Quasi-static in vivo elastography from internal displacement information only

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7 Supplementary materials

7.1 Obtaining the standard expression for the Virtual Fields Method

To go from equation (2) to equation (3) in the paper, the steps outlined in this section have to be taken. We repeat equation (2) to enhance readability:

$$\int_{\Omega} \vec{\eta} \cdot (\nabla \cdot \boldsymbol{\sigma}) \mathrm{d}V = 0. \tag{S.1}$$

Using the product rule and the fact that $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$ because of conservation of moment of momentum, we can derive the following expression:

$$\nabla \cdot (\vec{\eta} \cdot \boldsymbol{\sigma}) = \nabla \vec{\eta} : \boldsymbol{\sigma} + \vec{\eta} \cdot (\nabla \cdot \boldsymbol{\sigma}), \tag{S.2}$$

where " : " is the double dot operator. Using the result from (S.2) in (S.1) we arrive at:

$$\int_{\Omega} \nabla \cdot (\vec{\eta} \cdot \boldsymbol{\sigma}) \mathrm{d}V - \int_{\Omega} \nabla \vec{\eta} : \boldsymbol{\sigma} \mathrm{d}V = 0. \tag{S.3}$$

We can apply the divergence theorem on the first term of (S.3) such that the following expression emerges:

$$\int_{\partial\Omega} \vec{\eta} \cdot (\boldsymbol{\sigma} \cdot \hat{n}) \mathrm{d}A - \int_{\Omega} \nabla \vec{\eta} : \boldsymbol{\sigma} \mathrm{d}V = 0. \tag{S.4}$$

Using the linear constitutive relation: $\boldsymbol{\sigma} = \boldsymbol{C} : \boldsymbol{\epsilon}$ and the boundary condition $\boldsymbol{\sigma} \cdot \hat{n} = \vec{p}$ we arrive at:

$$\int_{\partial\Omega} \vec{\eta} \cdot \vec{p} \mathrm{d}A - \int_{\Omega} \nabla \vec{\eta} : (\boldsymbol{C} : \boldsymbol{\epsilon}) \mathrm{d}V = 0. \tag{S.5}$$

As derived in the Supplementary materials 7.2, the fundamental symmetries in $\boldsymbol{C}$ can be exploited to rewrite (S.5) into the general weak formulation for $\vec{u}$:

$$\int_{\Omega} \nabla \vec{\eta} : (\boldsymbol{C} : \nabla \vec{u}) \mathrm{d}V = \int_{\partial\Omega} \vec{\eta} \cdot \vec{p} \mathrm{d}A. \tag{S.6}$$

which corresponds to equation (3) in the paper.

7.2 Symmetry of $\boldsymbol{C}$ and its effect on the non-symmetric part of $\nabla\vec{u}$

All matrices or second order tensors can be uniquely decomposed into a symmetric and an anti-symmetric part:

$$\boldsymbol{A} = \boldsymbol{A}^s + \boldsymbol{A}^{as} \equiv \frac{1}{2}(\boldsymbol{A} + \boldsymbol{A}^T) + \frac{1}{2}(\boldsymbol{A} - \boldsymbol{A}^T). \tag{S.7}$$

The inherent symmetry of the elasticity tensor $\boldsymbol{C}$ can be summarized as:

$$\begin{aligned} C_{ijkl} &= C_{jikl} \\ C_{ijkl} &= C_{ijlk} \\ C_{ijkl} &= C_{klij}. \end{aligned} \tag{S.8}$$

Because of this symmetry in the fourth order tensor, the anti-symmetric part of any matrix disappears when using the double dot operation:

$$\begin{aligned} \boldsymbol{C} : \boldsymbol{A}^{as} &= C_{ijkl} A^{as}_{lk} \\ &= C_{ijlk} A^{as}_{kl} && \text{(Change letter order)} \\ &= -C_{ijlk} A^{as}_{lk} && \text{(Use anti-symmetry: } A^{as}_{kl} = -A^{as}_{lk}) \\ &= -C_{ijkl} A^{as}_{lk} && \text{(Use symmetry: } C_{ijkl} = C_{ijlk}) \end{aligned} \tag{S.9}$$

Comparing the first and last line of equation (S.9), it becomes clear that $C_{ijkl}A^{as}_{lk} = -C_{ijkl}A^{as}_{lk}$ which can only hold if $\boldsymbol{C} : \boldsymbol{A}^{as} = \boldsymbol{0}$.

Now as $\boldsymbol{\epsilon} = (\nabla\vec{u})^s = \nabla\vec{u} - (\nabla\vec{u})^{as}$ we find $\boldsymbol{C} : \boldsymbol{\epsilon} = \boldsymbol{C} : (\nabla\vec{u} - (\nabla\vec{u})^{as}) = \boldsymbol{C} : \nabla\vec{u}$.

7.3 Region-wise stiffness characterization

The derivation for the region-wise stiffness characterization follows almost exactly the same steps as the local, voxel-wise approach given in the paper, with as difference that we combine columns in the problem matrix $\boldsymbol{\mathcal{A}}$. The following figure might help to give some intuition.

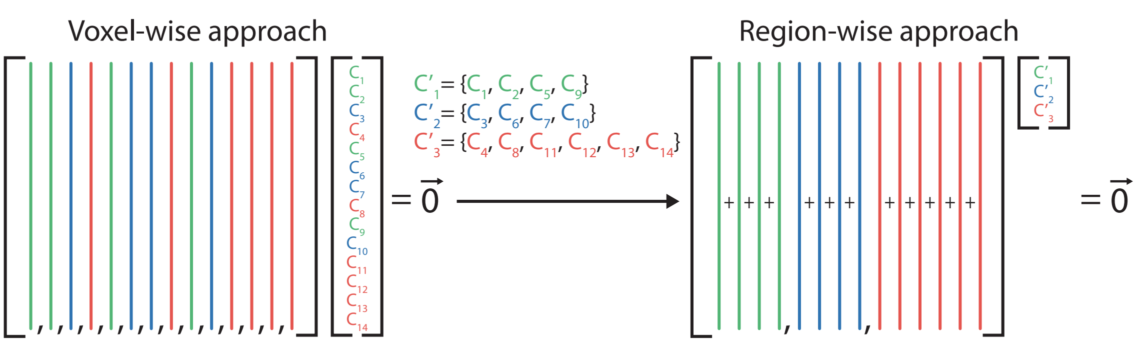

Figure S1: Matrix-vector problem for the voxel-wise and region-wise approach. In this example, the voxel-wise approach consists of 14 unknowns (voxels), but for realistic problem this number is higher.

Note that by assuming material properties to be region-wise constant, the amount of unknowns is drastically reduced (in the simple toy example from Fig. S1, from 14 unknowns to 3. In practice the voxel-wise approach has in the order of 1000 unknowns). Grouping a set of unknowns and assuming they have the same value results in an addition of the corresponding columns in the matrix. The grouping of unknowns (C_x) that refer to a piecewise linear shape function creates an unknown (C'_x) that still refers to a piecewise linear shape function which is the sum of the individual shape functions.

7.4 Time-resolved data

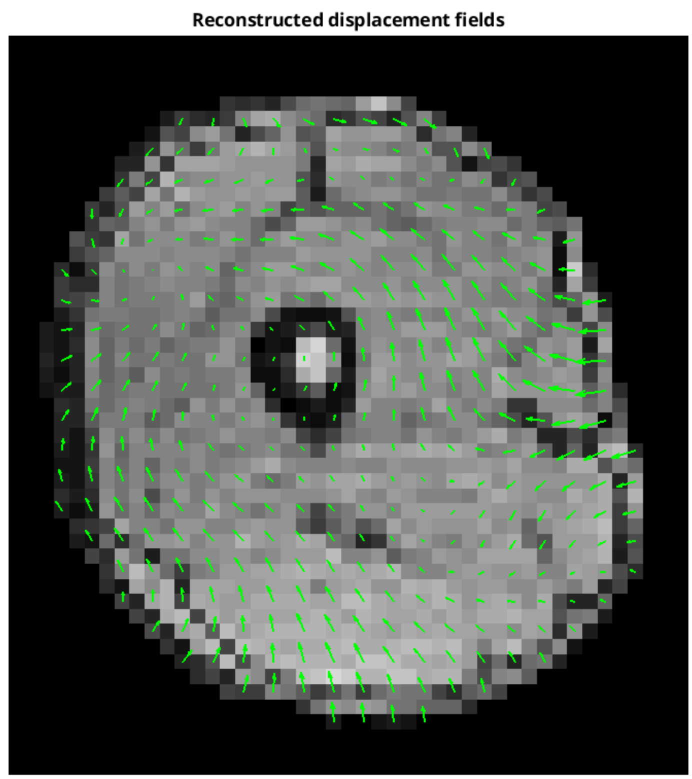

Figure S2: Single frame in the video for the estimates for the time-resolved displacement fields (green arrows) of the thigh muscle with a spatial resolution of 3.5mm isotropic and a temporal resolution of 345.6ms. The time-resolved images are obtained by performing an inverse Fast Fourier Transform (IFFT) on the estimated time-resolved k-space data reconstructed using Spectro-dynamic MRI. This is followed by a geometric correction to account for gradient nonlinearities. The full video can be found online.