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Asymptotic Tensor Rank Is Characterized by Polynomials

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Abstract

Asymptotic tensor rank, originally developed to characterize the complexity of matrix multiplication, is a parameter that plays a fundamental role in problems in mathematics, computer science and quantum information. This parameter is notoriously difficult to determine; indeed, determining its value for the 2×2 matrix multiplication tensor would determine the matrix multiplication exponent, a long-standing open problem.

Strassen's asymptotic rank conjecture, on the other hand, makes the bold statement that asymptotic tensor rank equals the largest dimension of the tensor and is thus as easy to compute as matrix rank. Recent works have proved strong consequences of Strassen's asymptotic rank conjecture in computational complexity theory. Despite tremendous interest, much is still unknown about the structural and computational properties of asymptotic rank; for instance whether it is computable.

We prove that asymptotic tensor rank is "computable from above", that is, for any real number r there is an (efficient) algorithm that determines, given a tensor T, if the asymptotic tensor rank of T is at most r. The algorithm has a simple structure; it consists of evaluating a finite list of polynomials on the tensor. Indeed, we prove that the sublevel sets of asymptotic rank are Zariski-closed (just like matrix rank). While we do not exhibit these polynomials explicitly, their mere existence has strong implications on the structure of asymptotic rank.

As one such implication, we find that the values that asymptotic tensor rank takes, on all tensors, is a well-ordered set. In other words, any non-increasing sequence of asymptotic ranks stabilizes ("discreteness from above"). In particular, for the matrix multiplication exponent (which is the base-2 logarithm of an asymptotic rank) there is no sequence of exponents of bilinear maps that approximates it arbitrarily closely from above without being eventually constant. In other words, any such upper bound on the matrix multiplication exponent that is close enough, will "snap" to it. Previously such discreteness results were only known for finite fields or for other tensor parameters (e.g., asymptotic slice rank). We obtain them for infinite fields like the complex numbers.

We prove our result more generally for a large class of functions on tensors, and in particular obtain similar properties for all functions in Strassen's asymptotic spectrum of tensors. We prove a variety of related structural results on the way. For instance, we prove that for any converging sequence of asymptotic ranks, the limit is also an asymptotic rank for some tensor. We leave open whether asymptotic rank is also discrete from below (which would be implied by Strassen's asymptotic rank conjecture).

CCS Concepts

• **Theory of computation → Algebraic complexity theory**.

Keywords

Matrix multiplication exponent, tensors, asymptotic tensor rank, asymptotic rank conjecture, semicontinuity

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1 Introduction

Asymptotic tensor rank is a fundamental parameter in algebraic complexity theory [5, 15]. Originally rooted in the study of the matrix multiplication exponent, a burst of recent works has put this tensor parameter, and specifically Strassen's asymptotic rank conjecture, at the forefront of a wide range of computational complexity problems [3, 4, 31]. These kinds of asymptotic tensor parameters more broadly play an important role in various fields, like additive combinatorics and quantum information theory (asymptotic slice rank, asymptotic subrank) [30, 40, 42].

Despite tremendous effort (resulting in new matrix multiplication algorithms [2, 23, 29, 44], barriers [1, 8, 17], new routes [9–11, 19–21], and fundamental theory [18, 27, 28, 36, 38, 43]), we

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are still far away from determining the matrix multiplication exponent, computing asymptotic ranks in general, or resolving the asymptotic rank conjecture. All in all, determining asymptotic rank has turned out very challenging and much is still unknown about the properties of this parameter.

In this paper we prove a *polynomial characterization* of asymptotic tensor rank: for any number r there are finitely many polynomials on tensors whose vanishing determines if the asymptotic rank is at most r (just like matrix rank is characterized by vanishing of determinants of submatrices). This characterization has many consequences regarding the computation and topological structure of this parameter. Indeed, for any r it leads to an algorithm to determine if asymptotic rank is at most r. We obtain from it that asymptotic rank is semi-continuous (like matrix rank) and that its values are well-ordered, that is, discrete from above: any non-increasing sequence of asymptotic ranks stabilizes. For the matrix multiplication exponent ω, this implies in particular (as we will explain more) that there is a constant $\varepsilon > 0$ such that no tensor has exponent between ω and $\omega + \varepsilon$.

We will discuss these results and their meaning in more detail. First we discuss the context of this work in complexity theory and mathematics.

Matrix Multiplication Exponent and Asymptotic Rank Conjecture. It is a fundamental open problem to determine the matrix multiplication exponent ω, which is defined as the infimum over all real numbers τ such that $n \times n$ matrices can be multiplied using $O(n^\tau)$ arithmetic operations, with current state of the art $2 \leq \omega \leq 2.371552$ [2, 23, 29, 44]. It is very well possible that $\omega = 2$.

The matrix multiplication problem can naturally be phrased in terms of tensors and asymptotic rank. Namely $2^\omega = \underset{\sim}{\mathrm{R}}(\langle 2,2,2\rangle)$, where the asymptotic rank $\underset{\sim}{\mathrm{R}}(T) = \lim_{n\to\infty} \mathrm{R}(T^{\boxtimes n})^{1/n}$ of a tensor T measures the rate of growth of tensor rank on Kronecker powers of T, and where $\langle 2,2,2\rangle \in \mathbb{F}^4 \otimes \mathbb{F}^4 \otimes \mathbb{F}^4$ is the so-called 2×2 matrix multiplication tensor. Whether $\omega = 2$ is thus tightly linked to the question whether asymptotic rank can take non-integer values or not.

Strikingly, it is possible that for *every* tensor $T \in \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$ the asymptotic rank is at most n, as opposed to the tensor rank, which can be $\Omega(n^2)$ (but we do not know such tensors explicitly [6, 32]). Strassen's asymptotic rank conjecture indeed states that this is true, and more precisely states that asymptotic rank equals the largest flattening rank of the tensor (matrix rank after grouping two legs of the tensor), and would thus imply not only that asymptotic rank is always an integer, but also that it is easy to compute and that $\omega = 2$.

Intriguingly, there is a partial converse to the above connection between the asymptotic rank conjecture and the matrix multiplication exponent, namely [37], for any tensor in $\mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$ the asymptotic rank is at most $n^{2\omega/3}$. In particular, if $\omega = 2$, then every asymptotic rank is at most $n^{4/3}$, which is "not far" from the claim of the asymptotic rank conjecture. In this sense, matrix multiplication almost acts as a "complete" instance for the asymptotic rank conjecture. In the same spirit, Kaski and Michałek proved a completeness result for the asymptotic rank conjecture for explicit sequences of tensors [27].

Recent works have found new strong connections between the asymptotic rank conjecture and a range of computational complexity problems related to the chromatic number, set partitioning and the set cover conjecture [3, 4, 31].

Structure of Asymptotic Ranks. The asymptotic rank conjecture naturally leads to the question: What are the possible values that the asymptotic rank can take,

$$\mathcal{R} = \{\underset{\sim}{\mathrm{R}}(T) : T \in \mathbb{F}^{d_1} \otimes \mathbb{F}^{d_2} \otimes \mathbb{F}^{d_3},\ d_1, d_2, d_3 \in \mathbb{Z}_{\geq 1}\}?$$

And more generally: What is the structure (geometric, algebraic, topological, computational) of this set of values $\mathcal{R}$? Is there anything we can say about $\mathcal{R}$ without resolving the asymptotic rank conjecture or determining ω? Little is known. Clearly, such questions can be asked much more broadly, for higher-order tensors, but also for other asymptotic tensor parameters (asymptotic slice rank, subrank).

One known structural property is that $\mathcal{R}$ is closed under applying any univariate polynomial with nonnegative integer coefficients [43, Theorem 4.8]. Thus $\mathcal{R}$ has "many" elements. On the other hand, a sequence of recent works have revealed a strikingly "discrete" structure of sets like $\mathcal{R}$ (for various notions of asymptotic ranks), and thus that they have "not too many" elements. In particular, it can be seen (with a simple proof) that over any finite field (or finite set of coefficients in any field), $\mathcal{R}$ is a discrete set [14]. Their techniques, however, do not apply to infinite fields like the complex numbers. This perhaps leaves the impression that discreteness may be a consequence of considering finitely many tensors in each format. (One of the main results of this paper, however, is that $\mathcal{R}$ is discrete from above over *infinite* fields, like the complex numbers).

More broadly, notions of discreteness were proven for a range of asymptotic tensor parameters, $\underset{\sim}{F}(T) = \lim_{n\to\infty} F(T^{\boxtimes n})^{1/n}$ for varying F. It was shown in [12] that for a class of functions over finite fields, which includes asymptotic (sub)rank and asymptotic slice rank, the set of values that they take is well-ordered (discrete from above). The work [18], which resolved a problem on Strassen's asymptotic spectrum [37], proved as a consequence that asymptotic slice rank over complex numbers takes only finitely many values per format (because of a deep result on the structure of moment polytopes in representation theory), and thus only countably many values in total. The work [13] proved that a class of tensor parameters over complex numbers, which again includes asymptotic (sub)rank and asymptotic slice rank, take only countably many values. [14] proved that asymptotic (sub)rank and asymptotic slice rank over any finite set of coefficients take only a discrete set of values. In [16, 24] explicit gaps were determined in the smallest values of these parameters.

Unlike for the asymptotic tensor rank (for which nothing is known), the computational complexity of tensor rank is well-understood. Namely, it is known that tensor rank is **NP**-hard over the rationals and **NP**-complete over any finite field [25] and this was extended by [26]. Recently, [33, 35] proved the stronger statement that tensor rank over any field has the same complexity as deciding the existential theory of that field, and [7, 39] proved that tensor rank is hard to approximate. These results, however, have no immediate implication for the computational complexity of asymptotic tensor rank (which, for all we know, may be in **P**).

New Results. In this paper:

- We prove that asymptotic tensor rank is computable from above: over "computable fields", for every upper bound r there is an algorithm that, given any $d \times d \times d$ tensor T, decides if its asymptotic tensor rank is at most r.
- As the core ingredient, we prove over any field $\mathbb{F}$ that the sublevel sets of asymptotic tensor rank are Zariski-closed. This means that each such set is precisely determined by the vanishing of a finite set of polynomials on tensors. It follows that the sublevel sets are also Euclidean-closed if $\mathbb{F}$ is a subfield of $\mathbb{C}$, and that $\underset{\sim}{\mathrm{R}}$ is lower-semi-continuous, just like matrix rank.
- Using the above, we prove for asymptotic tensor rank that the set of values it takes on all tensors is well-ordered (discrete from above), and when $\mathbb{F} = \mathbb{C}$ it is even closed (in the Euclidean topology). In particular, in the context of the matrix multiplication exponent ω, we find that there is a constant $\varepsilon > 0$ such that no tensor has exponent between ω and $\omega + \varepsilon$.
- We prove the above results in great generality, for tensors of any order, and for a large collection of tensor parameters that generalizes asymptotic rank, including Strassen's asymptotic spectrum.
- As a technical ingredient for the general version of our result we develop new lower bounds on a type of tensor-to-matrix transformation (max-rank) that may be of independent interest.

2 Polynomial Characterization

We now discuss our results and their context. We prove that upper bounds on asymptotic rank are computable:

Theorem 1 (Informal). *For any "computable" field $\mathbb{F}$ and $k \geq 3$, for any $r \in \mathbb{R}$, there is an algorithm that, given any k-tensor T over $\mathbb{F}$, decides if $\underset{\sim}{\mathrm{R}}(T) \leq r$.*

The algorithm in Theorem 1 is in fact efficient in the dimensions d_i of $T \in \mathbb{F}^{d_1} \otimes \cdots \otimes \mathbb{F}^{d_k}$, for simple reasons that we elaborate on in a moment.[1] The computability condition for the field is a natural one: we need to be able to write down (and compute with) the tensor and certain polynomials on the tensor space that the algorithm relies on. Theorem 1 almost directly follows from the following polynomial characterization of asymptotic rank:

Theorem 2. *For any field $\mathbb{F}$, $k \geq 3$, $d \in \mathbb{Z}^k_{\geq 1}$, and $r \in \mathbb{R}$, the sublevel set*

$$\{T \in \mathbb{F}^{d_1} \otimes \cdots \otimes \mathbb{F}^{d_k} : \underset{\sim}{\mathrm{R}}(T) \leq r\}$$

is Zariski-closed.

In other words, Theorem 2 says that (for every k, d, r) there is a finite set of polynomials $p_1, \ldots, p_\ell$ on $V = \mathbb{F}^{d_1} \otimes \cdots \otimes \mathbb{F}^{d_k}$ such that for every $T \in V$ we have $\underset{\sim}{\mathrm{R}}(T) \leq r$ if and only if all p_i vanish on T. Over suitably computable fields, like the algebraic closure of the rationals, these polynomials are also computable, and indeed lead to the algorithm of Theorem 1.[2]

We note that Theorem 2 has "practical" implications: if for some family of tensors one can prove an upper bound on the asymptotic tensor rank, then by Theorem 2 this upper bound directly extends to the Zariski-closure. Before our result this was only known on finite unions of GL-orbits.[3] In particular, it also follows from Theorem 2 that, over the complex numbers, for any sequence of tensors $T_1, T_2, \ldots$ converging to a tensor T in the Euclidean norm, if the asymptotic rank of all T_i is at most r, then the asymptotic rank of T is at most r (Euclidean lower-semicontinuity). For instance, we may apply this idea to tensors converging to a matrix multiplication tensor $\langle n, n, n \rangle$ in order to get upper bounds on ω.

As an ingredient in the proof of Theorem 2 we prove that for any subset $A \subseteq \mathbb{F}^{d_1} \otimes \cdots \otimes \mathbb{F}^{d_k}$ the supremum of $\underset{\sim}{\mathrm{R}}$ over the Zariski closure of A equals the supremum over A itself. This result we prove via a decomposition of powers of elements in the closure of A in terms of powers of elements in A combined with an asymptotic double-blocking analysis of asymptotic rank.

In fact, we prove Theorem 1 and Theorem 2 for a more general class of functions that includes all functions in Strassen's asymptotic spectrum of tensors. Via Strassen's duality [37], this implies that the set of all tensors that are an asymptotic restriction of a given tensor, is Zariski-closed.[4]

3 Discreteness from Above and Closedness

As a consequence of Theorem 2 we prove that the set of values that asymptotic rank takes, on all tensors (of fixed order, but arbitrary dimension), is well-ordered (discrete from above, every non-increasing sequence stabilizes):

Theorem 3. $\mathcal{R} = \{\underset{\sim}{\mathrm{R}}(T) : T \in \mathbb{F}^{d_1} \otimes \cdots \otimes \mathbb{F}^{d_k}, d \in \mathbb{Z}^k_{\geq 1}\}$ *is well-ordered.*

In particular, Theorem 3 says that there cannot be a sequence of tensors with asymptotic rank strictly larger than 2^ω that gets arbitrarily close to it. Indeed, if it comes arbitrarily close it must eventually "snap" to 2^ω. Previously this was only known over finite fields (where the proof is simple, see [14, Theorem 4.4]; however that proof strategy does not work over infinite fields).

We leave as an open problem to prove discreteness from below for asymptotic rank. That is, we do not know if there can be (non-constant) increasing converging sequences of asymptotic ranks of tensors.

Towards proving discreteness, as an intermediate result, we prove closedness of the set of values of asymptotic rank, when the base field is the complex numbers:

Theorem 4. *Let $\mathbb{F} = \mathbb{C}$. For any sequence in $\mathcal{R}$ that converges, the limit is in $\mathcal{R}$.*

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[1] This statement is, however, not uniform in r.

[2] The reason that this algorithm is efficient in the dimensions d_i of the input tensor $T \in \mathbb{F}^{d_1} \otimes \cdots \otimes \mathbb{F}^{d_k}$ is as follows. The flattening ranks are a lower bound on the asymptotic rank. So if a tensor is concise its dimensions will be at most the flattening rank. Thus for a given upper bound r on the asymptotic rank, and a tensor T, we first make T concise. It will then have one of finitely many formats with dimensions below r. We then evaluate the polynomials for that format and check if they all vanish.

[3] Indeed, it is known that for any S in the closure of $(\mathrm{GL}_{d_1} \times \cdots \times \mathrm{GL}_{d_k})T$ (i.e., S is a degeneration of T) the asymptotic rank $\underset{\sim}{\mathrm{R}}(S)$ is at most $\underset{\sim}{\mathrm{R}}(T)$; and the closure of a finite union of orbits equals the union of the closures.

[4] For two tensors S, T there is an asymptotic restriction $S \lesssim T$ if $S^{\boxtimes n} \leq T^{\boxtimes(n+o(n))}$. Strassen's duality says that $S \lesssim T$ if and only if for every F in the asymptotic spectrum $\Delta(\mathbb{F}, k)$ we have $F(S) \leq F(T)$. Then $\{S \in V : S \lesssim T\} = \cap_{F \in \Delta(\mathbb{F},k)} \{S : F(S) \leq F(T)\}$. Every sublevel set $\{S : F(S) \leq F(T)\}$ is closed, so their intersection is closed.

We note again that all of the above results are consistent with (and may be thought of as evidence towards) the asymptotic rank conjecture, since that would imply that asymptotic rank is computable as the matrix rank of a flattening of the tensor, and that the set of values that asymptotic rank takes is simply the natural numbers $\mathbb{N}$.

4 Asymptotic Spectrum and Tensor-to-Matrix Restrictions

Finally, we extend Theorem 3 to the tensor parameters in Strassen's asymptotic spectrum of tensors [37, 43], a collection of tensor parameters with special properties (namely they are restriction-monotone semiring homomorphisms with respect to direct sum and tensor product). We denote by $\Delta(\mathbb{F}, k)$ the asymptotic spectrum of k-tensors over $\mathbb{F}$.

Theorem 5. *For every $F \in \Delta(\mathbb{F}, k)$,*

$$\{F(T) : T \in \mathbb{F}^{d_1} \otimes \cdots \otimes \mathbb{F}^{d_k}, d \in \mathbb{Z}^k_{\geq 1}\}$$

is well-ordered.

In fact, by Strassen's duality [37], Theorem 5 implies Theorem 3, and moreover has discreteness implications for asymptotic transformation rates between tensors. The proof of Theorem 5 is more involved than Theorem 3. It relies not only on the Zariski-closedness of sublevel sets, but also on a more technical "growth" argument (which may be of independent interest) in order to obtain well-orderedness across all tensor formats.

This growth argument involves new lower bounds on a type of tensor-to-matrix restrictions. In quantum information these correspond to k-partite to bipartite entanglement transformations (under stochastic local operations and classical communication, SLOCC).

For any k-tensor $T \in V_1 \otimes V_2 \otimes \cdots \otimes V_k$, let $\mathrm{Q}_{1,2}(T)$ be the largest number r, such that there are $A_i \in \mathrm{GL}(V_i)$ such that $(A_1 \otimes \cdots \otimes A_k)T = \sum_{i=1}^r e_i \otimes e_i \otimes e_1 \otimes \cdots \otimes e_1$. The right hand side is essentially a rank r identity matrix on tensor legs 1 and 2. This definition extends to $\mathrm{Q}_{i,j}$ for $i \neq j \in [k]$ by placing this "identity matrix" at legs i and j. For any proper subset $I \subseteq [k]$, let $\mathrm{R}_I(T)$ be the matrix rank of the matrix obtained by grouping the legs in I together and grouping the remaining legs together. If $i \in I$ and $j \notin I$, then $\mathrm{R}_I(T) \geq \mathrm{Q}_{i,j}(T)$. We prove the following inequality in the opposite direction.

Theorem 6. *Let $I \subseteq [k]$ with $1 \leq |I| \leq k-1$, and let T be a k-tensor. If $|\mathbb{F}| > \mathrm{R}_I(T)$, then $\prod_{i \in I, j \in [k] \setminus I} \mathrm{Q}_{i,j}(T) \geq \mathrm{R}_I(T)$.*

Theorem 6 generalizes [14, Theorem 1.13], which covered $k = 3$. Using Theorem 6 we prove that any element F in the asymptotic spectrum is either essentially an element in the asymptotic spectrum for lower order tensors, or grows with the size of the tensor (which in turn is the right ingredient for proving Theorem 5).

5 Discussion and Open Problems

We discuss several natural directions and open problems in the context of our results, Strassen's asymptotic rank conjecture and the matrix multiplication exponent.

- **Discreteness from below.** We proved discreteness from above for the asymptotic rank (and a large class of other parameters) over (for instance) the complex numbers. One of the main problems that we leave open is whether this parameter is also discrete from below. Indeed, Strassen's asymptotic rank conjecture would imply this.
  As an intermediate result, we have shown that any converging sequence of asymptotic ranks has a limit which is also an asymptotic rank (Theorem 4). What remains to be shown is that any such sequence is eventually constant.
  Notions of discreteness for asymptotic parameters have been studied more broadly, in particular in the context of the Shannon capacity of graphs [34] and the related asymptotic spectrum of graphs [41, 45]; interestingly, in that setting, the asymptotic parameter of interest (Shannon capacity) is not discrete (neither from above or from below), which leads to a graph limit approach to determining Shannon capacity [22].
- **Geometric properties; irreducibility.** Given that the sublevel sets $\{T \in \mathbb{F}^{d_1} \otimes \cdots \otimes \mathbb{F}^{d_k} : \underline{\mathrm{R}}(T) \leq r\}$ of the asymptotic tensor rank are Zariski-closed (Theorem 2), it is natural to ask about the geometric properties of these sets. For instance, we may ask if they are irreducible (i.e., cannot be written as the union of two proper Zariski-closed subsets) whenever $\mathbb{F}$ is algebraically closed.
  Indeed, irreducibility is true for $k = 2$ (matrices), since then the asymptotic rank coincides with matrix rank, and the set of matrices of at most a given rank is an irreducible variety. For $k \geq 3$, irreduciblity is open. It would imply both that the set of achievable values of the asymptotic rank is discrete, and a weak form of Strassen's asymptotic rank conjecture: there exist at most $d_1 \cdots d_k + 1$ asymptotic ranks in format $d_1 \times \cdots \times d_k$. This follows from a dimension argument (topological dimension is the maximal length of a decreasing chain of irreducible subvarieties, and the dimension of $\mathbb{F}^{d_1 \times \cdots \times d_k}$ is $d_1 \cdots d_k$).
  To see that this is a weak form of Strassen's conjecture, note that Strassen's conjecture is equivalent to the statement that there are $d + 1$ distinct asymptotic ranks in $(\mathbb{F}^d)^{\otimes k}$ (namely $\{0, 1, \ldots, d\}$). Note that Strassen's conjecture also implies that the sets $\{T \in \mathbb{F}^{d_1} \otimes \cdots \otimes \mathbb{F}^{d_k} : \underline{\mathrm{R}}(T) \leq r\}$ are irreducible: r is integer, and $\mathbb{F}^{d_1 \times r} \times \cdots \times \mathbb{F}^{d_k \times r} \times \mathbb{F}^{r \times \cdots \times r}$ is irreducible and admits a surjective polynomial map onto the set of tensors with all flattening ranks at most r.
- **Computation and structure of asymptotic ranks.** Our results provide new tools to concretely understand the asymptotic rank of families of tensors and their relation to the matrix multiplication exponent ω, in the spirit of [27]. For instance (by Theorem 2), truth of the asymptotic rank conjecture on any subset of tensors by our result extends to the Zariski-closure.
  Another natural direction where our result plays a role is in the task of understanding the relation between asymptotic ranks of explicit (families of) tensors and the matrix multiplication exponent. For example, it is well-known that if the asymptotic rank of the small Coppersmith–Winograd tensor cw_2 equals 3, then $\omega = 2$. For which tensors can we prove this property?

It follows from Theorem 2 and known classifications that cw_2 has asymptotic rank at most the generic asymptotic rank of $3 \times 3 \times 3$ tensors with hyperdeterminant 0 (which is a codimension 1 variety). It also has at most the asymptotic rank of a generic tensor with support

$$\{(0,0,0),(1,1,1),(2,2,2),(0,1,2),(1,2,0),(2,0,1)\}.$$

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