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Large Deviations of the Giant Component in Scale-Free Inhomogeneous Random Graphs

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ABSTRACT

We study large deviations of the size of the largest connected component in a general class of inhomogeneous random graphs with iid weights, parametrized so that the degree distribution is regularly varying. We derive a large-deviation principle with logarithmic speed: the rare event that the largest component contains linearly more vertices than expected is caused by the presence of constantly many vertices with linear degree. Conditionally on this rare event, we prove distributional limits of the weight distribution and component-size distribution.

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1 | Introduction

This paper studies large deviations of the size of the largest connected component in a class of inhomogeneous random graphs (IRGs) in which the degree distribution is regularly varying. IRGs were introduced in a seminal paper by Bollobás et al. (2007). In short, each vertex $u \in [n] := \{1, \dots, n\}$ has a mark m_u from a mark set $\mathcal{M}$, such that the empirical mark-distribution converges weakly. Typically, marks are either independent and identically distributed (iid), or deterministic for each n. Given the marks, each pair of vertices uv is independently connected by an edge with probability $\min\big(\kappa(m_u, m_v)/n, 1\big)$, where κ is the so-called kernel function that encodes the influence of marks: it is non-negative, and symmetric in its arguments.

It is well-known that the size of the largest connected component $|\mathcal{C}_n^{(1)}|$ satisfies a weak law of large numbers as $n \to \infty$. The proportion of vertices in $\mathcal{C}_n^{(1)}$ converges in probability to θ: the probability that the branching process describing the local limit survives infinitely long, see (Bollobás et al. 2007; van der Hofstad 2024) for details. When this branching process is supercritical, the graph contains a *giant*, a linear-sized component, with probability tending to one as $n \to \infty$ (with high probability; whp). As its asymptotic size is determined by the local limit, the size of the giant is "almost local" (van der Hofstad 2021). Under regularity conditions, the giant (if it exists) is unique, and all other components are at most of logarithmic size whp (Bollobás et al. 2007; Chung and Lu 2002; Norros and Reittu 2006).

A large-deviation principle (LDP) of $|\mathcal{C}_n^{(1)}|/n$ (among other results) has been established recently by Andreis et al. (2023) for a subclass of IRGs: they consider *bounded* kernels, and *deterministic* marks. This setting leads to a graph in which the degree distribution has an *exponential* tail. They identify a non-negative rate function I_{det} such that for each Borel set $B \subseteq \mathbb{R}$ (writing $\overline{B}$ for its closure and B° for its interior),

$$-\inf_{\rho\in B^\circ} I_{\mathrm{det}}(\rho) \le \liminf_{n\to\infty} \frac{1}{n}\log \mathbb{P}\left(\frac{|\mathcal{C}_n^{(1)}|}{n}\in B\right) \le \limsup_{n\to\infty}\frac{1}{n}\log\mathbb{P}\left(\frac{|\mathcal{C}_n^{(1)}|}{n}\in B\right) \le -\inf_{\rho\in\overline{B}} I_{\mathrm{det}}(\rho). \tag{1.1}$$

Thus, (Andreis et al. 2023) proves an LDP of the giant with *linear* speed. Moreover, the rate-function I_{det} is determined uniquely by the local limit of the graph. The present paper proves an LDP for $|\mathcal{C}_n^{(1)}|/n$ in a complementary setting, in which the marks are *iid*, and the kernel is unbounded. In our setting, the mark distribution and kernel lead to a degree distribution that has a *regularly varying* tail, similar to many real-world networks (Newman 2003; Voitalov et al. 2019). We summarize our main results informally.

Meta-theorem 1.1 (Logarithmic speed and non-local rate function). *Consider an inhomogeneous random graph with iid marks and a regularly-varying degree distribution. Then,*

1. *$|\mathcal{C}_n^{(1)}|/n$ satisfies an LDP with logarithmic speed and rate function $I_{\mathrm{iid}}(\cdot)$.*
2. *the rate function of this LDP may be different for two inhomogeneous random graphs with the same local limit.*
3. *conditionally on the event $\{|\mathcal{C}_n^{(1)}| > \rho n\}$, for any $\rho\in(\theta,1)$, the component-size distribution converges to a random measure as $n\to\infty$, and $|\mathcal{C}_n^{(1)}|/n$ converges to a random variable.*

1.1 | Notation

If two vertices u and v are connected by an edge, we write $u \sim v$. Similarly, for sets of vertices A and B we write $u \sim B$ (resp. $A\sim B$) if u is connected to a vertex $v\in B$ (resp. there exist $u \in A$ and $v\in B$ such that $u\sim v$). We use standard Bachmann–Landau notation: we say that $f=o(g)$ if $|f(x)/g(x)|\to 0$, $f=\omega(g)$ if $g=o(f)$, $f=O(g)$ if $limsup_{x\to\infty}|f(x)/g(x)|<\infty$, $f=\Omega(g)$ if $g=O(f)$, and $f=\Theta(g)$ if both $f=O(g)$ and $f=\Omega(g)$. Similarly, we write $f\sim g$ if $f(x)/g(x)\to 1$, $f\lesssim g$ if $limsup f(x)/g(x)\le 1$, and $f\gtrsim g$ if $liminf f(x)/g(x) \ge 1$. We say that a random variable X stochastically dominates Y if $\mathbb{P}(X\ge x)\ge\mathbb{P}(Y\ge x)$ for all $x\in\mathbb{R}$, in which case we write $X \succcurlyeq Y$ or $Y \prec X$. We abbreviate $a\vee b := \max(a,b)$, $a\wedge b := \min(a,b)$, $[n] := \{1,\dots,n\}$, and write A^ℓ for the ℓ-fold Cartesian product of a set A.

In Appendix D we list further notation that we introduce later and use throughout the paper.

1.2 | Model Description

We formalize the model. We call a function $L(x)$ slowly varying if $L(cx)/L(x)\to 1$ as $x\to\infty$ for any constant $c>0$, while we say that $F(x)$ is regularly varying with index $-\alpha$ if there exists a slowly varying function $L(x)$ such that $F(x)=L(x)x^{-\alpha}$.

Definition 1.2 (Inhomogeneous scale-free random graph). Let $\alpha>1$, $\sigma\in\mathbb{R}$, and $q\in(0,1]$ be three constants. Consider the vertex set $\mathcal{V}_n := [n] := \{1,\dots,n\}$, and equip each vertex $u\in[n]$ with a weight W_u, which is an iid copy of the non-negative random variable W whose distribution is given by

$$1-F_W(w) := \mathbb{P}(W>w) = L(w)w^{-\alpha}, \qquad w\ge \underline{w}, \tag{1.2}$$

for some slowly varying function $L(w)$ and constant $\underline{w} := \inf_x\{x : F_W(x)>0\}>0$. Conditionally on all weights, two vertices u,v are independently connected by an edge in $\mathcal{G}_n=(\mathcal{V}_n,\mathcal{E}_n)$ with probability

$$p_{uv} := q\left(\frac{\kappa_\sigma(W_u,W_v)}{n}\wedge 1\right) := q\left(\frac{(W_u\vee W_v)(W_u\wedge W_v)^\sigma}{n}\wedge 1\right). \tag{1.3}$$

We denote the largest connected component in $\mathcal{G}_n$ by $\mathcal{C}_n^{(1)}$, and the component of vertex u by $\mathcal{C}_n(u)$.

Let us comment on the setup. Our assumption that $\underline{w} > 0$ with strict inequality ensures that $\lim_{w \downarrow \underline{w}} \kappa_\sigma(w, z) < \infty$ for all z when $\sigma < 0$. Moreover, it simplifies the technicalities when $\sigma \geq 0$, and we expect that all results carry through when one allows $\underline{w} = 0$ for non-negative σ. The kernel κ_σ was recently introduced for related random graph models that are embedded in Euclidean space with the additional restriction that $\sigma \geq 0$, see, for example, (Gracar et al. 2019; Jorritsma et al. 2025b). The more general setting here (including $\sigma < 0$) includes several models of interest: when $\sigma = 1$, the kernel $\kappa_\sigma(W_u, W_v)$ is simply a product, and we obtain a rank-one inhomogeneous random graph (Chung and Lu 2002; Norros and Reittu 2006). When $\sigma = \alpha - 1$, the graph is closely related to preferential attachment models (Dereich et al. 2012; Gracar et al. 2019; Jacob et al. 2019). The parameter σ can be seen as an (dis)*assortativity* parameter that tunes the correlation between the degrees of vertices incident to the same edges. Small or negative values of σ decrease the probability that high-weight vertices are connected by an edge, without necessarily affecting the tail of the degree distribution. Indeed, the parameters σ, α jointly determine the tail of the degree distribution of the resulting graph: it is regularly varying with index $-\alpha / \max(1, 1 + \sigma - \alpha)$, see for example, (Lüchtrath 2022) on a related model. Moreover, when σ is negative, the set of edges is no longer monotone in the weights under a natural coupling that encodes the presence of edges using uniform random variables. Hence, also the set of vertices in the giant is no longer monotone in the weights when $\sigma < 0$. The assumption $\alpha > 1$ ensures that the weight distribution has finite mean (so that the local limit below is well-defined). However, the degree distribution could still be tweaked to have infinite mean by setting $\sigma \geq 2\alpha - 1$. In this case, the number of edges grows superlinearly. Lastly, the parameter q makes the model closed under edge percolation.

The following multi-type branching process describes the local limit of the graph, that is, it describes the graph structure around a uniform vertex in $\mathcal{G}_n$ (see van der Hofstad 2024, for an introduction and references on local convergence).

Definition 1.3 (Associated multi-type branching process). Consider an inhomogeneous scale-free random graph $\mathcal{G}_n$ with kernel κ_σ, weight-distribution F_W, and percolation parameter q as in Definition 1.2. The associated multi-type branching process $\mathrm{BP} = \mathrm{BP}(\kappa_\sigma, F_W, q)$ is a branching process that starts with a single vertex $\emptyset$ with random type $w_\emptyset$ distributed according to F_W. In each generation, each particle v of type w_v gives independently birth to new particles according to a Poisson point process on $[\underline{w}, \infty)$ with intensity $q\kappa_\sigma(w_v, w)\mathrm{d}F_W(w)$. The atoms in the union of these Poisson point processes form the vertex types of the vertices in the next generation. We write T_q for the set of vertices in the progeny of the branching process, and define $\theta_q := \mathbb{P}\big(|T_q| = \infty\big)$.

By (Bollobás et al. 2007, theorems 3.1 and 9.1), the size of the largest connected component in $\mathcal{G}_n$ satisfies a Law of Large Numbers (LLN). By local convergence, also the number of vertices in components of size $\ell \in \mathbb{N}$ satisfy an LLN, see for instance (van der Hofstad 2021, 2024) and (A7) below. Combined, we have as $n \to \infty$

$$|\mathcal{C}_n^{(1)}| \Big/ n \xrightarrow{\mathbb{P}} \theta_q, \qquad S_{n,\ell} \Big/ n := \Big|\big\{u : |\mathcal{C}_n(u)| = \ell\big\}\Big| \Big/ n \xrightarrow{\mathbb{P}} \mathbb{P}\big(|T_q| = \ell\big). \tag{1.4}$$

Thus, the limits are uniquely determined by the local branching process. Our large-deviation principle that we formalize in the following section, implies a convergence rate for the first LLN. Contrary to the LLN, we will see shortly that the convergence rate is not uniquely determined by the local limit. In contrast with the exponential decay in (1.1) for IRGs with bounded and deterministic weights, the convergence rate of the LLN is polynomial when the weights are iid and regularly varying. We explain the reason for this polynomial decay: the most likely way to have a large $\mathcal{C}_n^{(1)}$ (resp. small $S_{n,\ell}$) is when vertices of weight $\Theta(n)$—called hubs—are present in the graph. Small components in the induced graph on non-hub vertices connect with constant probability by an edge to the hubs, increasing the size of the giant (that connect to the hubs with sufficiently high probability), and decreasing the number of small components. Since the weights are regularly varying, the probability of having h hubs is of the order $(n\mathbb{P}(W > n))^h = (L(n)n^{1-\alpha})^h$, that is, the decay is much slower than exponential. To motivate notation below, we next present a back-of-the-envelope calculation for the required number of hubs that realize the event $\{|\mathcal{C}_n^{(1)}| > \rho n\}$.

Back-of-the-envelope calculation for the number of hubs. Let $\mathcal{V}_n[a, b)$ denote the set of vertices with weight in the interval $[a, b)$, and let $\mathcal{C}_n(v)[a, b)$ denote the connected component of vertex v in the induced subgraph on $\mathcal{V}_n[a, b)$; write $A \sim B$ if there exists an edge between two sets of vertices A and B, and $A \nsim B$ otherwise. Instead of analyzing $\mathcal{C}_n^{(1)}$, we analyze the size of its complement $n - |\mathcal{C}_n^{(1)}|$. Assuming that the hubs (for this computation, these are the vertices with weight at least $\underline{w}^{-\sigma}n$) are part of the giant, small components of $\mathcal{G}_n[\underline{w}, \underline{w}^{-\sigma}n)$ do not merge with the giant in $\mathcal{G}_n$ if there is no edge between the component and the hubs. So,

$$n - |\mathcal{C}_n^{(1)}| \approx \sum_{v \in \mathcal{V}_n[1, \underline{w}^{-\sigma}n)} \mathbb{1}_{\{\mathcal{C}_n(v)[1, \underline{w}^{-\sigma}n) \nsim \mathcal{V}_n[\underline{w}^{-\sigma}n, \infty)\}} = \sum_{\ell \geq 1} \sum_{\mathcal{C} \in \mathcal{G}_n[1, \underline{w}^{-\sigma}n) : |\mathcal{C}| = \ell} \ell \cdot \mathbb{1}_{\{\mathcal{C} \nsim \mathcal{V}_n[\underline{w}^{-\sigma}n, \infty)\}}. \tag{1.5}$$

Each hub connects to any vertex with probability q by (1.3). Hence, there is no edge between a component of size ℓ and any of the hubs with probability $(1-q)^{h\ell}$. Moreover, as stated in (1.4), the number of vertices in components of size ℓ in $\mathcal{G}_n[1, \underline{w}^{-\sigma} n)$ is roughly $n \cdot \mathbb{P}(|T_q| = \ell)$. Therefore, the total number of size-ℓ components is about $(n/\ell) \cdot \mathbb{P}(|T_q| = \ell)$. Hence,

$$n - |C_n^{(1)}| \approx \sum_{\ell \geq 1} (n/\ell)\mathbb{P}(|T_q| = \ell) \cdot \ell \cdot (1-q)^{h\ell} = n\mathbb{E}\left[(1-q)^{h|T_q|}\right]. \tag{1.6}$$

For the event $\{|C_n^{(1)}| > \rho n\}$ to occur, the complement of the giant should contain at most $n(1-\rho)$ vertices. Thus, if the number of hubs h is the smallest integer satisfying

$$\mathbb{E}\left[(1-q)^{h|T_q|}\right] \leq 1 - \rho, \tag{1.7}$$

then we expect that the largest component has size at least ρn. In our proofs below, we formalize this reasoning and show that any other "strategy" is less effective in increasing the size of the giant. Our proof makes the above reasoning more precise: by controlling vertex weights in size-ℓ components, we estimate the impact of adding one hub with any weight yn as a function of y. As a result, we can analyze the joint distribution of the weights of the hubs that lead to a giant of size at least ρn, and show that the number of components of constant size decreases as described above. In the next section, we formalize the large-deviation principle for the giant; the organization of the remainder of the paper is given there.

2 | Main Results

Motivated by (1.7), we define for $z \in [0,1]$ the probability generating function of the associated branching process restricted to be finite:

$$H_{T_q}(z) := \mathbb{E}\left[z^{|T_q|}\mathbb{1}_{\{|T_q|<\infty\}}\right]. \tag{2.1}$$

This function is increasing, continuous, and has range $[0, 1-\theta_q]$ for $z \in [0,1]$. Hence, its inverse $y \mapsto H_{T_q}^{(-1)}(y)$ is well-defined for $y \in [0, 1-\theta_q]$. We define for $\rho \in [0,1)$,

$$\text{hubs}(\rho, q) := \begin{cases} \frac{\log H_{T_q}^{(-1)}(1-\rho)}{\log(1-q)}, & \text{if } \rho > \theta_q, \text{and } q < 1, \\ 1, & \text{if } \rho > \theta_q, \text{and } q = 1, \\ 0, & \text{if } \rho \leq \theta_q. \end{cases} \tag{2.2}$$

By the intuition leading to (1.7), $\lceil \text{hubs}(\rho, q) \rceil$ describes the least number of hubs required to increase the size of the giant from θ_q to above $\rho \in (\theta_q, 1)$. Though the formula in terms of the inverse of a generating function is rather implicit, its asymptotics are explicitly computable as shown in Lemma 2.7 below. The following set describes the weights (rescaled by a factor $1/n$) of the hubs that are jointly able to increase the proportion of the vertices in the giant from θ_q to $\rho \in (\theta_q, 1)$ with sufficiently large probability. Let, for $h \in \mathbb{N}$,

$$\mathcal{Y}_{\rho,q}(h) := \left\{ (y_1, \dots, y_h) \in (0, \infty)^h : \mathbb{E}\left[\prod_{x \in T_q, i \in [h]} \left(1 - q \cdot (y_i W_x^\sigma \wedge 1)\right) \right] \leq 1 - \rho \right\}. \tag{2.3}$$

The expectation in this definition is similar to the expectation in (1.7), and represents the probability that the component of a uniform vertex (represented by T_q, containing vertices with weight $(W_x)_{x \in T_q}$) does not connect to any of the hubs with weights $\{y_1 n, \dots, y_h n\}$, see the connection probability p_{uv} in (1.3). The proportion of vertices in a component not connecting to one of the hubs needs to be at most $1-\rho$ in order for the giant to have size at least ρn. In Lemma 4.5 below, we show that $\mathcal{Y}_{\rho,q}(\lceil \text{hubs}(\rho, q) \rceil)$ is non-empty, and that the set does not contain any points in a small neighborhood around the origin if $\text{hubs}(\rho, q) \notin \mathbb{N}$ or $q = 1$.

Let $\rho \in (\theta_q, 1)$, $q \in (0, 1]$, $h = \lceil \text{hubs}(\rho, q) \rceil$, and α as in (1.2). We define the constant

$$C_{\rho,q} := \frac{\alpha^h}{h!} \int_{y_1=0}^{\infty} \cdots \int_{y_h=0}^{\infty} \mathbb{1}\{(y_1, \dots, y_h) \in \mathcal{Y}_{\rho,q}(h)\} \cdot (y_1 \cdot \ldots \cdot y_h)^{-(\alpha+1)} \mathrm{d}y_1 \cdot \ldots \cdot \mathrm{d}y_h. \tag{2.4}$$

Our main result is the following theorem.

Theorem 2.1 (Upper tail for the giant). *Consider an inhomogeneous scale-free random graph as in Definition 1.2. Fix a constant $\rho \in (\theta_q, 1)$. If $\mathrm{hubs}(\rho, q) \notin \mathbb{N}$ or $q = 1$, then, as $n \to \infty$,*

$$\mathbb{P}\left(|\mathcal{C}_n^{(1)}| > \rho n\right) \sim C_{\rho,q}(n\mathbb{P}(W > n))^{\lceil \mathrm{hubs}(\rho,q) \rceil}. \tag{2.5}$$

If $\mathrm{hubs}(\rho, q) \in \mathbb{N}$ *and* $q < 1$, *there exists a constant* $c > 0$ *such that, as* $n \to \infty$,

$$c(n\mathbb{P}(W > n))^{\mathrm{hubs}(\rho,q)+1} \leq \mathbb{P}\left(|\mathcal{C}_n^{(1)}| > \rho n\right) \lesssim C_{\rho,q}(n\mathbb{P}(W > n))^{\mathrm{hubs}(\rho,q)}. \tag{2.6}$$

Thus, the lower bound and upper bound coincide up to smaller order terms when $\mathrm{hubs}(\rho, q) \notin \mathbb{N}$ or $q = 1$. At the discontinuity points of $\lceil \mathrm{hubs}(\rho, q) \rceil$, that is, when $\mathrm{hubs}(\rho, q) \in \mathbb{N}$ and $q < 1$, the decay rate of the upper and lower bound differ by a regularly-varying factor. The following theorem illustrates that the lower tail of large deviation decays exponentially fast.

Theorem 2.2 (Lower tail for the giant). *Consider an inhomogeneous scale-free random graph as in Definition 1.2. For all $\rho < \theta_q$, there exists a constant $c = c(\theta_q) > 0$ such that for all $n \geq 1$,*

$$\mathbb{P}\left(|\mathcal{C}_n^{(1)}| < \rho n\right) \leq \exp(-cn). \tag{2.7}$$

We remark that more precise results than (2.7) could be derived by adjusting the quenched LDP (i.e., with fixed weight sequence) from (Andreis et al. 2023) to the annealed setting with iid weights. We focus on the upper tail, and leave this open. Theorems 2.1 and 2.2 imply a large-deviation principle with logarithmic speed, and rate function

$$I_q(\rho) := \begin{cases} (\alpha - 1)\lceil \mathrm{hubs}(\rho, q) \rceil, & \text{if } \rho \in \left[\theta_q, 1\right), \\ +\infty, & \text{otherwise.} \end{cases} \tag{2.8}$$

Because $\lim_{\rho \uparrow 1} \mathrm{hubs}(\rho, q) = +\infty$, the function $I_q(\rho)$ is lower semi-continuous, and finite for $\rho \in [\theta_q, 1)$. Let B° and $\overline{B}$ denote the interior and closure of a Borel set $B \subseteq \mathbb{R}$.

Corollary 2.3 (Large-deviation principle). *Consider an inhomogeneous scale-free random graph as in Definition 1.2. Then for any Borel set $B \subseteq \mathbb{R}$,*

$$-\inf_{\rho \in B^\circ} I_q(\rho) \leq \liminf_{n\to\infty} \frac{1}{\log n} \log \mathbb{P}\left(\frac{|\mathcal{C}_n^{(1)}|}{n} \in B\right) \leq \limsup_{n\to\infty} \frac{1}{\log n} \log \mathbb{P}\left(\frac{|\mathcal{C}_n^{(1)}|}{n} \in B\right) \leq -\inf_{\rho \in \overline{B}} I_q(\rho).$$

Remark 2.4 (The rate function is global). By definition of $\mathrm{BP}(\kappa_\sigma, F_W, q)$ in Definition 1.3, there are multiple pairs (F_W, q) that lead to the same distribution of $\mathcal{T}_q$. A straightforward coupling yields an alternative construction for $\mathrm{BP}(\kappa_\sigma, F_W, q)$: first sample $\mathrm{BP}(\kappa_\sigma, F_W, 1)$, then remove each edge independently with probability q. The tree in this forest that contains $\emptyset$ is equal in distribution to $\mathrm{BP}(\kappa_\sigma, F_W, q)$. Moreover, the parameter q can be encapsulated by a reparametrization of F_W to some $F_{W,q}$ without affecting the distribution of the associated branching process, that is, $\mathrm{BP}(\kappa_\sigma, F_W, q) = \mathrm{BP}(\kappa_\sigma, F_{W,q}, 1)$ in distribution. Therefore, the two IRGs with the parametrizations (κ_σ, F_W, q) and $(\kappa_\sigma, F_{W,q}, 1)$ have the same local limit.

However, the IRGs with parameters (κ_σ, F_W, q) and $(\kappa_\sigma, F_{W,q}, 1)$ are different in distribution: for example, when $\sigma > 0$ and $q < 1$, vertices with weight $\omega(n^{1/(\sigma+1)})$ are not connected by an edge with probability $1 - q > 0$, while they are connected almost surely when $q = 1$. Therefore, the function $\mathrm{hubs}(\rho, q)$ and the rate function I do not agree for such parameter pairs, and we conclude that the rate function is not determined uniquely by the local limit, and is a global quantity instead. This contrasts the result that the *typical* size of the giant is uniquely described by its local limit (Bollobás et al. 2007; van der Hofstad 2024).

We end this section with two corollaries that describe the graph structure *conditionally* on the rare event $\{|\mathcal{C}_n^{(1)}| > \rho n\}$. They illustrate that the intuition given above is correct, and that the rare event is indeed caused by exactly $\lceil \mathrm{hubs}(\rho, q) \rceil$

hubs whose rescaled weights are in the set $\mathcal{Y}_{\rho,q}$; the precise values of these weights then determine the empirical component-size distribution. All other vertices have sublinear weight.

Corollary 2.5 (Conditional weight distribution of the hubs). *Consider an inhomogeneous scale-free random graph as in Definition 1.2. Fix $\rho \in (\theta_q, 1)$ such that $\mathrm{hubs}(\rho, q) \notin \mathbb{N}$ or $q = 1$. Let $h = \lceil \mathrm{hubs}(\rho, q) \rceil$. There exist constants $\varepsilon, \phi > 0$ such that the following holds. As $n \to \infty$,*

$$\mathbb{P}\Big(\big|\{v \in [n] : W_v \in [n^{1-\varepsilon}, \phi n)\}\big| = 0,\ \big|\{v \in [n] : W_v \in [\phi n, \infty)\}\big| = h \ \Big|\ |\mathcal{C}_n^{(1)}| > \rho n\Big) \to 1.$$

Let $W_{(1)} \geq W_{(2)} \geq \cdots$ denote vertex weights in decreasing order. Let $Y_1, \dots, Y_h$ be iid Pareto random variables following distribution $\mathbb{P}(Y_i \geq y) = (\phi/y)^{\alpha}$ for $y \geq \phi$, and let $Y_{(1)} \geq Y_{(2)} \geq \cdots$ denote their ordered values. Then, as $n \to \infty$,

$$\big(W_{(1)}/n, \dots, W_{(h)}/n\big) \ \Big|\ |\mathcal{C}_n^{(1)}| > \rho n \ \xrightarrow{d}\ \big(Y_{(1)}, \dots, Y_{(h)}\big) \Big|\ \big(Y_{(1)}, \dots, Y_{(h)}\big) \in \mathcal{Y}_{\rho,q}(h),$$

with convergence in the product topology on $(0, \infty)^h$.

The constant ϕ in Corollary 2.5 is chosen small such that $\mathcal{Y}_{\rho,q}(h) \subseteq [\phi, \infty)^h$. The above corollary does not describe the empirical weight distribution of the vertices with weight at most $n^{1-\varepsilon}$. Using methods similar to (van der Hofstad et al. 2024) it can be proven that the empirical weight distribution of these vertices, conditionally on $\{|\mathcal{C}_n^{(1)}| > \rho n\}$, converges weakly to F_W. We leave the proof open, as it would lead to many more technicalities below.

We proceed to the conditional component-size distribution, of which we show that it converges to a random sequence. Define for $h \in \mathbb{N}$ and $(y_1, \dots, y_h) \in (0, \infty)^h$,

$$g_{\ell}\big((y_i)_{i \leq h}\big) := \frac{1}{\ell}\mathbb{E}\Bigg[\mathbb{1}_{\{|T_q| = \ell\}} \prod_{x \in T_q, i \leq h} \big(1 - q(W_x^{\sigma} y_i \wedge 1)\big)\Bigg]. \tag{2.9}$$

We write $N_{n,\ell}$ for the number of components of size ℓ in $\mathcal{G}_n$. Let R^{∞} denote the space of all sequences $\mathbf{x} = (x_1, x_2, \dots)$ of real numbers, metrized by $d_{\infty}(\mathbf{x}, \mathbf{y}) = \sum_i (|x_i - y_i| \wedge 1)2^{-i}$. For background on this metric space and its use in extreme-value theory, we refer to (Resnick 2007, page 45).

Corollary 2.6 (Conditional component-size distribution). *Consider an inhomogeneous scale-free random graph as in Definition 1.2. Fix $\rho \in (\theta_q, 1)$ such that $\mathrm{hubs}(\rho, q) \notin \mathbb{N}$ or $q = 1$. Let $h = \lceil \mathrm{hubs}(\rho, q) \rceil$, and let $(Y_i)_{i \leq h}$ be independent copies of Y following distribution $\mathbb{P}(Y \geq y) = (\phi/y)^{\alpha}$ for $y \geq \phi$ with $\phi > 0$ a sufficiently small constant. As $n \to \infty$,*

$$\big(N_{n,\ell}/n : \ell \geq 1\big) \ \Big|\ |\mathcal{C}_n^{(1)}| > \rho n \ \xrightarrow{d}\ \big(g_{\ell}\big((Y_i)_{i \leq h}\big) : \ell \geq 1\big) \Big|\ \big(Y_i\big)_{i \leq h} \in \mathcal{Y}_{\rho,q}(h).$$

Here, the weak convergence takes place in R^{∞}. Moreover, conditionally on $|\mathcal{C}_n^{(1)}| > \rho n$, $|\mathcal{C}_n^{(1)}|/n$ converges in distribution to a random variable Q supported on $\big[\rho, \inf\{\rho' > \rho : \mathrm{hubs}(\rho', q) \in \mathbb{N}\}\big]$ with distribution

$$\mathbb{P}(Q > s) = \frac{C_{s,q}}{C_{\rho,q}}, \qquad s \in \big[\rho, \inf\{\rho' > \rho : \mathrm{hubs}(\rho', q) \in \mathbb{N}\}\big). \tag{2.10}$$

We finish this section with a lemma that illustrates some properties of hubs defined in (2.2). Only the first two items of the lemma will be required in the proofs of our main results. The proof is postponed to the appendix on page 31.

Lemma 2.7 (Properties of $\mathrm{hubs}(\rho, q)$). *Consider an inhomogeneous scale-free random graph as in Definition 1.2. Let $W_{\emptyset}, W$ be two independent random variables following distribution F_W, and assume $\rho > \theta_q$. Then,*

i. The following identity holds:

$$\mathrm{hubs}(\rho, q) = \inf\big\{h' > 0 : \mathbb{E}\big[(1-q)^{|T_q| h'}\big] \leq 1 - \rho\big\}. \tag{2.11}$$

ii. $\mathrm{hubs}(\rho, q)$ *is continuous, positive, non-decreasing in ρ, and decreasing in q.*

iii. If either ρ is fixed and $q \downarrow 0$, or $q < 1$ is fixed and $\rho \uparrow 1$, then $\mathrm{hubs}(\rho, q)$ *tends to infinity, and*

$$\mathrm{hubs}(\rho, q) \sim \frac{\log(1/(1-\rho))}{\log(1/(1-q))}.$$

iv. If $q < 1$ is fixed, as $\rho \uparrow 1$,

$$\mathrm{hubs}(\rho, q) = \frac{\log(1/(1-\rho)) - \log\left(\mathbb{E}\left[\exp\left(-q\mathbb{E}[\kappa_\sigma(W_\emptyset, W)|W_\emptyset]\right)\right]\right)}{\log(1/(1-q))} + o(1),$$

which simplifies for rank-one inhomogeneous scale-free random graphs, that is, $\sigma = 1$, to

$$\mathrm{hubs}(\rho, q) = \frac{\log(1/(1-\rho)) - \log\left(\mathbb{E}\left[\exp\left(-q\mathbb{E}[W]W_\emptyset\right)\right]\right)}{\log(1/(1-q))} + o(1), \qquad \text{as } \rho \uparrow 1.$$

2.1 | Discussion and Related Work

This paper provides the first large-deviation principle (LDP) for component sizes in random graph models with iid regularly varying weights, and enriches the emerging literature on LDPs for other graph properties. The rate function in Corollary 2.3 has the most interesting behavior when the edge-percolation parameter q is strictly smaller than 1; when $q = 1$, a single hub can increase the size of the giant to n.

An applied example where varying q is of interest can be found in epidemiology: the largest component in an edge-percolated graph corresponds to the final size of a Reed–Frost epidemic (Abbey 1952; Barbour and Utev 2004; Lefèvre 1990) on the largest component of the graph before percolation. The parameter q represents the probability that a disease is transmitted along an edge. Thus, Theorem 2.1 translates to the asymptotic probability that the epidemic has a significantly larger size than expected; Theorem 2.2 examplifies that a significantly smaller final size than expected is less likely to occur in IRGs.

The rate function in Corollary 2.3 behaves drastically different compared to the rate function of LDPs for the number of edges E_n (van der Hofstad et al. 2024; Kerriou and Mörters 2025; Stegehuis and Zwart 2023a, 2023b), and the number of triangles $\triangle_n$ (Stegehuis and Zwart 2023a). Similar to Theorem 2.1, the LDP for E_n requires constantly many hubs of linear weight to find more than *an* edges than expected, and yields a discontinuous rate function in a. The number of hubs scales linearly in a as $a \to \infty$ in (Hofstad et al. 2024; Stegehuis and Zwart 2023b), while in our setting $\mathrm{hubs}(\rho, q) = \Theta(\log(1/(1-\rho)))$ as $\rho \uparrow 1$. For $\triangle_n$ the situation is different (Stegehuis and Zwart 2023a): when the parameters are such that $\mathbb{E}[\triangle_n] = o(n)$, a single vertex of large (but sublinear) weight is required to find $a\mathbb{E}[\triangle_n]$ additional triangles, which occurs with a probability decaying polynomially in n; when $\mathbb{E}[\triangle_n] = \omega(n)$, polynomially many additional vertices of weight $\Omega(n^{1/(\sigma+1)})$ are required, which occurs with a probability that decays stretched exponentially.

The rate function $I_q(\rho)$ in (2.8) also appears in the related work on the size of the giant component in scale-free random graphs that are embedded in Euclidean space (Jorritsma et al. 2025a), where it is shown that $\mathbb{P}(|\mathcal{C}_n^{(1)}| > \rho n) = \Theta(n^{-I_q(\rho)})$ when $\mathrm{hubs}(\rho, q) \notin \mathbb{N}$. Parts of our proofs are related to the techniques from (Jorritsma et al. 2025a). However, the way that we obtain concentration bounds for the number of components of size ℓ in the graph without hubs is different: there is no geometry that we can leverage to obtain concentration via (almost independent) disjoint graphs in sub-boxes as in (Jorritsma et al. 2025a). Instead, we use truncation/discretization arguments in combination with the LDP from (Andreis et al. 2023). Compared to (Jorritsma et al. 2025a), Theorem 2.1 additionally provides the constant prefactor $C_{\rho,q}$, which requires substantial additional analysis. Deriving this constant allows us to prove the LDP in Corollary 2.1, and to give a detailed description of the graph conditionally on $\{|\mathcal{C}_n^{(1)}| > \rho n\}$.

A commonality among Theorem 2.1 and the above-mentioned works (van der Hofstad et al. 2026; Stegehuis and Zwart 2023a, 2023b), is that the lower bound and upper bound are non-matching on discontinuity points of $I_q(\rho)$. As argued in (van der Hofstad et al. 2026, section 2.4) and (Stegehuis and Zwart 2023b, section 3), a delicate analysis is required to find the correct scaling for such ρ, and the decay may heavily depend on the precise form of the connection probability in (1.3).

The setup of Definition 1.2 contains (a non-spatial version of) the age-dependent random connection model (ADCM) by setting $\sigma = \alpha - 1$ (Gracar et al. 2022), which in several contexts mimics preferential attachment models (PAMs) (Dereich et al. 2012; Gracar et al. 2019; Jacob et al. 2019). However, we expect that our results here do not extend to PAMs, and that $\mathbb{P}(|\mathcal{C}_n^{(1)}| > \rho n)$ decays exponentially. Informally speaking, the main difference is that the connection probability in ADCM depends on the *exact age* of vertices, while in PAMs only the *order of the ages* affects connection probabilities. On the contrary, we believe that the edge-percolated configuration model with iid regularly varying degrees behaves similar to IRGs, and that polynomial decay of $\mathbb{P}(|\mathcal{C}_n^{(1)}| > \rho n)$ can be proven using adaptations of our techniques. This would contrast the exponential decay from (Bhamidi et al. 2022) for the configuration model with given degrees in which no linear-sized hubs are allowed.

Theorems 2.1 and 2.2 provide the asymptotic probability that the size of the giant deviates by a constant factor from its expectation if the parameters of the IRG are such that $\theta_q > 0$, that is, when the model is supercritical. However, the theorems also apply when $\theta_q = 0$, in which case the model is subcritical or critical and the largest component grows sublinear in n. For such parameters, Theorem 2.1 describes deviations of $|\mathcal{C}_n^{(1)}|$ at a much larger scale than $\mathbb{E}[|\mathcal{C}_n^{(1)}|]$, and Theorem 2.2 is trivial. This poses a natural follow-up question: when $\theta_q = 0$, what is the decay of $\mathbb{P}\big(|\mathcal{C}_n^{(1)}| < \beta_n\big)$ and $\mathbb{P}\big(|\mathcal{C}_n^{(1)}| > \gamma_n\big)$ at intermediate scales, that is, when $1 \ll \beta_n \ll \mathbb{E}[|\mathcal{C}_n^{(1)}|] \ll \gamma_n \ll n$? For critical rank-one IRGs with infinite third moment ($\sigma = 1, \alpha \in (2,3)$) the component-size tails were studied in (Hofstad et al. 2018), but we are not aware of further results in this direction.

We describe the organization of the remainder of the paper via a proof sketch of Theorem 2.1.

2.2 | Outline of the Proof

In Section 3 we analyze the graph without hubs (vertices with weight at most ϕn for some small constant $\phi > 0$). Following the calculation on page 3, we have to show for each constant $\ell \in \mathbb{N}$ the proportion of vertices in size-ℓ components in this graph concentrates around the probability that the total progeny of the associated branching process has size ℓ. To estimate the precise effect of adding one hub with any weight at least ϕn, we additionally control the vertex weights in size-ℓ components.

To do so, we discretize the interval $[\underline{w}, \infty)$ into small intervals of length ε and categorize each component of size ℓ in $\mathcal{G}_n[\underline{w}, R)$ by the intervals that contain the weights of the ℓ vertices. For each possible category—that we call component type—we establish concentration of the proportion of vertices in such a component around the probability that the associated branching process has size ℓ, and that the weights of the vertices in the total progeny fall exactly in the same intervals. This discretization is at the core of the proof, and allows us to derive the explicit constant $C_{\rho,q}$ in Theorem 2.1. We achieve the concentration for finitely many couplings via a coupling with inhomogeneous random graphs with given weights, and then rely on a large-deviation principle from (Andreis et al. 2023) for inhomogeneous random graphs with bounded and given weights. Since in our case the weights are not bounded, we first truncate the weights at a large constant R to which we apply the LDP. Afterwards, we add the vertices with weight in $[R, \phi n)$, and argue that the number of components of each type is only mildly affected. The size of the largest component in this graph is still about $\theta_q n$.

In Section 4 we analyze the impact of vertices of weight at least ϕn to the graph. When $\text{hubs}(\rho, q) \notin \mathbb{N}$, we show that if there are exactly $\lceil \text{hubs}(\rho, q) \rceil$ vertices with weight in the set $\mathcal{Y}_{\rho+\delta,q}$ for any $\delta > 0$, then the size of the giant increases from $\theta_q n$ to at least ρn whp. Since $\mathcal{Y}_{\rho+\delta,q} \subset \mathcal{Y}_\rho$ by (2.3), we require slightly larger weights than explained before. On the contrary, if the number of hubs is not equal to $\lceil \text{hubs}(\rho, q) \rceil$, or if their weights are not contained in $\mathcal{Y}_{\rho-\delta,q} \supset \mathcal{Y}_{\rho,q}$, then the size of the giant is very unlikely to increase above ρn: the probability of this event is of the same order as the event that there are strictly more than $\lceil \text{hubs}(\rho, q) \rceil$ hubs. The presence of the small constant $\delta > 0$ mitigates the effect of the truncation and discretization from Section 3. At the end of Section 4 we show that we can take the limit $\delta \to 0$ when $\text{hubs}(\rho, q) \notin \mathbb{N}$, showing that $\mathbb{P}\big(|\mathcal{C}_n^{(1)}| > \rho n\big) \sim \mathbb{P}\big(|\mathcal{V}_n[\phi n, \infty)| = \lceil \text{hubs}(\rho, q) \rceil, \mathcal{V}_n[\phi n, \infty) \in \mathcal{Y}_{\rho,q}\big) \sim C_{\rho,q}(n\mathbb{P}(W > n))^{\lceil \text{hubs}(\rho,q) \rceil}$; proving the first bound in Theorem 2.1.

Section 4.2 formally verifies Theorems 2.1 and 2.2 and proves the LDP for the giant in Corollary 2.3. Lastly, in Section 5 we analyze the graph conditionally on the rare event $\{|\mathcal{C}_n^{(1)}| > \rho n\}$ to prove Corollaries 2.5 and 2.6. The appendix contains the proof of Lemma 2.7, and proofs of some technical lemmas from the following sections, and standard concentration bounds that we frequently use.

3 | The Graph Without Hubs

In this section we analyze the graph in which all vertices with weight at least ϕn are removed. Our main goal is to obtain concentration bounds for components of size ℓ, together with their weight configuration. We introduce notation to categorize components.

Definition 3.1 (Component of ε-type $\widetilde{\mathbf{w}}$). Fix $\ell \in \mathbb{N}$, a small constant $\varepsilon > 0$, and $R \in [\underline{w}, \infty]$ such that $(R-\underline{w})/\varepsilon \in \mathbb{N} \cup \{\infty\}$. Let

$$\mathrm{CT}_\ell(\varepsilon, R) := \left\{ \widetilde{\mathbf{w}}^{(\ell)} = (\tilde{w}_1, \dots, \tilde{w}_\ell) : \tilde{w}_1 \le \tilde{w}_2 \le \dots \le \tilde{w}_\ell, \quad \tilde{w}_i \in \{\underline{w} + j\varepsilon\}_{j=0}^{(R-\underline{w})/\varepsilon - 1} \forall i \in [\ell] \right\}.$$

We say that a connected component C of a vertex-weighted graph G has ε-type $\widetilde{\mathbf{w}}^{(\ell)}$, if its number of vertices is ℓ, and if its vertices $(v_1, \dots, v_\ell)$, ordered increasingly by weights, satisfy $\tilde{w}_i \le w_{v_i} < \tilde{w}_i + \varepsilon$ for all $i \in [\ell]$. So, $\mathrm{CT}_\ell(\varepsilon, R)$ denotes the set of (ε-discretized) component types of size ℓ with maximal vertex weight R. We abbreviate $\mathrm{CT}_\ell(\varepsilon) = \mathrm{CT}_\ell(\varepsilon, \infty)$.

Let $\overline{w} > \underline{w}$ be a weight threshold. We write $N_n(\widetilde{\mathbf{w}}^{(\ell)}, \varepsilon, \overline{w})$ for the number of components of ε-type $\widetilde{\mathbf{w}}^{(\ell)}$ in $\mathcal{G}_n[\underline{w}, \overline{w})$.

For the total progeny T_q of the associated branching process of $\mathcal{G}_n$, we define

$$\theta(\widetilde{\mathbf{w}}^{(\ell)}, \varepsilon) := \mathbb{P}\Big(T_q \text{ has } \varepsilon\text{-type } \widetilde{\mathbf{w}}^{(\ell)}\Big). \tag{3.1}$$

The following lemma establishes concentration bounds for the proportion of vertices in components of ε-type $\mathbf{w}^{(\ell)}$ when we truncate the weights at a large constant R. We postpone its proof to Appendix A as it follows from a (much more detailed) LDP for inhomogeneous random graphs with bounded non-random weights given by Andreis et al. (2023).

Lemma 3.2 (Concentration of size-ℓ components, constant weights). *Consider an inhomogeneous scale-free random graph as in Definition 1.2. For any constants $\psi, R, \ell_* > 0$, there exists a constant $c > 0$ such that for all $\varepsilon > 0$ such that $(R-\underline{w})/\varepsilon \in \mathbb{N}$, and all $n \ge 1$,*

$$\mathbb{P}\left(\sum_{\ell \le \ell_*} \sum_{\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon, R)} \left| \frac{\ell N_n(\mathbf{w}^{(\ell)}, \varepsilon, R)}{n} - \theta(\mathbf{w}^{(\ell)}, \varepsilon) \right| > \psi \right) \le \exp(-cn). \tag{3.2}$$

The next lemma shows that if we truncate at linear weights instead, the convergence rate is at least polynomial.

Lemma 3.3 (Concentration of size-ℓ components, without hubs). *Consider an inhomogeneous scale-free random graph as in Definition 1.2. For any constants $\psi, R, C, \ell_* > 0$, there exists a constant $\phi > 0$ such that for all $\varepsilon > 0$ such that $(R-\underline{w})/\varepsilon \in \mathbb{N}$, as $n \to \infty$,*

$$\mathbb{P}\left(\sum_{\ell \le \ell_*} \sum_{\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon, R)} \left| \frac{\ell N_n(\mathbf{w}^{(\ell)}, \varepsilon, \phi n)}{n} - \theta(\mathbf{w}^{(\ell)}, \varepsilon) \right| > \psi \right) = o\big(n^{-C}\big). \tag{3.3}$$

We proceed with a lemma for the lower tail of the largest component in truncated graphs, which follows again by the LDPs from (Andreis et al. 2023), and whose proof we postpone to Appendix A.

Lemma 3.4 (Lower tail of the truncated giant). *Consider an inhomogeneous scale-free random graph as in Definition 1.2. For any constant $\rho \in (0, \theta_q)$, there exist constants $R, c > 0$ such that for all $n \ge 1$ and all $\overline{w} \ge R$,*

$$\mathbb{P}\big(|\mathcal{C}_n^{(1)}[\underline{w}, \overline{w})| < \rho n\big) \le \exp(-cn). \tag{3.4}$$

In the remainder of the section we show how Lemma 3.3 follows from Lemma 3.2, for which we require an auxiliary lemma. To show that the number of components of a certain ε-type is not severely affected when vertices with weight in $[R, \phi n)$ are added, we first bound the number of edges between vertices with weight at least R and weight at most S for some $S < R$. We write $\deg_v[a, b)$ for the number of neighbors of v with weight in the interval $[a, b)$.

Lemma 3.5 (Excess edges). *Consider an inhomogeneous scale-free random graph as in Definition 1.2. For all constants $\psi, S, C > 0$ there exist constants $R_0, \phi_0 > 0$ such that for all $R \ge R_0$ and $\phi \in (0, \phi_0)$, as $n \to \infty$,*

$$\mathbb{P}\left(\sum_{v \in \mathcal{V}_n[R, \phi n)} \deg_v[\underline{w}, S) > \psi n \right) = o(n^{-C}). \tag{3.5}$$

Proof. Let $B(u,v), u,v \in [n]$ be a collection of Bernoulli random variables with success probability p_{uv} defined in (1.3). Observe that

$$\sum_{v\in\mathcal{V}_n[R,\phi n)} \deg_v[\underline{w}, S) \overset{d}{=} \sum_{u\in\mathcal{V}_n[R,\phi n)} \sum_{v\in\mathcal{V}_n[\underline{w},S)} B(u,v) =: E_n(S,R,\phi).$$

The mean of the right-hand side conditional upon the weights $(W_u : u\in[n])$, equals

$$K_n(S,R,\phi) := \frac{q}{n} \sum_{u\in\mathcal{V}_n[R,\phi n)} W_u \sum_{v\in\mathcal{V}_n[\underline{w},S)} W_v^\sigma \tag{3.6}$$

(making ϕ smaller if needed so that $\phi n R^\sigma \le n$ and we can drop the minimum operator in the definition of p_{uv}). Next, take $\varepsilon > 0$ and bound using the triangle inequality

$$\begin{aligned}
&\mathbb{P}(E_n(S,R,\phi) > \psi n)\\
&\quad = \mathbb{P}\big(\{E_n(S,R,\phi) > \psi n\} \cap \{|E_n(S,R,\phi) - K_n(S,R,\phi)| \le \varepsilon K_n(S,R,\phi)\}\big)\\
&\qquad + \mathbb{P}\big(\{E_n(S,R,\phi) > \psi n\} \cap \{|E_n(S,R,\phi) - K_n(S,R,\phi)| > \varepsilon K_n(S,R,\phi)\}\big)\\
&\quad \le \mathbb{P}\Big(K_n(S,R,\phi) > \frac{\psi}{1+\varepsilon} n\Big) + \mathbb{P}\big(|E_n(S,R,\phi) - K_n(S,R,\phi)| > \varepsilon K_n(S,R,\phi)\big).
\end{aligned} \tag{3.7}$$

As a first step, we show that the first term on the right-hand side is of the order $o(n^{-C})$ if R is sufficiently large and ϕ sufficiently small. Consequently,

$$\begin{aligned}
&\mathbb{P}\big(K_n(R,\phi) > \psi n/(1+\varepsilon)\big)\\
&\le \mathbb{P}\left(\sum_{v\in[n]} W_v \mathbb{1}_{\{W_v\in[R,\phi n)\}} > \frac{\psi}{(1+\varepsilon)2\mathbb{E}[W^\sigma \mathbb{1}_{\{W<S\}}]\mathbb{E}[W\mathbb{1}_{\{W>R\}}]} \mathbb{E}\left[W\mathbb{1}_{\{W>R\}}\right] n\right)\\
&\quad + \mathbb{P}\left(\frac{q}{n}\sum_{v\in[n]} W_v^\sigma \mathbb{1}_{\{W_v<S\}} > 2\mathbb{E}[W^\sigma \mathbb{1}_{\{W<S\}}]\right).
\end{aligned} \tag{3.8}$$

The second term decreases exponentially in n, regardless of the choice of R, due to Cramérs bound. For the first term, we first show that the ratio on the right-hand side between brackets diverges as $R\to\infty$. Indeed, using Potter's bound, $\mathbb{E}[W\mathbb{1}_{\{W\ge R\}}] = O(R^{1-\alpha+\varepsilon'}) = o(1)$ as $R\to\infty$, while $\mathbb{E}[W^\sigma \mathbb{1}_{\{W<S\}}]$ is a constant for fixed S, as we assume $\alpha > 1$ in Definition 1.2. Hence, we may choose R_0 such that the ratio between brackets on the second line in (3.8) is at least 2 for all $R \ge R_0$. Thus,

$$\mathbb{P}\big(K_n(R,\phi) > \psi n/(1+\varepsilon)\big) \le \mathbb{P}\left(\sum_{v\in[n]} W_v \mathbb{1}_{\{W_v\in[R,\phi n)\}} > 2\mathbb{E}\left[W\mathbb{1}_{\{W>R\}}\right] n\right) + \exp(-\Theta(n)).$$

Next, we apply a bound for sums of truncated heavy-tailed random variables (see Lemma C.2) to conclude that, given $C>0$, we can pick ϕ_0 suitably small so that the first term on the left-hand side is of order $o(n^{-C})$. Hence, the first term in (3.7) is of order $o(n^{-C})$.

We turn to the second term in (3.7). Conditional on $(W_u)_{u\in[n]}$, the $B(u,v), u<v$, are independent. Therefore, we can apply Lemma C.3 to obtain that

$$\mathbb{P}\big(|E_n(S,R,\phi) - K_n(S,R,\phi)| > \varepsilon K_n(S,R,\phi) | W_1,\dots,W_n\big) \le 2\mathrm{e}^{-K_n J(\varepsilon)}$$

almost surely, for some $J(\varepsilon) > 0$. We distinguish whether $K_n(R,\phi) \le \zeta n$ for some $\zeta > 0$:

$$\begin{aligned}
&\mathbb{P}\big(|E_n(S,R,\phi) - K_n(S,R,\phi)| > \varepsilon K_n(S,R,\phi)\big)\\
&\le \mathbb{P}\big(\{|E_n(S,R,\phi) - K_n(S,R,\phi)| > \varepsilon K_n(S,R,\phi)\} \cap \{K_n(S,R,\phi) > \zeta n\}\big) + \mathbb{P}(K_n(S,R,\phi) \le \zeta n)\\
&\le 2\mathrm{e}^{-\zeta n J(\varepsilon)} + \mathbb{P}(K_n(S,R,\phi) \le \zeta n).
\end{aligned} \tag{3.9}$$

Recall the definition of K_n from (3.6). By the union bound,

$$\mathbb{P}(K_n(S,R,\phi)\le \zeta n)\le \mathbb{P}\left(q\sum_{u\in[n]}W_u\mathbb{1}_{\{W_u\in[R,\phi n)\}}\le\sqrt{\zeta}n\right)+\mathbb{P}\left(\sum_{v\in[n]}W_v^{\sigma}\mathbb{1}_{\{W_v<S\}}\le\sqrt{\zeta}n\right).$$

Applying Cramér's theorem to each of the terms on the right-hand side yields that the right-hand side tends to zero exponentially fast for ζ sufficiently small depending on S, R, and ϕ. Therefore both terms in (3.9) decay exponentially fast, and also the second term in (3.7) is of order $o(n^{-C})$. □

Proof of Lemma 3.3. We start from Lemma 3.2, assuming that this bound holds with ψ replaced by $\psi/3$. The probability on the left-hand side in (3.3) is increasing in R. Therefore, we may assume without loss of generality that R is at least a sufficiently large constant.

We let r be a sufficiently large constant such that $\mathbb{E}[|\mathcal{V}_n[r,\infty)|]=\psi/(6\ell_*)$. Let $R=R(r)$ be a sufficiently large constant, and ϕ a small constant, both given by Lemma 3.5 with ψ replaced by $\psi/(3\ell_*)$. We first consider only component types that contain at least one weight in $[r,R)$. By the triangle inequality,

$$\begin{aligned}&\sum_{\ell\le\ell_*}\sum_{\mathbf{w}^{(\ell)}\in\mathrm{CT}_\ell(\varepsilon,R)\setminus\mathrm{CT}_\ell(\varepsilon,r)}|\ell N_n(\mathbf{w}^{(\ell)},\varepsilon,\phi n)-n\theta(\mathbf{w}^{(\ell)},\varepsilon)|\\&\quad\le\sum_{\ell\le\ell_*}\sum_{\mathbf{w}^{(\ell)}\in\mathrm{CT}_\ell(\varepsilon,R)\setminus\mathrm{CT}_\ell(\varepsilon,r)}\left(\ell\cdot|N_n(\mathbf{w}^{(\ell)},\varepsilon,\phi n)-N_n(\mathbf{w}^{(\ell)},\varepsilon,R)|+|\ell N_n(\mathbf{w}^{(\ell)},\varepsilon,R)-n\theta(\mathbf{w}^{(\ell)},\varepsilon)|\right).\end{aligned}$$

We analyze the double sum over the first term between brackets, in which we bound $\ell\le\ell_*$. If one first considers the graph on vertices with weight at most R, and then adds the vertices with weight in $[R,\infty)$, the number of components of ε-type $\mathbf{w}^{(\ell)}\in\mathrm{CT}_\ell(\varepsilon,R)$ cannot increase: by definition all vertices in a component of ε-type $\mathbf{w}^{(\ell)}$, have weight at most R by Definition 3.1. So, the first absolute value is trivially bounded from above by $N_n(\mathbf{w}^{(\ell)},\varepsilon,R)$. The total number of components with maximal weight in $[r,R)$ is bounded by the total number of vertices with weight in $[r,R)$. Hence,

$$\begin{aligned}&\sum_{\ell\le\ell_*}\sum_{\mathbf{w}^{(\ell)}\in\mathrm{CT}_\ell(\varepsilon,R)\setminus\mathrm{CT}_\ell(\varepsilon,r)}|\ell N_n(\mathbf{w}^{(\ell)},\varepsilon,\phi n)-n\theta(\mathbf{w}^{(\ell)},\varepsilon)|\\&\quad\le\ell_*|\mathcal{V}_n[r,R)|+\sum_{\ell\le\ell_*}\sum_{\mathbf{w}^{(\ell)}\in\mathrm{CT}_\ell(\varepsilon,R)\setminus\mathrm{CT}_\ell(\varepsilon,r)}|\ell N_n(\mathbf{w}^{(\ell)},\varepsilon,R)-n\theta(\mathbf{w}^{(\ell)},\varepsilon)|.\end{aligned}$$

We next consider the number of components for all ε-types $\mathbf{w}^{(\ell)}\in\mathrm{CT}_\ell(\varepsilon,r)$ in $\mathcal{G}[\underline{w},R)$, and iteratively add the edges incident to the vertices with weight in $[R,\phi n)$ to this graph. None of the vertices in $\mathcal{V}_n[R,\phi n)$ can be in a component of ε-type $\mathbf{w}^{(\ell)}\in\mathrm{CT}_\ell(\varepsilon,r)$. Therefore each edge in $\mathcal{E}_n[\underline{w},\phi n)\setminus\mathcal{E}_n[\underline{w},R)$ never increases the number of components of ε-type $\mathbf{w}^{(\ell)}$, and can decrease the number of at most one ε-type $\mathbf{w}^{(\ell)}\in\mathrm{CT}_\ell(\varepsilon,r)$ by at most one if its other vertex has weight at most r. Therefore,

$$\begin{aligned}&\sum_{\ell\le\ell_*}\sum_{\mathbf{w}^{(\ell)}\in\mathrm{CT}_\ell(\varepsilon,r)}|\ell N_n(\mathbf{w}^{(\ell)},\varepsilon,\phi n)-n\theta(\mathbf{w}^{(\ell)},\varepsilon)|\\&\quad\le\sum_{\ell\le\ell_*}\sum_{\mathbf{w}^{(\ell)}\in\mathrm{CT}_\ell(\varepsilon,S)}\left(\ell\cdot|N_n(\mathbf{w}^{(\ell)},\varepsilon,\phi n)-N_n(\mathbf{w}^{(\ell)},\varepsilon,R)|+|\ell N_n(\mathbf{w}^{(\ell)},\varepsilon,R)-n\theta(\mathbf{w}^{(\ell)},\varepsilon)|\right)\\&\quad\le\ell_*\cdot\sum_{v\in\mathcal{V}_n[R,\phi n)}\deg_v[\underline{w},r)+\sum_{\ell\le\ell_*}\sum_{\mathbf{w}^{(\ell)}\in\mathrm{CT}_\ell(\varepsilon,r)}|\ell N_n(\mathbf{w}^{(\ell)},\varepsilon,R)-n\theta(\mathbf{w}^{(\ell)},\varepsilon)|.\end{aligned}$$

We bound

$$\begin{aligned}&\mathbb{P}\left(\sum_{\ell\le\ell_*,\mathbf{w}^{(\ell)}\in\mathrm{CT}_\ell(\varepsilon,R)}|\ell N_n(\mathbf{w}^{(\ell)},\varepsilon,\phi n)/n-\theta(\mathbf{w}^{(\ell)},\varepsilon)|>\psi\right)\\&\quad\le\mathbb{P}\left(\sum_{\ell\le\ell_*,\mathbf{w}^{(\ell)}\in\mathrm{CT}_\ell(\varepsilon,R)}|\ell N_n(\mathbf{w}^{(\ell)},\varepsilon,R)/n-\theta(\mathbf{w}^{(\ell)},\varepsilon)|>\psi/3\right)+\mathbb{P}\left(\ell_*|\mathcal{V}_n[r,R)|\ge\psi/3\right)+\mathbb{P}\left(\ell_*\sum_{v\in\mathcal{V}_n[R,\phi n)}\deg_v[\underline{w},r)>\psi/3\right).\end{aligned}$$

The first term decays exponentially by (3.2). The second term decays exponentially as well, since $|\mathcal{V}_n[r,R)|$ is $\mathrm{Bin}(n,\mathbb{P}(W\in[r,R))$-distributed, and r was chosen such that $\mathbb{E}[|\mathcal{V}_n[r,R)|]\le\mathbb{E}[|\mathcal{V}_n[r,\infty)|]=\psi/(6\ell_*)$. The third term is $o(n^{-C})$ by Lemma 3.5 and our assumptions on r, R and ϕ.

4 | The Graph With Hubs

In this section we prove our main result, the upper tail $\mathbb{P}(|\mathcal{C}_n^{(1)}| > \rho n)$ stated in Theorem 2.1, whose proof we outline briefly. The core idea is that the most likely way to realize the event $|\mathcal{C}_n^{(1)}| > \rho n$ is to have exactly $\lceil \text{hubs}(\rho, q)\rceil$ vertices whose joint weights are in the set $\mathcal{Y}_{\rho,q}$, called hubs.

For the upper bound, we formalize that the probability that the giant increases above ρn is of smaller order than $(n\mathbb{P}(W > n))^{\lceil \text{hubs}(\rho,q)\rceil}$ if the number of hubs is not equal to $\lceil \text{hubs}(\rho, q)\rceil$ or if the weights are not contained in $\mathcal{Y}_{\rho-\delta,q}$ for arbitrarily small δ, leading to a lower bound for arbitrarily small δ. In formulas, assuming $|\mathcal{V}_n[\phi n, \infty)| = h$, we write $\mathcal{W}_n[\phi n, \infty) \in n \cdot \mathcal{Y}_{\rho,q}(h)$ if $(w_v/n)_{v\in\mathcal{V}_n[\phi n,\infty)} \in \mathcal{Y}_{\rho,q}(h)$. We will bound the terms on the right-hand side in the following bound from above:

$$\begin{aligned}\mathbb{P}\left(|\mathcal{C}_n^{(1)}| > \rho n\right) \le{}& \mathbb{P}\left(\{|\mathcal{C}_n^{(1)}| > \rho n\} \cap \neg\{|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in n \cdot \mathcal{Y}_{\rho-\delta,q}(h)\}\right)\\ &+ \mathbb{P}\left(|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in n \cdot \mathcal{Y}_{\rho-\delta,q}(h)\right).\end{aligned}$$

The second term will give the main contribution.

Afterwards, for the lower bound, we prove that conditionally on having exactly $\lceil \text{hubs}(\rho, q)\rceil$ hubs with weights in $\mathcal{Y}_{\rho+\delta,q}$, the size of the giant is at least ρn with high probability. Combining the upper and lower bound, we will obtain

$$\begin{aligned}&(1 - o(1)) \cdot \mathbb{P}\left(|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in n \cdot \mathcal{Y}_{\rho+\delta,q}(h)\right)\\ &\quad\le \mathbb{P}\left(|\mathcal{C}_n^{(1)}| > \rho n\right) \le (1 + o(1)) \cdot \mathbb{P}\left(|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in n \cdot \mathcal{Y}_{\rho-\delta,q}(h)\right)\end{aligned}$$

At the end of the section we show that we can take the limit of $\delta \to 0$ in our upper and lower bounds to prove Theorem 2.1.

The concentration of the number of components of a certain ε-type obtained in the previous section is key for the execution of this plan: knowing how many components of a certain ε-type $\mathbf{w}^{(\ell)}$ there are in the graph induced on vertices of weight at most ϕn, helps estimating the distribution of the number of such components that connect to vertices with weight in $[\phi n, \infty)$. We start with a lemma that analyzes the effect of discretizing the weights in Definition 3.1. Among others, it approximates the expectation in (2.3) with a version where the weights are discretized and truncated, so that we can consider a finite number of ε-types in the graph. Define

$$\overline{P}(\mathbf{w}^{(\ell)}, \mathbf{y}^{(h)}) := \prod_{j_1=1}^{\ell}\prod_{j_2=1}^{h}\left(1 - q\left(w_{j_1}^{\sigma} y_{j_2} \wedge 1\right)\right). \tag{4.1}$$

$\overline{P}$ corresponds to the probability that a component whose vertices $\{v_1, \dots, v_\ell\}$ have weight exactly $\{w_1, \dots, w_\ell\}$, do not connect to any of the h hubs that have weight exactly $\{y_1 n, \dots, y_h n\}$, see the connection probability p_{uv} in (1.3). Let $\theta(\mathbf{w}^{(\ell)}, \varepsilon)$ be the probability that the progeny of the associated branching process has ε-type $\mathbf{w}^{(\ell)}$, as defined in Definition 3.1. In the next lemma we implicitly show that

$$\mathbb{E}\left[\overline{P}\left((W_x)_{x\in T_q}, \mathbf{y}\right)\right] = \mathbb{E}\left[\prod_{\substack{x\in T_q,\\ i\in[h]}}\left(1 - q\cdot(y_i W_x^{\sigma} \wedge 1)\right)\right] = \lim_{\varepsilon\downarrow 0}\sum_{\ell=1}^{\infty}\sum_{\mathbf{w}^{(\ell)}\in \mathrm{CT}_\ell(\varepsilon)} \overline{P}(\mathbf{w}^{(\ell)}, \mathbf{y}^{(h)}) \cdot \theta(\mathbf{w}^{(\ell)}, \varepsilon).$$

We will control the rate of convergence in the above limit uniformly in the vector $\mathbf{y}^{(h)}$. We recall from Definition 1.3 that θ_q denotes the probability that the associated branching process survives. For $\phi > 0$ and a vector $\mathbf{y}^{(h)}$, we write $\mathbf{y}^{(h)} > \phi$ if the inequality holds for all elements of $\mathbf{y}^{(h)}$. We write $\mathcal{W}_n[\phi n, \infty)$ for the set of the weights of the vertices in $\mathcal{V}_n[\phi n, \infty)$.

Lemma 4.1 (Effect of truncation and discretization). *Consider an inhomogeneous scale-free random graph as in Definition 1.2. For all constants $\psi, h > 0$, there exist constants $\varepsilon, \ell_*, R > 0$ such that*

$$\sum_{\ell=1}^{\ell_*}\sum_{\mathbf{w}^{(\ell)}\in \mathrm{CT}_\ell(\varepsilon,R)} \theta(\mathbf{w}^{(\ell)}, \varepsilon) > 1 - \theta_q - \psi, \tag{4.2}$$

and for all $\phi > 0$,

$$\sup_{\mathbf{y}^{(h)}>\phi}\left|\mathbb{E}\left[\overline{P}\Big((W_x)_{x\in T_q},\mathbf{y}^{(h)}\Big)\right] - \sum_{\ell=1}^{\ell_*}\sum_{\mathbf{w}^{(\ell)}\in\mathrm{CT}_\ell(\varepsilon,R)}\overline{P}(\mathbf{w}^{(\ell)},\mathbf{y}^{(h)})\theta(\mathbf{w}^{(\ell)},\varepsilon)\right| < \psi, \tag{4.3}$$

and

$$\sup_{\substack{\mathbf{y}^{(h)}>\phi,\\ \mathbf{w}^{(\ell)}\in\mathrm{CT}_\ell(\varepsilon,R)}}\left|\frac{\mathbb{P}\left(\mathcal{C}\nsim\mathcal{V}_n[\phi n,\infty)\;\middle|\;\mathcal{C}\text{ has }\varepsilon\text{-type }\mathbf{w}^{(\ell)},\mathcal{W}_n[\phi n,\infty)=\{ny_1,\dots,ny_h\}\right)}{\overline{P}(\mathbf{w}^{(\ell)},\mathbf{y}^{(h)})}-1\right| < \psi. \tag{4.4}$$

We postpone the proof to the appendix on page 32. We proceed to a proposition that we will use to prove a lower bound on $\mathbb{P}(|\mathcal{C}_n^{(1)}| > \rho n)$. Let $\mathcal{Y}_{\rho,q}(h) \subseteq [0,\infty)^h$ as in (2.3). For a set of vectors $\mathcal{Z}$, we write $n\cdot\mathcal{Z}$ for the set in which all elements of the vectors are multiplied by n. Thus, if $\mathcal{V}_n[\phi n,\infty)$ has size h, we write $\mathcal{W}_n[\phi n,\infty)\in n\cdot\mathcal{Y}_{\rho,q}(h)$ if $(w_v/n)_{v\in\mathcal{V}_n[\phi n,\infty)}\in\mathcal{Y}_{\rho,q}(h)$.

Proposition 4.2 (Absence of hubs implies no large giant). *Consider an inhomogeneous scale-free random graph as in Definition 1.2. Fix a constant $\rho\in(\theta_q,1)$. There exists a constant $\phi_0>0$ such that for any $\phi\in(0,\phi_0)$ and $r\in(\theta_q,\rho)$, with $h=\lceil\mathrm{hubs}(r,q)\rceil$, as $n\to\infty$,*

$$\mathbb{P}\left(|\mathcal{C}_n^{(1)}|>\rho n\;\middle|\;\neg\big\{|\mathcal{V}_n[\phi n,\infty)|=h,\mathcal{W}_n[\phi n,\infty)\in n\cdot\mathcal{Y}_{r,q}(h)\big\}\right)=O\Big(\big(n\mathbb{P}(W_1>n)\big)^{h+1}\Big). \tag{4.5}$$

Proof. We describe the idea of the proof. In Step 1, we will distinguish whether there are at least $h+1$ vertices of weight at least ϕn—which occurs with probability at most the right-hand side in (4.5)—or at most h such vertices. In the latter case, we will argue that the size of the largest component remains smaller than ρn in Step 2, when intersected with two additional events. In Step 3, we analyze the probability of this intersection of events.

To argue so, we reveal the graph $\mathcal{G}_n$ in two stages: first we consider the graph $\mathcal{G}_n[\underline{w},\phi n)$, in which the number of size-ℓ components of finitely many ε-types concentrates using Lemma 3.3 (first additional event, Step 3a). In this graph, the complement of the giant has size approximately $(1-\theta_q)n$. Then we 'add' the vertices of weight at least ϕn to the graph, which we call hubs. If the number of hubs is smaller than h, or the set of their weights is not contained in $n\cdot\mathcal{Y}_{r,q}$ while there are exactly h many hubs, then enough components do *not* connect to one of the hubs with sufficiently high probability, which we show by standard concentration inequalities (second additional event, Step 3b). Consequently, the complement of the giant may become much smaller than $(1-\theta_q)n$, but remains larger than $(1-\rho)n$ as long as $r<\rho$.

Step 1. Rewriting the conditional probability. We now make this formal. Fix $r\in(\theta_q,\rho)$, and let $\psi=\psi(r)$ be a sufficiently small constant that we choose later. Given ψ, let $\varepsilon=\varepsilon(\psi)$, $\phi=\phi(\psi)$ be sufficiently small constants, and let $R=R(\psi)$, $\ell_*=\ell_*(\psi)$ be two large constants so that the bounds in Lemmas 3.3 and 4.1 apply. We define two subevents for the event on which we conditioned in (4.5):

$$\mathcal{A}_{\le\mathrm{hubs}} := \big\{|\mathcal{V}_n[\phi n,\infty)|<h\big\}\cup\big\{|\mathcal{V}_n[\phi n,\infty)|=h,\mathcal{W}_n[\phi n,\infty)\notin n\cdot\mathcal{Y}_{r,q}(h)\big\}, \tag{4.6}$$

$$\mathcal{A}_{>\mathrm{hubs}} := \big\{|\mathcal{V}_n[\phi n,\infty)|>h\big\}. \tag{4.7}$$

The event on which we conditioned in (4.5) corresponds to $\mathcal{A}_{\le\mathrm{hubs}}\cup\mathcal{A}_{>\mathrm{hubs}}$. We bound

$$\begin{aligned}\mathbb{P}\big(|\mathcal{C}_n^{(1)}|>\rho n|\mathcal{A}_{\le\mathrm{hubs}}\cup\mathcal{A}_{>\mathrm{hubs}}\big) \le{}& \mathbb{P}\big(\{|\mathcal{C}_n^{(1)}|>\rho n\}\cap(\neg\mathcal{A}_{>\mathrm{hubs}})|\mathcal{A}_{\le\mathrm{hubs}}\cup\mathcal{A}_{>\mathrm{hubs}}\big)\\ &+\mathbb{P}\big(\mathcal{A}_{>\mathrm{hubs}}|\mathcal{A}_{\le\mathrm{hubs}}\cup\mathcal{A}_{>\mathrm{hubs}}\big).\end{aligned}$$

Writing out the conditional probabilities and applying elementary operations, we obtain that

$$\mathbb{P}\big(|\mathcal{C}_n^{(1)}|>\rho n|\mathcal{A}_{\le\mathrm{hubs}}\cup\mathcal{A}_{>\mathrm{hubs}}\big)\le\mathbb{P}\big(|\mathcal{C}_n^{(1)}|>\rho n|\mathcal{A}_{\le\mathrm{hubs}}\big)+\frac{\mathbb{P}\big(\mathcal{A}_{>\mathrm{hubs}}\big)}{\mathbb{P}\big(\mathcal{A}_{\le\mathrm{hubs}}\big)}. \tag{4.8}$$

The denominator of the second term tends to 1 as $n\to\infty$, while the numerator is of order $O\Big(\big(n\mathbb{P}(W_1>n)\big)^{h+1}\Big)$ by (1.2) and the fact that the number of vertices with weight at least w is distributed as $\mathrm{Bin}(n,1-F_W(w))$. Thus, it suffices to show in the remainder that the first term on right-hand side is of smaller order.

Step 2. Small giant on three events. We introduce more notation. For each $\ell \le \ell_*$, we write $M_n(\mathbf{w}^{(\ell)}, \varepsilon)$ for the number of components of ε-type $\mathbf{w}^{(\ell)}$ in the induced subgraph $\mathcal{G}_n[\underline{w}, \phi n)$ that are *not* connected by an edge to the hubs in $\mathcal{G}_n$, $\tilde{h} := |\mathcal{V}_n[\phi n, \infty)|$ for the number of hubs, and $\mathbf{Y}^{(\tilde{h})} = \{y_1, \dots, y_{\tilde{h}}\}$ for the rescaled weights of the hubs, that is, $\mathcal{W}_n[\phi n, \infty) =: n \cdot \mathbf{Y}^{(\tilde{h})}$. We define two events:

$$\mathcal{A}_{\text{comp}} := \left\{\forall \ell \in [\ell_*], \mathbf{w}^{(\ell)} \in \text{CT}_\ell(\varepsilon, R) : N_n(\mathbf{w}^{(\ell)}, \varepsilon, \phi n) \ge (1-\psi)\theta(\mathbf{w}^{(\ell)}, \varepsilon) n/\ell\right\} \cap \{|\mathcal{C}_n^{(1)}[\underline{w}, \phi n)| \ge (\theta_q - \psi)n\}, \tag{4.9}$$

$$\mathcal{A}_{\text{conn}} := \left\{ \begin{array}{r} \forall \ell \in [\ell_*], \mathbf{w}^{(\ell)} \in \text{CT}_\ell(\varepsilon, R) : M_n(\mathbf{w}^{(\ell)}, \varepsilon) \ge (1-\psi)^3 \cdot (n/\ell) \cdot \theta(\mathbf{w}^{(\ell)}, \varepsilon) \\ \cdot \overline{P}(\mathbf{w}^{(\ell)}, \mathbf{Y}^{(\tilde{h})}) \mathbb{1}_{\{\overline{P}(\mathbf{w}^{(\ell)}, \mathbf{Y}^{(\tilde{h})}) \ge \psi\}} \end{array} \right\}. \tag{4.10}$$

We first show that $\{|\mathcal{C}_n^{(1)}| \le \rho n\} \subseteq \mathcal{A}_{\text{comp}} \cap \mathcal{A}_{\le\text{hubs}} \cap \mathcal{A}_{\text{conn}}$. To do so, we bound the size of the complement of the giant on this intersection. Since $\mathcal{G}_n$ contains a component of size at least $(\theta_q - \psi)n$ by $\mathcal{A}_{\text{comp}}$, the complement of the giant contains all components of size at most ℓ_*. On $\mathcal{A}_{\text{comp}} \cap \mathcal{A}_{\le\text{hubs}} \cap \mathcal{A}_{\text{conn}}$, we obtain by (4.3) in Lemma 4.1

$$\begin{aligned} 1 - \frac{|\mathcal{C}_n^{(1)}|}{n} &\ge \frac{1}{n}\sum_{\ell=1}^{\ell_*} \sum_{\mathbf{w}^{(\ell)} \in \text{CT}_\ell(\varepsilon, R)} \ell M_n(\mathbf{w}^{(\ell)}, \varepsilon) \\ &\ge (1-\psi)^3 \sum_{\ell=1}^{\ell_*} \sum_{\mathbf{w}^{(\ell)} \in \text{CT}_\ell(\varepsilon, R)} \overline{P}(\mathbf{w}^{(\ell)}, \mathbf{Y}^{(\tilde{h})}) \theta(\mathbf{w}^{(\ell)}, \varepsilon) \mathbb{1}_{\{\overline{P}(\mathbf{w}^{(\ell)}, \mathbf{Y}^{(\tilde{h})}) \ge \psi\}} \\ &\ge (1-\psi)^3 \left(\mathbb{E}[\overline{P}((W_x)_{x \in T_q}, \mathbf{Y}^{(\tilde{h})})] - \psi - \sum_{\ell=1}^{\ell_*} \sum_{\mathbf{w}^{(\ell)} \in \text{CT}_\ell(\varepsilon, R)} \overline{P}(\mathbf{w}^{(\ell)}, \mathbf{Y}^{(\tilde{h})}) \theta(\mathbf{w}^{(\ell)}, \varepsilon) \mathbb{1}_{\{\overline{P}(\mathbf{w}^{(\ell)}, \mathbf{Y}^{(\tilde{h})}) < \psi\}} \right). \end{aligned}$$

Because of the indicator, we may bound $\overline{P}(\mathbf{w}^{(\ell)}, \mathbf{Y}^{(\tilde{h})})$ in the sum from above by ψ. We also use that the probabilities $\theta(\mathbf{w}^{(\ell)}, \varepsilon)$, see (3.1), sum up to at most one, so that the sum is at most ψ. Therefore, on $\mathcal{A}_{\text{comp}} \cap \mathcal{A}_{\le\text{hubs}} \cap \mathcal{A}_{\text{conn}}$,

$$1 - \frac{|\mathcal{C}_n^{(1)}|}{n} \ge (1-\psi)^3 \left(\mathbb{E}\left[\overline{P}\left((W_x)_{x \in T_q}, \mathbf{Y}^{(\tilde{h})} \right) \right] - 2\psi \right).$$

When $\tilde{h} < \lceil \text{hubs}(r, q) \rceil = h$, the expectation is larger than $1 - r$ by (2.11). When $\tilde{h} = h$ and $\mathbf{Y}^{(\tilde{h})} \notin \mathcal{Y}_{r,q}(\tilde{h})$, the definition of $\mathcal{Y}_{r,q}(\tilde{h})$ in (2.3) implies that the expectation on the right-hand side is larger than $1 - r$. Thus, on $\mathcal{A}_{\text{comp}} \cap \mathcal{A}_{\le\text{hubs}} \cap \mathcal{A}_{\text{conn}}$,

$$1 - |\mathcal{C}_n^{(1)}|/n > (1-\psi)^3(1 - r - 2\psi).$$

As we fixed $r \in (\theta_q, \rho)$, the right-hand side is strictly larger than $1 - \rho$ if ψ is sufficiently small depending on r and ρ. Rearranging yields that $\mathcal{A}_{\text{comp}} \cap \mathcal{A}_{\le\text{hubs}} \cap \mathcal{A}_{\text{conn}} \subseteq \{|\mathcal{C}_n^{(1)}| \le \rho n\}$ for such values of ψ. This implies that $\{|\mathcal{C}_n^{(1)}| > \rho n\} \subseteq (\neg\mathcal{A}_{\text{comp}}) \cup (\neg\mathcal{A}_{\text{conn}})$ when working conditionally on $\mathcal{A}_{\le\text{hubs}}$. We bound

$$\begin{aligned} \mathbb{P}\left(|\mathcal{C}_n^{(1)}| > \rho n | \mathcal{A}_{\le\text{hubs}}\right) &\le \mathbb{P}\left(\neg\mathcal{A}_{\text{comp}} | \mathcal{A}_{\le\text{hubs}}\right) + \mathbb{P}\left((\neg\mathcal{A}_{\text{conn}}) \cap \mathcal{A}_{\text{comp}} | \mathcal{A}_{\le\text{hubs}}\right) \\ &\le \mathbb{P}\left(\neg\mathcal{A}_{\text{comp}}\right) / \mathbb{P}\left(\mathcal{A}_{\le\text{hubs}}\right) + \mathbb{P}\left((\neg\mathcal{A}_{\text{conn}}) \cap \mathcal{A}_{\text{comp}} | \mathcal{A}_{\le\text{hubs}}\right). \end{aligned}$$

Step 3a. Concentration for ε-type components. Recall the definition of $\mathcal{A}_{\text{comp}}$ from (4.9). By Lemmas 3.3 and 3.4 we obtain for $\phi = \phi(\psi)$ sufficiently small that

$$\mathbb{P}\left(\neg\mathcal{A}_{\text{comp}}\right) = O\left((n\mathbb{P}(W > n))^{h+1}\right).$$

Since $\mathbb{P}\left(\mathcal{A}_{\le\text{hubs}}\right) \to 1$, see below (4.8) and (1.2), it follows that

$$\mathbb{P}\left(|\mathcal{C}_n^{(1)}| > \rho n | \mathcal{A}_{\le\text{hubs}}\right) \le \mathbb{P}\left(\neg\mathcal{A}_{\text{conn}} | \mathcal{A}_{\le\text{hubs}} \cap \mathcal{A}_{\text{comp}}\right) + O\left((n\mathbb{P}(W > n))^{h+1}\right). \tag{4.11}$$

Step 3b. Edges from hubs to small components in restricted graph. We now condition on the graph $\mathcal{G}_n[\underline{w}, \phi n)$ satisfying $\mathcal{A}_{\text{comp}}$ and the realization of $\mathcal{W}_n[\phi n, \infty)$ being equal to $n\mathbf{Y}^{(\tilde{h})}$ and satisfying $\mathcal{A}_{\le\text{hubs}}$. We abbreviate

$$\mathbb{P}_{\mathbf{Y},\mathcal{G}}(\,\cdot\,) := \mathbb{P}\left(\cdot | \mathcal{G}_n[\underline{w}, \phi n), \mathcal{W}_n[\phi n, \infty) = n\mathbf{Y}^{(\tilde{h})}, \mathcal{A}_{\text{comp}} \cap \mathcal{A}_{\le\text{hubs}} \right).$$

Consequently,

$$\mathbb{P}\big(\neg\mathcal{A}_{\text{conn}} \mid \mathcal{A}_{\le \text{hubs}}\big) = \mathbb{E}\Big[\mathbb{1}_{\{\mathcal{A}_{\text{comp}}\}}\mathbb{P}_{\mathbf{Y},\mathcal{G}}\big(\neg\mathcal{A}_{\text{conn}}\big)\Big]. \tag{4.12}$$

We recall $\mathcal{A}_{\text{conn}}$ from (4.10). By a union bound over all component types, it follows that

$$\mathbb{P}_{\mathbf{Y},\mathcal{G}}\big(\neg\mathcal{A}_{\text{conn}}\big) \le \sum_{\ell=1}^{\ell_*}\sum_{\mathbf{w}^{(\ell)}\in\text{CT}_\ell(\varepsilon,R)} \mathbb{P}_{\mathbf{Y},\mathcal{G}}\Big(M_n(\mathbf{w}^{(\ell)},\varepsilon) < (1-\psi)^3(n/\ell)\theta(\mathbf{w}^{(\ell)},\varepsilon)\overline{P}(\mathbf{w}^{(\ell)},\mathbf{Y}^{(\tilde{h})})\mathbb{1}_{\{\overline{P}(\mathbf{w}^{(\ell)},\mathbf{Y}^{(\tilde{h})})\ge\psi\}}\Big).$$

Since $M_n(\mathbf{w}^{(\ell)},\varepsilon)$ counts the number of components that do not connect by an edge to the hubs, $M_n(\mathbf{w}^{(\ell)},\varepsilon)$ is nonnegative. Therefore, when the indicator inside the probability equals 0, the probability also equals 0. Therefore, we only need to consider the cases in which the indicator equals one, that is,

$$\mathbb{P}_{\mathbf{Y},\mathcal{G}}\big(\neg\mathcal{A}_{\text{conn}}\big) \le \sum_{\ell=1}^{\ell_*}\sum_{\mathbf{w}^{(\ell)}\in\text{CT}_\ell(\varepsilon,R)} \mathbb{1}_{\{\overline{P}(\mathbf{w}^{(\ell)},\mathbf{Y}^{(\tilde{h})})\ge\psi\}}\mathbb{P}_{\mathbf{Y},\mathcal{G}}\Big(M_n(\mathbf{w}^{(\ell)},\varepsilon) < (1-\psi)^3(n/\ell)\theta(\mathbf{w}^{(\ell)},\varepsilon)\overline{P}(\mathbf{w}^{(\ell)},\mathbf{Y}^{(\tilde{h})})\Big).$$

Components in $\mathcal{G}_n[\underline{w},\phi n)$ of ε-type $\mathbf{w}^{(\ell)}$ connect independently by an edge to the hubs. The probability that a component does not connect to the hubs, is at least $(1-\psi)\overline{P}(\mathbf{w}^{(\ell)},\mathbf{Y}^{(\tilde{h})})$ by (4.4). Thus, conditionally on $\mathcal{G}_n[\underline{w},\phi n)$ satisfying $\mathcal{A}_{\text{comp}}$ defined in (4.9), and $\mathcal{W}_n[\phi n,\infty) = n\mathbf{Y}^{(\tilde{h})}$,

$$\begin{aligned} M_n(\mathbf{w}^{(\ell)},\varepsilon) &\succcurlyeq \text{Bin}\Big(N_n(\mathbf{w}^{(\ell)},\varepsilon,R),(1-\psi)\overline{P}(\mathbf{w}^{(\ell)},\mathbf{Y}^{(\tilde{h})})\Big) \\ &\succcurlyeq \text{Bin}\Big((1-\psi)\theta(\mathbf{w}^{(\ell)},\varepsilon)n/\ell,(1-\psi)\overline{P}(\mathbf{w}^{(\ell)},\mathbf{Y}^{(\tilde{h})})\Big). \end{aligned}$$

We apply a Chernoff bound for each ε-type $\mathbf{w}^{(\ell)}$: there exists a constant $c > 0$ depending on ψ such that

$$\begin{aligned} \mathbb{P}_{\mathbf{Y},\mathcal{G}}\big(\neg\mathcal{A}_{\text{conn}}\big) &\le \sum_{\ell=1}^{\ell_*}\sum_{\mathbf{w}^{(\ell)}\in\text{CT}_\ell(\varepsilon,R)} \mathbb{1}_{\{\overline{P}(\mathbf{w}^{(\ell)},\mathbf{Y}^{(\tilde{h})})\ge\psi\}}\exp\Big(-c\theta(\mathbf{w}^{(\ell)},\varepsilon)(n/\ell)\overline{P}(\mathbf{w}^{(\ell)},\mathbf{Y}^{(\tilde{h})})\Big) \\ &\le \sum_{\ell=1}^{\ell_*}\sum_{\mathbf{w}^{(\ell)}\in\text{CT}_\ell(\varepsilon,R)} \exp\big(-c\psi\theta(\mathbf{w}^{(\ell)},\varepsilon)(n/\ell)\big). \end{aligned}$$

Since both ℓ_* and the number of considered component types for each ℓ are finite, the right-hand side decays exponentially fast in n. Thus, also (4.12) decays exponentially fast, and (4.11) is of order $O\big((n\mathbb{P}(W>n))^{h+1}\big)$. We substitute that bound into (4.8), which finishes the proof since the second term in (4.8) is of the same order. □

Next, we state and prove a proposition that shows that the presence of $\lceil \text{hubs}(r,q)\rceil$ hubs with weights in $\mathcal{Y}_{r,q}(\lceil \text{hubs}(\rho,q)\rceil)$ lead to a large giant for any $r > \rho$.

Proposition 4.3 (Presence of hubs implies a large giant). *Consider an inhomogeneous scale-free random graph as in Definition 1.2. Fix a constant $\rho \in (\theta_q, 1)$. For any $r \in (\rho, 1)$ there exists a constant $c > 0$ such that, with $h = \lceil \text{hubs}(r,q)\rceil$, for any $\mathbf{y}^{(h)} \in \mathcal{Y}_{r,q}(h)$ and $n \ge 1$,*

$$\mathbb{P}\left(|\mathcal{C}_n^{(1)}| > \rho n \;\middle|\; \mathcal{W}_n[\phi n,\infty) = n\mathbf{y}^{(h)}\right) \ge 1 - \exp(-cn). \tag{4.13}$$

Proof. The proof uses a similar construction as Proposition 4.2. In Step 1, we define three additional good events on which we show that the size of the giant exceeds ρn. To analyze the probability of the intersection of these events, we reveal the graph $\mathcal{G}_n$ in two stages: in Step 2a (corresponding to the first additional event), we consider the graph $\mathcal{G}_n[\underline{w},R)$ for a large constant $R > \underline{w}$ (this is opposed to the proof of Proposition 4.2, where we first revealed the graph up to ϕn). Lemma 3.3 yields concentration for the number of size-ℓ components of finitely many ε-types in this graph. The giant has size approximately $\theta_q n$ by Lemma 3.4. In Step 2b, we 'add' the vertices of weight at least ϕn to the graph, which we call hubs. Since we assume that there are exactly $\lceil \text{hubs}(r,q)\rceil$ many hubs, whose set of their weights is contained in $n\cdot\mathcal{Y}_{r,q}$, sufficiently many components connect to one of the hubs as shown in Step 2b(i) (second additional event). Step 2b(ii) shows that each hub connects with probability tending to one exponentially fast to one of the vertices in the giant of $\mathcal{G}[\underline{w},R)$ (third additional event). Consequently, the size of the giant becomes larger than ρn with sufficiently high probability.

We will not consider the effect of vertices with weight in $[R, \phi n)$: edges incident to those vertices can only increase the size of the largest component, but this effect is negligible.

Step 1. Large giant on three events. We now formalize this. Fix $r \in (\theta_q, \rho)$, let $\psi = \psi(r), \varepsilon = \varepsilon(\psi)$ be sufficiently small constants, and let $R = R(\psi), \ell_* = \ell_*(\psi)$ be two large constants. For each $\ell \le \ell_*$, we write $M_n(\mathbf{w}^{(\ell)}, \varepsilon)$ for the number of components of ε-type $\mathbf{w}^{(\ell)}$ in the induced subgraph $\mathcal{G}_n[\underline{w}, R)$ that are *not* connected by an edge to the hubs in $\mathcal{G}_n$. Let $h := |\mathcal{V}_n[\phi n, \infty)|$ denote the number of hubs, which is equal to $\lceil \mathrm{hubs}(r, q) \rceil$ by assumption. By the conditioning in (4.13) all rescaled weights in $\mathbf{y}^{(h)} = \{y_1, \dots, y_h\}$ are at least ϕ. Define

$$\mathcal{A}_{\mathrm{comp}} := \left\{ \forall \ell \in [\ell_*], \mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon, R) : \ell N_n(\mathbf{w}^{(\ell)}, \varepsilon, R) \Big/ (n\theta(\mathbf{w}^{(\ell)}, \varepsilon)) \ge 1 - \psi \right\}, \tag{4.14}$$

$$\cap \{ |\mathcal{C}_n^{(1)}[\underline{w}, \phi n)| \ge (\theta_q - \psi)n \}$$

$$\mathcal{A}_{\mathrm{conn}} := \left\{ \begin{array}{r} \forall \ell \in [\ell_*], \mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon, R) : M_n(\mathbf{w}^{(\ell)}, \varepsilon) \le (1+\psi)^2 \cdot \left(\psi + \overline{P}(\mathbf{w}^{(\ell)}, \mathbf{y}^{(h)}) \right) \\ \cdot N_n(\mathbf{w}^{(\ell)}, \varepsilon, R) \end{array} \right\}, \tag{4.15}$$

$$\mathcal{A}_{\mathrm{hubs}} := \left\{ \forall v \in \mathcal{V}_n[\phi n, \infty) : v \sim \mathcal{C}_n^{(1)}[\underline{w}, R) \right\}. \tag{4.16}$$

The events $\mathcal{A}_{\mathrm{comp}}$ and $\mathcal{A}_{\mathrm{conn}}$ are the counterparts of the events (having the same name) defined in (4.9) and (4.10). The event $\mathcal{A}_{\mathrm{hubs}}$ here ensures edges between the restricted giant and hubs, and its meaning differs from $\mathcal{A}_{\le \mathrm{hubs}}$ and $\mathcal{A}_{>\mathrm{hubs}}$ in (4.6) and (4.7).

We bound the size of the giant on the intersection of the three events from below. Since all vertices in $\mathcal{V}_n[\phi n, \infty)$ connect by an edge to the largest component in $\mathcal{G}_n[\underline{w}, R)$ on $\mathcal{A}_{\mathrm{hubs}}$, the size of the largest component $|\mathcal{C}_n^{(1)}|$ increases from $|\mathcal{C}_n^{(1)}[\underline{w}, R)|$ by at least the total number of vertices in a component of size at most ℓ_* in $\mathcal{G}_n[\underline{w}, R)$ that is connected by an edge to one of the hubs in $\mathcal{G}_n$. Thus, using first the definition of $\mathcal{A}_{\mathrm{conn}}$, and then the definition of $\mathcal{A}_{\mathrm{comp}}$,

$$\begin{aligned} |\mathcal{C}_n^{(1)}|/n &\ge \theta_q - \psi + \frac{1}{n} \sum_{\ell=1}^{\ell_*} \sum_{\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon, R)} \ell \left(N_n(\mathbf{w}^{(\ell)}, \varepsilon, R) - M_n(\mathbf{w}^{(\ell)}, \varepsilon) \right) \\ &\ge \theta_q - \psi + \frac{1}{n} \sum_{\ell=1}^{\ell_*} \sum_{\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon, R)} \ell N_n(\mathbf{w}^{(\ell)}, \varepsilon, R) \Big(1 - (1+\psi)^2 \Big(\psi + \overline{P}(\mathbf{w}^{(\ell)}, \mathbf{y}^{(h)}) \Big) \Big) \\ &\ge \theta_q - \psi + (1-\psi) \sum_{\ell=1}^{\ell_*} \sum_{\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon, R)} \theta(\mathbf{w}^{(\ell)}, \varepsilon) \\ &\quad - (1+\psi)^3 \psi \sum_{\ell=1}^{\ell_*} \sum_{\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon, R)} \theta(\mathbf{w}^{(\ell)}, \varepsilon) - (1+\psi)^3 \sum_{\ell=1}^{\ell_*} \sum_{\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon, R)} \theta(\mathbf{w}^{(\ell)}, \varepsilon) \overline{P}(\mathbf{w}^{(\ell)}, \mathbf{y}^{(h)}). \end{aligned}$$

We assume that ε, R, and ℓ_* satisfy the conditions of Lemma 4.1. We bound the first double sum via (4.2). The second double sum over the probabilities $\theta(\mathbf{w}^{(\ell)}, \varepsilon)$ is at most 1 by (3.1). We invoke (4.3) for the third double sum. Therefore, on $\mathcal{A}_{\mathrm{comp}} \cap \mathcal{A}_{\mathrm{hubs}} \cap \mathcal{A}_{\mathrm{conn}}$,

$$|\mathcal{C}_n^{(1)}|/n \ge \theta_q - \psi + (1-\psi)(1 - \theta_q - \psi) - \psi(1+\psi)^3 - (1+\psi)^3 \left(\mathbb{E}\left[\overline{P}\left((W_x)_{x \in T_q}, \mathbf{y}^{(h)} \right) \right] + \psi \right).$$

By assumption, we have $\mathbf{y}^{(h)} \in \mathcal{Y}_{r,q}$. Thus, the expectation is at most $1 - r$ by definition of $\mathcal{Y}_{r,q}$ in (2.3). Hence, for some constant $C > 0$,

$$|\mathcal{C}_n^{(1)}|/n \ge \theta_q - \psi + (1-\psi)(1-\theta_q-\psi) - \psi(1+\psi)^3 - (1+\psi)^3(1 - r + \psi) \ge r - C\psi.$$

Since $r > \rho$, we can make the right-hand side strictly larger than ρ when ψ is fixed sufficiently small. So, for ψ sufficiently small, and ε, R, and ℓ_* satisfying the conditions of Lemma 4.1,

$$\mathbb{P}\left(|\mathcal{C}_n^{(1)}| > \rho n \;\Big|\; \mathcal{W}_n[\phi n, \infty) = n\mathbf{y}^{(\ell)} \right) \ge \mathbb{P}\left(\mathcal{A}_{\mathrm{comp}} \cap \mathcal{A}_{\mathrm{hubs}} \cap \mathcal{A}_{\mathrm{conn}} \;\Big|\; \mathcal{W}_n[\phi n, \infty) = n\mathbf{y}^{(\ell)} \right).$$

In the remainder, we bound the probability on the right-hand side from below.

Step 2a. Concentration for ε-type components. Since the event $\mathcal{A}_{\mathrm{comp}}$ is defined on the graph induced on vertices with weight at most R, it is independent of the event on which we conditioned. Thus,

$$\begin{aligned}&\mathbb{P}\left(|\mathcal{C}_n^{(1)}| > \rho n \;\middle|\; \mathcal{W}_n[\phi n, \infty) = n\mathbf{y}^{(\ell)}\right)\\ &\quad\geq \mathbb{P}\big(\mathcal{A}_{\mathrm{comp}}\big)\cdot \mathbb{P}\left(\mathcal{A}_{\mathrm{hubs}}\cap\mathcal{A}_{\mathrm{conn}} \;\middle|\; \mathcal{W}_n[\phi n, \infty) = n\mathbf{y}^{(\ell)}, \mathcal{A}_{\mathrm{comp}}\right).\end{aligned}$$

By definition of $\mathcal{A}_{\mathrm{comp}}$ in (4.14),

$$\begin{aligned}\mathbb{P}(\neg\mathcal{A}_{\mathrm{comp}}) &\leq \mathbb{P}\big(\exists \ell\in[\ell_*], \mathbf{w}^{(\ell)}\in \mathrm{CT}_\ell(\varepsilon, R) : N_n(\mathbf{w}^{(\ell)}, \varepsilon, R) < (1-\psi)\theta(\mathbf{w}^{(\ell)}, \varepsilon)n/\ell\big)\\ &\quad+\mathbb{P}\left(|\mathcal{C}_n^{(1)}[\underline{w}, R)| < (1-\psi)\theta_q n\right).\end{aligned}$$

By Lemma 3.2 and Lemma 3.4, the two terms on the right-hand side decay exponentially in n. So, there exists a constant $c > 0$ such that

$$\begin{aligned}&\mathbb{P}(|\mathcal{C}_n^{(1)}| > \rho n \;\big|\; \mathcal{W}_n[\phi n, \infty) = n\mathbf{y}^{(\ell)})\\ &\quad\geq (1-\exp(-cn))\cdot \mathbb{P}(\mathcal{A}_{\mathrm{hubs}}\cap\mathcal{A}_{\mathrm{conn}} \;\big|\; \mathcal{W}_n[\phi n, \infty) = n\mathbf{y}^{(\ell)}, \mathcal{A}_{\mathrm{comp}}).\end{aligned}\tag{4.17}$$

Step 2b. Edges from the hubs. We condition on the graph $\mathcal{G}_n[\underline{w}, R)$ satisfying $\mathcal{A}_{\mathrm{comp}}$ and the realization of $\mathcal{W}_n[\phi n, \infty) = n\mathbf{y}^{(h)}$ satisfying $\mathcal{A}_{\mathrm{comp}}$. We abbreviate

$$\mathbb{P}_{\mathbf{y},\mathcal{G}}(\,\cdot\,) := \mathbb{P}(\,\cdot\,|\mathcal{G}_n[\underline{w}, R), \mathcal{W}_n[\phi n, \infty) = n\mathbf{y}^{(h)}, \mathcal{A}_{\mathrm{comp}}).\tag{4.18}$$

Since we condition on the weights of all vertices with weight in $[\underline{w}, R)\cup[\phi n, \infty)$ in (4.18), edges between the hubs and components in $\mathcal{G}[\underline{w}, R)$ are present independently. Thus, by definition of $\mathcal{A}_{\mathrm{hubs}}$ and $\mathcal{A}_{\mathrm{conn}}$ in (4.16) and (4.15),

$$\mathbb{P}_{\mathbf{y},\mathcal{G}}\big(\mathcal{A}_{\mathrm{hubs}}\cap\mathcal{A}_{\mathrm{conn}}\big) = \mathbb{P}_{\mathbf{y},\mathcal{G}}\big(\mathcal{A}_{\mathrm{conn}}\big)\cdot\mathbb{P}_{\mathbf{y},\mathcal{G}}\big(\mathcal{A}_{\mathrm{hubs}}\big).\tag{4.19}$$

Step 2b(i). Edges from hubs to small components in restricted graph. We next show that the first probability on the right-hand side tends to one. By definition of $\mathcal{A}_{\mathrm{conn}}$ in (4.15), we have to bound from above the number of components of ε-type $\mathbf{w}^{(\ell)}$ that do *not* connect by an edge to one of the hubs. We apply a union bound over all component types:

$$\mathbb{P}_{\mathbf{y},\mathcal{G}}\big(\neg\mathcal{A}_{\mathrm{conn}}\big) \leq \sum_{\ell=1}^{\ell_*}\sum_{\mathbf{w}^{(\ell)}\in\mathrm{CT}_\ell(\varepsilon,R)}\mathbb{P}_{\mathbf{y},\mathcal{G}}\Big(M_n(\mathbf{w}^{(\ell)}, \varepsilon) > (1+\psi)^2\Big(\psi + \overline{P}(\mathbf{w}^{(\ell)}, \mathbf{y}^{(h)})\Big)N_n(\mathbf{w}^{(\ell)}, \varepsilon, R)\Big).\tag{4.20}$$

Components in $\mathcal{G}_n[\underline{w}, R)$ of ε-type $\mathbf{w}^{(\ell)}$ connect independently by an edge to the hubs, and the probability that a component does not connect to the hubs, is at least $(1-\psi)\overline{P}(\mathbf{w}^{(\ell)}, \mathbf{y}^{(h)})$ by (4.4). Thus, under the conditional probability measure $\mathbb{P}_{\mathbf{y},\mathcal{G}}$,

$$\begin{aligned}M_n(\mathbf{w}^{(\ell)}, \varepsilon) &\preceq \mathrm{Bin}\Big(N_n(\mathbf{w}^{(\ell)}, \varepsilon, R), (1+\psi)\overline{P}(\mathbf{w}^{(\ell)}, \mathbf{y}^{(h)})\Big)\\ &\preceq \mathrm{Bin}\Big(N_n(\mathbf{w}^{(\ell)}, \varepsilon, R), (1+\psi)\Big(\psi + \overline{P}(\mathbf{w}^{(\ell)}, \mathbf{y}^{(h)})\Big)\Big).\end{aligned}$$

We apply the stochastic domination to all terms in (4.20) and apply a Chernoff bound for each component type. This yields for some $c = c(\psi) > 0$,

$$\begin{aligned}\mathbb{P}_{\mathbf{y},\mathcal{G}}\big(\neg\mathcal{A}_{\mathrm{conn}}\big) &\leq \sum_{\ell=1}^{\ell_*}\sum_{\mathbf{w}^{(\ell)}\in\mathrm{CT}_\ell(\varepsilon,R)}\exp\Big(-c(\overline{P}(\mathbf{w}^{(\ell)}, \mathbf{y}^{(h)}) + \psi)N_n(\mathbf{w}^{(\ell)}, \varepsilon, R)\Big)\\ &\leq \sum_{\ell=1}^{\ell_*}\sum_{\mathbf{w}^{(\ell)}\in\mathrm{CT}_\ell(\varepsilon,R)}\exp\big(-c\psi N_n(\mathbf{w}^{(\ell)}, \varepsilon, R)\big).\end{aligned}$$

The probability measure $\mathbb{P}_{\mathbf{y},\mathcal{G}}$ defined in (4.18) is conditional on the event $\mathcal{A}_{\text{comp}}$ defined in (4.14). On this event, each $N_n(\mathbf{w}^{(\ell)}, R, \varepsilon)$ increases linearly in n. Since the number of component types is finite, $\mathbb{P}_{\mathbf{y},\mathcal{G}}(\neg\mathcal{A}_{\text{conn}})$ decays exponentially in n. We substitute this bound into (4.19) and obtain for some other $c > 0$,

$$\mathbb{P}_{\mathbf{y},\mathcal{G}}\big(\mathcal{A}_{\text{hubs}} \cap \mathcal{A}_{\text{conn}}\big) \geq (1 - \exp(-cn)) \cdot \mathbb{P}_{\mathbf{y},\mathcal{G}}\big(\mathcal{A}_{\text{hubs}}\big). \tag{4.21}$$

Step 2b(ii). Edges from hubs to restricted giant. We bound the second factor on the right-hand side. Recall $\mathcal{A}_{\text{hubs}}$ from (4.16). On the event $\mathcal{A}_{\text{comp}}$ on which we conditioned, $|\mathcal{C}_n[\underline{w}, R]| \geq (\theta_q - \psi)n$, and each vertex of weight at least ϕn connects with probability at least $q(\phi\underline{w} \wedge 1) = \Theta(1)$ to each vertex in $\mathcal{C}_n^{(1)}[\underline{w}, R)$. Thus, the probability that a single vertex of weight at least ϕn does not connect by an edge to the giant, decays exponentially in n. By assumption, there are exactly $\lceil \text{hubs}(r, q) \rceil$ many vertices in $\mathcal{V}_n[\phi n, \infty)$. By a union bound over these constantly many hubs, also the second factor in (4.21) tends to one on the event $\mathcal{A}_{\text{comp}}$ exponentially fast. Substituting this limit into (4.17) finishes the proof for some other $c > 0$. That is,

$$\mathbb{P}\left(|\mathcal{C}_n^{(1)}| > \rho n \;\middle|\; \mathcal{W}_n[\phi n, \infty) = n\mathbf{y}^{(\ell)}\right) \geq 1 - \exp(-cn).$$

□

Via similar proof strategies as Propositions 4.2 and 4.3, we prove the following lemma in the appendix on page 34.

Lemma 4.4. *Consider an inhomogeneous scale-free random graph as in Definition 1.2. Moreover, for any constants $C, \psi > 0$ and $\ell, h \in \mathbb{N}$, there exists $\phi_1 > 0$ such that for any $\phi \in (0, \phi_1)$, as $n \to \infty$,*

$$\sup_{\mathbf{y}^{(h)} \geq \phi} \mathbb{P}\left(\left|N_{n,\ell}/n - \frac{1}{\ell}\mathbb{E}\left[\mathbb{1}_{\{|T_q|=\ell\}}\overline{P}\Big((W_x)_{x\in T_q}, \mathbf{y}^{(h)}\Big)\right]\right| > \psi \;\middle|\; \mathcal{W}_n[\phi n, \infty) = n\mathbf{y}^{(h)}\right) \leq o(n^{-C}).$$

4.1 | Asymptotics for the Probability of Having Hubs

In this section, we analyze the probabilities of the events on which we conditioned in Propositions 4.2 and 4.3. Let $C_{\rho,q}(h)$ be the constant from (2.4).

Lemma 4.5 (Leading constant). *Consider an inhomogeneous scale-free random graph as in Definition 1.2. Let $\rho \in (\theta_q, 1)$ and set $h = \lceil \text{hubs}(\rho, q) \rceil$. There exists $\phi_0 > 0$ such that for any $\phi \in (0, \phi_0)$, as $n \to \infty$,*

$$\frac{\mathbb{P}\left(|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in n \cdot \mathcal{Y}_{\rho,q}(h)\right)}{(n\mathbb{P}(W > n))^h} \to C_{\rho,q}(h) \in (0, \infty). \tag{4.22}$$

Proof. We start with analyzing the numerator of (4.22). Abbreviate $h = \lceil \text{hubs}(\rho, q) \rceil$. By Definition 1.2, all weights are iid and follow distribution $F_W(w) = 1 - L(w)w^{-\alpha}$ for a parameter $\alpha > 1$ and slowly varying function $L(w)$. Thus,

$$\mathbb{P}\left(|\mathcal{V}_n[\phi n, \infty)| = h\right) = \binom{n}{h}(L(\phi n)(\phi n)^{-\alpha})^h(1 - L(\phi n)(\phi n)^{-\alpha})^{n-h} \sim \frac{1}{h!}(nL(\phi n)(\phi n)^{-\alpha})^h.$$

Since L is slowly varying, $L(\phi n)n^{-\alpha} \sim L(n)n^{-\alpha} \sim \mathbb{P}(W > n)$. So,

$$\frac{\mathbb{P}\left(|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in n \cdot \mathcal{Y}_{\rho,q}(h)\right)}{(n\mathbb{P}(W > n))^h} \sim \frac{1}{\phi^{\alpha h}h!}\mathbb{P}\left(\frac{1}{n}\mathcal{W}_n[\phi n, \infty) \in \mathcal{Y}_{\rho,q}(h) \;\middle|\; |\mathcal{V}_n[\phi n, \infty)| = h\right).$$

Let $(W_1/n, \dots, W_h/n)$ denote the weights of the vertices with weight at least ϕn. Since the weights are iid and regularly varying with index α, conditionally on $W_i \geq \phi n$ for all $i \in [h]$,

$$(W_1/n, \dots, W_h/n) \xrightarrow{d} (Y_1, \dots, Y_h),$$

where $(Y_1, \dots, Y_h)$ are independent copies of Y following distribution $\mathbb{P}(Y \geq y) = (y/\phi)^{-\alpha}$ for $y \geq \phi$. Writing out the probability in (4.1) as an integral, the factor $\phi^{-\alpha h}$ cancels, that is,

$$\frac{\mathbb{P}\left(|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in n \cdot \mathcal{Y}_{\rho,q}(h)\right)}{(n\mathbb{P}(W > n))^h} \sim \frac{1}{h!}\int_{y_1=\phi}^{\infty} \cdots \int_{y_h=\phi}^{\infty} \frac{\alpha^h}{(y_1 \cdot \ldots \cdot y_h)^{\alpha+1}}\mathbb{1}\{(y_1, \dots, y_h) \in \mathcal{Y}_{\rho,q}\}\mathrm{d}y_1 \cdot \ldots \cdot \mathrm{d}y_h. \tag{4.23}$$

The only difference between the integrals here compared to the ones in the definition of $C_{\rho,q}$ in (2.4), is that the integrals in (2.4) start at the value 0, rather than ϕ. In the remainder of the proof we argue that the indicator above always equals 0 when there exists an index i such that $y_i < \phi$ for a small constant $\phi > 0$. Let $(y_1, \dots, y_h)$ be any vector such that $y_i < \phi$ for some i. By definition of $\mathcal{Y}_{\rho,q}$ in (2.3), we have to show that

$$\mathbb{E}\left[\prod_{x\in T_q, j\in[h]}\left(1 - q(W_x^\sigma y_j \wedge 1)\right)\right] > 1-\rho. \tag{4.24}$$

Bounding all factors with $j \neq i$ from below by $(1-q)$, and $y_i < \phi$, we obtain

$$\mathbb{E}\left[\prod_{x\in T_q, j\in[h]}\left(1 - q(W_x^\sigma y_j \wedge 1)\right)\right] \geq \mathbb{E}\left[(1-q)^{|T_q|(h-1)}\prod_{x\in T_q}\left(1 - q(W_x^\sigma \phi \wedge 1)\right)\right]. \tag{4.25}$$

The expectation is non-increasing in ϕ, so we evaluate the limit as $\phi \downarrow 0$. The argument of the expectation is a continuous and monotone function of ϕ. Therefore, by the Monotone Convergence Theorem,

$$\lim_{\phi\downarrow 0}\mathbb{E}\left[(1-q)^{|T_q|(h-1)}\prod_{x\in T_q}\left(1 - q(W_x^\sigma \phi \wedge 1)\right)\right] = \mathbb{E}\left[(1-q)^{|T_q|(h-1)}\right].$$

If $q = 1$, then $\text{hubs}(\rho, q) = 1$, and the expression equals one. By definition of $h = \lceil \text{hubs}(\rho, q)\rceil$ in (2.2), $\mathbb{E}\left[(1-q)^{|T_q|\text{hubs}(\rho,q)}\right] = 1-\rho$ when $q < 1$. Thus, for all values $q \in (0, 1]$,

$$\lim_{\phi\downarrow 0}\mathbb{E}\left[(1-q)^{|T_q|(h-1)}\prod_{x\in T_q}\left(1 - q(W_x^\sigma \phi \wedge 1)\right)\right] > \mathbb{E}\left[(1-q)^{|T_q|\text{hubs}(\rho,q)}\right] = 1-\rho.$$

Since the expectation on the right-hand side in (4.25) is a non-increasing function in ϕ, there exists $\phi > 0$ such that (4.24) holds if there exists $y_i < \phi$. This proves that $(y_1, \dots, y_h) \notin \mathcal{Y}_{\rho,q}$ if there exists $y_i < \phi$. Using this in (4.23), we obtain

$$\frac{\mathbb{P}\left(|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in n\cdot\mathcal{Y}_{\rho,q}(h)\right)}{(n\mathbb{P}(W > n))^h} \sim \frac{1}{h!}\int_{y_1=0}^{\infty}\cdots\int_{y_h=0}^{\infty}\frac{\alpha^h}{(y_1\cdot\ldots\cdot y_h)^{\alpha+1}}\mathbb{1}\{(y_1,\dots,y_h)\in\mathcal{Y}_{\rho,q}\}\mathrm{d}y_1\cdot\ldots\cdot\mathrm{d}y_h.$$

The integral on the right-hand side equals $C_{\rho,q}$ by its definition in (2.4), proving (4.22). □

As the bounds in Propositions 4.2 and 4.3 are not stated for $\rho \in (\theta_q, 1)$, but for r close to ρ, we analyze the scaling of $C_{\rho,q}(h)$.

Lemma 4.6 (Limit of the constant). *Consider an inhomogeneous scale-free random graph as in Definition 1.2. Let $\rho \in (\theta_q, 1)$, and set $h = \lceil \text{hubs}(\rho, q)\rceil$. Then $\rho \mapsto C_{\rho,q}(h)$ is strictly decreasing. Moreover,*

$$C_{\rho,q}(h) - \lim_{r\downarrow\rho} C_{r,q}(h) = \begin{cases} 0, & \text{if } \text{hubs}(\rho,q) \notin \mathbb{N} \text{ or } q = 1 \text{ or } \sigma < 0, \\ \underline{w}^{\alpha\sigma}/h!, & \text{if } \text{hubs}(\rho,q) \in \mathbb{N} \text{ and } q < 1 \text{ and } \sigma \geq 0, \end{cases} \tag{4.26}$$

and

$$\lim_{r\uparrow\rho} C_{r,q}(h) = C_{\rho,q}(h). \tag{4.27}$$

Proof. We first compute the limit of $(h!/\alpha^h)\left(C_{\rho,q}(h) - \lim_{r\downarrow\rho} C_{r,q}(h)\right)$. By definition of $C_{\rho,q}$ in (2.4), this corresponds to evaluating

$$\lim_{r\downarrow\rho}\int_{y_1=0}^{\infty}\cdots\int_{y_h=0}^{\infty}\mathbb{1}\{(y_1,\dots,y_h)\in\mathcal{Y}_{\rho,q}\setminus\mathcal{Y}_{r,q}\}\cdot(y_1\cdot\ldots\cdot y_h)^{-(\alpha+1)}\mathrm{d}y_1\cdot\ldots\cdot\mathrm{d}y_h. \tag{4.28}$$

Assume $r \in (\rho, 1)$. If $\mathbf{y} \in \mathcal{Y}_{\rho,q} \setminus \mathcal{Y}_{r,q}$, then by definition of $\mathcal{Y}_{\rho,q}$ in (2.3),

$$1 - r < \mathbb{E}\left[\prod_{x \in T_q, i \in [h]} \left(1 - q(W_x^\sigma y_i \wedge 1)\right)\right] \leq 1 - \rho.$$

Thus the indicator function in (4.28) is monotone in r. By the Monotone Convergence Theorem, the integral in (4.28) corresponds to

$$\int_{y_1=0}^{\infty} \cdots \int_{y_h=0}^{\infty} \mathbb{1}\left\{\mathbf{y} : \mathbb{E}\left[\prod_{x \in T_q, i \in [h]} \left(1 - q(W_x^\sigma y_i \wedge 1)\right)\right] = 1 - \rho\right\} \cdot (y_1 \cdot \ldots \cdot y_h)^{-(\alpha+1)} \mathrm{d}y_1 \cdot \ldots \cdot \mathrm{d}y_h. \tag{4.29}$$

Given $y_1, \ldots, y_{h-1}$, we compute the integral over y_h. To do so, we compare the function

$$f_h(y_h) := \mathbb{E}\left[\prod_{x \in T_q, i \in [h]} \left(1 - q(W_x^\sigma y_i \wedge 1)\right)\right]$$

to $1 - \rho$. The function $f_h(y_h)$ is non-increasing for $y_h \geq 0$. We first rule out that $f_h(0) = 1 - \rho$: regardless of $y_1, \ldots, y_{h-1}$,

$$f_h(0) \geq \mathbb{E}\left[(1-q)^{|T_q|(\lceil \mathrm{hubs}(\rho,q) \rceil - 1)}\right] > \mathbb{E}\left[(1-q)^{|T_q| \mathrm{hubs}(\rho,q)}\right] = 1 - \rho, \tag{4.30}$$

where the equality follows from the definition of hubs in (2.2).

Next, we argue that $f_h(y_h)$ is *strictly* decreasing for $y_h \in (0, \underline{w}^{-\sigma})$ when $\sigma \geq 0$ and for all $y_h \in (0, \infty)$ when $\sigma < 0$. Fix y_h', y_h in this σ-dependent domain with $y_h < y_h'$. If $\sigma \geq 0$, there exists $w_h > \underline{w}$ small enough such that $w_h^\sigma y_h' < 1$, so that $w^\sigma y_h < w^\sigma y_h' < 1$ for all $w \in [\underline{w}, w_h]$. If $\sigma < 0$, there exists $w_h > \underline{w}$ such that $w^\sigma y_h < w^\sigma y_h' < 1$ for all $w \geq w_h$. Let $I = [\underline{w}, w_h]$ if $\sigma \geq 0$, and $I = [w_h, 2w_h)$ if $\sigma < 0$. We bound the difference by only considering the event that the root has no offspring and weight in the interval I. That is,

$$\begin{aligned} f_h(y_h) - f_h(y_h') &= \mathbb{E}\left[\prod_{x \in T_q, i \in [h]} \left(1 - q(W_x^\sigma y_i \wedge 1)\right) - \prod_{x \in T_q, i \in [h]} \left(1 - q(W_x^\sigma y_i' \wedge 1)\right)\right] \\ &\geq q\left(y_h' - y_h\right) \mathbb{E}\left[\mathbb{1}_{\{|T_q|=1, W_\emptyset \in I\}} W_\emptyset^\sigma \prod_{i \in [h-1]} \left(1 - q(W_\emptyset^\sigma y_i \wedge 1)\right)\right] \\ &\geq q\left(y_h' - y_h\right) \cdot (1-q)^{h-1} \cdot \inf\{z^\sigma, z \in I\} \cdot \mathbb{P}\left(|T_q| = 1, W_\emptyset \in I\right). \end{aligned}$$

If $q < 1$, $(1-q)^{h-1}$ is strictly positive for any $h \geq 1$. If $q = 1$, then $h = \mathrm{hubs}(\rho, q) = 1$, and $(1-q)^{h-1} = 1$. The interval I is bounded, so the infimum is positive. So the right-hand side is positive for any q and σ. The event that the root of the branching process has weight $W_\emptyset \in I$ and no offspring has strictly positive probability. Therefore, $f_h(y_h)$ is strictly decreasing for $y_h \in (0, \underline{w}^{-\sigma})$ when $\sigma \geq 0$ and for all $y_h \in (0, \infty)$ when $\sigma < 0$.

When $\sigma < 0$, this implies that the indicator in (4.29) evaluates to one in at most a single point, and hence the integral vanishes. This proves (4.26) when $\sigma < 0$.

Assume next that $\sigma \geq 0$. By strict monotonicity, the indicator in the integral in (4.29) over y_h restricted to $[0, \underline{w}^{-\sigma})$ evaluates to one in at most a single point y_h^*, which is strictly positive by (4.30). Hence, the integral vanishes for $y < \underline{w}^{-\sigma}$. Thus, the integral in (4.29) is equal to

$$\int_{y_1=0}^{\infty} \cdots \int_{y_h=\underline{w}^{-\sigma}}^{\infty} \mathbb{1}\left\{\mathbf{y} : f_h(y_h) = 1 - \rho\right\} \cdot (y_1 \cdot \ldots \cdot y_h)^{-(\alpha+1)} \mathrm{d}y_1 \cdot \ldots \cdot \mathrm{d}y_h. \tag{4.31}$$

For $y_h \geq \underline{w}^{-\sigma}$, the function $f_h(y_h)$ is constant: each vertex $x \in T_q$ has weight at least $\underline{w}$ by Definition 1.3, so for all $y_h \geq \underline{w}^{-\sigma}$,

$$f_h(y_h) = \mathbb{E}\left[(1-q)^{|T_q|} \prod_{i\in[h-1], x\in T_q} \left(1 - q(W_x^\sigma y_i \wedge 1)\right) \right],$$

which only depends on $y_1, \dots, y_{h-1}$. Thus, also the indicator in (4.31) is a constant. Integrating over $y_h \geq \underline{w}^{-\sigma}$ yields that we have to evaluate

$$\frac{\underline{w}^{\sigma\alpha}}{\alpha} \int_{y_1=0}^{\infty} \cdots \int_{y_{h-1}=0}^{\infty} \mathbb{1}\{\mathbf{y} : \mathbb{E}[(1-q)^{|T_q|} \prod_{i\in[h-1], x\in T_q} \left(1 - q(W_x^\sigma y_i \wedge 1)\right)] = 1-\rho\} \cdot (y_1 \cdot \ldots \cdot y_{h-1})^{-(\alpha+1)} \mathrm{d}y_1 \cdot \ldots \cdot \mathrm{d}y_{h-1}. \quad (4.32)$$

Now we iterate the reasoning from (4.29) until (4.32) to compute the integral over the variables $y_{h-1}, \dots, y_1$. Thus, the limit in (4.28), corresponding to $(h!/\alpha^h)\left(C_{\rho,q}(h) - \lim_{r\downarrow\rho} C_{r,q}(h)\right)$, is equal to

$$\left(\frac{\underline{w}^{\sigma\alpha}}{\alpha}\right)^h \mathbb{1}\left\{\mathbb{E}\left[(1-q)^{|T_q|h}\right] = 1-\rho\right\}.$$

By definition of hubs(ρ, q) in (2.2), the indicator is one precisely when hubs$(\rho, q) \in \mathbb{N}$ and $q < 1$. This proves the limit in (4.26) for $\sigma \geq 0$.

We next prove that $\lim_{r\uparrow\rho} C_{r,q}(h) - C_{\rho,q}(h) = 0$. Similar to (4.28), this corresponds to evaluating

$$\lim_{r\uparrow\rho} \frac{\alpha^h}{h!} \int_{y_1=0}^{\infty} \cdots \int_{y_h=0}^{\infty} \mathbb{1}\{(y_1, \dots, y_h) \in \mathcal{Y}_{r,q} \setminus \mathcal{Y}_{\rho,q}\} \cdot (y_1 \cdot \ldots \cdot y_h)^{-(\alpha+1)} \mathrm{d}y_1 \cdot \ldots \cdot \mathrm{d}y_h. \quad (4.33)$$

Assume $r \in (\theta_q, \rho)$. If $\mathbf{y} \in \mathcal{Y}_{r,q} \setminus \mathcal{Y}_{\rho,q}$, then by definition of $\mathcal{Y}_{\rho,q}$ in (2.3),

$$1 - \rho < \mathbb{E}\left[\prod_{x\in T_q, i\in[h]} \left(1 - q(W_x^\sigma y_i \wedge 1)\right) \right] \leq 1 - r.$$

Thus, the indicator function in (4.33) is monotone in r, and converges to 0 for all y. This proves the limit in (4.27).

The statement that $\rho \mapsto C_{\rho,q}$ is strictly decreasing follows from the fact that the integral in (4.28) is strictly positive for every $r > \rho$. □

4.2 | Proofs of the Main Results

Given the above lemmas, we can prove the main results of the paper.

Proof of Theorem 2.1. We start with the upper bound. Let $\delta > 0$ be a small constant such that $\lceil \text{hubs}(\rho - \delta, q)\rceil = \lceil \text{hubs}(\rho, q)\rceil =: h$, which is possible by the continuity of hubs in (2.2), see also Lemma 2.7ii. Fix $\phi > 0$ smaller than ϕ_0 in Proposition 4.2 and ϕ_0 from Lemma 4.5. Recall that, assuming $|\mathcal{V}_n[\phi n, \infty)| = h$, we write $\mathcal{W}_n[\phi n, \infty) \in n \cdot \mathcal{Y}_{\rho,q}(h)$ if $(w_v/n)_{v\in\mathcal{V}_n[\phi n,\infty)} \in \mathcal{Y}_{\rho,q}(h)$.

We distinguish two cases for the set of the weights of the vertices of at least ϕn and apply Proposition 4.2 and Lemma 4.5:

$$\begin{aligned}
\mathbb{P}\left(|\mathcal{C}_n^{(1)}| > \rho n\right) &\leq \mathbb{P}\left(\{|\mathcal{C}_n^{(1)}| > \rho n\} \cap \neg\{|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in n \cdot \mathcal{Y}_{\rho-\delta,q}(h)\}\right) \\
&\quad + \mathbb{P}\left(|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in n \cdot \mathcal{Y}_{\rho-\delta,q}(h)\right) \\
&= O\left((n\mathbb{P}(W > n))^{h+1}\right) + (1 + o(1))C_{\rho-\delta,q}(h) \cdot (n\mathbb{P}(W > n))^h \sim C_{\rho-\delta,q}(h) \cdot (n\mathbb{P}(W > n))^h.
\end{aligned}$$

Since $\delta > 0$ was arbitrary, it follows by (4.27) in Lemma 4.6 that

$$\mathbb{P}\left(|\mathcal{C}_n^{(1)}| > \rho n\right) \lesssim C_{\rho,q}(h) \cdot (n\mathbb{P}(W > n))^{\lceil \text{hubs}(\rho,q)\rceil},$$

proving the upper bound of both (2.5) and (2.6). For the lower bound we invoke Proposition 4.3 and Lemma 4.5 to obtain that

$$\begin{aligned}&\mathbb{P}\left(|\mathcal{C}_n^{(1)}| > \rho n\right)\\ &\geq \mathbb{P}\left(\{|\mathcal{C}_n^{(1)}| > \rho n\} \cap \{|\mathcal{V}_n[\phi n, \infty)| = \lceil \mathrm{hubs}(\rho+\delta, q)\rceil, \mathcal{W}_n[\phi n, \infty) \ \in \ n \cdot \mathcal{Y}_{\rho+\delta,q}(\lceil \mathrm{hubs}(\rho+\delta,q)\rceil)\}\right)\\ &\quad \sim C_{\rho+\delta,q}(\lceil \mathrm{hubs}(\rho+\delta,q)\rceil)\cdot (n\mathbb{P}(W>n))^{\lceil \mathrm{hubs}(\rho+\delta,q)\rceil}.\end{aligned}$$

If $\mathrm{hubs}(\rho, q) \notin \mathbb{N}$ or $q = 1$, it follows that $\lceil \mathrm{hubs}(\rho, q)\rceil = \lceil \mathrm{hubs}(\rho-\delta, q)\rceil = \lceil \mathrm{hubs}(\rho+\delta, q)\rceil$ for any $\delta > 0$ sufficiently small. Hence, (2.5) follows by sending $\delta \to 0$ and invoking (4.26) in Lemma 4.6. If $\mathrm{hubs}(\rho, q) \in \mathbb{N}$, then $\lceil \mathrm{hubs}(\rho+\delta, q)\rceil = \mathrm{hubs}(\rho, q) + 1$ for any δ sufficiently small and (2.6) follows. □

Next, we formally prove also the upper bound for the lower tail as stated in Theorem 2.2.

Proof of Theorem 2.2. The bound immediately follows from Lemma 3.4, since the largest component in $\mathcal{G}_n$ has at least the size of the largest component in an induced subgraph $\mathcal{G}_n[\underline{w}, \overline{w})$. □

Next, we derive the large deviation principle in Corollary 2.3.

Proof of Corollary 2.3. We start with the upper bound. If $\theta_q \in \overline{B}$, then by (2.2),

$$\inf_{\rho\in\overline{B}} I_q(\rho) = (\alpha-1)\lceil \mathrm{hubs}(\theta_q, q)\rceil = 0,$$

and the upper bound is trivial. Assume $\theta_q \notin \overline{B}$, and assume B is such that $b_- := \max_{x<\theta_q}\{x \in \overline{B}\}$ and $b_+ := \min_{x>\theta_q}\{x \in \overline{B}\}$ exist. Then, for any $\varepsilon > 0$,

$$\mathbb{P}\left(|\mathcal{C}_n^{(1)}|/n \in B\right) \leq \mathbb{P}\left(|\mathcal{C}_n^{(1)}|/n \leq b_-\right) + \mathbb{P}\left(|\mathcal{C}_n^{(1)}|/n > b_+ - \varepsilon\right). \tag{4.34}$$

Since b_- and b_+ are strictly smaller (resp. larger) than θ_q, the first term decays exponentially in n by Theorem 2.2. If $b_+ < 1$, the second term is regularly varying with index $I_q(b_+)$ by Theorem 2.1 if ε is sufficiently small so that $\lceil \mathrm{hubs}(b_+, q)\rceil$ and $\lceil \mathrm{hubs}(b_+ - \varepsilon, q)\rceil$ agree, which is possible since $\rho \mapsto \mathrm{hubs}(\rho, q)$ is continuous and increasing. Thus, the second term dominates the right-hand side in (4.34) as $n \to \infty$, proving the upper bound if b_- and b_+ both exist and $b_+ < 1$. If $b_+ \geq 1$, we use that $\lim_{\rho\uparrow 1} \mathrm{hubs}(\rho, q) = \infty$ by Lemma 2.7, and the result follows by Theorem 2.1 and taking ε arbitrarily small. If b_- does not exist, only the second term on the right-hand side remains and the upper bound follows by the same reasoning. If b_+ does not exist, only the first term remains, which decays exponentially in n, so its logarithm tends to $-\infty$ much faster than $\log n$.

We turn to the lower bound. If $\theta_q \in B^\circ$, the proof is again trivial. Assume $\theta_q \notin B^\circ$. Let $b_+ := \inf_{x>\theta_q}\{x \in B^\circ\}$ as before. If b_+ does not exist or is at least 1, in which case $I_q(\rho) = \infty$ for all $\rho \in B^\circ$, the lower bound is trivial. Assume that $b_+ < 1$. Since B° is an open set, there exists $\varepsilon > 0$ such that $(b_+ + \varepsilon, b_+ + 2\varepsilon] \in B^\circ \cap (\rho, 1)$. Then,

$$\mathbb{P}\left(|\mathcal{C}_n^{(1)}|/n \in B\right) \geq \mathbb{P}\left(|\mathcal{C}_n^{(1)}|/n > b_+ + \varepsilon\right) - \mathbb{P}\left(|\mathcal{C}_n^{(1)}|/n > b_+ + 2\varepsilon\right).$$

By continuity and monotonicity of hubs, we may assume that ε is so small that

$$\lceil \mathrm{hubs}(b_+ + \varepsilon, q)\rceil = \lceil \mathrm{hubs}(b_+ + 2\varepsilon, q)\rceil = \inf_{\rho\in B^\circ} \lceil \mathrm{hubs}(\rho, q)\rceil,$$

and that for $b \in \{b_+ + \varepsilon, b_+ + 2\varepsilon\}$, $\mathrm{hubs}(b, q) \notin \mathbb{N}$. By Theorem 2.1,

$$\mathbb{P}\left(|\mathcal{C}_n^{(1)}|/n \in B\right) \geq (1+o(1))\Big(C_{b_+ + \varepsilon, q} - C_{b_+ + 2\varepsilon, q}\Big)(n\mathbb{P}(W>n))^{\inf_{\rho\in B^\circ}\lceil \mathrm{hubs}(\rho,q)\rceil}.$$

The constant factor is positive since $C_{b_+ + \varepsilon, q} > C_{b_+ + 2\varepsilon, q}$ by Lemma 4.6. The probability on the right-hand side is regularly varying with index $-\alpha$. Thus, the lower bound follows. □

5 | The Graph Conditional on a Large Giant

We prove the remaining corollaries from Section 2. We first start with a more general lemma.

Lemma 5.1. *Consider an inhomogeneous scale-free random graph as in Definition 1.2. Let $\rho \in (\theta_q, 1)$, set $h = \lceil \mathrm{hubs}(\rho, q) \rceil$, and assume $\mathrm{hubs}(\rho, q) \notin \mathbb{N}$ or $q = 1$. Then there exists a constant $\phi > 0$ such that, as $n \to \infty$,*

$$\mathbb{P}\big(|\mathcal{C}_n^{(1)}| > \rho n\big) \sim \mathbb{P}\left(|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in n \cdot \mathcal{Y}_{\rho,q}(h)\right). \tag{5.1}$$

Moreover, for a sequence $(\mathcal{A}_n)_{n\geq 1}$ of events,

$$\mathbb{P}\big(\mathcal{A}_n \big| |\mathcal{C}_n^{(1)}| > \rho n\big) = \mathbb{P}\left(\mathcal{A}_n \big| |\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in \mathcal{Y}_{\rho,q}\right) + o(1).$$

Proof. The first statement follows by reasoning analogous to the beginning of the proof of Theorem 2.1above: we combine Propositions 4.2 and 4.3, Lemmas 4.5 and 4.6, and take the limit $r \to \rho$. We prove the second statement. We start with a lower bound. Let $\delta > 0$ be an arbitrarily small constant. We write out the conditional expectation and apply Lemma 5.1 to obtain for some small constant $\phi > 0$,

$$\begin{aligned}\mathbb{P}\big(\mathcal{A}_n \big| |\mathcal{C}_n^{(1)}| > \rho n\big) &\gtrsim \frac{\mathbb{P}\left(\mathcal{A}_n \cap \{|\mathcal{C}_n^{(1)}| > \rho n\} \cap \{|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in \mathcal{Y}_{\rho+\delta,q}\}\right)}{\mathbb{P}\left(|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in \mathcal{Y}_{\rho,q}\right)} \\ &\geq \frac{\mathbb{P}\left(\mathcal{A}_n \cap \{|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in \mathcal{Y}_{\rho+\delta,q}\}\right)}{\mathbb{P}\left(|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in \mathcal{Y}_{\rho,q}\right)} \\ &\quad - \frac{\mathbb{P}\left(|\mathcal{C}_n^{(1)}| \leq \rho n\} \cap \{|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in \mathcal{Y}_{\rho+\delta,q}\}\right)}{\mathbb{P}\left(|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in \mathcal{Y}_{\rho,q}\right)}.\end{aligned}$$

The second term vanishes as $n \to \infty$ by Proposition 4.3 and Lemma 4.5. For the first term we use that $\mathcal{Y}_{\rho+\delta,q} \subseteq \mathcal{Y}_{\rho,q}$ by its definition in (2.3). Thus,

$$\begin{aligned}\mathbb{P}\big(\mathcal{A}_n \big| |\mathcal{C}_n^{(1)}| > \rho n\big) &\geq \frac{\mathbb{P}\left(\mathcal{A}_n \cap \{|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in \mathcal{Y}_{\rho,q}\}\right)}{\mathbb{P}\left(|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in \mathcal{Y}_{\rho,q}\right)} \\ &\quad - \frac{\mathbb{P}\left(|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in \mathcal{Y}_{\rho,q} \setminus \mathcal{Y}_{\rho+\delta,q}\right)}{\mathbb{P}\left(|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in \mathcal{Y}_{\rho,q}\right)} - o(1).\end{aligned}$$

The lower bound follows by rewriting the term on the first line as a conditional probability, and applying Lemma 4.5 and Lemma 4.6 to the first term on the second line which vanishes as $\delta \to 0$ under the assumption that $\mathrm{hubs}(\rho, q) \notin \mathbb{N}$.

For the upper bound we argue similarly. Writing out the conditional expectation and distinguishing whether the hubs have weight in the set $\mathcal{Y}_{\rho-\delta,q}$ yields

$$\begin{aligned}\mathbb{P}\big(\mathcal{A}_n \big| |\mathcal{C}_n^{(1)}| > \rho n\big) &\leq \frac{\mathbb{P}\left(\mathcal{A}_n \cap \{|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in \mathcal{Y}_{\rho-\delta,q}\}\right)}{\mathbb{P}\left(|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in \mathcal{Y}_{\rho,q}\right)} \\ &\quad + \frac{\mathbb{P}\left(\{|\mathcal{C}_n^{(1)}| > \rho n\} \cap \neg\{|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in \mathcal{Y}_{\rho-\delta,q}\}\right)}{\mathbb{P}\left(|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in \mathcal{Y}_{\rho,q}\right)}.\end{aligned}$$

If δ is sufficiently small, then $\lceil \mathrm{hubs}(\rho - \delta, q) \rceil = \lceil \mathrm{hubs}(\rho, q) \rceil$. By Lemma 4.5 and Proposition 4.2 the second term is of order $O(n\mathbb{P}(W > n)) = o(1)$. For the numerator in the first term we also distinguish whether the hubs are in $\mathcal{Y}_{\rho,q} \subseteq \mathcal{Y}_{\rho-\delta,q}$. This yields

$$\begin{aligned}\mathbb{P}\big(\mathcal{A}_n \big| |\mathcal{C}_n^{(1)}| > \rho n\big) &\leq \frac{\mathbb{P}\left(\mathcal{A}_n \cap \{|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in \mathcal{Y}_{\rho,q}\}\right)}{\mathbb{P}\left(|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in \mathcal{Y}_{\rho,q}\right)} \\ &\quad + \frac{\mathbb{P}\left(|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in \mathcal{Y}_{\rho-\delta,q} \setminus \mathcal{Y}_{\rho,q}\right)}{\mathbb{P}\left(|\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in \mathcal{Y}_{\rho,q}\right)} + o(1).\end{aligned}$$

We apply Lemmas 4.5 and 4.6 to the first term on the second line which vanishes as $\delta \to 0$ under the assumption that $\mathrm{hubs}(\rho, q) \notin \mathbb{N}$. The term on the first line can be rewritten as a conditional probability. This finishes the proof. □

Proof of Corollary 2.5. Let $h := \lceil \mathrm{hubs}(\rho, q) \rceil$. We first show that with high probability the number of vertices with weight at least $n^{1-\epsilon}$ is equal to h conditional on $\{|\mathcal{C}_n^{(1)}| > \rho n\}$. By Lemma 5.1,

$$\begin{aligned}
&\mathbb{P}\Big(\Big|\mathcal{V}_n[n^{1-\epsilon}, \infty)\Big| \neq h \;\Big|\; |\mathcal{C}_n^{(1)}| > \rho n\Big) \\
&\qquad = \mathbb{P}\Big(\Big|\mathcal{V}_n[n^{1-\epsilon}, \infty)\Big| \neq h \;\Big|\; |\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in \mathcal{Y}_{\rho,q}(h)\Big) + o(1) \\
&\qquad = \mathbb{P}\Big(\Big|\mathcal{V}_n[n^{1-\epsilon}, \infty)\Big| > h \;\Big|\; |\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in \mathcal{Y}_{\rho,q}(h)\Big) + o(1).
\end{aligned}$$

If the number of vertices with weight at least ϕn is exactly h, and there are more than h vertices with weight at least $n^{1-\epsilon}$, we have

$$\begin{aligned}
&\mathbb{P}\Big(\Big|\mathcal{V}_n[n^{1-\epsilon}, \infty)\Big| \neq h \;\Big|\; |\mathcal{C}_n^{(1)}| > \rho n\Big) \\
&\qquad = \mathbb{P}\Big(\Big|\mathcal{V}_n[n^{1-\epsilon}, \phi n)\Big| \geq 1 \;\Big|\; |\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in \mathcal{Y}_{\rho,q}(h)\Big) + o(1).
\end{aligned}$$

The weights of the $n-h$ vertices in $\mathcal{V}_n[\underline{w}, \phi n)$ are independent and conditioned to be at most ϕn. By Markov's inequality, and the definition of F_W in (1.2),

$$\begin{aligned}
&\mathbb{P}\Big(\Big|\mathcal{V}_n[n^{1-\epsilon}, \infty)\Big| \neq h \;\Big|\; |\mathcal{C}_n^{(1)}| > \rho n\Big) \\
&\qquad \leq n\frac{\mathbb{P}(W \in [n^{1-\epsilon}, \phi n))}{\mathbb{P}(W < \phi n)} + o(1) \\
&\qquad \leq (1+o(1))n\mathbb{P}\big(W \geq n^{1-\epsilon}\big) + o(1) \\
&\qquad \leq (1+o(1))L(n^{1-\epsilon})n^{1-(1-\epsilon)\alpha} + o(1).
\end{aligned} \tag{5.2}$$

As $\alpha > 1$ by assumption, the right-hand side tends to zero when $\epsilon \in (0, 1-1/\alpha)$. Thus, on the event $\{|\mathcal{C}_n^{(1)}| > \rho n\}$, with high probability, there are exactly h vertices with linear weight. We next analyze their joint distribution.

Let $W_{(1)} \geq \cdots \geq W_{(h)}$ denote the order statistics of the largest h weights. Let $(Y_i)_{i \leq h}$ be iid copies of a Pareto random variable Y that has distribution $\mathbb{P}(Y \geq y) = (\phi/y)^\alpha$ for $y \geq \phi$, and let $Y_{(1)} \geq Y_{(2)} \geq \cdots$ denote their order statistics. We have to show that, conditional on $\{|\mathcal{C}_n^{(1)}| > \rho n\}$, for any Borel set $\mathcal{B} \in (0, \infty)^h$

$$\mathbb{P}\Big((W_{(i)}/n)_{i \leq h} \in \mathcal{B} \;\Big|\; |\mathcal{C}_n^{(1)}| > \rho n\Big) \xrightarrow{n \to \infty} \mathbb{P}\Big((Y_{(i)})_{i \leq h} \in \mathcal{B} \;\Big|\; (Y_{(i)})_{i \leq h} \in \mathcal{Y}_{\rho,q}(h)\Big).$$

By Lemma 5.1, we can change the conditioning on $\{|\mathcal{C}_n^{(1)}| > \rho n\}$ on the left-hand side. That is, we have to show that for any Borel set $\mathcal{B} \subseteq (0, \infty)^\ell$,

$$\begin{aligned}
&\mathbb{P}\Big((W_{(i)}/n)_{i \leq h} \in \mathcal{B} \;\Big|\; |\mathcal{V}_n[\phi n, \infty)| = h, \mathcal{W}_n[\phi n, \infty) \in \mathcal{Y}_{\rho,q}(h)\Big) \\
&\qquad \xrightarrow{n \to \infty} \mathbb{P}\Big((Y_{(i)})_{i \leq h} \in \mathcal{B} \;\Big|\; (Y_{(i)})_{i \leq h} \in \mathcal{Y}_{\rho,q}(h)\Big).
\end{aligned} \tag{5.3}$$

Now we use that all weights of vertices with weight at least ϕn are independent with distribution F_W from (1.2). So for $y \geq \phi$,

$$\mathbb{P}\big(W_u \geq y \cdot \phi n | W_u \geq \phi n\big) = \frac{L(x\phi n)}{L(\phi n)}(y/\phi)^{-\alpha} \sim (\phi/y)^\alpha = \mathbb{P}\big(Y_1 \geq y\big).$$

Thus, the ordered collection $(W_{(i)}/n)_{i \leq h}$ conditional on a large giant, has asymptotically the same distribution as $(Y_{(i)})_{i \leq h}$ conditional on being in $\mathcal{Y}_{\rho,q}$. This finishes the proof. □

Proof of Corollary 2.6. Recall that R^∞ is the space of sequences of real numbers, metrized by $d_\infty(\mathbf{x}, \mathbf{y}) = \sum_i (|x_i - y_i| \wedge 1)2^{-i}$. By (Billingsley 2013, example 1.2) each probability measure on R^∞ is tight, and hence it is also relatively compact (Billingsley 2013, Theorem 3.1 by Theorem 5.1). Thus, it satisfies by (Billingsley 2013, Theorem 2.6) to show convergence of the finite-dimensional distributions. Fix $\ell_* \in \mathbb{N}$, and recall the functions g_ℓ from (2.9). Let $(x_\ell)_{\ell \leq \ell_*} \in$

$[0,\infty)^{\ell_*}$ be a continuity point of the distribution of $\left(g_\ell\left((\phi Y_i)_{i\le\lceil \mathrm{hubs}(\rho,q)\rceil}\right), \ell\in[\ell_*]\right)$ conditionally on $\left(\phi Y_i\right)_{i\le\lceil \mathrm{hubs}(\rho,q)\rceil}\in \mathcal{Y}_{\rho,q}$. Abbreviate $h=\lceil \mathrm{hubs}(\rho,q)\rceil$. By Lemma 5.1,

$$\begin{aligned}
&\mathbb{P}(\forall \ell\le \ell_* : N_{n,\ell}/n\le x_\ell \;\big|\; |C_n^{(1)}|>\rho n)\\
&\qquad = \mathbb{P}(\forall \ell\le \ell_* : N_{n,\ell}/n\le x_\ell \;\big|\; |\mathcal{V}_n[\phi n,\infty)|=h, \mathcal{W}_n[\phi n,\infty)\in \mathcal{Y}_{\rho,q}(h))+o(1).
\end{aligned}\tag{5.4}$$

We condition on $\mathcal{W}_n[\phi n,\infty)=n\mathbf{y}^{(h)}$ for some $\mathbf{y}^{(h)}\in\mathcal{Y}_{\rho,q}$ and establish a lower and upper bound on the probability on the right-hand side. We start with a lower bound. Let $\delta>0$ be an arbitrary small constant. Then,

$$\begin{aligned}
&\mathbb{P}\left(\forall \ell\le \ell_* : N_{n,\ell}/n\le x_\ell \;\Big|\; \mathcal{W}_n[\phi n,\infty)=n\mathbf{y}^{(h)}\right)\\
&\ge \mathbb{P}\left(\forall \ell\le \ell_* : N_{n,\ell}/n\le x_\ell, \left|N_{n,\ell}/n-\frac{1}{\ell}\mathbb{E}\left[\mathbb{1}_{\{|T_q|=\ell\}}\overline{P}\left((W_x)_{x\in T_q},\mathbf{y}^{(h)}\right)\right]\right|\le\delta \;\Big|\; \mathcal{W}_n[\phi n,\infty)=n\mathbf{y}^{(h)}\right)\\
&\ge \mathbb{P}\left(\forall \ell\le \ell_* : \frac{1}{\ell}\mathbb{E}\left[\mathbb{1}_{\{|T_q|=\ell\}}\overline{P}\left((W_x)_{x\in T_q},\mathbf{y}^{(h)}\right)\right]\le x_\ell-\delta \;\Big|\; \mathcal{W}_n[\phi n,\infty)=n\mathbf{y}^{(h)}\right)\\
&\quad -\mathbb{P}\left(\exists \ell\le \ell_* : \left|N_{n,\ell}/n-\frac{1}{\ell}\mathbb{E}\left[\mathbb{1}_{\{|T_q|=\ell\}}\overline{P}\left((W_x)_{x\in T_q},\mathbf{y}^{(h)}\right)\right]\right|>\delta \;\Big|\; \mathcal{W}_n[\phi n,\infty)=n\mathbf{y}^{(h)}\right).
\end{aligned}$$

The negative term tends to 0 by Lemma 4.4. So,

$$\begin{aligned}
&\mathbb{P}\left(\forall \ell\le \ell_* : N_{n,\ell}/n\le x_\ell \;\Big|\; |C_n^{(1)}|>\rho n\right)\\
&\ge \mathbb{P}\left(\forall \ell\le \ell_* : \frac{1}{\ell}\mathbb{E}\left[\mathbb{1}_{\{|T_q|=\ell\}}\overline{P}\left((W_x)_{x\in T_q},\mathbf{y}^{(h)}\right)\right]\le x_\ell-\delta \;\Big|\; |\mathcal{V}_n[\phi n,\infty)|=h, \mathcal{W}_n[\phi n,\infty)\in\mathcal{Y}_{\rho,q}(h)\right)-o(1).
\end{aligned}$$

Recall that by definition of $\overline{P}$ in (4.1), the expectation inside the probability corresponds to $g_\ell(\mathbf{y}^{(\ell)})$ defined in (2.9). By the weak convergence of the weights of the vertices in $\mathcal{V}_n[\phi n,\infty)$, see (5.3), we obtain

$$\begin{aligned}
&\mathbb{P}\left(\forall \ell\le \ell_* : N_{n,\ell}/n\le x_\ell \;\Big|\; |C_n^{(1)}|>\rho n\right)\\
&\ge \mathbb{P}\left(\forall \ell\le \ell_* : \frac{1}{\ell}\mathbb{E}\left[\mathbb{1}_{\{|T_q|=\ell\}}\overline{P}\left((W_x)_{x\in T_q},(\phi Y_i)_{i\le h}\right)\right]\le x_\ell-\delta \;\Big|\; (Y_i)_{i\le h}\in\mathcal{Y}_{\rho,q}(h)\right)-o(1),
\end{aligned}$$

where $(Y_i)_{i\ge1}$ are independent copies of Y, with distribution $\mathbb{P}(Y\ge y)=(\phi/y)^\alpha$ for $y\ge\phi$. Recall that $(x_\ell)_{\ell\le\ell_*}$ is a continuity point of the (finite-dimensional version of) the limiting distribution in Corollary 2.6. Since $\delta>0$ was arbitrary, for each $\varepsilon>0$ there exists $\delta>0$ such that

$$\begin{aligned}
&\mathbb{P}\left(\forall \ell\le \ell_* : \frac{1}{\ell}\mathbb{E}\left[\mathbb{1}_{\{|T_q|=\ell\}}\overline{P}\left((W_x)_{x\in T_q},(Y_i)_{i\le\lceil \mathrm{hubs}(\rho,q)\rceil}\right)\right]\le x_\ell-\delta \;\Big|\; (Y_i)_{i\le h}\in\mathcal{Y}_{\rho,q}(h)\right)\\
&\ge \mathbb{P}\left(\forall \ell\le \ell_* : \frac{1}{\ell}\mathbb{E}\left[\mathbb{1}_{\{|T_q|=\ell\}}\overline{P}\left((W_x)_{x\in T_q},(Y_i)_{i\le h}\right)\right]\le x_\ell \;\Big|\; (Y_i)_{i\le h}\in\mathcal{Y}_{\rho,q}(h)\right)-\varepsilon/2.
\end{aligned}$$

Thus, for any $\varepsilon>0$, when n is sufficiently large

$$\begin{aligned}
&\mathbb{P}\left(\forall \ell\le \ell_* : N_{n,\ell}/n\le x_\ell \;\Big|\; |C_n^{(1)}|>\rho n\right)\\
&\ge \mathbb{P}\left(\forall \ell\le \ell_* : \frac{1}{\ell}\mathbb{E}\left[\mathbb{1}_{\{|T_q|=\ell\}}\overline{P}\left((W_x)_{x\in T_q},(Y_i)_{i\le h}\right)\right]\le x_\ell \;\Big|\; (Y_i)_{i\le h}\in\mathcal{Y}_{\rho,q}(h)\right)-\varepsilon.
\end{aligned}$$

We leave it to the reader to prove an upper bound (almost analogously), so that weak convergence of the finite-dimensional distribution follows. By the reasoning above (5.4), this suffices for the proof of the conditional component-size distribution in Corollary 2.6. The proof of the conditional distribution of $|C_n^{(1)}|/n$ in (2.10) follows immediately from Theorem 2.1. □

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Appendix A

The Graph on Constant-Weight Vertices

In this section we will construct an "approximating" inhomogeneous random graph ((δ, R)-IRG) with discretized weights of at most a constant R. We denote this graph by $\mathcal{G}_n^{(\delta,R)} = \big(\mathcal{V}_n^{(\delta,R)}, \mathcal{E}_n^{(\delta,R)}\big)$. In this graph, the number of vertices is slightly smaller than n, and the weights are no longer random. Relying on the large-deviation principle for component sizes in inhomogeneous random graphs (IRGs) with bounded kernel and given weights from (Andreis et al. 2023), we show that the number of components of ε-type $\mathbf{w}^{(\ell)}$ concentrates around its expectation in $\mathcal{G}_n^{(\delta,R)}$ for all ε-types simultaneously.

Then, on the event that the weights in $\mathcal{G}_n[\underline{w}, R)$ satisfy a good event, we construct a coupling between the (δ, R)-IRG with the original graph $\mathcal{G}_n[\underline{w}, R)$. We show that the number of components of any ε-type changes compared to the (δ, R)-IRG only by a small number with sufficiently high probability under this coupling, so that Lemmas 3.4 and 3.2 follow.

Definition A.1 (Approximating inhomogeneous random graph and branching process). Let $\delta, \varepsilon > 0$ be small constants such that $\varepsilon/\delta \in \mathbb{N}$, let R be a large constant, and $n \in \mathbb{N}$. Let the weight-distribution F_W, kernel κ_σ, and percolation parameter q be as in Definition 1.2, and recall that $\underline{w} = \inf\{w : F_W(w) > 0\} > 0$. Let $z_i = \underline{w} + i\delta$ for $i \in \mathbb{N}_0$. Set

$$f_W^{(\delta)}(z_i) := \mathbb{P}\big(z_i \le W < z_{i+1}\big), \qquad \underline{n}_i := \lceil (1-\delta) n f_W(z_i) \rceil .$$

For $i \in [(R - \underline{w})/\delta)]$, we fix $\underline{n}_i$ to be the number of vertices of weight exactly z_i in $\mathcal{V}_n^{(\delta,R)}$, and write

$$\underline{n} = \underline{n}(\delta, R) := \sum_{i \in [(R-\underline{w})/\delta)]} \underline{n}_i \tag{A1}$$

for the total number of vertices in $\mathcal{G}_n^{(\delta,R)}$. For a weight $w \in [\underline{w}, R)$, let $\underline{w}^{(\delta)} := \sup_i\{z_i : z_i \le w\}$, $\overline{w}^{(\delta)} := \inf_i\{z_i : z_i > w\}$. Define for two weights $w_1, w_2 \in [\underline{w}, \infty)$ the approximating kernel as

$$\kappa_\sigma^{(\delta,R)}(w_1, w_2) := \mathbb{1}_{\{w_1 \le R, w_2 \le R\}} \inf\Big\{ \kappa_\sigma(\tilde{w}_1, \tilde{w}_2) : \tilde{w}_1 \in [\underline{w}_1^{(\delta)}, \overline{w}_1^{(\delta)}], \tilde{w}_2 \in [\underline{w}_2^{(\delta)}, \overline{w}_2^{(\delta)}] \Big\}. \tag{A2}$$

Two vertices u, v of weight w_u, w_v are connected by an edge in $\mathcal{G}_n^{(\delta,R)} := \big(\mathcal{V}_n^{(\delta,R)}, \mathcal{E}_n^{(\delta,R)}\big)$ with probability

$$p_{uv}^{(\delta,R)} := q\left(\frac{\kappa_\sigma^{(\delta,R)}(w_u, w_v)}{n} \wedge 1\right)$$

independently of other vertex pairs. The approximating associated branching process is denoted by $\mathrm{BP}^{(\delta,R)}$. In this branching process, the root ø has type $W_ø$ where

$$\mathbb{P}\big(W_ø = z_i\big) = f_W^{(\delta)}(z_i) \Big/ F_W(R).$$

In each generation, each particle v of weight w_v gives, for each $i \in [(R - \underline{w})/\delta]$, independently birth to $\mathrm{Poi}\Big(q \cdot \kappa_\sigma^{(\delta,R)}(w_v, w) \cdot f_W^{(\delta)}(z_i)\Big)$ many particles of weight z_i. We denote the set of types in the total progeny by $T_q^{(\delta,R)}$

We write $\mathcal{C}_n^{(1),(\delta,R)}$ for the largest connected component in $\mathcal{G}_n^{(\delta,R)}$, $N_n^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon)$ for the number of components of ε-type $\mathbf{w}^{(\ell)}$ in the (δ, R)-IRG, $\theta_q^{(\delta,R)}$ for the survival probability of the associated branching process ((δ, R)-BP) of the (δ, R)-IRG, and $\theta_q^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon)$ for the probability that the (δ, R)-BP has size ℓ and ε-type $\mathbf{w}^{(\ell)}$.

We state a lemma following from a large-deviation principle by Theorem 3.1 by Andreis et al. (2023).

Lemma A.2 (Size-ℓ components in the (δ, R)-IRG). *Consider the approximated inhomogeneous random graph and its associated branching process for some $R > 0$, $\delta > 0$. Fix $\ell_* \in \mathbb{N}$ and $\varepsilon > 0$. For each $\psi > 0$, there exist constants $c_{\psi,1}, \delta_0 > 0$ such that for all $\delta \in (0, \delta_0)$ such that $\varepsilon/\delta \in \mathbb{N}$, and $n \ge 1$,*

$$\mathbb{P}\left(\sum_{\ell \le \ell_*} \sum_{\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon,R)} \left| \frac{\ell N_n^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon)}{n} - \theta^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon) \right| > \psi \right) \le \exp\left(-c_{\psi,1} n\right). \tag{A3}$$

Moreover, there exists $c_{\psi,2} > 0$ such that for all $n \ge 1$

$$\mathbb{P}\Big(\Big| |\mathcal{C}_n^{(1),(\delta,R)}|/n - \theta_q^{(\delta,R)} \Big| > \psi \Big) \le \exp\big(-c_{\psi,2} n\big). \tag{A4}$$

Proof. By Definition A.1, the graph $\mathcal{G}_n^{(\delta,R)}$ contains $\underline{n} \le n$ vertices. Then,

$$\begin{aligned}
&\left\{ \sum_{\ell \le \ell_*} \sum_{\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon,R)} \left| \frac{\ell N_n^{(\delta,R)} \mathbf{w}^{(\ell)}}{n} - \theta^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon) \right| > \psi \right\} \\
&\quad = \left\{ \sum_{\ell, \mathbf{w}^{(\ell)}} \left| \frac{\ell N_n^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon)}{\underline{n}} - \theta^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon) \frac{n}{\underline{n}} \right| > \psi \frac{n}{\underline{n}} \right\} \\
&\quad \subseteq \left\{ \sum_{\ell, (\mathbf{w}^{(\ell)}, \varepsilon)} \left(\left| \frac{\ell N_n^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon)}{\underline{n}} - \theta^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon) \right| + \theta^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon) |1 - n/\underline{n}| \right) > \psi \frac{n}{\underline{n}} \right\},
\end{aligned}$$

where in the second and third line we take the sums over the same sets as in the first line. By definition of $\underline{n}$ in (A1), $\underline{n} \ge (1-\delta) n \mathbb{P}(W \le R)$. Thus, we can take δ sufficiently small and R sufficiently large such that $(n/\underline{n} - 1) \le \psi/4$. Since the sum over the probabilities $\theta^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon)$ is at most one, we obtain for these values of δ and R that

$$\begin{aligned}
&\mathbb{P}\left(\sum_{\ell \le \ell_*} \sum_{\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon,R)} \left| \frac{\ell N_n^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon)}{n} - \theta^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon) \right| > \psi \right) \\
&\quad \le \sum_{\ell \le \ell_*} \sum_{\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon,R)} \mathbb{P}\left(\left| \frac{\ell N_n^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon)}{\underline{n}} - \theta^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon) \right| > \frac{\psi}{2 \sum_{\ell \le \ell_*} |\mathrm{CT}_\ell(\varepsilon, R)|} \right). \qquad \text{(A5)}
\end{aligned}$$

By the same argumentation,

$$\mathbb{P}\left(\left| |C_n^{(1),(\delta,R)}|/n - \theta_q^{(\delta,R)} \right| > \psi \right) \le \mathbb{P}\left(\left| |C_n^{(1),(\delta,R)}|/\underline{n} - \theta_q^{(\delta,R)} \right| > \psi/2 \right). \qquad \text{(A6)}$$

We now use the results from (Andreis et al. 2023), that (rephrased to our notation) derives among others a large-deviations principle (LDP) with speed $\underline{n}$ for the vector $\left(N_n^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon)/n \right)_{\ell \ge 1, \mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon,R)}$ and $|C_n^{(1),(\delta,R)}|/n$. In particular, Theorem 3.1 in (Andreis et al. 2023) discusses an LRP with finitely many vertex types, as is the case in $\mathcal{G}_n^{(\delta,R)}$. We present a corollary of this LDP and omit the full description as it would require significantly more notation. The LDP implies laws of large numbers with exponential convergence rate: there exist constants $(\eta(\mathbf{w}^{(\ell)}, \varepsilon))_{\ell \ge 1, \mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon,R)}$ such that for each $\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon, R)$ and any $\psi' > 0$,

$$\limsup_{\underline{n} \to \infty} \frac{1}{\underline{n}} \log \mathbb{P}\left(\left| \frac{N_n^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon)}{\underline{n}} - \eta(\mathbf{w}^{(\ell)}, \varepsilon) \right| > \psi' \right) < 0.$$

We choose $\psi' := \psi/(\ell \sum_{\ell \le \ell_*} |\mathrm{CT}_\ell(\varepsilon, R)|)$, and thus obtain that there exists $c' > 0$ such that for all n sufficiently large

$$\sum_{\ell \le \ell_*} \sum_{\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon,R)} \mathbb{P}\left(\left| \frac{\ell N_n^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon)}{\underline{n}} - \ell \eta(\mathbf{w}^{(\ell)}, \varepsilon) \right| > \frac{\psi}{2 \sum_{\ell \le \ell_*} |\mathrm{CT}_\ell(\varepsilon, R)|} \right) \le \exp(-c' n).$$

Recalling (A5), (A3) follows if we show that

$$\frac{\ell N_n^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon)}{\underline{n}} \xrightarrow{\mathbb{P}} \theta^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon), \qquad \text{as } \underline{n} \to \infty, \qquad \text{(A7)}$$

as this implies that $\ell \eta(\mathbf{w}^{(\ell)}, \varepsilon) = \theta^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon)$. Showing (A7) is the goal of the remainder of the proof. We employ local convergence in probability for rooted vertex-marked graphs that we introduce briefly for finite mark sets (corresponding to the setting of Definition A.1). We refer to (van der Hofstad 2024) for references and more elaborate descriptions. A rooted vertex-marked graph is a couple (G, o) of a graph G and some, possibly random, distinguished vertex o of G, which we call the root of G. We assume that the vertices $v \in V$ are given marks m_v from a finite mark set $\mathcal{M}$. Let $\mathfrak{G}$ be the space of all vertex-marked rooted locally finite graphs. We call two vertex-marked graphs (G_1, o_1) and $(G_2, o_2) \in \mathfrak{G}$ isomorphic, that is, $(G_1, o_1) \simeq (G_2, o_2)$, if there exists a bijection $\phi : V((G_1, o_1)) \mapsto V((G_2, o_2))$ such that $\phi(o_1) = o_2$, $\{u, v\}$ is an edge in (G_1, o_1) if and only if $\{\phi(u), \phi(v)\}$ is an edge in (G_2, o_2), and $m_u = m_{\phi(u)}$ for all vertices u in G_1. We write $B_G(v, r)$ for the induced subgraph of G on all vertices that are at graph distance at most r from a vertex v. Define

$$\begin{aligned}
R\big((G_1, o_1), (G_2, o_2)\big) &:= \max\left\{ r \in \mathbb{N} : B_{G_1}(o_1, r) \simeq B_{G_2}(o_2, r) \right\}, \\
d_{\mathfrak{G}}\big((G_1, o_1), (G_2, o_2)\big) &:= 1/\big(R\big((G_1, o_1), (G_2, o_2)\big)\big).
\end{aligned}$$

Then, $(\mathfrak{G}, d_{\mathfrak{G}})$ constitutes a Polish space. We call a finite rooted graph (G, o) uniformly rooted, if o is chosen uniformly at random among the vertices of G. We say that a sequence of uniformly rooted graphs $(G_n, o_n)_{n \ge 1}$ converges locally in probability towards (G_∞, o) having law μ, if for every bounded and continuous function $h : \mathfrak{G} \mapsto \mathbb{R}$,

$$\mathbb{E}\big[h(G_n, o_n) | G_n\big] \xrightarrow{\mathbb{P}} \mathbb{E}_\mu[h(G_\infty, o)], \qquad \text{as } n \to \infty, \qquad \text{(A8)}$$

where the expectation on the left-hand side is only with respect to the uniform root o_n. Inhomogeneous random graphs as in Definition A.1 converge locally in probability to their associated branching process: it was essentially proven in (Bollobás et al. 2007), and a formal proof is given in (van der Hofstad 2024). We now use (A8) for the specific function $h(\mathcal{G}_n^{(\delta,R)}, o_n) = \mathbb{1}\left\{C_n^{(\delta,R)}(o_n) \text{ has } \varepsilon\text{-type } \mathbf{w}^{(\ell)}\right\}$. As the function only depends on the induced subgraph up to graph distance $\ell+1$ from the root, it follows that $h(G_1, o_1) = h(G_2, o_2)$ for any two rooted graphs that are within distance $1/(\ell+2)$ from each other. So $h(G_1, o_1)$ is continuous in the Polish space $(\mathfrak{G}, d_{\mathfrak{G}})$. It is clearly bounded, and therefore

$$\frac{|\{v \in [n] : C^{(\delta,R)}(v) \text{ has } \varepsilon\text{-type } \mathbf{w}^{(\ell)}\}|}{\underline{n}} = \frac{1}{|\mathcal{V}_n^{(\delta,R)}|} \sum_{v \in \mathcal{V}_n^{(\delta,R)}} h(G_n, v)$$
$$= \mathbb{E}[h(\mathcal{G}_n^{(\delta,R)}, o_n) | \mathcal{G}_n^{(\delta,R)}]$$
$$\xrightarrow{\mathbb{P}} \mathbb{P}\left(T_q^{(\delta,R)} \text{ has } \varepsilon\text{-type } \mathbf{w}^{(\ell)}\right) = \theta^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon).$$

The number of components of ε-type $\mathbf{w}^{(\ell)}$ is exactly a factor ℓ smaller than the left-hand side. This proves (A7) and therefore (A3) follows. The bound (A4) follows from (A6) and the LDP in (Andreis et al. 2023, Theorem 3.1), noting that the weak law of large numbers $|C_n^{(1),(\delta,R)}|/\underline{n} \to \theta_q^{(\delta,R)}$ was already proven in (Bollobás et al. 2007). ◻

The next lemma controls the relation between the IRG and the (δ, R)-IRG. Recall the definition of a component of ε-type $\mathbf{w}^{(\ell)}$ from Definition 3.1. By our assumption that $\varepsilon/\delta \in \mathbb{N}$, the sequence $(w_i)_{i\geq 0}$ from Definition 3.1 is a subsequence of the sequence $(z_i)_{i\geq 0}$ from Definition A.1. We define an event that ensures that the number of vertices in $\mathcal{G}_n$ with weight in $[z_i, z_{i+1})$ is close to its expectation: for fixed $\delta > 0$, let

$$\mathcal{A}_{\text{reg}} := \left\{\forall i \in [(R-\underline{w})/\delta] : |\mathcal{V}_n[z_i, z_{i+1})| \in \left((1-\delta)nf_W^{(\delta)}(z_i), (1+\delta)nf_W^{(\delta)}(z_i)\right)\right\}. \tag{A9}$$

Lemma A.3 (Coupling of the graphs). *Consider the inhomogeneous random graph from Definition 1.2, and its approximation from Definition A.1 given some $R > \underline{w}$ and $\delta > 0$. There exists a constant $c_1 > 0$ such that for all $n \geq 1$,*

$$\mathbb{P}\left(\mathcal{A}_{\text{reg}}\right) \geq 1 - \exp(-c_1 n). \tag{A10}$$

On the event $\mathcal{A}_{\text{reg}}$, there exists a coupling between $\mathcal{G}_n[\underline{w}, R)$ and $\mathcal{G}_n^{(\delta,R)}$ such that $\mathcal{G}_n[\underline{w}, R) \supseteq \mathcal{G}_n^{(\delta,R)}$ and $|\mathcal{V}_n[\underline{w}, R) \setminus \mathcal{V}_n^{(\delta,R)}| \leq 2\delta n$. If under the coupling a set of vertices is a component of both $\mathcal{G}_n[\underline{w}, R)$ and $\mathcal{G}_n^{(\delta,R)}$, then the ε-type of the component, cf. Definition 3.1, coincides in the two graphs. Moreover, for all $\psi, R > 0$ there exists $\delta_0, c_2 > 0$ such that for all $\delta \in (0, \delta_0)$ and $n \geq 1$, under this coupling,

$$\mathbb{P}\left(|\mathcal{E}_n[\underline{w}, R) \setminus \mathcal{E}_n^{(\delta,R)}| \geq \psi n | \mathcal{A}_{\text{reg}}\right) \leq \exp(-c_2 n). \tag{A11}$$

Proof. The number of vertices with weight in $[z_i, z_{i+1})$ is distributed as $\text{Bin}(n, f_W^{(\delta)}(z_i))$ by F_W in Definition 1.2 and $f_W^{(\delta)}$ in Definition A.1. Therefore, the exponential decay of $\mathbb{P}(\neg\mathcal{A}_{\text{reg}})$ in (A10) follows by a union bound and afterwards applying a Chernoff bound for all $i \in [(R-\underline{w})/\delta]$.

We will now construct the coupling between the two graphs. To do so, we work conditionally on the realization of the vertex set $\mathcal{V}_n[\underline{w}, R)$ that satisfies $\mathcal{A}_{\text{reg}}$. Fix a subset $\mathcal{V}_n^{\text{sub}}[\underline{w}, R) \subseteq \mathcal{V}_n[\underline{w}, R)$ such that $|\mathcal{V}_n^{\text{sub}}[\underline{w} + i\delta, \underline{w} + (i+1)\delta)| = \underline{n}_i$ for all i, which is possible by definition of $\underline{n}_i$ in Definition A.1 and $\mathcal{A}_{\text{reg}}$ in (A9). Then, the vertex set $\mathcal{V}_n^{\text{sub}}[\underline{w}, R)$ has almost the same weight distribution as $\mathcal{V}_n^{(\delta,R)}[\underline{w}, R)$. Indeed, with $(z_i)_{i\geq 0}$ as in Definition A.1, define $w_v^{(\delta)} := \max\{z_i : z_i \leq w_v\}$ for $w \geq \underline{w}$. Then,

$$\{w_v^{(\delta)} : v \in \mathcal{V}_n^{\text{sub}}[\underline{w}, R)\} = \{w_v : v \in \mathcal{V}_n^{(\delta,R)}\}. \tag{A12}$$

Thus, we set the weights of the vertices $[n]$ in $\mathcal{V}_n^{(\delta,R)}$ to be the set on the left-hand side for the coupling. We first show that the set of the remaining vertices from $\mathcal{V}_n^{\text{sprinkle}}[\underline{w}, R) := \mathcal{V}_n[\underline{w}, R) \setminus \mathcal{V}_n^{\text{sub}}[\underline{w}, R) = \mathcal{V}_n[\underline{w}, R) \setminus \mathcal{V}_n^{(\delta,R)}[\underline{w}, R)$ is small. Recall $\underline{n}_i$ from Definition A.1. On the event $\mathcal{A}_{\text{reg}}$,

$$|\mathcal{V}_n^{\text{sprinkle}}[\underline{w}, R)| \leq (1+\delta)\mathbb{P}(W \leq R)n - (1-\delta)n\mathbb{P}(W \leq R) < 2\delta n.$$

Now we simultaneously construct the edges in the induced subgraph $\mathcal{G}_n^{\text{sub}}[\underline{w}, R)$ and $\mathcal{G}_n^{(\delta,R)}$. For each pair of vertices u, v in $\mathcal{V}_n^{\text{sub}}[\underline{w}, R)$, we include the edge $\{u, v\}$ in both graphs with probability p_{uv} from Definition 1.2, but then independently remove the edge from $\mathcal{G}_n^{(\delta,R)}$ with probability

$$1 - \frac{\kappa_\sigma^{(\delta,R)}(w_u^{(\delta)}, w_v^{(\delta)})}{\kappa_\sigma(w_u, w_v)} = 1 - \frac{\kappa_\sigma^{(\delta,R)}(w_u, w_v)}{\kappa_\sigma(w_u, w_v)}.$$

Here, the equality follows from the definition of $\kappa_\sigma^{(\delta,R)}$ in (A2). The constructed edge set $\mathcal{E}_n^{(\delta,R)}$ has the same distribution as desired by Definition A.1. Moreover, if a set of vertices is a component in both $\mathcal{G}_n^{(\delta,R)}$ and $\mathcal{G}_n[\underline{w}, R)$, then the ε-type of this set is the same by the choice of $\mathcal{V}_n^{(\delta,R)}$ below (A12), and since we assume $\varepsilon/\delta \in \mathbb{N}$. It remains to prove (A11).

The probability that an edge is present in $\mathcal{G}_n^{\mathrm{sub}}[\underline{w}, R)$, but absent in $\mathcal{G}_n^{(\delta,R)}$, is at most

$$\sup_{w_1,w_2\in[\underline{w},R]} q\left(\kappa_\sigma(w_1,w_2) - \kappa_\sigma^{(\delta,R)}(w_1,w_2)\right) \Big/ n.$$

The kernel $\kappa_\sigma^{(\delta,R)}(w_1,w_2)$ converges uniformly to κ_σ for $w_1,w_2 \in [\underline{w}, R)$ by its definition in (A2) as $\delta \downarrow 0$. Therefore, the right-hand side is at most $\psi/(4n)$ for any sufficiently small δ. Let

$$\mathcal{A}_{\text{edge-1}} := \left\{|\mathcal{E}_n^{\mathrm{sub}}[\underline{w},R) \setminus \mathcal{E}_n^{(\delta,R)}[\underline{w},R)| \le (\psi/2)n\right\}.$$

There are at most n^2 edges that can be in the symmetric difference. By independence of the edge-removals, we obtain conditionally on the vertex set $\mathcal{V}_n[\underline{w},R)$ satisfying $\mathcal{A}_{\mathrm{reg}}$,

$$\mathbb{P}\left(\neg\mathcal{A}_{\text{edge-1}} | \mathcal{V}_n[\underline{w},R)\right) \le \mathbb{P}\left(\mathrm{Bin}\left(n^2, \psi/(4n)\right) \ge (\psi/2)n\right) = \exp(-\Omega(n)). \tag{A13}$$

Next, we bound the total number of edges incident to one of the vertices not in $\mathcal{V}_n^{\mathrm{sub}}[\underline{w},R)$, that is., incident to at least one vertex in $\mathcal{V}_n^{\mathrm{sprinkle}}[\underline{w},R) = \mathcal{V}_n[\underline{w},R) \setminus \mathcal{V}_n^{\mathrm{sub}}[\underline{w},R)$. Let

$$\mathcal{A}_{\text{edge-2}} := \left\{ \left| \{u,v \in \mathcal{V}_n^{\mathrm{sprinkle}}[\underline{w},R) \times \mathcal{V}_n[\underline{w},R) : u \sim v\} \right| \ge 4\delta n \sup_{x,y\in[\underline{w},R]} \kappa_\sigma(x,y) \right\}.$$

On the event $\mathcal{A}_{\mathrm{reg}}$, there are at most $2\delta n^2$ potential edges, each occurring with probability at most $\sup_{x,y\in[\underline{w},R]} \kappa_\sigma(x,y)/n$. By another application of the Chernoff bound,

$$\mathbb{P}\left(\mathcal{A}_{\text{edge-2}}\right) \le \mathbb{P}\left(\mathrm{Bin}\left(2\delta n^2, \sup_{x,y\in[\underline{w},R]} \kappa_\sigma(x,y)/n\right) \ge 4\delta n \sup_{x,y\in[\underline{w},R]} \kappa_\sigma(x,y)\right) \le \exp(-\Omega(n)).$$

Assume that $\delta = \delta(R,\psi)$ is so small that the constant factor $4\delta \sup_{x,y\in[\underline{w},R]} \kappa_\sigma(x,y)$ is at most $\psi/2$, then (A11) follows when this bound is combined with (A13). □

The next lemma compares the approximating branching process from Definition A.1 to the branching process from Definition 1.3.

Lemma A.4 (Coupling of the branching processes). *Consider the associated branching process of a scale-free inhomogeneous random graph from Definition 1.3, and its approximation from Definition A.1. Fix $\ell_* \in \mathbb{N}$. For all $\psi > 0$ there exist $\delta_0, R_0 > 0$ such that for each $\varepsilon > 0$, $\delta \in (0,\delta_0)$ such that $(\varepsilon/\delta) \in \mathbb{N}$, and $R > R_0$,*

$$\sum_{\ell \le \ell_*} \sum_{\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon)} \left|\theta(\mathbf{w}^{(\ell)},\varepsilon) - \theta^{(\delta,R)}(\mathbf{w}^{(\ell)},\varepsilon)\right| < \psi. \tag{A14}$$

Moreover, there exists R_0 such that for any $\delta \in (0,\delta_0)$ and $R > R_0$,

$$|\theta_q - \theta_q^{(\delta,R)}| < \psi. \tag{A15}$$

Proof. We will couple the branching process BP and $\mathrm{BP}^{(\delta,R)}$ until the branching processes die out or have size larger than ℓ_*. On a good event that we construct (and which holds with probability arbitrary close to 1 by choosing δ small and R large), the two branching processes have the same ε-type. A branching process has exactly one ε-type in $\mathrm{CT}_\ell(\varepsilon)$, so the sum on the left-hand side in (A14) is at most the probability that the branching processes have a different ε-type.

Approximating infinite-type branching process. We start with an observation. By definition of $f_W^{(\delta)}$ and $\kappa_\sigma^{(\delta,R)}$ in Definition A.1, the approximating *finite*-type branching process from Definition A.1can be coupled with the following infinite-type branching process that we denote by $\widetilde{\mathrm{BP}}^{(\delta,R)}$. The root of the branching process has type $\widetilde{W}_\emptyset$ with distribution satisfying $\mathbb{P}(\widetilde{W}_\emptyset > w) = \mathbb{P}(W > w | W < R)$. Here, W has distribution F_W defined in Definition 1.2. In each generation, each particle v of type w_v gives independently birth to new particles according to a Poisson point process (PPP) on $[\underline{w},\infty)$ with intensity $q\kappa_\sigma(w_v,w)\mathrm{d}F_W(w)$. The atoms in the union of these PPPs form the vertex types of the vertices in the next generation.

The coupling with the branching process $\mathrm{BP}^{(\delta,R)}$ from Definition A.1 follows from rounding down each type W to the largest $z_i \le W$ with z_i from Definition A.1, and by coupling the PPP determining the offspring (which has constant intensity in sets $[z_i, z_i+\delta) \times [z_j, z_j+\delta)$) with the Poisson random variables determining the offspring in Definition A.1. This rounding operation does not affect the ε-type of the total progeny of the branching process, since we assumed that $\varepsilon/\delta \in \mathbb{N}$, see Definitions 3.1 and A.1.

Now, we sample the original branching process BP from Definition 1.2. If it contains a vertex with weight at least R in one of the first ℓ generations, we say that the coupling has failed. If $R = R(\psi)$ is sufficiently large, this occurs with probability at most $\psi/2$. The branching process $\widetilde{\mathrm{BP}}^{(\delta,R)}$ can be created from BP as follows. Consider the particles of the PPP determining the offspring of the root. We remove each particle with probability $1 - \kappa_\sigma^{(\delta,R)}(W_j,W_\emptyset)/\kappa_\sigma(W_j,W_\emptyset)$ independently of the rest. The obtained offspring has the same distribution as in $\widetilde{\mathrm{BP}}^{(\delta,R)}$. If none of the at most $\ell_* - 1$ particles is removed, the coupling step of generation zero is successful. By the definition of $\kappa^{(\delta,R)}$ in (A2), we obtain for $\delta = \delta(\psi, \ell_*)$ sufficiently small that

$$1 - \kappa_\sigma^{(\delta,R)}(W_j, W_\emptyset)/\kappa_\sigma(W_j, W_\emptyset) < \psi/(2\ell_*^2).$$

Hence, with probability at most $\psi/(2\ell_*)$, one of the particles is removed. Iterating this procedure at most ℓ_* times yields that no particle is removed with probability at least $1 - \psi/2$. Thus, the coupling is successful with probability at least $1 - \psi$, and (A14) follows.

We turn to (A15). Let $\theta_q^{(0,R)}$ denote the survival probability of a slightly modified associated branching process compared to Definition 1.3: the new particles are formed by a PPP on $[\underline{w}, \infty)$ with intensity $\mathbb{1}_{\{w \le R\}} q \kappa_\sigma(w_v, w) \mathrm{d}F_W(w)$. By the triangle inequality

$$|\theta_q - \theta_q^{(\delta,R)}| \le |\theta_q - \theta_q^{(0,R)}| + |\theta_q^{(0,R)} - \theta_q^{(\delta,R)}|. \tag{A16}$$

Since $\mathbb{1}_{\{w \le R\}} \kappa_\sigma(w_v, w) \uparrow \kappa_\sigma(w_v, w)$ as $R \to \infty$, it follows by (Bollobás et al. 2007, Theorem 6.3) that $\theta_q^{(0,R)} \uparrow \theta_q$. Fix R sufficiently large that (A14) holds, and moreover, $|\theta_q - \theta_q^{(0,R)}| < \psi/2$. The same argument works for the second term on the right-hand side in (A16): by definition of $\kappa_\sigma^{(\delta,R)}$ in (A2), $\kappa_\sigma^{(\delta,R)}(w_u, w_v) \uparrow \mathbb{1}_{\{w_u, w_v \le R\}} \kappa_\sigma(w_u, w_v)$ as δ tends to 0, so $|\theta_q^{(\delta,R)} - \theta_q^{(0,R)}| \to 0$ as δ tends to 0. Hence, we can choose δ sufficiently small so that (A14) holds, and so that $|\theta_q^{(0,R)} - \theta_q^{(\delta,R)}| \le \psi/2$. □

Using Lemmas A.2–A.4 one can prove the bound (3.2) in Lemma 3.2. We postpone the proof to the end of the following subsection, after we analyzed the effect of the vertices with weight in $[R, \phi n)$ so that we can immediately prove the other bound in Lemma 3.2.

Proof of Lemma 3.2. We start with the proof of (3.2). We use the coupling of the IRG restricted to weights in $[\underline{w}, R)$, and its (δ, R)-approximation from Lemma A.3. Fix $\psi_{3.2}$ where δ is sufficiently small that Lemma A.3 holds with $\psi_{3.2}/4$, and (A14) in Lemma A.4 hold with $\psi = \psi_{3.2}/(16\ell_*)$, and $2\delta \le \psi_{3.2}/8$. We bound

$$\begin{aligned}\sum_{\ell, \mathbf{w}^{(\ell)}} |\ell N_n(\mathbf{w}^{(\ell)}, \varepsilon, R)/n - \theta(\mathbf{w}^{(\ell)}, \varepsilon)| \le{}& \sum_{\ell, \mathbf{w}^{(\ell)}} \ell \cdot |N_n(\mathbf{w}^{(\ell)}, \varepsilon, R) - N_n^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon, R)|/n \\ &+ \sum_{\ell, \mathbf{w}^{(\ell)}} |\ell N_n^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon, R)/n - \theta^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon)| \\ &+ \sum_{\ell, \mathbf{w}^{(\ell)}} |\theta^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon) - \theta(\mathbf{w}^{(\ell)}, \varepsilon)|.\end{aligned}$$

Here and in the remainder of the proof, all sums are for $\ell \le \ell_*$ and $\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon, R)$. For readability we omit this in the subscripts of the sums. The third term on the right-hand side is at most $\psi_{3.2}/4$ by Lemma A.4. With probability tending to one exponentially fast, the second term is also at most $\psi_{3.2}/4$ by Lemma A.2. The coupling is successful with probability tending to one exponentially fast by (A10). Thus, it remains to show on the event that the coupling is successful that the first sum is at most $\psi_{3.2}/2$ with probability tending to one exponentially fast, that is, for some $c > 0$,

$$\mathbb{P}\left(\sum_{\ell, \mathbf{w}^{(\ell)}} \ell |N_n(\mathbf{w}^{(\ell)}, \varepsilon, R) - N_n^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon, R)| > (\psi_{3.2}/4)n \,\middle|\, \mathcal{A}_{\mathrm{reg}} \right) \le \exp(-cn).$$

On the event $\mathcal{A}_{\mathrm{reg}}$, we add the vertices from $\mathcal{V}_n[1, R) \setminus \mathcal{V}_n^{(\delta,R)}$ and the edges from $\mathcal{E}_n[1, R) \setminus \mathcal{E}_n^{(\delta,R)}$, and evaluate how the sum changes. After adding the vertices, the sum of the differences changes by at most $|\mathcal{V}_n[1, R) \setminus \mathcal{V}_n^{(\delta,R)}|$. We add the edges iteratively: after each added edge, the summands corresponding to the two incident vertices changes its value by the size of the respective components. So the added edges change the sum in total by at most $2\ell_* |\mathcal{E}_n[1, R) \setminus \mathcal{E}_n^{(\delta,R)}|$. We conclude that

$$\begin{aligned}&\mathbb{P}\left(\sum_{\ell, \mathbf{w}^{(\ell)}} |N_n(\mathbf{w}^{(\ell)}, \varepsilon, R) - N_n^{(\delta,R)}(\mathbf{w}^{(\ell)}, \varepsilon, R)| > (\psi_{3.2}/4)n \,\middle|\, \mathcal{A}_{\mathrm{reg}} \right) \\ &\qquad \le \mathbb{P}(|\mathcal{V}_n[1, R) \setminus \mathcal{V}_n^{(\delta,R)}| + 2\ell_* |\mathcal{E}_n[1, R) \setminus \mathcal{E}_n^{(\delta,R)}| > (\psi_{3.2}/4)n \mid \mathcal{A}_{\mathrm{reg}}).\end{aligned}$$

Since on $\mathcal{A}_{\mathrm{reg}}$, $|\mathcal{V}_n[1, R) \setminus \mathcal{V}_n^{(\delta,R)}| \le 2\delta n \le (\psi_{3.2}/8)n$ by Lemma A.3 and the choice of δ at the beginning, the right-hand side is at most $\mathbb{P}\left(|\mathcal{E}_n[1, R) \setminus \mathcal{E}_n^{(\delta,R)}| > (\psi_{3.2}/16\ell_*)n \,\middle|\, \mathcal{A}_{\mathrm{reg}} \right)$. This probability decays exponentially fast by Lemma A.3 by the choice of ψ at the beginning of the proof. This finishes the proof of (3.2). □

We end this section by proving Lemma 3.4.

Proof of Lemma 3.4. Let $\mathcal{A}_{\mathrm{reg}}$ be as in (A9). Assume that δ and R are such that Lemmas A.2–A.4 apply with $\psi = (\theta_q - \rho)/2$. By Lemma A.3, we can couple the graphs $\mathcal{G}_n^{(\delta,R)}$ and $\mathcal{G}_n[\underline{w}, R)$ on the event $\mathcal{A}_{\mathrm{reg}}$ such that $\mathcal{G}_n[\underline{w}, R) \supseteq \mathcal{G}_n^{(\delta,R)}$. Therefore, the largest component of $\mathcal{G}_n[\underline{w}, R)$ has at least the size of the largest component of $\mathcal{G}_n^{(\delta,R)}$ provided that the coupling is successful, which happens with probability increasing to 1 exponentially fast. Therefore,

$$\mathbb{P}\left(|\mathcal{C}_n^{(1)}[\underline{w}, R)| < \rho n \right) \le \mathbb{P}\left(\left\{ |\mathcal{C}_n^{(\delta,R),(1)}| < (\theta_q - \psi)n \right\} \cap \mathcal{A}_{\mathrm{reg}} \right) + \exp(-\Theta(n)).$$

By Lemma A.2, $\mathbb{P}\Big(|C_n^{(\delta,R),(1)}| < (\theta_q^{(\delta,R)} - \psi/2)n\Big)$ decays exponentially fast. Moreover, by Lemma A.4, $|\theta_q^{(\delta,R)} - \theta_q| \le \psi/2$. Thus, Lemma 3.4 follows. □

Appendix B

Postponed Proofs

We present the proofs of Lemmas 2.7, 4.1, and 4.4.

Proof of Lemma 2.7. The first statement follows from a rearrangement of (2.2), and the fact that the generating function $H_{T_q}(z)$ is increasing in z.

We proceed to (ii). Continuity follows from continuity of the generating function $H_{T_q}(z)$. Assume q is fixed, then by (i) it follows that $\rho \mapsto \mathrm{hubs}(\rho, q)$ is non-decreasing. Assume ρ is fixed. The branching processes with different percolation parameters $q' < q$ can be coupled such that $\mathbb{P}\big(|T_q| \ge |T_{q'}|\big) = 1$: $T_{q'}$ is obtained from T_q by removing independently each edge and the entire subtree with probability q'/q. Hence,

$$\mathbb{E}\Big[(1-q)^{|T_q|h}\Big] \le \mathbb{E}\Big[(1-q)^{|T_{q'}|h}\Big] < \mathbb{E}\Big[(1-q')^{|T_{q'}|h}\Big].$$

As a result,

$$\inf\Big\{h' > 0 : \mathbb{E}\Big[(1-q)^{|T_q|h'}\Big] \le 1-\rho\Big\} < \inf\Big\{h' > 0 : \mathbb{E}\Big[(1-q')^{|T_{q'}|h'}\Big] \le 1-\rho\Big\},$$

and (ii) follows by (i).

We proceed to (iii) when $q \downarrow 0$. We analyze the generating function $\mathbb{E}[z^{|T_q|}\mathbb{1}_{\{|T_q|<\infty\}}]$ appearing in the definition of hubs in (2.2). As $q \downarrow 0$, the probability that the root of the branching process has degree 0, tends to 1. Thus, $\mathbb{P}\big(|T_q| = 1\big) \to 1$ as well. Therefore, for any z as $q \downarrow 0$,

$$H_{T_q}(z) = \mathbb{E}[z^{|T_q|}\mathbb{1}_{\{|T_q|<\infty\}}] = \sum_{k=0}^{\infty}\mathbb{P}\big(|T_q| = k\big)z^k = (1+o(1))z.$$

Inverting H_{T_q} yields that also $H_{T_q}^{(-1)}(z) = (1+o(1))z$ as $q \downarrow 0$. Substituting this limit into (2.2) yields (iii) when $q \downarrow 0$. When q is fixed and $\rho \uparrow 1$, the other statement in (iii) follows immediately from part (iv), which we prove now. We analyze the generating function $\mathbb{E}[z^{|T_q|}\mathbb{1}_{\{|T_q|<\infty\}}]$ as $z \downarrow 0$. Since the total progeny of a branching process is at least 1,

$$H_{T_q}(z) = \mathbb{E}[z^{|T_q|}\mathbb{1}_{\{|T_q|<\infty\}}] = z\mathbb{P}\big(|T_q| = 1\big)(1+o(1)) = z\mathbb{P}\big(D_\emptyset = 0\big)(1+o(1)), \qquad \text{as } z \downarrow 0,$$

where $D_\emptyset$ is the degree of the root of the branching process, see Definition 1.3. By definition, $D_\emptyset$ is a compound Poisson random variable with mean $qW_\emptyset\mathbb{E}[\kappa_\sigma(W_\emptyset, W)|W_\emptyset]$, where $W_\emptyset$ has distribution F_W. Therefore, $\mathbb{P}\big(D_\emptyset = 0\big) = \mathbb{E}\big[\exp\big(-qW_\emptyset\mathbb{E}[\kappa_\sigma(W_\emptyset, W)|W_\emptyset]\big)\big]$. Thus, as $z \downarrow 0$,

$$H_{T_q}(z) = z\mathbb{E}\big[\exp\big(-qW_\emptyset\mathbb{E}[\kappa_\sigma(W_\emptyset, W)|W_\emptyset]\big)\big](1+o(1)).$$

Inverting the formula, and substituting the limit into (2.2) yields the first limit in (iv). For the second limit, we use that $\kappa_1(W_\emptyset, W) = W_\emptyset W$ and that $W_\emptyset$ and W are independent. □

Proof of Lemma 4.1. We first give a probabilistic proof of the statement (4.2). By Definition 3.1, (4.2) is equivalent to showing that there exist constants R, ℓ_* such that

$$\mathbb{P}\left(|T_q| \le \ell_*, \max_{x\in T_q} W_x \le R\right) \ge \mathbb{P}\big(|T_q| < \infty\big) - \psi, \tag{B1}$$

which is equivalent to showing for some R, ℓ_* that

$$\mathbb{P}\big(\ell_* < |T_q| < \infty\big) + \mathbb{P}\left(|T_q| \le \ell_*, \max_{x\in T_q} W_x > R\right) \le \psi. \tag{B2}$$

The first term on the left-hand side tends to 0 as ℓ_* tends to 0. Let ℓ_0 be such that for any $\ell_* \ge \ell_0$ the left-hand side is at most $\psi/2$. To bound the second term, we use that

$$\mathbb{P}\left(|T_q| \le \ell_*, \max_{x\in T_q} W_x > R\right) \le \mathbb{P}\left(|T_q| \le \ell_* \big| \max_{x\in T_q} W_x > R\right). \tag{B3}$$

We argue now that the right-hand side tends to zero as $R \to \infty$. Let $x_{\ge R} \in T_q$ be a vertex that has weight at least R. Then the number of offspring of $x_{\ge R}$ with weight at most R, stochastically dominates a Poisson random variable with mean

$$qR\int_{\underline{w}}^{R} w^\sigma \mathrm{d}F_W(w) =: CR.$$

So, the probability that $x_{\geq R}$ has at least $(C/2)R$ offspring tends to 1 as $R \to \infty$. If $R > 2\ell_*/C$, the size of the total progeny T_q is also at least ℓ_* on this event. Thus, there exists $R_0 = R_0(\ell_0)$ such that for $\ell_* = \ell_0$ and $R = R_0$ also the right-hand side in (B3) is at most $\psi/2$. Thus, both terms in (B2) are at most $\psi/2$ for these values $\ell_* = \ell_0$ and $R = R_0$. This proves (4.2). Since the left-hand side in (B1), corresponding to the left-hand side in (4.2), is increasing in ℓ_*, and R, (4.2) also holds for any $R \geq R_0, \ell_* \geq \ell_0$.

To prove (4.3), we need to show that we can choose $\varepsilon, \ell_* \geq \ell_0, R \geq R_0$ such that

$$s(\varepsilon, \ell_*, R) := \sup_{\mathbf{y} > \phi \mathbf{1}} \left| \mathbb{E}\left[\overline{P}\left((W_x)_{x \in T_q}, \mathbf{y} \right) \right] - \sum_{\ell=1}^{\ell^*} \sum_{\mathbf{w} \in \mathrm{CT}_\ell(\varepsilon, R)} \overline{P}(\mathbf{w}, \mathbf{y}) \theta(\mathbf{w}, \varepsilon) \right| \tag{B4}$$

can be made arbitrarily small. We truncate the expectation on the right-hand side in (B4) by considering four events for the branching process. First define the constant

$$c := \frac{q}{2} \int_{\underline{w}}^{2\underline{w}} w^\sigma \mathrm{d}F_W(w) > 0.$$

Define the events

$$\begin{aligned}
\mathcal{A}_1(R) &:= \left\{ \max_{x \in T_q} W_x > R, \left| \{ x \in T_q : W_x \in [\underline{w}, 2\underline{w}] \} \right| < cR \right\}, \\
\mathcal{A}_2(R) &:= \left\{ \max_{x \in T_q} W_x > R, \left| \{ x \in T_q : W_x \in [\underline{w}, 2\underline{w}] \} \right| \geq cR \right\}, \\
\mathcal{A}_3(\ell_*, R) &:= \left\{ \max_{x \in T_q} W_x \leq R, |T_q| > \ell_* \right\}, \\
\mathcal{A}_4(\ell_*, R) &:= \left\{ \max_{x \in T_q} W_x \leq R, |T_q| \leq \ell_* \right\}.
\end{aligned}$$

Using that $\overline{P} \leq 1$ by definition in (4.1),

$$\begin{aligned}
\mathbb{E}\left[\overline{P}\left((W_x)_{x \in T_q}, \mathbf{y} \right) \right] &\leq \mathbb{P}\left(\mathcal{A}_1(R) \right) + \mathbb{E}\left[\overline{P}\left((W_x)_{x \in T_q}, \mathbf{y} \right) \mathbb{1}_{\{\mathcal{A}_2(R)\}} \right] \\
&\quad + \mathbb{P}\left(\mathcal{A}_3(\ell_*, R) \right) + \mathbb{E}\left[\overline{P}\left((W_x)_{x \in T_q}, \mathbf{y} \right) \mathbb{1}_{\{\mathcal{A}_4(\ell_*, R)\}} \right].
\end{aligned} \tag{B5}$$

The fourth term considers only the types of branching processes that are considered in the sums in (B4). We will show that it forms the main contribution to the right-hand side in (B5). We first analyze the three other terms. For the first term, we consider the offspring of the branching process of a vertex of weight at least R, assuming that $R \geq 2\underline{w}$. By Definition 1.3, the number of offspring in the interval $[\underline{w}, 2\underline{w}]$ dominates a Poisson random variable with mean $2cR$. By concentration of Poisson random variables (Lemma C.1), we obtain that as $R \to \infty$,

$$\mathbb{P}\left(\mathcal{A}_1(R) \right) \leq \mathbb{P}\left(|\{ x \in T_q : W_x \in [\underline{w}, 2\underline{w}] \}| < cR \;\middle|\; \max_{x \in T_q} W_x > R \right) \leq \mathbb{P}(\mathrm{Poi}(2cR) < cR) = o(1).$$

For the second term in (B5), we use that there are at least cR vertices with weight in $[\underline{w}, 2\underline{w}]$ on the event $\mathcal{A}_2(R)$. Let $c_1 = \min\left(\underline{w}^\sigma, (2\underline{w})^\sigma \right)$. By definition of $\overline{P}$ in (4.1), and that $y_i \geq \phi$ for all i, we obtain

$$\mathbb{E}\left[\overline{P}\left((W_x)_{x \in T_q}, \mathbf{y} \right) \mathbb{1}_{\{\mathcal{A}_2(R)\}} \right] \leq \left(1 - q(c_1 \phi \wedge 1) \right)^{cR} \to 0, \qquad \text{as } R \to \infty.$$

We turn to the third term in (B5) in which we assume that R is fixed so that the first two terms in (B5) are at most $\psi/4$. We consider again the branching process. The number of offspring with weight at least R generated from a particle with weight at most R, dominates a Poisson random variable with mean

$$q \min(\underline{w}^\sigma, R^\sigma) \int_R^\infty w \mathrm{d}F_W(w) =: c_R.$$

The offspring of the first (at least) ℓ_* particles of the branching process are formed by independent Poisson point processes. Hence, the probability that none of them generates a particle of at least R, decays exponentially in ℓ_*, that is,

$$\mathbb{P}\left(\mathcal{A}_3(\ell_*, R) \right) \leq \exp\left(-\ell_* c_R \right) \to 0, \qquad \text{as } \ell_* \to \infty.$$

This bounds the first three terms on the right-hand side in (B5). Substituting the bounds on (B5) into (B4) yields that for all ψ there exists a constant $R_1 \geq R_0$, such that for all $R_2 \geq R_1$ there exists a constant $\ell_1 = \ell_1(R_2) \geq \ell_0$ such that for all $\ell_2 \geq \ell_1$,

$$\inf_{\ell_*\geq 0, R\geq 0} s(\varepsilon, \ell_*, R)$$

$$\leq 3\psi/4 + \sup_{\mathbf{y}>\phi\mathbf{1}} \left| \mathbb{E}\left[\overline{P}\left((W_x)_{x\in T_q}, \mathbf{y} \right) \mathbb{1}_{\{|T_q|\leq \ell_2\}} \right] - \sum_{\ell=1}^{\ell_2} \sum_{\mathbf{w}\in \mathrm{CT}_\ell(\varepsilon, R_2)} \theta(\mathbf{w}, \varepsilon) \overline{P}(\mathbf{w}, \mathbf{y}) \right|$$

$$\leq 3\psi/4 + \sup_{\mathbf{y}>\phi\mathbf{1}} \sum_{\ell=1}^{\ell_2} \sum_{\mathbf{w}\in \mathrm{CT}_\ell(\varepsilon, R_2)} \theta(\mathbf{w}, \varepsilon) \cdot \left| \mathbb{E}\left[\overline{P}(\mathbf{w}, \mathbf{y}) - \overline{P}\left((W_x)_{x\in T_q}, \mathbf{y} \right) \;\middle|\; T_q \text{ has } \varepsilon\text{-type } \mathbf{w} \right] \right|. \tag{B6}$$

Next, we will find a suitable upper bound on the expectations in the third line. Define

$$\delta_\varepsilon := \sup_{\mathbf{w},\mathbf{y}} \left| \mathbb{E}\left[\frac{\overline{P}\left((W_x)_{x\in T_q}, \mathbf{y} \right)}{\overline{P}(\mathbf{w}, \mathbf{y})} \;\middle|\; T_q \text{ has } \varepsilon\text{-type } \mathbf{w} \right] - 1 \right|. \tag{B7}$$

Here the supremum runs over all component types in $\mathrm{CT}_\ell(\varepsilon, R_2)$ with $\ell \leq \ell_2$, and $\mathbf{y} \geq \phi$. We show that δ_ε tends to 0 as ε tends to 0. First assume that $\sigma \geq 0$. By construction of the ε-types in Definition 3.1, the numerator is always smaller than the denominator, and for each $i \in [\ell]$ there exists $x \in T_q$ such that $w_i \leq W_x < w_i + \varepsilon$. Using the definition of $\overline{P}$ in (4.1), we obtain that for all $\mathbf{y}, \mathbf{w}$,

$$1 \geq \mathbb{E}\left[\frac{\overline{P}\left((W_x)_{x\in T_q}, \mathbf{y} \right)}{\overline{P}(\mathbf{w}, \mathbf{y})} \right] \geq \prod_{i\in[\ell], j\in[h]} \frac{1 - q\left((w_i+\varepsilon)^\sigma y_j \wedge 1 \right)}{1 - q\left(w_i^\sigma y_j \wedge 1 \right)}$$

$$\geq \left(\inf_{w\in[\underline{w}, R_2], y\geq\phi} \frac{1 - q((w+\varepsilon)^\sigma y \wedge 1)}{1 - q(wy \wedge 1)} \right)^{\ell_2}. \tag{B8}$$

When $\sigma < 0$, these bounds hold in the opposite direction, replacing the infimum by a supremum. Since ℓ_2 is a constant, it is elementary to verify that the right-hand side tends to 1 as $\varepsilon \to 0$ for any $\sigma \in \mathbb{R}$. So δ_ε tends to 0 as $\varepsilon \to 0$. Thus, each expectation in the third line in (B6) is at most $\delta_\varepsilon \overline{P}(\mathbf{w}, \mathbf{y}) \leq \delta_\varepsilon$. This proves that

$$\inf_{\ell_*\geq 0, R\geq 0} s(\varepsilon, \ell_*, R) \leq 3\psi/4 + \delta_\varepsilon \sum_{\ell=1}^{\ell_2} \sum_{\mathbf{w}\in \mathrm{CT}_\ell(\varepsilon, R_2)} \theta(\mathbf{w}, \varepsilon)$$

$$= 3\psi/4 + \delta_\varepsilon \mathbb{P}\left(|T_q| \leq \ell_2, \max_{x\in T_q} W_x \leq R_2 \right) \leq 3\psi/4 + \delta_\varepsilon.$$

We conclude that for each $\psi > 0$, there exist choices of ℓ_*, R, ε such that the right-hand side is at most ψ, so that (4.3) holds. The bound (4.4) follows from the same reasoning as from (B7) until the lines below (B8). □

Proof of Lemma 4.4. Let $\tilde{\psi}$ be a sufficiently small constant depending on ψ. We first claim that for any $\ell \in \mathbb{N}$, if R is sufficiently large and ε sufficiently small that

$$\sup_{\mathbf{y}^{(h)}>\phi} \left| \mathbb{E}\left[\mathbb{1}_{\{|T_q|=\ell\}} \overline{P}\left((W_x)_{x\in T_q}, \mathbf{y}^{(h)} \right) \right] - \sum_{\mathbf{w}^{(\ell)}\in \mathrm{CT}_\ell(\varepsilon, R)} \overline{P}(\mathbf{w}^{(\ell)}, \mathbf{y}^{(h)}) \theta(\mathbf{w}^{(\ell)}, \varepsilon) \right| < \tilde{\psi}. \tag{B9}$$

We leave it to the reader to verify that this follows from the same reasoning as the proof of (4.3), starting from (B4).

Moreover, let $\ell_* \geq \ell$ and R be at least so large, and ε so small that we may apply Theorem 4.1 with $\psi_{4.1} = \tilde{\psi}$, and that $\mathbb{P}(|T_q| \leq \ell_*) \geq \mathbb{P}(|T_q| < \infty) - \psi' = 1 - \theta_q - \tilde{\psi}$. By definition of $\theta(\mathbf{w}^{(\ell)}, \varepsilon)$ in (3.1), $\mathbb{P}(|T_q| = \ell) = \sum_{\mathbf{w}^{(\ell)}\in \mathrm{CT}_\ell(\varepsilon)} \theta(\mathbf{w}^{(\ell)}, \varepsilon)$ for any $\varepsilon > 0$. Thus,

$$\sum_{k\in[\ell_*]} \sum_{\mathbf{w}^{(k)}\in \mathrm{CT}_k(\varepsilon, R)} \theta(\mathbf{w}^{(k)}, \varepsilon) = \mathbb{P}\left(|T_q| \leq \ell_* \right) - \sum_{k\in[\ell_*]} \sum_{\mathbf{w}^{(k)}\in \mathrm{CT}_k(\varepsilon)\setminus \mathrm{CT}_k(\varepsilon, R)} \theta(\mathbf{w}^{(k)}, \varepsilon).$$

Since the sum over $\theta(\mathbf{w}^{(k)}, \varepsilon)$ for all $k < \infty$ and $\mathbf{w}^{(k)} \in \mathrm{CT}_k(\varepsilon)$ is at most one, if R is sufficiently large, the double sum on the right-hand side is at most $\tilde{\psi}$. Thus,

$$\sum_{k\in[\ell_*]} \sum_{\mathbf{w}^{(k)}\in \mathrm{CT}_k(\varepsilon, R)} \theta(\mathbf{w}^{(k)}, \varepsilon) \geq 1 - \theta_q - 2\tilde{\psi}. \tag{B10}$$

We now adjust the proof of Proposition 4.3. Let $\phi = \phi(\psi, C)$ be a sufficiently small constant. We define

$$\mathcal{A}_{\text{comp}} := \left\{ \sum_{\substack{\ell \le \ell_*, \\ \mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon, R)}} \left| \frac{\ell N_n(\mathbf{w}^{(\ell)}, \varepsilon, \phi n)}{n} - \theta(\mathbf{w}^{(\ell)}, \varepsilon) \right| \le \tilde{\psi} \right\} \cap \left\{ \frac{|C_n^{(1)}[\underline{w}, \phi n)}{n} \ge \theta_q - \tilde{\psi} \right\}$$

$$\cap \left\{ \forall \ell \le \ell_*, \mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon, R) : \left(|\ell N_n(\mathbf{w}^{(\ell)}, \varepsilon, \phi n)/n \right) \Big/ \theta(\mathbf{w}^{(\ell)}, \varepsilon) - 1| \le \tilde{\psi} \right\}, \tag{B11}$$

$$\mathcal{A}_{\text{hubs}} := \left\{ \forall v \in \mathcal{V}_n[\phi n, \infty) : v \sim C_n^{(1)}[\underline{w}, \phi n) \right\}. \tag{B12}$$

For each $\ell \le \ell_*$, we write $M_n(\mathbf{w}^{(\ell)}, \varepsilon)$ for the number of components of ε-type $\mathbf{w}^{(\ell)}$ in the induced subgraph $\mathcal{G}_n[\underline{w}, \phi n)$ that are *not* connected by an edge to the hubs in $\mathcal{G}_n$. Let $h := |\mathcal{V}_n[\phi n, \infty)|$ denote the number of hubs, which is equal to $\lceil \mathrm{hubs}(r, q) \rceil$ by assumption. By the conditioning in (4.13) all rescaled weights in $\mathbf{y}^{(h)} = \{y_1, \dots, y_h\}$ are at least ϕ. Define

$$\mathcal{A}_{\text{conn}} := \left\{ \begin{aligned} \forall \ell \in [\ell_*], \mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon, \overline{R}) : M_n(\mathbf{w}^{(\ell)}, \varepsilon) \ge (1 - \tilde{\psi})^3 \cdot (n/\ell) \cdot \theta(\mathbf{w}^{(\ell)}, \varepsilon) \\ \cdot \overline{P}(\mathbf{w}^{(\ell)}, \mathbf{y}^{(h)}) \mathbb{1}_{\{\overline{P}(\mathbf{w}^{(\ell)}, \mathbf{y}^{(h)}) \ge \tilde{\psi}\}} \end{aligned} \right\}$$

$$\cap \left\{ \begin{aligned} \forall \ell \in [\ell_*], \mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon, \overline{R}) : M_n(\mathbf{w}^{(\ell)}, \varepsilon) \le (1 + \tilde{\psi})^2 \cdot \left(\tilde{\psi} + \overline{P}(\mathbf{w}^{(\ell)}, \mathbf{y}^{(h)}) \right) \\ \cdot N_n(\mathbf{w}^{(\ell)}, \varepsilon, \phi n) \end{aligned} \right\}.$$

We next analyze the number of size-ℓ components on the intersection $\mathcal{A}_{\text{comp}} \cap \mathcal{A}_{\text{hubs}} \cap \mathcal{A}_{\text{conn}}$. We first establish a lower bound. The number of remaining components of size ℓ is at least the number of components of size ℓ with ε-type in $\mathrm{CT}_\ell(\varepsilon, R)$ that does not connect by an edge to one of the hubs. Thus, we find for each $\ell \in [\ell_*]$,

$$\begin{aligned} N_{n,\ell} &\ge \sum_{\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon, R)} M_n(\mathbf{w}^{(\ell)}, \varepsilon) \ge \sum_{\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon, R)} (1 - \tilde{\psi})^3 \cdot (n/\ell) \cdot \theta(\mathbf{w}^{(\ell)}, \varepsilon) \cdot \overline{P}(\mathbf{w}^{(\ell)}, \mathbf{y}^{(h)}) \mathbb{1}_{\{\overline{P}(\mathbf{w}^{(\ell)}, \mathbf{y}^{(h)}) \ge \tilde{\psi}\}} \\ &\ge (1 - \tilde{\psi})^3 \cdot (n/\ell) \sum_{\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon, R)} \theta(\mathbf{w}^{(\ell)}, \varepsilon) \cdot \overline{P}(\mathbf{w}^{(\ell)}, \mathbf{y}^{(h)}) - \tilde{\psi}(1 - \tilde{\psi})^3 \cdot (n/\ell) \sum_{\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon, R)} \theta(\mathbf{w}^{(\ell)}, \varepsilon) \\ &\ge (1 - \tilde{\psi})^3 \cdot (n/\ell) \sum_{\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon, R)} \theta(\mathbf{w}^{(\ell)}, \varepsilon) \cdot \overline{P}(\mathbf{w}^{(\ell)}, \mathbf{y}^{(h)}) - \tilde{\psi} n. \end{aligned}$$

In the last step we used that the sum over the probabilities on the third line is at most one. Applying the bound from (B9), we obtain on $\mathcal{A}_{\text{comp}} \cap \mathcal{A}_{\text{hubs}} \cap \mathcal{A}_{\text{conn}}$ that

$$N_{n,\ell}/n \ge (1 - \tilde{\psi})^3 \cdot (1/\ell) \left(\mathbb{E}\left[\mathbb{1}_{\{|T_q| = \ell\}} \overline{P}\left((W_x)_{x \in T_q}, \mathbf{y}^{(h)} \right) \right] - \tilde{\psi} \right) - \tilde{\psi}.$$

We now subtract $(1/\ell)(\mathbb{E}\left[\mathbb{1}_{\{|T_q| = \ell\}} \overline{P}\left((W_x)_{x \in T_q}, \mathbf{y}^{(h)} \right) \right]$ from both sides, and use that the expectation on the right-hand side is at most 1 by definition of $\overline{P}$ in (4.1). As a result, for $\tilde{\psi}$ sufficiently small depending on ψ, we obtain that

$$N_{n,\ell}/n - (1/\ell) \mathbb{E}\left[\mathbb{1}_{\{|T_q| = \ell\}} \overline{P}\left((W_x)_{x \in T_q}, \mathbf{y}^{(h)} \right) \right] \ge (-3\tilde{\psi} + 3\tilde{\psi}^2 - \tilde{\psi}^3) \cdot (1/\ell) \mathbb{E}\left[\mathbb{1}_{\{|T_q| = \ell\}} \overline{P}\left((W_x)_{x \in T_q}, \mathbf{y}^{(h)} \right) \right] - (1 - \tilde{\psi})^3 \tilde{\psi}/\ell - \tilde{\psi} \ge -\psi, \tag{B14}$$

We next establish an upper bound on the number of size-ℓ components on $\mathcal{A}_{\text{comp}} \cap \mathcal{A}_{\text{hubs}} \cap \mathcal{A}_{\text{conn}}$. Since all hubs connect to a component of size at least $(\theta_q - \tilde{\psi})n$, no component of size less than ℓ in $\mathcal{G}_n[1, \phi n)$ is contained in a component of size exactly ℓ in $\mathcal{G}_n$. Thus, the number of components of size-ℓ is at most the number of components with ε-type in $\mathrm{CT}_\ell(\varepsilon, R)$ that do not connect by an edge to one of the hubs, plus the number of components of size ℓ with ε-type *not* in $\mathrm{CT}_\ell(\varepsilon, R)$, that is,

$$N_{n,\ell} \le \sum_{\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon, R)} M_n(\mathbf{w}^{(\ell)}, \varepsilon) + \sum_{\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon) \setminus \mathrm{CT}_\ell(\varepsilon, R)} N_n(\mathbf{w}^{(\ell)}, \varepsilon, \phi n). \tag{B15}$$

We first bound the second term from above, multiplied by a factor ℓ to count vertices. The number of vertices in components with ε-type in $\mathrm{CT}_\ell(\varepsilon) \setminus \mathrm{CT}_\ell$ is at most the number of components that is not in the largest component, and not in a component with ε-type in $\mathrm{CT}_k(\varepsilon, R)$ for some $k \in \mathbb{N}$. We obtain

$$\sum_{\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon) \setminus \mathrm{CT}_\ell(\varepsilon, R)} \ell N_n(\mathbf{w}^{(\ell)}, \varepsilon, \phi n) \le n - |C_n^{(1)}| - \sum_{k \in [\ell_*]} \sum_{\mathbf{w}^{(K)} \in \mathrm{CT}_k(\varepsilon, R)} k N_n(\mathbf{w}^{(k)}, \varepsilon, \phi n).$$

We use the bounds from the definition of $\mathcal{A}_{\text{comp}}$ in (B11), yielding

$$\sum_{\mathbf{w}^{(\ell)}\in \mathrm{CT}_\ell(\varepsilon)\setminus \mathrm{CT}_\ell(\varepsilon,R)} \ell N_n(\mathbf{w}^{(\ell)},\varepsilon,\phi n) \le (1-\theta_q+\tilde\psi)n + \tilde\psi n - \sum_{k\in[\ell_*]}\sum_{\mathbf{w}^{(K)}\in \mathrm{CT}_k(\varepsilon,R)} k\theta(\mathbf{w}^{(k)},\varepsilon)n.$$

By (B10), the double sum on the right-hand side is at least $(1-\theta_q-2\tilde\psi)n$, so that the right-hand side is in total at most $4\tilde\psi n$. We substitute this bound in (B15), and use the upper bound on $M_n(\mathbf{w}^{(\ell)},\varepsilon)$ from (B13) to obtain

$$\begin{aligned} N_{n,\ell} &\le 4\tilde\psi n + \sum_{\mathbf{w}^{(\ell)}\in \mathrm{CT}_\ell(\varepsilon,R)} (1+\tilde\psi)^2\cdot\left(\tilde\psi + \overline{P}(\mathbf{w}^{(\ell)},\mathbf{y}^{(h)})\right)\cdot N_n(\mathbf{w}^{(\ell)},\varepsilon,\phi n)\\ &\le 4\tilde\psi n + (1+\tilde\psi)^2\tilde\psi n + (1+\tilde\psi)^2 \sum_{\mathbf{w}^{(\ell)}\in \mathrm{CT}_\ell(\varepsilon,R)} \overline{P}(\mathbf{w}^{(\ell)},\mathbf{y}^{(h)})\cdot N_n(\mathbf{w}^{(\ell)},\varepsilon,\phi n)\end{aligned}$$

as the total number of components is at most n. Without loss of generality, we may assume that $\tilde\psi$ is at most one. Next, we use the upper bound on $N_n(\mathbf{w}^{(\ell)},\varepsilon,\phi n)$ by $\mathcal{A}_{\text{comp}}$ in (B11), which yields,

$$N_{n,\ell} \le 8\tilde\psi n + (1+\tilde\psi)^2(n/\ell)\sum_{\mathbf{w}^{(\ell)}\in \mathrm{CT}_\ell(\varepsilon,R)} \overline{P}(\mathbf{w}^{(\ell)},\mathbf{y}^{(h)})\cdot\left(\tilde\psi\theta(\mathbf{w}^{(\ell)},\varepsilon)+\theta(\mathbf{w}^{(\ell)},\varepsilon)\right).$$

Since $\overline{P}$ is at most 1 by (4.1), and the sum over all $\theta(\mathbf{w}^{(\ell)},\varepsilon)$ is at most one, we obtain by (B9),

$$\begin{aligned} N_{n,\ell} &\le 8\tilde\psi n + (1+\tilde\psi)^2\tilde\psi + (1+\tilde\psi)^2(n/\ell)\sum_{\mathbf{w}^{(\ell)}\in \mathrm{CT}_\ell(\varepsilon,R)} \overline{P}(\mathbf{w}^{(\ell)},\mathbf{y}^{(h)})\cdot\theta(\mathbf{w}^{(\ell)},\varepsilon)\\ &\le 12\tilde\psi n + (n/\ell)(1+\tilde\psi)^2\left(\mathbb{E}\left[\mathbb{1}_{\{|T_q|=\ell\}}\overline{P}\left((W_x)_{x\in T_q},\mathbf{y}^{(h)}\right)\right]+\tilde\psi\right).\end{aligned}$$

Thus, there exists a constant $c>0$ such that

$$\le N_{n,\ell}/n - (1/\ell)\mathbb{E}\left[\mathbb{1}_{\{|T_q|=\ell\}}\overline{P}\left((W_x)_{x\in T_q},\mathbf{y}^{(h)}\right)\right] \le c\psi.$$

We combine this upper bound with the lower bound in (B14), and obtain on the event $\mathcal{A}_{\text{comp}}\cap\mathcal{A}_{\text{hubs}}\cap\mathcal{A}_{\text{conn}}$

$$\left|N_{n,\ell}/n - (1/\ell)\mathbb{E}\left[\mathbb{1}_{\{|T_q|=\ell\}}\overline{P}\left((W_x)_{x\in T_q},\mathbf{y}^{(h)}\right)\right]\right| \le \max(c,1)\tilde\psi.$$

Thus, for proving Lemma 4.4, it suffices to show that

$$\mathbb{P}\left(\mathcal{A}_{\text{comp}}\cap\mathcal{A}_{\text{hubs}}\cap\mathcal{A}_{\text{conn}} \,|\, \mathcal{W}_n[\phi n,\infty) = n\mathbf{y}^{(h)}\right) = 1-o(n^{-C}). \tag{B16}$$

We argue similar as around (4.18). We condition on the graph $\mathcal{G}_n[\underline{w},\overline{R})$ satisfying $\mathcal{A}_{\text{comp}}$ and the realization of $\mathcal{W}_n[\phi n,\infty)=n\mathbf{y}^{(h)}$ satisfying $\mathcal{A}_{\text{comp}}$. We abbreviate

$$\mathbb{P}_{\mathbf{y},\mathcal{G}}(\,\cdot\,) := \mathbb{P}\left(\,\cdot\,|\mathcal{G}_n[\underline{w},\overline{R}),\mathcal{W}_n[\phi n,\infty)=n\mathbf{y}^{(h)},\mathcal{A}_{\text{comp}}\right). \tag{B17}$$

Then,

$$\mathbb{P}(\mathcal{A}_{\text{comp}}\cap\mathcal{A}_{\text{hubs}}\cap\mathcal{A}_{\text{conn}}|\mathcal{W}_n[\phi n,\infty)=n\mathbf{y}^{(h)}) = \mathbb{E}[\mathbb{1}_{\{\mathcal{A}_{\text{comp}}\}}\mathbb{P}_{\mathbf{y},\mathcal{G}}\left(\mathcal{A}_{\text{hubs}}\cap\mathcal{A}_{\text{conn}}\right)\,\Big|\,\mathcal{W}_n[\phi n,\infty)=n\mathbf{y}^{(\ell)}].$$

To bound the conditional probability, the same reasoning as below (4.18) applies, using for bounding the probability of the event $\mathcal{A}_{\text{conn}}$ also the reasoning from (4.12). As a result, the conditional probability is at least $1-\exp(-c'n)$ for some $c'>0$. Recalling $\mathcal{A}_{\text{comp}}$ from (B11), this leaves to show that

$$\begin{aligned}&\mathbb{P}\left(\sum_{\substack{\ell\le\ell_*,\\ \mathbf{w}^{(\ell)}\in \mathrm{CT}_\ell(\varepsilon,R)}}\left|\frac{\ell N_n(\mathbf{w}^{(\ell)},\varepsilon,\phi n)}{n}-\theta(\mathbf{w}^{(\ell)},\varepsilon)\right|>\tilde\psi\right)+\mathbb{P}\left(\frac{|C_n^{(1)}[\underline{w},\phi n)}{n}\ge\theta_q-\tilde\psi\right)\\ &+\mathbb{P}\left(\forall\ell\le\ell_*,\mathbf{w}^{(\ell)}\in \mathrm{CT}_\ell(\varepsilon,R):\left(|\ell N_n(\mathbf{w}^{(\ell)},\varepsilon,\phi n)/n\right)\Big/\theta(\mathbf{w}^{(\ell)},\varepsilon)-1|>\tilde\psi\right)=o(n^{-C}).\end{aligned}$$

The first two terms are $o(n^{-C})$ by Lemmas 3.3 and 3.4. The third term is also of order $o(n^{-C})$ by Lemma 3.3 by choosing ψ dependent on the finitely many $\theta(\mathbf{w}^{(\ell)},\varepsilon)$. This finishes the proof. □

Appendix C

Preliminaries

The following lemma is a straightforward application of the Chernoff bound.

Lemma C.1 (Concentration bounds). *Let X be Poisson or Binomial with mean μ. Then, for every $\delta > 0$ there exists a constant $c_\delta > 0$ such that $\mathbb{P}(|X-\mu| > \delta\mu) \le \mathrm{e}^{-c_\delta \mu}$.*

The next result provides a useful estimate for sums of truncated heavy-tailed random variables; it is a version of Lemma 3 of (Resnick and Samorodnitsky 1999) adapted to our setting.

Lemma C.2. *Let $(W_i)_{i\in[n]}$ be iid random variables with regularly varying distribution as in Definition 1.2. For every $C, R > 0$ there exists $\phi_0 > 0$ such that for all $\phi \in (0, \phi_0)$,*

$$\mathbb{P}\left(\sum_{i=1}^{n} W_i \mathbb{1}_{\{W\in[R,\phi n)\}} > 2\mathbb{E}[W\mathbb{1}_{\{W\ge R\}}]n\right) = o(n^{-C}). \tag{C1}$$

Proof. Let $(W_i')_{i\ge 1}$ be iid copies of W conditionally on $W \ge R$. Let $n_\delta := (1+\delta)n\mathbb{P}(W\ge R)$ for $\delta \in (0,1)$. Conditionally on $|\sum_{\iota\in[n]} \mathbb{1}_{\{W_i\ge R\}}| \le n_\delta$, $\sum_{i\in[n]} W_i \mathbb{1}_{\{W_i\ge R\}}$ is stochastically dominated by $\sum_{i\in[n_\delta]} W_i' \mathbb{1}_{\{W_i'<\phi n\}}$. Since $|\sum_{\iota\in[n]} \mathbb{1}_{\{W_i\ge R\}}| \sim \mathrm{Bin}(n, \mathbb{P}(W\ge R))$ and R is a constant, we obtain by a Chernoff bound,

$$\begin{aligned}\mathbb{P}\left(\sum_{i=1}^{n} W_i \mathbb{1}_{\{W_i\in[R,\phi n)\}} > 2\mathbb{E}[W\mathbb{1}_{\{W\ge R\}}]n\right) &\le \mathbb{P}\left(\sum_{i=1}^{n_\delta} W_i' \mathbb{1}_{\{W_i'<\phi n\}} > 2\mathbb{E}[W\mathbb{1}_{\{W\ge R\}}]n\right) + \mathbb{P}(|\mathcal{V}_n[R,\phi n)| > n_\delta)\\ &= \mathbb{P}\left(\sum_{i=1}^{n_\delta} W_i' \mathbb{1}_{\{W_i'<\phi n\}} > \frac{2}{1+\delta}\mathbb{E}[W']n_\delta\right) + \exp(-\Omega(n)).\end{aligned}$$

Let $(W_i^{(\phi)})_{i\ge 1}$ be iid copies of W' conditional upon $W' < \phi n$. Then $\sum_{i\in[n_\delta]} W_i' \mathbb{1}_{\{W_i'<\phi n\}}$ is stochastically dominated by $\sum_{i\in[n_\delta]} W_i^{(\phi)}$. As a result, by Lemma 3 in (Resnick and Samorodnitsky 1999), also the first term is of order $o(n_\delta^{-C}) = o(n^{-C})$ for any ϕ sufficiently small. □

Our final auxiliary result is a bound for sums of independent Bernoulli random variables.

Lemma C.3 ((Alon and Spencer 2016, Theorem A.1.4)). *Let $B_i, i \ge 1$, be a sequence of independent Bernoulli random variables with $p_i = \mathbb{P}(B_i = 1) = 1 - \mathbb{P}(B_i = 0)$. Set $\mu_n = \sum_{i=1}^n p_i$. For every $b > 0$ we have*

$$\mathbb{P}\left(\sum_{i=1}^{n} B_i > (1+b)\mu_n\right) \le \mathrm{e}^{-\mu_n I_B(b)}, \mathbb{P}\left(\sum_{i=1}^{n} B_i < (1-b)\mu_n\right) \le \mathrm{e}^{-\mu_n I_B(-b)}, \tag{C2}$$

with $I_B(b) = (1+b)\log(1+b) - b$.

Appendix D

List of Notation

$\mathcal{G}_n = (\mathcal{V}_n, \mathcal{E}_n)$	Inhomogeneous random graph on n vertices, see Definition 1.2.
$\mathcal{G}_n[a,b) = (\mathcal{V}_n[a,b), \mathcal{E}_n[a,b))$	$\mathcal{G}_n$ induced on the vertices with weight in the interval $[a,b)$.
$\mathcal{W}_n[a,b)$	Set of weights of vertices in $\mathcal{V}_n[a,b)$.
$\deg_v[a,b)$	number of neighbors of v in $\mathcal{G}_n$ with weight in $[a,b)$.
$\mathcal{C}_n^{(1)}$	Largest connected component in $\mathcal{G}_n$.
$\mathcal{C}_n^{(1)}[a,b)$	Largest connected component in $\mathcal{G}_n[a,b)$.
$\mathcal{C}_n(u)$	Connected component of the vertex u in $\mathcal{G}_n$.
$\mathrm{CT}_\ell(\varepsilon, R)$	Set of possible component types, with weights discretized in intervals of length ε, and maximal weight R, see Definition 3.1.
$N_{n,\ell}$	Number of components of size ℓ in $\mathcal{G}_n$.
$N_n(\mathbf{w}^{(\ell)}, \varepsilon, \overline{w})$	Number of components of ε-type $\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon,\infty)$ in $\mathcal{G}_n[\underline{w}, \overline{w})$.
$S_{n,\ell}$	Number of vertices in $\mathcal{G}_n$ in component of size ℓ.
$\overline{P}(\mathbf{w}^{(\ell)}, y^{(h)})$	Probability that a vertex set of size ℓ with weights exactly $\mathbf{w}^{(\ell)} = (w_1, \dots, w_\ell)$ does not connect by an edge to a vertex set of size h with weights $n\mathbf{y}^{(h)} = (y_1 n, \dots, n y_h)$.
$\mathrm{BP} = \mathrm{BP}(\kappa_\sigma, F_W, q)$	Associated multi-type branching process representing the local limit of $\mathcal{G}_n$, see Definition 1.3.
T_q	Total progeny of BP.
θ_q	Survival probability of BP.
$\theta(\widetilde{\mathbf{w}}^{(\ell)}, \varepsilon)$	Probability that T_q has ε-type $\mathbf{w}^{(\ell)} \in \mathrm{CT}_\ell(\varepsilon,\infty)$.

In Section A we extend all of the above notations to the approximating inhomogeneous random graph from Definition A.1 by adding a superindex (δ, R).