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Lissajous varieties

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Abstract: This paper studies affine algebraic varieties parametrized by sine and cosine functions, generalizing algebraic Lissajous figures in the plane. We show that, up to a combinatorial factor, the degree of these varieties equals the volume of a polytope. We deduce defining equations from rank constraints on a matrix with polynomial entries. We discuss applications of Lissajous varieties in dynamical systems, in particular the Kuramoto model. This leads us to study connections with convex optimization and Lissajous discriminants.

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1 Introduction

A matrix $A \in \mathbb{Q}^{d \times n}$ defines a linear space $\text{Row}(A) \subseteq \mathbb{C}^n$ by taking the \mathbb{C} -linear span of its rows. In this paper, we are interested in the algebraic varieties obtained by taking the coordinate-wise cosine of points in translates of $\text{Row}(A)$. Concretely, let us define the map

$$\cos : \mathbb{C}^n \longrightarrow \mathbb{C}^n, \quad (x_1, \dots, x_n) \longmapsto (\cos(x_1), \dots, \cos(x_n)). \quad (1)$$

For any vector $b \in \mathbb{C}^n$, we let $\mathcal{L}_{A,b}$ be the image of the affine-linear space $L_{A,b} = \text{Row}(A) - b\pi/2 = \{x - b\pi/2 : x \in \text{Row}(A)\} \subseteq \mathbb{C}^n$ under the map \cos . In symbols, we set

$$\mathcal{L}_{A,b} = \cos(L_{A,b}). \quad (2)$$

Notice that $\mathcal{L}_{A,0} = \cos(\text{Row}(A))$ and $\mathcal{L}_{A,1} = \sin(\text{Row}(A))$, where $\mathbf{0} \in \mathbb{C}^n$ and $\mathbf{1} \in \mathbb{C}^n$ are the all-zeros and the all-ones vector respectively, and \sin is the coordinate-wise sine map, analogous to (1). It is convenient to write $\mathcal{C}_A = \mathcal{L}_{A,0}$ and $\mathcal{S}_A = \mathcal{L}_{A,1}$ for these special cases.

The requirement that A has rational entries ensures that $\mathcal{L}_{A,b}$ is an irreducible affine variety of dimension $\text{rank}(A)$. In particular, the set $\cos(L_{A,b}) \subseteq \mathbb{C}^n$ is Zariski closed in \mathbb{C}^n ; see Lemma 2.1. Our first goal is to determine its degree and defining equations. We present some familiar examples and illustrate some features of $\mathcal{L}_{A,b}$.

Example 1.1. The plane curves obtained from $A \in \mathbb{Q}^{1 \times 2}$ and $b \in \mathbb{C}^2$ are known in the literature as *Lissajous curves*, which motivates the title of our paper. Such curves describe two objects driven in simple harmonic motion along the x - and y -axis; cf. [11]. In that context, one usually allows real entries for A , in which case $\mathcal{L}_{A,b}$ is not necessarily an algebraic curve. Moreover, one restricts to real vectors $b \in \mathbb{R}^2$ and focuses on the real points of $\mathcal{L}_{A,b}$.

Example 1.2. The real points of $X = \{(x, y) \in \mathbb{C}^2 : x^2 + y^2 = 1\}$ form a circle of radius one, centered at the origin. The curve X is parametrized by $(\cos(t), \sin(t)) = (\cos(t), \cos(t - \frac{\pi}{2}))$. It is the Lissajous variety $X = \mathcal{L}_{A,b}$ with $A = \begin{pmatrix} 1 & 1 \end{pmatrix} \in \mathbb{Q}^{1 \times 2}$ and $b = (0, 1)$.

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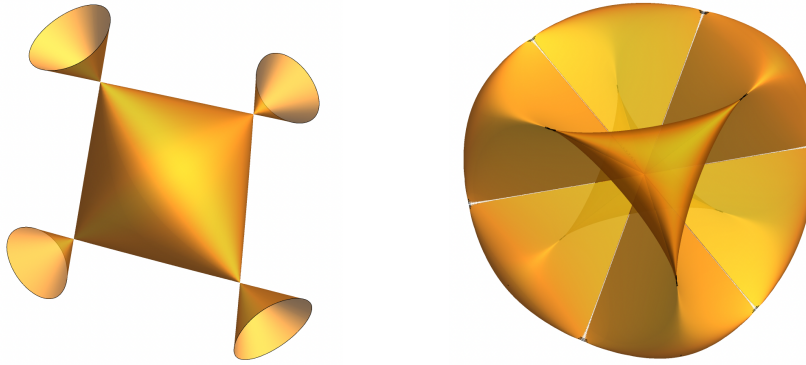


Figure 1: The surfaces \mathcal{C}_A (left) and \mathcal{S}_A (right) from Example 1.3.

Example 1.3. Let $A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$. The surfaces \mathcal{C}_A and \mathcal{S}_A are parametrized as follows:

$$\mathcal{C}_A : (\cos(t_1 - t_2), \cos(t_2), \cos(t_1)), \quad \mathcal{S}_A : (\sin(t_1 - t_2), \sin(t_2), -\sin(t_1)).$$

Figure 1 shows these surfaces. We compute that \mathcal{C}_A is given by $1 + 2xyz - x^2 - y^2 - z^2 = 0$. This is Cayley’s cubic surface with four nodes. It is the algebraic boundary of the ellipsope

$$\mathcal{E}_3 = \left\{ (x, y, z) \in \mathbb{R}^3 : \begin{pmatrix} 1 & x & y \\ x & 1 & z \\ y & z & 1 \end{pmatrix} \text{ is positive semi-definite} \right\},$$

a standard example of a spectrahedron in semi-definite programming; cf. [27]. The surface \mathcal{S}_A is

$$x^4 + 4x^2y^2z^2 - 2x^2y^2 - 2x^2z^2 + y^4 - 2y^2z^2 + z^4 = 0. \tag{3}$$

It has degree six. As the parameter b varies, the Lissajous varieties $\mathcal{L}_{A,b}$ form a family of surfaces. A generic fiber is reduced and irreducible of degree six. The special fiber for $b = \mathbf{0}$ is the cubic surface \mathcal{C}_A with multiplicity two. This is a general phenomenon, see Section 2.

In the next few paragraphs we summarize our main contributions and sketch the outline of the paper. Since $\mathcal{L}_{A,b}$ only depends on the affine space $L_{A,b}$, one may replace A by a matrix with the same row span without altering $\mathcal{L}_{A,b}$. Therefore, it is not restrictive to assume that A is of rank d , and we may clear denominators so that A has integer entries. Additionally, we make the following technical but equally non-restrictive assumption. The lattice \mathbb{Z}^d is an abelian group with entry-wise addition. Let $\mathbb{Z}A \subseteq \mathbb{Z}^d$ be the subgroup generated by the columns $a_1, \dots, a_n \in \mathbb{Z}^d$ of A . We assume that $\mathbb{Z}A = \mathbb{Z}^d$. The following statement, proved and stated in more detail in Section 2, gives the degree of $\mathcal{L}_{A,b}$, i.e., the number of intersection points of $\mathcal{L}_{A,b}$ with a generic affine-linear space of complementary dimension $n - \dim(\mathcal{L}_{A,b})$.

Theorem 1.4. Let $A = (a_1 \ a_2 \ \dots \ a_n) \in \mathbb{Z}^{d \times n}$ be such that $\text{rank}(A) = d$ and $\mathbb{Z}A = \mathbb{Z}^d$. Let $P_A \subset \mathbb{R}^d$ be the polytope obtained as the convex hull of the lattice points $\{\pm a_1, \dots, \pm a_n\} \subset \mathbb{Z}^d$.

Then for any $b \in \mathbb{C}^n$, the variety $\mathcal{L}_{A,b}$ is irreducible of dimension d . Moreover, for generic $b \in \mathbb{C}^n$ the degree of $\mathcal{L}_{A,b}$ is $\deg(\mathcal{L}_{A,b}) = 2^{-\text{CL}_A} d! \text{vol}(P_A)$, where $\text{vol}(\cdot)$ denotes the euclidean volume and CL_A is the number of zero entries of a generic vector in the kernel of $A : \mathbb{R}^n \rightarrow \mathbb{R}^d$.

The quantity $d! \text{vol}(P_A)$ is called the normalized volume of P_A . The polytope P_A in Example 1.2 is the line segment $[-1, 1]$, with normalized volume two, which is the degree of the circle. The polygon P_A in Example 1.3 is a hexagon with normalized volume $6 = \deg \mathcal{S}_A$. The notion of “generic b ” in Theorem 1.4 will be made more precise in Section 2, and we give a formula for $\deg(\mathcal{L}_{A,b})$ which holds for any A, b , but requires more notation (Theorem 2.3). We show that the hypersurface \mathcal{C}_A obtained from the incidence matrix A of the n -cycle graph C_n is the cycle polynomial from [25, Section 4.2]. Using Theorem 2.3 and a result from [5], we prove a formula for the degree of the cycle polynomial (Proposition 2.6).

In Section 3, we construct a matrix with polynomial entries whose maximal minors form a set of set-theoretic defining equations of $\mathcal{L}_{A,b}$ (Theorem 3.1). In particular, a point $x^* \in \mathbb{C}^n$ belongs to $\mathcal{L}_{A,b}$ if and only if that matrix is not of full rank when evaluated at x^* .

Section 4 establishes the role of Lissajous varieties of type \mathcal{S}_A in computing steady state angles for the Kuramoto equations of coupled oscillators. In this case, A is the incidence matrix of a graph G which encodes the coupling. We will see that the equilibrium angles correspond to the intersection points of \mathcal{S}_A with an affine-linear space of the form $Ax = \omega$. In particular, the degree of \mathcal{S}_A bounds the number of isolated solutions.

In Section 5, we generalize the Kuramoto equations and construct dynamical systems whose steady state varieties are linear sections $Ax = \omega$ of a Lissajous variety. Under certain assumptions on ω , we show that one of the intersection points in $\mathcal{L}_{A,b} \cap \{Ax = \omega\}$ corresponds to a stable equilibrium. It is the unique solution to a convex optimization problem. For varying ω , these equilibria parametrize a subset of the Lissajous variety, called its positive part.

In Section 6, we study the discriminant of the equilibrium equations $\mathcal{L}_{A,b} \cap \{Ax = \omega\}$ in the parameters ω . We call this the Lissajous discriminant. We bound its degree and, when A comes from a graph G , we describe its symmetries in terms of those of G . This is a first step in the bifurcation analysis of our dynamical system introduced in Section 5, see Example 6.6.

Related work. Replacing “cos” by “exp” in (2), we obtain the affine toric variety Y_A of the matrix A up to scaling the coordinates by $\exp(b_1), \dots, \exp(b_n)$. Such scaled toric varieties appear in our study of the defining equations of $\mathcal{L}_{A,b}$, see Sections 3 and 4. The variety $\mathcal{C}_A = \mathcal{L}_{A,0}$ is called a *Chebyshev variety* associated with A in [1, Section 5]. This is motivated by the fact that for $d = 1$, the curve \mathcal{C}_A is alternatively obtained from a parametrization by Chebyshev polynomials of the first kind. Previous work on the algebraic geometry of Kuramoto equations includes [5; 12; 18]. These works use a different “algebraic geometrization” of the equations, see Equation (3) in [18] and Definition 1.4 in [12]. Some Lissajous discriminants were studied using machine learning techniques in [2, Section 5.3].

2 Dimension and degree

As mentioned in the Introduction, if the matrices $A_1 \in \mathbb{Q}^{d_1 \times n}, A_2 \in \mathbb{Q}^{d_2 \times n}$ have the same row span, then we have an equality of Lissajous varieties $\mathcal{L}_{A_1,b} = \mathcal{L}_{A_2,b}$. In particular, after clearing denominators, we may assume that the matrix A has integer entries. We fix $A \in \mathbb{Z}^{d \times n}$ and $b \in \mathbb{C}^n$.

Choosing coordinates on $\text{Row}(A)$, we can parametrize $\mathcal{L}_{A,b}$ as follows:

$$\varphi_{A,b} : \mathbb{C}^d \rightarrow \mathbb{C}^n, \quad t = (t_1, \dots, t_d) \mapsto (\cos(a_1 \cdot t - b_1 \frac{\pi}{2}), \dots, \cos(a_n \cdot t - b_n \frac{\pi}{2})).$$

Here $a_j \in \mathbb{Q}^d$ is the j -th column of A and $a_j \cdot t$ is the standard dot product. We clearly have $\mathcal{L}_{A,b} = \text{im}\varphi_{A,b}$. We obtain a rational parametrization of our Lissajous variety as follows. Set

$$\beta_\ell = e^{-ib_\ell \pi/2} \text{ for } \ell = 1, \dots, n, \quad \text{with } i = \sqrt{-1}.$$

Using Euler’s identity $\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$, we see that $\varphi_{A,b}(t) = \psi_{A,b}(e^{it_1}, \dots, e^{it_d})$ with

$$\psi_{A,b} : (\mathbb{C}^*)^d \rightarrow \mathbb{C}^n, \quad v = (v_1, \dots, v_d) \mapsto \left(\frac{\beta_1 v^{a_1} + \beta_1^{-1} v^{-a_1}}{2}, \dots, \frac{\beta_n v^{a_n} + \beta_n^{-1} v^{-a_n}}{2} \right). \tag{4}$$

Here $v^{a_j} = \prod_{k=1}^d v_k^{a_{kj}}$. We show below that $\text{im}\varphi_{A,b} = \text{im}\psi_{A,b}$, and therefore $\mathcal{L}_{A,b} = \text{im}\psi_{A,b}$.

Lemma 2.1. *The set $\mathcal{L}_{A,b} = \text{im}\psi_{A,b} = \text{im}\varphi_{A,b} \subseteq \mathbb{C}^n$ is a closed affine subvariety of \mathbb{C}^n . Moreover, $\mathcal{L}_{A,b}$ is irreducible of dimension $\text{rank}(A)$.*

Proof. The equality $\text{im}\varphi_{A,b} = \text{im}\psi_{A,b}$ follows from the fact that $\varphi_{A,b} : \mathbb{C}^d \rightarrow \mathbb{C}^n$ is the composition of $\psi_{A,b} : (\mathbb{C}^*)^d \rightarrow \mathbb{C}^n$ with the surjective map $\mathbb{C}^d \rightarrow (\mathbb{C}^*)^d$ given by $t \mapsto (e^{it_1}, \dots, e^{it_d})$. The image $\text{im}\varphi_{A,b} = \text{im}\psi_{A,b}$ only

depends on the row span of A . After applying integer row operations and column permutations to A , and after dropping rows whose entries are all zero, we obtain a matrix \tilde{A} satisfying $\text{im}\psi_{A,b} = \text{im}\psi_{\tilde{A},b}$ of the form

$$\tilde{A} = \begin{pmatrix} a_{11} & 0 & \cdots & 0 & a_{1,r+1} & \cdots & a_{1,n} \\ 0 & a_{22} & \cdots & 0 & a_{2,r+1} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & a_{rr} & a_{r,r+1} & \cdots & a_{r,n} \end{pmatrix} \in \mathbb{Z}^{r \times n},$$

for some positive integers $a_{jj}, j = 1, \dots, r$. Here $r \leq d$ is the rank of A . We want to show that the image of $\psi_{\tilde{A},b}: (\mathbb{C}^*)^r \rightarrow \mathbb{C}^n$ is Zariski closed. Since \mathbb{C} is algebraically closed, the Zariski closure of $\text{im}\psi_{\tilde{A},b}$ equals its Euclidean closure. Hence, if $x^* \in \mathbb{C}^n$ lies in the closure of $\text{im}\psi_{\tilde{A},b}$, there is a smooth path $\{x^*(t) : t \in (0, 1]\}$ contained in $\text{im}\psi_{\tilde{A},b}$ whose limit for $t \rightarrow 0$ is x^* . This lifts to a smooth path $v(t) = (v_1(t), \dots, v_r(t))$ in $(\mathbb{C}^*)^r$ with $\psi_{\tilde{A},b}(v(t)) = x^*(t)$. We have

$$x_j^* = \lim_{t \rightarrow 0} \frac{1}{2}(\beta_j v_j(t)^{a_{jj}} + \beta_j^{-1} v_j(t)^{-a_{jj}}) \quad \text{for } j = 1, \dots, r. \tag{5}$$

Since the limit (5) lies in \mathbb{C} , we have $\lim_{t \rightarrow 0} v_j(t) \in \mathbb{C}^*$. Indeed, if the limit $\lim_{t \rightarrow 0} v_j(t)$ does not exist in \mathbb{C}^* , then $v_j(t)$ approaches 0 or ∞ for $t \rightarrow 0$. In both cases, $x_j(t)$ would not converge in \mathbb{C} . Hence, we have that $x^* = \lim_{t \rightarrow 0} \psi_{\tilde{A},b}(v(t)) = \psi_{\tilde{A},b}(\lim_{t \rightarrow 0} v(t)) \in \text{im}\psi_{\tilde{A},b}$.

We have now shown that $\mathcal{L}_{A,b} = \text{im}\psi_{A,b} = \text{im}\psi_{\tilde{A},b}$ is closed in \mathbb{C}^n . Since $\mathcal{L}_{A,b}$ is unirational, it is irreducible. It is also clear that the projection of $\text{im}\psi_{\tilde{A},b}$ to the first r coordinates is surjective onto \mathbb{C}^r . Hence, we have $\dim \mathcal{L}_{A,b} = r = \text{rank}(A)$. \square

The equality $\mathcal{L}_{A,b} = \text{im}\psi_{A,b}$ implies an algorithm for computing the defining ideal of $\mathcal{L}_{A,b}$:

- (1) Work in the ring $\mathbb{C}[v_1, \dots, v_d, w_1, \dots, w_d, x_1, \dots, x_n]$ with $2d + n$ variables.
- (2) For each column a_i of A , write $a_i = a_i^+ - a_i^-$ with $a_i^+, a_i^- \in \mathbb{N}^d$.
- (3) Define $I = \langle 2x_j - \beta_1 v_1^{a_j^+} w_1^{a_j^-} - \beta_1^{-1} v_1^{a_j^-} w_1^{a_j^+}, v_k w_k - 1 : j = 1, \dots, n, k = 1, \dots, d \rangle$.
- (4) Compute the elimination ideal $I \cap \mathbb{C}[x_1, \dots, x_n]$. The result equals $I(\mathcal{L}_{A,b})$.

The ideal I in step 3 has $n + d$ generators. Adding variables w_k and imposing $v_k w_k - 1$ is an effective way of working in the Laurent polynomial ring $\mathbb{C}[v_1^{\pm 1}, \dots, v_d^{\pm 1}]$: w_k plays the role of v_k^{-1} . The correctness of this algorithm follows from the parametrization $\psi_{A,b}$. Note that for the ground field we can use any field extension of \mathbb{Q} containing β_1, \dots, β_n . In particular, if β_ℓ is rational for $\ell = 1, \dots, n$, then one can replace \mathbb{C} by \mathbb{Q} in step 1. This happens when $b = \mathbf{0}$ and $\mathcal{L}_{A,b} = \mathcal{C}_A$. If $b = \mathbf{1}$ and $\mathcal{L}_{A,b} = \mathcal{S}_A$, then one can work over the field $\mathbb{Q}[i] = \mathbb{Q}[z]/\langle z^2 + 1 \rangle$.

The integer matrix A defines an affine toric variety $Y_A \subseteq \mathbb{C}^n$ obtained as the closure of the image of the Laurent monomial map $v \mapsto (v^{a_1}, \dots, v^{a_n}), v \in (\mathbb{C}^*)^d$, cf. [6; 26]. Its ideal is

$$I_A = \langle y^u - y^w : u, w \in \mathbb{N}^n \text{ and } A(u - w) = 0 \rangle \subseteq \mathbb{C}[y_1, \dots, y_n].$$

A modified or *scaled* toric variety $Y_{A,\beta}$ is obtained from $A \in \mathbb{Z}^{d \times n}, \beta \in (\mathbb{C}^*)^n$ as follows:

$$Y_{A,\beta} = \{(\beta_1 y_1, \dots, \beta_n y_n) : y \in Y_A\} \subseteq \mathbb{C}^n.$$

This variety is parametrized by $v \mapsto (\beta_1 v^{a_1}, \dots, \beta_n v^{a_n})$. One checks that if $I_A = I(Y_A) = \langle y^{u_1} - y^{w_1}, \dots, y^{u_r} - y^{w_r} \rangle$, then $I_{A,\beta} = I(Y_{A,\beta})$ is generated by $\beta^{w_k} y^{u_k} - \beta^{u_k} y^{w_k}, k = 1, \dots, r$.

Theorem 2.2. *The Lissajous variety $\mathcal{L}_{A,b} \subseteq \mathbb{C}^n$ is the image of the variety*

$$\mathcal{Y}_{A,b} = \{(x, y) \in \mathbb{C}^n \times (\mathbb{C}^*)^n : y \in Y_{A,\beta} \text{ and } x_j = \frac{1}{2}(y_j + y_j^{-1}) \text{ for } j = 1, \dots, n\}$$

under the coordinate projection $\pi_{A,b} : \mathcal{Y}_{A,b} \rightarrow \mathbb{C}^n$. Here $\beta = (e^{-ib_1\pi/2}, \dots, e^{-ib_n\pi/2}) \in (\mathbb{C}^)^n$.*

Proof. The proof is an adaptation of the proof of [1, Theorem 5.2]. The map $y \mapsto (\frac{1}{2}(y_1 + y_1^{-1}), \dots, \frac{1}{2}(y_n + y_n^{-1}), y)$ induces an isomorphism $Y_{A,\beta} \cap (\mathbb{C}^*)^n \simeq \mathcal{Y}_{A,b}$. Hence, we know from basic toric geometry that $\mathcal{Y}_{A,b}$ is a torus of dimension $\text{rank}(A)$, cf. [6; 26]. Composing this map with the coordinate projection $\pi_{A,b}$ and setting $y_j = \beta_j v^{a_j}$, we obtain precisely the map $\psi_{A,b}$ with image $\mathcal{L}_{A,b}$. This implies that $\mathcal{L}_{A,b} = \pi_{A,b}(\mathcal{Y}_{A,b})$. \square

Deleting spurious rows if necessary, we may assume that A has rank d . To state a degree formula for $\mathcal{L}_{A,b}$, let $[\mathbb{Z}^d : \mathbb{Z}A]$ be the index of the lattice $\mathbb{Z}A$ generated by the columns of A in the ambient lattice \mathbb{Z}^d . The *degree* $\text{deg } \varphi$ of a dominant morphism $\varphi : X \rightarrow Y$ between irreducible varieties of dimension d is the cardinality of a generic fiber.

Theorem 2.3. *Let $A \in \mathbb{Z}^{d \times n}$ be such that $\text{rank}(A) = d$. Let $P_A \subset \mathbb{R}^d$ be the polytope obtained as the convex hull of the lattice points $\{\pm a_1, \dots, \pm a_n\} \subset \mathbb{Z}^d$. For any $b \in \mathbb{C}^n$ we have*

$$\text{deg}(\mathcal{L}_{A,b}) = \frac{d! \text{vol}(P_A)}{\text{deg } \pi_{A,b} \cdot [\mathbb{Z}^d : \mathbb{Z}A]}$$

where $\text{vol}(\cdot)$ denotes the euclidean volume and $\pi_{A,b} : \mathcal{Y}_{A,b} \rightarrow \mathcal{L}_{A,b}$ is as in Theorem 2.2.

Proof. This is a generalization of [1, Theorem 5.3]. The degree of $\mathcal{L}_{A,b}$ is the cardinality of

$$S_X = \{x \in \mathcal{L}_{A,b} : c_{j0} + c_{j1}x_1 + \dots + c_{jn}x_n = 0, j = 1, \dots, d\}$$

for generic complex coefficients c_{jk} . We compare the set S_X to the set S_V given by

$$S_V = \{v \in (\mathbb{C}^*)^d : c_{j0} + \frac{c_{j1}}{2}(\beta_1 v^{a_1} + \beta_1^{-1} v^{-a_1}) + \dots + \frac{c_{jn}}{2}(\beta_n v^{a_n} + \beta_n^{-1} v^{-a_n}) = 0, j = 1, \dots, d\}.$$

By Lemma 2.1, we have $\psi_{A,b}(S_V) = S_X$ for any choice of c_{jk} . Moreover, this correspondence is generically $(\text{deg } \psi_{A,b})$ -to-one. The denominator $\text{deg } \pi_{A,b} \cdot [\mathbb{Z}^d : \mathbb{Z}A]$ in our degree formula is the degree of $\psi_{A,b}$, since $\psi_{A,b}$ is the composition of the map $v \mapsto (\psi_{A,b}(v), \beta_1 v^{a_1}, \dots, \beta_n v^{a_n})$ with $\pi_{A,b}$ (Theorem 2.2). Indeed, the first map has the same degree as $v \mapsto (\beta_1 v^{a_1}, \dots, \beta_n v^{a_n})$, which parametrizes $Y_{A,\beta}$ and whose degree is $[\mathbb{Z}^d : \mathbb{Z}A]$, see [26].

It remains to show that S_V consists of $d! \text{vol}(P_A)$ points. This is an application of Kushnirenko’s theorem [13, Théorème III’], which predicts the maximal (and expected) number of solutions to a system of Laurent polynomial equations with identical monomial support. In order to apply this theorem, we must ensure that our equations are *non-degenerate* with respect to the polytope P_A . This means that none of the facial subsystems have a solution in $(\mathbb{C}^*)^d$. By our assumption that $\text{rank}(A) = d$, the centrally symmetric polytope P_A is full-dimensional and it contains the origin in its interior. This implies that no face of P_A , except P_A itself, contains both the lattice points a_j and $-a_j$. Therefore, there are no dependencies among the coefficients appearing in any facial subsystem. Hence, for generic c_{jk} , our equations are indeed non-degenerate and Kushnirenko’s upper bound $|S_V| \leq d! \text{vol}(P_A)$ is attained. \square

To complete the proof of Theorem 1.4, we must introduce some more notation. We write $\text{Circ}(A)$ for the set of *circuits* of A . That is, $\text{Circ}(A)$ is the set of all subsets of $[n] = \{1, \dots, n\}$ indexing a minimal set of linearly dependent columns of A . For any integer vector $m = (m_1, \dots, m_n) \in \mathbb{Z}^n$, we define the *support* of m as follows: $\text{supp}(m) = \{j \in [n] : m_j \neq 0\}$. For each circuit $C \in \text{Circ}(A)$ there is a unique (up to sign) integer vector $m^C = (m_1^C, \dots, m_n^C) \in \mathbb{Z}^n$ of minimal length satisfying $A \cdot m^C = 0$ and $\text{supp}(m^C) = C$. This vector encodes the unique linear relation between the columns indexed by the circuit C . A *coloop* of the matrix A is an element $j \in [n]$ which does not belong to any circuit. We denote the set of coloops by $\text{Coloops}(A) \subseteq [n]$, and its cardinality by $\text{CL}_A = \#\text{Coloops}(A)$. The integer CL_A is the number of zero entries of a generic vector in the kernel of A , as in Theorem 1.4.

Theorem 2.4. *Assume that $\mathbb{Z}A = \mathbb{Z}^d$, so that $\text{rank}(A) = d$ and $[\mathbb{Z}^d : \mathbb{Z}A] = 1$. If for each circuit $C \in \text{Circ}(A)$ the vector $b = (b_1, \dots, b_n) \in \mathbb{C}^n$ has the property that*

$$m^C \cdot b = \sum_{j \in C} m_j^C b_j \quad \text{is not an even integer,} \tag{6}$$

then $\text{deg } \pi_{A,b} = 2^{\text{CL}_A}$ and the formula from Theorem 2.3 simplifies to $\text{deg}(\mathcal{L}_{A,b}) = d! \text{vol}(P_A) 2^{-\text{CL}_A}$.

Proof. To investigate the degree of $\pi_{A,b}$, pick a generic point $(x, y) \in \mathcal{Y}_{A,b} \subseteq \mathbb{C}^n \times (\mathbb{C}^*)^n$ and note that the only candidates for the points in the fiber $\pi_{A,b}^{-1}(\pi_{A,b}(x, y))$ are the 2^n points $(x, y_1^{\pm 1}, \dots, y_n^{\pm 1})$. For any subset $J \subseteq [n]$, let $\tilde{y} \in (\mathbb{C}^*)^n$ be given by

$$\tilde{y}_j = \begin{cases} y_j & j \notin J \\ y_j^{-1} & j \in J. \end{cases}$$

To test whether $(x, \tilde{y}) \in \pi_{A,b}^{-1}(\pi_{A,b}(x, y))$, we need to check whether $\tilde{y} \in Y_{A,\beta} \cap (\mathbb{C}^*)^n$. The variety $Y_{A,\beta} \cap (\mathbb{C}^*)^n$ is given by one binomial equation for each circuit of A :

$$Y_{A,\beta} \cap (\mathbb{C}^*)^n = \{y \in (\mathbb{C}^*)^n : y^{m^C} = \beta^{m^C} \text{ for all } C \in \text{Circ}(A)\}.$$

If $J \subseteq \text{Coloops}(A)$, then the coordinates y_j with $j \in J$ do not appear in our binomial equations and \tilde{y} lies on $Y_{A,\beta} \cap (\mathbb{C}^*)^n$. If $J \not\subseteq \text{Coloops}(A)$ (in particular, $J \neq \emptyset$), pick a circuit C such that $J \cap C \neq \emptyset$. If both y and \tilde{y} lie on $Y_{A,\beta}$, then

$$y^{m^C} \cdot \tilde{y}^{m^C} = \prod_{j \in C \setminus J} y_j^{2m_j^C} = \beta^{2m^C}. \tag{7}$$

The second equality in (7) holds for generic $y \in Y_{A,\beta}$ if and only if, for all $v \in (\mathbb{C}^*)^d$, we have

$$\prod_{j \in C \setminus J} (\beta_j v^{a_j})^{2m_j^C} = \beta^{2m^C}, \quad \text{which implies} \quad v^{2 \sum_{j \in C \setminus J} m_j^C a_j} = \prod_{j \in J \cap C} \beta_j^{2m_j^C}.$$

In particular, we must have $\sum_{j \in C \setminus J} m_j^C a_j = 0$. If $J \cap C \subsetneq C$, then this contradicts the fact that C is a circuit. If $J \cap C = C$, then the equality fails if $\beta^{2m^C} \neq 1$. Taking the logarithm on both sides of this inequality, we obtain precisely the condition (6). We have shown that, under the condition (6) for every circuit $C \in \text{Circ}(A)$, the points in $\pi_{A,b}^{-1}(\pi_{A,b}(x, y))$ are the points of the form \tilde{y} corresponding to $J \subseteq \text{Coloops}(A)$. Hence the cardinality is 2^{CL_A} . \square

Remark 2.5. When $b = 0$, the condition (6) is never satisfied. In fact, in this case we have $\deg \pi_{A,b} \geq 2$, as the fiber $\pi_{A,0}^{-1}(\pi_{A,0}(x, y))$ contains (x, y) and (x, y^{-1}) ; see [1, Remark 5.4]. The second point is obtained as \tilde{y} for $J = [n]$ in the notation of the proof of Theorem 2.4.

We end the section by applying our degree formula to a family of Chebyshev hypersurfaces which arises when studying elliptopes of cycle graphs; cf. [23; 25]. This is relevant in semidefinite completion problems, see for instance [15]. Rephrasing [25, Equation (28)] in our notation, the n -th cycle polynomial Γ'_n is the defining equation of the Chebyshev hypersurface \mathcal{C}_{A_n} , with

$$A_n = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \end{pmatrix} \in \mathbb{Z}^{(n-1) \times n}. \tag{8}$$

Note that Γ'_n is defined up to scale. Its degree was computed up to $n = 11$ in [25, Table 1]. In the paragraph preceding Conjecture 4.9 in [25], it is stated that a closed formula for $\deg(\Gamma'_n)$ is not known. The next proposition provides such a formula.

Proposition 2.6. *The degree of the n -th cycle polynomial Γ'_n is given by*

$$\deg(\Gamma'_n) = \frac{n}{2} \binom{n-1}{\lfloor \frac{1}{2}(n-1) \rfloor}. \tag{9}$$

Proof. By Theorem 2.3 we have $\deg(\Gamma'_n) = (\deg \pi_{A_n,0})^{-1} (n-1)! \text{vol}(P_{A_n})$. Let $y \in Y_{A_n} \cap (\mathbb{C}^*)^n = \{y \in (\mathbb{C}^*)^n : y_1 \cdots y_{d+1} = 1\}$ be a generic point on the affine toric hypersurface associated to A_n . The degree of $\pi_{A,0}$ is the number of points among $\{(y_1^{\pm 1}, \dots, y_n^{\pm 1})\}$ lying on Y_{A_n} . That number is 2: only (y_1, \dots, y_n) and $(y_1^{-1}, \dots, y_n^{-1})$ lie on Y_{A_n} . The number $(n-1)! \text{vol}(P_{A_n})$ was computed in [5, Theorem 14], and multiplying with $1/2$ gives (9). \square

3 Determinantal equations

Our goal in this section is to derive defining equations for $\mathcal{L}_{A,b}$ from rank conditions on a matrix with entries in $\mathbb{C}[x]$. Motivated by the Equations $2x_j = y_j + y_j^{-1}$ appearing in Theorem 2.2, we work in the following setup. Let $K = \overline{\mathbb{C}(x)}$ be the algebraic closure of the field of rational functions in $x = (x_1, \dots, x_n)$, and consider the following ideal:

$$J = \langle 2x_1 - y_1 - y_1^{-1}, \dots, 2x_n - y_n - y_n^{-1} \rangle \subset K[y^{\pm}] = K[y_1^{\pm}, \dots, y_n^{\pm}].$$

The affine variety $V(J)$ defined by J in $(K^*)^n$ is zero-dimensional and consists of 2^n distinct points, each with multiplicity one. Hence, J is radical and the quotient $\mathcal{A} = K[y^{\pm}]/J$ is a K -vector space of dimension 2^n . Let $f_j = 2x_j - y_j - y_j^{-1}$ be the j -th generator of J . We have

$$\mathcal{A} = \frac{K[y^{\pm}]}{\langle f_1, \dots, f_n \rangle} \simeq \frac{K[y_1^{\pm}]}{\langle f_1 \rangle} \otimes_K \dots \otimes_K \frac{K[y_n^{\pm}]}{\langle f_n \rangle}. \tag{10}$$

Below we write $[h]$ for the residue class of $h \in K[y^{\pm}]$ in \mathcal{A} . Multiplication with an element $g \in K[y^{\pm}]$ gives a K -linear endomorphism $M_g : \mathcal{A} \rightarrow \mathcal{A}$, where $M_g([h]) = [gh]$. Once we fix a K -basis for \mathcal{A} , such a map is represented by a $2^n \times 2^n$ matrix with entries in K . We shall now describe such matrices for a basis compatible with the tensor product structure (10).

For a moment, set $n = 1$. Let us fix the K -basis $\{[1], [y_1]\}$ for the algebra $\mathcal{A} = K[y_1^{\pm}]/\langle f_1 \rangle$. We claim that, in this basis, multiplication with y_1 is given by the 2×2 matrix

$$M_{y_1} = \begin{pmatrix} 0 & -1 \\ 1 & 2x_1 \end{pmatrix}.$$

Indeed, the first column reads $[y_1 \cdot 1] = 0 \cdot [1] + 1 \cdot [y_1]$, and the second column reads

$$[y_1 \cdot y_1] = [y_1^2] = -1 \cdot [1] + 2x_1 \cdot [y_1],$$

where the second equality follows from $[y_1 f_1] = 0$. Multiplication with a general element $g \in K[y_1^{\pm}]$ is given by $M_g = g(M_{y_1})$, where $g(M_{y_1})$ denotes the matrix obtained by substituting M_{y_1} for y_1 in the monomial expansion of g .

By (10), a general element $a \in \mathcal{A}$ is given by a finite sum $a = \sum_k a_1^{(k)} \otimes \dots \otimes a_n^{(k)}$, where

$$a_j^{(k)} \in \mathcal{A}_j = K[y_j^{\pm}]/\langle f_j \rangle.$$

Multiplication by a single variable, say y_1 , satisfies

$$M_{y_1}(a) = M_{y_1} \left(\sum_k a_1^{(k)} \otimes \dots \otimes a_n^{(k)} \right) = \sum_k M_{y_1}(a_1^{(k)}) \otimes a_2^{(k)} \otimes \dots \otimes a_n^{(k)}.$$

Hence, fixing the basis $\{[1], [y_j]\}$ for \mathcal{A}_j and the corresponding tensor product basis for $\mathcal{A} = \mathcal{A}_1 \otimes_K \dots \otimes_K \mathcal{A}_n$, our previous observation gives

$$M_{y_1} = \begin{pmatrix} 0 & -1 \\ 1 & 2x_1 \end{pmatrix} \otimes \text{id}_2 \otimes \dots \otimes \text{id}_2,$$

where id_2 is the 2×2 identity matrix and \otimes is the Kronecker product of matrices. The matrices M_{y_2}, \dots, M_{y_n} are found in an analogous way. By definition, our matrices M_{y_1}, \dots, M_{y_n} are pairwise commuting. Hence, for any Laurent polynomial $g \in K[y^{\pm}]$, the matrix $g(M_{y_1}, \dots, M_{y_n})$ obtained by substituting M_{y_j} for y_j in the monomial expansion of g is well-defined and represents the map M_g . Since the entries of both M_{y_j} and $M_{y_j}^{-1}$ are polynomials in x_j , we have that $M_g \in \mathbb{C}[x]^{2^n \times 2^n}$ for any $g \in \mathbb{C}[y^{\pm}] \subset K[y^{\pm}]$. Here is the main theorem of this section.

Theorem 3.1. Fix $A \in \mathbb{Z}^{d \times n}$ and $b \in \mathbb{C}^n$. Let $\beta \in (\mathbb{C}^*)^n$ as usual, and let $Y_{A,\beta} \subseteq \mathbb{C}^n$ be the corresponding scaled affine toric variety. If g_1, \dots, g_r generate the ideal of $Y_{A,\beta} \cap (\mathbb{C}^*)^n$ in $\mathbb{C}[y_1^{\pm}, \dots, y_n^{\pm}]$, then

$$\mathcal{L}_{A,b} = \{x \in \mathbb{C}^n : \text{rank}(M_{g_1} | \dots | M_{g_r}) < 2^n\}, \tag{11}$$

where M_{g_k} is a $2^n \times 2^n$ -matrix with entries in $\mathbb{C}[x]$ representing multiplication by g_k in \mathcal{A} .

Remark 3.2. It is easier to obtain generators for the ideal of $Y_{A,\beta} \cap (\mathbb{C}^*)^n$ than for the toric ideal of $Y_{A,\beta}$. First, one computes a \mathbb{Z} -kernel of A . This can be found, for instance, via the Smith normal form of A . If this kernel is generated by $m_1, \dots, m_r \in \mathbb{Z}^n$, with $r = n - \text{rank}(A)$, then the binomials g_1, \dots, g_r in Theorem 3.1 are given by $y^{m_k} - \beta^{m_k}$, $k = 1, \dots, r$; see [26]. To avoid computing matrix inverses, one may clear denominators by writing $m_k = u_k - w_k$ with $u_k, w_k \in \mathbb{N}^n$ and use $g_k = \beta^{w_k} y^{u_k} - \beta^{u_k} y^{w_k}$ instead.

By Theorem 3.1 a set of defining equations for $\mathcal{L}_{A,b}$ is given by the maximal minors of the $2^n \times (r2^n)$ -matrix $(M_{g_1} | \dots | M_{g_r})$ obtained by concatenating the matrices M_{g_j} as block columns. In particular, if A has rank $n - 1$, then $Y_{A,b}$ is a hypersurface and its ideal $I_{A,b} = \langle g \rangle$ is principal. Hence, Theorem 3.1 provides a determinantal representation for $\mathcal{L}_{A,b}$. Before proving Theorem 3.1, we illustrate the statement in our running example.

Example 3.3. Let A be as in Example 1.3. Following the discussion above, we construct matrices representing multiplication with the variables in $K[y_1^{\pm 1}, y_2^{\pm 1}, y_3^{\pm 1}] / \langle f_1, f_2, f_3 \rangle$:

$$M_{y_1} = \begin{pmatrix} 0 & -1 \\ 1 & 2x_1 \end{pmatrix} \otimes \text{id}_2 \otimes \text{id}_2, \quad M_{y_2} = \text{id}_2 \otimes \begin{pmatrix} 0 & -1 \\ 1 & 2x_2 \end{pmatrix} \otimes \text{id}_2 \quad \text{and} \quad M_{y_3} = \text{id}_2 \otimes \text{id}_2 \otimes \begin{pmatrix} 0 & -1 \\ 1 & 2x_3 \end{pmatrix}.$$

The toric ideal of $Y_{A,\beta}$ is generated by $g = y_1 y_2 y_3 - \beta_1 \beta_2 \beta_3$. Evaluating this at our multiplication operators, we find $M_g = M_{y_1} M_{y_2} M_{y_3} - \beta_1 \beta_2 \beta_3 \text{id}_{8 \times 8}$, which gives the following result:

$$M_g = \begin{pmatrix} -\beta_1 \beta_2 \beta_3 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & -\beta_1 \beta_2 \beta_3 & 0 & 0 & 0 & 0 & 1 & 2x_3 \\ 0 & 0 & -\beta_1 \beta_2 \beta_3 & 0 & 0 & 1 & 0 & 2x_2 \\ 0 & 0 & 0 & -\beta_1 \beta_2 \beta_3 & -1 & -2x_3 & -2x_2 & -4x_2 x_3 \\ 0 & 0 & 0 & 1 & -\beta_1 \beta_2 \beta_3 & 0 & 0 & 2x_1 \\ 0 & 0 & -1 & -2x_3 & 0 & -\beta_1 \beta_2 \beta_3 & -2x_1 & -4x_1 x_3 \\ 0 & -1 & 0 & -2x_2 & 0 & -2x_1 & -\beta_1 \beta_2 \beta_3 & -4x_1 x_2 \\ 1 & 2x_3 & 2x_2 & 4x_2 x_3 & 2x_1 & 4x_1 x_3 & 4x_1 x_2 & 8x_1 x_2 x_3 - \beta_1 \beta_2 \beta_3 \end{pmatrix}$$

Setting $b = \mathbf{0}$ (i.e., $\beta = (1, 1, 1)$), the determinant evaluates to

$$(\det M_g)_{|\beta=(1,1,1)} = 16(x_1^2 - 2x_1 x_2 x_3 + x_2^2 + x_3^2 - 1)^2.$$

This is the square of the defining equation of Cayley’s cubic surface $\mathcal{C}_A = \mathcal{L}_{A,\mathbf{0}}$. This computation shows that the equality (11) holds only set-theoretically; the maximal minors of $(M_{g_1} | \dots | M_{g_r})$ do not necessarily generate a radical ideal. The equation (3) for $\mathcal{S}_A = \mathcal{L}_{A,\mathbf{1}}$ is obtained by evaluating $\det M_g$ at $b = \mathbf{1}$ or $\beta = (-i, -i, -i)$. In this case, and for generic β , $\det M_g$ is an irreducible polynomial of degree six. The case $\beta = (1, 1, 1)$ is exceptional.

To prove Theorem 3.1, we first state a classical theorem, which describes the eigenvalues and left eigenvectors of a multiplication map. We refer to [10, Théorème 4.23] for details.

Theorem 3.4 (Eigenvalue, eigenvector theorem). *Let \mathcal{A} be the coordinate ring of a zero-dimensional scheme with support $V = \{z_1, \dots, z_\delta\}$. The eigenvalues of the multiplication map $M_g: \mathcal{A} \rightarrow \mathcal{A}$ are $g(z_1), \dots, g(z_\delta)$, and the multiplicity of the eigenvalue $g(z_j)$ equals the multiplicity of z_j . Moreover, for each $j = 1, \dots, \delta$, the evaluation map $\text{ev}_{z_j}: \mathcal{A} \rightarrow \mathbb{C}$ given by $\text{ev}_{z_j}([h]) = h(z_j)$ is a left eigenvector corresponding to the eigenvalue $g(z_j)$.*

Once a basis for the vector space \mathcal{A} is fixed, the evaluation map $\text{ev}_{z_j}: \mathcal{A} \rightarrow \mathbb{C}$ is represented by a row vector of length $\dim \mathcal{A}$. The eigenvalue relation is $\text{ev}_{z_j} M_g = g(z_j) \text{ev}_{z_j}$.

Proof of Theorem 3.1. Let W be the righthand side in (11). For $x^* \in \mathbb{C}^n$, let us write $J_{x^*} \subseteq \mathbb{C}[y^{\pm 1}]$ for the ideal generated by $(f_1)_{|x=x^*}, \dots, (f_n)_{|x=x^*}$. The variety $V(J_{x^*}) \subset (\mathbb{C}^*)^n$ consists of at most 2^n points. Recall from Theorem 2.2 that $\mathcal{L}_{A,b} = \pi_{A,b}(\mathcal{Y}_{A,b})$ and

$$\begin{aligned} x^* \in \pi_{A,b}(\mathcal{Y}_{A,b}) &\iff \text{there is } z \in Y_{A,\beta} \cap (\mathbb{C}^*)^n \text{ such that } (f_j)_{|x=x^*, y=z} = 0 \text{ for } j = 1, \dots, n \\ &\iff \text{there is } z \in V(J_{x^*}) \text{ such that } z \in Y_{A,\beta} \cap (\mathbb{C}^*)^n \\ &\iff \text{there is } z \in V(J_{x^*}) \text{ such that } g_1(z) = \dots = g_r(z) = 0. \end{aligned}$$

The ideal $J = \langle f_1, \dots, f_n \rangle \subset K[y^{\pm 1}]$ is zero-dimensional and radical. By Theorem 3.4, the eigenvalues of M_{g_j} are given by $\{g_j(z) : z \in V(J)\} \subset \bar{K}$. If there exists $z \in V(J_{x^*})$ such that $g_1(z) = \dots = g_r(z) = 0$, then $\text{ev}_z : \mathbb{C}[y^{\pm 1}]/J_{x^*} \rightarrow \mathbb{C}[y^{\pm 1}]/J_{x^*}$, $[h] \mapsto h(z)$ is a common left eigenvector of $(M_{g_1})|_{x=x^*}, \dots, (M_{g_r})|_{x=x^*}$ with eigenvalue zero. Hence, ev_z is represented by a row vector of length 2^n , which is a left kernel vector of the concatenated matrix $(M_{g_1} | \dots | M_{g_r})|_{x=x^*}$. This shows the inclusion $\pi_{A,b}(\mathcal{Y}_{A,b}) \subseteq W$, and hence $\mathcal{L}_{A,b} \subseteq W$.

For the reverse inclusion, suppose that $\text{rank}(M_{g_1} | \dots | M_{g_r})|_{x=x^*} < 2^n$ for some $x^* \in \mathbb{C}^n$. Then there exists a left null vector $v^t \in \mathbb{C}^{2^n}$. Applying column operations to the matrix $(M_{g_1} | \dots | M_{g_r})|_{x=x^*}$, we may replace each g_j by a random \mathbb{C} -linear combination \tilde{g}_j of g_1, \dots, g_r . This has the effect that for each $z \in V(J_{x^*})$ and each $j = 1, \dots, r$, we have $\tilde{g}_j(z) = 0$ if and only if $g_1(z) = \dots = g_r(z) = 0$, i.e., $\tilde{g}_j(z) = 0 \Leftrightarrow z \in Y_{A,\beta} \cap (\mathbb{C}^*)^n$. Since $v^t \cdot (M_{\tilde{g}_1} | \dots | M_{\tilde{g}_r})|_{x=x^*} = 0$, each of the matrices $M_{\tilde{g}_j}$ has a zero eigenvalue. Theorem 3.4 implies that for each j , $\tilde{g}_j(z^{(j)}) = 0$ for some $z^{(j)} \in V(J_{x^*})$. We conclude that $V(J_{x^*}) \cap Y_{A,\beta} \cap (\mathbb{C}^*)^n \neq \emptyset$ and $x^* \in \pi_{A,b}(\mathcal{Y}_{A,b})$. \square

Proposition 3.5. *In the situation of Theorem 3.1, if A has rank $n - 1$ and $I_{A,\beta} = \langle g \rangle$, then*

$$\mathcal{L}_{A,b} = \{x \in \mathbb{C}^n : \det(M_g) = 0\}. \quad (12)$$

Moreover, if the prime ideal of $\mathcal{L}_{A,b}$ is $\langle f \rangle \subset \mathbb{C}[x]$, then $\det(M_g) = cf^{\deg \pi_{A,b}}$ for some $c \in \mathbb{C}^*$.

Proof. Equation (12) is an immediate consequence of Theorem 3.1. To compute the order of vanishing of $\det(M_g)$ along $\mathcal{L}_{A,b}$, we introduce a small parameter t and evaluate $M_g(x)$ at the point $x^* + t \cdot x_0$ for a generic point $x^* \in \mathcal{L}_{A,b}$ and a generic point $x_0 \in \mathbb{C}^n$. By (12) we have

$$\det M_g(x^* + t \cdot x_0) = cf(x^* + t \cdot x_0)^k = c_1 t^k + O(t^{k+1}) \quad (13)$$

for some positive k and $c_1 \in \mathbb{C}^*$. By Theorem 3.4, the eigenvalues of $M_g(x^* + t \cdot x_0)$ are the values $g(z(t))$ for $z(t) \in V(J_{x^*+t \cdot x_0})$. Therefore, we have $\det M_g(x^* + t \cdot x_0) = \prod_{z(t) \in V(J_{x^*+t \cdot x_0})} g(z(t))$. By genericity of $x^* \in \mathcal{L}_{A,b}$, there are $\deg \pi_{A,b}$ eigenvalues for which $g(z(0)) = 0$. Hence, to conclude that $\det M_g(x^* + t \cdot x_0) = c_1 t^{\deg \pi_{A,b}} + O(t^{\deg \pi_{A,b}+1})$, it remains to show that for each eigenvalue with $g(z(0)) = 0$, we have $g(z(t)) = \tilde{c}_1 t + O(t^2)$ for some $\tilde{c}_1 \neq 0$. The coordinates of $z(t)$ satisfy $1 - 2(x_j^* + tx_{0j})z_j(t) + z_j(t)^2 = 0$. Solving this explicitly yields $z_j(t) = z_j(0) + tx_{0j}(1 + x_j^*((x_j^*)^2 - 1)^{-1/2}) + O(t^2)$. By genericity of x_0 , the curve parametrized by $t \mapsto z(t)$ intersects $Y_{A,\beta}$ transversally at $t = 0$. This implies that $g(z(t))$ has vanishing order one at $t = 0$. We conclude that $k = \deg \pi_{A,b}$ in (13), as desired. \square

4 Kuramoto oscillators

The Kuramoto model is a system of ordinary differential equations, widely used to describe systems of coupled phase oscillators, cf. [14]. In this section, we explain how Lissajous varieties of type \mathcal{S}_A show up in the study of its steady states. Let $G = (V, E)$ be a graph representing the coupling of a system of m oscillators, where $m = |V|$. Let $n = |E|$ be the number of edges and, for $k = 1, \dots, m$ let V_k be the set of vertices $v \in V$ adjacent to v_k . The *Kuramoto model* is a system of m ordinary differential equations in m unknown functions $\theta_k : \mathbb{R} \rightarrow \mathbb{R}$:

$$\dot{\theta}_k = \omega_k + \sum_{j \in V_k} K_{kj} \sin(\theta_j - \theta_k), \quad k = 1, \dots, m. \quad (14)$$

The function $\theta_k(t)$ is the angle at vertex k at time t , $\dot{\theta}_k$ is its derivative, $\omega_k \in \mathbb{R}$ is the natural frequency of the k -th oscillator, and $K_{kj} \in \mathbb{R}_+$ is the coupling strength of the edge $(k, j) \in E$.

Example 4.1. Let $G = C_3$ be the triangle graph. Here $m = n = 3$, and Equation (14) reads:

$$\begin{aligned} \dot{\theta}_1 &= K_{12} \sin(\theta_2 - \theta_1) + K_{13} \sin(\theta_3 - \theta_1) + \omega_1, \\ \dot{\theta}_2 &= K_{12} \sin(\theta_1 - \theta_2) + K_{23} \sin(\theta_3 - \theta_2) + \omega_2, \\ \dot{\theta}_3 &= K_{13} \sin(\theta_1 - \theta_3) + K_{23} \sin(\theta_2 - \theta_3) + \omega_3. \end{aligned}$$

The spring network associated with this graph is shown in Figure 2. This is a mechanical illustration of the Kuramoto model in [9]. The vertices of the graph are constrained to lie on a circle and are connected by spring-like edges. The Kuramoto model describes the angular velocity $\dot{\theta}_k$ of the particle at vertex k as it moves around the circle. One sees from the equations that, for an equilibrium point to exist, we must have $\omega_1 + \omega_2 + \omega_3 = 0$.

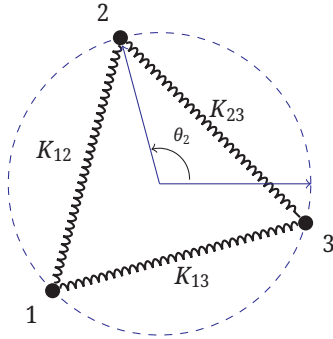


Figure 2: The spring network of the triangle graph C_3 .

We are interested in the steady states of Equation (14). That is, we want to solve the trigonometric equations $\dot{\theta}_k = 0$. For this we rephrase these equations as algebraic equations and use tools from computational algebraic geometry. This is much in the spirit of [5; 12; 18].

The approach taken in [5; 12; 18] and references therein is to substitute $x_k = \sin(\theta_k)$ and $y_j = \cos(\theta_j)$. The formula $\sin(\theta_j - \theta_k) = \sin(\theta_j) \cos(\theta_k) - \cos(\theta_j) \sin(\theta_k)$ turns (14) into

$$f_k = \omega_k + \sum_{v_j \in V_k} K_{kj}(x_j y_k - x_k y_j). \tag{15}$$

Let $I_\theta = \langle x_1^2 + y_1^2 - 1, \dots, x_m^2 + y_m^2 - 1 \rangle \subset \mathbb{C}[x, y]$ and $I_G = \langle f_1, \dots, f_m \rangle \subset \mathbb{C}[x, y]$. In [12], the Kuramoto ideal is $I_K = I_\theta + I_G$ and the Kuramoto variety is its vanishing set $V(I_K) \subseteq \mathbb{C}^{2m}$.

In this section we explore a different algebraic perspective. We assume that G is a simple and connected graph and identify $V = [m]$. Let $A_G \in \mathbb{Z}^{m \times n}$ be the incidence matrix of G , defined as follows. If $k \rightarrow j$ is the l -th edge of G with respect to an arbitrary fixed orientation and an arbitrary fixed ordering of the edges, then the l -th column of A_G is $a_l = (e_k - e_j)^T$, where e_k is the k -th standard basis vector of \mathbb{R}^n . Since G is connected, the rank of A_G is $m - 1$. The Lissajous variety $\mathcal{L}_{A_G, 1}$ depends only on the row span of A_G . Therefore, unless otherwise specified, we let $A(G)$ be the matrix obtained by removing the last row of the incidence matrix A_G , and we set $d = m - 1$. We refer to $A(G)$ as the reduced incidence matrix of G . Below, unless specified otherwise, we fix G and write $A = A(G) \in \mathbb{Z}^{d \times n}$ for short. In particular, unless specified otherwise, $\mathcal{S}_A = \mathcal{L}_{A(G), 1} = \mathcal{L}_{A_G, 1}$. We write $\omega \in \mathbb{R}^d$ for the vector of natural frequencies after dropping ω_m . For simplicity, we shall assume in what follows that all constants K_{ij} are equal to $K \in \mathbb{R}_+$. All statements are easily generalized to arbitrary K_{ij} .

With this setup, studying the steady states of Equation (14) amounts to studying the intersection of the Lissajous variety \mathcal{S}_A with an affine linear space. Indeed, after substituting $x_l = \sin(\theta_k - \theta_j)$ in (14), the resulting equations are affine-linear. We obtain

$$x \in \mathcal{S}_A \quad \text{and} \quad Ax = \omega/K. \tag{16}$$

A steady state is recovered from x satisfying (16) by computing the fiber $\varphi_{A,1}^{-1}(x)$. A different approach is to set $v_j = e^{i\theta_j}$ and solve the following nonlinear equations in (v_1, \dots, v_d) :

$$\frac{\omega_j}{K} + \frac{a_{j1}}{2i}(v^{a_1} - v^{-a_1}) + \dots + \frac{a_{jn}}{2i}(v^{a_n} - v^{-a_n}) = 0, \quad j = 1, \dots, d. \tag{17}$$

The relation between these two approaches was exploited in the proof of Theorem 2.3, where the solution sets were denoted by \mathcal{S}_x and \mathcal{S}_v respectively, and we have $\psi_{A,1}(\mathcal{S}_v) = \mathcal{S}_x$.

Recall that the degree of S_A is given by the number of intersection points with a generic affine space of complementary dimension. The affine linear space $Ax = \omega/K$ has codimension d , which matches the dimension of S_A . However, it is not generic, as both the variety and the affine space depend on the matrix A . On the other hand, if $Ax = \omega/K$ and S_A intersect in a finite number of isolated solutions, then this number is bounded above by the degree. The volume in Theorem 2.3 appeared as a bound on the number of isolated equilibria in [5].

Example 4.2. Let $G = C_3$ be the triangle graph, as in Example 4.1. For cycle graphs, we choose the following edge ordering and orientation: $1 \rightarrow 2, 2 \rightarrow 3, \dots, n \rightarrow 1$. In the case $n = 3$, the reduced incidence matrix A with respect to this ordering coincides with A from Example 1.3, hence yielding the same Lissajous variety S_A . Since $\text{rank}(A)$ is $m - 1$, one of the equations in (14) can be dropped. For feasibility, we must have $\sum_{l=1}^m \omega_l = 0$. We set the coupling strength K to 1, and choose $\omega = (\frac{1}{10}, \frac{1}{5})$. We also fix $\theta_m = 0$ and use $\theta_1, \dots, \theta_{m-1} = \theta_d$ as coordinates on $\text{Row}(A)$. With these choices, Equation (16) reads

$$x^4 + 4x^2y^2z^2 - 2x^2y^2 - 2x^2z^2 + y^4 - 2y^2z^2 + z^4 = 10x - 10z - 1 = -5x + 5y - 1 = 0.$$

This system has six distinct real solutions, attaining the degree of S_A . The (x, y, z) -coordinates of these solutions determine the sines of the steady state angles: $y = \sin(\theta_2), z = -\sin(\theta_1)$. The cosines can be found by substituting these values appropriately in $f_k = 0$, where f_k is as in (15), and solving the resulting system of linear equations. Alternatively, one solves two Laurent polynomial equations in two unknowns given by (17), and finds $\theta_j = -i \log(v_j)$.

Remark 4.3. Let G be the cycle graph C_n . With the choices of Example 4.2, its (reduced) incidence matrix has the same row span as the matrix A_n from (8). The cycle polynomial Γ_n^t is the defining equation of the Lissajous variety $\mathcal{C}_{A_n} = \mathcal{L}_{A_n, 0}$.

We interpret the equations (16) from an optimization perspective. For this discussion, the matrix $A \in \mathbb{Z}^{d \times n}$ has rank d , and it does not necessarily come from a graph. We define a submanifold $S_A^+ \subset S_A$ as follows. Consider the d -dimensional convex polytope $P = \text{Row}(A) \cap [-\frac{\pi}{2}, \frac{\pi}{2}]^n$ obtained by intersecting a hypercube with the row span of A . We set

$$S_A^+ = \sin(\text{int}(P)) \subset S_A. \tag{18}$$

That is, S_A^+ is the image under $y \mapsto (\sin(y_1), \dots, \sin(y_n))$ of the interior of $P \subset \text{Row}(A)$. That map is an isomorphism of manifolds $\text{int}(P) \simeq S_A^+$ with inverse given by the coordinate-wise arcsine function $\arcsin: (-1, 1)^n \rightarrow (-\frac{\pi}{2}, \frac{\pi}{2})^n$ given by $\arcsin(x_1, \dots, x_n) = (\arcsin(x_1), \dots, \arcsin(x_n))$.

Theorem 4.4. Let $A \in \mathbb{Z}^{d \times n}$ be of rank d . We have $x^* \in S_A^+ \cap \{AKx = \omega\}$ if and only if x^* is the unique minimizer of the following convex optimization problem:

$$\text{minimize } \sum_{j=1}^n (x_j \arcsin(x_j) + \sqrt{1 - x_j^2}), \quad \text{subject to } Ax = \omega/K \text{ and } x \in (-1, 1)^n. \tag{19}$$

Proof. The function $g(t) = t \arcsin(t) + \sqrt{1 - t^2}$ is strictly convex on the open interval $(-1, 1)$. Hence the objective function $\sum_{j=1}^n g(x_j)$ is strictly convex on the feasible region of our optimization problem. If a minimizer $x^* \in (-1, 1)^n$ exists, then it is unique, and it satisfies the first order optimality conditions

$$\frac{\partial \text{Lag}}{\partial x_j} = \frac{\partial}{\partial x_j} \left(\sum_{j=1}^n g(x_j) - \lambda^t (Ax - \omega/K) \right) = 0 \quad \text{and} \quad Ax = \omega/K.$$

Here Lag is the Lagrangian and $\lambda = (\lambda_1, \dots, \lambda_d)$ are the Lagrange multipliers. The derivative of $g(t)$ is $\arcsin(t)$. Hence, the equations coming from partial derivatives of Lag with respect to x_j are equivalent to $\arcsin(x) \in \text{Row}(A)$, which implies $\arcsin(x) \in \text{int}(P)$. Taking the coordinate-wise sine on both sides, we see that this is equivalent to $x \in S_A^+$. □

Motivated by Theorem 4.4, we introduce the following notation for the A -projection of S_A^+ :

$$\Omega^+ = \{Ax \in \mathbb{R}^d : x \in S_A^+\} = A(S_A^+).$$

The optimization problem (19) has a unique minimizer if and only if $\omega/K \in \Omega^+$. The projection $A : S_A^+ \rightarrow \Omega^+$ is one-to-one, and the inverse is given by $\omega \mapsto S_A^+ \cap \{Ax = \omega\}$.

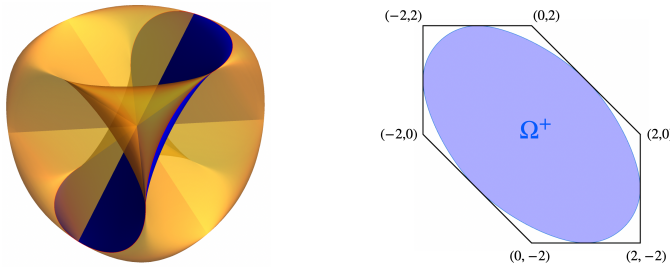


Figure 3: Positive regions for $x \in \mathcal{S}_A$ (left) and $\omega \in \mathbb{R}^2$ (right).

Example 4.5. We set $K = 1$ and use the matrix $A = \begin{pmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$ associated with $G = C_3$. The manifold \mathcal{S}_A^+ is shown in blue in the left part Figure 3. Its projection to \mathbb{R}^2 via $x \mapsto (x_1 - x_3, -x_1 + x_2)$ is Ω^+ , seen in Figure 3 (right). This set is contained in a hexagon Q , obtained as the A -projection of the cube $[-1, 1]^3$. For $\omega \in \text{int}(Q)$, the feasible region of (19) is an open line segment $\ell = \{Ax = \omega\} \cap (-1, 1)^n$. For $\omega \in Q \setminus \Omega^+$, the objective function does not attain a minimum on ℓ . For $\omega \in \Omega^+$, there is a unique minimizer given by $\ell \cap \mathcal{S}_A^+$.

We note that a different nonlinear (transcendental) optimization approach for $\omega = 0$ is discussed in [16]. In Section 5, we shall establish a more general connection between Lissajous varieties and optimization. Before that, we conclude the present section with a remark on the stability of equilibria. Write the system of ODEs in Equation (14) as $\dot{\theta} = \Phi(\theta)$. Let $J_\Phi(\theta) = \left(\frac{\partial \Phi_i}{\partial \theta_j}(\theta)\right)_{ij}$ be the Jacobian matrix of Φ . The matrix $J_\Phi(\theta)$ is symmetric, hence its eigenvalues are real. Because of the linear relation between the equations (14) pointed out above, $J_\Phi(\theta)$ is rank deficient — one of its eigenvalues is always zero. A solution θ^* to $\Phi(\theta) = 0$ is called *linearly stable* if all other eigenvalues of $J_\Phi(\theta^*)$ are negative. If $\omega/K \in \Omega^+$, then the equilibrium θ^* corresponding to the minimizer x^* from Theorem 4.4 is linearly stable.

Proposition 4.6. Let A be the reduced incidence matrix of G and let $\omega/K \in \Omega^+$. Then the unique minimizer described in Theorem 4.4 yields a linearly stable equilibrium θ^* of the corresponding Kuramoto model. This is the unique vector θ^* satisfying $A^t \theta^* = \arcsin(x^*)$.

Proof. If $\omega/K \in \Omega^+$, then $A^t \theta^* \in \text{int}(P) \subset \text{Row}(A)$. In particular, $\cos(a_l \cdot \theta^*) > 0$ for all $l = 1, \dots, n$. The statement then follows from [12, Lemma 3.2]. □

While solving the equations (17) leads to all potential equilibria, there may be many “spurious” solutions which have no physical meaning. Proposition 4.6 says that if $\omega/K \in \Omega^+$, then a linearly stable solution can be found more efficiently via convex optimization. We now consider a larger graph to illustrate the computational range of this technique.

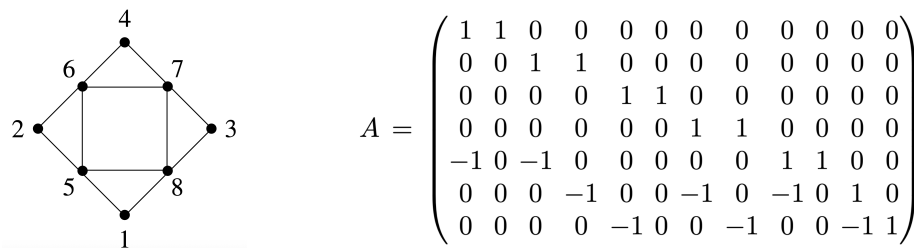


Figure 4: A graph and its reduced incidence matrix.

Example 4.7. The graph shown in Figure 4 appears in [12, Example 3.1]. The corresponding Lissajous variety \mathcal{S}_A has degree 832. For $\omega = (0.1, \dots, 0.1)$, the unique minimizer in (19) is

$$x^* = (-0.050, 0.150, 0.083, 0.017, -0.050, 0.150, 0.017, 0.083, -0.067, 0.200, 0.067, 0.200),$$

here rounded to three decimal places. This can be computed using, for instance, `JuMP.jl`, see [17]. The optimum is found in less than 0.01 seconds. Solving $A^t \theta^* = \arcsin(x^*)$ we find that

$$\theta^* = (0.151, 0.285, 0.151, 0.285, 0.201, 0.268, 0.201),$$

which is a linearly stable equilibrium of the Kuramoto equations. We verify that $x \in \mathbb{S}_A^+ \cap \{Ax = \omega\}$ and $\omega \in \Omega^+$ by checking that $\omega = A \sin(A^t \theta^*)$, and $A^t \theta^* \in [-\frac{\pi}{2}, \frac{\pi}{2}]^{12}$.

5 Positive points and convex optimization

By Theorem 4.4 the open subset $\mathbb{S}_A^+ \subset \mathbb{S}_A = \mathcal{L}_{A,1}$ parametrizes all solutions of the optimization problem (19) for $\omega \in \Omega^+$. In this section we prove a similar characterization for an arbitrary Lissajous variety $\mathcal{L}_{A,b}$. In particular, we generalize Theorem 4.4 and Proposition 4.6. The Kuramoto model is generalized by the dynamical system

$$\dot{\theta} = A \varphi_{A,b}(\theta) - \omega, \tag{20}$$

where $A \in \mathbb{Q}^{d \times n}$ has rank d , $b \in \mathbb{R}^n$, and $\omega \in \mathbb{R}^d$. The map $\varphi_{A,b} : [-\pi, \pi]^d \rightarrow [1, 1]^n$ is

$$\varphi_{A,b}(\theta) = (\cos(a_1 \cdot \theta - b_1 \pi/2), \dots, \cos(a_n \cdot \theta - b_n \pi/2)).$$

Investigating the equilibria of this dynamical system leads us to solve $A \varphi_{A,b}(\theta) - \omega = 0$. We complexify $\varphi_{A,b} : \mathbb{C}^d \rightarrow \mathbb{C}^n$ and refer to the solutions $\theta \in \mathbb{C}^d$ as *steady states* or *equilibria*. The steady states correspond to points in the intersection $\mathcal{L}_{A,b} \cap \{Ax = \omega\}$. We shall define $\mathcal{L}_{A,b}^+ \subset \mathcal{L}_{A,b} \cap (-1, 1)^n$ such that there is at most one intersection point in $\mathcal{L}_{A,b}^+ \cap \{Ax = \omega\}$. Moreover, if such a point exists, then it is the solution to a convex optimization problem.

The affine linear space $L_{A,b}$ is defined as $L_{A,b} = \text{Row}(A) - b\pi/2$. Its image under the coordinate-wise cosine is $\mathcal{L}_{A,b}$. The appropriate generalizations of \mathbb{S}_A^+ and Ω^+ are as follows:

$$\mathcal{L}_{A,b}^+ = \cos(L_{A,b} \cap (0, \pi)^n) \subset \mathcal{L}_{A,b}, \quad \Omega_{A,b}^+ = \{Ax \in \mathbb{R}^d : x \in \mathcal{L}_{A,b}^+\} = A(\mathcal{L}_{A,b}^+).$$

One checks that this is consistent with the previous section, in that $\mathcal{L}_{A,1}^+ = \mathbb{S}_A^+$ and $\Omega_{A,1}^+ = \Omega^+$.

Restricting to the real points of $L_{A,b}$, we have that $\cos(L_{A,b} \cap \mathbb{R}^n) = \cos(L_{A,b} \cap [-\pi, \pi]^n)$. This is a subset of $\mathcal{L}_{A,b}(\mathbb{R})$. The ‘+’ in our notation is motivated by the fact that $\mathcal{L}_{A,b}^+$ is the image under the cosine map of all “positive” tuples of angles in $L_{A,b} \cap (0, \pi)^n$.

Theorem 5.1. *Let $A \in \mathbb{Q}^{d \times n}$ be of rank d . We have $x^* \in \mathcal{L}_{A,b}^+ \cap \{Ax = \omega\}$ if and only if x^* is the unique minimizer of the following convex optimization problem:*

$$\text{minimize } \frac{-\pi}{2} b^t x - \sum_{j=1}^n (x_j \arccos(x_j) - \sqrt{1 - x_j^2}), \quad \text{such that } Ax = \omega \text{ and } x \in (-1, 1)^n. \tag{21}$$

In particular, a minimizer exists if and only if $\omega \in \Omega_{A,b}^+$, and in that case it is unique.

Proof. The first and second order derivatives of $g(t) = -t \arccos(t) + \sqrt{1 - t^2}$ are

$$g'(t) = -\arccos(t) \quad \text{and} \quad g''(t) = (1 - t^2)^{-1/2}.$$

Hence, the objective function in (21) is strictly convex on $(-1, 1)^n$, and so is its restriction to the feasible region. The rest of the proof is analogous to that of Theorem 4.4. The method of Lagrange multipliers gives the first order optimality conditions $\arccos(x) \in \text{Row}(A) - \pi b/2$ and $Ax = \omega$, which is equivalent to $x \in \mathcal{L}_{A,b}^+ \cap \{Ax = \omega\}$. \square

Remark 5.2. Theorem 5.1 is inspired by analogous convex optimization problems in which the constraint $x \in (-1, 1)^n$ is replaced by $x \in \mathbb{R}_+^n$ and the objective function is a strictly convex function $G(x)$ on the positive orthant. Often G is of the form $G(x) = \sum_{j=1}^n g(x_j)$. Appropriate choices of g lead naturally to semi-algebraic descriptions of the unique minimizer, similar to $x^* \in \mathcal{L}_{A,b} \cap \{Ax = \omega\}$ in Theorem 5.1. For $g(t) = \log(t)$, the Lissajous variety is replaced by a *reciprocal linear space*; cf. [7]. The function $g(t) = t \log(t) - t$ naturally leads to *positive toric varieties*, see [24]. If the universal barrier function $G(x)$ of the feasible polytope is minimized instead, then one intersects $\{Ax = \omega\}$ with the *Santaló patchwork*, see [21].

Remark 5.3. The coordinates of the minimizer x^* of (21) are algebraic functions of ω . Their minimal polynomial in $\mathbb{Q}(\omega)[x_j]$ has degree at most $\deg(\mathcal{L}_{A,b})$, see the formula stated in Theorem 2.3. This follows from the fact that $x^* \in \mathcal{L}_{A,b} \cap \{Ax = \omega\}$ by Theorem 5.1.

Write the equations (20) as $\dot{\theta} = \Phi_{A,b}(\theta)$ and let $J_{\Phi_{A,b}}(\theta)$ be the $d \times d$ Jacobian matrix. We say that a steady state solution θ^* is *linearly stable* if all eigenvalues of $J_{\Phi}(\theta^*)$ are negative.

Proposition 5.4. *Let $\omega \in \Omega_{A,b}^+$ and let $x^* \in \mathcal{L}_{A,b}^+$ be the unique minimizer of the optimization problem (21). The unique solution θ^* of the linear equations $A^t \theta^* = \arccos(x^*) + b\pi/2$ is a linearly stable steady state solution of the dynamical system (20).*

Proof. Note that θ^* is a steady state solution by construction: $A\varphi_{A,b}(\theta^*) = Ax^* = \omega$. The Jacobian matrix $J_{\Phi_{A,b}}$ has the following explicit expression:

$$J_{\Phi_{A,b}}(\theta) = -A \operatorname{diag}(\sin(a_1 \cdot \theta - b_1\pi/2), \dots, \sin(a_n \cdot \theta - b_n\pi/2))A^t.$$

Since $\arccos(x^*) = A^t \theta^* - b\pi/2 \in (0, \pi)^n$, the diagonal matrix in this expression has positive diagonal entries for $\theta = \theta^*$. Hence $J_{\Phi_{A,b}}(\theta^*)$ is negative definite. \square

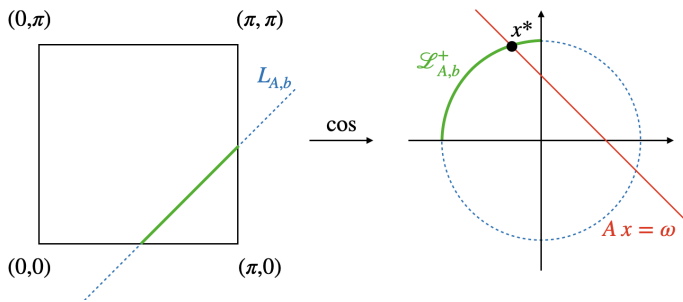


Figure 5: The line $x + y = \omega$ has precisely one intersection point with $\mathcal{L}_{A,b}^+$ for $\omega \in (-1, 1)$.

Example 5.5. The Lissajous curve $\mathcal{L}_{A,b}$ for the data $A = \begin{pmatrix} 1 & 1 \end{pmatrix}$, $b = (0, 1)$ is the circle with defining equation $x^2 + y^2 = 1$, see Example 1.2. The semi-algebraic set $\mathcal{L}_{A,b}^+$ is the segment of the curve contained in the $(-, +)$ quadrant, see Figure 5. The projection $\Omega_{A,b}^+ = A(\mathcal{L}_{A,b}^+)$ is the open line segment $(-1, 1)$. For $\omega \in \Omega_{A,b}^+$, the line $x + y = \omega$ has one intersection point with $\mathcal{L}_{A,b}^+$. This is the unique solution to (21). The differential equation

$$\dot{\theta} = \cos(\theta) + \sin(\theta) - \omega, \quad \omega \in \Omega_{A,b}^+$$

has two steady state solutions θ^* . Only one of them is stable. For concreteness, set $\omega = 0.6$. The stable equilibrium is $\theta^* \approx 1.918$. Its image $(\cos(\theta^*), \sin(\theta^*))$ is $\mathcal{L}_{A,b}^+ \cap \{x + y = 0.6\}$.

6 Lissajous discriminants

Varying the natural frequencies ω affects the number of real solutions to the systems of polynomial equations (16) and (17). In the context of dynamical systems, varying ω influences the nature of the equilibria, which is the topic of bifurcation analysis. Investigating this leads us to study a discriminant locus in the ω parameters. More precisely, we shall define a variety $\nabla_{A,b} \subset \mathbb{C}^d$, which is expected to be a hypersurface, such that the number of real solutions to

$$\frac{1}{2} a_{j1} (\beta_1 v^{a_1} + \beta_1^{-1} v^{-a_1}) + \dots + \frac{1}{2} a_{jn} (\beta_n v^{a_n} + \beta_n^{-1} v^{-a_n}) - \omega_j = 0, \quad j = 1, \dots, d \tag{22}$$

is constant for ω in each connected component of $\mathbb{R}^d \setminus \nabla_{A,b}$. For $b = \mathbf{1}$ ($\beta = -i \cdot \mathbf{1}$), these are the equations (17). Taking cues from standard discriminant analysis, $\nabla_{A,b}$ should consist of points $\omega \in \mathbb{C}^d$ for which two complex solutions of (22) collide. We now make this precise.

We continue to assume that the matrix $A \in \mathbb{Z}^{d \times n}$ has full rank d , and we write the equations (22) in a compact way as follows: $A\psi_{A,b}(v) = \omega$, where $\psi_{A,b}$ is as in (4). Consider the *incidence variety* $W_{A,b} = \{(v, \omega) \in (\mathbb{C}^*)^d \times \mathbb{C}^d : A\psi_{A,b}(v) = \omega\}$. The fiber of

$$\text{pr}_\omega : W_{A,b} \longrightarrow \mathbb{C}^d, \quad (v, \omega) \longmapsto \omega \tag{23}$$

over $\omega \in \mathbb{C}^d$ consists of the solutions v to (22) for that fixed value of ω . The *toric Jacobi matrix* of $v \mapsto A\psi_{A,b}(v)$ is the $d \times d$ matrix given by

$$J_{A,b}(v) = \left(v_j \frac{\partial}{\partial v_j} (A\psi_{A,b}(v))_k \right)_{1 \leq j, k \leq d} = \frac{1}{2} A \text{diag}(\beta_1 v^{a_1} - \beta_1^{-1} v^{-a_1}, \dots, \beta_n v^{a_n} - \beta_n^{-1} v^{-a_n}) A^t.$$

This is the usual Jacobi matrix, with $\frac{\partial}{\partial v_j}$ replaced by the Euler operator $v_j \frac{\partial}{\partial v_j}$. One checks that for a point $v \in \text{pr}_\omega^{-1}(\omega)$, the toric Jacobian determinant $\det J_{A,b}(v)$ vanishes if and only if the usual Jacobian determinant vanishes. We prefer the toric version because of the elegant expression $J_{A,b}(v) = \frac{1}{2} AD(v)A^t$, where $D(v)$ is the diagonal $n \times n$ matrix shown above.

Lemma 6.1. *The toric Jacobian $\det J_{A,b}(v)$ is not identically zero as a Laurent polynomial in β and v . Moreover, for generic $\beta \in (\mathbb{C}^*)^n$ and generic $\omega \in \mathbb{C}^d$, the fiber $\text{pr}_\omega^{-1}(\omega)$ is finite. That is, for generic β, ω the equations (22) have finitely many solutions $v \in (\mathbb{C}^*)^d$.*

Proof. If $v = \mathbf{1} \in (\mathbb{C}^*)^d$ and $\beta = -i \cdot \mathbf{1} \in (\mathbb{C}^*)^n$, then $\det J_{A,1}(\mathbf{1}) = \det(-iAA^t) \neq 0$ as A has full rank. This shows the first statement. For the second part of the lemma, since β is generic we may assume that $\det J_{A,b}(v)$ is not identically zero as a Laurent polynomial in v . Pick $v_0 \in (\mathbb{C}^*)^d$ such that $\det J_{A,b}(v_0) \neq 0$ and let $\omega_0 = A\psi_{A,b}(v_0)$. By construction, v_0 is isolated in $\text{pr}_\omega^{-1}(\omega_0)$. Since pr_ω is a dominant morphism of irreducible d -dimensional varieties, this implies that its generic fiber is finite by [19, Chapter 1, § 8, Theorem 2 and Corollary 1]. \square

Remark 6.2. For fixed b , the toric Jacobian $\det J_{A,b}$ might be identically zero. This happens for $A = \begin{pmatrix} 1 & 1 \end{pmatrix}$ when $\beta_1 + \beta_2 = 0$. The equations (22) have no solutions for generic ω .

Definition 6.3. The *ramification locus* $R_{A,b} \subseteq W_{A,b}$ is the divisor

$$R_{A,b} = \{(v, \omega) \in W_{A,b} : \det J_{A,b}(v) = 0\}.$$

The *Lissajous discriminant* $\nabla_{A,b}$ is the associated branch locus: $\nabla_{A,b} = \overline{\text{pr}_\omega(R_{A,b})} \subseteq \mathbb{C}^d$. If $\nabla_{A,b}$ is a hypersurface, then its defining equation is denoted by $\Delta_{A,b} \in \mathbb{C}[\omega_1, \dots, \omega_d]$. If $\nabla_{A,b}$ has codimension greater than one, then we set $\Delta_{A,b} = 1$.

Note that the polynomial $\Delta_{A,b}$ is defined up to scaling by a nonzero complex number.

Remark 6.4. The Lissajous discriminant can be viewed as the branch locus of the linear projection of the Lissajous variety $\mathcal{L}_{A,b}$ given by the matrix A . It is an analog of the *entropic discriminant*, see [22], for which $\mathcal{L}_{A,b}$ is replaced by the reciprocal linear space of $\text{Row}(A)$.

By definition, the Lissajous discriminant is the variety of the elimination ideal

$$\langle A\psi_{A,b}(v) - \omega, \det(J_{A,b}) \rangle \cap \mathbb{C}[\omega]. \tag{24}$$

Here we start from an ideal with $d + 1$ generators in $\mathbb{C}[v^{\pm 1}, \omega]$. We compute three examples.

Example 6.5. For $A = \begin{pmatrix} 1 & 2 \end{pmatrix} \in \mathbb{Z}^{1 \times 2}$, Equation (24) reads

$$\langle \beta v + \beta^{-1} v^{-1} + 2(\beta^2 v^2 + \beta^{-2} v^{-2}) - 2\omega, \beta v - \beta^{-1} v^{-1} + 4(\beta^2 v^2 - \beta^{-2} v^{-2}) \rangle \cap \mathbb{C}[\omega].$$

Setting $b = 1, \beta = -i$, we compute $\Delta_{A,1} = 256\omega^4 - 2367\omega^2 + 3375$, which has four real roots. A root $\omega^{(j)}$ of $\Delta_{A,1}$ corresponds to a tangent line $x + 2y = \omega^{(j)}$ of $\mathcal{S}_A = \mathcal{L}_{A,1}$; see Figure 6 below. For $b = 0$, the discriminant is $\Delta_{A,0} = 16\omega^3 - 31\omega^2 - 84\omega + 99 = (\omega - 1)(\omega - 3)(16\omega + 33)$.

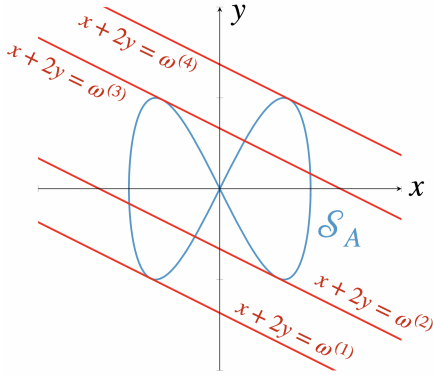


Figure 6: For $A = \begin{pmatrix} 1 & 2 \end{pmatrix}$, each of the four real roots $\omega^{(1)}, \dots, \omega^{(4)}$ of $\Delta_{A,1}$ corresponds to a tangent line $x + 2y = \omega^{(i)}$ (in red) of \mathcal{S}_A (in blue).

Example 6.6. As mentioned above, studying the Lissajous discriminant of the matrix A is closely related to the bifurcation analysis of the dynamical system (20). Indeed, in view of linear stability analysis, an eigenvalue of the Jacobian matrix evaluated at a critical point can change sign only if its determinant vanishes. This happens when ω lies on the discriminant. For instance, the ramification locus $R_{A,b}$ in Example 5.5 is defined by the equations

$$(v + v^{-1}) - i(v - v^{-1}) - 2\omega = v - v^{-1} - i(v + v^{-1}) = 0,$$

which imply that $\nabla_{A,b} = \{\pm\sqrt{2}\}$. These discriminant points correspond to the two red lines $\{x + y = \pm\sqrt{2}\}$ tangent to the circle $\mathcal{L}_{A,b}(\mathbb{R})$, see Figure 5. The bounded discriminant chamber $(-\sqrt{2}, \sqrt{2})$ contains $\mathcal{L}_{A,b}^+ = (-1, 1)$, and for ω in that chamber the dynamical system has one stable and one unstable equilibrium. For $\omega^2 > 2$, the system is unstable.

Example 6.7. Let A be as in Example 1.3. We compute the Lissajous discriminants of $\mathcal{L}_{A,b}$ for $b = 0$ and $b = 1$. The curve $\nabla_{A,0}$ has degree 6, and $\nabla_{A,1}$ has degree 12. The equations are

$$\begin{aligned} \Delta_{A,0} &= -8\omega_1^5 + 4\omega_1^4\omega_2^2 - 20\omega_1^4\omega_2 - 23\omega_1^4 + 8\omega_1^3\omega_2^3 - 8\omega_1^3\omega_2^2 - 46\omega_1^3\omega_2 + 4\omega_1^3 + 4\omega_1^2\omega_2^4 \\ &\quad + 8\omega_1^2\omega_2^3 - 69\omega_1^2\omega_2^2 + 6\omega_1^2\omega_2 + 36\omega_1^2 + 20\omega_1\omega_2^4 - 46\omega_1\omega_2^3 - 6\omega_1\omega_2^2 + 36\omega_1\omega_2 \\ &\quad + 8\omega_2^5 - 23\omega_2^4 - 4\omega_2^3 + 36\omega_2^2, \\ \Delta_{A,1} &= 64e_2^5 + 399e_2^4 + 840e_2^3 + 376e_2^2e_3^2 + 766e_2^2 + 3056e_2e_3^2 + 288e_2 - 16e_3^4 + 5812e_3^2 + 27, \end{aligned}$$

where $e_2 = \omega_1\omega_2 + \omega_1\omega_3 + \omega_2\omega_3$, $e_3 = \omega_1\omega_2\omega_3$ are the elementary symmetric polynomials and $\omega_3 = -\omega_1 - \omega_2$. We will explain the symmetric structure of $\Delta_{A,1}$ in Proposition 6.11. Figure 7 shows our two discriminant curves. They are the branch loci of the projection of Figure 1 given by A . The reference [2, Section 5.3] studies $\nabla_{A,1}$ via machine learning techniques.

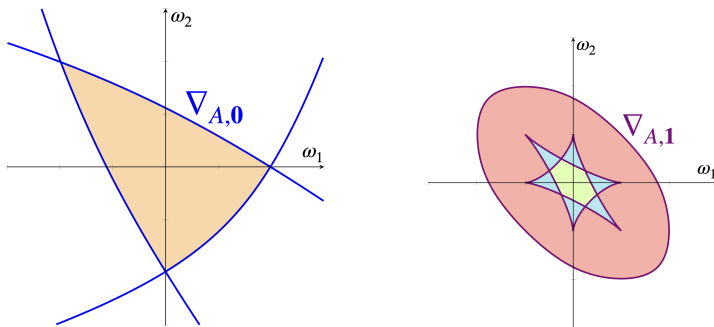


Figure 7: Lissajous discriminants from Example 6.7.

Different regions in Figure 7 correspond to different numbers of real solutions to (22), or different numbers of real intersection points in $\mathcal{L}_{A,b} \cap \{Ax = \omega\}$. For ω in the connected component of $\mathbb{R}^2 \setminus \nabla_{A,0}$ highlighted in orange, the line $Ax = \omega$ intersects \mathcal{C}_A in three real points. In all other connected components, there is only one real intersection point. On the right, the green, blue and red components correspond to ω with six, four and two real intersection points, respectively. We note that the convex hull of the real points of $\nabla_{A,1}$ contains the blue region Ω^+ in Figure 3 (right), but the two do not coincide. Figure 8 shows a comparison. This is similar to the inclusion $(-1, 1) \not\subset (-\sqrt{2}, \sqrt{2})$ in Example 6.6.

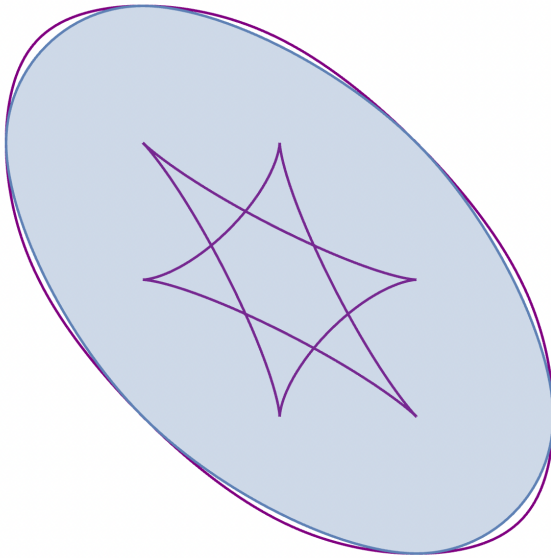


Figure 8: Comparing Ω^+ (blue) with the Lissajous discriminant (purple).

Theorem 6.8. *If the Lissajous discriminant variety $\nabla_{A,b}$ is a hypersurface, then it has degree at most $(\deg \text{pr}_\omega)^{-1} \cdot d \cdot d! \text{vol}(P_A)$, where pr_ω denotes the restriction $(\text{pr}_\omega)|_{\mathbb{R}^{A,b}}$ and $\text{vol}(P_A)$ is as in Theorem 2.3.*

Proof. For generic $\omega_0, \omega_1 \in \mathbb{C}^d$, the degree of $\nabla_{A,b}$ is the number of intersection points of $\nabla_{A,b}$ with the line parametrized by $\omega_0 + t\omega_1$. This is at most the number of solutions $(v, t) \in (\mathbb{C}^*)^{d+1}$ to the system of $d + 1$ Laurent polynomial equations

$$A\psi_{A,b}(v) = \omega_0 + t\omega_1, \quad \det J_{A,b}(v) = 0. \tag{25}$$

Let us denote this number by δ . In fact, it is clear that if (v, t) is a solution to these equations, then so is (v', t) for any $v' \in \text{pr}_\omega^{-1}(\omega_0 + t\omega_1)$. Hence, we have $\deg(\nabla_{A,b}) = (\deg \text{pr}_\omega)^{-1} \cdot \delta$.

By Bernstein’s theorem, the number δ is bounded by the mixed volume of the $d + 1$ Newton polytopes of the equations; see [3]. The Newton polytope of each of the first d equations is contained in the $(d + 1)$ -dimensional polytope $\hat{P}_A = \text{conv}(P_A \cup e_{d+1})$. Here we add the $(d + 1)$ -st standard basis vector to P_A because of the term $t\omega_1$. The Newton polytope of the toric Jacobian determinant $\det J_{A,b}(v)$ is contained in the d -dimensional polytope $d \cdot P_A$. Hence, we have $\delta \leq \text{MV}(\hat{P}_A, \dots, \hat{P}_A, d \cdot P_A)$, where \hat{P}_A is repeated d times and $\text{MV}(\cdot)$ denotes the mixed volume. By definition, this mixed volume equals the coefficient of $\lambda_0^d \lambda_1$ in $\text{vol}(\lambda_0 \hat{P}_A + \lambda_1 dP_A)$. The Minkowski sum $\lambda_0 \hat{P}_A + \lambda_1 dP_A$ is obtained from the pyramid $(\lambda_0 + d\lambda_1) \hat{P}_A$ with volume $(\lambda_0 + d\lambda_1)^{d+1} (d + 1)^{-1} \text{vol}(P_A)$ by “chopping off” the top of the pyramid $\lambda_0 e_{d+1} + d\lambda_1 \hat{P}_A$ with volume $(d\lambda_1)^{d+1} (d + 1)^{-1} \text{vol}(P_A)$. The coefficient standing with $\lambda_0^d \lambda_1$ in the difference of these expressions is $d \cdot d! \text{vol}(P_A)$. We have shown that $\delta \leq d \cdot d! \text{vol}(P_A)$, and since $\deg(\nabla_{A,b}) = (\deg \text{pr}_\omega)^{-1} \cdot \delta$ the theorem follows. \square

Example 6.9. The degree of the Lissajous discriminant can be computed by solving (25) for generic ω_0, ω_1 . There are δ solutions, whose image under pr_ω consists of $\deg \nabla_{A,b}$ many points. To compute all δ solutions, we use the package `HomotopyContinuation.jl` (v2.15.0) in Julia (v1.11), see [4]. To compute the lattice volume $d! \text{vol}(P_A)$, we use the function `lattice_volume` from `Oscar.jl` (v1.1.2), see [8; 20]. We apply these methods for matrices A coming from cyclic and complete graphs. The results are reported in Table 1 below. The column $d \cdot d! \text{vol}(P_A)$ is computed using the command `d*lattice_volume(convex_hull(transpose([A -A])))`. By [5, Theorem 14], these numbers for C_n are given by the sequence $n(n-1)\binom{n-1}{\lfloor \frac{1}{2}(n-1) \rfloor}$. The other entries can be reproduced using the following Julia code snippet, which uses standard tools:

```

using HomotopyContinuation 1
A = ... # input a matrix A of full row rank, e.g., A = [1 0 -1; -1 1 0] 2
b = ... # input a vector b, e.g., b = [0;0;0] 3
d, n = size(A) 4
β = exp(-im*b*pi/2); βinv = β.^(-1); 5
@var v[1:d] w[1:d] t # declare variables 6
Ad = [[maximum([aa,0]) for aa in A]; -[minimum([aa,0]) for aa in A]] 7
y = [prod([v;w].^Ad[:,i]) for i = 1:n] 8
yinv = [prod([w;v].^Ad[:,i]) for i = 1:n] 9
ψ = [1/2*(β[i]*y[i] + βinv[i]*yinv[i]) for i = 1:n] 10
D = det(A*diagm([1/2*(β[i]*y[i]-βinv[i]*yinv[i]) for i = 1:n])*transpose(A)) 11
myω0 = randn(ComplexF64,d); myω1 = randn(ComplexF64,d); 12
eqs = System([A*ψ - myω0+t*myω1; [v[i]*w[i]-1 for i = 1:d]; D], variables = [v;w;t]) 13
R = HomotopyContinuation.solve(eqs) 14
δ = length(solutions(R)) 15
degdisc = length(unique_points([[sol[end]] for sol in solutions(R)])) 16
degpr = δ/degdisc 17

```

The variables in line 6 are our variables v_1, \dots, v_d and the variable t in Equation (25). The variables w_k play the role of the inverses of the v_k , as in Section 2. The columns of the matrix Ad in line 7 are the concatenations of nonnegative integer vectors a_j^+ and a_j^- satisfying $a_j = a_j^+ - a_j^-$. Lines 8–13 construct the system of equations (25), and line 14 solves it. We emphasize that numerical homotopy continuation methods are based on heuristics and do not provide a proof that the numbers in the table are correct. However, we expect they are. Our goal here is to exemplify the (non-)tightness of the bound in Theorem 6.8.

A	$d \cdot d! \text{vol}(P_A)$	b generic		$b = 0$		$b = 1$	
		$\deg(\nabla_{A,b})$	$\deg(\text{pr}_\omega)$	$\deg(\nabla_{A,0})$	$\deg(\text{pr}_\omega)$	$\deg(\nabla_{A,1})$	$\deg(\text{pr}_\omega)$
$A(C_3)$	12	12	1	6	2	12	1
$A(C_4)$	36	36	1	12	2	12	2
$A(C_5)$	120	120	1	60	2	120	1
$A(C_6)$	300	300	1	130	2	150	2
$A(C_7)$	840	840	1	420	2	840	1
$A(C_8)$	1960	1960	1	910	2	910	2
$A(K_4)$	60	60	1	26	2	48	1
$A(K_5)$	280	280	1	90	2	140	1
$A(K_6)$	1260	1260	1	276	2	360	1

Table 1: $\deg(\nabla_{A,b})$ for different reduced incidence matrices $A = A(G)$. Numbers in bold indicate that the upper bound from Theorem 6.8 is *not* attained.

Theorem 6.8 and Example 6.9 show that Lissajous discriminants may have large degrees, which makes them challenging to compute. In some cases, the polynomial $\Delta_{A,b}$ exhibits some symmetries, and this can be exploited in the computation. Here is an example.

Proposition 6.10. *The discriminant $\Delta_{A,1}$ satisfies $\Delta_{A,1}(\omega) = \alpha\Delta_{A,1}(-\omega)$, where $\alpha = \pm 1$. In particular, the degree of each monomial appearing in $\Delta_{A,1}$ is even if $\alpha = 1$, and odd if $\alpha = -1$.*

Proof. The Lissajous discriminant is stable under changing the sign of ω . Indeed, we have

$$A\psi_{A,1}(v) = \omega \text{ and } \det J_{A,1}(v) = 0 \iff A\psi_{A,1}(v^{-1}) = -\omega \text{ and } \det J_{A,1}(v^{-1}) = 0.$$

This implies that $\Delta_{A,1}(\omega) = \alpha\Delta_{A,1}(-\omega)$ for some $\alpha \in \mathbb{C}^*$. Changing signs twice, we find $\Delta_{A,1}(\omega) = \alpha\Delta_{A,1}(-\omega) = \alpha^2\Delta_{A,1}(\omega)$, hence $\alpha = \pm 1$. Finally, write $\Delta_{A,1} = \Delta_e + \Delta_o$ where Δ_e is the sum of the monomials of $\Delta_{A,1}$ with even degree, and Δ_o contains those with odd degree. If $\alpha = 1$ then $2\Delta_o = \Delta_{A,1}(\omega) - \Delta_{A,1}(-\omega) = 0$. Similarly, if $\alpha = -1$ then $\Delta_e = 0$. \square

If A comes from a graph G , as in Section 4, then the Lissajous discriminant respects the symmetries of G . We shall now make this more precise. Recall that an automorphism of G is a permutation of its vertices which preserves adjacency. These permutations constitute the automorphism group $\text{Aut}(G)$, a subgroup of the symmetric group of order $n!$. For instance, the automorphism group of the complete graph K_n is $\text{Aut}(K_n) = S_n$, the full symmetric group. The automorphism group of the cycle graph C_n is the dihedral group of order $2n$.

Previously, we have worked with the reduced incidence matrix $A(G) \in \mathbb{Z}^{d \times n}$, which has full rank d . To describe the symmetries of the discriminant, it is more natural to work with the full incidence matrix $A_G \in \mathbb{Z}^{(d+1) \times n}$ of rank d and to consider the equivalent equations

$$A_G\psi_{A_G,1}(v) = \omega, v_{d+1} = 1 \quad \text{and} \quad \text{rank}(J_{A_G,b}(v)) < d.$$

Since the rows of A_G sum to zero, these equations imply that $\omega_1 + \dots + \omega_d + \omega_{d+1} = 0$. The Lissajous discriminant $\nabla_{A_G,1}$ is (expected to be) a hypersurface *inside the hyperplane* $\omega_1 + \dots + \omega_d + \omega_{d+1} = 0$. Its projection onto the first d coordinates is $\nabla_{A(G),1}$.

Proposition 6.11. *Let G be a graph with $m = d + 1$ vertices and let $\sigma \in \text{Aut}(G)$. We have $\omega \in \nabla_{A_G,1} \subset \mathbb{C}^{d+1}$ if and only if $\sigma(\omega) \in \nabla_{A_G,1}$, where σ acts on ω by permuting coordinates.*

Proof. The group $\text{Aut}(G)$ acts on $(\mathbb{C}^*)^{d+1}$ and on \mathbb{C}^{d+1} by permuting coordinates. The proposition follows from the observation that the map $f_G : (\mathbb{C}^*)^{d+1} \rightarrow \mathbb{C}^{d+1}$ given by $v \mapsto A_G\psi_{A_G,1}(v)$ is equivariant with respect to these actions, meaning that $f_G(\sigma(v)) = \sigma(f_G(v))$ for $\sigma \in \text{Aut}(G)$. To prove this, write the k -th coordinate of $f_G(v)$ as

$$f_G(v)_k = \sum_{(k,j)>0} \frac{1}{2i}(v_k v_j^{-1} - v_k^{-1} v_j) - \sum_{(k,j)<0} \frac{1}{2i}(v_j v_k^{-1} - v_j^{-1} v_k),$$

where $(k, j) > 0$ denotes a sum over all edges (k, j) oriented from k to j , and similarly for $(k, j) < 0$. This simplifies to $f_G(v)_k = \sum_{(k,j) \in E_k} \frac{1}{2i}(v_k v_j^{-1} - v_k^{-1} v_j)$, where E_k is the set of all edges adjacent to the vertex k . Now the equality $f_G(\sigma(v))_k = f_G(v)_{\sigma(k)}$ is clear from

$$\sum_{(k,j) \in E_k} \frac{1}{2i}(v_{\sigma(k)} v_{\sigma(j)}^{-1} - v_{\sigma(k)}^{-1} v_{\sigma(j)}) = \sum_{(\sigma(k), \sigma(j)) \in E_{\sigma(k)}} \frac{1}{2i}(v_{\sigma(k)} v_{\sigma(j)}^{-1} - v_{\sigma(k)}^{-1} v_{\sigma(j)}),$$

since $(k, j) \in E_k$ if and only if $(\sigma(k), \sigma(j)) \in E_{\sigma(k)}$, by definition of $\text{Aut}(G)$. \square

Example 6.12. Expanding $\Delta_{A(C_3),1}$ from Example 6.5 as a polynomial in ω_1 and ω_2 , we obtain 41 terms, all of even degree. The fact that $\Delta_{A(C_3),1}(\omega_1, \omega_2, \omega_3)$ can be expressed in terms of elementary symmetric polynomials mirrors the fact that $\text{Aut}(C_3) = S_3$.

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