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Path decompositions of oriented graphs

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ABSTRACT

We consider the problem of decomposing the edges of a digraph into as few paths as possible. A natural lower bound for the number of paths in any path decomposition of a digraph D is $\frac{1}{2}\sum_{v\in V(D)}|d^+(v)-d^-(v)|$; any digraph that achieves this bound is called consistent. Alspach et al. 1976 conjectured in 1976 that every tournament of even order is consistent and this was recently verified for large tournaments by Girão et al. 2023. A more general conjecture of Pullman (Reid and Wayland, 1987) states that for odd d, every orientation of a d-regular graph is consistent. We prove that the conjecture holds for random d-regular graphs with high probability i.e. for fixed odd d and as $n \to \infty$ the conjecture holds for almost all d-regular graphs. Along the way, we verify Pullman's conjecture for graphs whose girth is sufficiently large (as a function of the degree).

1. Introduction

Decomposing an object into smaller parts that share some common structure is a frequent theme in mathematics. Many classical and fundamental problems in graph theory can be phrased as *decomposition* problems, i.e. problems of partitioning the edge set of a graph into parts with certain properties, where the goal is usually to minimize the number of parts.

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A famous result of Lovász [24] says that every n-vertex graph can be decomposed into at most $\lfloor n/2 \rfloor$ paths and cycles. This result was inspired by a conjecture of Gallai (see [24]) which says that every connected n-vertex graph can be decomposed into at most $\lceil n/2 \rceil$ paths, and this is sharp for complete graphs with an odd number of vertices. The conjecture remains open, where currently the best general bound is $\lfloor 2n/3 \rfloor$ due to Dean and Kouider [14] and independently Yan [32]. The conjecture is also known to hold for several graph classes (see e.g. [6] and references therein).

In this paper, we are interested in the problem of decomposing *directed* graphs into as few *paths* as possible. The obvious analogue of Gallai's conjecture for directed graphs is false because there are directed graphs that need many more than $\lceil n/2 \rceil$ paths to decompose them. For example $\lfloor n^2/4 \rfloor$ paths are needed to decompose the balanced complete bipartite digraph that sends all its edges from one vertex class to the other since each edge must be a path; in fact $\lfloor n^2/4 \rfloor$ paths are enough to decompose any digraph on at least four vertices as shown by O'Brien [28] (building on work of Alspach and Pullman [5]). However, whereas Gallai's conjecture is close to sharp for many graphs (e.g. graphs where all degrees are odd require at least $n/2$ paths to decompose them), $\lfloor n^2/4 \rfloor$ is far from sharp for most digraphs (e.g. a bound of $n^2/4$ is meaningless for sparse digraphs). Therefore one is interested in more subtle bounds for digraphs.

In this direction, Alspach and Pullman [5] gave a natural lower bound on the number of paths needed to decompose a digraph in terms of the degrees of the vertices. Let $\mathrm{pn}(D)$ be the minimum number of paths required to decompose the edges of a digraph D. As observed in [5], $\mathrm{pn}(D)$ can be bounded below by the excess of D, which is defined to be

$$\mathrm{ex}(D) := \frac{1}{2} \sum_{v \in V(D)} |d^+(v) - d^-(v)|.$$

Indeed, we have $\mathrm{pn}(D) \geq \mathrm{ex}(D)$ because, for any path decomposition of D, the number of paths that start (resp. end) at a vertex v is at least $\max(d^+(v) - d^-(v), 0)$ (resp. $\max(d^-(v) - d^+(v), 0)$), and summing this over all vertices counts each path exactly twice. Any digraph satisfying $\mathrm{pn}(D) = \mathrm{ex}(D)$ is said to be *consistent*. We say an undirected graph G is *strongly consistent* if every orientation of G is consistent.

It is clear that not every digraph is consistent (e.g. a directed cycle or any Eulerian digraph has excess zero), and in fact it is NP-complete to determine whether a digraph is consistent; see [29] or [12]. A conjecture of Alspach, Mason, and Pullman from 1976 states that every tournament of even order is consistent; in our terminology the conjecture can be stated as follows.

Conjecture 1.1 (*[4]*)**.** *For each odd d, the complete graph on $d + 1$ vertices is strongly consistent.*

Girão, Granet, Kühn, Lo, and Osthus [16] recently resolved Conjecture 1.1 for large d, building on earlier work of the second author with Lo, Skokan, and Talbot [23]. In fact, Conjecture 1.1 is a generalization of Kelly's conjecture about decomposing regular tournaments into Hamilton cycles, which was solved by Kühn and Osthus [21] for large tournaments using their robust expanders technique. The following conjecture attributed to Pullman (see [30]) considerably generalizes Conjecture 1.1 from complete graphs to all regular graphs.

Conjecture 1.2. *For odd d, every d-regular graph is strongly consistent.*

We note that Conjecture 1.2 is immediate for $d = 1$, and was verified for $d = 3$ by Reid and Wayland [30], but is wide open beyond that. (Note that Eulerian digraphs show that Conjecture 1.2 cannot hold for even d.) Our main result is to verify Conjecture 1.2 for random regular graphs.

Theorem 1.3. *For each fixed odd d, a uniformly random n-vertex, d-regular graph is strongly consistent with probability tending to 1 as $n \to \infty$.*

Secondly, we verify Conjecture 1.2 for graphs with no short cycles. Write $g(G)$ for the girth of a graph G.

Theorem 1.4. *For odd d, every d-regular graph G with $g(G) \geq 200d^2$ is strongly consistent.*

In fact, our results are more general. We prove that for any graph (not necessarily regular), every orientation without short cycles is consistent, provided that $d^+(v) \neq d^-(v)$ for every vertex v (see Theorem 3.1). Then Theorem 1.4 follows as an immediate corollary. Furthermore, we show that it is possible to allow some vertices satisfying $d^+(v) = d^-(v)$, as long as they are sufficiently far apart from each other (see Theorem 3.2).

While the robust expanders technique was used heavily in the previous related works [16,21,23] mentioned above, it is not applicable in our setting of sparse graphs. Instead, we develop a new absorption idea tailored to the sparse setting of Pullman's Conjecture.

Further related work

Path and cycle decomposition problems are among the most widely studied problems in extremal graph theory. Such problems go as far back as the 19th century with Walecki [25] showing that the edges of a complete graph of odd order can be decomposed into Hamilton cycles. There are many beautiful problems and results in the area, and we mention only a few of them first for graphs and then for digraphs.

We have already mentioned Gallai's conjecture about decomposing graphs into paths. For cycles, Erdős and Gallai [15] conjectured that any n-vertex graph can be decomposed into $O(n)$ cycles and edges. There has been extensive work on this conjecture (see e.g. references in [9]) including recent progress of Conlon, Fox and Sudakov [11] giving a bound of $O(n \log\log n)$ cycles and currently the best bound of $O(n \log^* n)$ cycles due to Bucić and Montgomery [9] (here $\log^* n$ is the iterated logarithm function). Another important problem, due to Akiyama, Exoo and Harary [1], is the *Linear Arboricity Conjecture*, which asks for a decomposition of a d-regular graph into at most $\frac{d+1}{2}$ linear forests (a linear forest is a vertex disjoint union of paths). Again, this conjecture has received considerable attention; we mention here only that it is known to hold asymptotically as shown by Alon [2], and currently the best bounds are $\frac{d}{2} + 3\sqrt{d}\log^4 d$ due to Lang and Postle [22], and $\frac{d}{2} + O(\log n)$ due to Christoph, Draganić, Girão, Hurley, Michel, and Müyesser [10], where the comparison between the two best bounds depends on whether $d = \Omega(\log^2 n)$ or not. In a related direction, one can ask for a path decomposition in which path lengths are specified; for example Kotzig [20] asked which d-regular graphs (for odd d) can be decomposed into paths of length d. Recently Montgomery, Müyesser, Pokrovskiy, and Sudakov [26] have made progress on this problem by showing all d-regular graphs can be almost decomposed into paths of length roughly d.

In the directed setting, we have already mentioned the long-standing conjecture of Kelly (now settled for large tournaments), which eventually led to Conjectures 1.1 and 1.2 a few years later. A directed analogue of the Erdős-Gallai Conjecture says that any directed Eulerian graph[3] can be decomposed into $O(n)$ cycles (in fact an exact number of cycles was conjectured by Jackson [18] and later updated by Dean [13].) The best bound here is $O(n \log n)$ [19]. A directed analogue of the linear arboricity conjecture was first mentioned by Nakayama and Peroché [27] who conjectured that every d-regular digraph can be decomposed into at most $d+1$ linear diforests. He, Li, Bai, and Sun [17] observed that the conjecture does not hold for complete digraphs on three or five vertices, but conjectured that these are the only exceptions. The conjecture is known to be true for digraphs with large girth due to Alon [3], and for complete digraphs due to He, Li, Bai, and Sun [17], but not much is known beyond this. There does not seem to be any formulation of Kotzig's question in the directed setting, but it is worth noting that while Kotzig's question for (even) cliques asks to decompose the clique into Hamilton paths, the obvious analogue in the oriented setting would ask to decompose even tournaments that are almost regular into Hamilton paths; this is a much harder problem which is in fact equivalent to Kelly's conjecture.

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[3] Note that the Erdős-Gallai Conjecture is equivalent to the conjecture that every Eulerian graph can be decomposed into $O(n)$ cycles since every graph can be written as an edge-disjoint union of an Eulerian graph and a forest. Here we use a slightly general definition for Eulerian (directed) graphs; we say a graph (resp. directed graph) G is Eulerian if $d(v)$ is even (resp. $d^+(v) = d^-(v)$) for all $v \in V(G)$.

The basic idea

Given an orientation D of some graph (with some further properties), we wish to decompose D into $\text{ex}(D)$ paths. By iteratively removing cycles from D until it is no longer possible, we can write D as the edge-disjoint union $D = E \cup A$, where E is an Eulerian digraph and A is an acyclic digraph. In fact, $\text{ex}(A) = \text{ex}(D)$ (and $\text{ex}(E) = 0$) because each removal of a cycle preserves $|d^+(v) - d^-(v)|$ for every vertex v. Furthermore, it is easy to show that any acyclic digraph is consistent simply by iteratively removing maximal paths and observing that the excess of the digraph decreases by one with each removal. Putting all this together, we see that D can be decomposed into an Eulerian digraph E together with a family $\mathcal{P}$ of $\text{ex}(D) = \text{ex}(A)$ paths. While we have the correct number of paths, not all of the edges (namely those in E) have been covered by these paths. We know E can be decomposed into a set of cycles $\mathcal{C}$ and the challenge now is to absorb each cycle in $\mathcal{C}$ into some of the paths in $\mathcal{P}$. More precisely, for each cycle $C \in \mathcal{C}$, we try to find paths $P_1, \dots, P_k \in \mathcal{P}$ so that $C \cup P_1 \cup \cdots \cup P_k$ can be rewritten as the edge-disjoint union of k new paths $P'_1 \cup \cdots \cup P'_k$; in this way the total number of paths remains $\text{ex}(D)$. By repeating this, we eventually use up all the edges in E and end up with the correct number (i.e. $\text{ex}(D)$) of paths.

In order to make the absorption step work, we must choose $\mathcal{P}$ and $\mathcal{C}$ carefully. In fact $\mathcal{P}$ (as described above) is not mentioned explicitly in our proofs and is constructed somewhat indirectly. On the other hand, $\mathcal{C}$ is given explicitly. One might expect that it is desirable to have as few cycles as possible in $\mathcal{C}$ so that less absorption is needed; however, surprisingly, we actually maximize the number of cycles in $\mathcal{C}$ because it gives us some structural control over the cycles.

2. Preliminaries

Throughout the paper, we use standard graph theory notation and terminology. For $k \in \mathbb{N}$, we sometimes denote the set $\{1, 2, \dots, k\}$ by $[k]$. Throughout, a *digraph* D is a directed graph without loops where, for any distinct vertices u and v in D, there are at most two edges between u and v, at most one in each direction. A digraph D is an *oriented* graph if it has at most one edge between any two distinct vertices, i.e. D can be obtained by orienting the edges of an undirected graph. For a (di)graph G, we denote the vertex set and the edge set of G by $V(G)$ and $E(G)$, respectively. For $a, b \in V(G)$, we write ab for the (directed) edge from a to b. For an oriented graph D, we denote the *underlying undirected graph of* D by $D^\diamond$. We say an oriented graph D is a *tournament* if $D^\diamond$ is a complete graph, that is $ab \in E(D^\diamond)$ for every distinct $a, b \in V(D^\diamond)$. We write $H \subseteq G$ to mean H is a *sub(di)graph* of the (di)graph G, that is $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The *(di)graph induced* by $F \subseteq E(G)$, denoted by $G[F]$, is the sub(di)graph of G with vertex set consisting of those vertices incident to edges in F and with edge set F. Similarly, the *(di)graph induced* by $S \subseteq V(G)$, denoted by $G[S]$, is the sub(di)graph of G with the vertex set S and edge set consisting of those edges which have both endpoints in S. For simplicity, we sometimes write F instead of $G[F]$. A subset $S \subseteq V(G)$ is called an *independent set* if $G[S]$ has no edges. For a collection $\mathcal{Q}$ of (di)graphs, we write $V(\mathcal{Q}) = \bigcup_{Q \in \mathcal{Q}} V(Q)$ and $E(\mathcal{Q}) = \bigcup_{Q \in \mathcal{Q}} E(Q)$. For a collection $\mathcal{Q}$ of sub(di)graphs of G, we write $G - \mathcal{Q}$ for the (di)graph obtained from G by deleting all the edges in $E(\mathcal{Q})$. We say a graph G is *bipartite with bipartition* (A, B) if $V(G) = A \cup B$ and $E(G) \subseteq \{ab : a \in A, b \in B\}$, and we indicate G is such a bipartite graph by writing $G = G(A, B)$.

Neighbourhoods and degrees: For a graph G and a vertex $v \in V(G)$, $N_G(v)$ denotes the set of *neighbours* of v, that is the set of vertices $w \in V(G)$ for which $wv \in E(G)$. We denote the *degree* of a vertex v by $d_G(v)$, that is $d_G(v) = |N_G(v)|$. Similarly, for a digraph D and $v \in V(D)$, let $N^+_D(v)$ and $N^-_D(v)$ denote the set of *outneighbours* and *inneighbours* of v, that is the set of vertices $w \in V(D)$ for which $vw \in E(D)$ and $wv \in E(D)$, respectively. We denote the *outdegree* and *indegree* of v by $d^+_D(v)$ and $d^-_D(v)$, respectively, that is, $d^+_D(v) = |N^+_D(v)|$ and $d^-_D(v) = |N^-_D(v)|$. For $S \subseteq V(D)$, we write $N^+_D(S) := \bigcup_{v \in S} N^+(v)$ and $N^-_D(S) := \bigcup_{v \in S} N^-(v)$. We often omit subscripts when they are clear from the context. For directed graphs, we define $d(v) := d^+(v) + d^-(v)$ as the *total degree* of v. The *minimum degree* of a graph G is $\delta(G) := \min\{d(v) : v \in V(G)\}$. Similarly, the *maximum degree* of G is $\Delta(G) := \max\{d(v) : v \in V(G)\}$. For a digraph D, the *minimum semi-degree* of D is $\delta^0(D) := \min\{\min\{d^+(v), d^-(v)\} : v \in V(D)\}$, and, the *maximum semi-degree* of D is defined as

$\Delta^0(D) := \max\{\max\{d^+(v), d^-(v)\} : v \in V(D)\}$. We say a graph (resp. digraph) G is *k-regular* if $d(v) = k$ (resp. $d^+(v) = d^-(v) = k$) for all vertices $v \in V(G)$.

Paths and cycles: A *directed path P* in a digraph D is a subdigraph of D such that $V(P) = \{v_1, \dots, v_k\}$ for some $k \in \mathbb{N}$ and $E(P) = \{v_1v_2, v_2v_3, \dots, v_{k-1}v_k\}$. We denote such a directed path by its vertices in order, i.e. we write $P = v_1v_2 \dots v_k$. The *length* of a path P is denoted by $|P|$, that is the number of edges in P. A *directed cycle* in D is the same except that it also includes the edge v_kv_1. We sometimes omit the word "directed" and only say path or cycle if it is obvious from the context. We write $\ell(P)$ and $r(P)$ for the starting and ending point of a path P, that is $\ell(P) = v_1$ and $r(P) = v_k$. We sometimes treat an edge ab as a path with $\ell(P) = a$ and $r(P) = b$. For a directed path $P = v_1v_2 \dots v_k$ and $1 \le i < j \le k$, we write v_iPv_j for the directed path $v_iv_{i+1} \dots v_{j-1}v_j$, that is the unique directed path from v_i to v_j along P. Similarly, for a directed cycle C and $x, y \in V(C)$, we write xCy for the unique directed path from x to y along C. For two directed paths P and P' with $\ell(P) = a$, $r(P) = \ell(P') = b$, $r(P') = c$, and $V(P) \cap V(P') = \{b\}$, we write $aPbP'c$ for the directed path that is obtained by *gluing* paths P and P' at the vertex b. For a (di)graph G and $x, y \in V(G)$, we denote the *distance* from x to y by $\mathrm{dist}_G(x, y)$, that is the length of a shortest path P in G with $\ell(P) = x$ and $r(P) = y$. For a (di)graph G, the length of the shortest (directed) cycle is called *girth*, and denoted by $g(G)$. We say a (di)graph G is *acyclic* if it has no (directed) cycle, and in that case, we set $g(G)$ to be infinity. A collection of edge-disjoint (directed) paths (resp. cycles) $\mathcal{Q}$ in a (di)graph G is called a *path family* (resp. *cycle family*) in G. The definitions of *path* and *cycle* extend to graphs in the obvious ways.

The following are non-standard definitions but will be used frequently in the paper.

Tangent and incident paths: We say a path P is *incident* to a vertex v if $v \in \{\ell(P), r(P)\}$. We say P is *incident* to a cycle C (at v) if $E(C) \cap E(P) = \emptyset$ and there exists $v \in V(C)$ such that P is incident to v. We say P is *tangent* to C if either $V(P) \cap V(C) = \{\ell(P)\}$ or $V(P) \cap V(C) = \{r(P)\}$.

We now discuss some basic results we require. We postpone the preliminaries on random graphs to the start of Section 4. The first proposition is a standard application of Hall's theorem.

Proposition 2.1. *Let $G = G(A, B)$ be a bipartite graph. Assume there exists a positive integer t such that $d(a) \ge t \cdot d(b)$ for all $a \in A$ and $b \in B$. Then, there exists a collection $\{R_a : a \in A\}$ of pairwise disjoint sets such that $R_a \subseteq N(a)$ and $|R_a| = t$ for all $a \in A$.*

We will use the following form of Lovász Local Lemma.

Lemma 2.2 (*see e.g. [31]*)**.** *Let $\{A_i : i \in [t]\}$ be a set of events such that each event occurs with probability at most p and such that each event is independent of all the other events except for at most d of them. If $epd \le 1$, then there is a positive probability that none of the events occurs.*

Path decomposition in digraphs

For a digraph D, we say a path family $\mathcal{P}$ is a *path decomposition* for D if $\bigcup_{P\in\mathcal{P}} E(P) = E(D)$. Using this notation, the *path number* of D is given by

$$\mathrm{pn}(D) := \min\{r : \text{there exists a path decomposition } \mathcal{P} \text{ of } D \text{ with } |\mathcal{P}| = r\}.$$

For a vertex $v \in V(D)$, recall that the *excess* of v is

$$\mathrm{ex}_D(v) := \max\{\mathrm{ex}^+_D(v), \mathrm{ex}^-_D(v)\},$$

where $\mathrm{ex}^+_D(v) = \max\{d^+_D(v) - d^-_D(v), 0\}$ and $\mathrm{ex}^-_D(v) = \max\{d^-_D(v) - d^+_D(v), 0\}$. The *excess* of D is

$$\mathrm{ex}(D) := \sum_{v\in V(D)} \mathrm{ex}^+_D(v) = \sum_{v\in V(D)} \mathrm{ex}^-_D(v) = \frac{1}{2} \sum_{v\in V(D)} \mathrm{ex}_D(v).$$

We say v is a *plus* (resp. *minus*) vertex if $ex^+(v) > 0$ (resp. $ex^-(v) > 0$). Let $V^+(D)$ and $V^-(D)$ be the set of plus and minus vertices in D, respectively. Similarly, let $V^0(D) := \{v \in V(D) : \mathrm{ex}(v) = 0\}$ be the set of *excess-zero* vertices. We say a path P is a *plus-minus path (in D)* if $\ell(P) \in V^+(D)$ and $r(P) \in V^-(D)$. For convenience, we state the observation discussed in Section 1 formally, and expand on it.

Observation 2.3 (*See [5]*). *Let D be a digraph, and $\mathcal{P}$ be any path decomposition for D. Then, for all $v \in V(D)$, we have $|\{P \in \mathcal{P} : \ell(P) = v\}| \geq ex^+(v)$ and $|\{P \in \mathcal{P} : r(P) = v\}| \geq ex^-(v)$, which in particular implies that $|\mathcal{P}| \geq \mathrm{ex}(D)$. Moreover, if $|\mathcal{P}| = \mathrm{ex}(D)$, then we have*

(i) $|\{P \in \mathcal{P} : \ell(P) = v\}| = ex^+(v)$ and $|\{P \in \mathcal{P} : r(P) = v\}| = ex^-(v)$ for all $v \in V(D)$;
(ii) Every $P \in \mathcal{P}$ is a plus-minus path in D.

We say a path family $\mathcal{P}$ is a *perfect path decomposition* for D if $\mathcal{P}$ is a path decomposition for D with $|\mathcal{P}| = \mathrm{ex}(D)$. A digraph that admits a perfect path decomposition is said to be *consistent*. Not all graphs are consistent, e.g. a directed cycle has excess zero, but it is easy to see that every acyclic digraph is consistent by successively removing maximal paths.

Proposition 2.4 (*[5, Theorem 4]*). *Any acyclic digraph is consistent.*

We say a path family $\mathcal{P}$ is a *partial path decomposition* for D if $|\{P \in \mathcal{P} : \ell(P) = v\}| \leq ex^+(v)$ and $|\{P \in \mathcal{P} : r(P) = v\}| \leq ex^-(v)$ for all $v \in V(D)$. Note that, if $\mathcal{P}$ is a partial path decomposition, then every $P \in \mathcal{P}$ is a plus-minus path.

Proposition 2.5. *Let D be a digraph, and $\mathcal{Q}$ be a partial path decomposition for D. If $D - \mathcal{Q}$ is consistent, then D has a perfect path decomposition $\mathcal{P}$ with $\mathcal{Q} \subseteq \mathcal{P}$.*

Proof. Let $D' := D - \mathcal{Q}$. Note that for all $v \in V^+(D)$, $\mathrm{ex}^+_{D'}(v) = \mathrm{ex}^+_D(v) - |\{Q \in \mathcal{Q} : \ell(Q) = v\}| \geq 0$ as $\mathcal{Q}$ is a partial path decomposition. Hence, summing over all $v \in V(D')$ gives $\mathrm{ex}(D') = \mathrm{ex}(D) - |\mathcal{Q}|$. If D' has a path decomposition $\mathcal{P}_0$ with exactly $\mathrm{ex}(D')$ many paths, then the result follows by setting $\mathcal{P} = \mathcal{Q} \cup \mathcal{P}_0$. ■

Remark 2.6. If A is an acyclic digraph, Propositions 2.4 and 2.5 imply that any partial path decomposition for A can be completed to a perfect path decomposition.

3. Graphs with no short cycles

In this section, we prove the following theorem, from which Theorem 1.4 follows immediately.

Theorem 3.1. *Let $d \in \mathbb{N}$, and D be a digraph with $\Delta^0(D) \leq d$ and $g(D) \geq 200d^2$ such that $\mathrm{ex}_D(v) \neq 0$ for all $v \in V(D)$. Then, D is consistent.*

In fact, it is possible to allow excess-zero vertices in Theorem 3.1 as long as they are far away from each other. For a digraph D, we call $S \subseteq V(D)$ *k-sparse* in D if for all $u \in S$, there is at most one vertex $v \in S \setminus \{u\}$ satisfying $\mathrm{dist}_{D^\diamond}(u, v) \leq k$. Our general result is the following.

Theorem 3.2. *Let $d \in \mathbb{N}$, and D be a digraph with $\Delta^0(D) \leq d$ and $g(D) \geq 1000d^2$ such that $V^0(D)$ is k-sparse in D for some $k \geq 20d^2 + 2$. Then, D is consistent.*

We note that Theorem 3.2 will be an important part in the proof of Theorem 1.3. In Section 4, we use Theorem 3.2 in the proof of a stronger result Theorem 4.8, from which the proof of Theorem 1.3 will be immediate. We first need a proposition that has a similar flavour to finding an independent transversal.

Proposition 3.3. *Let $d \in \mathbb{N}$, and G be a graph with $\Delta(G) \leq d$. Let $\{X_i : i \in [t]\}$ be a collection of disjoint subsets of $V(G)$ such that $|X_i| \geq 25d$ for each $i \in [t]$. Then G has an independent set X of size $2t$ with $|X \cap X_i| = 2$ for all $i \in [t]$.*

Proof. Without loss of generality, we can assume that $|X_i| = 25d$ for each $i \in [t]$ (or else we can remove some vertices from X_i), and that $V(G) = \bigcup_{i \in [t]} V(X_i)$. For each $i \in [t]$, pick two vertices $a_i, b_i \in X_i$ uniformly at random (without replacement) , where the choices are made independently of one another. Let $X = \bigcup_{i \in [t]} \{a_i, b_i\}$. For any edge $f \in E(G)$, let A_f be the event that both endpoints of f lie in X. Note that X is an independent set if and only if none of the events $\{A_f : f \in E(G)\}$ occur. Hence, it suffices to show that $\mathbb{P}\left(\bigcap_{f\in E(G)} A_f^c\right) > 0$.

Consider an edge $f = uv \in E(G)$ with $u \in X_i$ and $v \in X_j$ for some $i, j \in [t]$. Note that if $i \neq j$, we have

$$\mathbb{P}(A_f) = \mathbb{P}(u, v \in X) = \left(1 - \frac{\binom{25d-1}{2}}{\binom{25d}{2}}\right)^2 = \frac{4}{625d^2}.$$

Similarly, if $i = j$, we have

$$\mathbb{P}(A_f) = \mathbb{P}(u, v \in X) = \frac{1}{\binom{25d}{2}} = \frac{2}{25d(25d-1)} \le \frac{4}{625d^2}.$$

As a result, we have $\mathbb{P}(A_f) \le 4d^{-2}/625$. Moreover, if A_f and A_g are dependent for some $g \in E(G)$, then g should be incident to a vertex in $X_i \cup X_j$. This means that the event A_f is independent of all the other events $\{A_g : g \in E(G)\}$ except for at most $2 \cdot 25d \cdot d = 50d^2$ of them. By Lemma 2.2, the result follows as $e \cdot (4d^{-2}/625) \cdot 50d^2 = 8e/25 < 1$. ■

The following will be the key lemma in the proofs of Theorems 3.1 and 3.2. We say a cycle family $\mathcal{C}$ in a digraph D is *maximal* if $D - \mathcal{C}$ is acyclic.

Lemma 3.4. *Let $d \in \mathbb{N}$, and D be a digraph with $\Delta^0(D) \le d$. Suppose that there exists a maximal cycle family $\mathcal{C} = \{C_i : i \in [t]\}$ in D and a path family $\{Q_{ij} : i \in [t], j \in [100d]\}$ of plus-minus paths in $D - \mathcal{C}$ such that Q_{ij} is tangent to C_i for all $i \in [t]$ and $j \in [100d]$. Then, D is consistent.*

Proof. Let $A := D - \mathcal{C}$, and z_{ij} be the unique vertex in $V(Q_{ij}) \cap V(C_i)$ for $i \in [t]$ and $j \in [100d]$. Since $\mathcal{C}$ is a cycle family, we have $\text{ex}_D(v) = \text{ex}_A(v)$ for all $v \in V(D)$, which, in particular, implies that $\text{ex}(D) = \text{ex}(A)$. Note that A is acyclic as $\mathcal{C}$ is maximal. We emphasize that $\{Q_{ij} : i \in [t], j \in [100d]\}$ is not necessarily a partial path decomposition for A. We claim that for each $i \in [t]$, there exist distinct $a_i, b_i \in [100d]$ such that

(a) $\bigcup_{i\in[t]}\{Q_{ia_i}, Q_{ib_i}\}$ is a partial path decomposition for A,

(b) for all $i \in [t]$, we have $z_{ia_i} \neq z_{ib_i}$, and either $z_{ia_i}, z_{ib_i} \in V^+(D)$ or $z_{ia_i}, z_{ib_i} \in V^-(D)$.

For each $i \in [t]$, let $\mathcal{Q}_i^+ = \{Q_{ij} : z_{ij} \in V^+(D), j \in [100d]\}$ and $\mathcal{Q}_i^- = \{Q_{ij} : z_{ij} \in V^-(D), j \in [100d]\}$. Note that either $|\mathcal{Q}_i^+| \ge 50d$ or $|\mathcal{Q}_i^-| \ge 50d$. Let $\mathcal{Q}_i$ be the largest one among $\mathcal{Q}_i^+$ and $\mathcal{Q}_i^-$. Next, we define an undirected graph G with vertex set $\mathcal{Q} := \bigcup_{i\in[t]} \mathcal{Q}_i$ and such that for any distinct $Q, Q' \in \mathcal{Q}$, we have $QQ' \in E(G)$ if and only if either $\ell(Q) = \ell(Q')$ or $r(Q) = r(Q')$. It is easy to see that $\Delta(G) \le 2d-2$ as $\Delta^0(D) \le d$. Since $|\mathcal{Q}_i| \ge 50d$ for all $i \in [t]$, by applying Proposition 3.3 (where $2d$ plays the role of d), we find an independent set $\mathcal{X}$ of size $2t$ in G such that $|\mathcal{X} \cap \mathcal{Q}_i| = 2$ for all $i \in [t]$. Letting $\mathcal{X} \cap \mathcal{Q}_i = \{Q_{ia_i}, Q_{ib_i}\}$, we see that (a) holds as $\mathcal{X}$ is an independent set in G. Moreover, we have $z_{ia_i} \neq z_{ib_i}$ for all $i \in [t]$. Then, (b) holds by the definition of $\mathcal{Q}_i$, so the claim follows.

Using (a), A has a perfect path decomposition $\mathcal{P}_0$ with $\mathcal{X} \subseteq \mathcal{P}_0$ by Remark 2.6. Then, $\mathcal{P}_0$ is a partial path decomposition for D as $\text{ex}_D(v) = \text{ex}_A(v)$ for all $v \in V(D)$, and hence so is $\mathcal{P}_0 \setminus \mathcal{X}$. By Proposition 2.5, it suffices to prove that the digraph D' induced by the edges $\bigcup_{i\in[t]}\left(E(C_i) \cup E(Q_{ia_i}) \cup E(Q_{ib_i})\right)$ is consistent. It is clear that $\text{ex}(D') = 2t$, and we will show that for each $i \in [t]$, $C_i \cup \{Q_{ia_i}, Q_{ib_i}\}$ can be decomposed into two paths. Fix $i \in [t]$. Using (b), we have either $z_{ia_i}, z_{ib_i} \in V^+(D)$ or $z_{ia_i}, z_{ib_i} \in V^-(D)$. Then, if $z_{ia_i}, z_{ib_i} \in V^+(D)$, we define $U_i := z_{ia_i}C_iz_{ib_i}Q_{ib_i}r(Q_{ib_i})$ and $V_i := z_{ib_i}C_iz_{ia_i}Q_{ia_i}r(Q_{ia_i})$. Similarly, if $z_{ia_i}, z_{ib_i} \in V^-(D)$, we define $U_i := \ell(Q_{ia_i})Q_{ia_i}z_{ia_i}C_iz_{ib_i}$ and $V_i := \ell(Q_{ib_i})Q_{ib_i}z_{ib_i}C_iz_{ia_i}$. In either case, $\{U_i, V_i\}$ decomposes $C_i \cup \{Q_{ia_i}, Q_{ib_i}\}$, so the result follows. ■

The following proposition gives us a cycle family with a simple but crucial structural property that is needed for our absorption method.

Proposition 3.5. *Let D be digraph. Then, there exists a maximal cycle family $\mathcal{C}$ in D such that, writing $A := D - \mathcal{C}$, we have $xy \notin E(A)$ for every $C \in \mathcal{C}$ and distinct $x, y \in V(C)$.*

Proof. Let $t \geq 0$ be the maximum integer such that D has t many edge-disjoint cycles. Then, take a cycle family $\mathcal{C}$ with $|\mathcal{C}| = t$ such that $\sum_{C\in\mathcal{C}} |E(C)|$ is as small as possible. Since $\mathcal{C}$ has t cycles, it is clear that A is acyclic. Assume for a contradiction that there exists a cycle $C_0 \in \mathcal{C}$ and distinct vertices $x, y \in V(C_0)$ such that $xy \in E(A)$. As there are no double edges, we have $xy \notin E(C_0)$. Let C_1 be the cycle obtained from C_0 by replacing the arc xC_0y by the edge xy. It is clear that $|E(C_1)| < |E(C_0)|$. Then, $(\mathcal{C} - C_0) \cup C_1$ is another cycle family consisting of t many cycles, having strictly less than $\sum_{C\in\mathcal{C}} |E(C)|$ edges in total, a contradiction. ■

We will use the following notation in the proofs of Theorems 3.1 and 3.2. For a cycle C and a plus-minus path P that is incident to C at v in a digraph D, we define the *precise part of P with respect to C through v*, denoted by $\mathrm{pp}(P, C, v)$, as follows. If $v \in V^+(D)$ (so if P starts at v), we define $\mathrm{pp}(P, C, v)$ as zPy where y is the first minus vertex along P and $z \neq y$ is the last vertex along vPy lying in C (possibly $z = v$). Similarly, if $v \in V^-(D)$ (so if P ends at v), we define $\mathrm{pp}(P, C, v)$ as yPz where y is the last plus vertex along P and $z \neq y$ is the first vertex along yPv lying in C (possibly $z = v$). Note by the definition that $\mathrm{pp}(P, C, v) \subseteq P$, and that either $\mathrm{pp}(P, C, v)$ is tangent to C (if $y \notin V(C)$) or $V(\mathrm{pp}(P, C, v)) \cap V(C) = \{y, z\}$. Moreover, $\mathrm{pp}(P, C, v) \subseteq P$ is a plus-minus path in D unless $z \in V^0(D)$, so $\mathrm{pp}(P, C, v)$ will always be a plus-minus path in the proof of Theorem 3.1.

Proof of Theorem 3.1. If D is acyclic, then the result follows by Proposition 2.4. Otherwise, using Proposition 3.5, let us take a cycle family $\mathcal{C} = \{C_i : i \in [t]\}$ in D such that $A := D - \mathcal{C}$ is acyclic, and

$$xy \notin E(A) \text{ for all } i \in [t] \text{ and distinct } x, y \in V(C_i). \qquad (*)$$

By Proposition 2.4, A is consistent. Let $\mathcal{P} = \{P_j : j \in [r]\}$ be a perfect path decomposition for A where $r := \mathrm{ex}(A) = \mathrm{ex}(D)$. We define an undirected bipartite graph $G = G(\mathcal{C}, \mathcal{P})$ such that for any $i \in [t]$ and $j \in [r]$, we have $C_iP_j \in E(G)$ if and only if P_j is incident to C_i. Note that $d_G(P_j) \leq 2d$ for every $j \in [r]$ as $\Delta^0(D) \leq d$. Moreover, by Observation 2.3, for every $i \in [t]$, we have $d_G(C_i) \geq g(D) \geq 200d^2$ as $V^0(D) = \emptyset$. Hence, by Proposition 2.1, there exists a collection $\{\mathcal{R}_i : i \in [t]\}$ of pairwise disjoint subsets of $[r]$ such that for all $i \in [t]$ and $j \in \mathcal{R}_i$, we have $|\mathcal{R}_i| = 100d$, and P_j is incident to C_i. For notational convenience, for any $i \in [t]$ and $j \in \mathcal{R}_i$, write $P_{ij} = P_j$. Note that $\{P_{ij} : i \in [t], j \in \mathcal{R}_i\}$ is a path family. Hence, by Lemma 3.4, it suffices to find plus-minus paths $Q_{ij} \subseteq P_{ij}$ such that Q_{ij} is tangent to C_i for all $i \in [t]$ and $j \in \mathcal{R}_i$. Fix $i \in [t]$ and $j \in \mathcal{R}_i$, and take $v_{ij} \in V(C_i)$ such that P_{ij} is incident to v_{ij}. Let $\ell(\mathrm{pp}(P_{ij}, C_i, v_{ij})) = x$ and $r(\mathrm{pp}(P_{ij}, C_i, v_{ij})) = y$. Note that $xP_{ij}y$ is a plus-minus path as $V^0(D) = \emptyset$, and that $E(xP_{ij}y) \cap E(C_i) = \emptyset$. Moreover, either $xP_{ij}y$ is tangent to C_i or $V(C_i) \cap V(xP_{ij}y) = \{x, y\}$. In the former case, we define Q_{ij} to be $xP_{ij}y$. In the latter case, by (*), there exists a vertex $z \in V(xP_{ij}y)$ other than x and y. If $z \in V^+(D)$, we define Q_{ij} to be $zP_{ij}y$, and if $z \in V^-(D)$, we define Q_{ij} to be $xP_{ij}z$. In every case, Q_{ij} is a plus-minus path that is tangent to C_i, so the result follows. ■

We need a simple observation for the proof of Theorem 3.2. Recall the definition of a k-sparse set at the beginning of this section.

Observation 3.6. *Let D be a digraph, and S be a k-sparse set in D for some $k \in \mathbb{N}$. Then, we have $|V(C) \cap S| < 2|V(C)|/k$ for any cycle C in D.*

Proof. Let $\{a_1, \ldots, a_t\}$ be the vertices from S lying on C in this order clockwise. Letting $b_i := |a_iCa_{i+1}|$, we have $b_i + b_{i+1} > k$ for each $i \in [t]$ (with the convention $a_{t+1} = a_1$ and $b_{t+1} = b_1$), because otherwise we would have $\mathrm{dist}_{D^\diamond}(a_i, a_{i+1}) + \mathrm{dist}_{D^\diamond}(a_{i+1}, a_{i+2}) \leq k$, which contradicts S being k-sparse. Hence, the result follows by summing up all these inequalities for $i \in [t]$. ■

Proof of Theorem 3.2. If D is acyclic, then the result follows by Proposition 2.4. Otherwise, using Proposition 3.5, let us take a cycle family $\mathcal{C} = \{C_i : i \in [t]\}$ in D such that $A := D - \mathcal{C}$ is acyclic, and (*) holds. By Proposition 2.4, A is consistent. Let $\mathcal{P} = \{P_j : j \in [r]\}$ be a perfect path decomposition for A where $r := \mathrm{ex}(A) = \mathrm{ex}(D)$. For each $i \in [t]$, we will construct a subset $\mathcal{R}_i \subseteq \mathcal{P}$ such that

(a′) $|\mathcal{R}_i| \geq 200d^2$,
(b′) for each $P_j \in \mathcal{R}_i$, there is a plus-minus path $\mathcal{Q}_{ij} \subseteq P_j$ that is tangent to C_i.

Fix $i \in [t]$. Write $V^\rho := V^\rho(D) \cap V(C_i)$ for $\rho \in \{+, -, 0\}$ for simplicity. Without loss of generality, assume that $|V^+| \geq |V^-|$. Let $V^b := \{v \in V^+ : N_D^+(v) \cap V^0(D) \neq \emptyset\}$. Let $\mathcal{R}_i$ be the set of paths $P_j \in \mathcal{P}$ such that $V(P_j) \cap (V^0 \cup V^b) = \emptyset$ and $\ell(P_j) \in V^+$. Note that each $P_j \in \mathcal{R}_i$ is a plus-minus path incident to C_i at $\ell(P_j)$. We will prove that (a′) holds. Using Observation 2.3, there are at least $|V^+|$ paths $P \in \mathcal{P}$ with $\ell(P) \in V^+$. Using $\Delta^0(D) \leq d$, each $v \in V^0 \cup V^b$ lies on at most d paths in $\mathcal{P}$, so it suffices to show that $|V^+| - d|V^0 \cup V^b| \geq 200d^2$. Let $|V(C_i)| = g$. By Observation 3.6, we have $|V^0| < 2g/k$ as $V^0(D)$ is k-sparse. Let p be the maximum integer such that there exist distinct $x_1, \dots, x_p \in V^b$ and distinct $z_1, \dots, z_p \in V^0(D)$ such that $x_jz_j \in E(D)$ for all $j \in [p]$. Note that $|V^b| \leq dp$ as $\Delta^0(D) \leq d$. Since $\mathrm{dist}_{D^\diamond}(z_j, z_{j'}) \leq \mathrm{dist}_{D^\diamond}(x_j, x_{j'}) + 2$ for all distinct $j, j' \in [p]$, we see that $\{x_1, \dots, x_p\}$ is a $(k-2)$-sparse set in D. By Observation 3.6, we have $p < 2g/(k-2)$. Therefore, we obtain $|V^b| < 2gd/(k-2)$. Hence, we have

$$|V^+| - d|V^0 \cup V^b| \geq \frac{g - |V^0|}{2} - d(|V^0| + |V^b|) > g \cdot \left(\frac{1}{2} - \frac{2d+1}{k} - \frac{2d^2}{k-2}\right) \geq 200d^2,$$

where the last inequality holds since $g \geq 1000d^2$ and $k \geq 20d^2 + 2$.

Next, we will show that (b′) holds. Fix some $P_j \in \mathcal{R}_i$. Let us write $X_j := \mathrm{pp}(P_j, C_i, \ell(P_j))$, $\ell(X_j) = u_j$ and $r(X_j) = w_j$. Recall that w_j is the first minus vertex along P_j and $u_j \neq w_j$ is the last vertex along $\ell(P_j)P_jw_j$ lying in C_i (possibly $u_j = \ell(P_j)$). Since $V(P_j) \cap V^0 = \emptyset$, we have $u_j \in V^+$. Now, if $w_j \notin V(C_i)$, then X_j is tangent to C_i. In that case, we define $\mathcal{Q}_{ij}$ to be X_j. Otherwise, by (*), we have $|X_j| \geq 2$. Let z_j be the unique outneighbour of u_j along X_j. Since $V(P_j) \cap V^b = \emptyset$, we have $z_j \notin V^0(D)$. If $z_j \in V^+(D)$, we define $\mathcal{Q}_{ij}$ to be $z_jP_jw_j$, and if $z_j \in V^-(D)$, we define $\mathcal{Q}_{ij}$ to be u_jz_j. In all cases, we have $\mathcal{Q}_{ij} \subseteq P_j$ is a plus-minus path that is tangent to C_i.

Let us construct an undirected bipartite graph $G = G([t], [r])$ such that for all $i \in [t]$ and $j \in [r]$, $ij \in E(G)$ if and only if $P_j \in \mathcal{R}_i$. By the definition of $\mathcal{R}_i$, $ij \in E(G)$ implies P_j is incident to C_i, so we have $d_G(j) \leq 2d$ for all $j \in [r]$. On the other hand, by (a′), we have $d_G(i) \geq 200d^2$ for all $i \in [t]$. Hence, by Proposition 2.1, there exists a collection $\{\mathcal{S}_i : i \in [t]\}$ of pairwise disjoint subsets of $\mathcal{P}$ such that $\mathcal{S}_i \subseteq \mathcal{R}_i$ and $|\mathcal{S}_i| = 100d$ for all $i \in [t]$. Then, by (b′), for all $P_j \in \mathcal{S}_i$, there exists a plus-minus path $\mathcal{Q}_{ij} \subseteq P_j$ that is tangent to C_i. Since $\{P_j : j \in [r]\}$ is a path family, we can conclude that $\{\mathcal{Q}_{ij} : i \in [t],\ P_j \in \mathcal{S}_i\}$ is a path family. Hence, the result follows by Lemma 3.4. ■

4. Random regular graphs

In this section we prove Theorem 1.3 by showing a more general result, namely Theorem 4.8. We first show that a random d-regular graph has certain properties with high probability (see Definition 4.5 and Proposition 4.6). Then, we prove that for every graph with these properties, every orientation without excess-zero vertices is consistent, where the main ingredient of the proof uses Proposition 4.7 and Theorem 3.2.

Throughout the section, we use standard probability theory notation and terminology. We say a random variable X has *Poisson distribution with parameter* λ if $\mathbb{P}(X = k) = \lambda^k e^{-k}/k!$ for all nonnegative integers k, denoted by $X \sim \mathrm{Po}(\lambda)$. It is well-known that $\mathbb{E}[X] = \lambda$ for such a random variable. Moreover, for independent random variables $X_i \sim \mathrm{Po}(\lambda_i)$, we have $\sum_i X_i \sim \mathrm{Po}(\lambda)$ where $\lambda = \sum_i \lambda_i$. Given n and d, we denote the set of all d-regular graphs on n vertices by $\mathbb{G}_{n,d}$. We write $\mathcal{G}_{n,d}$ for the random d-regular graph on n vertices, that is a graph uniformly selected from $\mathbb{G}_{n,d}$. We denote the set of all d-regular *multigraphs* on n vertices by $\mathbb{S}_{n,d}$, where we allow loops and more than one edges between two vertices. The *configuration model*, introduced by Bollobás [7], provides a model for such graphs . Each partition $\mathcal{P}$ of $[n] \times [d]$ into $nd/2$ pairs (equivalently each perfect matching in K_{nd}) naturally corresponds to a graph G in $\mathbb{S}_{n,d}$ by including an edge between $a \in [n]$ and $a' \in [n]$ in G if and only if $(a, b), (a', b') \in [n] \times [d]$ are matched with each other in $\mathcal{P}$ for some $b, b' \in [d]$. We write $\mathcal{S}_{n,d}$ for the random d-regular multigraph on n vertices corresponding to a pairing of $[n] \times [d]$ sampled uniformly at random. Given a (di)graph property Q and a sequence of random (di)graphs $\{G_n\}_{n>0}$ with $|V(G_n)| \to \infty$ as $n \to \infty$, we say G_n *has* Q *with high probability* if $\mathbb{P}(G_n \text{ has Q}) \to 1$ as $n \to \infty$. The configuration model is quite useful to work on $\mathcal{G}_{n,d}$ because of the following folklore result.

Proposition 4.1 (*See e.g.* [8] , *Corollary 2.18*)**.** *For given $d \geq 2$, any graph property that holds with high probability for $\mathcal{S}_{n,d}$ holds with high probability for $\mathcal{G}_{n,d}$.*

For a graph G, let $Y_i(G)$ denote the number of cycles of length i. The number of cycles of a fixed length in a random regular graphs is known to have a Poisson distribution.

Proposition 4.2 (*see e.g.* [8] , *Corollary 2.19*)**.** *For fixed $k \geq 3$ and $d \geq 2$, let $\{Z_i : 3 \leq i \leq k\}$ be independent random variables with $Z_i \sim \mathrm{Po}((d-1)^i/2i)$, and let $Y_i^n := Y_i(\mathcal{G}_{n,d})$ for $3 \leq i \leq k$. Then, $(Y_3^n, \dots, Y_k^n)_{n>0}$ converges in distribution to $(Z_3, \dots, Z_k)$.*

Recall Markov's inequality which states that for a nonnegative random variable X and a constant $a > 0$, we have $\mathbb{P}(X \geq a) \leq \mathbb{E}[X]/a$. It is easy to see that $\mathcal{G}_{n,d}$ contains only a few short cycles.

Proposition 4.3. *For fixed $k \geq 3$ and $d \geq 2$, as $n \to \infty$, we have*

$$\mathbb{P}\left(\sum_{i=3}^{k} Y_i(\mathcal{G}_{n,d}) \leq \log\log\log n\right) \to 1.$$

Proof. Let $Y := \sum_{i=3}^k Y_i(\mathcal{G}_{n,d})$ and $Z = \sum_{i=3}^k Z_i$ where $\{Z_i : 3 \leq i \leq k\}$ are independent random variables with $Z_i \sim \mathrm{Po}((d-1)^i/2i)$. Note that $Z \sim \mathrm{Po}(\lambda)$ where $\lambda := \sum_{i=3}^k (d-1)^i/2i$. Since k is fixed, by Proposition 4.2, Y converges in distribution to Z. Using Markov's inequality, we have $\mathbb{P}(Z \geq \log\log\log n) \leq \lambda/\log\log\log n$, so $\mathbb{P}(Z \geq \log\log\log n) \to 0$ as $n \to \infty$ as k and d are fixed. Since $\mathbb{P}(Y \geq \log\log\log n) \to \mathbb{P}(Z \geq \log\log\log n)$, the result follows. ■

Next, we give a bound for the probability of the existence of a fixed graph in $\mathcal{S}_{n,d}$.

Proposition 4.4. *Let $k \in \mathbb{N}$. For a graph G_0 with $|V(G_0)| = k$ and $|E(G_0)| = k+1$, we have*

$$\mathbb{P}(\mathcal{S}_{n,d} \text{ has a copy of } G_0) \leq (2d)^{k+1}/n$$

for all natural numbers $d < n$ with dn even and $dn \geq 4k+2$.

Proof. Recall that there are d copies of each vertex in the configuration model. For each vertex $v \in [n]$, let v^j denote the jth copy of v for $j \in [d]$. For simplicity, we write $[u^i - v^j]$ to mean u^i is paired up with v^j. Note that for distinct $u, v \in [n]$, we have $uv \in \mathcal{S}_{n,d}$ if and only if $[u^i - v^j]$ for some $i, j \in [d]$. Let $p := dn/2$, and consider a fixed collection of edges $\{e_1, \dots, e_{k+1}\}$ on the vertex set $[n]$. For $i \in [k+1]$, let us write $e_i = x_iy_i$ for some $x_i, y_i \in [n]$. Given a complete graph on $2r$ vertices, it is well-known that the number of pairings is $(2r-1)(2r-3)\cdots 3\cdot 1$. Hence, using the union bound, we can write

$$\begin{aligned}
\mathbb{P}(e_1, \dots, e_{k+1} \in E(\mathcal{S}_{n,d})) &= \mathbb{P}([x_1^{i_1} - y_1^{j_1}], \dots, [x_{k+1}^{i_{k+1}} - y_{k+1}^{j_{k+1}}] \text{ for some } i_1, j_1, \dots, i_{k+1}, j_{k+1} \in [d]) \\
&= \mathbb{P}\left(\bigcup_{i_1,j_1,\dots,i_{k+1},j_{k+1}\in[d]} \left([x_1^{i_1} - y_1^{j_1}] \wedge \cdots \wedge [x_{k+1}^{i_{k+1}} - y_{k+1}^{j_{k+1}}]\right)\right) \\
&\leq \sum_{i_1,j_1,\dots,i_{k+1},j_{k+1}\in[d]} \mathbb{P}\left([x_1^{i_1} - y_1^{j_1}] \wedge \cdots \wedge [x_{k+1}^{i_{k+1}} - y_{k+1}^{j_{k+1}}]\right) \\
&\leq d^{2k+2} \cdot \frac{(2p-2k-3)(2p-2k-5)\cdots 3\cdot 1}{(2p-1)(2p-3)\cdots 3\cdot 1} \\
&= d^{2k+2} \cdot ((2p-1)(2p-3)\cdots(2p-2k-1))^{-1} \\
&= \frac{d^2}{dn-1} \cdot \frac{d^2}{dn-3} \cdots \frac{d^2}{dn-2k-1}
\end{aligned}$$

$$\leq (2d/n)^{k+1},$$

where the last inequality holds as $dn \geq 4k+2$. Then, as there are at most $\binom{n}{k}k!$ copies of G_0 in the complete graph on n vertices, the expected number of copies of G_0 in $\mathcal{S}_{n,d}$ can be bounded above by $\binom{n}{k}k!(2d/n)^{k+1} \leq (2d)^{k+1}/n$. The result follows by Markov's inequality. ■

Definition 4.5. For $n, d, p \in \mathbb{N}$, we say a graph G is *(n, d, p)-discrete* if $|V(G)| = n$, $\Delta(G) \leq d$, and, letting $\mathcal{C}$ be the set of all cycles of length at most p, the following hold:

(i) $|\mathcal{C}| \leq \log\log\log n$;
(ii) $V(C) \cap V(C') = \emptyset$ for all distinct $C, C' \in \mathcal{C}$;
(iii) $\mathrm{dist}_{G-\mathcal{C}}(u, v) \geq \log\log n$ for all distinct $u, v \in V(\mathcal{C})$.

We prove that a d-regular random graph on n vertices is (n, d, p)-discrete with high probability.

Proposition 4.6. *For fixed $p \geq 3$ and $d \geq 2$, $\mathcal{G}_{n,d}$ is (n, d, p)-discrete with high probability.*

Proof. By Proposition 4.3, we know that List (i) holds for $\mathcal{G}_{n,d}$ with high probability. Let n be a sufficiently large natural number and $\mathcal{C}$ be the collection of all cycles in $\mathcal{G}_{n,d}$ of length at most p. For $x, y \geq 2$ and $z \geq 1$, $F_{x,y,z}$ denotes the (unique) unlabelled graph obtained from three internally vertex disjoint paths of length x, y, z between two fixed vertices. Similarly, for $x, y \geq 3$ and $z \geq 0$, $H_{x,y,z}$ denotes the (unique) unlabelled graph obtained from two cycles of length x and y by taking one vertex from each cycle and joining them via a path of length z. Note that each $F_{x,y,z}$ and $H_{x,y,z}$ has $x + y + z - 1$ vertices and $x + y + z$ edges. Let $\mathcal{F} = \{F_{x,y,z} : x, y \leq p,\ z \leq \log\log n\}$ and $\mathcal{H} = \{H_{x,y,z} : x, y \leq p,\ z \leq \log\log n\}$. Observe that if two distinct cycles $C, C' \in \mathcal{C}$ have a unique common vertex, then $\mathcal{G}_{n,d}$ has a copy of $H_{|C|,|C'|,0}$. If $C, C' \in \mathcal{C}$ have more than one vertices in common, then $\mathcal{G}_{n,d}$ has a copy $F_{a,b,c}$ for some $a, b, c \leq p$. Similarly, if $\mathrm{dist}_{\mathcal{G}_{n,d}-\mathcal{C}}(u, v) \leq \log\log n$ for some distinct $u, v \in V(\mathcal{C})$, it is easy to see that $\mathcal{G}_{n,d}$ contains a graph from $\mathcal{F} \cup \mathcal{H}$. As a result, if $\mathcal{G}_{n,d}$ contains no graph from $\mathcal{F} \cup \mathcal{H}$, then properties (ii) and (iii) hold. Also note that $|\mathcal{F} \cup \mathcal{H}| \leq p^2 \log\log n + p^2(\log\log n + 1) \leq 3p^2 \log\log n$, and that each graph in $\mathcal{F} \cup \mathcal{H}$ has at most $2p + \log\log n - 1 \leq 2\log\log n - 1$ vertices. Hence, by Proposition 4.4, we obtain

$$\mathbb{P}(\mathcal{S}_{n,d} \text{ contains a graph from } \mathcal{F} \cup \mathcal{H}) \leq 3p^2 \log\log n (2d)^{2\log\log n}/n \leq 1/\sqrt{n},$$

where the last inequality holds for sufficiently large n. Therefore, with high probability, $\mathcal{S}_{n,d}$ does not contain any graph from $\mathcal{F} \cup \mathcal{H}$. By Proposition 4.1, the same holds for $\mathcal{G}_{n,d}$, so properties (ii) and (iii) hold with high probability for $\mathcal{G}_{n,d}$, which completes the proof. ■

We need the following definitions in the proofs of Proposition 4.7 and Theorem 4.8. We say a path family $\mathcal{Q}$ (in a digraph) of plus-minus paths is *distinctive* if $\ell(Q) \neq \ell(Q')$ and $r(Q) \neq r(Q')$ for all distinct $Q, Q' \in \mathcal{Q}$. For a digraph D and a vertex $v \in V(D)$, we define the set $\mathrm{PM}(D, v)$ as the set of vertices w for which there exists a plus-minus path P in D such that $\{\ell(P), r(P)\} = \{v, w\}$. Note that $\mathrm{PM}(D, v) \neq \emptyset$ for all $v \in V(D) \setminus V^0(D)$. We also define the *distance to sign change* of a vertex $v \in V(D) \setminus V^0(D)$, denoted by $\mathrm{SD}(D, v)$, as the minimum length of a plus-minus path P in D that is incident to v. The following proposition helps us to find a lower bound for $|\mathrm{PM}(D, v)|$ when $|V^0(D)|$ is small compared to $\mathrm{SD}(D, v)$.

Proposition 4.7. *Let $d \geq 2$ be a natural number. Then, for every digraph D with $\Delta^0(D) \leq d$ and $\mathrm{ex}(D) \neq 0$, and for every vertex $v \in V(D) \setminus V^0(D)$ with $\mathrm{SD}(D, v) > 3|V^0(D)|$, we have*

$$|\mathrm{PM}(D, v)| \geq \frac{1}{d}\left(1 + \frac{1}{2d}\right)^{\mathrm{SD}(D,v)/6}.$$

Proof. Due to symmetry, we may assume without loss of generality that $v \in V^+(D)$. For simplicity, write $|V^0(D)| = \ell$ and $\mathrm{SD}(D, v) = t$. Let $A_0 = \{v\}$, and for every integer $k \geq 0$, define

$$A_{k+1} = \big(N^+(A_k) \cap (V^+(D) \cup V^0(D))\big) \setminus B_k,$$

where $B_k = \bigcup_{i=0}^{k} A_i$. For every integer $k \geq 1$, we now construct a set $E_k \subseteq E(D)$ as follows. By definition, for every $w \in A_k$, we can find an edge $w'w \in E(D)$ with $w' \in A_{k-1}$. We pick one such edge for every $w \in A_k$ and put it in E_k. Letting T be the digraph induced by $\bigcup_{k\geq 1} E_k$, we can see by the construction that for every $w \in V(T)$, there exists a unique path P with $\ell(P) = v$ and $r(P) = w$. In fact, T is an oriented out-tree rooted at v.

If $t = 1$, then $(1 + 1/2d)^{1/6} \leq (1 + 1/2d) \leq d$ as $d \geq 2$, so the result follows as $|\mathrm{PM}(D, v)| \geq 1$. Suppose $t \geq 2$. By the definition of $\mathrm{SD}(D, v)$, we have $|A_i| \geq 1$ for all $0 \leq i \leq t - 1$. This, in particular, implies that $|B_{2\ell}| \geq 2\ell + 1$ as $2\ell < t$. Now, fix some $2\ell \leq i \leq t - 2$; we will prove that $|B_{i+1}| \geq (1 + 1/2d)|B_i|$. Since $B_i \subseteq V^+(D) \cup V^0(D)$, we have $\sum_{v\in B_i} \mathrm{ex}(v) \geq |B_i| - \ell \geq |B_i|/2$ as $|B_i| \geq |B_{2\ell}| \geq 2\ell + 1$. Hence, we can find $|B_i|/2$ edges $f \in E(D)$ with $\ell(f) \in B_i$ and $r(f) \notin B_i$. Therefore, since $i \leq t - 2$, it is necessarily the case that $\ell(f) \in A_i$ and $r(f) \in A_{i+1}$. This implies $|A_{i+1}| \geq |B_i|/2d$ as $\Delta^0(D) \leq d$. Therefore, we obtain $|B_{i+1}| \geq (1 + 1/2d)|B_i|$. Inductively, we obtain

$$|B_{t-1}| \geq |B_{2\ell}|(1 + 1/2d)^{t-1-2\ell} \geq (2\ell + 1)(1 + 1/2d)^{(t-1)/3} \geq \ell + (1 + 1/2d)^{t/6},$$

where the last inequality holds as $\ell \geq 0$ and $t \geq 2$, and the penultimate inequality holds as $\ell \leq (t - 1)/3$ and $|B_{2\ell}| \geq 2\ell + 1$. By the definition of T, note that for any edge $f \in E(D)$ with $\ell(f) \in V(T)$ and $r(f) \notin V(T)$, we have $r(f) \in V^-(D)$. Since $V(T) \subseteq V^+(D) \cup V^0(D)$ and $B_{t-1} \subseteq V(T)$, we have $\sum_{v\in V(T)} \mathrm{ex}(v) \geq |B_{t-1}| - \ell$. Then, we can find $|B_{t-1}| - \ell$ many edges $f \in E(D) \setminus E(T)$ with $\ell(f) \in V(T)$ and $r(f) \notin V(T)$. For every such edge, we have $r(f) \in \mathrm{PM}(D, v)$. Since $\Delta^0(D) \leq d$, we obtain

$$|\mathrm{PM}(D, v)| \geq (|B_{t-1}| - \ell)/d \geq (1 + 1/2d)^{t/6}/d,$$

so the result follows. ■

Now, we are ready to prove our main result in this section.

Theorem 4.8. *For natural numbers $d \geq 2$ and p with $p \geq 1000d^2$, there exists an integer $n_0 = n_0(d, p)$ such that the following holds. Let G be an (n, d, p)-discrete graph with $n \geq n_0$. Then, every orientation D of G satisfying $\mathrm{ex}_D(v) \neq 0$ for all $v \in V(D)$ is consistent.*

Proof. Fix natural numbers $d \geq 2$ and $p \geq 1000d^2$. Let $k := 20d^2 + 2$, and n be sufficiently large. Let G be an (n, d, p)-discrete graph, and D be an orientation of G satisfying $\mathrm{ex}_D(v) \neq 0$ for all $v \in V(D)$. Let $\mathcal{C}_0$ be the collection of all cycles in G of length at most p, and write $G_0 := G - \mathcal{C}_0$. Similarly, let $\mathcal{C} = \{C_i : i \in [t]\}$ be the collection of all directed cycles in D of length at most p, and write $D_0 := D - \mathcal{C}$. Using Proposition 2.5, it suffices to find a partial path decomposition $\mathcal{P}$ for D in which $D - \mathcal{P}$ is consistent. It is clear that $\Delta^0(D - \mathcal{P}) \leq d$ and $V^0(D - \mathcal{P}) \subseteq \bigcup_{P\in\mathcal{P}}\{\ell(P), r(P)\}$ for any partial path decomposition $\mathcal{P}$ for D. Then, by Theorem 3.2, it is enough to find a partial path decomposition $\mathcal{P}$ for D such that $g(D - \mathcal{P}) \geq 1000d^2$ and that $\bigcup_{P\in\mathcal{P}}\{\ell(P), r(P)\}$ is k-sparse in $D - \mathcal{P}$. We will ensure the condition $g(D - \mathcal{P}) \geq 1000d^2$ by choosing $\mathcal{P}$ in such a way that at least one edge from each $C_i \in \mathcal{C}$ is used in $\mathcal{P}$. As a first step, we prove that $\mathcal{C}$ satisfies certain properties as a consequence of G being (n, d, p)-discrete.

Claim 4.9. *The following hold:*

(i′) $t \leq \log\log\log n$;
(ii′) *for any distinct $C, C' \in \mathcal{C}$, $V(C) \cap V(C') = \emptyset$;*
(iii′) $\mathrm{dist}_{D_0^\diamond}(u, v) \geq \log\log n$ *for all distinct $u, v \in V(\mathcal{C})$.*

Proof of claim. Recall Definition 4.5. Since every cycle in $\mathcal{C}$ is an orientation of a cycle in $\mathcal{C}_0$, (i) and (ii) immediately give (i′) and (ii′), respectively. Then, assume for a contradiction that, for some distinct $u, v \in V(\mathcal{C})$, there exists a path P in $D_0^\diamond$ between u and v such that $|E(P)| < \log\log n$. By (iii), we have $\mathrm{dist}_{G_0}(u, v) \geq \log\log n$, which implies P has an edge $xy \in E(\mathcal{C}_0)$ with $xy, yx \notin E(\mathcal{C})$. Let x_0y_0 be the first such edge (i.e. satisfying $x_0y_0 \in E(\mathcal{C}_0)$ and $x_0y_0, y_0x_0 \notin E(\mathcal{C})$) encountered when travelling from u to v along P, with x_0 being closest to u. By (ii), we have $x_0 \in V(\mathcal{C}_0) \setminus V(\mathcal{C})$, which in particular implies that $x_0 \neq u$. Then, we see that $\mathrm{dist}_{G_0}(u, x_0) < \log\log n$ by the choice of x_0y_0, which contradicts (iii). Hence, (iii′) follows. ■

Next, note that $\mathrm{ex}_{D_0}(v) = \mathrm{ex}_D(v) \neq 0$ for all $v \in V(D)$. Then, for every $i \in [t]$, we have either $|V(C_i) \cap V^+(D)| \geq 2$ or $|V(C_i) \cap V^-(D)| \geq 2$. Therefore, we can choose distinct $a_i, b_i \in V(C_i)$ for all $i \in [t]$ such that either $\{a_i, b_i\} \subseteq V^+(D)$ or $\{a_i, b_i\} \subseteq V^-(D)$. Recall the definition of distinctive path family just before Proposition 4.7.

Claim 4.10. *There exists a distinctive path family $\mathcal{Q}$ in D_0 such that*

(1) *there exists a bijection $f : [t] \to \mathcal{Q}$ such that $\{\ell(f(i)), r(f(i))\} \cap V(\mathcal{C}) = \{a_i\}$ for all $i \in [t]$,*
(2) $\mathrm{dist}_{D_0^\diamond}(u, v) > k$ *for all distinct $u, v \in \left(\bigcup_{Q \in \mathcal{Q}} \{\ell(Q), r(Q)\}\right) \cup V(\mathcal{C})$ unless possibly $\{u, v\} = \{\ell(Q), r(Q)\}$ for some $Q \in \mathcal{Q}$.*

Proof of claim. We begin with a brief explanation of our proof strategy. Note that if for all $i \in [t]$, we manage to find a "short" plus-minus path Q_i that is incident to a_i, then Claim 4.9 implies (2). We start the proof by taking the largest collection of such "short" plus-minus paths Q_i. If we fail to find a path for some a_i, we can conclude that a_i has large distance to sign change, which means there exist a "large" number of plus-minus paths incident to a_i using Proposition 4.7. Hence, we have a lot of choice for the other endpoint of Q_i. In this way, (2) can be satisfied as there are only a "small" number of vertices which are close to the endpoints of the paths previously chosen (as $\Delta(G) \leq d$).

Set $\theta := 6(\log\log\log n + \log d)/\,(\log(2d+1) - \log 2d)$ so that

$$\frac{1}{d}\left(1 + \frac{1}{2d}\right)^{\theta/6} = \log\log n. \tag{4.1}$$

Since n is sufficiently large, we have that

$$6\log\log\log n < \theta < (\log\log n)/3. \tag{4.2}$$

Consider the largest subset $S \subseteq [t]$ that admits a distinctive path family $\mathcal{Q}$ in D_0 satisfying (1) and (2) (where S plays the role of $[t]$) such that $|E(Q)| \leq \theta$ for all $Q \in \mathcal{Q}$. If $S = [t]$, we are done, so assume $[t] \setminus S$ is nonempty. Let $D_0' := D_0 - \mathcal{Q}$ and $F_0 := \bigcup_{Q \in \mathcal{Q}} \{\ell(Q), r(Q)\}$. For any $i \in [t] \setminus S$, we have $a_i \notin F_0$ using (1). Also, it is clear that $V^0(D_0') \subseteq F_0$, so $\mathrm{SD}(D_0', a_i)$ is well-defined for all $i \in [t] \setminus S$. Suppose for a contradiction that $\mathrm{SD}(D_0', a_i) \leq \theta$ for some $i \in [t] \setminus S$. Then, we can choose a plus-minus path Q_i in D_0' of length $\mathrm{SD}(D_0', a_i)$ that is incident to a_i. Let x_i be the other endpoint of Q_i. Since $|E(Q)| \leq \theta$ for all $Q \in \mathcal{Q}$ and $|E(Q_i)| \leq \theta$, by (iii′) and (4.2), we see that for all $x \in (F_0 \cup V(\mathcal{C})) \setminus \{a_i\}$,

$$\mathrm{dist}_{D_0^\diamond}(x_i, x) \geq \log\log n - 2\theta > (\log\log n)/3 > k.$$

In particular, this gives $x_i \notin F_0 \cup V(\mathcal{C})$. By (ii′), we have $\mathcal{Q} \cup \{Q_i\}$ is a distinctive path family in D_0 satisfying (1) and (2) (where $S \cup \{i\}$ plays the role of $[t]$) such that $|E(Q)| \leq \theta$ for all $Q \in \mathcal{Q} \cup \{Q_i\}$, which contradicts S being the largest such subset of $[t]$. As a result,

$$\mathrm{SD}(D_0', a_i) > \theta \text{ holds for all } i \in [t] \setminus S. \tag{4.3}$$

Then, consider the largest subset $T \subseteq [t]$ that admits a distinctive path family $\mathcal{Q}'$ satisfying (1) and (2) (where T plays the role of $[t]$) such that $\mathcal{Q} \subseteq \mathcal{Q}'$. We will complete the proof of the claim by showing that $T = [t]$. Assume for a contradiction that there exists $i \in [t] \setminus T$. Let $D_1' := D_0 - \mathcal{Q}'$ and $F_1 := \bigcup_{Q \in \mathcal{Q}'} \{\ell(Q), r(Q)\}$. We have $a_i \notin F_1$ using (1). Also, it is clear that $V^0(D_1') \subseteq F_1$, so $\mathrm{SD}(D_1', a_i)$ is well-defined. Moreover, we have $\mathrm{SD}(D_0', a_i) \leq \mathrm{SD}(D_1', a_i)$, which, in particular, implies that $\mathrm{SD}(D_1', a_i) > \theta$ using (4.3). On the other hand, by (4.2) and (i′), we have

$$|V^0(D_1')| \leq |F_1| \leq 2t \leq 2\log\log\log n < \theta/3 < \mathrm{SD}(D_1', a_i)/3. \tag{4.4}$$

Hence, by using (4.4) to apply Proposition 4.7, we have $|\mathrm{PM}(D_1', a_i)| \geq \log\log n$ using (4.1). Moreover, by (i′) and (4.4), we obtain $|F_1 \cup V(\mathcal{C})| \leq (p+2)\log\log\log n$. Note that, using $\Delta(G) \leq d$, for any $x \in V(D_0)$ and $j \in [k]$, the number of vertices $w \in V(D_0)$ satisfying $\mathrm{dist}_{D_0^\diamond}(w, x) = j$ is at most $d(d-1)^{j-1}$. As $d(d-1)^{j-1} \le d(d-1)^{k-1}$ for any $j \in [k]$, the number of vertices $w \in V(D_0)$ satisfying $\mathrm{dist}_{D_0^\diamond}(w,x) \le k$ for some $x \in F_1 \cup V(\mathcal{C})$ is at most

$$(1+kd(d-1)^{k-1})(p+2)\log\log\log n < \log\log n \le |\mathrm{PM}(D_1', a_i)|,$$

as n is sufficiently large. Hence, we can find $x_i \in \mathrm{PM}(D_1', a_i)$ and a plus-minus path Q_i in D_1' with $\{\ell(Q_i), r(Q_i)\} = \{a_i, x_i\}$ such that $\mathrm{dist}_{D_0^\diamond}(x_i, x) > k$ for all $x \in F_1 \cup V(\mathcal{C})$. This, in particular, implies that $x_i \notin F_1 \cup V(\mathcal{C})$, so we have $\mathcal{Q}' \cup \{Q_i\}$ is a distinctive path family in D_0 satisfying (1) and (2) (where $T \cup \{i\}$ playing the role of $[t]$), which contradicts T being the largest such subset of $[t]$. We conclude that $T = [t]$, so the result follows. ■

By Claim 4.10, let $\mathcal{Q} := \{Q_i : i \in [t]\}$ be a distinctive path family in D_0 satisfying (1) (with $f(i) = Q_i$ for $i \in [t]$) and (2). For $i \in [t]$, let x_i be the other endpoint of Q_i. By (1), note that $x_i \notin V(\mathcal{C})$. Recall that for every $i \in [t]$, either we have $a_i, b_i \in V^+(D)$ or $a_i, b_i \in V^-(D)$. Now, for each $i \in [t]$, we will modify Q_i to obtain another plus-minus path P_i in D_0. If $a_i \in V^+(D)$, we take the last vertex a_i' along Q_i such that $a_i' \in V(C_i)$. If $a_i' \neq b_i$, we define $P_i = b_i C_i a_i' Q_i x_i$, otherwise we define $P_i = a_i C_i b_i Q_i x_i$. Similarly, if $a_i \in V^-(D)$, we take the first vertex a_i' along Q_i such that $a_i' \in V(C_i)$. If $a_i' \neq b_i$, we define $P_i = x_i Q_i a_i' C_i b_i$, otherwise we define $P_i = x_i Q_i b_i C_i a_i$. It is easy to check that for each $i \in [t]$, we have that $E(P_i) \cap E(C_i) \neq \emptyset$ and that $\{\ell(P_i), r(P_i)\} = \{x_i, y_i\}$ where $y_i \in \{a_i, b_i\}$. Hence, $\mathcal{P} := \{P_i : i \in [t]\}$ is a distinctive path family in D_0 since $\{x_i, a_i, b_i : i \in [t]\}$ are distinct using that $\mathcal{Q}$ is a distinctive path family, $x_i \notin V(\mathcal{C})$, and (ii′). Since $\mathrm{ex}_{D_0}(v) = \mathrm{ex}_D(v) \neq 0$ for all $v \in V(D)$, we can conclude that $\mathcal{P}$ is a partial path decomposition for D.

Let $F = D - \mathcal{P}$. By Proposition 2.5, it suffices to prove that F is consistent. Note that $E(P_i) \cap E(C_i) \neq \emptyset$ for all $i \in [t]$, so we have $g(F) > p \ge 1000d^2$. Also, it is clear that $\Delta^0(F) \le d$. Hence, by Theorem 3.2, it is enough to prove that $V^0(F)$ is k-sparse in F as $k = 20d^2 + 2$. Note that $E(F) \setminus E(D_0) \subseteq E(D) \setminus E(D_0) = E(\mathcal{C})$ and that $V^0(F) \subseteq \bigcup_{i \in [t]} \{x_i, y_i\}$ where $y_i \in \{a_i, b_i\}$ for all $i \in [t]$. Consider a path P in $F^\diamond$ between some distinct $u, v \in V^0(F)$ with $|E(P)| \le k$; we will prove that $\{u, v\} = \{x_i, y_i\}$ for some $i \in [t]$ (which shows $V^0(F)$ is k-sparse in F). Note that $a_i \in V^0(F)$ (resp. $b_i \in V^0(F)$) implies that $y_i = a_i$ (resp. $y_i = b_i$) as we have $\mathrm{ex}_F(v) \neq 0$ for all $v \notin \bigcup_{j \in [t]} \{\ell(P_j), r(P_j)\}$. Moreover, $\{u, v\} = \{a_i, b_i\}$ is not possible as at most one of a_i and b_i belongs to $V^0(F)$. Hence, it suffices to prove that $u, v \in \{a_i, b_i, x_i\}$. If $E(P) \cap E(\mathcal{C}^\diamond) = \emptyset$, then $\mathrm{dist}_{D_0^\diamond}(u, v) \le k$. Hence, we have $\{u, v\} = \{a_i, x_i\}$ for some $i \in [t]$ using (2), so we are done. Suppose $E(P) \cap E(\mathcal{C}^\diamond) \neq \emptyset$. Let $z_1 z_1' \in E(P) \cap E(\mathcal{C}^\diamond)$ be the first such edge encountered when travelling from u to v along P, with z_1 being the closest vertex to u (possibly $z_1 = u$ and $z_1' = v$). By (ii′), there exists a unique $i \in [t]$ with $z_1, z_1' \in V(C_i)$. Note that $uPz_1 \subseteq D_0^\diamond$ and $|E(uPz_1)| \le k$. Then, by (2), we have either $z_1 = u$ or $\{z_1, u\} = \{a_{i'}, x_{i'}\}$ for some $i' \in [t]$. Using $z_1 \in V(C_i)$ and $u \in V^0(F) \subseteq \bigcup_{j \in [t]} \{a_j, b_j, x_j\}$, we can conclude that either $u = z_1 \in \{a_i, b_i\}$ or $i' = i$, $u = x_i$, $z_1 = a_i$. This, in particular, implies that $u \in \{x_i, a_i, b_i\}$. Let $z_2 \in V(P)$ be the closest vertex to v along $z_1 P v$ satisfying $E(z_1 P z_2) \subseteq E(C_i^\diamond)$ (possibly $z_2 = z_1'$ and/or $z_2 = v$). It is clear that $z_2 \in V(C_i)$. Moreover, by (ii′), we have $z_2 \notin V(C_j)$ for all $j \in [t] \setminus \{i\}$. Hence, if $v = z_2$, we obtain $v \in \{a_i, b_i\}$ since $v \in V^0(F) \subseteq \bigcup_{j \in [t]} \{a_j, b_j, x_j\}$, so we are done. Suppose that $v \neq z_2$.

Next, assume for a contradiction that $E(z_2 P v) \cap E(\mathcal{C}^\diamond) \neq \emptyset$. Let $z_3 z_3' \in E(P) \cap E(\mathcal{C}^\diamond)$ be the first such edge encountered when travelling from z_2 to v along P, with z_3 being closest vertex to z_2 (a priori possibly $z_3 = z_2$ and $z_3' = v$). Note that $z_2 P z_3 \subseteq D_0^\diamond$ and $|E(z_2 P z_3)| \le k$. Then, by (2), we have either $z_2 = z_3$ or $\{z_2, z_3\} = \{a_{i'}, x_{i'}\}$ for some $i' \in [t]$. Using $z_2, z_3 \in V(\mathcal{C})$ and $x_{i'} \notin V(\mathcal{C})$, the latter case is impossible, so we find $z_2 = z_3$. However, using $z_2 \notin V(C_j)$ for all $j \in [t] \setminus \{i\}$, this implies that $z_3 z_3' \in E(C_i)$, which is a contradiction due to the definition of z_2. As a result, we have $E(z_2 P v) \cap E(\mathcal{C}^\diamond) = \emptyset$. Then, by (2), we have $\{z_2, v\} = \{a_{i'}, x_{i'}\}$ for some $i' \in [t]$. Again, using $v \in V^0(F) \subseteq \bigcup_{j \in [t]} \{a_j, b_j, x_j\}$ and $z_2 \notin V(C_j)$ for all $j \in [t] \setminus \{i\}$, we obtain $i' = i$, $z_2 = a_i$, $v = x_i$, so we are done. ■

Now, the proof of Theorem 1.3 is immediate.

Proof of Theorem 1.3. If $d = 1$, then any orientation of $\mathcal{G}_{n,d}$ is just a matching, so the result trivially follows. Let $d \ge 3$ be an odd integer. By Proposition 4.6, $\mathcal{G}_{n,d}$ is $(n, d, 1000d^2)$-discrete with high probability. Moreover, any orientation D of $\mathcal{G}_{n,d}$ satisfies $\mathrm{ex}_D(v) = |d^+(v) - d^-(v)| \neq 0$ for all $v \in V(D)$ as $d^+(v) - d^-(v)$ has the same parity as d. Hence, the result follows by Theorem 4.8. ■

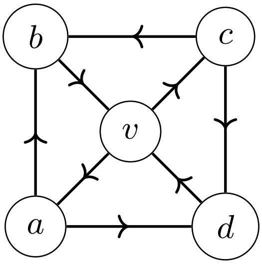

Fig. 1. The oriented graph D_0 on five vertices with $\mathrm{ex}(D_0) = 2$ and $\mathrm{pn}(D_0) = 4$.

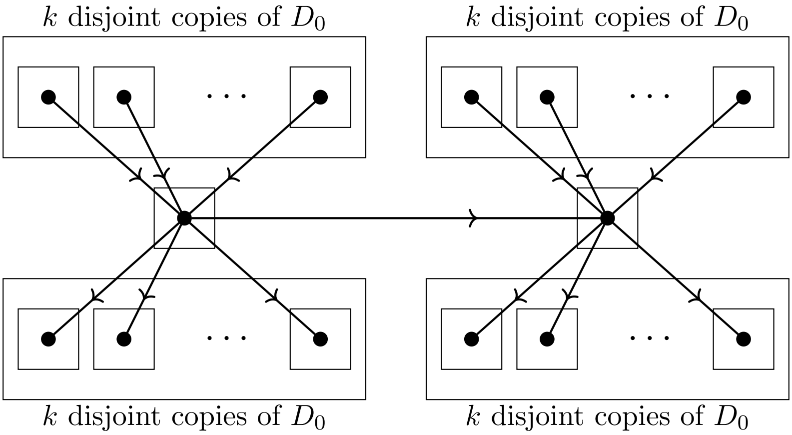

Fig. 2. The (inconsistent) oriented graph G_k on $20k + 10$ vertices with $\mathrm{ex}(v) = 1$ for all $v \in V(G_k)$.

5. Conclusion

In this section, we point out possible directions for further research. Recall that Theorem 3.1 does not require the graph to be regular and so Theorem 1.4 holds not just for regular graphs but also for graphs G with all degrees odd. One might wonder if Pullman's Conjecture (Conjecture 1.2) could also be true for graphs with all degrees odd. However, an example is given in [30] of a graph with degrees 1, 3, or 5 which has an orientation D with $\mathrm{pn}(D) = 4$ and $\mathrm{ex}(D) = 3$. Here, we generalize that example by showing that the difference $\mathrm{pn}(D) - \mathrm{ex}(D)$ can be made arbitrarily large even when $\mathrm{ex}(v) \neq 0$ for all $v \in V(D)$.

Proposition 5.1. *For each integer $k \geq 0$, there exists an oriented graph G_k on $20k + 10$ vertices with $\Delta(G_k^\diamond) = 2k + 5$ such that $d_{G_k^\diamond}(v)$ is odd for all $v \in V(G_k)$ and $\mathrm{pn}(G_k) - \mathrm{ex}(G_k) \geq 2k + 2$. In particular, G_k is not consistent.*

Proof. First, consider the oriented graph D_0 with $\mathrm{ex}(D_0) = 2$, depicted in Fig. 1.

It can be easily checked that $\mathrm{pn}(D_0) = 4$. Consider the oriented graph G_k which consists of $4k+2$ copies of D_0 with additional 4k+1 edges, depicted in Fig. 2, where the unique excess-zero vertices in each copy of D_0 are shown as black circles.

Note that vertices in the middle copies have degree $2k + 5$ in $G_k^\diamond$. Moreover, it is clear from the picture that all the vertices in G_k have excess one, so $\mathrm{ex}(G_k) = 10k + 5$. On the other hand, since $\mathrm{pn}(D_0) = 4$, we have $\mathrm{pn}(G_k) \geq 4(4k + 2) - (4k + 1) = 12k + 7$, which completes the proof. □

We note that the girth condition in Theorem 3.1 is unlikely to be optimal, although Proposition 5.1 shows that we cannot drop the girth condition altogether. It would be interesting to try and improve the girth condition in Theorem 3.1 perhaps even down to a constant independent of the maximum degree.

Of course Conjecture 1.2 remains wide open. We have verified the conjecture in a sparse setting by imposing girth conditions on our graph. Another direction for the sparse setting is to consider small values of the degree. Recall that Conjecture 1.2 is known to be true for $d = 3$ [30] using an inductive argument. However, the method does not seem to extend well to the case $d = 5$, as the complexity of the structural analysis increases dramatically. A better understanding of the case $d = 5$ could provide more insights about the sparse setting.

Conjecture 1.2 is also wide open in the dense setting with the only solved case being that of the complete graph (as far as we are aware), and this required considerable effort relying heavily on the robust expansion method. It would be interesting to determine whether $K_{2n+1,2n+1}$ is strongly consistent.

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