we prove **Lemmas 2.2** and **2.3**. **Lemma 2.3** will not play a formal role in the proofs of our main results. We include its proof to explain what **Lemma 2.2** obtains with the natural choice $f_G(\beta) = Z_G(\beta)$, and to provide a simple demonstration of the second moment calculations that arise in our method.

PROOF OF LEMMA 2.2. Note that (deterministically)

$$\begin{aligned} & \#\{\omega \in \mathbb{D}(0, r) : f_G(\omega) = 0\} \\ & \leq \left(\log \frac{R}{r} \right)^{-1} \sum_{\omega \in \mathbb{D}(0, r) : f_G(\omega) = 0} \log \left(\frac{R}{|\omega|} \right) \\ & \leq \frac{1}{\varepsilon} \sum_{\omega \in \mathbb{D}(0, R) : f_G(\omega) = 0} \log \left(\frac{R}{|\omega|} \right) \\ & = \frac{1}{\varepsilon} \mathbb{E}_\theta \left[\log \left| \frac{f_G(Re^{i\theta})}{f_G(0)} \right| \right], \end{aligned}$$

where the final equality is by **Theorem 2.1**. Taking expectation over G we obtain

$$\begin{aligned} \mathbb{E}_G [\#\{\omega \in \mathbb{D}(0, r) : f_G(\omega) = 0\}] & \leq \frac{1}{\varepsilon} \mathbb{E}_G \mathbb{E}_\theta \left[\log \left| \frac{f_G(Re^{i\theta})}{f_G(0)} \right| \right] \\ & \leq \frac{1}{2\varepsilon} \mathbb{E}_\theta \log \mathbb{E}_G \left[\left| \frac{f_G(Re^{i\theta})}{f_G(0)} \right|^2 \right], \end{aligned}$$

by Jensen's inequality. $\square$

PROOF OF LEMMA 2.3. We first compute the inner expectation by leveraging the independence of the entries of G . For notational convenience, we write $\beta = Re^{i\theta} \in \mathbb{C}$, which we treat as fixed for the moment. Observe that

$$\begin{aligned} & \mathbb{E}_G \left[\left| \frac{Z_G(\beta)}{Z_G(0)} \right|^2 \right] \\ & = \mathbb{E}_{\tau, \sigma \sim \varrho} \left[\mathbb{E}_G \left[\exp \left(\beta \cdot \mathcal{H}_G(\tau) + \bar{\beta} \cdot \mathcal{H}_G(\sigma) \right) \right] \right] \\ & = \mathbb{E}_{\tau, \sigma \sim \varrho} \left[\exp \left(\frac{1}{2} \cdot \mathbb{E}_G \left[\left(\beta \cdot \mathcal{H}_G(\tau) + \bar{\beta} \cdot \mathcal{H}_G(\sigma) \right)^2 \right] \right) \right] \\ & = \mathbb{E}_{\tau, \sigma \sim \varrho} \left[\exp \left(\frac{n}{2} \cdot \left(\beta^2 + \bar{\beta}^2 \right) \cdot \xi(1) + n \cdot |\beta|^2 \cdot \xi \left(\frac{\langle \tau, \sigma \rangle}{n} \right) \right) \right] \\ & = \exp \left(n \cdot \xi(1) \cdot \operatorname{Re} \beta^2 \right) \cdot Z_{\text{CW}}(|\beta|^2). \end{aligned}$$

Here we used the Gaussian moment-generating function, **Eq. (2)**, and symmetry of the hypercube/sphere. Since $|\beta|^2 = R^2$, plugging this back in yields the desired identity, up to an extra additive error term of

$$n \cdot \xi(1) \cdot \mathbb{E}_\theta \operatorname{Re} \left(R^2 e^{2i\theta} \right).$$

This is clearly zero, concluding the proof. $\square$

2.4 Preliminaries

Throughout this paper, we use $c, C, K, L > 0$ to denote various constants that may change depending on the context; unless otherwise specified, these constants are universal, i.e. independent of all other parameters (especially n). All logarithms are base e . As a complex function, we take the branch of the logarithm that is real on the positive real line. We write $\mathbb{D}(z, r)$ for the open disk of radius- r around z in the complex plane $\mathbb{C}$; we abbreviate $\mathbb{D}$ for $\mathbb{D}(0, 1)$.

Lemma 2.4. We have $\beta_{2\text{nd}}^2 \cdot \xi''(0) \leq 1$.

PROOF. In either case of the definition of $h(m)$ in **Definition 1**, $h''(0) = -1$. Let φ be defined as in **Definition 1** with $\beta = \beta_{2\text{nd}}$. The definition of $\beta_{2\text{nd}}$ implies

$$0 \geq \varphi''(0) = \beta_{2\text{nd}}^2 \cdot \xi''(0) - 1,$$

from which the result follows. $\square$

3 Evaluation of Reweighted Second Moment

In this section, we formally define the reweighting factor $A_G(\beta)$. We will then prove **Proposition 3.1** below, which controls the upper bound from **Lemma 2.2** with $f_G(\beta) = X_G(\beta)$ defined in **Eq. (9)**, assuming as input **Propositions 3.2** and **3.3** which are proved in the full version of this paper. **Theorem 1.2** is immediate from **Proposition 3.1**, and its proof appears at the end of this section. We recall that we write

$$R = \left(1 - \frac{\varepsilon}{2} \right) \beta_{2\text{nd}}$$

throughout.

Proposition 3.1. Let $X_G(\beta) = A_G(\beta) \cdot Z_G(\beta)$, where A_G is defined in Eq. (15) below. Then,

$$\mathbb{E}_G \left[\left| \frac{X_G(\beta)}{X_G(0)} \right|^2 \right] \leq 1 + o_n(1), \quad \forall \beta \in \overline{\mathbb{D}(0, R)},$$

where the error term $o_n(1)$ is uniform over β in this domain.

3.1 Definition of Reweighting Factor

We begin by defining $A_G(\beta)$. Note that

$$\begin{aligned} \nabla^2 \mathcal{H}_G(0) &= \frac{\gamma_2}{\sqrt{n}} (G_{i,j} + G_{j,i} : 1 \leq i, j \leq n) \\ &= \frac{\xi''(0)^{1/2}}{\sqrt{2n}} (G_{i,j} + G_{j,i} : 1 \leq i, j \leq n). \end{aligned} \quad (10)$$

We will write throughout this paper

$$M = \nabla^2 \mathcal{H}_G(0). \quad (11)$$

Let $\binom{[n]}{2}$ denote the set of unordered pairs $\{i, j\}$ with $1 \leq i < j \leq n$. We will think of elements of $\binom{[n]}{2}$ as edges on the complete graph K_n . For $e = \{i, j\} \in \binom{[n]}{2}$, we write $M_e = M_{i,j}$. Let

$$\zeta = \zeta(\beta) = \beta \xi''(0)^{1/2}. \quad (12)$$

Note that for all $\beta \in \overline{\mathbb{D}(0, R)}$,

$$|\zeta| \leq R \xi''(0)^{1/2} = \left(1 - \frac{\varepsilon}{2}\right) \beta_{2nd} \cdot \xi''(0)^{1/2} \leq 1 - \frac{\varepsilon}{2}, \quad (13)$$

by Lemma 2.4. Let $K = \lfloor \log \log n \rfloor$. Then, define

$$\text{UC}(n) = \left\{ \Gamma \subseteq \binom{[n]}{2} : \begin{array}{l} \Gamma \text{ is the edge set of a vertex-disjoint} \\ \text{union of cycles and } |\Gamma| \leq K \end{array} \right\}. \quad (14)$$

For $\Gamma \in \text{UC}(n)$, let $c(\Gamma)$ denote the number of connected components of Γ viewed as a graph. Finally define the reweighting factor

$$\begin{aligned} A_G(\beta) &= (1 - \zeta^2)^{-1/2} \exp \left(-\frac{n\beta^2}{2} \xi(1) + \frac{n\zeta^2}{4} - \frac{\zeta^2}{2} - \frac{\zeta^4}{8} \right) \\ &\quad \times \exp \left(-\frac{\beta}{2} \text{Tr}(M) - \frac{\beta^2}{2} \sum_{1 \leq i < j \leq n} M_{i,j}^2 \right) \\ &\quad \times \left(\sum_{\Gamma \in \text{UC}(n)} (-1)^{c(\Gamma)} \prod_{e \in \Gamma} (\beta M_e) \right). \end{aligned} \quad (15)$$

We explain the motivation for this definition in the full version of the paper. In light of Eq. (13), it is possible to pick a branch of the square root such that $(1 - \zeta^2)^{-1/2}$ is analytic for β in an open neighborhood $\Omega \supseteq \overline{\mathbb{D}(0, R)}$. Thus $A_G(\beta)$ is analytic in Ω .

3.2 Second Moment Calculation

We next introduce the two inputs that are proved in the full version of the paper, and use them to prove Proposition 3.1. Let $\text{ov}(\varrho) \in \mathcal{P}([-1, 1])$ denote the distribution of the overlap $\langle \sigma, \tau \rangle / n$ for independent $\sigma, \tau \sim \varrho$.

Proposition 3.2. Let $\alpha = 0.01$. The following estimates hold for all $\beta \in \overline{\mathbb{D}(0, R)}$.

$$\mathbb{E}_{q \sim \text{ov}(\varrho)} \left[\mathbb{I} \left[|q| \leq n^{-1/2+\alpha} \right] \exp \left(n |\beta|^2 \xi(q) \right) \right] \quad (16)$$

$$= \frac{1}{\sqrt{1 - |\beta|^2 \xi''(0)}} + o_n(1), \quad (17)$$

$$\mathbb{E}_{q \sim \text{ov}(\varrho)} \left[\mathbb{I} \left[|q| > n^{-1/2+\alpha} \right] \exp \left(n |\beta|^2 \xi(q) \right) \right] \leq e^{-\Omega(n^{2\alpha})}. \quad (18)$$

To provide intuition, we give a heuristic calculation justifying Proposition 3.2 under two simplifying assumptions:

- A Gaussian approximation to the overlap distribution $\text{ov}(\varrho)$. That is, replace $\text{ov}(\varrho)$ by $\mathcal{N}(0, 1/n)$. This is reasonable because in the spherical setting, the actual overlap distribution is a high dimensional sphere projected onto a one dimensional subspace, while in the Ising setting, the overlap distribution is a binomial distribution – both of which are well-approximated by a Gaussian.
- A quadratic approximation to $\xi(q)$. That is, pretend that $\xi(q) = \frac{1}{2} \xi''(0) q^2$. This is essentially saying that the higher degree terms will only contribute terms which are lower order in n .

With these assumptions, the first part of the proposition follows via a Gaussian integral.

$$\begin{aligned} &\mathbb{E}_{q \sim \text{ov}(\varrho)} \left[\mathbb{I} \left[|q| \leq n^{-1/2+\alpha} \right] \exp \left(n |\beta|^2 \xi(q) \right) \right] \\ &\approx \mathbb{E}_{q \sim \mathcal{N}(0, 1/n)} \left[\exp \left(\frac{1}{2} n |\beta|^2 \xi''(0) q^2 \right) \right] \\ &= \int_{-\infty}^{\infty} \sqrt{\frac{n}{2\pi}} \exp \left(-\frac{n}{2} \left(1 - |\beta|^2 \xi''(0) \right) q^2 \right) dq \\ &= \frac{1}{\sqrt{1 - |\beta|^2 \xi''(0)}}. \end{aligned}$$

The second part of the proposition is similarly easy with this simplification, because it just ends up being a Gaussian tail bound.

For $q \in \text{supp } \text{ov}(\varrho)$, let $\varrho_2(q)$ be the distribution of $(\sigma, \tau) \sim \varrho^{\otimes 2}$ conditioned on $\langle \sigma, \tau \rangle / n = q$.

Proposition 3.3. Let $\alpha = 0.01$. Consider any $\beta \in \overline{\mathbb{D}(0, 1 - R)}$ and $q \in \text{supp } \text{ov}(\varrho)$. Let

$$\begin{aligned} \text{Val}(\beta, q) &= \mathbb{E}_{(\sigma, \tau) \sim \varrho_2(q)} \frac{\mathbb{E}_G [\exp (\beta \mathcal{H}_G(\sigma) + \bar{\beta} \mathcal{H}_G(\tau)) |A_G(\beta)|^2]}{\mathbb{E}_G [\exp (\beta \mathcal{H}_G(\sigma) + \bar{\beta} \mathcal{H}_G(\tau))]} \\ &\quad \times \exp \left(n \text{Re}(\beta^2) \xi(1) \right). \end{aligned}$$

If $|q| \leq n^{-1/2+\alpha}$, then

$$\text{Val}(\beta, q) = (1 - |\beta|^2 \xi''(0))^{1/2} + o_n(1), \quad (19)$$

for an error $o_n(1)$ uniform over such q . Moreover, there exists $C = C(\varepsilon)$ such that for all other q ,

$$|\text{Val}(\beta, q)| \leq C \cdot 2^{4K}. \quad (20)$$

We next outline some of the main ideas of the proof of Proposition 3.3. First, we show the expectation ratio

$$\frac{\mathbb{E}_G [\exp (\beta \mathcal{H}_G(\sigma) + \bar{\beta} \mathcal{H}_G(\tau)) |A_G(\beta)|^2]}{\mathbb{E}_G [\exp (\beta \mathcal{H}_G(\sigma) + \bar{\beta} \mathcal{H}_G(\tau))]} \quad (21)$$

equals $\mathbb{E}_{G^{\text{pl}}} [|A_{G^{\text{pl}}}(\beta)|^2]$ for disorder G^{pl} drawn from the following planted model:

$$\mathcal{H}_{G^{\text{pl}}}(x) = \mathcal{H}_G(x) + \beta n \xi \left(\frac{\langle x, \sigma \rangle}{n} \right) + \bar{\beta} n \xi \left(\frac{\langle x, \tau \rangle}{n} \right). \quad (22)$$

Indeed, if β were real, Eq. (21) can be interpreted as re-weighting the gaussian density of G by the factor $\exp(\beta \mathcal{H}_G(\sigma) + \bar{\beta} \mathcal{H}_G(\tau))$ (where we treat σ, τ as fixed), which results in the planted law Eq. (22) by a simple gaussian change of measure calculation. For complex β , arguing this requires more care, and we refer the reader to the full version of this paper.

We have thus reduced Proposition 3.3 to evaluating the expectation $\mathbb{E}_{G^{\text{pl}}} [|A_{G^{\text{pl}}}(\beta)|^2]$ under the planted law Eq. (22). We achieve this by expanding $|A_{G^{\text{pl}}}(\beta)|^2$, whose main term (recalling the definition Eq. (15) of $A_G(\beta)$) is a sum over pairs of subgraphs $\Gamma^1, \Gamma^2 \in \text{UC}(n)$. We then identify the (Γ^1, Γ^2) that provide the dominant contribution to the expectation and carefully track the total contribution of such pairs. This calculation occupies a large part of the full version of this paper, and we refer the reader to the full version for details.

PROOF OF PROPOSITION 3.1. We calculate

$$\begin{aligned} \mathbb{E}_G \left[\left| \frac{X_G(\beta)}{X_G(0)} \right|^2 \right] &= \mathbb{E}_G [Z_G(\beta) Z_G(\bar{\beta}) |A_G(\beta)|^2] \\ &= \mathbb{E}_{\sigma, \tau \sim \mathcal{Q}} \mathbb{E}_G [\exp(\beta \mathcal{H}_G(\sigma) + \bar{\beta} \mathcal{H}_G(\tau)) |A_G(\beta)|^2] \\ &= \mathbb{E}_{q \sim \text{ov}(\mathcal{Q})} \mathbb{E}_{(\sigma, \tau) \sim \mathcal{Q}_2(q)} \left[\frac{\mathbb{E}_G [\exp(\beta \mathcal{H}_G(\sigma) + \bar{\beta} \mathcal{H}_G(\tau)) |A_G(\beta)|^2]}{\mathbb{E}_G [\exp(\beta \mathcal{H}_G(\sigma) + \bar{\beta} \mathcal{H}_G(\tau))]} \right. \\ &\quad \left. \times \mathbb{E}_G [\exp(\beta \mathcal{H}_G(\sigma) + \bar{\beta} \mathcal{H}_G(\tau))] \right]. \end{aligned}$$

The last expectation can be written as (similarly to the proof of Lemma 2.3)

$$\mathbb{E}_G [\exp(\beta \mathcal{H}_G(\sigma) + \bar{\beta} \mathcal{H}_G(\tau))] = \exp(n \text{Re}(\beta^2) \xi(1) + n |\beta|^2 \xi(q)).$$

Note that this depends on (σ, τ) only through their overlap q . So,

$$\mathbb{E}_G \left[\left| \frac{X_G(\beta)}{X_G(0)} \right|^2 \right] = \mathbb{E}_{q \sim \text{ov}(\mathcal{Q})} [\exp(n |\beta|^2 \xi(q)) \text{Val}(\beta, q)],$$

that can be written as the sum (I) + (II), where (recall $\alpha = 0.01$)

$$\begin{aligned} \text{(I)} &= \mathbb{E}_{q \sim \text{ov}(\mathcal{Q})} \left[\mathbb{I} \left[|q| \leq n^{-1/2+\alpha} \right] \exp(n |\beta|^2 \xi(q)) \text{Val}(\beta, q) \right], \\ \text{(II)} &= \mathbb{E}_{q \sim \text{ov}(\mathcal{Q})} \left[\mathbb{I} \left[|q| > n^{-1/2+\alpha} \right] \exp(n |\beta|^2 \xi(q)) \text{Val}(\beta, q) \right]. \end{aligned}$$

By Propositions 3.2 and 3.3,

$$\begin{aligned} \text{(I)} &\leq \left(\sqrt{1 - |\beta|^2 \xi''(0)} + o_n(1) \right) \\ &\quad \times \mathbb{E}_{q \sim \text{ov}(\mathcal{Q})} \left[\mathbb{I} \left[|q| \leq n^{-1/2+\alpha} \right] \exp(n |\beta|^2 \xi(q)) \right] \\ &\leq \left(\sqrt{1 - |\beta|^2 \xi''(0)} + o_n(1) \right) \left(\frac{1}{\sqrt{1 - |\beta|^2 \xi''(0)}} + o_n(1) \right) \\ &= 1 + o_n(1). \end{aligned}$$

Moreover,

$$\begin{aligned} |(\text{II})| &\leq 2^{4K} \cdot \mathbb{E}_{q \sim \text{ov}(\mathcal{Q})} \left[\mathbb{I} \left[|q| > n^{-1/2+\alpha} \right] \exp(n |\beta|^2 \xi(q)) \right] \\ &\leq 2^{4 \log \log n} \cdot e^{-\Omega(n^{2\alpha})} = o_n(1). \end{aligned}$$

This concludes the proof. $\square$

3.3 Proof of Main Zero-Freeness Result

We now derive Theorem 1.2 from Proposition 3.1.

PROOF OF THEOREM 1.2. As discussed just below Eq. (15), A_G is analytic in an open neighborhood $\Omega \supset \overline{\mathbb{D}(0, R)}$. Therefore, X_G is also analytic in Ω . Setting $f_G(\beta) = X_G(\beta)$ in Lemma 2.2 and combining with Proposition 3.1 yields

$$\begin{aligned} \mathbb{E} [\#\{\omega \in \mathbb{D}(0, r) : X_G(\omega) = 0\}] &\leq \frac{1}{2\varepsilon} \mathbb{E}_\theta \log \mathbb{E}_G \left[\left| \frac{X_G(Re^{i\theta})}{X_G(0)} \right|^2 \right] \\ &\leq \frac{1}{2\varepsilon} \mathbb{E}_\theta \log(1 + o_n(1)) = o_n(1). \end{aligned}$$

By Markov's inequality,

$$\mathbb{P} [\#\{\omega \in \mathbb{D}(0, r) : X_G(\omega) = 0\} = 0] = 1 - o_n(1).$$

The result follows because deterministically

$$\#\{\omega \in \mathbb{D}(0, r) : Z_G(\omega) = 0\} \leq \#\{\omega \in \mathbb{D}(0, r) : X_G(\omega) = 0\}.$$

$\square$

4 Discussion and Future Directions

We conclude this paper with some open problems. The first pertains to algorithms with better running time.

Open Problem 1 (Faster algorithms). *Design a genuine polynomial time algorithm for estimating the partition function $Z_G(\beta)$ up to the second moment threshold.*

We note that for partition functions defined on bounded degree graphs one can design genuine polynomial time algorithms [60] based on appropriate zero-freeness. Thus far it is not clear how to improve the quasi-polynomial running time for mean field models.

The next two open problems pertain to extending the range in which we obtain efficient approximation algorithms.

Open Problem 2 (The Replica-Symmetric Phase). *Design an algorithm for estimating the partition function $Z_G(\beta)$ in the entire replica-symmetric phase, as discussed in Section 1.2.*

Open Problem 3 (External Fields). *For the Sherrington–Kirkpatrick model, design an algorithm for estimating the partition function in the presence of an external field (either in the direction 1 or in a random direction $g \sim \mathcal{N}(0, I)$), up to the de Almeida–Thouless line [27].*

While a direct application of the second moment method fails to resolve Open Problem 2 and Open Problem 3, it is conceivable that a suitably truncated version of the argument still works, perhaps taking inspiration from e.g. [22, 45, 68].

Another natural problem is to establish a zero-free region for the partition function in terms of external fields, rather than the inverse temperature.

Open Problem 4 (Lee–Yang Zeros). *Establish a zero-free region for the function*

$$Z_{G,\beta}(\lambda) \stackrel{\text{def}}{=} \mathbb{E}_{\sigma \sim \varrho} [\exp(\beta \cdot \mathcal{H}_G(\sigma) + \lambda \cdot \langle \sigma, 1 \rangle)],$$

and its multivariate analog.

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