Orthogonal series for si- and related processes, Karhunen-Loève decompositions

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Abstract

This paper reproduces results from Chapter 11 of the forthcoming book [11]. It discusses series expansions of processes with stationary increments (si-processes) and certain associated processes. Making use of de Branges theory of Hilbert spaces of entire functions, it sheds new light on the existing literature and makes available some new results. In particular, it provides some new decompositions of the Karhunen-Loève type.

1 Introduction

This paper makes use of some material from our previous work [13]–[16] in collaboration with Harry van Zanten. One of the prime topics we were interested in was series expansions of *fractional Brownian motion* (FBM) and, in general, of *si-processes*. The latter acronym stands for *processes with stationary increments* which are defined on a Gaussian probability space $(\Omega, \mathcal{F}, P)$ and are squire integrable in the metric of this space. They are assumed to be mean squire continuous, centered, and to emanate from the origin. Denoted by $(X_t)_{t\geq 0}$ such process has stationary increments in the sense that for every u the increment process $(X_{t+u} - X_u)_{t\geq 0}$ has the same mean and covariance function as process X. The covariance function $r(s,t) = \mathbb{E}(X_s X_t)$ is determined by variance $v(t) = \mathbb{E}|X_t|^2$ in this way $r(s,t) = \frac{1}{2}\big(v(s) + v(t) - v(t-s)\big)$ for all $s, t \geq 0$, where $v(t)$ is extended to negative values of the argument by the convention $v(t) = v(-t)$. Indeed, $v(t-s) = \mathbb{E}|X_{t-s}|^2 = \mathbb{E}|X_t - X_s|^2$ for $t > s$ and this in turn is equal to $v(s) + v(t) - 2r(s,t)$. Since X is mean squire continuous and starts from the origin, the variance function is a continuous function and starts from the origin, moreover $v(t) > 0$ if $t > 0$. Conversely, if a centered process X has a covariance function of the preceding form for a continuous function v starting from the origin, then $X_0 = 0$ a.s. and for every u the process $(X_{t+u} - X_u)_{t\geq 0}$ has the same covariance function as the process X. Hence X is an si-process.

According to [23, Theorem 2] the variance function permits the representation

$$v(t) = i\gamma t + \int \Big(1 - e^{i\lambda t} + \frac{i\lambda t}{1+\lambda^2}\Big)\frac{\mu(d\lambda)}{\lambda^2}$$

with $v(t) = v(-t)$ for $t \in \mathbb{R}$, where γ is a real constant determined uniquely by the variance function and μ is a non-negative Borel measure on the real line, squire integrable in the sense of

$$\int \frac{\mu(d\lambda)}{1+\lambda^2} < \infty. \tag{1.1}$$

The measure μ is defined essentially uniquely. Conversely, any function v defined by Kreĭn's presentation is a variance function of an si-process.

Due to Kreĭn's presentation, the covariance function of an si-process is represented as

$$r(s,t) = \int \big(1 - e^{i\lambda s}\big)\big(1 - e^{-i\lambda t}\big)\frac{\mu(d\lambda)}{\lambda^2} \tag{1.2}$$

for $s, t \in \mathbb{R}_+$.

We will be also interested in double-sided si-processes $(X_t)_{t\in\mathbb{R}}$ defined by the covariance function of the form (1.2) but with the arguments s, t taking their values on the whole real line $\mathbb{R}$. A process emanates from the origin and combines two identically distributed si-processes which run to the left and to the right of the origin. For convenience, the process to the left of the origin will be assumed reflected across the x-axes, and in this way we deal with two single-sided processes $(X_t)_{t\geq 0}$ and $(-X_{-t})_{t\geq 0}$. Moreover, we take the mean of these two processes at each positive value of t, i.e.

$$X^e_t := \frac{1}{2}\big(X_t + (-X_{-t})\big),$$

and call it the *even part* of the reflected si-process (this explains the upper index e). The corresponding odd part will be

$$X^o_t := \frac{1}{2}\big(X_t - (-X_{-t})\big)$$

and process X will be represented by its even and odd parts, in this manner

$$X_t = \operatorname{sign}(t)\, X^e_{|t|} + X^o_{|t|}.$$

A convenience of such a splitting is in the form of the covariance functions

$$\mathbb{E}(X_s^e X_t^e) = \int_{\mathbb{R}} \frac{\sin s\lambda}{\lambda} \frac{\sin t\lambda}{\lambda} \mu(d\lambda)$$
$$\mathbb{E}(X_s^o X_t^o) = \int_{\mathbb{R}} \frac{\cos s\lambda - 1}{\lambda} \frac{\cos t\lambda - 1}{\lambda} \mu(d\lambda) \qquad (1.3)$$

for $s, t \geq 0$ and mutual independence of two processes: we have $\mathbb{E}(X_s X_t) = \mathbb{E}(X_s^e X_t^e) + \mathbb{E}(X_s^o X_t^o)$, since $\mathbb{E}(X_{-s} X_t) = \mathbb{E}(X_s X_{-t})$.

Let us overview shortly the content of the paper.

There are various kinds of series expansions of si-processes, e.g. in pioneering works of R. E. Paley and N. Wiener [31] and P. Lévi [27] or popular textbooks such as M. Loève [28, Section 34.5] or I.I. Gikhman and A.V. Skorokhod [19, Section 5.1]. Based on their characterization of finite Fourier transforms as entire functions f of exponential type at most $a > 0$ which satisfy identity

$$\int |f(\lambda)|^2 d\lambda = \frac{\pi}{a} \sum_n \left| f\left(\frac{\theta + n\pi}{a} \right) \right|^2, \qquad (1.4)$$

Paley and Wiener obtained a series expansion and took it as the definition for the standard Brownian motion (which they call the "fundamental random function", cf. [27], Section 13 of the complement). They first considered the series $\sum_{n\in\mathbb{Z}} e^{int} Z_n$ for $t \in [0, 2\pi]$, where Z_n are i.i.d. complex valued standard Gaussian random variables. This series correspond to white noise, but the series does not converge in the usual sense. So, instead they considered its formal integral $\sum_{n\in\mathbb{Z}} \frac{e^{int} - 1}{in} Z_n$. The latter series is shown to converge almost surely and is taken as the definition for the complex valued Brownian motion. In [13] and [16] one can find extension of this result to a complex valued fractional Brownian motion of arbitrary Hurst index, not necessarily $H = 1/2$. The basic tools in the latter paper and the subsequent one [15] are Kreĭn's spectral theory and theory of vibrating strings. Combining the latter theory with de Branges' theory of reproducing Hilbert spaces of entire functions, H. Dym and H. P. McKean [10] discuss so-called "sampling formula": on p. 302 one can find a generalization of identity (1.4). Dym and McKean took this over from De Brange's theorem 22, see Section 3.1, in particular Theorem 3.1 The theorem lay in the basis for the proofs of Theorem 3.2 and its Corollary 3.3 which extend the Paley–Wiener series expansion (PW-series) to si-processes. In Section 3.3 the particular case of fractional Brownian motion receives special attention. Section 3.4 considers moving average representations of si-processes in the form of Wiener integrals with respect to the so-called *fundamental martingales*, processes with martingale properties, adapted to the filtration of the given si-processes and producing equal filtration. Theorem 3.7 provides the PW-series expansion for fundamental martingales. We briefly discuss also

PW-series expansion for stationary processes, in particular the *Ornstein–Uhlenbeck* and *autoregressive* processes.

The series of [28, Section 34.5] is of a completely different nature. A random process defined on a finite interval, is decomposed into series of orthogonal functions which represent eigenfunctions of its covariance. This requires solving the Fredholm integral equation associated with the covariance kernel of the process. There are no restrictions on processes, except the mean squire continuity. This makes the *Karhunen-Loève expansion*, as it is usually called, a powerful tool. But explicit solutions to the required equations run into usual difficulties and up to now only some simple examples are known (a list of references can be found in the recent survey [30]). We shall discuss some of these examples and investigate more possibilities in Section 4.

Besides their mathematical appeal, series expansions are of a clear practical significance. Truncated at some point, they may be used for simulation of processes in question (as regards the FBM, there is a number of good papers discussing various simulation techniques, such as e.g. [29] or [1], to mention a few). The level at which the series should be truncated depends of course on the intended accuracy of the approximation. It is therefore desirable in each concrete application to get an idea about the rate of convergence of the series. We will discuss this question at hand of of a number of examples.

2 Preliminaries

2.1 De Branges spaces

As is said in the introduction, the proofs utilize de Branges theory of Hilbert spaces of entire functions [6]. Let us briefly review the necessary facts from this theory. The generalization of the Paley–Wiener space begins with replacing the exponential functions, fundamental for Fourier analyses, with more general entire functions which we shall call *de Branges functions* and denote by $E(z)$. Like the exponential functions e^{-iaz} of a complex variable $z = x + iy$, De Branges functions satisfy the inequality

$$|E(x - iy)| < |E(x + iy)|$$

for $y > 0$ (equivalently, $|E^\sharp(z)| < |E(z)|$ for all z in the upper half-plane, where $E^\sharp(z) = \overline{E(\bar{z})}$). Each such function generates a linear space of entire functions, called the *De Branges space* and denoted by $\mathcal{H}(E)$. By definition, it consists of entire functions f which satisfy the following conditions.
– The norm of an element f is defined by

$$\|f\|_E^2 = \int \Big|\frac{f(\lambda)}{E(\lambda)}\Big|^2 d\lambda < \infty.$$

– The restrictions of the quotients f/E and $f^\sharp/E$ to the upper half-plane are of bounded type and have non-positive mean type in the half-plane.

The space $\mathcal{H}(E)$ is a Hilbert space with the inner product

$$\langle f, g\rangle_E = \int \frac{f(\lambda)\overline{g}(\lambda)}{|E(\lambda)|^2}d\lambda.$$

Moreover, it is reproducing kernel Hilbert space with the reproducing kernel

$$K(w,z) = \frac{\overline{E(w)}E(z) - \overline{E^\sharp(w)}E^\sharp(z)}{-2\pi i(z-\bar{w})} = \frac{A(\bar{w})B(z) - B(\bar{w})A(z)}{\pi(z-\bar{w})}, \tag{2.1}$$

which is the point evaluator in the sense that for every complex number w and every element f of the space

$$f(w) = \langle f, K_w\rangle_E = \int \frac{f(\lambda)\overline{K_w(\lambda)}}{|E(\lambda)|^2}d\lambda.$$

In the definition of the reproducing kernel the two entire functions

$$A(z) = \frac{E^\sharp(z) + E(z)}{2} \qquad B(z) = \frac{E^\sharp(z) - E(z)}{2i}$$

are real for real z, and called *even* and *odd* components of the de Branges function, since $A^\sharp(z) = A(z)$ and $B^\sharp(z) = B(-z)$. In applications like in [9], an important specific feature of the space is symmetry about the origin, meaning that if $f(z)$ belongs to the space, then $f(-z)$ belongs to the space as well and its norm is equal to that of $f(z)$. Any space $\mathcal{H}(E)$ generated by de Branges function such that $E^\sharp(z) = E(-z)$ has this property, as is easily verified at hand of its definition. Following [9], we say that a symmetric de Branges function satisfies the *reality condition*. Obviously, the reality condition is equivalent to the condition that $A(z)$ and $B(z)$ are even and odd entire functions, respectively, since $A^\sharp(z) = A(z) = A(-z)$ and $B^\sharp(z) = B(z) = -B(-z)$. Theory of symmetric spaces is developed in [5] and [6, Section 47]. It is shown, in particular, that if the symmetric space $\mathcal{H}(E)$ contains an element with a non-zero value at the origin, then the generating de Branges function can be chosen so that $E(0) = 1$. In applications that we shall discuss the si-processes will be real. Therefore their spectral measures will be symmetric about the origin (the spectral function $\mu(\lambda) = \mu(-\infty, \lambda]$ is odd $\mu(\lambda) = -\mu(-\lambda)$) and all de Branges spaces contained isometrically in $L^2(\mu)$ will satisfy the reality conditions (see [6, Problem 176] or [11, Theorem 6.3.3]).

Observe that different de Branges spaces can have the same reproducing kernel. To see this, rewrite the numerator in the form

$$[A(z), B(z)]g_{\pi/2}[A(w), B(w)]^* = [A(z), B(z)]Ug_{\pi/2}U^\top[A(w), B(w)]^*$$

with the help of matrix

$$g_\theta = \begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix} \tag{2.2}$$

which defines the group of rotations about the origin of the 2-dimensional Cartesian plane through angle θ, see e.g. [33, Section II.1.2] for the unitary representation of the rotation group $SO(2)$. For $\theta = \pi/2$, the latter matrix rotates the plane through angle $\pi/2$ about the origin and matrix U in this representation of the numerator represents any $g_{\pi/2}$-unitary matrix, i.e. one that satisfies $U g_{\pi/2} U^{\top} = g_{\pi/2}$. It is then clear that every space $\mathcal{H}(E)$ based on the De Branges functions with the components $[A(z), B(z)]U$ have the same reproducing kernel. For instance, the matrices (2.2) are $g_{\pi/2}$-unitary and the reproducing kernels are rotation invariant. Denote

$$[A_\theta(z), B_\theta(z)] = [A(z), B(z)]\, g_\theta. \tag{2.3}$$

As $[A_{\pi/2}(z), B_{\pi/2}(z)] = [B(z), -A(z)]$. the rotation by angle $\pi/2$ leads to interchanging the roles of A and B,

In terms of [9], a de Branges space $\mathcal{H}(E)$ is called *short* if it is invariant with respect to the backward shift operator[1] for some (and hence every) complex number α, more precisely

$$\frac{f(z) - f(\alpha)}{z - \alpha} \in \mathcal{H}(E)$$

whenever f belongs to $\mathcal{H}(E)$. Propositions 1 and 2 in [6, Section 6.2] characterize such spaces. It is stated, in particular, that $\mathcal{H}(E)$ is short if and only if E is of exponential type and satisfies inequality

$$\int \frac{d\lambda}{(1+\lambda^2)\,|E(\lambda)|^2} < \infty.$$

Such a de Branges function E (and its components A and B) does satisfy

$$\lim_{y \nearrow \infty} \frac{1}{y} \log |E(iy)| = p$$

and for $0 < \theta < \pi$

$$\lim_{r \nearrow \infty} \frac{1}{r} \log |E(re^{i\theta})| = p \sin\theta$$

with some positive constant p.

In [9, Sections 4.8 and 6.4] short de Branges spaces are associated with *Kreĭn's alternative* which regards space $L^2(\mu)$ and its subspace which is the linear span of $(e^{i\lambda t} - 1)/\lambda$ for $-r \le t \le r$, closed in the metric of $L^2(\mu)$. There are only two possibilities: the span, denoted as

$$\overline{sp}\big((e^{izt} - 1)/z : |t| \le r\big)_\mu, \tag{2.4}$$

[1]For a fixed complex number α the *backward shift operator* $R_\alpha f$ is defined at every point z by

$$R_\alpha f(z) = \begin{cases} \dfrac{f(z) - f(\alpha)}{z - \alpha} & \text{if } z \ne \alpha \\ f'(\alpha) & \text{if } z = \alpha. \end{cases}$$

$r > 0$, is eider whole of $L^2(\mu)$ or a proper subspace of $L^2(\mu)$. In the latter case, it comprises class $\mathbf{I}_r(\mu)$ of entire functions which are of exponential type at most r and squire integrable with respect to the spectral measure μ. This class, in its turn, is equivalent to the de Branges space $\mathcal{H}(E)$ of type $r > 0$ which is contained isometrically in $L^2(\mu)$.

The spectral analysis of si-processes via theory of de Branges spaces is based on the following statement which is formulated as Problem 127 in [6] and proved by approximation of $\mathcal{H}(E)$ with finite dimensional spaces.[2]

Theorem 2.1. *(i) Given an arbitrary spectral measure μ of property* (1.1)*, there exists a short de Branges space $\mathcal{H}(E)$ of exponential type which is contained isometrically in $L^2(\mu)$.*

Moreover, in [6] De Branges proves Theorem 40 which says that there exists whole *chain of de Branges spaces*

$$\mathcal{H}(E_t),\ t \in [0, \infty), \tag{2.5}$$

which is contained isometrically in $L^2(\mu)$. In general, any finite or infinite interval on the real line is allowed, but since si-processes emanate from the origin and run unboundedly to the left or right, we restrict our attention to $[0, \infty)$. Also, there may be isolated points at which isometry is spoiled, but this will be excluded in the sequel. Taking this into consideration, we provide only a weakened version of [6, Theorem 40], suited to our purposes. Moreover, it will be assumed throughout that the basic de Branges function has no real zeros and is normalized $E_t(0) = 1$.

Theorem 2.1 (continued.) *(ii) Let a short de Branges space $\mathcal{H}(E)$ of exponential type $r > 0$ be contained isometrically in $L^2(\mu)$, where μ is a spectral measure of a real si-process. Then there exists a chain of spaces* (2.5) *and a diagonal structure function $m_t = diag\,[\alpha_t, \gamma_t]$, $t \in [0, \infty)$, with the following properties*

– *$E(z) = E_c(z)$ for some number c such that $\tau_c = r$.*

– *The spaces $\mathcal{H}(E_t)$ of type τ_t are contained isometrically in $L^2(\mu)$.*

– *$E_t(w)$ is a continuous function of t for every complex w, and the integral equation*

$$[A_b(w), B_b(w)]g_{\pi/2} - [A_a(w), B_a(w)]g_{\pi/2} = w \int_a^b [A_t(w), B_t(w)]dm_t \tag{2.6}$$

holds for $0 \le a < b < \infty$.

– *The kernel $K_t(w, z) = \dfrac{A_t(\bar{w})B_t(z) - B_t(\bar{w})A_t(z)}{\pi(z - \bar{w})}$ in space $\mathcal{H}(E_t)$ is such that $\lim_{t \searrow 0} K_t(w, w) = 0$ for every complex number w, and*

$$K_b(w, z) - K_a(w, z) = \frac{1}{\pi} \int_a^b [A_t(z), B_t(z)]dm_t[A_t(w), B_t(w)]^* \tag{2.7}$$

[2] In [11] this statement is formulated and proved as Theorems 4.3.4 and 4.5.4.

for $0 \leq a < b < \infty$ *and all complex numbers* w*,* z*.*

The notions of the structure function and the type defined by this structure function trough the so-called *type integral*

$$\tau_b - \tau_a = \int_a^b \sqrt{\alpha' \gamma'} d\sigma \tag{2.8}$$

need further explanation (the entries α and γ on the diagonal of the structure function are dominated by a certain function σ and differentiated with respect to this dominating function). In general, the structure function is a non-decreasing matrix valued function whose entries are continuous real valued functions of argument t (of the form as in [6, Theorem 40]), but the reality condition restricts the off-diagonal entries to constants. Therefore we can ignore the off-diagonal entries and let the structure function to be diagonal. In [6, Problem 127] the type integral is defined as the integral of $\sqrt{\det m'}$ with respect to the dominating function σ, which in our case reduces to the type integral mentioned above. Occasionally, the determinant may vanish but in the sequel τ_t always will be strictly increasing, $\tau_t' > 0$.

In [6, Section 43] the structure function defines a Hilbert space $L^2([0,\infty), m)$ of vector valued Borel measurable functions on the interval, say $h = [\varphi, \psi]$, such that $\|h\|_m^2 = \int h \, dm \, h^*$. In our case this so-called *structure space* falls into two parts: $L^2([0,\infty), \alpha)$ and $L^2([0,\infty), \gamma)$ spaces of scalar valued functions with squire norms $\|\varphi\|_\alpha^2 = \int |\varphi|^2 d\alpha$ and $\|\psi\|_\gamma^2 = \int |\varphi|^2 d\gamma$. Since

$$q(t,w) := [A_t(w), B_t(w)] \, 1_{(0,c)}(t) \tag{2.9}$$

belongs to space $L^2([0,\infty), m)$ as a function of t for every number $c > 0$ and every complex number w, the integrals in (2.6) and (2.7) are well-defined.

The next theorem from [6] which is essential for our purposes is Theorem 44 on the eigenfunction expansion of elements of a de Branges space which belong to the given chain of spaces (2.5) contained isometrically in $L^2(\mu)$ where μ is the spectral measure of a given si-process. It defines generalized Fourier transforms which are the cosine and sine transforms in the classical case of the Paley–Wiener spaces [6, Sections 16–18] and the Hankel transforms in the case of the forthcoming Section 3.3. We again provide only a weakened version of [6, Theorem 44], adapted to the present purposes.

Theorem 2.2. *At some point* $c > 0$*, let a de Branges space* $\mathcal{H}(E_c)$ *belong to the chain of spaces* (2.5) *which is contained isometrically in* $L^2(\mu)$ *where* μ *is the symmetric spectral measure of* X*. Let its diagonal structure function with an unboundedly growing trace define a strictly increasing type through the type integral* (2.8).[3] *Then*
– *If for each element* $h = [\varphi, \psi]$ *of* $L^2([0,\infty), m)$ *which vanishes outside of* $[0, c]$*, a function* f *is defined by*

$$\pi f(w) = \int_0^\infty h_t \, dm_t \, q(t,w)^\top \tag{2.10}$$

[3] In terms of Definition 2.5, the chain (2.5) determines the first chaos associated with an si-process.

for all complex numbers w, then f is an entire function, it belongs to $\mathcal{H}(E_c)$ and

$$\pi \int_{-\infty}^{\infty} |f(\lambda)/E_c(\lambda)|^2 d\lambda = \int_0^{\infty} h\, dm_t\, h^*. \tag{2.11}$$

– *If g is in $\mathcal{H}(E_c)$, then $g = f$ for some such choice of $h = [\varphi, \psi]$ in $L^2(m)$.*

Since $q(t,w)$ in (2.10) is defined by (2.9), we have

$$\pi f(w) = \int_0^c \varphi_t A_t(w) d\alpha_t + \int_0^c \psi_t B_t(w) d\gamma_t. \tag{2.12}$$

In (2.11) the left-hand side represents π multiple of the squire norm of f in the metric of space $\mathcal{H}(E_c)$. The right-hand side is the sum of the squire norms $\|\varphi\|_\alpha^2 + \|\psi\|_\gamma^2$.

In the sequel, we shall distinguish two kinds of the kernels in the integral representation (2.12).

Definition 2.3. A kernel $h = [\varphi, \psi]$ in the space $L^2([0,\infty), dm)$ that does vanish outside of the interval $[0,c]$ is *singular* of degree $n > 0$ if it is orthogonal to the first n coefficients in the formal power series expansion of the entire function $q_0(t,z) = [A_t(z), B_t(z)]\, 1_{(0,c)}(t)$. If $\langle h, c_0\rangle_m \neq 0$, then h is *non-singular*; $c_0 = [1,0]$ is the free coefficient.

Consider again, the chain of short de Branges spaces $\mathcal{H}(E_t)$, $t \in [0,\infty)$, which satisfies the conditions of Theorem 2.2, $E_t(0) = 1$. On discussing single-sided si-processes we shall need to shift each space $\mathcal{H}(E_c)$ of entire functions of type at most $r = \tau_c$ by multiplying every element $f(z)$ of the space by the exponential e^{-izr}. The mapping $f(z) \mapsto e^{-izr} f(z)$ forms a set which we shall denote by $e^{-izr}\mathcal{H}(E_c)$. The eigenfunction expansion of elements of the latter set is just e^{-izr} multiple of (2.12), of course.

The set $e^{-izr}\mathcal{H}(E_c)$, endowed with the metric of $\mathcal{H}(E_c)$ is a reproducing Hilbert space but fails to be a de Branges space due to its asymmetry. If $K_c(w,z)$ is a kernel of the de Branges space $\mathcal{H}(E_c)$, then the reproducing kernel in space $e^{-izr}\mathcal{H}(E_c)$ will be

$$\mathsf{K}_c(w,z) = e^{-i(z-\bar{w})r} K_c(w,z). \tag{2.13}$$

As is shown in [16, Section 6.2.3], it is useful to represent this kernel as

$$\pi \mathsf{K}_c(w,z) = \int_0^c \overline{\mathcal{E}_t(w)} \mathcal{E}_t(z) d\alpha_t \tag{2.14}$$

for all $c > 0$, where

$$\mathcal{E}_c(z) = e^{-izr}\big(A_c(z) + \frac{d(\beta_c - i\,\tau_c)}{d\alpha_c} B_c(z)\big)$$

is a de Branges function. Here β_c is the off-diagonal entry in the structure function m_c, therefore it vanishes in the present situation. To see that the right-hand side of (2.14) is equal to the π multiple of the right-hand side of (2.13), check that

$$\int_0^c \overline{\mathcal{E}_t(w)}\mathcal{E}_t(z)d\alpha_t = \int_0^c e^{-i\tau_t(z-\bar{w})}[A_t(\bar{w}), B_t(\bar{w})]d(m_t - i\tau_t g_{\pi/2})[A_t(z), B_t(z)]^\top$$

and that this is π times the integral of $e^{-i\tau_t(z-\bar{w})}dK_t(w,z)+K_t(w,z)de^{-i\tau_t(z-\bar{w})}$. In the latter display $g_{\pi/2}$ is matrix (2.2) that rotates the plane through angle $\theta = \pi/2$ about the origin. Since $E_c(0) = 1$, also $\mathcal{E}_c(0) = 1$ and $\mathsf{K}_c(0, z) = \mathsf{B}_c(z)/(\pi z)$ where $\mathsf{B}_c(z) = e^{-izr}B_c(z)$. Each element $f(z)$ of the shifted space $e^{-izr}\mathcal{H}(E_c)$ has the eigenfunction expansion

$$\pi f(z) = \int_0^c k_t\, \mathcal{E}_t(z)\, d\alpha_t \tag{2.15}$$

for some choice of a kernel k_t which vanishes outside of interval $[0, c]$ and is squire integrable with respect to $d\alpha$.[4] The norm of f in the metric of the shifted space is the same as in the metric of $\mathcal{H}(E_c)$ and equals to

$$\pi\|f\|^2 = \int_0^\infty |k_t|^2 d\alpha_t. \tag{2.16}$$

Let us formulate these results regarding shifted spaces as the second part of Theorem 2.2. For the sake of brevity, the shifted space $e^{-izr}\mathcal{H}(E_c)$, $r = \tau_c$, will also be denoted as $H(\mathcal{E}_c)$, indicating its relationship with the de Branges function $\mathcal{E}_c$.

Theorem 2.2 (continued). *Let $H(\mathcal{E}_t), t \in [0, \infty)$, be the chain of shifted spaces which satisfy the conditions of the first part of the theorem. At point $c > 0$, the reproducing kernel* (2.13) *is representable in the form* (2.14)*. Each element of space $H(\mathcal{E}_c)$ has the eigenfunction representation* (2.15) *for some choice of a function k_t which vanishes outside of interval $[0, c]$ and is squire integrable with respect to $d\alpha$. The norm of this element in the metric of the shifted space is given by* (2.16).

Analogously to Definition 2.3, the kernel k in the integral representation (2.15) is called *singular* of degree $n > 0$ if it is orthogonal to the first n coefficients in the formal power series expansion of the de Branges function $\mathcal{E}$ in the metric of $L^2([0, \infty], d\alpha)$. If $\langle k, 1\rangle_\alpha \neq 0$, then k is *non-singular.*

[4]This claim is clearly true for the specific choice $f(z) = \mathsf{K}_c(0, z)$, since in this case $k_t = 1_{(0,c)}(t)$ in view of (2.14) and the normalisation $\mathcal{E}_t(0) = 1$. It is equally simple to check the statement for any finite linear combination of the type $\sum c_j \mathsf{K}_c(w_j, z)$ which is an element of the shifted space whose eigenfunction expansion is given by $\sum c_j \mathcal{E}_t(w_j) 1_{(0,c)}(t)$. The general statement regarding closed span of such linear combinations can be deduced by arguing as in the course of proving [16, Corollary 8.2.6].

2.2 Isometry

The spectral representation (1.2) gives rise to an isometric relationship between random variables X_t and integrals of exponential functions $\widehat{1}_t(z) = \int_0^t e^{izu}du$, the Fourier transforms of the indicator function $1_{(0,t)}$ of the set $(0,t)$. With this notation (1.2) is written as the inner product $\langle \widehat{1}_s, \widehat{1}_t \rangle_\mu$ in the metric of space $L^2(\mu)$. Let us express this isometric connection of random variables X_t with functions of a complex variable z in this manner

$$\mathfrak{I}_1 \, \widehat{1}_{(0,t)} = X_t. \tag{2.17}$$

The isometry extends in a natural way to closed linear manifolds of random variables and functions of complex variable z

$$\mathfrak{N}_{(a,b)} = \overline{sp}\{\int_0^t e^{izu}du : a \leq t \leq b\} \qquad \mathfrak{M}_{(a,b)} = \overline{sp}\{X_t : a \leq t \leq b\}$$

where the closure of the linear span of random variables $\mathfrak{M}_{(a,b)}$ is in mean squire and the linear span $\mathfrak{N}_{(a,b)}$ is closed in $L^2(\mu)$, where μ is the spectral measure of process X. Note that the present closed linear manifold $\mathfrak{N}_{(-r,r)}$ coincides with the linear span (2.4) mentioned earlier in connection with Kreĭn's alternative. On discussing single-sided si-processes, it is to be taken into consideration that e^{-irz} multiple of manifold $\mathfrak{N}_{(-r,r)}$ turns into the shifted manifold $\mathfrak{N}_{(0,2r)}$, symbolically $e^{-irz}\,\mathfrak{N}_{(-r,r)} = \mathfrak{N}_{(0,2r)}$.[5]

The spectral isometry can be now extended by linearity and continuity to the foregoing linear manifolds

$$\mathfrak{I}_1 : \mathfrak{N}_{(a,b)} = \overline{sp}\{\int_0^t e^{izu}du : a \leq t \leq b\} \rightarrow \mathfrak{M}_{(a,b)} = \overline{sp}\{X_t : a \leq t \leq b\} \tag{2.18}$$

for all intervals (a,b) within which processes are defined. Hence, at the right-hand side of this map one associates with process X the linear span of the random variables $\mathfrak{M}_{(0,r)} = \overline{sp}\{X_t : 0 \leq t \leq r\}$ in single-sided case and $\mathfrak{M}_{(-r,r)} = \overline{sp}\{X_t : -r \leq t \leq r\}$ in the double-sided case. The closure in $L^2(\Omega, \mathcal{F}, P)$ of the span will be called the *first chaos associated with the process* X and denoted by $\mathfrak{M}_1(r)$.[6] Clearly, all random variables in the first chaos are centered and Gaussian. Since an si-process is mean squire continuous, its first chaos is separable in the metric of $L^2(\Omega, \mathcal{F}, P)$. The set of elements of $\mathfrak{M}_1(r)$ which are linear combinations of random variables of the form $\sum c_j X_j$ with c_j and t_j rational, is a countable, dense subset. Space $\mathfrak{M}_1(r)$ endowed with the $L^2(\Omega, \mathcal{F}, P)$ norm and $\mathfrak{N}_r(\mu) = \overline{sp}\{\widehat{1}_t(z) : 0 \leq t \leq r\}_\mu$ endowed with the $L^2(\mu)$ norm are by construction Hilbert spaces. In these terms, the spectral isometry (2.18) is the linear map

$$\mathfrak{I}_1 : \mathfrak{N}_r(\mu) \rightarrow \mathfrak{M}_1(r) \tag{2.19}$$

[5]see Note 1 to [11, Section 3.6].

[6]The term is borrowed from Wiener's works [37] and [36]. The first chaos $\mathfrak{M}_1(r)$ is clearly related to the flow of σ-algebras $\mathcal{F}_{t\geq 0}$ with which the given probability space $(\Omega, \mathcal{F}, P)$ is equipped, $r = \tau_t$.

for each $r > 0$ and $\mathbb{E}|\mathfrak{I}_1 f|^2 = \|f\|_\mu^2$ for every element f of space $\mathfrak{N}_r(\mu)$.

If an si-process is double-sided, then the closed linear span $\mathfrak{N}_r(\mu) = \overline{sp}\{\widehat{1}_t(z) : -r \le t \le r\}$ on the lift hand-side of (2.19) is the same as (2.4) which is a proper subspace of $L^2(\mu)$, according to Kreĭn's alternative. As is said above, it comprises class $\mathbf{I}_r(\mu)$ of entire functions which are of exponential type at most r and squire integrable with respect to the spectral measure μ. This class, in its turn, is equivalent to the de Branges space $\mathcal{H}(E)$ of type $r > 0$ which is contained isometrically in $L^2(\mu)$. On the right-hand side of (2.19) the first chaos $\mathfrak{M}_1(r) = \overline{sp}\{X_t : -r \le t \le r\}$ is associated with a double-sided si-process which may be represented as the sum of two independent single-sided processes X^e and X^o with the covariance functions (1.3). Although both components are not si-processes, there is a natural way to associate with both processes corresponding first chaoses and define the spectral isometry between
1) $\mathfrak{N}_r^e(\mu)$ and $\mathfrak{N}_r^o(\mu)$ which are the closure in $L^2(\mu)$ of the linear span of functions $\{\sin tz/z : |t| \le r\}$ and $\{(\cos tz - 1)/z : |t| \le r\}$, respectively, and
2) $\mathfrak{M}_1^e(r)$ and $\mathfrak{M}_1^o(r)$ which are the closure in $L^2(\Omega, \mathcal{F}, P)$ of the linear span of the random variables $\{X_t^e : 0 \le t \le r\}$ and $\{X_t^o : 0 \le t \le r\}$, respectively.
We first define isometry $\mathfrak{I}_1 \widehat{1}_t^e = X_t^e$ and $\mathfrak{I}_1 \widehat{1}_t^o = X_t^o$ for every $t \ge 0$, where $\widehat{1}_t^e(z)$ and $\widehat{1}_t^o(z)$ are cosine and sine transforms of the indicator function $1_{(0,t)}$, respectively, and then extend this by linearity and continuity to

$$\mathfrak{I}_1 : \mathfrak{N}_r^e(\mu) \to \mathfrak{M}_1^e(r) \qquad \mathfrak{I}_1 : \mathfrak{N}_r^o(\mu) \to \mathfrak{M}_1^o(r). \tag{2.20}$$

Since the processes X^e and X^o are mutually independent, the sets $\mathfrak{M}_1^e(r)$ and $\mathfrak{M}_1^o(r)$ are mutually independent. Clearly, we have the orthogonal decomposition $\mathfrak{N}_r(\mu) = \mathfrak{N}_r^e(\mu) \oplus \mathfrak{N}_r^o(\mu)$, orthogonality is in $L^2(\mu)$. Likewise, decomposition $\mathfrak{M}_1(r) = \mathfrak{M}_1^e(r) \oplus \mathfrak{M}_1^o(r)$ is orthogonal in the metric of $L^2(\Omega, \mathcal{F}, P)$. The spectral isometry is then reformulated in this manner.

Theorem 2.4. *For $r > 0$, let $\mathfrak{M}_1(r)$ be the first chaos associated with a given double-sided si-process X. There exists a de Branges space of entire functions of exponential type at most r, contained isometrically in $L^2(\mu)$ where μ is a spectral measure of the given process. It can be identified with one of the spaces in a chain of spaces* (2.5)*, say $E(z) = E_c(z)$ for a certain point $c > 0$ which is related to type r by the type integral* (2.8)*, $\tau_c = r$. Then* (2.19) *is equivalent to*

$$\mathfrak{I}_1 : \mathcal{H}(E_c) \to \mathfrak{M}_1(\tau_c) \tag{2.21}$$

at point $c > 0$ such that $\tau_c = r$ (in terms of Definition 2.5, chain (2.5) *determines* the first chaos).

By equivalence, $\mathbb{E}|\mathfrak{I}_1 f|^2 = \|f\|_E^2 = \|f\|_\mu^2$ for every element f of the de Branges space $\mathcal{H}(E) = \mathcal{H}(E_c)$.

Since the process is real, its spectral measure is symmetric about the origin and there exists a symmetric short de Branges space contained isometrically in $L^2(\mu)$.

Therefore, the de Branges space $\mathcal{H}(E) = \mathcal{H}(E_c)$ is symmetric about the origin, $E^\sharp(z) = E(-z)$, i.e, $A(z) = A(-z)$ and $B(z) = -B(-z)$.

If an si-process is single-sided, the closed linear span $\mathfrak{N}_r(\mu)$ in (2.19) is also single-sided $\mathfrak{N}_r(\mu) = \overline{sp}\{\widehat{1}_t(z) : 0 \le t \le r\}$. It is obtained as $e^{-irz/2}$ multiple of manifold $\mathfrak{N}_{(-r/2,r/2)}$ which we denote as $e^{-irz/2}\mathfrak{N}_{(-r/2,r/2)} = \mathfrak{N}_{(0,r)}$. In this case we have to modify our arguments by introducing e^{-izr} multiple of $\mathcal{H}(E_c)$ which is as before a de Branges space of entire functions of exponential type at most $r = \tau_c$, contained isometrically in $L^2(\mu)$ where μ is a spectral measure of the process. The set obtained by multiplying elements of this de Branges space by e^{-izr} is denoted as $e^{-izr}\mathcal{H}(E_c)$ or, alternatively, as $H(\mathcal{E}_c)$. It is easily seen that
1) the set $H(\mathcal{E}_c)$ is contained in the single-sided span $\mathfrak{N}^\sharp_{2r}(\mu)$, $r = \tau_c$, where

$$\mathfrak{N}^\sharp_r(\mu) := \overline{sp}\Big(\frac{1 - e^{-izt}}{z} : 0 \le t \le r\Big), \tag{2.22}$$

2) if $\mathfrak{N}^\sharp_{2r}(\mu)$ is not the whole of $L^2(\mu)$, then it is contained in set $H(\mathcal{E}_c)$.

As is said above, set $H(\mathcal{E}_c)$, endowed with the metric of $\mathcal{H}(E)$ is a reproducing Hilbert space but fails to be a de Branges space due to its asymmetry. In these terms, the statement of Theorem 2.4 regarding the double-sided case extends to the single-sided case as follows.[7]

Theorem 2.4 (continued). *For $r > 0$, let $\mathfrak{M}_1(r)$ denote the first chaos associated with a given single-sided si-process X. Let $\mathcal{H}(E_c)$ be a de Branges space of entire functions of exponential type at most $r/2$, contained isometrically in $L^2(\mu)$ where μ is a spectral measure of the given process. This space, shifted by multiplying with e^{-izr}, $r = \tau_c$, and denoted by $H(\mathcal{E}_c)$ defines the first chaos as the following $\mathfrak{I}_1$-map*

$$\mathfrak{I}_1 : H^\sharp(\mathcal{E}_c) \to \mathfrak{M}_1(2r) \tag{2.23}$$

at any point c such that $r = \tau_c$.

2.3 Moving average w.r.t. fundamental martingales

2.3.1 Fundamental martingales

Let the first chaos for a double-sided si-process X be *determined* by a chain of de Branges spaces (2.5) in the sense of

[7]For more details, see [11, Section 9.3.1]. For example, in the classical case of the exponential de Branges function e^{-izr} space $\mathcal{H}(E)$ is the Paley–Wiener space of squire integrable entire functions of type at most $r > 0$, with the reproducing kernel $k_r(w, z) = \dfrac{\sin(z - w)}{\pi(z - w)}$. Therefore, the kernel of the shifted space $e^{-izr}\mathcal{H}(E)$ becomes $\mathsf{K}_r(w, z) = \dfrac{1 - e^{-i2(z-w)r}}{\pi(z - w)}$.

Definition 2.5. We say that chain (2.5) *determines* the first chaos for an si-process X, if it is contained isometrically in $L^2(\mu)$ where μ is the symmetric spectral measure of X and its diagonal structure function with an unboundedly growing trace defines a strictly increasing type through the type integral (2.8).

Theorem 2.2 establishes one-to-one correspondence between the elements of the first chaos $\mathfrak{M}_1(r)$, $r = \tau_c$, and the elements of $L^2([0,\infty), m)$ that vanish outside of interval $[0, c]$. This relationship is displayed through equation (2.10) in which $f(z)$ is an element of $\mathcal{H}(E_c)$, $h_u = [\varphi_u, \psi_u]$ is an element of $L^2([0,\infty), m)$ that vanishes outside of interval $0 \leq u \leq c$ and $[A_u(z), B_u(z)]$ are the components of the de Branges functions $E_u(z)$, $0 \leq u \leq c$.

In this section we are interested in two particular examples. For a fixed t from interval $[0, c]$, in the first case $\pi[\varphi_u, \psi_u] = [1_{[0,t]}(u), 0]$ and in the second case $\pi[\varphi_u, \psi_u] = [0, 1_{[0,t]}(u)]$, where $1_{[0,t]}(u)$ denotes the indicator function of interval $[0, t]$. On the right-hand side of (2.10) appears π multiple of $\int_0^t A_u(z) d\alpha_u = B_t(z)/z$ in the first case and $\int_0^t B_u(z) d\gamma_u = (A_t(z) - 1)/z$ in the second case. For each $t \in [0, c]$, both entire functions belong to space $\mathcal{H}(E_c)$ and by Theorem 2.4 their $\mathfrak{I}_1$-map belong to the first chaos $\mathfrak{M}_1(r)$, $r = \tau_c$. Moreover, the first one is an even function of z and the second one is an odd function of z, therefore

$$M_t^e := \mathfrak{I}_1 R_0 B_t \qquad M_t^o := -\mathfrak{I}_1 R_0 A_t \tag{2.24}$$

are in $\mathfrak{M}_1^e(r)$ and $\mathfrak{M}_1^o(r)$, respectively, the even and odd parts of the first chaos. For the sake of brevity, we make use of the *backward shift operator*, i.e. $R_0 B_t(z) = B_t(z)/z$ and $R_0 A_t(z) = (A_t(z) - 1)/z$. Note that in view of the integral representation (2.7), the first of definitions (2.24) is equivalent to $M_t^e = \pi\, \mathfrak{I}_1 K_t(0, \cdot)$. Clearly, the definition is independent of the right endpoint $c > 0$ and $(M_t^e)_{t\geq 0}$ and $(M_t^o)_{t\geq 0}$ can be viewed as squire integrable processes defined on the same Gaussian probability space $(\Omega, \mathcal{F}, P)$ as X, with the second order properties

$$\mathbb{E} M_s^e\, M_t^e = \pi \alpha_{s\wedge t} \qquad \mathbb{E} M_s^o\, M_t^o = \pi \gamma_{s\wedge t} \tag{2.25}$$

for all non-negative s, t. Hence, both processes are mutually independent squire integrable martingales with respect to their own filtrations $\mathcal{F}_{t\geq 0}^{M^e}$ and $\mathcal{F}_{t\geq 0}^{M^o}$, the σ-algebras generated by M^e and M^o, respectively, since

$$\mathbb{E}(M_t^e - M_s^e | M_u^e : 0 \leq u \leq s) = 0 \qquad a.s. \tag{2.26}$$

for $s \leq t$ (the same is true for the odd martingale, of course). Moreover, the increments are orthogonal not only to of M_s^e, but also to all elements of the even part of the first chaos $\mathfrak{M}(\tau_s)$. Let us define the filtrations

$$\mathcal{F}_{t\geq 0}^e := \mathfrak{M}_1^e(\tau_t) \qquad \mathcal{F}_{t\geq 0}^o := \mathfrak{M}_1^o(\tau_t) \tag{2.27}$$

for even and odd parts of a given si-process and formulate the forgoing results as

Theorem 2.6. *Let X be a real valued double-sided si-process and $\mathfrak{M}_1(r)$, $r > 0$, its first chaos determined by a chain of de Branges spaces* (2.5) *in the sense of Definition 2.5. Then* (2.24) *defines two mutually independent squire integrable martingales $(M, \mathcal{F}^e)_{t\geq 0}$ and $(M, \mathcal{F}^o)_{t\geq 0}$ adapted to the filtrations* (2.27)*, with the quadratic variations $\langle M^e\rangle_t = \pi\alpha_t$ and $\langle M^o\rangle_t = \pi\gamma_t$.*

For each fixed t, filtration $\mathcal{F}^e_{t\geq 0}$ is larger than $\mathcal{F}^{M^e}_{t\geq 0}$ (the same is true for the odd filtrations, of course). But in fact, the filtrations coincide as is not difficult to check.[8]

The second part of Theorem 2.4 regards a single-sided si-process. The spectral isometry is displayed through (2.23). On the right-hand side $\mathfrak{M}_1(r)$ is the first chaos associated with a given single-sided si-process at each instant $r > 0$, and on the left-hand side we have $e^{-izr/2}$ multiple of a de Branges space $\mathcal{H}(E_c)$ of exponential type $r/2$, $r = \tau_c$. The reproducing kernel in this shifted space is given by (2.13). Like in the case of even martingale, we define process

$$M_t := \pi\mathfrak{I}_1\mathsf{K}_t(0,\cdot) = \mathfrak{I}_1 R_0\,\mathsf{B}_t \tag{2.28}$$

for $t \geq 0$, where $\mathsf{B}_t(z) = e^{-iz\tau_t}B_t(z)$, which is again martingale with respect to filtration

$$\mathcal{F}^X_{t\geq 0} := \mathfrak{M}_1(2\tau_t) \tag{2.29}$$

(the first chaos on the right-hand side is associated with the given single-sided si-process X so as in the second part of Theorem 2.4). The second order moments are $\mathbb{E}M_s\,\overline{M}_t = \pi\alpha_{s\wedge t}$ for all non-negative s, t.

Theorem 2.6 (continued). *If in the first part of the theorem the given si-process X is single-sided, then process $(M, \mathcal{F}^X)_{t\geq 0}$ defined by* (2.28) *possesses property*

$$\mathbb{E}(M_t - M_s | X_u : 0 \leq u \leq \tau_s) = 0 \qquad a.s.$$

for $s \leq t$ and hence, it is a squire integrable martingale with the quadratic variation $\langle M\rangle_t = \pi\alpha_t$.

Example 2.7. In [13] process X is a *fractional Brownian motion* (FBM) of Hurst index $0 < H < 1$ with the covariance function

$$r(s,t) = \frac{1}{2}\big(s^{2H} + t^{2H} - |t-s|^{2H}\big) \tag{2.30}$$

and the spectral measure

$$\mu(d\lambda) = c_H|\lambda|^{1-2H}\,d\lambda \qquad c_H = \frac{\Gamma(1+2H)\sin H\pi}{2\pi}. \tag{2.31}$$

[8] see [11, Section 10.1].

The arguments s, t are allowed to take values on the whole real axis, so that X is a double-sided si-process. The corresponding even and odd processes have the covariance functions

$$4\mathbb{E}(X_s^e X_t^e) = |s+t|^{2H} - |s-t|^{2H} \quad 4\mathbb{E}(X_s^o X_t^o) = 2(s^{2H}+t^{2H}) - |s+t|^{2H} - |s-t|^{2H}.$$

The latter process is known in applied literature under the name *sub-fractional Brownian motion*. The respective spectral representations are

$$\mathbb{E}(X_s^e X_t^e) = c_H \int_0^\infty \frac{\sin\lambda s\, \sin\lambda t}{\lambda^{1+2H}}\, d\lambda \quad \mathbb{E}(X_s^o X_t^o) = c_H \int_0^\infty \frac{(\cos\lambda s - 1)\,(\cos\lambda t - 1)}{\lambda^{1+2H}}\, d\lambda.$$

As is shown in [13], the even and odd fundamental martingales M^e and M^o, adapted to the filtrations of $(X_t^e)_{t\geq 0}$ and $(X_t^o)_{t\geq 0}$, respectively, are given by

$$M_r^e = C\int_0^r (r^2-t^2)^{\frac{1}{2}-H}\, dX_t^e \qquad M_r^o = \frac{C}{\alpha_r}\int_0^r (r^2-t^2)^{\frac{1}{2}-H} \frac{d\big(\alpha_t X_t^o\big)}{\alpha_t'} \tag{2.32}$$

with constant $1/C = (1-H)\, B(\frac{1}{2}, \frac{3}{2}-H)$ ($B(x,y)$ is the beta function). For $H < 1/2$, the odd martingale can be written in a less complicated form

$$M_r^o = \frac{2r^{2H}}{B(\frac{1}{2}, \frac{1}{2}-H)} \int_0^r (r^2-t^2)^{-\frac{1}{2}-H}\, X_t^o\, dt. \tag{2.33}$$

The second order moments of these two squire integrable martingales are

$$\mathbb{E}|M_t^e|^2 = \pi\alpha_t = \pi\frac{t^{2-2H}}{2-2H} \qquad \mathbb{E}|M_t^o|^2 = \pi\gamma_t = \pi\frac{t^{2H}}{2H}, \tag{2.34}$$

The relations inverse to (2.32) are shown to be

$$X_r^e = C_1\int_0^r \varphi_r(t) dM_t^e \qquad X_r^o = C_1\int_0^r \psi_r(t) dM_t^o \tag{2.35}$$

where $C_1 = 2/B(1-H, H+1/2)$ and

$$\varphi_r(t) = (r^2-t^2)^{H-\frac{1}{2}} \qquad \psi_r(t) = t^{2-2H}\Big((r^2-t^2)^{H-\frac{1}{2}} + \int_t^r (u^2-t^2)^{H-\frac{1}{2}} \frac{du}{u^2}\Big).$$

Usually, the FBM is defined as a single-sided process, a real valued Gaussian process $(X_t)_{t\geq 0}$ with the covariance function (2.30) of $s, t \geq 0$, and the power spectral measure (2.31). It is well-known that in this case the fundamental martingale is defined as

$$M_{r/2} = \frac{C}{2}\int_0^r t^{\frac{1}{2}-H}(r-t)^{\frac{1}{2}-H}\, dX_t \tag{2.36}$$

and that this relationship is invertible

$$X_{2r} = 2C_1\int_0^r \Big(r^{H-\frac{1}{2}}(r-t)^{H-\frac{1}{2}} - \int_t^r (u-t)^{H-\frac{1}{2}}\, du^{H-\frac{1}{2}}\Big) dM_t. \tag{2.37}$$

The proofs can be found e.g. in [12]. □

2.3.2 Moving average

The integral representations (2.35) and (2.37) are called *moving average*. The general definition is as follows. Let $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a stochastic basis equipped with the filtration $\mathbb{F} = \mathcal{F}^X_{t\geq 0}$ which is defined by the first chaos of an si-process X through (2.27) or (2.29). Define also filtration $\mathcal{F}^M_{t\geq 0}$ generated by the random variables $(M_u : 0 \leq u \leq t)$ where M is the squire integrable martingale fundamental for the si-process X. The filtrations are equal, as was emphasized above.

Definition 2.8. The integral representations of the elements of the first chaos $\mathfrak{M}_1(r)$, $r = \tau_c$, in the form of Wiener integrals with respect to the fundamental martingales are called *moving average* if, and only if, the kernels involved are *non-singular*.

The definition needs further explanation. The basic fact is that every element of the first chaos has the required integral representation. This is a consequence of Theorem 2.2 and the spectral isometry. In the case of a double-sided si-process X, if f is an element of the first chaos $\mathfrak{M}_1(r)$, $r = \tau_c$, with the integral representation (2.10), then the spectral isometry represents the corresponding element $\mathfrak{I}_1 f := X_r(h)$ of the first chaos as

$$X_r(h) = \int_0^c \varphi\, dM^e + \int_0^c \psi\, dM^o \tag{2.38}$$

where M^e and M^o are the squire integrable, mutually independent martingales (2.24) and $h = [\varphi, \psi]$ is the same as in (2.10). In the case of a single-sided si-process X, if f is an element of the first chaos $\mathfrak{M}_1(2r)$, $r = \tau_c$, with the integral representation (2.15), then the spectral isometry represents the corresponding element $\mathfrak{I}_1 f := X_r(k)$ of the first chaos as

$$X_r(k) = \int_0^c k\, dM \tag{2.39}$$

where M is the squire integrable martingale (2.28) and k is the same as in (2.15). Definition 2.8 requires *non-singularity* of the kernels in the sense of Section 2.1: for h see Definition 2.5 and for k the last parapraph of that section. Let us reformulate this as a separate theorem.

Theorem 2.9. *In the situation of Theorem 2.2 the integral representations* (2.10) *and* (2.15) *for the elements of the first chaoses associated with si-processes* (double- and single-sided, respectively) *are moving average if, and only if,* $\int_0^c \varphi d\alpha \neq 0$ *and* $\int_0^c k d\alpha \neq 0$*, respectively.*

In particular, the integral representations

$$X^e_r = \int_0^c \varphi dM^e \qquad X^o_r = \int_0^c \psi dM^o \tag{2.40}$$

are moving average, where the kernels come from the eigenfunction expansions

$$\frac{\sin zr}{z} = \int_0^c \varphi_t A_t(z) d\alpha_t \qquad \frac{\cos zr - 1}{z} = \int_0^c \psi_t B_t(z) d\gamma_t$$

in space $\mathcal{H}(E_c)$ of exponential type $r = \tau_c$.

Likewise, the integral representation

$$X_{2r} = \int_0^c kdM \tag{2.41}$$

is moving average, where the kernel comes from the eigenfunction expansion

$$\frac{e^{-2izr} - 1}{iz} = \int_0^c k_t \mathcal{E}_t(z) d\alpha_t$$

in the shifted space $H(\mathcal{E}_c)$, $r = \tau_c$.

Clearly, not all integral representations for the elements of the first chaoses with respect to the fundamental martingales are moving average. Methods for determining such representations for all elements of the first chaoses can be found in [16, Section 10.3].

3 Paley-Wiener series

It is shown in this section how to extend the Paley–Wiener series expansion (PW-series) of Brownian motion to a wide class of si-processes. For example, if the motions are fractional with Hurst index $0 < H < 1$, then the PW-series

$$\sum_{n\in\mathbb{Z}} \frac{e^{2in\pi t} - 1}{2in\pi} Z_n \qquad t \in [0, 1] \tag{3.1}$$

which we had for $H = \frac{1}{2}$ retains this form for $H \neq 1/2$ too, however the real zeros $n\pi$ of $\sin z$ are substituted by the real zeros of the Bessel function $J_{1-H}(z)$ of the first kind (since $\sin z = \sqrt{\pi z/2} J_{1/2}(z)$, the Bessel function $J_{1/2}(z)$ and the sine function have common zeros). This result and its extension to si-processes are obtained in [16] (see also the Ph.D. theses [39]) by methods developed along the lines of Kreĭn's theory of strings so as in [9]. The present approach is based on the De Branges theory of orthogonal sets of [6, Section 22]. The following section reproduces some of these basic results.

3.1 Sampling formula

With each de Branges function $E(z)$ one associates the so-called phase function φ such that $\pi K(\lambda, \lambda) = |E(\lambda)|^2 \varphi'(\lambda) > 0$ at each real point λ, where $K(w, z)$ is the reproducing kernel (2.1) in the de Branges space $\mathcal{H}(E)$. It is shown in [6, Section 22] that if (λ_n) are roots of equation $\varphi(\lambda_n) = \theta + n\pi$ for a fixed real number θ, then the entire functions $K(\lambda_n, z)/\bar{E}(\lambda_n)$ comprise orthogonal set in $\mathcal{H}(E)$ so that $\pi \|K(\lambda_n, \cdot)/\bar{E}(\lambda_n)\|^2 = \varphi'(\lambda_n)$.[9] Moreover, the set is complete whenever the number θ

[9] In [6, Section 22] this is formulated as Problem 49; the proof is given in [11, Lemma 4.2.1, item iv].

is chosen so that rotation (2.3) of the components of the de Branges function yields B_θ that is not in the space $\mathcal{H}(E)$. One can always make this choice and, moreover, one can choose $\theta = 0$ without loss of generality. With the latter choice the λ_n become zeros of the odd component $B(z)$ (as was observed above, rotation by angle $\pi/2$ interchanges the roles of even and odd components and makes possible to work with zeros of component $A(z)$). Therefore, it will be assumed throughout we have in our disposal a complete basis in space $\mathcal{H}(E)$, which is defined by the set of real zeros of $B(z)$. It is then possible to expand every element f in terms of this basis.

Theorem 3.1. ([6, Theorem 22]). *Let $\mathcal{H}(E)$ be a given de Branges space and let φ be the phase function associated with E. Let (λ_n) be a sequence of real numbers such that $\varphi(\lambda_n) = \theta$ modulo π, with some fixed real number θ. Then the functions $K_{\lambda_n}/\bar{E}(\lambda_n)$ comprise an orthogonal set in $\mathcal{H}(E)$. The only elements of $\mathcal{H}(E)$ which are orthogonal to the functions $K_{\lambda_n}/\bar{E}(\lambda_n)$ for every n are constant multiples of B_θ. If this function does not belong to the space $\mathcal{H}(E)$, then*

$$\int \left|\frac{f(\lambda)}{E(\lambda)}\right|^2 d\lambda = \sum_n \left|\frac{f(\lambda_n)}{E(\lambda_n)}\right|^2 \frac{\pi}{\varphi'(\lambda_n)} \tag{3.2}$$

for every element f of the space.

By Theorem 3.1 series

$$\sum_n f(\lambda_n)\frac{K(\lambda_n, z)}{K(\lambda_n, \lambda_n)} \tag{3.3}$$

does converge in mean square to $f(z)$, in the metric of $\mathcal{H}(E)$. Parseval's identity estimates the norm

$$\|f\|_E^2 = \sum_n \frac{|f(\lambda_n)|^2}{K(\lambda_n, \lambda_n)}. \tag{3.4}$$

The coefficients $f(\lambda_n)/\sqrt{K(\lambda_n, \lambda_n)}$ are calculated by making use of the reproducing property of the kernel. This is the *sampling formula* in terms of Dym and McKean [10, p. 302].

Throughout this paper, the de Branges space $\mathcal{H}(E)$ is identified with one in the chain of spaces $\mathcal{H}(E_t)$, $0 \le t < \infty$, at certain point $t = c$ and space $\mathcal{H}(E_c)$ (and the whole chain) is contained isometrically in $L^2(\mu)$, where μ is the spectral measure of an si-process in question. Therefore

$$\Big\| f - \sum_{|n|\le N} f(\lambda_n)\frac{K_c(\lambda_n, \cdot)}{K_c(\lambda_n, \lambda_n)}\Big\|_\mu \to 0 \tag{3.5}$$

as $N \to \infty$ whatever element f of space $\mathcal{H}(E_c)$ with the reproducing kernel $K_c(w, z)$. The λ_n are zeros of the odd component $B_c(z)$.

As regards mapping (2.23), we note that for elements of the shifted space $e^{-izr}\mathcal{H}(E_c)$, $r=\tau_c$, say $\mathsf{f}=e^{-izr}f$ where $f\in\mathcal{H}(E_c)$, it holds that

$$\Big\|\mathsf{f}-\sum_{|n|\le N}\mathsf{f}(\lambda_n)\frac{\mathsf{K}_c(\lambda_n,\cdot)}{\mathsf{K}_c(\lambda_n,\lambda_n)}\Big\|_\mu=\Big\|f-\sum_{|n|\le N}f(\lambda_n)\frac{K_c(\lambda_n,\cdot)}{K_c(\lambda_n,\lambda_n)}\Big\|_\mu\to 0 \tag{3.6}$$

as $N\to\infty$, where $\mathsf{K}_c(w,z)$ is the kernel in the shifted space $e^{-izr}\mathcal{H}(E_c)$ and the λ_n are zeros of the odd component $B_c(z)$. This is clear, as the metric does not change by shifting and $\mathsf{K}_c(\lambda,\lambda)=K_c(\lambda,\lambda)$ on the real diagonal.

3.2 PW-series

The sampling formula can be used for calculating covariance functions. If one has the set of zeros (λ_n), obtained so as is described above, then the covariance function of a double-sided si-process X is calculated in this manner

$$\mathbb{E}(X_s\overline{X}_t)=\sum_n\frac{\widehat{1}_s(\lambda_n)\overline{\widehat{1}_t(\lambda_n)}}{K_c(\lambda_n,\lambda_n)} \tag{3.7}$$

for all $-r\le s,t\le r$. This follows directly from (3.4) applied to the particular element $\widehat{1}_t$ of the space $\mathcal{H}(E_c)$, as $|t|\le r$. If X is single-sided, then we have again formula (3.7) for all $0\le s,t\le 2r$ in this case, since $\mathsf{K}_c(\lambda,\lambda)=K_c(\lambda,\lambda)$, as we know.

Now, $\mathfrak{I}_1$-map of the entire functions $K_c(\lambda_n,z)/K_c(\lambda_n,\lambda_n)$ defines the independent Gaussian random variables Z_n with zero mean and variance $\mathbb{E}|Z_n|^2=1/K_c(\lambda_n,\lambda_n)$. Since X_t is $\mathfrak{I}_1$-map of $\widehat{1}_t(z)$ by definition, the spectral isometry gives the covariances

$$\mathbb{E}(X_t\overline{Z}_n)=\frac{\overline{\widehat{1}_t(\lambda_n)}}{K_c(\lambda_n,\lambda_n)}.$$

But then the finite linear combination $\Sigma_t^N:=\sum_{|n|\le N}\widehat{1}_t(\lambda_n)\,Z_n$ is so that

$$\mathbb{E}\Big|X_t-\sum_{|n|\le N}\frac{e^{i\lambda_n t}-1}{i\lambda_n}\,Z_n\Big|^2\to 0 \tag{3.8}$$

as $N\to\infty$ for each $t\in[-r,r]$ or $t\in[0,2r]$ depending whether X is double- or single-sided. For definiteness, let us focus our attention to the case of a double-sided X, the singles-sided case will be handled analogously. We call $\Sigma_t^N=\sum_{|n|\le N}\widehat{1}_t(\lambda_n)\,Z_n$ the *partial sum* and consider partial processes $\Sigma^N=(\Sigma_t^N)_{t\in[-r,r]}$. It follows from the preceding that the finite dimensional distributions of Σ^N converge weakly to the finite dimensional distributions of X. To extend the Paley and Wiener result beyond the Brownian motion, one need to show that the partial processes converge in $C[-r,r]$ with probability one, where $C[-r,r]$ is the space of continuous functions on $[-r,r]$, endowed with the supremum metric. But this is guaranteed by the Lévy-Itô-Nisio

Theorem [21, Theorem 4.1] on the convergence of sums of symmetric Banach space valued random variables, cf. also [26, Theorem 2.4]. Hence, we have the following extension of the Paley–Wiener theorem.

Theorem 3.2. *Let $\cdots < \lambda_{-1} < \lambda_0 = 0 < \lambda_1 < \cdots$ be the real valued zeros of the component $B_r(z)$ of the de Branges function generating space $\mathcal{H}(E_r)$ which is contained isometrically in the symmetric space $L^2(\mu)$ where μ is the spectral measure of the double-sided si-process X in question. Then the process, restricted to interval $-r \leq t \leq r$, has the series representation*

$$X_t = \sum_n \frac{e^{i\lambda_n t} - 1}{i\lambda_n} Z_n$$

where the Z_n are mutually independent Gaussian random variables with zero mean and variance

$$\mathbb{E}|Z_n|^2 = \sigma_n^2 = \frac{1}{K_r(\lambda_n, \lambda_n)}.$$

The series converges in mean square for each $t \in [-r, r]$. If process X admits a continuous version, then with probability one, the series converge uniformly in $[-r, r]$.

Due to the definitions $X_t^e = (X_t - X_t)/2$ and $X_t^o = (X_t + X_t)/2$, $t \geq 0$, for the even and odd parts of a double sided si-process X, the theorem implies

Corollary 3.3. *The even and odd parts of the si-process of Theorem 3.2 expand in the series*

$$X_t^e = tZ + 2\sum_{n>0} \frac{\sin \lambda_n t}{\lambda_n} Z_n^e \qquad X_t^o = 2\sum_{n>0} \frac{\cos \lambda_n t - 1}{\lambda_n} Z_n^o$$

for each $t \in [0, r]$, in terms of independent Gaussian random variables with variances

$$\mathbb{E}|Z|^2 = \sigma_0^2 \qquad \mathbb{E}|Z_n^e|^2 = \mathbb{E}|Z_n^o|^2 = \frac{1}{2}\sigma_n^2$$

where $\sigma_n^2 = 1/K_c(\lambda_n, \lambda_n)$ is the same as in the theorem.

Proof. One only needs to check that on calculating even and odd parts there occur random variables

$$Z = Z_0 \qquad Z_n^e = \frac{Z_n + Z_{-n}}{2} \qquad Z_n^o = \frac{Z_n - Z_{-n}}{2}$$

for $n > 0$ which are independent and Gaussian with zero mean and indicated variances. □

For applications, for instance, for simulation purposes, it is important to know how quickly the remainder term

$$R_t^N := X_t - \sum_{|n|\le N} \frac{e^{i\lambda_n t}-1}{i\lambda_n} Z_n = \sum_{|n|>N} \frac{e^{i\lambda_n t}-1}{i\lambda_n} Z_n \tag{3.9}$$

vanishes as $N \to \infty$. It depends, of course, on the behaviour of the variances of the random variables Z_n for large indices n. To characterized this behaviour it will be assumed in the sequel that there exists a positive number $a > -1$ such that

$$\mathbb{E}|Z_n|^2 \le c\, n^{-a} \tag{3.10}$$

for all $n \ge N$, with N large enough; constant c can be any positive number.

Remark. Jumping ahead, we refer to Lemma 3.5 which will prove that in the case of an FBM with Hurst index H it holds that $\mathbb{E}|Z_n|^2 \sim n^{1-2H}$. Therefore the variances decline for $H > 1/2$, stay constant for an ordinary BM and increase for $H < 1/2$. The proof is based on the well-known fact that the n^{th} zero of the Bessel function $J_{\nu+1}(rz)$ (and hence of the odd component $B_r(z)$, cf. (3.16)) is such that $|\lambda_n| = (n\pi/r)(1+o(1))$ far out; see e.g. [35, p. 509]. By definition $\mathbb{E}|Z_n|^2 = 1/K_c(\lambda_n, \lambda_n)$, condition (3.10) is equivalent to

$$K_c(\lambda_n, \lambda_n) \ge c|\lambda_n|^a \tag{3.11}$$

for $n \ge N$. The exponent a will determine the speed at which the remainder term R^N vanishes as $N \to \infty$. It varies, in principal, with the choice of the metric with which the difference between the process X and the partial sum Σ^N is measured. For instance, one may calculate the expected values of squired pointwise deviations, i.e. $\mathbb{E}|R_t^N|^2$, and take maximum over the whole interval $[-r, r]$. The following estimate rather straightforward

$$\sup_{t\in[-r,r]} \mathbb{E}|R_t^N|^2 \simeq N^{-(1+a)}, \tag{3.12}$$

where the relation symbol $\simeq$ is used to designate there exists a positive constant such that the left-hand side is less or equal to the constant multiple of the right hand-side. Indeed, the variances of the remainder terms for $-r < t \le r$ are uniformly bounded

$$\mathbb{E}|R_t^N|^2 \le \sum_{|n|>N} \frac{1}{|\lambda_n|^2 K_c(\lambda_n, \lambda_n)}.$$

When $|\lambda_n|$ is of order $(2n\pi/r)$ for n large enough, so as in the case of an FBM, condition (3.11) yields the result

$$\sup_{t\in[-r,r]} \mathbb{E}|R_t^N|^2 \simeq \sum_{n>N} n^{-(2+a)} \simeq \int_N^\infty x^{-(2+a)} dx = N^{-(1+a)}/(1+a).$$

which shows how quickly the remainder term vanishes. □

Turning back to the general case, we shall follow [14] and make use of the uniform norm $\|R^N\|_\infty$. The objective is to obtain an upper bound

$$\mathbb{E}\|R^N\|_\infty \simeq (N^{-(a+1)}\,\ln N)^{1/2} \tag{3.13}$$

([14] discuses fractional Brownian sheets, however the method of proof is applicable to the present more general case too).

Proof of (3.13). For a positive integer m, define the difference of the partial sums

$$\Delta_m(t) := \Sigma_t^{2^m} - \Sigma_t^{2^{m-1}}$$

and estimate the expectation of the supremum over interval $[-r,r]$ in this manner. Let the interval be covered with smaller intervals $I_1,\cdots,I_l$ with centers $t_1,\cdots,t_l$ and length ϵ, where the length and the covering number are related as $l \simeq 1/\epsilon$ (length ϵ will later go down with increase of the number m). Clearly, $I_k = (t_k - \epsilon/2, t_k + \epsilon/2)$. The supremum over interval $[-r,r]$ of the m^{th} difference of the partial sums can be bounded by two terms

$$\sup_{t\in[-r,r]} |\Delta_m(t)| \le \sup_{k=1,\cdots,l} |\Delta_m(t)| + \sup_{k=1,\cdots,l}\,\sup_{s,t\in I_k} |\Delta_m(t) - \Delta_m(s)|.$$

The expectation of the first term can be estimated by aid of a standard maximal inequality for Gaussian sequences which can be found, e.g., in [32, Lemma 2.2.2]. It says

$$\mathbb{E}\sup_{k=1,\cdots,l} |\Delta_m(t)| \simeq \sqrt{1+\ln l}\,\sup_{k=1,\cdots,l}\sqrt{\mathbb{E}|\Delta_m(t_k)|^2}.$$

The variance of the sum of independent random variables is easily calculated and bounded as follows

$$\mathbb{E}|\Delta_m(t_k)|^2 = \sum_{2^{m-1}<j\le 2^m} \frac{|\widehat{1}_{(0,t_k]}(\lambda_j)|^2}{K_c(\lambda_j,\lambda_j)} \simeq \sum_{2^{m-1}<j\le 2^m} \frac{1}{|\lambda_j|^2 K_c(\lambda_j,\lambda_j)}.$$

For the rest, we can argue as in the case of (3.12) and obtain

$$\sum_{2^{m-1}<j\le 2^m} \frac{1}{|\lambda_j|^2 K_c(\lambda_j,\lambda_j)} \simeq 1/2^{m(a+1)/2},$$

therefore the expectation of the first term on the right-hand side of the inequality for $\sup_{t\in[-r,r]} |\Delta_m(t)|$ is bounded by

$$\mathbb{E}\sup_{k=1,\cdots,l} |\Delta_m(t)| \simeq \sqrt{1+\ln l}/2^{m(a+1)/2}.$$

The expectation of the second term is bounded by

$$\mathbb{E} \sup_{k=1,\cdots,l} \sup_{s,t\in I_k} \sum_{2^{m-1}<j<2^m} |Z_j| \frac{|\widehat{1}_{(0,t)}(\lambda_j) - \widehat{1}_{(0,t)}(\lambda_j)|}{K_c(\lambda_j,\lambda_j)}$$

where $|\widehat{1}_{(0,t)}(\lambda_j) - \widehat{1}_{(0,t)}(\lambda_j)| = |\int_s^t e^{i\lambda_j u} du| \le |t-s| \le \epsilon$ for $s,t \in I_k$. By (3.11), kernel $K_r(w,z)$ at $w = z = \lambda_n$ is bounded below by a multiple of n^a, therefore

$$\mathbb{E} \sup_{k=1,\cdots,N} \sup_{s,t\in I_k} |\Delta^m(t) - \Delta^m(s)| \simeq \epsilon \sum_{2^{m-1}<j\le 2^m} j^{-a/2} \le \epsilon\, 2^{\frac{m-1}{1-a/2}}.$$

We have estimated the expectations of both terms in the foregoing inequality for $\mathbb{E}\sup_{t\in[-r,r]} |\Delta_m(t)|$. By combining these two results, we obtain

$$\mathbb{E} \sup_{t\in[-r,r]} |\Delta_m(t)| \simeq \sqrt{1+\ln N}\, 2^{-m(a+1)/2} + \epsilon\, 2^{\frac{m-1}{1-a/2}}.$$

The covering number and the length of intervals are in relation $l \simeq 1/\epsilon$. We shall now specify the length so as to make the second term on the right-hand side of the latter display of lower order than the first. This is achieved by $\epsilon = 2^{-2m}$. Hence

$$\mathbb{E} \sup_{t\in[-r,r]} |\Delta_m(t)| \simeq \sqrt{m}\, 2^{-m(1+a)/2}. \tag{3.14}$$

Now, let us fix the number N of terms in the partial sum and select m so that $2^{m-1} < N \le 2^m$. At each point $t \in [-r,r]$ the remainder term is bounded by the sum of two terms

$$|R_t^N| \le \Big| \sum_{N<j\le 2^m} \frac{\widehat{1}_{(0,t]}(\lambda_j)}{K_c(\lambda_j,\lambda_j)} Z_j \Big| + \sum_{j>m} |\Delta_j(t)|. \tag{3.15}$$

The second one is bounded as follows

$$\sum_{j>m} \mathbb{E} \sup_{t\in[-r,r]} |\Delta_j(t)| \simeq \sum_{j>m} \sqrt{j}\, 2^{-j(1+a)/2} \simeq \sqrt{m+1}\, 2^{(m+1)(a+1)/2}$$
$$\simeq N^{-(a+1)/2} \sqrt{\ln N}$$

(number m is selected so that $2^{2m+1} < 4N$). To prove this we first apply (3.14) and then the fact that $\sum_{j>m} \sqrt{j}\, 2^{-j\,\alpha} \simeq \sqrt{m+1}\, 2^{(m+1)\,\alpha}$ for any positive number α. As for the first term on the right-hand side of (3.15), we can make again use of (3.14) to get

$$\mathbb{E} \sup_{t\in[-r,r]} \Big| \sum_{N<j\le 2^m} \frac{\widehat{1}_{(0,t]}(\lambda_j)}{K_c(\lambda_j,\lambda_j)} Z_j \Big| \simeq \sqrt{m}\, 2^{-m(a+1)/2} \simeq N^{-(a+1)/2} \sqrt{\ln N}.$$

Hence

$$\mathbb{E} \sup_{t\in[-r,r]} |R_t^N| \simeq N^{-(a+1)/2} \sqrt{\ln N}$$

which is the required estimate (3.13). □

As is pointed out earlier, Theorem 3.2 regards double-sided si-processes, but is easily extendable to single-sided processes. Because, if the independent random variables Z_n are defined as $\mathfrak{I}_1$-map of element $\mathsf{K}_c(\lambda_n, z)/\mathsf{K}_c(\lambda_n, \lambda_n)$ of the space $e^{-izr}\mathcal{H}(E_c)$, $r = \tau_c$ (which is $e^{-i(z-\lambda_n)r}$ multiple of $K_c(\lambda_n, z)/K_c(\lambda_n, \lambda_n)$), then their Gaussian distribution is the same as that of the previous Z_n. Clearly, they are again centered and have variances $\mathbb{E}|\mathsf{Z}_n|^2 = 1/K_c(\lambda_n, \lambda_n)$. We have mentioned already that if X is single-sided, then the convergence in (3.8) holds for $t \in [0, 2r]$, and this is true with Z_n substituted by Z_n.

Theorem 3.2 (continued). *In the situation of the first part of the theorem, let μ be the spectral measure of a single-sided si-process X. Then the process, restricted to interval $0 \leq t \leq 2r$, has the series representation*

$$X_t = \sum_n \frac{e^{i\lambda_n t} - 1}{i\lambda_n} Z_n$$

where the Z_n are mutually independent Gaussian random variables with zero mean and variance

$$\mathbb{E}|Z_n|^2 = \sigma_n^2 = \frac{1}{K_c(\lambda_n, \lambda_n)}.$$

The series converges in mean square for each $t \in [0, 2r]$.
If process X admits a continuous version, then with probability one, the series converge uniformly in $[0, 2r]$.
If process X is real, then the series turns into

$$X_t = tZ + 2\sum_{n>0} \frac{\sin \lambda_n t}{\lambda_n} Z_n^{(1)} + 2\sum_{n>0} \frac{\cos \lambda_n t - 1}{\lambda_n} Z_n^{(2)}$$

for each $t \in [0, 2r]$, with independent Gaussian random variables, centered and having variances

$$\mathbb{E}|Z|^2 = \sigma_0^2 \qquad \mathbb{E}|Z_n^{(1)}|^2 = \mathbb{E}|Z_n^{(2)}|^2 = \frac{1}{2}\sigma_n^2.$$

Proof. The proof follows that of the first part of theorem, except the statement regarding the real case. To proof the latter, note that in formula (3.7) for the covariance function the terms indexed by n and $-n$ are complex conjugated. Therefore

$$\mathbb{E}X_s\overline{X}_t = \sigma_0^2 + 2\sum_{n>0} \sigma_n^2 \mathrm{Re}\left(\widehat{1}_s(\lambda_n)\overline{\widehat{1}_t(\lambda_n)}\right)$$

with

$$\mathrm{Re}\left(\widehat{1}_s(\lambda_n)\overline{\widehat{1}_t(\lambda_n)}\right) = \frac{\sin \lambda_n s}{\lambda_n}\frac{\sin \lambda_n t}{\lambda_n} + \frac{\cos \lambda_n s - 1}{\lambda_n}\frac{\cos \lambda_n t - 1}{\lambda_n}.$$

This shows that the series for real X converges in mean squire sense and that the resulting process has the same distribution as Brownian motion. On calculating covariances we make use of the relations

$$Z = Z_0 \qquad Z_n^{(1)} = \frac{Z_n + Z_{-n}}{2} \qquad Z_n^{(2)} = \frac{Z_n - Z_{-n}}{2}$$

for $n > 0$. □

3.3 PW-series for FBM

(a) In this section the general results of the preceding one are applied to fractional Brownian motions of Hurst index $0 < H < 1$. It will be shown below that these particular si-processes are *determined* by the chain of homogeneous[10] de Branges spaces which are based on de Branges functions $E_t(z)$, $0 \le t < \infty$, with the components

$$[A_t(z), B_t(z)] = \Gamma(1-H)\Big(\frac{tz}{2}\Big)^H [J_{-H}(tz), t^{1-2H} J_{1-H}(tz)]. \tag{3.16}$$

Let us show that these components indeed define a de Branges function.

Proposition 3.4. *For $\nu > -1$ and $r > 0$ the entire function $E_r(z) = A_r(z) - iB_r(z)$ with $-H = \nu$ is a De Branges function, normalized at the origin $E_r(0) = 1$. It is of exponential type r.*

Proof. The normalization is clear, since $z^{-\nu} J_\nu(z) \to (1/2)^\nu/\Gamma(\nu+1)$ as $z \to 0$ for $\nu > -1$. For future use, we shall prove more general statement: for each $r > 0$ the following entire function of two complex variables w and z has the integral representation

$$\begin{aligned} K_r(w, z) :=& \frac{B_r(z)A_r(\bar w) - A_r(z)B_r(\bar w)}{\pi(z - \bar w)} \\ =& \frac{1}{\pi}\int_0^r [A_t(\bar w), B_t(\bar w)]\, dm_t\, [A_t(z), B_t(z)]^\top \end{aligned} \tag{3.17}$$

where the integration is carried out with respect to a diagonal matrix-valued function $m_t = \mathrm{diag}\,[\alpha_t, \gamma_t]$ with the entries $\alpha_t = t^{2\nu+2}/(2\nu+2)$ and $\gamma_t = t^{-2\nu}/(-2\nu)$ and the density

$$m_t' = \mathrm{diag}\left[t^{1+2\nu},\ (1/t)^{1+2\nu}\right] \tag{3.18}$$

[10]This term is used on the basis of the identities

$$[A_a(z), B_a(z)] = [A_1(az),\ a^{1+2\nu} B_1(az)]$$
$$K_a(w, z) = a^{2+2\nu} K_1(aw, az)$$

which show homogeneous nature of these functions.

for $t > 0$. To prove (3.17), calculate the integrals with the help of the integration and differentiation rules, cf. e.g. [25, formulas 5.3.5-5.3.7]. Since $\nu > -1$, one obtains

$$\int_0^r A_t(\bar{w})A_t(z)t^{1+2\nu}\,dt + \int_0^r B_t(\bar{w})B_t(z)(1/t)^{1+2\nu}\,dt$$
$$= \Gamma^2(\nu+1)\Big(\frac{4}{\bar{w}z}\Big)^{\nu}\,\frac{J_{\nu+1}(rz)J_\nu(r\bar{w}) - J_\nu(rz)J_{\nu+1}(a\bar{w})}{(z-\bar{w})/r}. \tag{3.19}$$

It is easily seen that the right hand-side is equal to $\pi K_a(w,z)$. It follows that $K_r(w,w) > 0$ and

$$\pi K_r(w,w) = \frac{|E_r(w)|^2 - |E_r^\sharp(w)|^2}{4y} > 0$$

for each w in the upper half-plane, $r > 0$. Hence $|E_r(w)| > |E_r^\sharp(w)|$ in the upper half-plane and $E_r(z)$ is a de Branges function.

We deal here with a generalization of the classical example $E_r(z) = e^{-irz}$, for if $\nu = -1/2$, then $A_r(z) = \cos rz$ and $B_r(z) = \sin rz$. As in this particular case, the positive number r is a type parameter. It follows from the inequalities from Section 3.3 of Watson's book [35] that $E_r(z)$ is of exponential type r. Indeed, those inequalities show

$$|E_r(w)| \leq \big(1 + |w|\,r^{2\nu+2}/(2\nu+2)\big)e^{r|y|}$$

for every $w = x + iy$. The proof is complete. □

The proposition determines the reproducing kernels of the present chain of spaces and the structure function through (3.17) and (3.18), respectively. Indirectly, it defines also the spectral measure μ such that the whole chain is contained isometrically in $L^2(\mu)$. To see this, associate with the present diagonal structure function the *structure space* $L^2([0,\infty), m)$ which consists of all pairs $h = [\varphi, \psi]$ of functions squire integrable with respect to dm. Each such pair that vanishes outside of interval $[0, r]$ defines an element f of the de Branges space $\mathcal{H}(E_r)$, since the expansion theorem [6, Section 44] says

$$\pi f(w) = \int_0^\infty h_t\,dm_t q(t,w)^\top \tag{3.20}$$

for all complex numbers, where $q(t,w) = [A_t(w), B_t(w)]1_{(0,r)}(t)$ so as in (2.10). Note use of the identity $\tau_t = t$ which is clear in view of (3.18). The norm of element f in the metric of $\mathcal{H}(E_r)$ is given by Parseval's identity (2.11) with $c = r$. Now, it is to be shown that the right-hand side of the latter identity, which is the squire norm of h in the metric of $L^2([0,\infty), m)$, is calculated also as

$$\int_0^\infty h\,dm\,h^\top = 2\pi \int_0^\infty |f(\lambda)|^2 \mu(d\lambda) \tag{3.21}$$

where $\mu(d\lambda)$ is given by

$$\mu(d\lambda) = \mu'(1)|\lambda|^{1+2\nu}d\lambda \qquad\qquad \mu'(1) = \frac{\pi}{2^{1+2\nu}\Gamma^2(1+\nu)}. \tag{3.22}$$

This is achieved by considering the eigenfunction representation (3.20) (and its particular case (3.17) for $f(z) = K_a(w, z)$ and $h_t = q(t, w)$) as a certain Hankel transform by spelling it out as follows

$$\frac{\pi(z/2)^\nu}{\Gamma(\nu+1)} f(z) = \int_0^r \varphi_t J_\nu(tz)t^{\nu+1}dt + \int_0^r \psi_t J_{\nu+1}(tz)t^{-\nu}dt. \tag{3.23}$$

This shows that the entire function $\pi(z/2)^{\nu+\frac{1}{2}} f(z)/\Gamma(\nu+1)$ forms the pair of Hankel transforms of order ν and $\nu+1$ with the functions which vanish outside of the interval $[0, r]$ and are defined within the interval as $t^{\nu+\frac{1}{2}}\varphi_t/\sqrt{2}$ and $\psi_t/(\sqrt{2}\, t^{\nu+\frac{1}{2}})$, respectively. By Parseval's theorem for Hankel transforms one obtains (3.21).

Hence the squire norm in the metric of de Branges spaces $\mathcal{H}(E_r)$ is equal to that of $L^2(\mu)$ with the spectral measure μ defined by (3.22). For Hurst index $0 < H < 1$, parameter ν is confined to interval $(-1, 0)$ and so the spectral measure is of property (1.1). Upon comparing the latter with the spectral measure (2.31) for the standard FBM, we see the difference of constant multipliers. Actually, the spectral measure (2.31) defines a non-standard FBM, with $r(1, 1) = \mu'(1)\Gamma(1 - 2H)\cos H\pi/H$. The necessary changes caused by this difference are made by virtue of rule 1 in [9, Section 6.6] and [16, Section 7.2.1]. But this will be unnecessary in the sequel.

The space $\mathcal{H}(E_r)$ is defined by the components (3.16) with $t = r$ and the homogeneity order $\nu > -1$ is related to Hurst index $0 < H < 1$ by $\nu = -H$. In the sequel, the zeros λ_n will be that of the second component in (3.16), i.e. zeros of the Bessel function $J_{1-H}(z)$ (we remark that alternatively one can choose to work with the zeros of the first component – one can easily verify that all consequent results of this chapter remain true, like in the case $H = 1/2$ in which expansions are obtainable in terms of the zeros of the sine or cosine, cf. [38, Section 26.1]). Regarding the zeros of the Bessel function of the first kind of order $\nu > -1$, see [17, Section 7.9]. The Bessel function $J_\nu(z)$ has a countable number of positive zeros that can be arranged in ascending order of magnitude. We denote them by $\lambda_{\nu,1} < \lambda_{\nu,2} < \cdots$. For positive ν, the Bessel function vanishes at the origin $J_\nu(0) = 0$, and its negative zeros are given by $-\lambda_{\nu,1} > -\lambda_{\nu,2} > \cdots$. Hence, for $\nu \geq 0$ the zeros can be ordered as $\cdots < \lambda_{\nu,-1} < \lambda_{\nu,0} = 0 < \lambda_{\nu,1} < \cdots$.

Making use of these properties of the zeros and the form of the reproducing kernel (3.19), we can verify condition (3.11).

Lemma 3.5. *If the de Branges space $\mathcal{H}(E_r)$ is defined by the components* (3.16) *and if $\cdots \lambda_{-1} < \lambda_0 = 0 < \lambda_1 < \cdots$ are the zeros of $J_{1-H}(rz)$, then condition* (3.11) *is satisfied with $a = 1 - 2H$.*

Proof. The reproducing kernel in (3.11) is $K_r(w,z)$, since in the present case $\tau_t = t$. On the diagonal $w = z = \lambda$, with λ real, the reproducing kernel is equal to $A_r(\lambda)B_r'(\lambda) - A_r'(\lambda)B_r(\lambda)$ divided through π, e.g.

$$\begin{aligned}\frac{K_r(\lambda,\lambda)}{K_r(0,0)} &= (2-2H)\Gamma^2(1-H)(r\lambda/2)^{2H} \\ &\times \left(J_{1-H}^2(r\lambda) - \frac{1-2H}{r\lambda} J_{-H}(r\lambda)J_{1-H}(r\lambda) + J_{-H}^2(r\lambda)\right)\end{aligned}$$

by (3.19) and the properties of the Bessel functions. Substituting λ by the n^{th} zero of the Bessel function J_{1-H} we get

$$\frac{K_r(\lambda_n,\lambda_n)}{K_r(0,0)} = (2-2H)\Gamma^2(1-H)(r\lambda_n/2)^{2H} J_{-H}^2(r\lambda_n). \tag{3.24}$$

It is shown in [35, Section 7.1] that the initial terms in the expansion of the Bessel functions of the first kind are such that for order $\nu > -1$ and $\nu + 1$ it holds that

$$J_\nu^2(z) + J_{\nu+1}^2(z) \sim \frac{2}{\pi z},$$

see Lommel's formula on p. 200. This asymptotic behaviour for large $|z|$ shows that if we substitute z by the n^{th} positive zero λ_n of the Bessel function $J_\nu(z)$, then $J_{\nu+1}(\lambda_n) \sim 2/(\pi\lambda_n)$ for $n \to \infty$, as the zeros λ_n tend to infinity. Hence (3.24) implies $K_r(\lambda_n,\lambda_n) \sim c\,|\lambda_n|^{2H-1}$ with c a constant. The proof is complete. $\square$

The lemma provides a key argument for extending the PW-expansion to fractional Brownian motions. It proves that the series of the next theorem converges towards an FBM with rate that is optimal in the sense of Kühn and Linde [24, Theorem 7.4].

Theorem 3.6. *For an arbitrary Hurst index $0 < H < 1$, let $\cdots < \lambda_{-1} < \lambda_{-1} = 0 < \lambda_1 < \cdots$ be zeros of the Bessel function $J_{1-H}(rz)$. Let the Z_n be independent, complex valued Gaussian random variables with mean zero and variance $\mathbb{E}|Z_n|^2 = \sigma_n^2$ where the $\sigma_n^2 = 1/K_r(\lambda_n,\lambda_n)$ are given by* (3.24). *Then, with probability one, the series*

$$\sum_n \frac{e^{i\lambda_n t} - 1}{i\lambda_n} Z_n$$

converges uniformly in $t \in [-r,r]$ (or $t \in [0,2r]$) *and defines a complex valued double-sided* (or single-sided) *FBM with Hurst index H.*

The remainder term (3.9) *satisfies* (3.12) *and* (3.13) *with $a = 2H-1$, i.e.*

$$\begin{aligned}\sup_t \mathbb{E}|R_t^N| &\simeq N^{-H} \\ \mathbb{E}\|R^N\|_\infty &\simeq \sqrt{\ln N}\, N^{-H},\end{aligned}$$

hence the convergence is rate-optimal.

With increase of Hurst index, sample paths of an FBM become smoother and the above estimates tell us that the convergence rate does improve and less number of terms are needed to achieve a desired level of approximation.

3.4 PW-series for fundamental martingales

Section 2.3 introduces so-called *fundamental martingales* associated with si-processes X. In the case of a double-sided X, $\mathfrak{I}_1$-map (2.24) defines two martingales, the even and odd one, belonging to the even and odd part of the first chaos $\mathfrak{M}_1^e(r)$ and $\mathfrak{M}_1^o(r)$, respectively. Both processes M^e and M^o, restricted to interval $[0,r]$, expand in PW-series. This is clear from the general considerations of Section 3.2. The same can be said regarding the single-sided case in which the fundamental martingale is defined as $\mathfrak{I}_1$-map (2.28). Restricted to interval $[0,2r]$, process M expands in PW-series, as well. In view of (3.5) and (3.6) we can make the following statement.

Theorem 3.7. *In the situation of Theorem 3.2*
(i) The even and odd fundamental martingales associated with the given double-sided si-process X by (2.24)*, expand in series*

$$M_t^e = \sum_n \frac{B_t(\lambda_n)}{\lambda_n} Z_n \qquad M_t^o = \sum_n \frac{A_t(\lambda_n) - 1}{\lambda_n} Z_n$$

for $t \in [0,r]$, with the zeros λ_n and the random variables Z_n as defined in Theorem 3.2.
(ii) If X is single-sided, then the associated fundamental martingale defined by (2.28) *expands in series*

$$M_t = \sum_n \frac{\mathsf{B}_t(\lambda_n)}{\lambda_n} Z_n$$

for $t \in [0,2r]$, with the zeros λ_n, the random variables Z_n as above, and $\mathsf{B}_t(z)/z = e^{iz\tau_t} B_t(z)/z = \pi\, \mathsf{K}_t(0,z)$.

In the forthcoming Section 4 we shall be interested in so-called *martingale bridges* associated with fundamental martingales like in [18]. For instance, the bridge associated with an even fundamental martingale is defined and expanded in PW-series as follows

$$M(t;r) \coloneqq M_t^e - \frac{\langle M^e\rangle_t}{\langle M^e\rangle_r} M_r^e = 2\sum_{n=1}^{\infty} \frac{B_t(\lambda_n)}{\lambda_n} Z_n^e$$

where $Z_n^e = (Z_n + Z_{-n})/2$ and $\langle M^e\rangle_t = \alpha_t$. To deduce this from Theorem 3.7, note that in the expansion for M_t^e the term indexed $n=0$ is equal to $\alpha_t Z_0$, since $B_t(\lambda_0)/\lambda_0 = \alpha_t$, as $\lambda_0 = 0$. Now, evaluate both sides of the expansion at $t=r$ to get $M_r^e = \alpha_r Z_0$. Hence, in the expansion the random variable Z_0 can be substituted by M_r^e/α_r. The right side of the display follows by the symmetry about the origin of the zeros $\lambda_0 = 0$, $\lambda_{-n} = \lambda_n$, as the entire function $B_t(z)/z$ is even.

3.5 Stationary processes

The sampling method applies to stationary processes $(Y_t)_{t\in\mathbb{R}}$ without essential changes. Since the covariance function $\mathbb{E}(Y_sY_t) = r(s,t)$ in this case depends only on the difference $t-s$ and has the spectral representation

$$r(s,t)=\int e^{-i\lambda(t-s)}\mu(d\lambda) \qquad s,t\in\mathbb{R}$$

with a finite spectral measure μ, the random variable Y_0 has a non-zero variance equal to $\mu(\mathbb{R})<\infty$.

In analogy to (2.17), the random variable Y_t is brought in connection with exponential e^{itz} in the metric of $L^2(\mu)$ by isometry $\mathfrak{I}_1 e^{it\cdot}=Y_t$, and the first chaos $\mathfrak{M}_1(r)$ in connection with $\mathfrak{I}_1$-map of the closed span $\overline{sp}\{e^{izt}:0\le t\le r\}_\mu$. This single-sided span of exponentials coincides with the shifted space $e^{-izr/2}\mathcal{H}(E_{r/2})$, where $\mathcal{H}(E_r)$ is a de Branges function of exponential type $r>0$ contained isometrically in $L^2(\mu)$ (since spectral measures are finite, constants belong to space $\mathcal{H}(E_r)$ and type $r=0$ is not excluded; but this is irrelevant in the current case). Let $B_r(z)$ be the odd component of the de Branges function generating space $\mathcal{H}(E_r)$. Let $\mathsf{K}_r(w,z)$ be kernel (2.13) of the shifted space $e^{-izr}\mathcal{H}(E_r)$ (for simplicity, we restrict out attention to the particular case $\tau_t=t$, since this will be the only case in the examples we have in mind). In this situation, the sampling formula applies to each element e^{-izt}, $t\in[0,2r]$, of the shifted space $e^{-izr}\mathcal{H}(E_r)$

$$e^{-izt}=\sum_n e^{-it\lambda_n}\frac{\mathsf{K}_r(\lambda_n,z)}{\mathsf{K}_r(\lambda_n,\lambda_n)}$$

where the λ_n are zeros of $B_r(z)$. Now we can make use of spectral isometry like in Theorem 3.2 and obtain the series expansion

$$Y_t=\sum_n e^{i\lambda_n t}\,\mathsf{Z}_n \tag{3.25}$$

where the Z_n are mutually independent Gaussian random variables with zero mean and variance

$$\mathbb{E}|Z_n|^2=\sigma_n^2=\frac{1}{\mathsf{K}_r(\lambda_n,\lambda_n)}.$$

The series converges in mean square for each $t\in[0,2r]$. Let us consider two examples.

3.5.1 OU process

Let Y be the *Ornstein-Uhlenbeck* (OU) process, centered stationary Gaussian process Y having the covariance function

$$\mathbb{E}(Y_sY_t)=\frac{\sigma^2}{2\theta}e^{-\theta|t-s|} \tag{3.26}$$

and the density of the spectral measure

$$\mu(d\lambda)=\frac{\sigma^2}{2\pi}\frac{d\lambda}{\lambda^2+\theta^2} \tag{3.27}$$

with the parameters $\sigma^2, \theta > 0$. The space $L^2(\mu)$ generated by this finite spectral measure contains isometrically all spaces based on the de Branges functions

$$E_t(z) = \sqrt{2\pi/\sigma^2}(\theta - iz)e^{-izt} \qquad t \geq 0. \tag{3.28}$$

The identity of the norms in the metric of all spaces $\mathcal{H}(E_t)$ for $t \geq 0$ and $L^2(\mu)$ is obvious. At the origin space $\mathcal{H}(E_0)$ is non-trivial and to construct the chain of spaces which starts from a trivial one, [11, Section 7.2.4] lets the chain to start from a certain negative point $t_- > 0$ and lets $E_t(z)$ to be $\sqrt{2\pi/\sigma^2}$ multiple of $(\theta - iz\,\frac{t-t_-}{0-t_-})\,1_{(t_-,0)}(t)$. This part of the chain is not contained isometrically in $L^2(\mu)$, but this is beyond our concern, since our objective is to specify series (3.25) for positive values of t. For this purpose one needs to show that the shifted space $e^{-izr}\mathcal{H}(E_r)$ which is contained isometrically in $L^2(\mu)$ has the reproducing kernel

$$\pi \mathsf{K}_a(w,z) = \pi \mathsf{K}_0(w,z) + \int_0^a \overline{\mathcal{E}_t(w)}\mathcal{E}_t(z)d\alpha_t \tag{3.29}$$

(in contrast with (2.14) there occur the additional term $\pi\mathsf{K}_0(w,z)$, due to non-triviality of the de Branges space at the origin) and that in the present case this gives

$$\mathsf{K}_{r/2}(w,z) = \frac{2\theta}{\sigma^2} + \frac{(\theta + i\bar{w})(\theta - iz)}{\sigma^2}\frac{e^{-i(z-\bar{w})r} - 1}{-i(z-\bar{w})} \tag{3.30}$$

for $r \geq 0$. Check this by substituting (3.28) into definition (2.1). It follows that

$$\sigma^2\,\mathsf{K}_r(\lambda,\lambda) = 2\theta^2 + (\theta^2 + \lambda^2)\,r.$$

Due to the form of the odd component $B_r(z)$ of the de Branges function (3.28), the zeros λ_n are the roots of equation

$$\theta\tan(\lambda r) + \lambda = 0.$$

With these zeros, the OU-process expands in series (3.25) for $t \in [0, 2r]$ where the Z_n are mutually independent Gaussian random variables with zero mean and variance

$$\sigma_n^2 = \frac{\sigma^2}{2\theta^2 + (\theta^2 + \lambda_n^2)\,r}.$$

The variances vanish with rate n^{-2}. This gives estimates

$$\sup_t \mathbb{E}|R_t^N| \simeq N^{-1/2}$$
$$\mathbb{E}\|R^N\|_\infty \simeq \sqrt{\ln N}\,N^{-1/2}$$

and shows the rate is optimal, see Section 5, Note 1.

3.5.2 Autoregressive process

The *autoregressive processes* Y of n^{th}-order is a cantered stationary Gaussian process with the spectral measure whose density is

$$\mu(d\lambda) = \frac{\sigma^2}{2\pi} \frac{d\lambda}{|\Theta(i\lambda)|^2} \tag{3.31}$$

where

$$\Theta(iz) = (iz - \phi_1) \cdots (iz - \phi_n) \tag{3.32}$$

is a polynomial of degree n. Its zeros have negative real part and, moreover, $\phi_k > 0$, since process Y is real. Like in the previous example, it is easily seen that space $L^2(\mu)$ generated by this finite spectral measure contains isometrically all spaces based on the de Branges functions

$$E_t(z) = \sqrt{2\pi/\sigma^2}\Theta(iz)e^{-izt} \qquad t \geq 0. \tag{3.33}$$

For $r > 0$, the odd component $B_r(z)$ is such that its zeros λ_l are the roots of equation

$$\tan(r\lambda) \operatorname{Re} \Theta(i\lambda) = \operatorname{Im} \Theta(i\lambda).$$

The reproducing kernel (3.29) in the shifted space $e^{-izr}\mathcal{H}(E_r)$ is calculated like (3.30) for $n = 1$. The later generalises to all positive integers as

$$\begin{aligned}
\sigma^2 \mathsf{K}_0(w,z) &= \frac{\Theta^\sharp(iz)\Theta(i\overline{w}) - \Theta(iz)\Theta^\sharp(i\bar{w})}{-i(z-\bar{w})} \\
\sigma^2 \mathsf{K}_r(w,z) &= \sigma^2 \mathsf{K}_0(w,z) + \Theta(i\overline{w})\Theta^\sharp(iz)\, \frac{e^{-i(z-\bar{w})r} - 1}{-i(z-\bar{w})},
\end{aligned}$$

therefore

$$\sigma^2 \mathsf{K}_r(\lambda_l, \lambda_l) = \sigma^2 \mathsf{K}_0(\lambda_l, \lambda_l) + |\Theta(i\lambda_l)|^2\, r$$

for $r \geq 0$. Since $\phi_k > 0$ by assumption, we have $|\Theta(i\lambda_l)|^2 = (\lambda_l^2 + \phi_1^2) \cdots (\lambda_l^2 + \phi_n^2)$ and

$$\sigma^2 \mathsf{K}_0(\lambda_l, \lambda_l) = 4 \operatorname{Re} \Theta(i\lambda_l) \operatorname{Im} \Theta'(i\lambda_l).$$

The series expansion

$$Y_t = \sum_l e^{i\lambda_l t}\, \mathsf{Z}_l$$

for $0 \leq t \leq r$ is in terms of mutually independent Gaussian random variables Z_l with zero mean and variance

$$\sigma_n^2 = \frac{\sigma^2}{4 \operatorname{Re} \Theta(i\lambda_n) \operatorname{Im} \Theta'(i\lambda_n) + |\Theta(i\lambda_n)|^2\, r}.$$

4 The Karhunen–Loève expansion

The generalized PW-series for fundamental martingales provided in Section 3.4 posses an extra feature which will be brought forward in the present section. To get the idea, let us apply Theorem 3.7 to a standard Brownian motion. In the theorem the λ_n are zeros of $\cos(rz)$, therefore the positive zeros are $\lambda_n = (n + \frac{1}{2})\pi/r$. The independent Gaussian random variables Z_n are $N(0, 1/r)$, since in this particular case $\pi K_r(\lambda, \lambda) = 1/r$ for all real λ. As the zeros of $\cos(rz)$ are symmetrically spread about the origin, Theorem 3.7 gives

$$W_t = \sqrt{2/r} \sum_{n=0}^{\infty} \frac{\sin((n + \frac{1}{2})\pi t/r)}{(n + \frac{1}{2})\pi/r} \xi_n \tag{4.1}$$

for $0 \leq t \leq r$, where the ξ_n are mutually independent standard normal random variables. If λ_n are zeros of $\sin(rz)$, i.e. $\lambda_n = n\pi/r$, then the remark following Theorem 3.7 entails

$$W_t - \frac{t}{r} W_r = \sqrt{2/r} \sum_{n=1}^{\infty} \frac{\sin(n\pi t/r)}{n\pi/r} \xi_n \tag{4.2}$$

for $0 \leq t \leq r$, with the same ξ_n as above. The same results follow from Theorem 3.6 with $H = 1/2$, of course. The extra feature we are speaking about is *bi-orthogonality* of these expansions as $H = 1/2$, in the sense that set (ξ_n) of random variables form an orthogonal basis in $L^2(\Omega, \mathcal{F}, P)$ and the trigonometric functions in both expansions form an orthogonal basis in $L^2[0, r]$. This is well-known fact, on p. 37 of [20] e.g. one can find two examples of such sets of functions

$$\{1/\sqrt{r},\ \sqrt{2/r}\cos(n\pi t/r),\ n = 1, 2, \cdots\}$$
$$\{\sqrt{2/r}\sin(n\pi t/r),\ n = 1, 2, \cdots\}.$$

The expansions (4.1) and (4.2) are also well-known. They are classical examples of the *Karhunen–Loève expansion*, cf. for instance, [19, Section V.2, formulas (17) and (18)]. This is how Loève formulates his theorem.

Theorem 4.1. (Loève [28, Section 34.5 (b)]). *A random process X_t, continuous in the mean squire on a closed interval $[a, b]$, has an orthogonal decomposition*

$$X_t = \sum_n \omega_n \varphi_n(t) \xi_n \tag{4.3}$$

if and only if $|\omega_n|^2$ and $\varphi_n(t)$ are eigenvalues and eigenfunctions of its covariance function. The series converge in the mean squire uniformly on $[a, b]$.

The random variables ξ_n are uncorrelated $\mathbb{E}\xi_m \bar{\xi}_n = \delta_{mn}$. If $k(s, t)$ is the covariance function, then

$$\int_a^b k(s, t)\varphi_n(s) ds = |\omega_n|^2 \varphi_n(t)$$

and the series

$$k(s,t) = \sum_n |\omega_n|^2 \varphi_n(s)\overline{\varphi_n(t)}$$

converges on $[a,b] \times [a,b]$ uniformly and absolutely. The proof is based on the so-called *Mercer's theorem* (cf. e.g. [34]), which will be discussed in the next section.

Let us turn back to the examples (4.1) and (4.2). The covariance functions of the Brownian motion and Brownian bridge are $k(t,s) = s \wedge t$ and $k(t,s) = s \wedge t - st/r$, respectively. Since these real processes are defined on interval $[0,r]$, the integral equation of Theorem 4.1 with these covariance functions is

$$\int_0^r k(s,t)\varphi_n(s)ds = \omega_n^2\, \varphi_n(t)$$

where the ω_n are positive numbers. The functions φ_n obey the boundary condition $\varphi_n(0) = 0$. Another boundary condition is found upon differentiating both sides with respect to dt. It will be seen that if $k(t,s) = s \wedge t$, then $\varphi_n'(r) = 0$, and if $k(t,s) = s \wedge t - ts/r$, then $\varphi_n(r) = 0$. Differentiating once more we arrive at the second order differential equation $\omega_n^2\, \varphi_n''(t) = -\varphi_n(t)$. The normalized solution of this equation with the preceding boundary conditions are well-known

$$\begin{array}{lll} \sqrt{2/r}\sin\left((n+\frac{1}{2})\frac{\pi t}{r}\right), & \omega_n^{-1} = (n+\frac{1}{2})\pi, & n = 0,1,\cdots \\ \sqrt{2/r}\sin\left(n\frac{\pi t}{r}\right), & \omega_n^{-1} = n\pi, & n = 1,2,\cdots \end{array}$$

respectively. This obtains the Karhunen–Loève expansions (4.1) and (4.2).

The arguments which extend the preceding results to a class of fundamental martingales and martingale bridges are quite similar. As is said at the beginning of this section, we have already generalized PW-series for these real valued processes and it is to be shown they are KL-expansions. To describe this class, we first invoke the definitions of Section 2.3. For a given si-process X and its first chaos $\mathfrak{M}_1(r)$, $r > 0$, an associated fundamental martingale is defined in terms of a chain of de Branges spaces of exponential type which *determines* the given si-process and its first chaos. The chain is assumed to be contained isometrically in $L^2(\mu)$, where μ is the symmetric spectral measure of X. It will be assumed throughout the present section that the following proposition holds true.

Theorem 4.2. *Let α_t be a continuously differentiable strictly increasing function, $\alpha_0 = 0$ and $\alpha_t' > 0$. Then the system of integral equations*

$$1 - A_t(z) = z\int_0^t B_u(z)d\gamma_u \qquad B_t(z) = z\int_0^t A_u(z)d\alpha_u \tag{4.4}$$

with $\gamma_t' = 1/\alpha_t'$, defines a chain of de Branges functions $E_t(z) = A_t(z) - iB_t(z), t \geq 0$, starting from $E_0(z) = 1$, normalized at the origin $E_t(0) = 1$, and such that the *reproducing kernel in the corresponding chain of spaces satisfies*

$$\pi K_t(w,z) = \frac{B_t(z)A_t(\bar{w}) - A_t(z)B_t(\bar{w})}{z-\bar{w}}$$
$$= \int_0^t A_u(\bar{w})A_u(z)\,d\alpha_u + \int_0^t B_u(\bar{w})B_u(z)\,d\gamma_u \tag{4.5}$$

for $t \geq 0$ and all complex numbers w and z. Both integrals on the right-hand side of the latter equation satisfy the so-called Lagrange identities

$$\int_0^t A_u(\bar{w})A_u(z)\,d\alpha_u = \frac{zA_t(\bar{w})B_t(z) - \bar{w}A_t(z)B_t(\bar{w})}{z^2-\bar{w}^2}$$
$$\int_0^t B_u(\bar{w})B_u(z)\,d\gamma_u = \frac{\bar{w}B_t(z)A_t(\bar{w}) - zB_t(\bar{w})A_t(z)}{z^2-\bar{w}^2} \tag{4.6}$$

for all complex numbers w and z. If these numbers are real and equal, then

$$\int_0^t |A_u(w)|^2\,d\alpha_u = \frac{w\big(A_t(w)\dot{B}_t(w) - \dot{A}_t(w)B_t(w)\big) + A_t(w)B_t(w)}{2w}$$
$$\int_0^t |B_u(w)|^2\,d\gamma_u = \frac{w\big(A_t(w)\dot{B}_t(w) - \dot{A}_t(w)B_t(w)\big) - A_r(w)B_t(w)}{2w} \tag{4.7}$$

for $w \neq 0$.

The derivatives $\dot{A}_t(w)$ and $\dot{B}_t(w)$ are with respect to the variable w.

Proof. For the integral equations (4.4) and representation (4.5) of the reproducing kernel in the de Branges space $\mathcal{H}(E_r)$, see (2.6) and (2.7). It is taken here into consideration that under the present conditions the chain of symmetric de Branges spaces starts from a trivial space and $E_t(0) = 1$ for all $t \geq 0$.[11]

For the *Lagrange identities*, see [22, Lemma 1.1]. It suffices to provide the proof of the first of the equations (4.6), since the second one follows from (4.5). Making use of $[dB_t(z), -dA_t(z)] = z[A_t(z)\,d\alpha_t, B_t(z)\,d\gamma_t]$ and integrating by parts, we get

$$-z^2\int_0^t A_u(\bar{w})A_u(z)\,d\alpha_u = -z\int_0^t A_u(\bar{w})\,dB_u(z)$$
$$= -zA_u(\bar{w})B_u(z)\Big|_0^t + z\int_0^t B_u(z)dA_u(\bar{w})$$
$$= -zA_u(\bar{w})B_u(z)\Big|_0^t + z\bar{w}\int_0^t B_u(z)B_u(\bar{w})d\gamma_u.$$

[11] In [10] H. Dym discusses the equivalence of the integral equations (4.4) to the second order differential equations

$$\frac{d^2A_t(z)}{d\alpha_t\,d\gamma_t} = -z^2A_t(z) \qquad \frac{d^2B_t(z)}{d\gamma_t\,d\alpha_t} = -z^2B_t(z) \tag{4.8}$$

relating the latter to classical differential equations of Sturm–Liouville type. We shall utilize this relationship in the course of proving Theorem 4.5.

Obviously, we also have

$$-\bar{w}^2 \int_0^t A_u(\bar{w})A_u(z)\,d\alpha_u = -\bar{w}A_u(z)B_u(\bar{w})\big|_0^t + z\bar{w}\int_0^t B_u(z)B_u(\bar{w})d\gamma_u$$

which follows from the preceding equation by interchanging the arguments z and $\bar{w}$. It remains to subtract the first of these equations (4.6) from the second one. The formulas (4.7) follow from (4.6) by l'Hôpital's rule. □

Clearly, the present chain of de Branges functions is of exponential type and its diagonal structure function defines type τ_t, $0 \le t < \infty$, so that $\tau_t' = \alpha_t'\gamma_t' = 1$. If X is a fractional Brownian motion of Hurst index $0 < H < 1$, for instance, the chain is defined by (3.16) and $\alpha_t' = 1/\gamma_t' = t^{1-2H}$.

Remark 4.3. For later use we make connection with the De Branges' theorem [6, Section 37] which proves relationship (4.4) between the even and odd components of the de Branges functions and more. It regards the chain of matrices of entire functions which are real for a real z, called usually as *De Branges matrices* and denoted as

$$M_t(z) = \begin{bmatrix} A_t(z) & B_t(z) \\ C_t(z) & D_t(z) \end{bmatrix} \qquad 0 \le t < \infty.$$

They are normalized so that $M_t(0) = I$. The chain of matrices is related to the structure function of the chain of de Branges spaces (2.5) by $m_t = \dot{M}_t(0)g_{\pi/2}$. Here the chain of de Branges spaces (2.5) is generated by the upper row of the de Branges matrices and $\dot{M}_t(z)$ denotes the derivative of $M(z)$ with respect to variable z. De Branges [6] Theorem 37 provides the following integral equation

$$M_r(z)g_{\pi/2} - g_{\pi/2} = z\int_0^r M_t(z)dm_t \tag{4.9}$$

which in our particular case implies (4.4), as well as [12]

$$D_t(z) - 1 = z\int_0^t C_u(z)d\alpha_u \qquad - C_t(z) = z\int_0^t D_u(z)d\gamma_u. \tag{4.11}$$

The Lagrange identities (4.6) turn now into

$$\begin{aligned} \int_0^t D_u(\bar{w})D_u(z)\,d\gamma_u &= \frac{\bar{w}D_t(z)C_t(\bar{w}) - zD_t(\bar{w})C_t(z)}{z^2 - \bar{w}^2} \\ \int_0^t C_u(\bar{w})C_u(z)\,d\alpha_u &= \frac{zC_t(\bar{w})D_t(z) - \bar{w}C_t(z)D_t(\bar{w})}{z^2 - \bar{w}^2} \end{aligned} \tag{4.12}$$

[12] Parallel to (4.8), we have

$$\frac{d^2C_t(z)}{d\alpha_t\,d\gamma_t} = -z^2C_t(z) \qquad \frac{d^2D_t(z)}{d\gamma_t\,d\alpha_t} = -z^2D_t(z). \tag{4.10}$$

for all complex numbers w and z.

In conclusion, we point out that by rotation of the de Branges matrices

$$g_{\pi/2}^{\top} M_t(z) g_{\pi/2} = \begin{bmatrix} D_t(z) & -C_t(z) \\ -B_t(z) & A_t(z) \end{bmatrix} \tag{4.13}$$

the roles of the components $A_t(z)$ and $B_t(z)$ are given to $D_t(z)$ and $-C_t(z)$. respectively. The latter even and odd entire functions form kernel

$$Q_t(w,z) = \frac{D_t(z)C_t(w) - C_t(z)D_t(w)}{\pi(z - \bar{w})} \tag{4.14}$$

which is introduced in [6, Sections 27 and 28] for characterizing bilinear J-forms of de Branges matrices. Applied to the present particular situation, the results of these sections make available a new chain of de Branges functions $\tilde{E}_t(z) = D_t(z) + iC(z), t > 0$, of the same exponential type as the former chain $E_t(z), t > 0$. The reproducing kernel in this new space is (4.14).

4.1 Mercer's theorem

A key argument proving the Karhunen–Loève theorem is provided by Mercer's theorem, as is said above. In the present section the latter theorem is restricted to symmetric positive definite kernels

$$k : [a,b] \times [a,b] \to \mathbb{R},$$

symmetry means that $k(x,y) = k(y,x)$ and positive definiteness that

$$\sum_{i,j} c_j c_k k(x_i, x_j) \geq 0$$

for all finite number of points $x_j \in [a,b]$ and all choices of finite real numbers c_j. With each such kernel, one associates the Hilbert-Schmidt operator

$$\mathcal{K}\phi(t) := \int_a^b k(s,t)\varphi(s)ds$$

on functions $\varphi \in L^2[a,b]$, and defines the homogeneous Fredholm integral equation

$$\int_a^b k(s,t)\varphi(s)ds = \epsilon\, \varphi(t),$$

with squire integrable kernels

$$\int_a^b \int_a^b |k(s,t)|^2 ds dt < \infty.$$

Since $\mathcal{K}$ is a linear operator, there is an orthonormal basis (φ_n) in $L^2[a,b]$ where φ_n are eigenfunctions of the operator which correspond to non-negative eigenvalues (ϵ_n) that solve the integral equation

$$\int_a^b k(s,t)\varphi_n(s)ds = \epsilon_n\, \varphi_n(t).$$

The eigenvalues are assumed to be arranged in ascending order of magnitude. Mercer's theorem states an expansion of kernel k in terms of this basis.

Theorem 4.4. *Let k be a continuous symmetric positive definite kernel. Its eigenfunctions φ_n corresponding to positive eigenvalues ϵ_n are continuous on $[a,b]$ and k expands as*

$$k(s,t) = \sum_{n=1}^{\infty} \epsilon_n\, \varphi_n(s)\varphi_n(t) \tag{4.15}$$

where the series converges absolutely and uniformly.

4.1.1 Kernels (4.16) and (4.17)

It is assumed in this and subsequent subsections that a chain of De Branges spaces $\mathcal{H}(E_t)$, $t \geq 0$, is given and satisfies the conditions of Section 2.3. The objective is to formulate and prove Mercer's type theorems for a number of positive definite kernels which are defined in terms of a diagonal structure function of the chain $m_t = \mathrm{diag}[\alpha_t, \gamma_t]$, $t \geq 0$. The next theorem and Theorem 4.13 regard the upper entry α of the structure function. Theorem 4.12 and Theorem 4.10 regard the lower entry γ.

Theorem 4.5. *Let a given chain of de Branges spaces $\mathcal{H}(E_t), 0 \leq t \leq r$, satisfy the conditions of Theorem 4.2.*
(i) If points on the real axes $0 < \lambda_1 < \lambda_2 < \cdots$ are such that $A_r(\lambda_n) = 0$, then the kernel

$$k(s,t) = \frac{\alpha_{s\wedge t}}{\sqrt{\alpha'_s}\sqrt{\alpha'_t}} \tag{4.16}$$

is expanded in the series (4.15) with $\epsilon_n = \lambda_n^{-2}$ and

$$\varphi_n(t) = \frac{B_t(\lambda_n)}{\sqrt{\alpha'_t}}\sqrt{\frac{2}{\pi K_r(\lambda_n,\lambda_n)}}$$

where $\pi K_r(\lambda_n,\lambda_n) = -\dot{A}_r(\lambda_n)B_r(\lambda_n)$.
(ii) If points on the real axes $0 < \lambda_1 < \lambda_2 < \cdots$ are such that $B_r(\lambda_n) = 0$, then the kernel

$$k(s,t) = \frac{\alpha_{s\wedge t} - \dfrac{\alpha_s\alpha_t}{\alpha_r}}{\sqrt{\alpha'_s}\sqrt{\alpha'_t}} \tag{4.17}$$

is expanded in the series (4.15) with ϵ_n and $\varphi_n(t)$ of the same form as above but with $\pi K_r(\lambda_n,\lambda_n) = A_r(\lambda_n)\dot{B}_r(\lambda_n)$.

In both items the expressions for $K_r(\lambda_n, \lambda_n)$ stem from the following estimate of the reproducing kernel at the real diagonal

$$\pi K_r(w,w) = A_r(w)\dot{B}_r(w) - \dot{A}_r(w)B_r(w), \tag{4.18}$$

the derivatives are with respect to variable w. If w is not real, then $K_r(w,w) > 0$, but if is real, then the latter inequality holds true if, and only if, $E_r(w) \neq 0$.

It follows that the zeros of A and B are always real and separate each other, A and B cannot have common zeros, since E has no zeros at the origin. Moreover, A has no zero at the origin and B has a simple zero. Their non-zero zeros are always real, simple and interlacing.

Indeed, the zeros are necessarily real, because for non-real w it holds that $K(w,w) > 0$, i.e. $A(\bar{w})B(w) - B(\bar{w})A(w) > 0$ and neither A nor B can vanish. If E has no real zeros, then $|E(x)|^2 = A^2(x) + B^2(x) > 0$ on the real line, so that A and B cannot have common zeros. It follows from preceding that we again have $K(x,x) > 0$, i.e. $A(x)B'(x) - A'(x)B(x) > 0$ and only simple zeros are admitted. Moreover,

$$\frac{d}{dx}\frac{B(x)}{A(x)} = \frac{A(x)B'(x) - A'(x)B(x)}{A^2(x)} > 0$$

and the quotient B/A is strictly increasing when finite. Hence its zeros and poles are interlacing. [13]

Proof. (i) In view of Mercer's Theorem 4.4, it is required to prove $\phi_n(t)$ are eigenfunctions of the present linear operator that correspond to eigenvalues ϵ_n, i.e. the eigenfunctions and eigenvalues of the homogeneous Fredholm integral equation

$$\int_0^r k(s,t)\phi(s)\,ds = \epsilon\,\phi(t) \tag{4.19}$$

(assume $\epsilon \neq 0$ to exclude trivial solutions). The present kernel (4.16) turns this equation into

$$\int_0^r \alpha_{s\wedge t}\,b_s\,d\gamma_s = \epsilon\,b_t,$$

where $\sqrt{\alpha'_t}\phi(t) = b_t$ and $d\gamma_s = ds/\alpha'_s$ by assumption. Upon integrating the left-hand side by parts

$$\int_0^r \alpha_{s\wedge t}\,b_s\,d\gamma_s = \int_0^t \alpha_s\,b_s\,d\gamma_s + \alpha_t\int_t^r b_s\,d\gamma_s = \int_0^t d\alpha_u\int_u^r b_s d\gamma_s$$

we obtain

$$\int_0^t d\alpha_u\int_u^r b_s d\gamma_s = \epsilon\,b_t. \tag{4.20}$$

[13] If E does have real zeros, then these are zeros of A and B as well, since $|E(x)|^2 = A^2(\lambda) + B^2(\lambda)$ for real x. As is noticed above, in this case inequality $K(w,w) > 0$ is not necessarily strict and only conclusion we can draw is that zeros separate each other and multiplicities can differ by at most 1.

Differentiating the latter integral equation first with respect to $d\alpha$ and then $d\gamma$, we obtain

$$\frac{d^2 b_t}{d\gamma_t d\alpha_t} = -\frac{1}{\epsilon}\, b_t, \tag{4.21}$$

a differential equation of the Sturm–Liouville type. The solutions to this equation are sought under the boundary conditions $b_0 = 0$ and $b'_r = 0$. At hand of the differential equations (4.8) the solutions are available in terms of the underlying chain of de Branges functions $E_t(z) = A_t(z) - iB_t(z)$, $0 \le t \le r$, in the form $b_t = cB_t(z)$, with a suitable choice of argument z and rescaling constant $c \ne 0$. Any choice will satisfy the boundary condition $b_0 = 0$, since $E_0(z) = 1$ by assumption. The condition $b_r = 1$ will be satisfied by suitable choice of rescaling.

In order to determine required roots of the differential equation (4.21), restrict equation (4.8) to the real axis and select points $0 < \lambda_1 < \lambda_2 < \cdots$ such that $A_r(\lambda_n) = 0$. With this choice, (4.21) turns into

$$\frac{d^2 b_n(t)}{d\gamma_t d\alpha_t} = -\frac{1}{\epsilon_n}\, b_n(t) \tag{4.22}$$

where $1/\epsilon_n = \lambda_n^2$ and $b_n(t) = c_n B_t(\lambda_n)$, with non-zero constants c_n such that $b_n(r) = 1$. Hence $c_n = 1/B_r(\lambda_n)$. It can be verified that

$$\int_0^r b_m(t) b_n(t)\, d\gamma_t = -\delta_{mn}\frac{c_n^2}{2} B_r(\lambda_n)\dot{A}_r(\lambda_n) = -\frac{\delta_{mn}}{2}\frac{\dot{A}_r(\lambda_n)}{B_r(\lambda_n)}.$$

Indeed, the second equations in (4.6) and (4.7) show that since λ_n are zeros of $A_r(z)$, the elements $B_t(\lambda_n)$ of $L^2([0,r], d\gamma)$ are mutually orthogonal, with the squire norm

$$\|B(\lambda_n)\|_\gamma^2 := \int_0^r |B_t(\lambda_n)|^2\, d\gamma_t = -\frac{1}{2} B_r(\lambda_n)\dot{A}_r(\lambda_n).$$

Recall now relation $\sqrt{\alpha'_t}\phi(t) = b_t$ between the solutions to the integral equation (4.19) and the differential equation (4.21). Since we are looking for a countable number of solutions and therefore make use of index n, the relation is to be rewritten as $\sqrt{\alpha'_t}\phi_n(t) = b_n(t)$. The orthogonality property of the set of functions on the right-hand side of the latter identity, just verified, shows that the set of functions $\phi_n(t)$ are mutually orthogonal in $L^2[0,r]$, with the squire norm

$$\|\phi_n\|^2 := \int_0^r |\phi_n(t)|^2\, dt = -\frac{c_n^2}{2} B_r(\lambda_n)\dot{A}_r(\lambda_n) = \frac{K_r(\lambda_n,\lambda_n)}{2B_r^2(\lambda_n)} = -\frac{\dot{A}_r(\lambda_n)}{2B_r(\lambda_n)}.$$

Upon normalization $\varphi_n(t) = \phi_n(t)/\|\phi_n\|$, we get

$$\int_0^r k(s,t)\varphi_n(s) ds = \lambda_n^{-2}\, \varphi_n(t).$$

The proof is complete by verifying that the $\varphi_n(t)$ are of the required form, since

$$\varphi_n(t) = \frac{\phi_n(t)}{\|\phi_n(t)\|} = \frac{b_n(t)/\sqrt{\alpha_t'}}{\|\phi_n(t)\|} = \frac{c_n B_t(\lambda_n)}{\|\phi_n(t)\|\sqrt{\alpha_t'}}$$

with $c_n = 1/B_r(\lambda_n)$.
(ii) Differentiating the Fredholm equation (4.19) with kernel (4.17), first with respect to $d\alpha$ and then $d\gamma$, leads again to (4.21). If $0 < \lambda_1 < \lambda_2 < \cdots$ are such that $B_r(\lambda_n) = 0$, then equation (4.22) retains its form but with ϵ_n and $b_n(t)$ depending on these new points, i.e. $\epsilon_n = \lambda_n^{-2}$ and $b_n(t) = c_n\, B_t(\lambda_n)$. The boundary conditions to be satisfied are $b_n(0) = b_n(r) = 0$. They are automatically satisfied in view of the present choice of the points λ_n. The rescaling constants $c_n \neq 0$ will be used for normalization. Invoke again the Lagrange identities (4.6) to obtain

$$\int_0^r b_m(t) b_n(t)\, d\gamma_t = \delta_{mn} \frac{c_n^2}{2} A_r(\lambda_n) \dot{B}_r(\lambda_n).$$

Hence, in this case

$$\|\phi_n\|^2 := \int_0^r |\phi_n(t)|^2\, dt = \frac{c_n^2}{2} A_r(\lambda_n) \dot{B}_r(\lambda_n) = \frac{c_n^2}{2} K_r(\lambda_n, \lambda_n).$$

Choosing $c_n^{-2} = K_r(\lambda_n, \lambda_n)/2$, we identify $\varphi_n(t) = \phi_n(t)$ and write

$$\int_0^r k(s,t) \varphi_n(s) ds = \lambda_n^{-2}\, \varphi_n(t).$$

It is easily verified that the $\varphi_n(t)$ are of the required form

$$\varphi_n(t) = b_n(t)/\sqrt{\alpha_t'} = c_n B_t(\lambda_n)/\sqrt{\alpha_t'}.$$

The proof is complete. □

The expansions of Theorem 4.5 can be rewritten entirely in terms of the reproducing kernel of the underlying chain of spaces which under the present conditions is given by (4.5).

Corollary 4.6. *In the situation of Theorem 4.5 the reproducing kernels $K_t(w,z)$, $0 \le t \le r$, of the underlying chain of spaces satisfy*

$$K_{s\wedge t}(0,0) = \sum_n \frac{K_s(0,\lambda_n) K_t(\lambda_n, 0)}{K_r(\lambda_n, \lambda_n)} \tag{4.23}$$

where the λ_n are either zeros of $A_r(z)$ or $B_r(z)$ symmetrically spread about the origin.

Proof. The form (4.16) of the kernel $k(s,t)$ entails

$$\alpha_{s\wedge t} = 2\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2} \frac{B_s(\lambda_n)B_t(\lambda_n)}{\pi K_r(\lambda_n,\lambda_n)}$$

with zeros λ_n of $A_r(z)$, and since $\pi K_t(0,0) = \alpha_t$ and $\pi K_t(0,z) = B_t(z)/z$, this is equivalent to (4.23). To see that the latter holds true also with zeros of $B_r(z)$, one only needs to notice that 0 is one of the zeros and the term corresponding to this zero equals to $\alpha_s\alpha_t/\alpha_r$. □

4.1.2 Kernel (4.24)

In the present subsection Theorem 4.5 will be generalized. Working under the conditions of the latter theorem, we extend the statement to the kernels of the form

$$\sqrt{\alpha'_s}\sqrt{\alpha'_t}\, k(s,t) = \alpha_{s\wedge t} - \frac{< 1_s, \kappa >_\alpha < 1_t, \kappa >_\alpha}{\|\kappa\|_\alpha^2} \tag{4.24}$$

where κ's are non-constant continuously differentiable kernels in $L^2([0,r], d\alpha)$ (as $\kappa = 0$ brings us back to (4.16) and a non-zero constant κ to (4.17)). In the sequel we make use of the notation

$$g_t := < 1_t, \kappa >_\alpha \qquad g'_t := \frac{dg_t}{d\alpha_t} \tag{4.25}$$

just to simplify the exposition. On proving main Theorem 4.13, we shall follow the same succession of arguments as before. First we shall show that in the present case of kernel (4.24) the linear operator $\mathcal{K}$ of Mercer's Theorem 4.4 is determined by a Fredholm equation which is reducible to the non-homogeneous second order differential equation (4.26), subject to the boundary conditions (4.27). Then we shall show how the eigenvalues and eigenfunctions of the latter determine those of operator $\mathcal{K}$. To prepare the way for this, we begin with characterizing equation (4.26).

The next theorem will provide necessary information. The first part of this theorem gives the *Lagrange type* identity (4.28) for two independent solutions. The second part will describe eigenvalues and eigenfunctions in terms of underline chain of de Branges spaces, the same as in Theorem 4.5.

Theorem 4.7. *Let $m_t = diag[\alpha_t, \gamma_t]$, $t \in [0,r]$, be a diagonal structure function. Given a twice differentiable function g on interval $[0,r]$, define the non-homogeneous second order differential equation*

$$\frac{d^2 b_t}{d\gamma_t\, d\alpha_t} = -w^2\Big(b_t + \frac{d^2 g_t}{d\gamma_t\, d\alpha_t}\frac{< g, b >_\gamma}{\|g'\|_\alpha^2}\Big) \tag{4.26}$$

subject to the boundary conditions

$$b_0 = 0 \qquad \int_0^r b_t \, d\gamma_t = 0. \tag{4.27}$$

If $b_t(w)$ *and* $b_t(z)$ *are two eigenfunctions of* (4.26) *corresponding to the two different* (real) *eigenvalues* w *and* z*, then*

$$\int_0^r b_t(w)\, b_t(z)\, d\gamma_t = \frac{b_r(z)\,\beta_r(w) - b_r(w)\,\beta_r(z)}{z^2 - w^2} \tag{4.28}$$

where $\beta_r(w) = b_r'(w) + w^2 g_r' \dfrac{< g, b(w) >_\gamma}{\|g'\|_\alpha^2}$ *with* $b_r' = \dfrac{db_t}{d\alpha_r}$ *and* $g_r' = \dfrac{dg_t}{d\alpha_r}$.

If $w = z$*, then*

$$\|b(w)\|_\gamma^2 = \frac{1}{2w} b_r(w) \dot{\beta}_r(w)$$

where $\dot{\beta}_r(w) = db_r(w)/dw$.

Proof. It follows from (4.26) that

$$b_t'(w) = -w^2 \Big(\int_0^t b_u(w) d\gamma_u + g_t' \frac{< g, b(w) >_\gamma}{\|g'\|_\alpha^2} \Big).$$

Therefore

$$-w^2 \int_0^t b_u(w)\, d\gamma_u = b_t'(w) + w^2 g_t' \frac{< g, b(w) >_\gamma}{\|g'\|_\alpha^2} =: \beta_t(w)$$

and

$$\begin{aligned} -z^2 \int_0^r b_t(w)\, b_t(z)\, d\gamma_t &= \int_0^r b_t(w)\, d\beta_t(z) \\ &= b_t(w)\, \beta_t(z) \Big|_0^r - \int_0^r \beta_t(z) b_t'(w) d\alpha_t \end{aligned}$$

by integrating by parts. Invoking the boundary conditions, we obtain

$$-z^2 \int_0^r b_t(w)\, b_t(z)\, d\gamma_t = b_r(w)\, \beta_r(z) - \int_0^r b_t'(z) b_t'(w) d\alpha_t.$$

Indeed, identity $< \beta(z), b'(w) >_\alpha = < b'(z), b'(w) >_\alpha$ comes from the second of boundary conditions in this manner

$$\begin{aligned} < g', b'(w) >_\alpha &= -w^2 \Big(\int_0^r g_t' d\alpha_t \int_0^t b_u(w) d\gamma_u + < g, b(w) >_\gamma \Big) \\ &= -w^2 \Big(\int_0^r b_u(w)(g_r - g_u) d\gamma_u + < g, b(w) >_\gamma \Big) = 0. \end{aligned}$$

Interchanging w and z we also obtain

$$-w^2 \int_0^r b_t(z)\, b_t(w)\, d\gamma_t = b_r(z)\, \beta_r(w) + \int_0^r b_t'(w) b_t'(z) d\alpha_t,$$

so that (4.28) follows by taking the difference of these two equations. The norm of b_t in the metric of $L^2([0,r], d\gamma)$ is obtained from (4.28) by l'Hôpital's rule. □

By definition of $\beta_t(w)$, the second of boundary conditions (4.27) is equivalent to

$$\beta_r(w) = b_r'(w) + w^2 g_r' \frac{< g, b(w) >_\gamma}{\|g'\|_\alpha^2} = 0. \tag{4.29}$$

As is said, the second part of the preceding theorem will provide eigenvalues and eigenfunctions in terms of the chain of de Branges spaces $\mathcal{H}(E_t), 0 \le t \le r$, the same as in the preceding section. The density of its diagonal structure function m is such that $\alpha_t' = 1/\gamma_t'$. Moreover, it was pointed out that the structure function determines a chain of de Branges matrices $M_t(z)$ of unit determinant, $0 \le t \le r$, cf. equation (4.9), whose entries in the upper and lower row are related to each other by (4.4) and (4.11), respectively.

The proof of the next statement makes use of the second equations in (4.8) and (4.10) regarding the linearly independent entire functions $B_t(z)$ and $D_t(z)$, real for a real z. Due to these equations, both satisfy the homogeneous part of equation (4.26) and their linear combination (4.30) satisfies the second order inhomogeneous equation (4.31).

Lemma 4.8. *If $M_t(z), 0 \le t \le r$, is the chain of de Branges matrices associated as above with the underlying chain of de Branges spaces, then the linear combination*

$$G_t(z) = D_t(z) \int_0^t A_u(z) dg_u - B_t(z) \int_0^t C_u(z) dg_u, \tag{4.30}$$

satisfies

$$\frac{d^2 G_t(z)}{d\gamma_t d\alpha_t} = -z^2 G_t(z) + \frac{d^2 g_t}{d\gamma_t d\alpha_t}. \tag{4.31}$$

Proof. Since $\det M_t(z) = 1$ and the derivatives of $B_t(z)$ and $D_t(z)$ with respect to $d\alpha_t$ are equal to $zA_t(z)$ and $zC_t(z)$, respectively, the derivative of $G_t(z)$ with respect to $d\alpha_t$ is equal to

$$\begin{aligned} & g_t' + z\Big(C_t(z) \int_0^t A_u(z) dg_u - A_t(z) \int_0^t C_u(z) dg_u\Big) \\ & = g_t' - z\Big(C_t(z) \int_0^t g_u dA_u(z) - A_t(z) \int_0^t g_u dC_u(z)\Big). \end{aligned} \tag{4.32}$$

Differentiating the latter with respect to $d\gamma$, one obtains $\dfrac{d^2 g_t}{d\gamma_t d\alpha_t}$ minus z times

$$\frac{dC_t(z)}{d\gamma_t}\int_0^t g_u dA_u(z) - \frac{dA_t(z)}{d\gamma_t}\int_0^t g_u dC_u(z) - g_t\Big(C_t(z)\frac{dA_t(z)}{d\gamma_t} - A_t(z)\frac{dC_t(z)}{d\gamma_t}\Big).$$

The last of these tree terms is equal to $zg_t\big(C_t(z)B_t(z) - A_t(z)D_t(z)\big) = -zg_t$, since the determinant of the de Branges matrix is equal to 1. The sum of the other two terms is equal to z times

$$\begin{aligned}
&-D_t(z)\int_0^t g_u dA_u(z) + B_t(z)\int_0^t g_u dC_u(z)\\
&= -D_t(z)\Big(g_t A_t(z) - \int_0^t A_u(z) dg_u\Big) + B_t(z)\Big(g_t C_t(z) - \int_0^t C_u(z) dg_u\Big)\\
&= -g_t + G_t(z).
\end{aligned}$$

Combining these results, we obtain (4.31). □

The next statement makes use of the following

Corollary 4.9. *In the situation of the lemma*

$$\begin{aligned}
-z < 1, G(z) >_\gamma &= C_r(z)\int_0^r A_t(z) dg_t - A_r(z)\int_0^r C_t(z) dg_t\\
-z < g, G(z) >_\gamma &= \int_0^r g_t dC_t(z)\int_0^t A_u(z) dg_u - \int_0^r g_t dA_t(z)\int_0^t C_u(z) dg_u.
\end{aligned}$$

Proof. By (4.31)

$$-z^2 < 1, G(z) >_\gamma = < 1, \frac{d^2 G(z)}{d\gamma d\alpha} >_\gamma - < 1, \frac{d^2 g}{d\gamma d\alpha} >_\gamma = G'_r(z) - g'_r$$

where G' is the derivative of G with respect to $d\alpha$. The first of the required formulas follows from (4.32). The second one follows from

$$\begin{aligned}
-z < g, G(z) >_\gamma &= \int_0^r g_t d\Big(C_t(z)\int_0^t A_u(z) dg_u - A_t(z)\int_0^t C_u(z) dg_u\Big)\\
&= -z\Big(D_t(z)\int_0^t A_u(z) dg_u - B_t(z)\int_0^t C_u(z) dg_u\Big)
\end{aligned}$$

by definition (4.30). □

Now, we are in a position to complete Theorem 4.7.

Theorem 4.7 (continued). *Let* $w_1, w_2, \cdots$ *be positive roots of the determinantal equation*

$$w^2 \begin{vmatrix} < 1, G(w) >_\gamma & < g, G(w) >_\gamma \\ < 1, B(w) >_\gamma & < g, B(w) >_\gamma \end{vmatrix} = \|g'\|_\alpha^2 < 1, B(w) >_\gamma \tag{4.33}$$

arranged in ascending order of magnitude. They define eigenvalues $\epsilon_n = 1/w^2$ *for equation* (4.26)*, subject to the boundary conditions* (4.27)*. The corresponding eigenfunctions* $b_t(w_n)$ *are given by*

$$b_t(w) = c(w)\Big(B_t(w) - \frac{< 1, B(w) >_\gamma}{< 1, G(w) >_\gamma} G_t(w)\Big) \tag{4.34}$$

where c *is a normalization constant, making* b_t *of unit norm in the metric of* $L^2([0, r], d\gamma)$*, i.e.*

$$1/c^2(w) = \int_0^r \big(B_t(w) - \frac{< 1, B(w) >_\gamma}{< 1, G(w) >_\gamma} G_t(w)\big)^2 d\gamma_t.$$

Proof. The determinantal equation (4.33) is well-defined, since $B_t(w)$ and $D_t(w)$ are linearly independent, as well as 1 and g (by assumption made in the beginning of the present section that g is not a constant). Note that $w = 0$ is not a root, since $M_t(0) = I$ and $G_t(0) = g_t$ identically. Indeed, the right-hand side of (4.33), divided through w^2, is a positive number, since $w^{-2} < 1, B(w) >_\gamma$ tends to $\int_0^r \alpha_t d\gamma_t$ as $w \searrow 0$, while the left-hand side vanishes. The boundary conditions (4.27) are easily verified. Due to (4.31), differentiating twice both sides of (4.34) yields

$$\begin{aligned} \frac{d^2 b_t(w)}{d\gamma_t d\alpha_t} &= c(w)\Big(\frac{d^2 G_t(w)}{d\gamma_t d\alpha_t} - \frac{< 1, B(w) >_\gamma}{< 1, G(w) >_\gamma} \frac{d^2 G_t(w)}{d\gamma_t d\alpha_t}\Big) \\ &= -w^2 b_t(w) - c(w) \frac{< 1, B(w) >_\gamma}{< 1, G(w) >_\gamma} \frac{d^2 g_t}{d\gamma_t d\alpha_t}. \end{aligned}$$

Compare this with (4.31). It is seen we need prove identity

$$w^2 \frac{< g, b >_\gamma}{\|g'\|_\alpha^2} = c(w) \frac{< 1, B(w) >_\gamma}{< 1, G(w) >_\gamma}$$

whenever w satisfies (4.33). But this is easily obtained by calculating $< g, b >_\gamma$ from (4.34) in this manner

$$\frac{< g, b >_\gamma}{c(w)} =< g, B(w) >_\gamma - \frac{< 1, B(w) >_\gamma}{< 1, G(w) >_\gamma} < g, G(w) >$$

and this, combined with (4.33), proves the identity. □

Theorem 4.7 provides a key argument proving the next statement of Mercer's type.

Theorem 4.10. *The kernel of the current subsection that is defined by* (4.24) *on* $[0,r] \times [0,r]$ *does expand in series* (4.15) *with* $\epsilon_n = 1/w_n^2$*, where* w_n *are positive roots of the determinantal equation* (4.33)*, and with* $\varphi_n(t) = b_t(w_n)/\sqrt{\alpha'_t}$*, where* $b_t(w)$ *is given by* (4.34)*. The series converges absolutely and uniformly.*

Proof. The linear operator $\mathcal{K}$ of Mercer's Theorem 4.4 is now determined by means of the Fredholm integral equation

$$\int_0^r k(s,t)\varphi_s ds = \epsilon\varphi_t$$

with kernel

$$\sqrt{\alpha'_s}\sqrt{\alpha'_t}\, k(s,t) = \alpha_{s\wedge t} - \frac{< 1_s, g' >_\alpha < 1_t, g' >_\alpha}{\|g'\|_\alpha^2},$$

where g is so as in (4.25). With the notation $b_t := \sqrt{\alpha'_t}\,\varphi_t$, rewrite the latter equation as

$$\int_0^r \sqrt{\alpha'_s}\sqrt{\alpha'_t}\, k(s,t)\, b_s\, d\gamma_s = \epsilon b_t.$$

By making use of (4.20) we obtain

$$\int_0^t d\alpha_u \int_u^r b_s d\gamma_s = \epsilon\, b_t + \frac{g_t}{\|g'\|_\alpha^2} < g, b >_\gamma .$$

Set $t = 0$ to verify the first of boundary conditions $b_0 = 0$ ($g_0 = 0$ by assumption). Differentiate both sides with respect to $d\alpha_t$

$$\int_t^r b_s d\gamma_s = \epsilon\, b'_t + g'_t \frac{< g, b >_\gamma}{\|g'\|_\alpha^2}$$

where $b'_t = db_t/d\alpha_t$ as before. Set $t = r$ to verify the second boundary condition (4.29).

Differentiate once more with respect to $d\gamma_t$, this time. We obtain

$$-b_t = \epsilon\, \frac{d^2 b_t}{d\gamma_t\, d\alpha_t} + \frac{d^2 g_t}{d\gamma_t d\alpha_t} \frac{< g, b >_\gamma}{\|g'\|_\alpha^2}$$

or

$$\frac{d^2 b_t}{d\gamma_t\, d\alpha_t} = -w^2 \Big(b_t + \frac{d^2 g_t}{d\gamma_t\, d\alpha_t} \frac{< g, b >_\gamma}{\|g'\|_\alpha^2} \Big)$$

with $\epsilon = w^{-2}$. This second order inhomogeneous differential equation is (4.26), of course. Therefore, Theorem 4.7 determines the eigenvalues and corresponding eigenfunctions for the present linear operator $\mathcal{K}$ and our claim becomes a particular case of the general Theorem 4.4. □

In general, the entire functions (4.30) depend rather complicatedly on kernel κ defining the positive definite kernel (4.24). As is assumed at the beginning of this subsection, constant κ's are excluded and this excludes the case $dg = d\alpha$. Matters are relatively simple when $\kappa_t = \gamma_t$.

Example 4.11. Consider a positive definite kernel on $[0, r] \times [0, r]$

$$\sqrt{\alpha'_s}\sqrt{\alpha'_t}\, k(s,t) = \alpha_{s\wedge t} - \frac{< 1_s, \gamma >_\alpha < 1_t, \gamma >_\alpha}{\|\gamma\|_\alpha^2}.$$

By definition (4.25), in this case $dg_t/d\alpha_t = \gamma_t$. Therefore the second term on the right-hand side of (4.31) is independent of t and equals to 1. Definition (4.30) reduces to

$$G_t(z) = \frac{1 - D_t(z)}{z^2}, \tag{4.35}$$

which certainly satisfies the differential equation (4.31). Indeed, calculate

$$\begin{aligned} G_t(z) &= D_t(z)\int_0^t \gamma_u A_u(z) d\alpha_u - B_t(z)\int_0^t \gamma_u C_u(z) d\alpha_u \\ &= \frac{1}{z}\Big(D_t(z)\int_0^t \gamma_u dB_u(z) - B_t(z)\int_0^t \gamma_u d(D_u(z) - 1)\Big) \end{aligned}$$

by integrating by parts

$$\begin{aligned} zG_t(z) &= D_t(z)\big(B_t(z)\gamma_t - \int_0^t B_u(z) d\gamma_u\big) \\ &- B_t(z)\big((D_t(z) - 1)\gamma_t - \int_0^t (D_u(z) - 1) d\gamma_u\big) \\ &= D_t(z)\big(B_t(z)\gamma_t - \frac{1 - A_t(z)}{z}\big) - B_t(z)\big((D_t(z) - 1)\gamma_t + \frac{C_t(z)}{z} + \gamma_t\big). \end{aligned}$$

This proves (4.35). Since $g_t = \int_0^t \gamma_u d\alpha_u$ by definition, we have

$$g_t = \frac{D_t(z) - 1}{z^2}\Big|_{z=0} = -G_t(0)$$

and the determinantal equation (4.33) takes the form

$$w^2 \begin{vmatrix} < 1, G(w) >_\gamma & - < G(0), G(w) >_\gamma \\ < 1, B(w) >_\gamma & - < G(0), B(w) >_\gamma \end{vmatrix} = \|\gamma\|_\alpha^2 < 1, B(w) >_\gamma .$$

In the upper row $w^2 < 1, G(w) >_\gamma = \gamma_r + C_r(w)/w$ and $w^2 < G(0), G(w) >_\gamma = \int_0^r (G_t(0)(1 - D_t(w)) d\gamma_t$. □

4.1.3 Kernels (4.39) and (4.40)

In order to apply similar arguments also to the lower entry of the structure function, the roles of α and γ are to be interchanged. This is described with the help of rotations through angle $\pi/2$ about the origin so that

$$\mathrm{diag}[\gamma_t, \alpha_t] = g_{\pi/2}^{\top} \mathrm{diag}[\alpha_t, \gamma_t] g_{\pi/2}. \tag{4.36}$$

As is pointed out in Remark 4.3, such rotation (4.13) of the de Branges matrices transfers the role of the entries in the upper row $A_t(z)$ and $B_t(z)$ to those of the lower row $D_t(z)$ and $-C_t(z)$. The latter entire functions form the chain of de Branges functions $\tilde{E}_t(z) = D_t(z) + iC_t(z), t > 0$, and the reproducing kernels (4.14) in the corresponding spaces. With the system of integral equations (4.11) in stead of (4.4), the statement of Theorem 4.2 will regard kernel (4.14), in stead of (4.5). The Lagrange identities (4.6) will turn into (4.12) for all complex numbers w and z.

In terms of this new kernel (4.14) one may rephrase Corollary 4.6 in this manner.
In the situation of Theorem 4.5 the kernels $Q_t(w, z)$, $0 \le t \le r$, associated with the underlying chain of spaces by (4.14)*, satisfy*

$$Q_{s\wedge t}(0,0) = \sum_n \frac{Q_s(0,\lambda_n) Q_t(\lambda_n, 0)}{Q_r(\lambda_n, \lambda_n)} \tag{4.37}$$

where the λ_n are either zeros of $D_r(z)$ or $C_r(z)$ symmetrically spread about the origin.

This claim will be confirmed at hand of the next theorem. In both items of the theorem kernel (4.14) is evaluated at the real diagonal, say at a real point w, and it holds that

$$\pi Q_r(w,w) = \dot{D}_r(w) C_r(w) - D_r(w) \dot{C}_r(w) \tag{4.38}$$

where the derivatives are with respect to variable w, like in (4.18). Since in the present situation the de Branges functions $\tilde{E}_t(z)$, $t > 0$, have no zeros at the origin, the characterization of the zeros of A and B just after formula (4.18) applies to the zeros of D and C as well. The former has no zero at the origin and the latter has a simple zero. Their non-zero zeros are always real, simple and interlacing.

Theorem 4.12. *Let a given chain of de Branges spaces $\mathcal{H}(E_t), 0 \le t \le r$, satisfy the conditions of Theorem 4.2.*
(i) If points on the real axes $0 < \lambda_1 < \lambda_2 < \cdots$ are such that $D_r(\lambda_n) = 0$, then kernel

$$k(s,t) = \frac{\gamma_{s\wedge t}}{\sqrt{\gamma'_s}\sqrt{\gamma'_t}} \tag{4.39}$$

is expanded in series (4.15) *with $\epsilon_n = \lambda_n^{-2}$ and*

$$\varphi_n(t) = \frac{C_t(\lambda_n)}{\sqrt{\gamma'_t}} \sqrt{\frac{2}{\pi Q_r(\lambda_n, \lambda_n)}}$$

where $\pi Q_r(\lambda_n,\lambda_n) = \dot{D}_r(\lambda_n)C_r(\lambda_n)$.
(ii) If points on the real axes $0<\lambda_1<\lambda_2<\cdots$ *are such that* $C_r(\lambda_n)=0$, *then kernel*

$$k(s,t) = \frac{\gamma_{s\wedge t} - \dfrac{\gamma_s\gamma_t}{\gamma_r}}{\sqrt{\gamma_s'}\sqrt{\gamma_t'}} \tag{4.40}$$

is expanded in series (4.15) *with* ϵ_n *and* $\varphi_n(t)$ *of the same form as above but with* $\pi Q_r(\lambda_n,\lambda_n) = -D_r(\lambda_n)\dot{C}_r(\lambda_n)$.

In fact, this is a reformulation of Theorem 4.5 in accord with the rotations (4.13) and (4.36). The proof requires same adaptations and is not difficult to carry out.

Proof. (i) Upon differentiating first with respect to $d\gamma$ and then $d\alpha$, the homogeneous Fredholm integral equation (4.19) with kernel (4.39) turns into the differential equation of the Sturm–Liouville type

$$\frac{d^2b_t}{d\alpha_t d\gamma_t} = -\frac{1}{\epsilon}\, b_t, \tag{4.41}$$

where b_t is rescaling $\sqrt{\gamma_t'}\phi(t) = b_t$ of function $\phi(t)$ involved in the Fredholm equation. The solutions to this equation are sought under the boundary conditions $b_0=0$ and $b_r'=0$. The first of equations (4.8) offers solution $A_t(z)$, but this fails to satisfy the boundary conditions. We shall see in a moment that another possibility, the first of equations (4.10), gives a required solution. Looking for a solution to (4.41) in the form $b_t = c\,C_t(z)$, with $\epsilon = 1/z^2$, we shall choose suitably argument z and the rescaling constant $c\neq 0$. Any choice will satisfy the boundary condition $b_0=0$, since $\tilde{E}_0(z)=1$ by assumption. Another condition $b_r=1$ is to be guaranteed by rescaling. In the first of equations (4.10) restrict argument z to real axis and select points $0<\lambda_1<\lambda_2<\cdots$ such that $D_r(\lambda_n)=0$. With this choice, (4.41) turns into

$$\frac{d^2b_n(t)}{d\alpha_t d\gamma_t} = -\frac{1}{\epsilon_n}\, b_n(t) \tag{4.42}$$

where $1/\epsilon_n = \lambda_n^2$ and $b_n(t) = c_nC_t(\lambda_n)$, with non-zero constants c_n such that $b_n(r)=1$. Hence $c_n = 1/C_r(\lambda_n)$. It follows from (4.12) that

$$\int_0^r b_m(t)b_n(t)\,d\alpha_t = \delta_{mn}\frac{c_n^2}{2}C_r(\lambda_n)\dot{D}_r(\lambda_n) = \frac{\delta_{mn}}{2}\frac{\dot{D}_r(\lambda_n)}{C_r(\lambda_n)}.$$

Recall now relation $\sqrt{\gamma_t'}\phi(t) = b_t$ between the solutions to the integral equation (4.19) and the differential equation (4.41). Since we are looking for a countable number of solutions and therefore make use of index n, the relation is to be rewritten as $\sqrt{\gamma_t'}\phi_n(t) = b_n(t)$. The orthogonality property of the set of functions on the right-hand side of this identity (seen from the latter display) shows that the set of functions $\phi_n(t)$ are mutually orthogonal in $L^2[0,r]$, with the squire norm

$$\|\phi_n\|^2 = \int_0^r |\phi_n(t)|^2dt = \int_0^r |b_n(t)|^2\,d\alpha_t = \frac{\dot{D}_r(\lambda_n)}{2C_r(\lambda_n)}.$$

Upon normalization $\varphi_n(t) = \phi_n(t)/\|\phi_n\|$, we get

$$\int_0^r k(s,t)\varphi_n(s)ds = \lambda_n^{-2}\,\varphi_n(t).$$

The proof is complete, since the $\varphi_n(t)$ are of the required form

$$\varphi_n(t) = \frac{\phi_n(t)}{\|\phi_n(t)\|} = \frac{b_n(t)/\sqrt{\gamma_t'}}{\|\phi_n(t)\|} = \frac{C_t(\lambda_n)/\sqrt{\gamma_t'}}{C_r(\lambda_n)}\sqrt{\frac{2C_r(\lambda_n)}{\dot{D}_r(\lambda_n)}}.$$

(ii) Differentiating the Fredholm equation (4.19) with kernel (4.40), first with respect to $d\alpha$ and then $d\gamma$, leads again to (4.41). If $0 < \lambda_1 < \lambda_2 < \cdots$ are such that $C_r(\lambda_n) = 0$, then equation (4.42) retains its form but with ϵ_n and $b_n(t)$ depending on these new points, i.e. $\epsilon_n = \lambda_n^{-2}$ and $b_n(t) = c_n\, C_t(\lambda_n)$. The boundary conditions $b_n(0) = b_n(r) = 0$ are automatically satisfied. The rescaling constants $c_n \neq 0$ will be used for normalization. Invoke again the Lagrange identities (4.12) to obtain

$$\int_0^r b_m(t)b_n(t)\,d\alpha_t = -\delta_{mn}\frac{c_n^2}{2}D_r(\lambda_n)\dot{C}_r(\lambda_n).$$

Hence, in this case

$$\|\phi_n\|^2 := \int_0^r |\phi_n(t)|^2\,dt = -\frac{c_n^2}{2}D_r(\lambda_n)\dot{C}_r(\lambda_n) = \frac{c_n^2}{2}Q_r(\lambda_n,\lambda_n).$$

Choosing $c_n^{-2} = Q_r(\lambda_n,\lambda_n)/2$, we identify $\varphi_n(t) = \phi_n(t)$ and write

$$\int_0^r k(s,t)\varphi_n(s)ds = \lambda_n^{-2}\,\varphi_n(t).$$

It is easily verified that the $\varphi_n(t)$ are of the required form

$$\varphi_n(t) = b_n(t)/\sqrt{\gamma_t'} = c_n C_t(\lambda_n)/\sqrt{\gamma_t'}.$$

The proof is complete. $\square$

4.1.4 **Kernel** (4.43)

As in the preceding subsection, the idea is to substitute in (4.43) the upper entry of the structure function with the lower entry. So, Mercer's Theorem 4.13 proved below will regard the kernel

$$\sqrt{\gamma_s'}\sqrt{\gamma_t'}\,k(s,t) = \gamma_{s\wedge t} - \frac{<1_s,\kappa>_\gamma<1_t,\kappa>_\gamma}{\|\kappa\|_\gamma^2} \tag{4.43}$$

where κ's are now non-constant continuously differentiable kernels in $L^2([0,r],d\gamma)$. Accordingly, the notations (4.25) will turn into

$$g_t := < 1_t, \kappa >_\gamma \qquad g_t' := \frac{dg_t}{d\gamma_t}. \tag{4.44}$$

Moreover, in the forthcoming proof all differentials with respect to $d\gamma$ will receive the similar abbreviation, say $db_t/d\gamma_t = b_t'$ and so forth.

Theorem 4.13. *The kernel of the current subsection defined by* (4.43) *on* $[0,r]\times[0,r]$ *does expand in the absolutely and uniformly convergent series* (4.15) *with* ϵ_n *and* $\varphi_n(t)$ *defined as follows:*
(a) $\epsilon_n = 1/w_n^2$*, where the* w_n *are positive roots of the determinantal equation*

$$w^2 \begin{vmatrix} < 1, \tilde{G}(w) >_\alpha & < g, \tilde{G}(w) >_\alpha \\ < 1, C(w) >_\alpha & < g, C(w) >_\alpha \end{vmatrix} = \|g'\|_\gamma^2 < 1, C(w) >_\alpha \tag{4.45}$$

arranged in ascending order of magnitude and the entire functions $\tilde{G}_t(z)$ *are defined by*

$$\tilde{G}_t(z) = A_t(z)\int_0^t D_u(z)dg_u - C_t(z)\int_0^t B_u(z)dg_u. \tag{4.46}$$

(b) $\varphi_n(t) = b_t(w_n)/\sqrt{\alpha_t'}$*, where* $b_t(w)$ *is given by*

$$b_t(w) = c(w)\Big(C_t(w) - \frac{< 1, C(w) >_\alpha}{< 1, \tilde{G}(w) >_\alpha}\tilde{G}_t(w)\Big) \tag{4.47}$$

with a normalization constant c, making b_t *of unit norm in the metric of* $L^2([0,r],d\alpha)$*, i.e.*

$$1/c^2(w) = \int_0^r \big(C_t(w) - \frac{< 1, C(w) >_\alpha}{< 1, \tilde{G}(w) >_\alpha}\tilde{G}_t(w)\big)^2 d\alpha_t.$$

Proof. The entire functions (4.46) are derived from (4.30) by the replacements $A_t(z) \leftrightarrow D_t(z)$ and $B_t(z) \leftrightarrow -C_t(z)$. Lemma 4.8 suggests

$$\frac{d^2\tilde{G}_t(z)}{d\alpha_t d\gamma_t} = -z^2\tilde{G}_t(z) + \frac{d^2 g_t}{d\alpha_t d\gamma_t}, \tag{4.48}$$

cf. (4.31). To confirm this, take the derivative of $\tilde{G}_t(z)$ with respect to $d\gamma$ to get

$$g_t' - z\Big(B_t(z)\int_0^t g_u dD_u(z) - D_t(z)\int_0^t g_u dB_u(z)\Big)$$

which coincides with (4.32) upon the required replacements. Differentiating the latter with respect to $d\alpha$, one obtains $\dfrac{d^2 g_t}{d\alpha_t d\gamma_t}$ minus z times

$$\frac{dB_t(z)}{d\alpha_t}\int_0^t g_u dD_u(z) - \frac{dD_t(z)}{d\alpha_t}\int_0^t g_u dB_u(z) - g_t\Big(B_t(z)\frac{dD_t(z)}{d\alpha_t} - D_t(z)\frac{dB_t(z)}{d\alpha_t}\Big).$$

The last of these tree terms is equal to $zg_t\big(C_t(z)B_t(z) - A_t(z)D_t(z)\big) = -zg_t$, since the determinant of the de Branges matrix is equal to 1. The sum of the other two terms is equal to z times

$$\begin{aligned} &-A_t(z)\int_0^t g_u dD_u(z) + C_t(z)\int_0^t g_u dB_u(z) \\ &= - A_t(z)\Big(g_t D_t(z) - \int_0^t D_u(z)dg_u\Big) + C_t(z)\Big(g_t B_t(z) - \int_0^t B_u(z)dg_u\Big) \\ &= - g_t + \tilde{G}_t(z). \end{aligned}$$

So, (4.48) is confirmed. It provides a key argument for the remainder of the proof.

The linear operator $\mathcal{K}$ of Mercer's Theorem 4.4 is now determined by means of the Fredholm integral equation

$$\int_0^r k(s,t)\varphi_s ds = \epsilon\varphi_t$$

with kernel (4.43) which, rewritten in terms of (4.44), takes the form

$$\sqrt{\gamma_s'}\sqrt{\gamma_t'}\,k(s,t) = \gamma_{s\wedge t} - \frac{< 1_s, g' >_\gamma < 1_t, g' >_\gamma}{\|g'\|_\gamma^2}.$$

With the notation $b_t := \sqrt{\gamma_t'}\,\varphi_t$, rewrite the Fredholm equation as

$$\int_0^r \sqrt{\gamma_s'}\sqrt{\gamma_t'}\,k(s,t)\,b_s\,d\alpha_s = \epsilon b_t.$$

By making use of (4.20) we obtain

$$\int_0^t d\gamma_u \int_u^r b_s d\alpha_s = \epsilon\, b_t + \frac{g_t}{\|g'\|_\gamma^2} < g, b >_\alpha .$$

Set $t = 0$ to verify the first of boundary conditions $b_0 = 0$ ($g_0 = 0$ by assumption). Differentiate both sides with respect to $d\gamma$

$$\int_t^r b_s d\alpha_s = \epsilon\, b_t' + g_t'\frac{< g, b >_\alpha}{\|g'\|_\gamma^2}$$

where $b_t' = db_t/d\gamma_t$ as before. Set $t = r$ to verify the second boundary condition.

Differentiate once more with respect to $d\alpha$, this time. We obtain

$$-b_t = \epsilon\,\frac{d^2 b_t}{d\alpha_t\, d\gamma_t} + \frac{d^2 g_t}{d\alpha_t d\gamma_t}\frac{< g, b >_\gamma}{\|g'\|_\alpha^2}$$

or

$$\frac{d^2 b_t}{d\alpha_t\, d\gamma_t} = -w^2\Big(b_t + \frac{d^2 g_t}{d\alpha_t\, d\gamma_t}\frac{< g, b >_\alpha}{\|g'\|_\gamma^2}\Big)$$

with $\epsilon = w^{-2}$. This second order inhomogeneous differential equation is (4.26) with α and γ interchanged. Therefore, an adjusted version of Theorem 4.7 determines the eigenvalues and corresponding eigenfunctions for the present linear operator $\mathcal{K}$. Indeed, an adaptation entails substituting (4.31) with (4.48) and two equations (4.33) and (4.34) determining the eigenvalues and eigenfunctions with (4.45) and (4.47). The proof is complete. $\square$

4.2 Fundamental martingales

The preceding results are intended to derive the Karhunen-Loève expansions for certain Gaussian processes associated with double-sided si-processes. A double-sided si-process X is defined by the spectral representation (1.2) of its covariance function, where the arguments s and t are allowed to take any value on a whole real axis. Therefore process X can be represented by its even and odd parts $X_t = \operatorname{sign}(t)\, X^e_{|t|} + X^o_{|t|}$. These even and odd processes emerge from the origin and develop to the right, having moving average representations in the sense of Theorem 2.9. The Wiener integrals (2.40) of this theorem is now simplified to

$$X^e_t = \int_0^t \varphi dM^e \qquad X^o_t = \int_0^t \psi dM^o,$$

$t \geq 0$, since in Theorem 4.2 and thereafter a chain of de Branges spaces of exponential type which *determines* the given si-process has a structure function such that $\alpha'_t = 1/\gamma'_t$, which means the corresponding type is $\tau_t = t$. The even and odd fundamental martingales M^e and M^o that appear in the latter display are defined with the help of the spectral isometry (2.24) – they are mutually independent Gaussian processes defined on the same probability space as X, generating the same filtration as X^e and X^o, respectively, with the second order properties

$$\begin{aligned} \mathbb{E}(M^e_s M^e_t) &= \int \frac{B_s(\lambda)B_t(\lambda)}{\lambda^2}\mu(d\lambda) = \pi\alpha_{s\wedge t} \\ \mathbb{E}(M^o_s M^o_t) &= \int \frac{\big(A_s(\lambda)-1\big)\big(A_t(\lambda)-1\big)}{\lambda^2}\mu(d\lambda) = \pi\gamma_{s\wedge t}, \end{aligned} \tag{4.49}$$

cf. (2.25). The present section obtains KL-series expansions for processes allied to both M^e and M^o, first to the even and then to the odd one.

4.2.1 Even martingales

In Section 3.4 we apply the sampling formulas to obtain the generalized PW-series for both fundamental martingales. The result is formulated as Theorem 3.7, item (i). The expansion for the even fundamental martingale of this theorem is in a direct relation with the KL-expansion of the following statement.

Theorem 4.14. *Let X be a double-sided si-process determined by a chain of de Branges spaces of exponential type so as in Theorem 4.2. Let M^e be the fundamental martingale which defines the moving average representation of the even part of X, with the second order moments* (4.49). *Then process $M^e_t/\sqrt{\alpha'_t}$, $0 \le t \le r$, decomposes in KL-series* (4.3)

$$\frac{1}{\sqrt{\alpha'_t}} M^e_t = \sum_{n=1}^{\infty} \omega_n \varphi_n(t)\, \xi_n$$

where the ξ_n are i.i.d. Gaussian random variables, $\{\varphi_n(t) = B_t(\lambda_n)/\sqrt{\alpha'_t}\}$ is an orthonormal basis in $L^2[0,r]$, and

$$\omega_n = \frac{1}{\lambda_n}\sqrt{\frac{2}{\pi K_r(\lambda_n, \lambda_n)}}$$

with the positive zeros λ_n of $A_r(z)$.

Proof. In virtue of Theorem 4.1, it suffices to verify the covariance function of process $M^e_t/\sqrt{\alpha'_t}$ is of the form (4.16). But this is clear by the second order property (2.25) of the even fundamental martingale. □

The theorem just proved generalizes the classical Karhunen–Loève expansion (4.1). Due to Theorem 4.5, item *(ii)*, it is equally easy to generalize also expansion (4.2) of Brownian bridge.

Theorem 4.15. *In the situation of Theorem 4.14, the martingale bridge*

$$B^e_t = M^e_t - \frac{\alpha_t}{\alpha_r} M^e_r, \tag{4.50}$$

$t \in [0,r]$, *divided through $\sqrt{\alpha'_t}$, decomposes in KL-series* (4.3)

$$\frac{1}{\sqrt{\alpha'_t}} B^e_t = \sum_{n=1}^{\infty} \omega_n \varphi_n(t)\, \xi_n$$

where the ξ_n are i.i.d. Gaussian random variables, ω_n and $\{\varphi_n\}$ are of the same form as in Theorem 4.14 but now λ_n are the positive zeros of $B_t(z)$.

Proof. By Theorem 4.1, the covariance function of the process in question is of the form (4.17). □

This is reproving of the statement made just after Theorem 3.7 that regards PW-series for even martingale bridges.

The martingale bridge is an even fundamental martingale M^e centered by its conditional expectation, given its value at the right endpoint r. Section 4.1 contains the subsection on kernel (4.24), the results of which make available an extension of Theorem 4.15 to an even fundamental martingale M^e centered by its conditional expectation, given the value of a martingale

$$N_t^e = \int_0^t \kappa_u dM_u^e$$

at the right endpoint r. The kernel κ is assumed to be non-constant, otherwise we would turn back to an already handled martingale bridge. The Gaussian process so defined

$$B_t^e = M_t^e - \mathbb{E}(M_t^e | N_r^e) \tag{4.51}$$

$t \in [0, r]$, has the covariance function

$$r(s,t) = \pi\Big(\alpha_{s\wedge t} - \frac{< 1_s, \kappa >_\alpha < 1_t, \kappa >_\alpha}{\|\kappa\|_\alpha^2}\Big).$$

In Section 4.1 we have introduced the notations (4.25) and assumed smoothness of the kernel κ as to satisfy Theorem 4.7. This theorem provides a key argument for proving Mercer's type Theorem 4.10 which implies

Theorem 4.16. *In the situation of Theorem 4.10 process* (4.51)*, divided through* $\sqrt{\alpha_t'}$*, decomposes in KL-series* (4.3)

$$\frac{1}{\sqrt{\alpha_t'}} B_t^e = \sum_{n=1}^{\infty} \omega_n \varphi_n(t)\, \xi_n$$

where the ξ_n *are i.i.d. Gaussian random variables,* $\{\varphi_n(t) = b_t(w_n)/\sqrt{\alpha_t'}\}$ *is an orthonormal basis in* $L^2[0, r]$*, and*

$$\omega_n = \frac{1}{w_n c(w_n)}.$$

Here $b_t(w)$ *is given by* (4.34)*, the* w_n *are positive zeros of the determinantal equation* (4.33) *and* $c(w)$ *is a normalization constant in* (4.34)*.*

Proof. The claim that Theorem 4.10 implies the present statement follows from the comparison of the covariance function $r(s, t)$ displayed above with kernel (4.24). □

4.2.2 Odd martingales

Preceding results are adapted to odd fundamental martingales with the help of Theorem 4.12 and Theorem 4.13. They stem from Theorem 4.5 and Theorem 4.10, due to the rotations (4.36) and (4.13). Besides $\alpha \leftrightarrow \gamma$, the entries $A_t(z)$ and $B_t(z)$ in the upper row of the chain of de Branges matrices are interchanged with the entries $D_t(z)$ and $-C_t(z)$ in the lower row. The role of the reproducing kernel $K_t(w, z)$ is taken over by kernel (4.14). In this manner, Theorems 4.14 – 4.16 are reformulated as follows.

Theorem 4.17. *Let X be a double-sided si-process determined by a chain of de Branges spaces of exponential type so as in Theorem 4.2. Let M^o be the fundamental martingale which defines the moving average representation of the odd part of X, with the second order moments* (4.49)*. Then process $M_t^o/\sqrt{\gamma_t'}$, $t \in [0, r]$, decomposes in KL-series* (4.3)

$$\frac{1}{\sqrt{\gamma_t'}} M_t^o = \sum_{n=1}^{\infty} \omega_n \varphi_n(t)\, \xi_n$$

where the ξ_n are i.i.d. Gaussian random variables, $\{\varphi_n(t) = C_t(\lambda_n)/\sqrt{\gamma_t'}\}$ is an orthonormal basis in $L^2[0, r]$, and

$$\omega_n = \frac{1}{\lambda_n} \sqrt{\frac{2}{\pi Q_n(\lambda_n, \lambda_n)}}$$

with the positive zeros λ_n of $D_t(z)$.
(ii) The martingale bridge

$$B_t^o = M_t^o - \frac{\gamma_t}{\gamma_r} M_r^o, \tag{4.52}$$

$0 \le t \le r$, divided through $\sqrt{\gamma_t'}$, decomposes in KL-series (4.3)

$$\frac{1}{\sqrt{\gamma_t'}} B_t^o = \sum_{n=1}^{\infty} \omega_n \varphi_n(t)\, \xi_n$$

where the ξ_n are i.i.d. Gaussian random variables, ω_n and $\{\varphi_n\}$ are of the same form as before but now λ_n are the positive zeros of $C_t(z)$.

Recall formula (4.38) for kernel $Q(w, z)$ according to which $Q_n(\lambda_n, \lambda_n) = C_r(\lambda_n)\dot{D}_r(\lambda_n)$ in item (i) and $Q_n(\lambda_n, \lambda_n) = -\dot{C}_r(\lambda_n)D_r(\lambda_n)$ in item (ii).

Let the martingale bridge of the preceding statement be centered by its conditional expectation, given the value of a martingale

$$N_t^o = \int_0^t \kappa_u dM_u^o$$

at the right endpoint r. The kernel κ is assumed to be non-constant. The Gaussian process so defined

$$B_t^o = M_t^o - \mathbb{E}(M_t^o | N_r^o) \tag{4.53}$$

$t \in [0, r]$, has the covariance function

$$r(s,t) = \pi\Big(\gamma_{s\wedge t} - \frac{< 1_s, \kappa >_\gamma < 1_t, \kappa >_\gamma}{\|\kappa\|_\gamma^2}\Big).$$

Theorem 4.18. *In the situation of Theorem 4.13 process* (4.53)*, divided through* $\sqrt{\gamma_t'}$*, decomposes in KL-series* (4.3)

$$\frac{1}{\sqrt{\gamma_t'}} B_t^o = \sum_{n=1}^{\infty} \omega_n \varphi_n(t)\, \xi_n$$

where the ξ_n *are i.i.d. Gaussian random variables,* $\{\varphi_n(t) = b_t(w_n)/\sqrt{\gamma_t'}\}$ *is an orthonormal basis on* $L^2[0, r]$*, and*

$$\omega_n = \frac{1}{w_n c(w_n)}.$$

Here $b_t(w)$ *is given by* (4.47)*, the* w_n *are positive zeros of the determinantal equation* (4.45) *and* $c(w)$ *is a normalization constant in* (4.47)*.*

4.3 Processes associated with an FBM

(a) In this section we apply the results of the preceding section to obtain KL-expansions for processes associated with a centered Gaussian H-self-similar processes $(X_t^e)_{t\geq 0}$ and $(X_t^o)_{t\geq 0}$ of Hurst index $0 < H < 1$ which are defined in Example 2.7. Based on the spectral isometry, it is shown that the martingales are expressible in terms of X^e and X^o by (2.32). The converse relations (2.35) are moving average representations. The second order moment of the martingales are given by (2.34). In this example the de Branges matrices

$$M_t(z) = d_t(z) \begin{bmatrix} J_{-H}(tz) & J_{1-H}(tz)\alpha'(a) \\ -J_H(tz)/\alpha'(t) & J_{H-1}(tz) \end{bmatrix} \tag{4.54}$$

are defined for $t \geq 0$, where z is a complex variable and $d_t(z)$ the diagonal matrix

$$d_t(z) = \mathrm{diag}\big[\Gamma(1-H)(tz/2)^H, \Gamma(H)(tz/2)^{1-H}\big]. \tag{4.55}$$

Check that the determinant is equal to 1 at hand of the Wronskian $J_{1-\nu}(z)J_\nu(z) + J_{-\nu}(z)J_{\nu-1}(z) = \frac{2}{\pi z}\sin \nu\pi$ [17, formula 60 on p. 12].

Recall formula (3.24) for kernel $K_r(\lambda,\lambda)$ on the real diagonal. Formula for kernel $Q_r(\lambda,\lambda)$, cf. (4.38), is analogously calculated

$$\begin{aligned}\frac{Q_r(\lambda,\lambda)}{Q_r(0,0)} &= 2H\Gamma^2(H)(r\lambda/2)^{2-2H} \\ &\times \Big(J_H^2(r\lambda) - \frac{1-2H}{r\lambda} J_H(r\lambda)J_{H-1}(r\lambda) + J_{H-1}^2(r\lambda)\Big) \qquad (4.56)\end{aligned}$$

by using properties of the Bessel functions.

(b) The general results of the preceding section that regard even and odd fundamental martingales are applied to the present case as follows.

Theorem 4.19. *Let X be a double-sided FBM of Hurst index $0 < H < 1$. Then*
(i) Process $t^{H-\frac{1}{2}}M_t^e$, $0 \le t \le r$, cf. (2.32)*, decomposes in KL-series* (4.3) *where the ξ_n are i.i.d. Gaussian random variables,*

$$t^{H-\frac{1}{2}}\varphi_n(t) = \Gamma(1-H)\Big(\frac{t\lambda_n}{2}\Big)^H J_{1-H}(t\lambda_n)$$

and

$$\omega_n = \frac{\sqrt{2}}{\lambda_n r^{1-H}\Gamma(1-H)(r\lambda_n/2)^H J_{1-H}(r\lambda_n)}$$

where the λ_n are positive roots of the Bessel function $J_{-H}(r\lambda_n) = 0$.
(ii) Process $t^{\frac{1}{2}-H}M_t^o$, $0 \le t \le r$, cf. (2.32)*, decomposes in KL-series* (4.3) *where the ξ_n are i.i.d. Gaussian random variables,*

$$\varphi_n(t) = -\Gamma(H)\Big(\frac{t\lambda_n}{2}\Big)^{1-H} J_H(t\lambda_n)$$

and

$$\omega_n = \frac{\sqrt{2}}{\lambda_n r^H\Gamma(H)(r\lambda_n/2)^{1-H} J_H(r\lambda_n)}$$

cf. (4.56)*, where the λ_n are positive roots of the Bessel function $J_{H-1}(r\lambda_n) = 0$.*

In fact, item (ii) is a direct consequence of item (i), for the martingales in question depend on Hurst index H so that $M^e(H) \overset{Law}{=} M^o(1-H)$, equality in law.

Similar results are available for martingale bridges.

Theorem 4.20. *In the situation of Theorem 4.19*
(i) Process $t^{H-\frac{1}{2}}B_t^e$, $0 \le t \le r$, cf. (4.50)*, decomposes in KL-series* (4.3) *where the ξ_n are i.i.d. Gaussian random variables, ω_n and $t^{H-\frac{1}{2}}\varphi_n(t)$ is of the same form as in Theorem 4.19, item (i), but the λ_n are now positive roots of the Bessel function $J_{1-H}(r\lambda_n) = 0$.*
(ii) Process $t^{\frac{1}{2}-H}B_t^o$, $0 \le t \le r$, cf. (4.52)*, decomposes in KL-series* (4.3) *where the*

ξ_n are i.i.d. Gaussian random variables, ω_n and $t^{H-\frac{1}{2}}\varphi_n(t)$ is of the same form as in Theorem 4.19, item (ii), but the λ_n are now positive roots of the Bessel function $J_H(r\lambda_n) = 0$.

In fact, item (ii) is a direct consequence of item (i), for the bridges in question depend on Hurst index H so that $B^e(H) \overset{Law}{=} B^o(1-H)$, equality in law.

4.4 Autoregressive processes

In this section we discuss the KL decompositions of *autoregressive processes* Y of n^{th}-order whose spectral measures have densities of the form (3.31) where $\Theta(iz)$ is a polynomial (3.32) of degree n whose zeros have negative real part.

In the simplest case of $n = 1$ which is the so-called *Ornstein–Uhlenbeck process*, the covariance function $\mathbb{E}(Y_sY_t) = r(s,t)$ is of the exponential form (3.26) with the parameters $\sigma^2, \theta > 0$ and the characteristic equation of the Fredholm type with this kernel

$$\int_0^r r(s,t)\varphi_l(s)ds = \lambda_l\varphi_l(t) \tag{4.57}$$

is not very difficult to solve. Under the boundary conditions

$$\begin{aligned} \varphi_l'(0) - \theta\varphi_l(0) = 0 \\ \varphi_l'(r) + \theta\varphi_l(r) = 0 \end{aligned} \tag{4.58}$$

one obtains

$$\begin{aligned} \lambda_l &= \frac{\sigma^2}{w_l^2 + \theta^2} \\ \varphi_l(t) &= \sqrt{\frac{2w_l}{r(w_l^2+\theta^2)+2\theta}}\cos(w_lt) + \sqrt{\frac{2\theta^2}{r(w_l^2+\theta^2)+2\theta}}\sin(w_lt) \end{aligned} \tag{4.59}$$

where the w_l are positive roots of the determinantal equation

$$\begin{vmatrix} \theta & -w \\ \theta - w\tan(rw) & w + \theta\tan(rw) \end{vmatrix} = 0 \tag{4.60}$$

arranged in ascending order of magnitude. The proof can be found in the Ph.D. theses [34], Example 3 on p. 23 (cf. also [4]). We shall give more details on this example in a separate subsection below. In what follows we extend the method of this particular OU case to arbitrary autoregressive processes of order n.

4.4.1 Mercer's theorem

We apply Theorem 4.4 to the autoregressive process Y with the covariance function

$$r(s,t) = \frac{\sigma^2}{2\pi}\int e^{i\lambda(s-t)}\frac{d\lambda}{|\Theta(i\lambda)|^2} \tag{4.61}$$

with the density of the spectral measure of the form (3.31). As before, $\Theta(iz)$ is a polynomial (3.32) of degree n and its zeros have negative real part and, moreover, $\phi_k > 0$, since process Y is real.

Theorem 4.21. *Consider the characteristic equation with kernel* (4.61)

$$\int_0^r r(s,t)\varphi_l(s)ds = \lambda_l \varphi_l(t) \tag{4.62}$$

and let the eigenfunctions $\varphi_l(t)$*, vanishing outside of interval* $[0, r]$*, be* $2n$ *times continuously differentiable. Then eigenfunctions and eigenvalues may be given the form*

$$\varphi_l(t) = a_l \cos w_l t + b_l \sin w_l t \tag{4.63}$$

and

$$\lambda_l = \frac{\sigma^2}{(w_l^2 + \phi_1^2)\cdots(w_l^2 + \phi_n^2)}. \tag{4.64}$$

Under the boundary conditions

$$\begin{aligned} \Theta(d/dt)\varphi_l(t)|_{t=0} &= 0 \\ \Theta(-d/dt)\varphi_l(t)|_{t=r} &= 0 \end{aligned} \tag{4.65}$$

positive numbers w_l *are roots of the determinantal equation*

$$\begin{vmatrix} Re\,\Theta(iw) & Re\,\Theta(iw)\cos(rw) + Im\,\Theta(iw)\sin(rw) \\ Im\,\Theta(iw) & Re\,\Theta(iw)\sin(rw) - Im\,\Theta(iw)\cos(rw) \end{vmatrix} = 0 \tag{4.66}$$

and the coefficients a_l *and* b_l *are roots of the system of equations*

$$\begin{aligned} a\,Re\,\Theta(iw) + b\,Im\,\Theta(iw) &= 0 \\ \int_0^r \big(a\cos(wt) + b\sin(wt)\big)^2 dt &= 1. \end{aligned} \tag{4.67}$$

Note that the eigenvalues are 2π multiple of the spectral density at points w_l.

Proof. By applying $|\Theta(d/dt)|^2$ to both sides of the characteristic equation (4.62), one obtains

$$\sigma^2\varphi_l(t) = \lambda_l |\Theta(d/dt)|^2 \varphi_l(t). \tag{4.68}$$

Indeed, since

$$|\Theta(d/dt)|^2 e^{-i\lambda t} = |\Theta(i\lambda)|^2 e^{-i\lambda t},$$

the spectral representation (4.61) turns the left-hand side of the characteristic equation into

$$\frac{\sigma^2}{2\pi}\int_0^r e^{-i\lambda t}\hat{\varphi}_l(i\lambda)d\lambda$$

where $\hat{\varphi}_l(i\lambda)$ is the Fourier transform of φ_l, i.e. $\hat{\varphi}_l(i\lambda) = \int_0^r e^{it\lambda}\varphi_l(t)dt$, and the required expression is obtained by taking the inverse Fourier transform.

If the eigenfunctions are given the form (4.63), then we have

$$|\Theta(d/dt)|^2 \sin(\lambda t) = \prod_{j=1}^{n} \Big(-\frac{d^2}{dt^2} + \phi_j^2 \Big) \sin(\lambda t) = \prod_{j=1}^{n} \big(\lambda^2 + \phi_j^2\big) \sin(\lambda t)$$

and the same for cosine, therefore equation (4.68) becomes

$$\sigma^2 \varphi_l(t) = \lambda_l |\Theta(iw_l)|^2 \varphi_l(t)$$

which determines the eigenvalues (4.64).

To establish the first of boundary conditions (4.65) apply $\Theta(d/dt)$ to both sides of the spectral representation using $\Theta(d/dt)e^{izt} = \Theta(iz)e^{izt}$. Since

$$\Theta(d/dt)r(s,t) = \frac{\sigma^2}{2\pi} \int e^{i\lambda(s-t)} \frac{d\lambda}{\Theta(i\lambda)}$$

vanishes at $t = 0$ by the residue theorem we obtain the first of boundary conditions (4.65). Similarly, for $0 \le s \le r$

$$\Theta(-d/dt)r(s,t) = \frac{\sigma^2}{2\pi} \int e^{i\lambda(s-t)} \frac{d\lambda}{\Theta(-i\lambda)}$$

vanishes at $t = r$ and we obtain the second of boundary conditions (4.65).

With the eigenfunctions of the form (4.63), the boundary conditions give the following system of equations

$$a \operatorname{Re} \Theta(iw) + b \operatorname{Im} \Theta(iw) = 0$$
$$a\big(\operatorname{Re} \Theta(iw)\cos(rw) + \operatorname{Im} \ \sin(rw)\big) + b\big(\operatorname{Re} \Theta(iw)\sin(rw) - \operatorname{Im} \ \cos(rw)\big) = 0$$

whose non-zero solution for a and b requires a vanishing determinant which is (4.66). So, the w_n are roots of this determinantal equation, for their characterisation see below Lemma 4.22.

To calculate the coefficients a_l and b_l make use of the first of boundary conditions and the requirement that the eigenfunctions have to be of unit norm in the metric of $L^2[0, r]$. □

To compute the integral in (4.67), note

$$2\varphi^2(t) = \frac{d}{dt}\big((a^2 + b^2)t - w^{-2}\varphi(t)\varphi'(t)\big)$$

since $\varphi''(t) = -w^2\varphi(t)$ and $\varphi^2(t) = a^2 + b^2 - \big(\varphi'(t)/w\big)^2$. Therefore

$$\begin{aligned} 2\int_0^r \varphi(t)^2 dt &= (a^2 + b^2)r - w^{-2}\varphi(t)\varphi'(t)\big|_0^r \\ &= (a^2 + b^2)r + (a^2 - b^2)\frac{\sin(wr)\cos(wr)}{w} + 2ab\frac{\sin^2(wr)}{w}. \end{aligned} \tag{4.69}$$

For more details see concluding subsection where Theorem 4.23 is presented and proved.

For illustration, we apply the general Theorem 4.21 to two simplest cases. The case $n = 1$ is discussed in the next subsection and $n = 2$ in Section 5, Note 6.

4.4.2 OU process

With kernel (3.26) the characteristic equation (4.57) becomes

$$\frac{\sigma^2}{2\theta}\Big(\int_0^t e^{-\theta(t-s)}\varphi_l(s)ds + \int_t^r e^{\theta(t-s)}\varphi_l(s)ds\Big) = \lambda_l\varphi_l(t).$$

Differentiating both sides yields

$$\frac{\sigma^2}{2}\Big(-\int_0^t e^{-\theta(t-s)}\varphi_l(s)ds + \int_t^r e^{\theta(t-s)}\varphi_l(s)ds\Big) = \lambda_l\varphi_l'(t).$$

Evaluate first at $t = 0$ and then at $t = r$ to get the boundary conditions (4.58). Differentiating once more yields

$$\lambda_l\varphi_l''(t) = \frac{\sigma^2}{2}\Big(\theta\int_0^r e^{-\theta|t-s|}\varphi_l(s)ds - 2\varphi_l(t)\Big) = \varphi_l(t)(\theta^2\lambda_l - \sigma^2),$$

hence

$$\varphi_l''(t) = -w_l^2\varphi_l(t)$$

where $w_l^2 = (\sigma^2 - \theta^2\lambda_l)/\lambda_l$. Solving the latter with respect to λ_l gives the first equation (4.59).

The preceding second order differential equation is solved by a linear combination of the sine and cosine $\varphi(t) = a\cos(wt) + b\sin(wt)$. To determine the eigenvalues w_l of this differential equation use the boundary conditions (4.58) which give the system of equations

$$\begin{aligned} &a\theta - bw = 0 \\ &a(\theta - w\tan(rw)) + b(\theta\tan(rw) + w) = 0 \end{aligned}$$

whose non-zero solution is guaranteed by the determinantal equation (4.60). As for the coefficients a and b, which are related by $b = (\theta/w)a$, we can use the normalization requirement $\int_0^r \varphi^2(t)dt = 1$. Since

$$\tan(rw) = \frac{2\theta w}{w^2 - \theta^2} \tag{4.70}$$

by (4.60), formula (4.69) gives

$$a = \sqrt{\frac{2w^2}{r(w^2+\theta^2)+2\theta}} \qquad b = \sqrt{\frac{2\theta^2}{r(w^2+\theta^2)+2\theta}}.$$

which in tern gives expression (4.59) for the eigenfunctions.

We need to bring OU processes in connection with the chain of de Branges spaces $\mathcal{H}(E_t)$, $t \geq 0$, contained isometrically in $L^2(\mu)$ with the spectral measure of the processes (3.27). As is said above, the entire functions (3.28) are de Branges functions and all spaces they generate are contained isometrically in $L^2(\mu)$. Let us normalize the de Branges function (3.28) at the origin. The quotient

$$E_t^0(z) = E_t(z)/E_t(0) = (1 - iz/\theta)e^{-itz} \tag{4.71}$$

has the components

$$A_t(z) = \cos(tz) - (z/\theta)\sin(tz) \qquad B_t(z) = \sin(tz) + (z/\theta)\cos(tz).$$

Section 3.5 makes use of sampling points λ_n which are positive zeros of $B_r(z)$ and satisfy equation $\tan(\lambda_n r) = \lambda_n/\theta$. The present sampling points that come from the identity (4.70) are positive zeros of an odd component of another de Branges function which is $(1 - iz/\theta)$ multiple of $E_r(z)$, i.e.

$$E_r^1(z) := (1 - iz/\theta)^2 e^{-irz}. \tag{4.72}$$

Its even and odd components are

$$A_r^1(z) = (1 - (z/\theta)^2)\cos(rz) - 2(z/\theta)\sin(rz)$$
$$B_r^1(z) = (1 - (z/\theta)^2)\sin(rz) + 2(z/\theta)\cos(rz).$$

Clearly, the positive zeros of $B_r^1(z)$ satisfy

$$(1 - (w/\theta)^2)\sin(rw) + 2(w/\theta)\cos(rw) = 0$$

which matches (4.70). It is seen from (4.59) that at points w_l which are roots of the determinantal equation (4.60) and in the same time positive zeros of $B_r^1(z)$, the functions $B_t(w_l) = \sin tw_l + (w_l/\phi)\cos tw_l$ become constant multiples of the eigenfunctions, i.e.

$$\varphi_l(t) = \sqrt{\frac{2\theta^2}{r(w_l^2 + \theta^2) + 2\theta}}\, B_t(w_l).$$

In this manner the set of functions $\{B_t(w_1), B_t(w_2), \cdots\}$ form a basis in $L^2[0, r]$. This fact is confirmed by the Lagrange formula (4.6) for two different roots w_j and w_k

$$\int_0^r B_u(w_l)B_u(w_k)\, du = \frac{w_j B_t(w_k)A_t(w_j) - w_k B_t(w_j)A_t(w_k)}{w_k^2 - w_j^2}\Big|_0^r \tag{4.73}$$

which is equal to 0 at point $t = 0$, since $A_0(z) = 1$ and $B_0(z) = z/\theta$, and at point $t = r$, since

$$A_r^2(w) = A_0^2(w) \qquad B_r^2(w) = B_0^2(w) \tag{4.74}$$

if $B_r^1(w) = 0$. These two identities are obtained by simple algebra. For instance,

$$\begin{aligned}\theta^2 B_r^2(w) &= (\theta\sin(rw) + w\cos(rw))^2\\ &= (\theta^2 - w^2)\sin^2(tw) + 2\theta w\sin(rw)\cos(rw) + w^2 = \theta^2 B_0^2(w).\end{aligned}$$

4.4.3 AR processes (continued)

The relationship of an OU process to a chain of de Branges spaces just discussed does extend to autoregressive processes. To show this, we consider the de Branges function (3.33) for $t \geq 0$ which generates a space contained isometrically in $L^2(\mu)$ with the spectral measure (3.31). Like in (4.71), normalize this de Branges function at the origin

$$E_r^0(z) = E_t(z)/E_t(0) = \Theta(iz)\, e^{-izr}.$$

This normalized de Branges function has even and odd components

$$\begin{aligned} A_r(z) &= \text{Re}\;\Theta(iz)\cos(rz) + \text{Im}\;\Theta(iz)\sin(rz) \\ B_r(z) &= \text{Re}\;\Theta(iz)\sin(rz) - \text{Im}\;\Theta(iz)\cos(rz). \end{aligned} \tag{4.75}$$

Like in (4.72) we consider another de Branges function

$$E_t^1(z) = \Theta^2(iz)\, e^{-izt}$$

with the even and odd components

$$\begin{aligned} A_t^1(z) &= \left((\text{Re}\;\Theta(iz))^2 - (\text{Im}\;\Theta(iz))^2\right)\cos(tz) + 2\text{Re}\;\Theta(iz)\text{Im}\;\Theta(iz)\sin(tz) \\ B_t^1(z) &= \left((\text{Re}\;\Theta(iz))^2 - (\text{Im}\;\Theta(iz))^2\right)\sin(tz) - 2\text{Re}\;\Theta(iz)\text{Im}\;\Theta(iz)\cos(tz). \end{aligned} \tag{4.76}$$

The latter, being an odd component of a de Branges function, has a simple interlacing zeros symmetrically spread about the origin. Clearly, they satisfy the determinantal equation (4.66). Moreover, if w_l is the l^{th} positive root of the latter, then the odd component (4.75) evaluated at this point $B_t(w_l)$ is a constant multiple of the l^{th} eigenfunction (4.63), see Theorem 4.23 below. The proof of this theorem is based on the identities (4.74) which hold true in general.

Lemma 4.22. *Let w be a zero of the odd component $B_r^1(z)$ of the form* (4.76)*, i.e. $B_r^1(w) = 0$. Then the components* (4.75) *evaluated at this point satisfy the identities* (4.74)*.*

If w_j and w_k are two different zero of $B_r^1(z)$, then

$$\int_0^r B_t(w_j)B_t(w_k)dt = 0. \tag{4.77}$$

Proof. The proof of (4.74) is as simple as in the case $n = 1$. Moreover, the statement stay true even if polynomial $\Theta(iz)$ is substituted by any de Branges function, say $e(z) = a(z) - ib(z)$. Indeed, the even and odd components $a(z)$ and $b(z)$ replace $\text{Re}\;\Theta(iz)$ and $-\text{Im}\;\Theta(iz)$, respectively, and if

$$E_t^1(z) = e^2(z)\, e^{-izt}$$

then its even and odd components are

$$A_t^1(z) = \big(a^2(z) - b^2(z)\big)\cos(tz) - 2a(z)b(z)\sin(tz)$$
$$B_t^1(z) = \big(a^2(z) - b^2(z)\big)\sin(tz) + 2a(z)b(z)\cos(tz).$$

Now, if w is a zero of the latter component at the right endpoint r, i.e. $B_r^1(w) = 0$, then

$$\begin{aligned} A_r^2(w) &= a^2(w)\cos^2(rw) + b^2(w)\sin^2(rw) - 2a(w)\, b(w)\sin(rw)\cos(rw) \\ &= a(w)^2\cos^2(rw) + b^2(w)\sin^2(rw) + \big(a^2(w) - b^2(w)\big)\sin^2(rw) = A_0^2(w) \end{aligned}$$

and

$$\begin{aligned} B_r^2(w) &= a^2(w)\sin^2(rw) + b^2(w)\cos^2(rw) + 2a(w)\, b(w)\sin(rw)\cos(rw) \\ &= a^2(w)\sin^2(rw) + b^2(w)\cos^2(rw) - \big(a^2(w) - b^2(w)\big)\sin^2(rw) = B_0^2(w). \end{aligned}$$

The proof of (4.77) is based on the Lagrange identity (4.73), precisely as in the previous case of $n = 1$. □

The preceding results can be summarized as follows.

Theorem 4.23. *The eigenfunctions and eigenvalues are of the form*

$$\varphi_l(t) = \frac{B_t(w_l)}{\|B(w_l)\|} \qquad \lambda_l = 2\pi \frac{\mu(d\lambda)}{d\lambda}\Big|_{\lambda = w_l}$$

where w_l are positive zeros of $B_r^1(z)$, cf. (4.76), *and*

$$2\|B(w_l)\|^2 = \big((\text{Re}\,\Theta(iw_l))^2 + (\text{Im}\,\Theta(iw_l))^2\big)r + 2\text{Re}\,\Theta(iw_l)\,\text{Im}\,\Theta(iw_l).$$

The norm is calculated by (4.67) with $a = \text{Re}\,\Theta(iw_l)$ and $b = \text{Im}\,\Theta(iw_l)$.

5 Notes and addenda

1. Regarding the rate of convergence of Paley–Wiener series, discussed in Section 3.2, see [39, Theorem 4.1.4]. In Chapter 5 of the latter thesis the results of the theorem are applies to estimate small ball probabilities for si- and stationary processes. Much more far-reaching developments of theory of small ball asymptotics can be found in the recent survey [30].
2. In [3] Theorem 4.16 is proved in the particular case of the martingale bridge (4.51) where M^e is a standard Brownian motion, so that kernel (4.24) is simply $k(s,t) = s \wedge t - \int_0^s \kappa_u du \int_0^t \kappa_u du$ (assuming κ is of unit norm, for simplicity).
3. The bridging property $B_0^e = B_t^e = 0$ of an even martingale bridge (4.50) suggests notion of an *inverse bridge* $\overleftarrow{B}_t^e = B_{r-t}^e$ which is a cantered Gaussian process with the covariance function
$$\mathbb{E}(\overleftarrow{B}_s^e \overleftarrow{B}_t^e) = \alpha_{r-s\vee t} - \frac{\alpha_{r-s}\alpha_{r-t}}{\alpha_r}.$$
In these terms the following inverse version of Theorem 4.15 holds true.

Theorem 5.1. *In the situation of Theorem 4.14, the inverse martingale bridge* $\overleftarrow{B}^e_t = B^e_{r-t}$*,* $0 \le t \le r$*, divided through* $\sqrt{\alpha'_{r-t}}$*, decomposes in KL-series* (4.3)

$$\frac{1}{\sqrt{\alpha'_{r-t}}} \overleftarrow{B}^e_t = \sum_{n=1}^{\infty} \omega_n \varphi_n(t)\, \xi_n$$

where the ξ_n *are i.i.d. Gaussian random variables,* $\{\varphi_n(t) = B_{r-t}(\lambda_n)/\sqrt{\alpha'_{r-t}}\}$ *and*

$$\omega_n = \frac{1}{\lambda_n} \sqrt{\frac{2}{\pi K_r(\lambda_n, \lambda_n)}}$$

with the positive zeros λ_n *of* $A_r(z)$*.*

4. The main result of the paper [2] applies Theorem 5.1 to a fractional Brownian motion, more precisely, to an inverse version of an associated even martingale bridge. This is not immediately seen, since a Gaussian process that is studied in [2], is defined on interval $[0, r]$ as a Wiener integral with respect to a standard Brownian motion

$$X^{(\alpha)}_t = \int_0^t \left(\frac{r-t}{r-s}\right)^{\alpha} dW_s$$

and called *alpha-Wiener bridge*. In the present text symbol α is used for other purposes and to avoid ambiguities in this note the positive exponent α is substituted by $\frac{3}{2} - H$. We will discuss only the case $0 < H < 1$ and shown that process $X^{(\frac{3}{2}-H)}$ is associated in a certain way with a fractional Brownian bridge of Hurst index H. More precisely, a relation will be sought to an even martingale (2.32) whose quadratic variation is given by (2.34), with $\alpha_t = t^{2-2H}/(2-2H)$. It is easily seen that the normalized process

$$X_t := (r-t)^{\frac{1}{2}-H} X^{(\frac{3}{2}-H)}_t$$

is an *anticipative martingale bridge* for a martingale

$$M^r_t := \int_0^t (r-u)^{\frac{1}{2}-H} dW_u$$

where W is a standard Brownian motion (the term is taken over from [18]). By definition, process M^r_t on interval $[0, r]$ defines the anticipative martingale bridge $M_1(t, r)$ by

$$M_1(t, r) = \int_0^t \frac{< M^r >_r - < M^r >_t}{< M^r >_r - < M^r >_t} dM^r_u$$

where $< M^r >$ is the quadratic variation of the martingale M^r and equals to $< M^r >_t = \alpha_r - \alpha_{r-t}$, since

$$\mathbb{E}(M^r_s M^r_t) = \alpha_r - \alpha_{r-s\wedge t}.$$

Simple algebra shows identity $X_t = M_1(t, r)$ and the following form of the covariance function

$$\mathbb{E}(X_s X_t) = \alpha_{r-s\vee t} - \frac{\alpha_{r-s}\alpha_{r-t}}{\alpha_r}.$$

Compare this with the covariance function $\mathbb{E}(\overleftarrow{B}^e_s \overleftarrow{B}^e_t)$ of an inverse even martingale bridge provided in the preceding note. The covariances coincide. Hence, the present normalized alpha-Wiener bridge $\{X\}_{t\in[0,r]}$ is equal in law to the inverse bridge $\{\overleftarrow{B}^e_t = B^e_{r-t}\}_{t\in[0,r]}$ where B^e is as in Theorem 4.20. Theorem 5.1 can be applied with $\alpha_t = t^{2-2H}/(2-2H)$ and the components $A_t(z)$ and $B_t(z)$ specified in terms of the Bessel functions as in Section 4.3.

5. Theorem 4.19 is proved in [8] (cf. [7], Fact 1.2).

6. A second order autoregressive process has spectral density of the form

$$\mu(d\lambda) = \frac{\sigma^2 d\lambda}{2\pi(z^2+\phi_1^2)(z^2+\phi_2^2)}$$

with parameters $\sigma^2 > 0$, $\phi_1 > 0$, $\phi_2 > 0$. According to Lemma 4.21, the covariance function $r(s,t) = \mathbb{E}(X_s X_t)$ on interval $[0,r]$ has the Karhunen–Loève expansion

$$r(s,t) = \sum_{n=1}^{\infty} \lambda_n \varphi_n(s)\varphi_n(t)$$

with the eigenfunctions and eigenvalues which are of the form

$$\varphi_n(t) = a_n \cos w_n t + b_n \sin w_n t$$

and

$$\lambda_n = \frac{\sigma^2}{(w_n^2+\phi_1^2)(w_n^2+\phi_2^2)}$$

for certain numbers w_n and certain coefficients a_n and b_n, both depending on w_n. To determine the numbers w_n, consider the boundary conditions

$$(d/dt - \phi_1)(d/dt - \phi_2)\varphi_n(0) = 0 \qquad (d/dt + \phi_1)(d/dt + \phi_2)\varphi_n(1) = 0$$

which for the eigenfunctions of the preceding form imply the system of equations

$$\begin{aligned} a_n(\phi_1\phi_2 - w^2) - b_n(\phi_1+\phi_2)w = 0 \\ a_n\big((\phi_1\phi_2 - w^2)\cos wr - (\phi_1+\phi_2)w \sin wr\big) \\ + b_n\big((\phi_1\phi_2 - w^2)\sin wr + (\phi_1+\phi_2)w\cos wr\big) = 0. \end{aligned}$$

A non-zero solution with respect to a_n and b_n requires a vanishing determinant

$$\begin{aligned} &\begin{vmatrix} (\phi_1\phi_2 - w^2) & (\phi_1\phi_2 - w^2)\cos wr - (\phi_1+\phi_2)w\sin wr \\ -(\phi_1+\phi_2)w & (\phi_1\phi_2 - w^2)\sin wr + (\phi_1+\phi_2)w\cos wr \end{vmatrix} \\ &= \big((\phi_1\phi_2 - w^2)^2 - (\phi_1+\phi_2)^2 w^2\big)\sin w + 2w(\phi_1+\phi_2)(\phi_1\phi_2 - w^2)\cos w = 0. \end{aligned}$$

Let the numbers w_n be positive roots of this determinantal equation (it will be seen in a moment that these are positive zeros of the entire function $B^1_r(z)$ given by (5.2)). To determine the numbers a_n and b_n, make use of the first of boundary equations

$$a_n(\phi_1\phi_2 - w_n^2) = b_n(\phi_1+\phi_2)w_n$$

and the fact that the eigenfunctions are normalized

$$\int_0^r \varphi_n^2(t)dt = \int_0^r (a_n\cos w_n t + b_n \sin w_n t)^2 dt.$$

To shed more light on these results we consider space $L^2(\mu)$ with the present spectral measure which contains isometrically the part of the chain of de Branges spaces restricted to interval $[0, r]$ and is generated by the de Branges functions

$$E_t(z) = (iz - \phi_1)(iz - \phi_2)e^{-izt},$$

The corresponding odd and even components are

$$\begin{aligned} A_t(z) &= (\phi_1\phi_2 - z^2)\cos zt - (\phi_1 + \phi_2)z\sin zt \\ B_t(z) &= (\phi_1\phi_2 - z^2)\sin zt + (\phi_1 + \phi_2)z\cos zt. \end{aligned} \tag{5.1}$$

Clearly, $A_0(z) = \phi_1\phi_2 - z^2$ and $B_0(z) = (\phi_1 + \phi_2)z$. We shell show that at certain points w on the real axis $A_r^2(w) = A_0^2(w)$ and $B_r^2(w) = B_0^2(w)$. To this end in view, consider the following de Branges function

$$E_t^1(z) = (iz - \phi_1)^2(iz - \phi_2)^2 e^{-izt}$$

with the even and odd components

$$\begin{aligned} A_t^1(z) &= \big(A_0^2(z) - B_0^2(z)\big)\cos zt - 2A_0(z)B_0(z)\sin zt \\ B_t^1(z) &= \big(A_0^2(z) - B_0^2(z)\big)\sin zt + 2A_0(z)B_0(z)\cos zt \end{aligned} \tag{5.2}$$

As is said above, the even and odd components $A(z)$ and $B(z)$ of a de Branges function $E(z) = A(z) - iB(z)$ have only simple interlacing real zeros. The following statement makes use of this fact.
At the right endpoint $t = r$, let $w_1, w_2, \cdots$ be positive zeros of $B_1^1(z)$ of ascendant degree, i.e. the roots of the equation

$$\big((\phi_1\phi_2 - w^2)^2 - (\phi_1 + \phi_2)^2 w^2\big)\sin wr + 2(\phi_1 + \phi_2)(\phi_1\phi_2 - w^2)w\cos wr = 0.$$

For any such number w

$$A_r^2(w) = (\phi_1\phi_2 - w^2)^2 = A_0^2(w) \qquad B_r^2(w) = (\phi_1 + \phi_2)^2 w^2 = B_0^2(w).$$

The right-hand equations are obvious by definition (5.1). Calculations for the left-hand equations are straightforward

$$\begin{aligned} B_r^2(w) &= (\phi_1\phi_2 - w^2)^2\sin^2 wr + (\phi_1 + \phi_2)^2 w^2\cos^2 wr \\ &\quad + 2(\phi_1\phi_2 - w^2)(\phi_1 + \phi_2)w\sin wr\cos wr \\ &\quad = (\phi_1\phi_2 - w^2)^2\sin^2 wr + (\phi_1 + \phi_2)^2 w^2\cos^2 wr \\ &\quad - \big((\phi_1\phi_2 - w^2)^2 - (\phi_1 + \phi_2)^2 w^2\big)\sin^2 wr = B_0^2(w) \end{aligned}$$

and

$$\begin{aligned} A_r^2(w) &= (\phi_1\phi_2 - w^2)^2\cos^2 wr + (\phi_1 + \phi_2)^2 w^2\sin^2 wr \\ &\quad - 2(\phi_1\phi_2 - w^2)(\phi_1 + \phi_2)w\sin wr\cos wr \\ &\quad = (\phi_1\phi_2 - w^2)^2\cos^2 wr + (\phi_1 + \phi_2)^2 w^2\sin^2 wr \\ &\quad + \big((\phi_1\phi_2 - w^2)^2 - (\phi_1 + \phi_2)^2 w^2\big)\sin^2 wr = A_0^2(w). \end{aligned}$$

This statement implies the orthogonality of the sequence of functions $B_t(w_n)$. More precisely, we can claim the following

Let w_j and w_k be zeros of $B_1^1(z)$. Then

$$\int_0^r B_t(w_j)B_t(w_k)dt = 0.$$

This is simply verified. Indeed, if $f'' = -w^2 f$ and $g'' = -z^2 g$, then

$$\int fgdt = \frac{gf' - g'f}{z^2 - w^2}$$

since the derivative of the right-hand side is equal to $(gf'' - g''f)/(z^2 - w^2)$ and this in turn is equal to fg. Hence

$$\int_0^r B_t(w_j)B_t(w_k)dt = \frac{w_j A_t(w_j)B_t(w_k) - w_k A_t(w_k)B_t(w_j)}{w_j^2 - w_k^2}\Big|_0^r$$

and by preceding statement this is equal to zero.

In summary we have

Theorem 5.2. *The eigenfunctions and eigenvalues are of the form*

$$\phi_n(t) = \frac{B_t(w_n)}{\|B(w_n)\|} = \sqrt{\frac{2}{(a_n + b_n)^2 r + 2a_n b_n}}\left(a_n \cos w_n t + b_n \sin w_n t\right)$$

and

$$\lambda_n = \frac{\sigma^2}{a_n^2 + b_n^2}$$

where w_n are as in Lemma 4.22 and $a_n = (\phi_1 + \phi_2)w_n$ and $b_n = \phi_1\phi_2 - w_n^2$.

The required norm is calculated as in (4.69).

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