Random Structures & Algorithms

WILEY

RESEARCH ARTICLE OPEN ACCESS

Optimal Zero-Free Regions for the Independence Polynomial of Bounded Degree Hypergraphs

Ferenc Bencs[1] | Pjotr Buys[2]

[1]Centrum Wiskunde & Informatica, Amsterdam, the Netherlands | [2]Korteweg de Vries Institute for Mathematics, University of Amsterdam, Amsterdam, the Netherlands

Correspondence: Pjotr Buys (pjotr.buys@gmail.com)

Received: 9 July 2023 | **Revised:** 24 April 2025 | **Accepted:** 11 June 2025

Funding: Funded by the Netherlands Organisation of Scientific Research (NWO): VI.Vidi.193.068 and VI.Veni.222.303. Funded by the Netherlands Organisation of Scientific Research (NWO) Vidi grant (639.032.614).

Keywords: hypergraphs | independence polynomial | zero-freeness

ABSTRACT

In this paper, we investigate the distribution of zeros of the independence polynomial of hypergraphs of maximum degree Δ. For graphs, the largest zero-free disk around zero was described by Shearer as having radius $\lambda_s(\Delta) = (\Delta-1)^{\Delta-1}/\Delta^{\Delta}$. Recently, it was shown by Galvin et al. that for hypergraphs the disk of radius $\lambda_s(\Delta+1)$ is zero-free; however, it was conjectured that the actual truth should be $\lambda_s(\Delta)$. We show that this is indeed the case. We also show that there exists an open region around the interval $[0, (\Delta-1)^{\Delta-1}/(\Delta-2)^{\Delta})$ that is zero-free for hypergraphs of maximum degree Δ, which extends the result of Peters and Regts from graphs to hypergraphs. Finally, we determine the radius of the largest zero-free disk for the family of bounded degree k-uniform linear hypertrees in terms of k and Δ.

1 | Introduction

Let $\mathcal{H} = (V, E)$ be a hypergraph, where V denotes a finite set of vertices and E is a set of nonempty subsets of V. An independent set is a subset of the vertices $U \subseteq V$ such that $e \nsubseteq U$ for any $e \in E$. We let $\mathcal{I}(\mathcal{H})$ denote the set of independent sets of $\mathcal{H}$. If we assign to every vertex $v \in V$ a complex variable λ_v, we can define the multivariate independence polynomial of $\mathcal{H}$ as

$$Z(\mathcal{H}; \boldsymbol{\lambda}) = \sum_{U \in \mathcal{I}(\mathcal{H})} \prod_{v \in U} \lambda_v$$

We let $Z(\mathcal{H}; \lambda)$ denote the univariate specialization, that is, where all λ_v are set to λ.

The location of the zeros of the independence polynomial in the complex plane within the class of bounded degree graphs has been a topic gleaning a lot of interest [1–3]. Recently, this interest has been extended to the study of bounded degree hypergraphs [4]. The following two theorems contain our main contribution.

The first result concerns the largest zero-free disk around zero. Let $\lambda_s(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{\Delta^\Delta}$.

Theorem 1.1. *Let $\Delta \geq 2$. For any hypergraph $\mathcal{H}$ with maximum degree at most Δ and $\lambda \in \mathbb{C}^{V(\mathcal{H})}$ with $|\lambda_u| \leq \lambda_s(\Delta)$ for all $u \in V(\mathcal{H})$ we have $Z(\mathcal{H}; \lambda) \neq 0$.*

For graphs this bound was proved by Shearer [5]; see also [6]. For hypergraphs our result improves the bound from [4] from $\lambda_s(\Delta+1)$ to $\lambda_s(\Delta)$. This bound is the best possible since it is proved in [5, 6] that zeros of truncated $(\Delta-1)$-ary trees accumulate on $-\lambda_s(\Delta)$.

The second result concerns the largest possible open zero-free region around the positive real axis. Let $\lambda_c(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^\Delta}$.

Theorem 1.2. *Let $\Delta \geq 3$. There exists an open neighborhood U of the interval $(0, \lambda_c(\Delta))$ such that for any hypergraph $\mathcal{H}$ of maximum degree Δ and $\lambda \in U$ we have $Z(\mathcal{H}; \lambda) \neq 0$.*

For graphs this was conjectured by Sokal [7] and subsequently proved by Peters and Regts [8]. It is also proved in [8] that zeros of truncated $(\Delta-1)$-ary trees accumulate on $\lambda_c(\Delta)$. In fact, it is proved in [9], using results from [10], that zeros of graphs of maximum degree at most Δ accumulate on any real parameter in $(-\infty, -\lambda_s(\Delta)] \cup [\lambda_c(\Delta), \infty)$. Therefore, if we let $\mathcal{U}_{\Delta,2} \subseteq \mathbb{C}$ denote the maximal open zero-free set for graphs with maximum degree at most Δ and we let $\mathcal{U}_{\Delta,\geq 2} \subseteq \mathbb{C}$ denote the analogous set for hypergraphs with maximum degree at most Δ, we can conclude the following corollary.

Corollary 1.3. *Let $\Delta \geq 3$. We have*

$$\mathcal{U}_{\Delta,2} \cap \mathbb{R} = \mathcal{U}_{\Delta,\geq 2} \cap \mathbb{R} = \left(-\lambda_s(\Delta), \lambda_c(\Delta)\right)$$

In [4, Conjecture 6] it is conjectured that the connected component of $\mathcal{U}_{\Delta,2}$ containing 0 is equal to the connected component of $\mathcal{U}_{\Delta,\geq 2}$ containing 0. This conjecture is false for $\Delta = 1, 2$ because graphs of maximum degree at most two are real-rooted, while the independence polynomial of the hypergraph consisting of a single hyperedge with three vertices is $3\lambda^2 + 3\lambda + 1$, which has non-real roots. The conjecture is also false for $\Delta = 3$; see Corollary 6.9. We will however show that the conjecture is true in the $\Delta \to \infty$ limit.

Theorem 1.4. *The rescaled zero-free regions converge to the same limit object in terms of Hausdorff distance[1], that is,*

$$\lim_{\Delta\to\infty} \Delta \cdot \mathcal{U}_{\Delta,2} = \lim_{\Delta\to\infty} \Delta \cdot \mathcal{U}_{\Delta,\geq 2}$$

A priori it is not clear that either limit is well defined. In [11], the two authors of the present paper together with Han Peters investigated the left-hand limit and showed that it does indeed converge to a non-trivial limit set.

1.1 | Uniform Linear Hypertrees

A hypergraph is called linear if for any pair of distinct edges e_1, e_2, we have $|e_1 \cap e_2| \leq 1$. A path from vertex v_1 to vertex v_2 in a linear hypergraph is a tuple $(v_1 = u_1, e_1, u_2, e_2, \ldots, e_{n-1}, u_n = v_2)$, where $u_1, \ldots, u_n$ is a sequence of distinct vertices and $e_1, \ldots, e_{n-1}$ is a sequence of distinct edges such that $\{u_i, u_{i+1}\} \subseteq e_i$ for all $i = 1, \ldots, n-1$. A linear hypergraph is called a linear hypertree if there is a unique path between any pair of vertices. A hypergraph is called k-uniform if every hyperedge has size k.

In [4], the largest possible zero-free disk for bounded degree uniform linear hypertrees is investigated. They give an explicit radius $r_{d,b}$ such that for any $b+1$ uniform linear hypertree $\mathcal{T}$ with maximum degree at most $d+1$ and fugacities λ with $|\lambda_v| \leq r_{d,b}$ it is the case that $Z(\mathcal{T}; \lambda) \neq 0$; see [4, Theorem 4]. As $d \to \infty$ one has that $r_{d,b} = e^{-1} d^{-1/b} + \mathcal{O}(d^{-2/b})$. They also show that a disk of radius $\mathcal{O}((\log(d)/d)^{1/b})$ is not zero-free; see [4, Proposition 5]. In this paper, we determine the exact maximal zero-free disk for uniform bounded degree linear hypertrees.

Theorem 1.5. *Let $d \geq 2$, $b \geq 1$ and define the function $f_\lambda(z) = f_{\lambda,b,d}(z)$ as*

$$f_\lambda(z) = \lambda\left(1 - \left(\frac{z}{1+z}\right)^b\right)^d$$

Let $\rho_{d,b}$ be the largest $\rho \geq 0$, such that there exists a disk B_R of radius R around 0 that is mapped into itself by f_ρ. To be precise,

$$\rho_{d,b} = \max\{\rho \geq 0 \mid \text{there exists an } R > 0 \text{ such that } f_\rho(B_R) \subseteq B_R\}$$

Zero-freeness *If $\mathcal{T}$ is a $(b+1)$-uniform linear hypertree with maximum degree at most $d+1$ and $\lambda \in \mathbb{C}^{V(\mathcal{T})}$ is such that $|\lambda_v| \leq \rho_{d,b}$ for all $v \in V(\mathcal{T})$, then $Z(\mathcal{T};\lambda) \neq 0$.*

Maximality *There is a sequence of $(b+1)$-uniform linear hypertrees $\{\mathcal{T}_n\}$ with maximum degree at most $d+1$ and a sequence of parameters $\{\lambda_n\}$ such that $|\lambda_n|$ converges to $\rho_{d,b}$ and $Z(\mathcal{T}_n;\lambda_n) = 0$ for all n.*

Formula

$$\rho_{d,b} = \min\left\{ \left|(1-w^b)^{-d}\frac{w}{(1-w)}\right| : w \in \mathbb{C} \text{with } -bd\frac{w^b(1-w)}{(1-w^b)} = 1 \right\}$$

Asymptotics *For fixed b*

$$\rho_{d,b} = (ebd)^{-1/b} + \mathcal{O}(d^{-2/b})$$

as $d \to \infty$.

Contrary to the proof of Theorem 1.1, where we use the explicit formula for $\lambda_s(\Delta)$, we will not prove Theorem 1.5 using the formula. Instead, we will show that if a natural sufficient zero-freeness condition, namely forward invariance of the map f_λ, is not met this implies that there is a λ' of the same magnitude as λ for which $f_{\lambda'}$ has neutral fixed point. This will allow us to prove that zeros accumulate on λ' and to give an explicit formula for λ' and its magnitude.

Question 1.6. A few days before appearance of the present paper it was shown by Zhang [12] that for each $k \geq 3$ there exists a sequence of k-uniform linear hypergraphs with zeros of magnitude $\mathcal{O}(\log(\Delta)/\Delta)$. It follows that in general the disk of radius $\rho_{d,b}$ does not remain zero-free if one moves from uniform linear hypertrees to uniform linear hypergraphs. Describing the true asymptotics of the maximal zero-free disk for uniform bounded degree linear hypergraphs remains an open problem. The Shearer disk [5] shows that it is at least of size $\lambda_s(\Delta) = (e\Delta)^{-1} + \mathcal{O}(\Delta^{-2})$.

1.2 | Algorithmic Consequence

Let $\mathcal{F}_{\Delta,k}$ be the family of hypergraph of maximum degree at most Δ and of hyperedge size at most k. In the paper [4], an FPTAS[2] was established for computing $Z(\mathcal{H};\lambda)$ for the family $\mathcal{F}_{\Delta,k}$ and $|\lambda| < \lambda_s(\Delta+1)$. The proof of [4, Theorem 10] is based on 3 ingredients: Barvinok's interpolation method [13], the work of Liu, Sinclair and Srivastava [14] and Patel and Regts [15], and the zero-free region given in [4, Theorem 1]. In generality, they proved the following.

Theorem 1.7 ([4]). *Let $\Delta \geq 2$ and $k \geq 2$. Assume that there is a connected open set $0 \in U \subseteq \mathbb{C}$, such that for any $\mathcal{H} \in \mathcal{F}_{\Delta,k}$ and $\lambda \in U$ we have $Z(\mathcal{H};\lambda) \neq 0$. Then for any $\lambda \in U$ there is an algorithm of running time $(n/\epsilon)^{O_{k,\Delta}(1)}$ such that it computes an ϵ-relative approximation to $Z(\mathcal{H};\lambda)$ for any hypergraph $\mathcal{H} \in \mathcal{F}_{\Delta,k}$ on n vertices.*

There are several works on the complexity of counting independent sets in hypergraphs, that is, calculating $Z(\mathcal{H};1)$; see [4, 16–20]. By combining the previous theorem with Theorems 1.2 and 1.1 we obtain the following corollary.

Corollary 1.8. *Let $\Delta \geq 2$ and $k \geq 2$. Let $\lambda \in \mathbb{C}$, such that either $|\lambda| < \lambda_s(\Delta)$ or $0 < \lambda < \lambda_c(\Delta)$. Then there is an algorithm of running time $(n/\epsilon)^{O_{k,\Delta}(1)}$ such that it computes an ϵ-relative approximation to $Z(\mathcal{H};\lambda)$ for any hypergraph $\mathcal{H} \in \mathcal{F}_{\Delta,k}$ on n vertices.*

This result recovers the result of Liu and Lu [19], where they give an FPTAS for the number of independent sets (i.e., $\lambda = 1$) for the class of hypergraphs of maximum degree at most 5. More generally, this corollary extends the well-known result of the existence of an FPTAS for $\lambda \in (0, \lambda_c(\Delta))$ by Weitz [21] from bounded degree graphs to bounded degree hypergraphs. It is known that for $\lambda < -\lambda_s(\Delta)$ and $\lambda > \lambda_c(\Delta)$ approximating the independent set polynomial for bounded degree graphs is NP-hard; see [10] and [22] for the two regimes. We thus show that the computational dichotomy for real λ that was known for bounded degree graphs extends to bounded degree hypergraphs.

Question 1.9. Let $\mathcal{U}_{\Delta,\geq k}$ denote the maximal open zero-free set for hypergraphs with maximum degree at most Δ with hyperedges of size at least k and let $U_{\Delta,\geq k} \subseteq \mathcal{U}_{\Delta,\geq k}$ denote the component containing 0.

In [17], it is shown that there is an FPTAS for calculating the number of independent sets among hypergraphs of degree at most 6 and minimum hyperedge size 3. Is it possible to recover this result using zero-freeness? Concretely, do we have $1 \in U_{6,\geq 3}$? More generally: what is $\mathbb{R} \cap U_{\Delta,\geq k}$ for $k \geq 3$ and is it strictly increasing in k? See also [20] for recent zero-freeness results involving uniform bounded degree hypergraphs.

A different consequence of our zero-freeness result concerns the mixing time of a Markov chain known as Glauber dynamics. For positive real λ Glauber dynamics are used to approximately sample from the Gibbs distribution, that is, the distribution on $\mathcal{I}(\mathcal{H})$ where the probability of sampling U is proportional to $\lambda^{|U|}$. Anari et al. [23] proposed a sufficient condition for the rapid mixing of the Glauber dynamics known as spectral independence, which was investigated for special Gibbs distributions [24, 25]. Furthermore, it was shown in [26, Theorem 11] that zero-freeness in a neighborhood of λ implies spectral independence for a large class of models on bounded degree (hyper)graphs, including the hard-core model on hypergraphs. These results together with Theorem 1.2 prove rapid mixing of the Glauber dynamics for bounded degree hypergraphs for $\lambda \in (0, \lambda_c(\Delta))$.

1.3 | Overview of the Paper

The proofs of the main theorems rely on the dynamical behavior of ratios associated to the independence polynomials. For a hypergraph $\mathcal{H}$ and a vertex $v \in V(\mathcal{H})$, we can define the ratio as

$$R_v(\mathcal{H}, \lambda) = \frac{Z_v^{\text{in}}(\mathcal{H}, \lambda)}{Z_v^{\text{out}}(\mathcal{H}, \lambda)}$$

where $Z_v^{\text{in}}(\mathcal{H}, \lambda)$ is the sum of the terms of the partition function of $Z(\mathcal{H}; \lambda)$ with v contained in the independent set and $Z_v^{\text{out}}(\mathcal{H}; \lambda) = Z(\mathcal{H}; \lambda) - Z_v^{\text{in}}(\mathcal{H}; \lambda)$.

In Section 2, we build on the reduction of Weitz [21] to reduce the analysis of the ratios of hypergraphs to the analysis of ratios of linear hypertrees. It turns out that the ratios of linear hypertrees can be calculated recursively from smaller subhypertrees using appropriate multivariate functions.

In Section 3, we use this reduction to show that establishing forward invariant sets in the complex plane for these (multivariate) functions is a sufficient condition for zero-freeness. We gather results of this form for both the univariate and the multivariate case. This section also contains the proof of Theorem 1.1.

In Section 4, we show that we can satisfy the sufficient conditions for zero-freeness in the univariate case laid out in Lemma 3.5 for $\lambda \in (0, \lambda_c(\Delta))$. This leads to a proof of Theorem 1.2.

In Section 2, we use a characterization of the limit set $\mathcal{U} = \lim_{\Delta\to\infty} \Delta \cdot \mathcal{U}_{\Delta,2}$ given in [11]. This allows us to show that for $\Lambda \in \mathcal{U}$ and Δ large enough there exists a region A_Δ satisfying the conditions of Lemma 3.5 for the parameter Λ/Δ. This leads to a proof of Theorem 1.4.

In Section 6, we use Lemma 3.3 to establish a zero-free disk for k-uniform linear hypertrees. In fact, we show that the smallest disk to which we cannot apply Lemma 3.3 implies that the map f_λ given in Equation (9) has a neutral fixed point. This will allow us both to prove that zeros accumulate on λ, following the method in [27], and to establish a formula for λ, proving Theorem 1.5. In this section, we rely on methods from the field of complex dynamics; in particular, we use the theory of normal families and holomorphic motion.

2 | Preliminaries

2.1 | Independence Polynomial of Hypergraphs

For a hypergraph $\mathcal{H}$ and its vertex $v \in V(\mathcal{H})$ let us define the ratio of $R_v(\mathcal{H}, \lambda)$ as the multivariate rational function

$$R_v(\mathcal{H}; \lambda) = \frac{Z_v^{\text{in}}(\mathcal{H}; \lambda)}{Z_v^{\text{out}}(\mathcal{H}; \lambda)}$$

where $Z_v^{\text{in}}(\mathcal{H};\lambda)$ (resp. $Z_v^{\text{out}}(\mathcal{H};\lambda)$) is the sum of those terms of the partition function of $Z(\mathcal{H};\lambda)$ where v is in (resp. out) the independent set of U. To be precise,

$$[[spispace]]Z_v^{\text{in}}(\mathcal{H};\lambda)=\sum_{v\in U\in\mathcal{I}(\mathcal{H})}\prod_{u\in I}\lambda_u$$

and $Z_v^{\text{out}}(\mathcal{H};\lambda)=Z(\mathcal{H};\lambda)-Z_v^{\text{in}}(\mathcal{H};\lambda)$. In general, for $S\subseteq V$ we can define

$$Z_S^{\text{in}}(\mathcal{H};\lambda)=\sum_{S\subseteq U\in\mathcal{I}(\mathcal{H})}\prod_{u\in U}\lambda_u$$

Before establishing basic recursions, we need the following notations and conventions for the rest of this section.

Definition 2.1. Let $\mathcal{H}$ be a hypergraph on the vertex set V. We can then consider the following ways to obtain subhypergraphs of $\mathcal{H}$.

- If $S\subseteq V$, then $\mathcal{H}-S$ is the induced subhypergraph on $V\setminus S$. That is, a hypergraph with vertex set $V\setminus S$ and with hyperedges
$$\{e\in E\mid e\cap S=\emptyset\}$$
- If $S\subseteq V$, then $\mathcal{H}\ominus S$ is the hypergraph obtained by removing all vertices of S from each hyperedge. If the hypergraph obtains empty hyperedges, then we delete those. Formally, $\mathcal{H}\ominus S$ is a hypergraph on $V\setminus S$ with hyperedges
$$\{e\setminus S\mid e\in E \text{ s.t. } e\cap S\neq e\}$$
- If $F\subseteq E$, then $\mathcal{H}-F$ is the hypergraph obtained by removing the hyperedges F from the hypergraph. That is, a hypergraph with vertex set V and with hyperedges
$$\{e\in E\mid e\notin F\}$$

A hypergraph $\mathcal{H}'$ is a subhypergraph of $\mathcal{H}$ if $\mathcal{H}'$ can be obtained from $\mathcal{H}$ by applying any of the previous operations and/or their combinations. We will use the convention in this case that $Z(\mathcal{H}';\lambda)$ denotes $Z\left(\mathcal{H}';\lambda\big|_{V(\mathcal{H}')}\right)$.

Now, let us collect basic relations following from the definition of independent sets. We refer to [28] for their proofs.

Lemma 2.2. *Let $\mathcal{H}=(V,E)$ be a hypergraph and $v\in V$ such that $\{v\}\notin E$. Then*

- $Z_v^{\text{in}}(\mathcal{H},\lambda)=\lambda_v Z(\mathcal{H}\ominus v,\lambda)$.
- $Z_v^{\text{out}}(\mathcal{H},\lambda)=Z(\mathcal{H}-v,\lambda)=Z(\mathcal{H}-E(v)-v,\lambda)$, *where* $E(v)=\{e\in E\mid v\in e\}$.
- $Z(\mathcal{H};\lambda)=Z(\mathcal{H}-\{e\};\lambda)-Z_e^{\text{in}}(\mathcal{H}-\{e\};\lambda)$.

A short observation of the previous lemma is that the independence polynomial of a hypergraph is actually a product of independence polynomials of connected hypergraphs where no hyperedge is contained in an other, moreover each hyperedge has size at least 2.

Lemma 2.3. *Let $\mathcal{H}=(V,E)$ be a hypergraph and let $V_1=\{v\in V\mid\{v\}\in E\}$ and $E'=\{e\in E\mid\exists f\in E \text{ s.t. } f\subset e\}\cup\{e\in E\mid |e|=1\}$. Then the hypergraph $\mathcal{H}-E'-V_1$ is disjoint union of hypergraphs of hyperedge size at least 2 and there is no hyperedge that contains an other one, moreover*

$$Z(\mathcal{H};\lambda)=Z(\mathcal{H}-E'-V_1;\lambda)$$

The next lemma is the generalized form of the tree recursion of the independence polynomial of trees for hypertrees.

Lemma 2.4. *Let $\mathcal{T}=(V,E)$ be a linear hypertree and v a vertex. Suppose that $E(v)=\{e_1,\dots,e_d\}\subseteq E$ is the set of the incident edges at v and $e_i=\{v,v_1^{(i)},\dots,v_{b_i}^{(i)}\}$ for $i=1,\dots,d$. Let $\mathcal{T}_{i,j}$ be the connected component of $v_j^{(i)}$ in $\mathcal{T}-v$. Then we have*

$$R_v(\mathcal{T};\lambda) = \lambda_v \prod_{i=1}^{d}\left(1 - \prod_{j=1}^{b_i} \frac{R_{v_j^{(i)}}(\mathcal{T}_{i,j};\lambda)}{1 + R_{v_j^{(i)}}(\mathcal{T}_{i,j};\lambda)}\right)$$

Proof. Let us recall that $Z(\mathcal{T};\lambda) = Z_v^{\text{in}}(\mathcal{T};\lambda) + Z_v^{\text{out}}(\mathcal{T};\lambda)$. If $\{v\} \in E(\mathcal{T})$, then $R_v(\mathcal{T};\lambda) = 0$ by definition.

Thus for the rest let us assume that $\{v\} \notin E$. Then by Lemma 2.2 we have

$$R_v(\mathcal{T};\lambda) = \frac{Z_v^{\text{in}}(\mathcal{T};\lambda)}{Z_v^{\text{out}}(\mathcal{T};\lambda)} = \frac{\lambda_v Z(\mathcal{T} \ominus \{v\};\lambda)}{Z(\mathcal{T} - E(v) - v;\lambda)}$$

Observe that the hypergraphs $\mathcal{T}_1 = \mathcal{T} \ominus \{v\}$ and $\mathcal{T}_0 = \mathcal{T} - E(v) - v$ are defined on the same vertex set and their edge set differs only in $E' = \{e \setminus \{v\} \mid e \in E(v)\}$. Since $\mathcal{T}$ is a linear hypertree each connected component of $\mathcal{T}_1$ and $\mathcal{T}_0$ is a linear hypertree. The hyperforest $\mathcal{T}_0$ is isomorphic to the disjoint union of the hypertrees $\mathcal{T}_{i,j}$ for $i = 1, \dots, d$ and $j = 1, \dots, b_i$. Thus, $\mathcal{T}_1$ has exactly d connected components, that is, for every $i = 1, \dots, d$ we can obtain the connected component as the disjoint union of $\mathcal{T}_{i,j}$ for $j = 1, \dots, b_i$ and adding the edge $e_i' = e_i \setminus \{v\}$. Let us call these connected components $\mathcal{T}_i$.

Thus, we have that

$$R_v(\mathcal{T},\lambda) = \lambda_v \prod_{i=1}^{d} \frac{Z(\mathcal{T}_i,\lambda)}{Z(\mathcal{T}_i - \{e_i'\};\lambda)} = \lambda_v \prod_{i=1}^{d} \frac{Z(\mathcal{T}_i - \{e_i'\},\lambda) - Z_{e_i'}^{\text{in}}(\mathcal{T}_i - \{e_i\},\lambda)}{Z(\mathcal{T}_i - \{e_i'\};\lambda)}$$

To finish the proof let us observe that for each $i = 1, \dots, d$, we have

$$\begin{aligned}
\frac{Z(\mathcal{T}_i - \{e_i'\},\lambda) - Z_{e_i'}^{\text{in}}(\mathcal{T}_i - \{e_i\},\lambda)}{Z(\mathcal{T}_i - \{e_i'\};\lambda)} &= 1 - \frac{Z_{e_i'}^{\text{in}}(\mathcal{T}_i - \{e_i\},\lambda)}{Z(\mathcal{T}_i - \{e_i'\};\lambda)} \\
&= 1 - \frac{\prod_{j=1}^{b_i} Z_{v_i^{(j)}}^{\text{in}}(\mathcal{T}_{i,j};\lambda)}{\prod_{j=1}^{b_i} Z(\mathcal{T}_{i,j};\lambda)} \\
&= 1 - \prod_{j=1}^{b_i} \frac{R_{v_j^{(i)}}(\mathcal{T}_{i,j};\lambda)}{1 + R_{v_j^{(i)}}(\mathcal{T}_{i,j};\lambda)}
\end{aligned}$$

□

2.2 | Reduction to Hypertrees

In the next definition, we propose a hypergraph version of Weitz self-avoiding path tree [21] that allows us to reduce the problem of understanding zeros of hypergraphs to linear hypertrees. This idea was implicitly used in [19].

Definition 2.5 (Weitz-hypertrees). Let $\mathcal{H}$ be a hypergraph and $v \in V(\mathcal{H})$. We will define the set $\mathcal{W}(\mathcal{H}, v)$ whose elements are tuples $(\mathcal{T}, r, \pi)$, such that $\mathcal{T}$ is a linear hypertree with root $r \in V(\mathcal{T})$ and $\pi : V(\mathcal{T}) \to V(\mathcal{H})$ is a labeling function with $\pi(r) = v$. The set is defined inductively as follows.

- If the connected component $\mathcal{H}'$ of v consists of a single vertex, then $\mathcal{W}(\mathcal{H}, v) = \{(\mathcal{H}', v, \text{id}_{V(\mathcal{H}')})\}$.
- Otherwise, each element of $\mathcal{W}(\mathcal{H}, v)$ is constructed as
  1. Choose an ordering $\{e_1, \dots, e_d\}$ of $E(v)$.
  2. For each $i = 1, \dots, d$ choose an ordering $\{v_0^{(i)}, \dots, v_{b_i}^{(i)}\}$ of e_i with $v_0^{(i)} = v$.
  3. For each $i = 1, \dots, d$ and $j = 1, \dots, b_i$ let $\mathcal{H}_j^{(i)}$ be the connected component of $v_j^{(i)}$ of $\mathcal{H} - \{e_1, \dots, e_i\} \ominus \{v_0^{(i)}, \dots, v_{j-1}^{(i)}\}$.
  4. For each $i = 1, \dots, d$ and $j = 1, \dots, b_i$ choose $(\mathcal{T}_j^{(i)}, r_j^{(i)}, \pi_j^{(i)}) \in \mathcal{W}(\mathcal{H}_j^{(i)}, v_j^{(i)})$.
  5. Take disjoint union of the hypertrees $\mathcal{T}_j^{(i)}$, then add a new vertex r and add the hyperedges $e_i' := \{r, r_1^{(i)}, \dots, r_{b_i}^{(i)}\}$ for every $i = 1, \dots, d$. Denote the constructed graph by $\mathcal{T}$. Let $\pi : V(\mathcal{T}) \to V(\mathcal{H})$ be defined as

$$\pi(u) = \begin{cases} v & \text{if } u = r \\ \pi_j^{(i)}(u) & \text{if } u \in V(\mathcal{T}_j^{(i)}) \end{cases}$$

Any possible choice in steps 1, 2, and 4 yields an element $(\mathcal{T}, r, \pi) \in \mathcal{W}(\mathcal{H}, v)$.

Remark 2.6. Given a rooted hypergraph $(\mathcal{H}, v)$ we will sometimes refer to its Weitz-hypertree $\mathcal{T}$ whenever we refer to an arbitrary element $(\mathcal{T}, r, \pi) \in \mathcal{W}(\mathcal{H}, v)$. If $\mathcal{H}$ has degree at most Δ and edge-size at most $b+1$, then the same is true for its Weitz-hypertree. If $\mathcal{H}$ is a linear hypertree, then its Weitz-hypertree is equal to $\mathcal{H}$ itself, that is, $\mathcal{W}(\mathcal{H}, v) = \{(\mathcal{H}, v, \text{id})\}$. We also note that $\mathcal{W}(\mathcal{H}, v) = W(\mathcal{H}', v)$, where $\mathcal{H}'$ is the connected component of $\mathcal{H}$ containing v.

Definition 2.5 gives a recursive way to obtain elements of $\mathcal{W}(\mathcal{H}, v)$. In Section 2.3, we will describe an explicit element of $\mathcal{W}(\mathcal{H}, v)$ whose vertices (and hyperedges) correspond to stable-paths in the bipartite representation of $\mathcal{H}$; see also Figure 1.

Theorem 2.7. *Let $\mathcal{H}$ be a hypergraph with vertex $v \in V(\mathcal{H})$, then for any $(\mathcal{T}, r, \pi) \in \mathcal{W}(\mathcal{H}, v)$ we have*

$$\frac{Z_v^{\text{in}}(\mathcal{H}; \lambda)}{Z_v^{\text{out}}(\mathcal{H}; \lambda)} = \frac{Z_r^{\text{in}}(\mathcal{T}, \lambda')}{Z_r^{\text{out}}(\mathcal{T}, \lambda')}$$

as rational functions, where for any $u \in V(\mathcal{T})$ we let $\lambda'_u = \lambda_{\pi(u)}$.

Proof. Proof by induction on the number of vertices of $\mathcal{H}$.

First of all we may assume that $\mathcal{H}$ is connected, since otherwise we could take $\mathcal{H}'$ to be the connected component of v for which the ratio is the same function allowing us to conclude the theorem by induction.

If $|V(\mathcal{H})| = 1$ the theorem holds because $\mathcal{H}$ is isomorphic to $\mathcal{T}$.

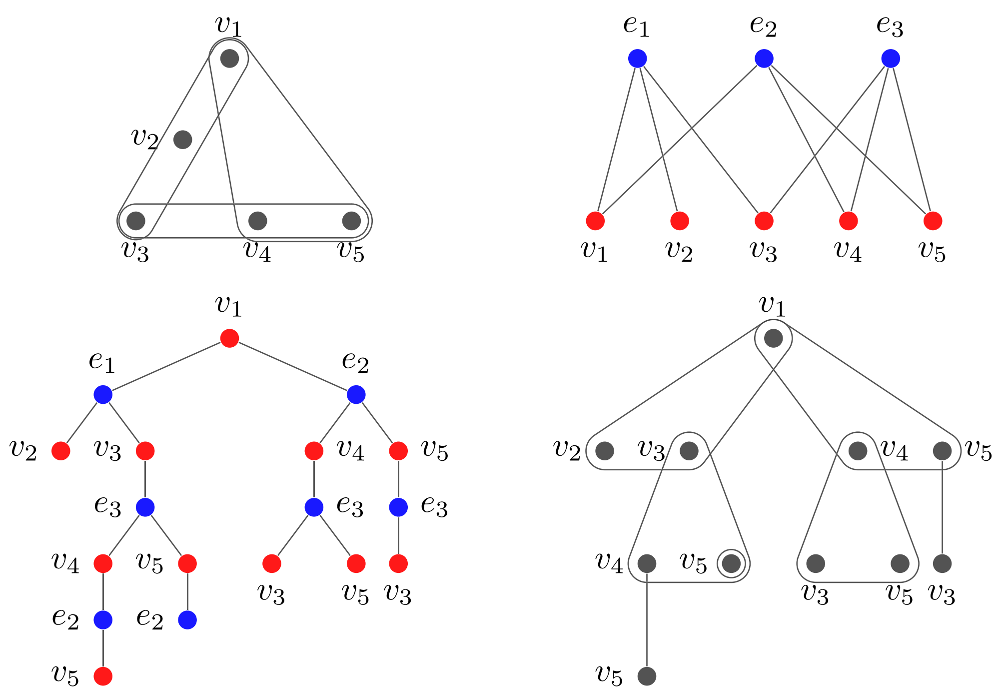

FIGURE 1 | The top left contains an example of a hypergraph $\mathcal{H}$. The top right contains its bipartite representation $G_{\mathcal{H}}$. The bottom left the stable-path tree of $G_{\mathcal{H}}$ rooted at v_1 if we order the vertices of $G_{\mathcal{H}}$ as $v_1 \leq v_2 \leq v_3 \leq v_4 \leq v_5 \leq e_1 \leq e_2 \leq e_3$. The bottom right represents the hypergraph corresponding to this bipartite representation. This is an element $(\mathcal{T}, r, \pi)$ of $\mathcal{W}(\mathcal{H}, v)$. In this case, if we let $\lambda_{v_i} = \lambda_i$, then $Z(\mathcal{T}; \lambda') = Z(\mathcal{H}; \lambda')(1 + \lambda_3)(1 + \lambda_3 + \lambda_5)(1 + \lambda_4 + \lambda_5)$.

Now let us assume that for any connected hypergraph on $n \geq 1$ vertices the statement holds. Let $\mathcal{H}$ be a connected hypergraph on $n+1$ vertices where v has degree $d \geq 1$ and let $(\mathcal{T}, r, \pi) \in \mathcal{W}(\mathcal{H}, v)$. Using the same notation as in items 1 and 5 of Definition 2.5 we define the sets $E_i = \{e_1, \dots, e_i\} \subseteq E(v) \subseteq E(\mathcal{H})$ and $E'_i = \{e'_1, \dots, e'_i\} \subseteq E(r) \subseteq E(\mathcal{T})$.

Then

$$\frac{Z_v^{\text{in}}(\mathcal{H}; \lambda)}{Z_v^{\text{out}}(\mathcal{H}; \lambda)} = \frac{\lambda_v Z(\mathcal{H} \ominus v; \lambda)}{Z(\mathcal{H} - E(v) \ominus v; \lambda)} = \lambda_v \prod_{i=1}^{d} \frac{Z(\mathcal{H} - E_{i-1} \ominus v; \lambda)}{Z(\mathcal{H} - E_i \ominus v; \lambda)}$$

and similarly we have

$$\frac{Z_r^{\text{in}}(\mathcal{T}; \lambda')}{Z_r^{\text{out}}(\mathcal{T}; \lambda')} = \lambda_v \prod_{i=1}^{d} \frac{Z(\mathcal{T} - E'_{i-1} \ominus v; \lambda')}{Z(\mathcal{T}' - E'_i \ominus v; \lambda')}$$

Thus it is sufficient to prove that for each $i = 1, \dots, d$ we have

$$\frac{Z(\mathcal{H} - E_{i-1} \ominus v; \lambda)}{Z(\mathcal{H} - E_i \ominus v; \lambda)} = \frac{Z(\mathcal{T} - E'_{i-1} \ominus r; \lambda')}{Z(\mathcal{T} - E'_i \ominus r; \lambda')}$$

Let us further reduce this formula by using the same notations as the Definition 2.5, thus let us define the sets $S_j = \{v_1^{(i)}, \dots, v_{j-1}^{(i)}\}$ for $j = 1, \dots, b_i + 1$. Let $\mathcal{H}' = \mathcal{H} - E_{i-1} \ominus v$, then

$$\frac{Z(\mathcal{H} - E_{i-1} \ominus v; \lambda)}{Z(\mathcal{H} - E_i \ominus v; \lambda)} = \frac{Z(\mathcal{H}'; \lambda)}{Z(\mathcal{H}' - \{e_i\}; \lambda)} = \frac{Z(\mathcal{H}' - \{e_i\}; \lambda) - Z_{e_i}^{\text{in}}(\mathcal{H}' - \{e_i\}; \lambda)}{Z(\mathcal{H}' - \{e_i\}, \lambda)}$$
$$= 1 - \frac{Z_{e_i}^{\text{in}}(\mathcal{H}' - \{e_i\}, \lambda)}{Z(\mathcal{H}' - \{e_i\}, \lambda)}$$

and similarly we have

$$\frac{Z(\mathcal{T} - E'_{i-1} \ominus r; \lambda')}{Z(\mathcal{T} - E'_i \ominus r; \lambda')} = 1 - \frac{Z_{e'_i}^{\text{in}}(\mathcal{T}_j^{(i)} - \{e'_i\}, \lambda')}{Z(\mathcal{T}_j^{(i)} - \{e'_i\}, \lambda')}$$

This means that it is sufficient to prove that

$$\frac{Z_{e_i}^{\text{in}}(\mathcal{H}' - \{e_i\}, \lambda)}{Z(\mathcal{H}' - \{e_i\}, \lambda)} = \frac{Z_{e'_i}^{\text{in}}(\mathcal{T}_j^{(i)} - \{e'_i\}, \lambda')}{Z(\mathcal{T}_j^{(i)} - \{e'_i\}, \lambda')}$$

Now we claim that for any $i = 1, \dots, d$ we have

$$\frac{Z_{e_i}^{\text{in}}(\mathcal{H}' - \{e_i\}, \lambda)}{Z(\mathcal{H}' - \{e_i\}, \lambda)} = \prod_{j=1}^{b_i} \frac{Z_{v_j^{(i)}}^{\text{in}}(\mathcal{H}_j^{(i)}; \lambda)}{Z(\mathcal{H}_j^{(i)}; \lambda)}$$

as rational functions. If $\mathcal{H}'$ does not have a hyperedge f such that $f \subseteq e_i$, then it holds, because in this case for any $S \subseteq e_i$ we have $Z_S^{\text{in}}(\mathcal{H}' - \{e_i\}; \lambda) = Z(\mathcal{H}' - \{e_i\} \ominus S)$ and thus

$$\frac{Z_{e_i}^{\text{in}}(\mathcal{H}' - \{e_i\}, \lambda)}{Z(\mathcal{H}' - \{e_i\}, \lambda)} = \prod_{j=1}^{b_i} \frac{Z_{S_j}^{\text{in}}(\mathcal{H}' - \{e_i\}; \lambda)}{Z_{S_{j-1}}^{\text{in}}(\mathcal{H}' - \{e_i\}; \lambda)} = \prod_{j=1}^{b_i} \frac{Z_{v_j^{(i)}}^{\text{in}}(\mathcal{H}_j^{(i)}; \lambda)}{Z(\mathcal{H}_j^{(i)}; \lambda)}$$

as rational functions. Otherwise, if there is a hyperedge contained in e_i, then the left-hand side is 0. But in this case the right-hand side is 0 as well. To see this, let us choose j to be the smallest integer such that S_j contains a hyperedge of $\mathcal{H}' - \{e_i\}$. Then by construction $\mathcal{H}_j^{(i)}$ contains the hyperedge $\{v_j^{(i)}\}$, which means that

$$\frac{Z_{v_j^{(i)}}^{\text{in}}(\mathcal{H}_j^{(i)}; \lambda)}{Z(\mathcal{H}_j^{(i)}; \lambda)} = 0$$

proving the claim.

To finish the proof observe that by definition and the induction hypothesis for any $i = 1, \dots, d$ and $j = 1, \dots, b_i$ we have

$$\frac{Z^{\text{in}}_{v_j^{(i)}}(\mathcal{H}_j^{(i)}; \lambda)}{Z^{\text{out}}_{v_j^{(i)}}(\mathcal{H}_j^{(i)}; \lambda)} = \frac{Z^{\text{in}}_{r_j^{(i)}}(\mathcal{T}_j^{(i)}; \lambda')}{Z^{\text{out}}_{r_j^{(i)}}(\mathcal{T}_j^{(i)}; \lambda')}$$

thus

$$\frac{Z^{\text{in}}_{e_i}(\mathcal{H}' - \{e_i\}, \lambda)}{Z(\mathcal{H}' - \{e_i\}, \lambda)} = \prod_{j=1}^{b_i} \frac{Z^{\text{in}}_{v_j^{(i)}}(\mathcal{H}_j^{(i)}; \lambda)}{Z(\mathcal{H}_j^{(i)}; \lambda)} = \prod_{j=1}^{b_i} \frac{Z^{\text{in}}_{r_j^{(i)}}(\mathcal{T}_j^{(i)}; \lambda')}{Z(\mathcal{T}_j^{(i)}; \lambda')} = \frac{Z^{\text{in}}_{e_i}(\mathcal{T}_j^{(i)} - \{e_i'\}, \lambda')}{Z(\mathcal{T}_j^{(i)} - \{e_i'\}, \lambda')}$$

□

Remark 2.8. During the construction of the Weitz-hypertree, it could occur that there are hyperedges of size 1 in $\mathcal{T}$. We claim that the theorem remains valid for some linear hypertree $\mathcal{T}'$ where each hyperedge has size at least 2 if $\{v\} \notin E(\mathcal{H})$. To see this let $(\mathcal{T}, r)$ be a Weitz-hypertree of $\mathcal{H}$ with starting vertex $v \in V(\mathcal{H})$. Let $V_1 = \{v \in V(\mathcal{T}) \mid \{v\} \in E(\mathcal{T})\}$ and let $E_1 = \{e \in E(\mathcal{T}) \mid e \cap V_1 \neq \emptyset\}$. By definition any independent set of $\mathcal{T}$ has to avoid V_1, thus $Z(\mathcal{T}; \lambda) = Z(\mathcal{T} - E_1 - V_1; \lambda)$. The hypergraph $\mathcal{T} - E_1 - V_1$ is disjoint union of linear hypertrees $\mathcal{T}_i$ of hyperedge size at least 2 for $i = 1, \dots, k$. Assume that $\mathcal{T}_1$ is the connected component of r in $\mathcal{T} - E_1 - V_1$. Now as rational functions we have that

$$R_v(\mathcal{H}; \lambda) = R_r(\mathcal{T}; \lambda) = R_r(\mathcal{T} - E_1 - V_1; \lambda) = R_r(\mathcal{T}_1; \lambda)$$

In the next theorem, we will reveal a useful algebraic property of the Weitz-hypertree. It is an extension of an analogous divisibility relation for graphs by the first author of the present paper [29, Proposition 2.7].

Theorem 2.9. *Let $\mathcal{H}$ be a connected hypergraph with vertex v and let $(\mathcal{T}, r, \pi) \in \mathcal{W}(\mathcal{H}, v)$. Then*

$$Z(\mathcal{H}; \lambda) \mid Z(\mathcal{T}; \lambda')$$

where $\lambda'_u = \lambda_{\pi(u)}$ for $u \in V(\mathcal{T})$. In particular, if $Z(\mathcal{H}; \lambda) = 0$ for some $\lambda \in \mathbb{C}^{V(\mathcal{H})}$, then $Z(\mathcal{T}; \lambda') = 0$.

Moreover, there exists subhypergraphs $\{\mathcal{H}_j\}_{j=1}^{l}$ of $\mathcal{H}$ with vertex set contained in $V(\mathcal{H}) \setminus \{v\}$ and integers $k_j \geq 1$ for $j = 1, \dots, l$, such that

$$Z(\mathcal{T}; \lambda') = Z(\mathcal{H}; \lambda) \prod_{j=1}^{l} Z(\mathcal{H}_j; \lambda)^{k_j}$$

Proof. We know that $Z_v^{\text{out}}(\mathcal{H})$ is the independence polynomial of the hypergraph $\mathcal{H}' = \mathcal{H} - v$. Let $\mathcal{H}_1', \dots, \mathcal{H}_l'$ be the connected components of $\mathcal{H}'$. For each connected component of $\mathcal{H}_k$ let e_{i_k} be the incident edge at v with the smallest index such that e_{i_k} share a common vertex with $\mathcal{H}_k'$. The edge $e_{i_k} = \{v, v_1^{(i_k)}, \dots, v_{b_{i_k}}^{(i_k)}\}$ and let j_k denote the smallest index such that $v_{j_k}^{(i_k)} \in V(\mathcal{H}_k')$. By construction we know that $\mathcal{H}_{j_k}^{(i_k)}$ from Definition 2.5is isomorphic to $\mathcal{H}_k'$. Now by induction we have that $Z(\mathcal{H}_k'; \lambda) | Z(\mathcal{T}_{j_k}^{(i_k)}; \lambda')$ for each $k = 1, \dots, l$ and therefore

$$Z_v^{\text{out}}(\mathcal{H}; \lambda) = Z(\mathcal{H}'; \lambda) = \prod_{k=1}^{l} Z(\mathcal{H}_k'; \lambda) \,\Big|\, \prod_{k=1}^{l} Z(\mathcal{T}_{j_k}^{(i_k)}; \lambda') \,\Big|\, Z_r^{\text{out}}(\mathcal{T}; \lambda')$$

On the other hand, we know that as rational functions

$$Z(\mathcal{T}; \lambda') = \frac{Z_r^{\text{out}}(\mathcal{T}; \lambda')}{Z_v^{\text{out}}(\mathcal{H}; \lambda)} Z(\mathcal{H}; \lambda)$$

and thus $Z(\mathcal{H}; \lambda) | Z(\mathcal{T}; \lambda')$ because $\frac{Z_r^{\text{out}}(\mathcal{T}; \lambda')}{Z_v^{\text{out}}(\mathcal{H}; \lambda)}$ is a polynomial.

The second part follows by induction from the previous equation. □

Remark 2.10. It is not hard to see that if $\mathcal{H}$ is a connected hypergraph and there is no hyperedge $e \in E$, that is contained in an other hyperedge of $\mathcal{H}$, then the rooted hypertree $(\mathcal{T}', r)$ obtained in Remark 2.8 also satisfies the divisibility relation of the previous theorem, that is,

$$Z(\mathcal{H}; \lambda) \,\Big|\, Z(\mathcal{T}'; \lambda')$$

In particular, $\mathcal{T}'$ is a linear hypertree of hyperedge size at least 2.

2.3 | An Explicit Construction of a Weitz-Hypertree via Stable-Paths

In this section, we give an example of an element of $\mathcal{W}(\mathcal{H}, v)$ that reveals a connection to the tree construction given in [21, 29] by unfolding the recursive nature of Definition 2.5. One could derive a complete description of $\mathcal{W}(\mathcal{H}, v)$ using σ-stable-path trees from [29], however, the most general proof would be notationally cumbersome. Instead, we present the following example that captures the essential idea. It is not necessary to read this section to understand the rest of the paper.

For the rest of this section, let us fix an ordering of the disjoint union of vertices and hyperedges of $\mathcal{H}$. Any hypergraph $\mathcal{H}$ has a bipartite representation $G_{\mathcal{H}}$, that is, the bipartite graph with bipartition $(V(\mathcal{H}), E(\mathcal{H}))$, and edge set $\{(v, e) \in V(\mathcal{H}) \times E(\mathcal{H}) \mid v \in e\}$. Observe that the ordering of the vertices and hyperedges of $\mathcal{H}$ induces an ordering on the vertices of $G_{\mathcal{H}}$.

For an arbitrary graph G with an ordering on the vertices of G (e.g., as in $G_{\mathcal{H}}$) a *stable-path* from v is a path $(v_0, v_1, \dots, v_\ell)$ such that $v = v_0$ and if $v_i \sim v_j$ for $j > i$, then $v_j \geq v_{i+1}$. We can obtain a tree structure on the stable-paths starting from v by connecting two paths if their length differs by 1 and one is a subpath of the other. Let us denote this tree by $T_{G,v}$. Also, let us assign a root vertex to $T_{G,v}$, that is, (v) and a labeling $\pi : V(T_{G,v}) \to V(G)$, which maps any stable path to its last vertex.

This tree also has a recursive definition: if v is a singleton of G, then $T_{G,v} = \{v\}$. Otherwise, let $N_G(v) = \{v_1, \dots, v_d\}$ the neighbors in order. For each $i = 1, \dots, d$ let G_i be the connected component of v_i in $G - \{v, v_1, \dots, v_{i-1}\}$ and let $T_i = T_{G_i, v_i}$ with root r_i. Now the tree $T_{G,v}$ is obtained by the disjoint union of the trees $T_1, \dots, T_d$ and then adding a new vertex r and edges $\{(r, r_1), \dots, (r, r_d)\}$.

Proposition 2.11. *The stable-path tree of $G_{\mathcal{H}}$ from v is a bipartite representation of a Weitz hypertree of $\mathcal{H}$ from v.*

Proof. Let us prove the statement by induction on the number of vertices.

If $|V(\mathcal{H})| = 1$, then the statement holds trivially. Now, let us assume that the statement holds for hypergraphs of size at most n and let $\mathcal{H}$ be a hypergraph on n vertices. If $\mathcal{H}$ is not connected, then let $\mathcal{H}'$ be the connected component of v, that has size at most n. Thus, by induction, the stable-path tree of $G_{\mathcal{H}'}$ is a bipartite representation of a Weitz hypertree of $\mathcal{H}'$ from v. On the other hand, the stable-path tree of $G_{\mathcal{H}}$ from v is exactly the stable-path tree of $G_{\mathcal{H}'}$ from v and the Weitz hypertree of $\mathcal{H}$ from v is the same as the Weitz hypertree of $\mathcal{H}'$ from v, thus proving the statement in this case.

Now let us assume that $\mathcal{H}$ is connected. Using the global ordering on the vertices of $G_{\mathcal{H}}$, we enumerate the incident hyperedges at v as $E(v) = \{e_1, \dots, e_d\}$, and for each $i = 1, \dots, d$, we enumerate $e_i \setminus \{v\} = \{v_1^{(i)}, \dots, v_{b_i}^{(i)}\}$. Let us expand the definition of $T_{G_{\mathcal{H}},v}$. By definition, $T_{G_{\mathcal{H}},v}$ consists of a new vertex r, trees $T^{(i)}$, and additional edges $\{(r, r_i) \mid i = 1, \dots, d\}$, where $T^{(i)}$ is the stable-path tree of $G_{\mathcal{H}} - \{v, e_1, \dots, e_{i-1}\}$ from e_i with root r_i. (Note that $e_1, \dots, e_d$ correspond to vertices in $G_{\mathcal{H}}$.)

Now, for each $i = 1, \dots, d$, let us apply the definition for $T^{(i)}$ again. Thus, $T^{(i)}$ consists of a new vertex r_i and trees $T_j^{(i)}$, where $T_j^{(i)}$ is the stable-path tree of $G_j^{(i)} = G_{\mathcal{H}} - \{v, e_1, \dots, e_i\} - \{v_1^{(i)}, \dots, v_{j-1}^{(i)}\}$ from $v_j^{(i)}$ for $j = 1, \dots, b_i$.

Observe that the connected component of $v_j^{(i)}$ in $G_j^{(i)}$ is isomorphic to $G_{\mathcal{H}_j^{(i)}}$ as defined in Definition 2.5. Thus, by induction, $T_j^{(i)}$ is a bipartite representation of a Weitz-hypertree of $\mathcal{H}_j^{(i)}$ from $v_j^{(i)}$, which we denote by $\mathcal{T}_j^{(i)}$.

For each $i = 1, \dots, d$, the vertex r_i is connected to $r, r_1^{(i)}, \dots, r_{b_i}^{(i)}$. Therefore, T is a bipartite representation of a hypergraph that consists of the disjoint union of $\mathcal{T}_j^{(i)}$ for all $i = 1, \dots, d$ and $j = 1, \dots, b_i$, together with hyperedges $\{r, r_1^{(i)}, \dots, r_{b_i}^{(i)}\}$ for each $i = 1, \dots, d$. By Definition 2.5, this hypergraph is indeed a Weitz hypertree as claimed. □

Remark 2.12. One might wonder: what is the Weitz hypertree of a graph? This is a hypertree with hyperedges of size 1 and 2. As we saw in Remark 2.8 we can reduce this tree by deleting vertices contained in hyperedges of size 1. The resulting tree is the stable-path tree of the graph.

3 | Ratios and Conditions for Zero-Freeness

In the previous section, we saw that a zero of a hypergraph yields a zero for a corresponding linear hypertree. In this section, we will show that in fact a zero of a hypergraph implies that there is a rooted linear hypertree with ratio equal to -1. It follows from Lemma 2.4 that the ratios of linear hypertrees can be calculated recursively. Roughly, to show zero-freeness for hypergraphs, it is therefore sufficient to show that the ratios of an appropriate class of rooted linear hypertrees stay trapped in a region of the complex plane under application of the appropriate set of maps. We will make this more precise for the multivariate case in Section 3.2, where we will also prove Theorem 1.1, and for the univariate case in Section 3.3.

The key relation between zeros of hypergraphs and -1 values of ratios of linear hypertrees is given by the following lemma. This lemma is a generalization of [9, Lemma 2.1], which contains a similar statement for graphs.

Lemma 3.1. *Let $A \subseteq \mathbb{C} \setminus \{-1\}$ be fixed. Then the following are equivalent*:

- *There exists a hypergraph $\mathcal{H}$ of degree at most Δ and hyperedge size at most $b+1$ and $\lambda \in A^{V(\mathcal{H})}$ such that $Z(\mathcal{H};\lambda) = 0$.*
- *There exists a linear hypertree $\mathcal{T}$ of degree at most Δ and hyperedge size at most $b+1$ and $\lambda \in A^{V(\mathcal{T})}$ such that $Z(\mathcal{T};\lambda) = 0$.*
- *There exists a linear hypertree $\mathcal{T}$ of degree at most Δ and hyperedge size at most $b+1$, $\lambda \in A^{V(\mathcal{T})}$ and a vertex $r \in V(\mathcal{T})$ of degree 1 such that $R_r(\mathcal{T};\lambda) = -1$.*

Moreover, if $\mathcal{H}$ is a $(b+1)$–uniform linear hypertree, then $\mathcal{T}$ can be chosen to be $(b+1)$-uniform as well.

Proof. $(1 \Rightarrow 2)$ Assume $\mathcal{H}$ has the stated properties and let v be any vertex of $\mathcal{H}$. By Theorem 2.9 we know that the rooted hypertree $\mathcal{T}$ given by Definition 2.5 satisfies that

$$Z(\mathcal{T};\lambda') = 0$$

where $\lambda'_u = \lambda_{\pi(u)} \in A$. By construction this is a linear hypertree that has degree at most Δ and hyperedge size at most $b+1$.

If $\mathcal{H}$ is a linear uniform hypertree, this implication is trivial.

$(2 \Rightarrow 3)$ Now let $\mathcal{T}'$ be a linear hypertree on a minimal number of vertices such that it has degree at most Δ and hyperedge size at most $b+1$ and there exists $\lambda \in A^{V(\mathcal{T}')}$ such that $Z(\mathcal{T}',\lambda) = 0$. Let r be a leaf vertex of $\mathcal{T}'$ and let e be the unique incident hyperedge at r. Since

$$0 = Z(\mathcal{T}';\lambda) = Z_r^{\text{in}}(\mathcal{T}';\lambda) + Z_r^{\text{out}}(\mathcal{T}';\lambda)$$

and $Z_r^{\text{out}}(\mathcal{T}';\lambda) = Z(\mathcal{T}' - \{e\} - \{r\};\lambda)$ is not zero by the choice of $\mathcal{T}'$, therefore

$$R_r(\mathcal{T}';\lambda) = -1$$

Observe that if $\mathcal{T}'$ would be a uniform linear hypertree, then $\mathcal{T}'' = \mathcal{T}' - \{e\} - r$ would consist of isolated vertices and a uniform linear hypertree, thus $Z(\mathcal{T}'';\lambda)$ is again not zero similarly implying $R_r(\mathcal{T}';\lambda) = -1$.

$(3 \Rightarrow 1)$ Let $\mathcal{T}$ be a linear hypertree that satisfies the properties in (3). We claim that $Z(\mathcal{T};\lambda) = 0$. If $Z_r^{\text{out}}(\mathcal{T};\lambda) = 0$, then $Z_r^{\text{in}}(\mathcal{T};\lambda)$ has to be 0 as well, thus

$$Z(\mathcal{T};\lambda) = Z_r^{\text{in}}(\mathcal{T};\lambda) + Z_r^{\text{out}}(\mathcal{T};\lambda) = 0$$

If $Z^{\text{out}}(\mathcal{T};\lambda) \neq 0$, then

$$Z(\mathcal{T};\lambda) = Z_r^{\text{out}}(\mathcal{T};\lambda)\left(1 + \frac{Z_r^{\text{in}}(\mathcal{T};\lambda)}{Z_r^{\text{out}}(\mathcal{T};\lambda)}\right) = 0$$

□

3.1 | Notation and the Grace–Walsh–Szegő Theorem

Fix $d \in \mathbb{Z}_{\geq 2}$ for the remainder of this section. Let $b_1, \ldots, b_d \in \mathbb{Z}_{\geq 1}$ and for $i = 1, \ldots, d$ let $v^i \in (\mathbb{C} \setminus \{-1\})^{b_i}$. We define the map F_λ by

$$F_\lambda(v^1, \dots, v^d) = \lambda \cdot \prod_{i=1}^{d} \left[1 - \prod_{j=1}^{b_i} \frac{v_j^i}{1+v_j^i} \right]$$

We say that a closed region $A \subseteq \mathbb{C} \setminus \{-1\}$ is strictly forward invariant for F_λ if there exists a closed subset $\tilde{A} \subset \mathrm{int}(A)$ such that for any $b_1, \dots, b_d$ and any $v^i \in A^{b_i}$ we have that

$$F_\lambda(v^1, \dots, v^d) \in \tilde{A}$$

We denote by $f_{\lambda,b}$ the following univariate specialization of F_λ

$$f_{\lambda,b}(z) = F_\lambda(v(z), \dots, v(z))$$

where $v(z) = z \cdot (1, \dots, 1) \in \mathbb{C}^b$. We will usually drop the subscript b and, unless some other b is specified, f_λ will denote $f_{\lambda,1}$. These maps are rational maps, which we will consider as a holomorphic map from the Riemann sphere $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to itself. Since we will be using it often, we also define the Möbius transformation

$$\mu(z) = \frac{z}{1+z}$$

We say that a closed region $A \subseteq \hat{\mathbb{C}}$ is strictly forward invariant for a rational map g if $g(A) \subset \mathrm{int}(A)$.

A *generalized circle* is a circle in $\hat{\mathbb{C}}$, that is, either a circle or a straight line in $\mathbb{C}$. A *circular region* is a subset of either $\hat{\mathbb{C}}$ or $\mathbb{C}$ bounded by a generalized circle that is either open or closed. A circular region in $\mathbb{C}$ is thus either a disk, the complement of a disk or a half-plane. We recall that Möbius transformation are comformal maps that send generalized circles to generalized circles and thus also circular regions to circular regions. We will denote the closed disk centered around 0 of radius r by B_r and the open unit disk by $\mathbb{D}$. The following theorem by Grace, Walsh and Szegő [30–32] will be used multiple times throughout the remainder of this paper. Its formulation was taken from [33].

Theorem 3.2 (Grace–Walsh–Szegő). *Let $f \in \mathbb{C}[z_1, \dots, z_n]$ be a multiaffine polynomial that is invariant under all permutations of the variables. Let $A \subseteq \mathbb{C}$ be a convex circular region. For any $(\zeta_1, \dots, \zeta_n) \in A^n$ there is a $\zeta \in A$ such that*

$$f(\zeta_1, \dots, \zeta_n) = f(\zeta, \dots, \zeta)$$

3.2 | Zero-Free Disks for the Multivariate Independence Polynomial

Let us recall the notation from the introduction

$$\lambda_s(\Delta) = \frac{(\Delta-1)^{\Delta-1}}{(\Delta-2)^\Delta}$$

Now we will establish one of our main theorems.

Theorem 1.1. *Let $\Delta \geq 2$. For any hypergraph $\mathcal{H}$ with maximum degree at most Δ and $\lambda \in \mathbb{C}^{V(\mathcal{H})}$ with $|\lambda_u| \leq \lambda_s(\Delta)$ for all $u \in V(\mathcal{H})$ we have $Z(\mathcal{H}; \lambda) \neq 0$.*

Proof. By Lemma 3.1 it is sufficient to prove that there is no rooted linear hypertree $(\mathcal{T}, r)$ and $\lambda \in B_{\lambda_s(\Delta)}^{V(\mathcal{T})}$, such that

$$R_r(\mathcal{T}; \lambda) = -1$$

where $\mathcal{T}$ has of degree at most Δ and hyperedge size at most $b+1$, and the degree of r is 1. By Lemma 2.3 and Remark 2.10 it is sufficient to further restrict to linear hypertrees with hyperedge size at least 2. Thus for the rest of the proof we will consider hypertrees to have hyperedge size at least 2.

We claim that for any rooted linear hypertree $(\mathcal{T}, r)$ of down degree at most $\Delta - 1$ we have that $R_r(\mathcal{T}, \lambda)$ is well-defined and is an element of $B_{1/\Delta}$. We will prove this by induction on the number of vertices of the hypertree. If $|V(\mathcal{T})| = 1$ then $R_r(\mathcal{T}, \lambda_r) = \lambda_r \in B_{\lambda_s(\Delta)} \subseteq B_{1/\Delta}$. Now let us assume that we know that any ratio of hypertrees on $n \geq 1$ vertices with fugacities from $B_{\lambda_s(\Delta)}$ is well-defined and an element of $B_{1/\Delta}$. Let $(\mathcal{T}, r)$ be a hypertree on $n+1$ vertices with down-degree at most d and let $\lambda \in B_{\lambda_s(\Delta)}^{V(\mathcal{T})}$. Then by Lemma 2.4 we know that

$$R_r(\mathcal{T};\lambda) = \lambda_r \prod_{e:r\in e}\left(1 - \prod_{v\in e\setminus\{r\}} \frac{R_v(\mathcal{T}_v;\lambda)}{1+R_v(\mathcal{T}_v;\lambda)}\right)$$

where $\mathcal{T}_v$ is the connected component of v in $\mathcal{T} - E(r)$. By induction, we know that $R_v(\mathcal{T}_v;\lambda) \neq -1$, thus $R_r(\mathcal{T};\lambda)$ is well-defined. Also we know by induction that for any neighbor v of r

$$\left|R_v(\mathcal{T}_v;\lambda)\right| \leq \frac{1}{\Delta}$$

thus

$$\left|\frac{R_v(\mathcal{T}_v;\lambda)}{1+R_v(\mathcal{T}_v;\lambda)}\right| \leq \frac{1}{\Delta-1}$$

therefore for any edge e that is incident to v we have

$$\left|\prod_{v\in e\setminus\{r\}} \frac{R_v(\mathcal{T}_v;\lambda)}{1+R_v(\mathcal{T}_v;\lambda)}\right| \leq \frac{1}{\Delta-1}$$

since $1/(\Delta-1) \leq 1$. Thus,

$$\left|R_r(\mathcal{T};\lambda)\right| = |\lambda_r| \prod_{e:r\in e}\left|1 - \prod_{v\in e\setminus\{r\}} \frac{R_v(\mathcal{T}_v;\lambda)}{1+R_v(\mathcal{T}_v;\lambda)}\right| \leq \lambda_s(\Delta)\left(1+\frac{1}{\Delta-1}\right)^{\Delta-1} \leq \frac{1}{\Delta-1}$$

□

Let us remark that in the previous argument we actually proved that for any $\lambda \in B_{\lambda_s(\Delta)}$ the disk $B_{1/\Delta}$ is F_λ forward invariant (where $\Delta = d+1$). In the next lemma we have a similar statement, but for $(b+1)$ uniform hypertrees.

Lemma 3.3. *Let $b \geq 1$. Suppose there exists an $0 < R < 1$ such that $f_{\lambda,b}(B_R) \subseteq B_R$. Then for any $(b+1)$-uniform hypertree $\mathcal{T}$ of maximum degree $\Delta = d+1$ and for any $\lambda \in B_{|\lambda|}^{V(\mathcal{T})}$ we have $Z(\mathcal{T};\lambda) \neq 0$.*

Proof. As in the previous proof, it is sufficient to prove that $R_r(\mathcal{T};\lambda)$ is well-defined and not -1 for some $r \in V(\mathcal{T})$ leaf. To be more precise we claim that all the ratios of $(b+1)$-uniform hypertrees of maximum degree Δ and for any leaf r the corresponding ratio is contained in B_R. To prove this we show that the one vertex hypertree has ratio in B_R and $\underbrace{B_R \times \cdots \times B_R}_{b}$ is mapped by $F_{\lambda'}$ into B_R for any $\lambda' \in B_{|\lambda|}$. This is sufficient, since now by induction on the number of vertices it follows that any ratio is well-defined and contained in B_R.

If $\mathcal{T}$ is an isolated vertex, then $R_r(\mathcal{T};\lambda) = \lambda_v \in B_{|\lambda|}$. To prove the second part let us consider the product

$$\prod_{j=1}^{b} \frac{v_j^i}{1+v_j^i}$$

where $v_j^i \in B_R$. Since $\mu(z) = \frac{z}{1+z}$ is a Möbius transformation, therefore $\mu(v_j^i)$ is an element of the disk $\mu(B_R)$ for $j = 1, \ldots, b$ and $i = 1, \ldots, d$. As the map $g_1(z_1, \ldots, z_b) = z_1 \ldots z_b$ is a symmetric polynomial, therefore by Theorem 3.2 we know that there exists $w_i \in \mu(B_R)$ such that

$$\prod_{j=1}^{b} \frac{v_j^i}{1+v_j^i} = (w_i)^b$$

that is,

$$F_{\lambda'}(v^1, \ldots, v^d) = \lambda' \prod_{i=1}^{d}(1 - w_i^b)$$

Now let $w \in \mu(B_R)$ be chosen such that it maximizes $|1-z^b|$ over $\mu(B_R)$. Thus, if $v = \mu^{-1}(w) \in B_R$, then

$$\left|F_{\lambda'}(v^1, \ldots, v^d)\right| \leq |\lambda'| \left|\prod_{i=1}^{d}(1-w^b)\right| \leq |\lambda| \left|\prod_{i=1}^{d}\left(1 - \left(\frac{v}{1+v}\right)^b\right)\right| = |f_{\lambda,b}(v)| \leq R$$

□

3.3 | A Zero-Freeness Condition for the Univariate Independence Polynomial

In this section we describe a sufficient condition for showing that λ cannot be a zero of a bounded degree hypergraph. We recall that $d \in \mathbb{Z}_{\geq 2}$ is assumed to be fixed and that the maps F and f are defined in Section 3.1.

Lemma 3.4. *Suppose there is a closed strictly forward invariant region $A \subseteq \{z \in \mathbb{C} : \Re(z) \geq -\frac{1}{2}\}$ for F_{λ_0} for some $\lambda_0 \in \mathbb{C} \setminus \{0\}$. Then there is a neighborhood U of λ_0 such that A is strictly forward invariant for F_λ for all $\lambda \in U$.*

Proof. Observe that $\mu(\{z \in \mathbb{C} : \Re(z) \geq -\frac{1}{2}\})$ is equal to the closed unit disk $\overline{\mathbb{D}}$. Therefore, if we let $g(w) = \lambda_0 \cdot (1-w)^d$, we see that

$$F_{\lambda_0}(v^1, \dots, v^d) \in g(\overline{\mathbb{D}})$$

for every choice $v^i \in A^{b_i}$. This is a bounded region, which implies that the region $\tilde{A}$, described in Equation (3.1), can be chosen to be bounded. It follows that there is an open neighborhood V of 1 such that $t \cdot \tilde{A}$ is strictly contained in A for every $t \in V$. Thus, if we let $U = \lambda_0 \cdot V$, we see that for every $\lambda \in U$

$$F_{\lambda_0}(v^1, \dots, v^d) = \frac{\lambda}{\lambda_0} F_\lambda(v^1, \dots, v^d) \in \frac{\lambda}{\lambda_0}\tilde{A}$$

which is a strict subset of A because $\frac{\lambda}{\lambda_0} \in V$. □

We say that a set $A \subseteq \mathbb{C} \setminus \mathbb{R}_{<0}$ is *log-convex* if $\log(A \setminus \{0\})$ is convex, where log is taken to be the principle branch of the logarithm defined on $\mathbb{C} \setminus \mathbb{R}_{\leq 0}$.

A set $B \subseteq \mathbb{C}$ is called a *multiplicative semigroup* if for every $w_1, w_2 \in B$ also $w_1 \cdot w_2 \in B$.

Lemma 3.5. *Let $\lambda_0 \in \mathbb{C}$ and suppose a closed set $A \subseteq \mathbb{C} \setminus \mathbb{R}_{\leq -1}$ satisfies the following properties*:

- $1 + A$ *is* log-*convex*;
- $\mu(A)$ *is a multiplicative semigroup*;
- *A is strictly forward invariant for the univariate map f_{λ_0}*

Then A is strictly forward invariant for F_{λ_0}.

Proof. Let $v^i \in A^{b_i}$ for $i = 1, \dots, d$. By the second property, we know that there exists $w_i \in A$ for $i = 1, \dots, d$ such that

$$\prod_{j=1}^{b_i} \frac{v^i_j}{1 + v^i_j} = \frac{w_i}{1 + w_i}$$

By the first property, we know that there exists a $\tilde{w} \in A$ such that

$$(1 + \tilde{w})^d = \prod_{i=1}^{d} (1 + w_i)$$

Therefore

$$F_{\lambda_0}(v^1, \dots, v^d) = \lambda_0 \prod_{i=1}^{d} \left[1 - \prod_{j=1}^{b_i} \frac{v^i_j}{1 + v^i_j}\right] = \lambda_0 \prod_{i=1}^{d} \frac{1}{1 + w_i} = \frac{\lambda_0}{(1 + \tilde{w})^d} = f_{\lambda_0}(\tilde{w})$$

And thus any $F_{\lambda_0}(v^1, \dots, v^d)$ lies in $f_{\lambda_0}(A)$, which is a strict subset of A by the third property. □

Corollary 3.6. *Suppose $\lambda_0 \in \mathbb{C}$ and $A \subseteq \{z \in \mathbb{C} : \Re(z) \geq -\frac{1}{2}\}$ satisfy the properties of Lemma 3.5 and $0 \in A$. Then there is an open neighborhood U of λ_0 such that for any hypergraph $\mathcal{H}$ of maximum degree $d + 1$ and $\lambda \in U$ we have $Z(\mathcal{H}; \lambda) \neq 0$.*

Proof. By Lemmas 3.6 and 3.5, we know that there is an open neighborhood U of λ_0 such that A is strictly forward invariant for F_λ for any $\lambda \in U$. Now let us fix a $\lambda \in U$. Let $\mathcal{T}$ be an arbitrary hypertree of maximum degree at most $d + 1$ and let r be a leaf. Then by the forward invariance of A and the fact that $-1 \notin A$ we see by Lemma 2.4 that $R_r(\mathcal{T}, \lambda) \neq -1$. Thus, by Lemma 3.1, we see that there is no hypergraph of degree at most $d + 1$ such that $Z(\mathcal{H}, \lambda) = 0$. □

4 | The Maximal Zero-Free Region for Hypergraphs Around the Positive Real Axis

This section is dedicated to proving Theorem 1.2. Again we will always fix $d \in \mathbb{Z}_{\geq 2}$ and let $f_\lambda(z) = \lambda/(1+z)^d$. We let

$$\lambda_c := \lambda_c(d+1) = \frac{d^d}{(d-1)^{d+1}}$$

We can relate λ_c to the dynamical behaviour of f_λ. The following lemma can also be found in [8].

Lemma 4.1. *For any $\lambda > 0$ the map f_λ has a unique positive fixed point p_λ. If $0 < \lambda < \lambda_c$ then this fixed point is attracting, that is, $|f_\lambda'(p_\lambda)| < 1$.*

Proof. Suppose f_λ has a fixed point $p \in \mathbb{R}_{\geq 0}$. Then it follows that $\lambda = \lambda_p = p(1+p)^d$. The map $p \mapsto \lambda_p$ is increasing on $\mathbb{R}_{\geq 0}$ and therefore invertible with inverse p_λ.

We calculate that $f'_{\lambda_p}(p_\lambda) = -dp_\lambda/(1+p_\lambda)$. It, therefore, follows that the fixed point p_λ of f_{λ_p} is attracting if $0 < p_\lambda < 1/(d-1)$, that is, if $0 < \lambda_p < \lambda_c$. □

To prove Theorem 1.2, we will define a family of regions $A(x, x_0, \epsilon)$ depending on three parameters, and we will show that for $0 < \lambda < \lambda_c$ the parameters can be chosen in such a way that the region satisfies the conditions of Lemma 3.5.

4.1 | Defining the Relevant Regions

For any $x \in \mathbb{R}$ and $\epsilon \in (0, \pi)$ we define $C(x, \epsilon)$ as the closed cone with vertex x that is symmetric around the real axis and has internal angle 2ϵ, that is,

$$C(x, \epsilon) = \{z \in \mathbb{C} : |\mathrm{Arg}(z-x)| \leq \epsilon\} \cup \{x\}$$

Let $x, x_0 \in \mathbb{R}$ with $-1 < x < x_0$ and let $\epsilon \in (0, \pi/2)$. Let $\tilde{\epsilon}$ be such that the intersections of the boundaries of $C(x, \epsilon)$ and $C(-1, \tilde{\epsilon})$ have real part x_0, that is, $\tilde{\epsilon} = \tan^{-1}(\frac{x_0 - x}{x_0 + 1}\tan(\epsilon))$. We define

$$A(x, x_0, \epsilon) = C(x, \epsilon) \cap C(-1, \tilde{\epsilon})$$

For $y < 1$ let a_ϵ be the circular arc between y and 1 with internal angle ϵ. And let $D(y, \epsilon)$ be the closed region in between a_ϵ and $\overline{a_\epsilon}$. For $y_0 \in (y, 1)$ let $\tilde{\eta}$ be such that the real part of the intersections of the boundary between the cone $-C(-1, \tilde{\eta})$ and $D(y, \epsilon)$ is y_0. We define

$$B(y, y_0, \epsilon) = D(y, \epsilon) \cap (-C(-1, \tilde{\eta}))$$

See Figure 2 for an example.

Recall that $\mu(z) = z/(1+z)$.

Lemma 4.2. *For $-1 < x < x_0$ we have*

$$\mu(A(x, x_0, \epsilon)) = B(\mu(x), \mu(x_0) + o(1), \epsilon) \setminus \{1\}$$

where $o(1)$ denotes some function that converges to 0 as ϵ converges to 0.

Proof. The map μ is a Möbius transformation that sends $\mathbb{R} \cup \{\infty\}$ to itself with $\mu(-1) = \infty$ and $\mu(\infty) = 1$. A Möbius transformation is a conformal map that sends generalized circles to generalized circles and therefore, for any $-1 < x$, we have that $\mu(C(x, \epsilon)) = D(\mu(x), \epsilon) \setminus \{1\}$. Moreover, it follows that $\mu(C(-1, \tilde{\epsilon})) = -C(-1, \tilde{\epsilon}) \setminus \{1\}$. If we let I_ϵ be the point on the boundary of $A(x, x_0, \epsilon)$ in the upper half-plane with real part x_0 we can thus conclude that

$$\mu(A(x, x_0, \epsilon)) = B(\mu(x), \Re(\mu(I_\epsilon)), \epsilon) \setminus \{1\}$$

As $\epsilon \to 0$ the point I_ϵ converges to x_0 and thus, by continuity, $\mu(I_\epsilon)$ converges to $\mu(x_0)$. □

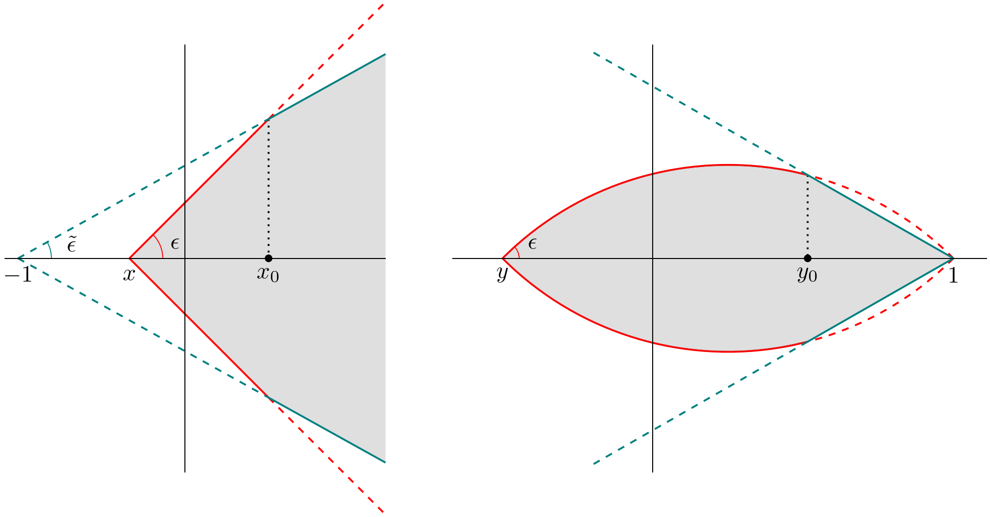

FIGURE 2 | Example of the regions $A(x, x_0, \epsilon)$ and $B(y, y_0, \epsilon)$.

4.2 | Logarithmic Convexity

In this section we prove that $A(x, x_0, \epsilon)$ satisfies the first condition of Lemma 3.5.

Lemma 4.3. *For any x, x_0 with $-1 < x < x_0$ and $\epsilon \in (0, \pi/2)$ the region $1 + A(x, x_0, \epsilon)$ is* log-*convex.*

Proof. Because $A(x, x_0, \epsilon)$ is the intersection of two cones it is sufficient to show that any cone of the form $C(y, \epsilon)$ with $y \geq 0$ and $\epsilon \in (0, \pi/2)$ is log-convex. Furthermore, such a cone is the intersection of the two closed half-planes $y - e^{i\epsilon}\mathbb{H}_{\geq 0}$ and $y + e^{-i\epsilon}\mathbb{H}$, where $\mathbb{H}$ denotes the closed upper half-plane. It is thus sufficient to show that the half-plane $H(y, \epsilon) := y + e^{-i\epsilon}\mathbb{H}$ is log-convex.

If $y = 0$ then

$$\log(H(y,\epsilon) \setminus \{0\}) = \log(e^{-i\epsilon}\mathbb{H}_{\geq 0} \setminus \{0\}) = \{w \in \mathbb{C} : -\epsilon \leq \Im(w) \leq \pi - \epsilon\}$$

which is convex.

Now suppose $y > 0$ and take $w_1, w_2 \in \log(H(y, \epsilon))$. By Theorem 3.2, there exists a $z \in H(y, \epsilon)$ such that $e^{w_1}e^{w_2} = z^2$. It follows that $(w_1 + w_2)/2 = \log(z) + ik\pi$ for some $k \in \mathbb{Z}$. Note that for any $w \in \log(H(y, \epsilon))$ we have $-\epsilon < \Im(w) < \pi - \epsilon$ and thus both $-\epsilon < \Im((w_1 + w_2)/2) < \pi - \epsilon$ and $-\epsilon < \Im(\log(z)) < \pi - \epsilon$. It follows that $k = 0$, that is, $(w_1 + w_2)/2 = \log(z)$, which shows that $\log(H(y, \epsilon))$ is midpoint convex. Because $\log(H(y, \epsilon))$ is also closed we can conclude that $H(y, \epsilon)$ is convex. □

4.3 | Multiplicative Invariance

In this section we prove that for the right choice of parameters the set $A(x, x_0, \epsilon)$ satisfies the second condition of Lemma 3.5.

Lemma 4.4. *Let $y \in (0, 1)$, and let r be a real-valued function such that $\lim_{\epsilon\to 0} r(\epsilon) = y_0$ with $y_0 \in (y^2, \sqrt{y})$. Then for ϵ sufficiently small $B(-y, r(\epsilon), \epsilon)$ is a multiplicative semigroup.*

Proof. Let $\mathcal{B}_\epsilon = B(-y, r(\epsilon), \epsilon)$. We have to show that for ϵ sufficiently small and $z_1, z_2 \in \mathcal{B}_\epsilon$ the product $z_1 z_2 \in \mathcal{B}_\epsilon$. Because $\mathcal{B}_\epsilon$ is convex and contains 0 we can assume that $z_1, z_2 \in \partial\mathcal{B}_\epsilon$. The boundary of $\mathcal{B}_\epsilon$ is the union of the closed circular arcs a_ϵ and $\overline{a}_\epsilon$ and the closed straight line segments b_ϵ and $\overline{b_\epsilon}$. We claim that we can assume that either $z_1, z_2 \in a_\epsilon$ or $z_1 \in a_\epsilon$ and $z_2 \in \overline{a_\epsilon}$.

To show this let us denote the unique element of the intersection $a_\epsilon \cap b_\epsilon$ by I_ϵ. Assume that z_1 is an element of either b_ϵ or $\overline{b_\epsilon}$, say $z_1 \in b_\epsilon$. The set $z_2 \cdot b_\epsilon$ is a straight line segment between z_2 and $I_\epsilon z_2$ containing $z_1 z_2$. Because $z_2 \in \mathcal{B}_\epsilon$ and $\mathcal{B}_\epsilon$ is convex it follows that $z_1z_2 \in \mathcal{B}_\epsilon$ if $I_\epsilon z_2 \in \mathcal{B}_\epsilon$. We can thus assume that z_1 is either an element of a_ϵ or of $\overline{a_\epsilon}$. Analogously, we can conclude the same for z_2. By symmetry of $\mathcal{B}_\epsilon$ under complex conjugation it suffices to only consider the two cases $z_1, z_2 \in a_\epsilon$ or $z_1 \in a_\epsilon$ and $z_2 \in \overline{a_\epsilon}$ as indicated in the claim.

We will now focus on the case where z_1 and z_2 are both an element of a_ϵ. The curve a_ϵ is a circular arc and can thus be written as $\{t + f_\epsilon(t)i : t \in [-y, r(\epsilon)]\}$ for a real valued function f_ϵ depending on ϵ. Note that this function naturally extends to the whole interval $[-y, 1]$ and that $f_\epsilon(t) \in \mathcal{O}(\epsilon)$ uniformly over all $t \in [-y, 1]$. Fix $y_u \in (y_0, \sqrt{y})$ such that $y_u^2 < y_0$ and in what follows let ϵ be sufficiently small such that $y_u^2 < r(\epsilon) < y_u$. We will show that for ϵ sufficiently small the product of two elements on the curve $\{t + f_\epsilon(t)i : t \in [-y, y_u]\}$ lies in $\mathcal{B}_\epsilon$. This implies that $z_1z_2 \in \mathcal{B}_\epsilon$. Consider the product of two such arbitrary elements

$$(t_1 + f_\epsilon(t_1)i) \cdot (t_2 + f_\epsilon(t_2)i) = t_1t_2 + (t_1f_\epsilon(t_2) + t_2f_\epsilon(t_1))i + \mathcal{O}(\epsilon^2)$$

We will show that there is a constant $C \in (0, 1)$ such that for ϵ sufficiently small $t_1t_2 + (t_1f_\epsilon(t_2) + t_2f_\epsilon(t_1))i$ is an element of

$$S_\epsilon = \{a + bi : -yy_u \le a \le y_u^2 \text{ and } |b| \le C \cdot f_\epsilon(a)\}$$

for every $t_1, t_2 \in [-y, y_u]$. This is sufficient because $S_\epsilon \subset \mathcal{B}_\epsilon$ and the distance between ∂S_ϵ and $\partial \mathcal{B}_\epsilon$ is lower bounded by a constant times ϵ. We will first argue why this is the case.

As we will see below the function $x \mapsto f_\epsilon(x)/\epsilon$ on the interval $[y, 1]$ converges to a non-zero quadratic function that has zeros at y and 1 and is strictly positive in between. It follows that there is a constant $K_1 > 0$ such that, for all $x_1, x_2 \in [-y, 1]$ and ϵ sufficiently small we have that $|f_\epsilon(x_1) - f_\epsilon(x_2)| \le K_1\epsilon \cdot |x_1 - x_2|$. It also follows that there is a constant $K_2 > 0$ such that $f_\epsilon(x) > K_2\epsilon$ for all $x \in [-yy_u, y_u^2]$. We find that the distance of a $z_1 \in \partial\mathcal{B}_\epsilon$ of the form $z_1 = x_1 + i f_\epsilon(x_1)$ and a $z_2 \in \partial S_\epsilon$ of the form $x_2 + iCf_\epsilon(x_2)$ with $x_2 \in [-yy_u, y_u^2]$ is lower bounded by $\max\{|\Re(z_1) - \Re(z_2)|, |\Im(z_1) - \Im(z_2)|\}$ which is at least

$$\max\{|x_1 - x_2|, f(x_2)(1 - C) - K_1\epsilon|x_1 - x_2|\} \ge \max\{|x_1 - x_2|, (K_2(1 - C) - K_1|x_1 - x_2|)\epsilon\}$$

For ϵ sufficiently small, this is lower bounded by a constant times ϵ.

For $t_1, t_2 \in [-y, y_u]$ we have that t_1t_2 is an element of $[-yy_u, y_u^2] \cup [-yy_u, y^2] = [-yy_u, y_u^2]$. What remains to be shown is that there exists a constant $C \in (0, 1)$ such that for small ϵ we have

$$|t_1f_\epsilon(t_2) + t_2f_\epsilon(t_1)| \le C \cdot f_\epsilon(t_1t_2) \tag{1}$$

To prove this inequality, we may replace the function $t \mapsto f_\epsilon(t)$ by a function $t \mapsto \gamma_\epsilon f_\epsilon(t)$, where γ_ϵ is any positive valued function of ϵ. Let $\gamma_\epsilon = \frac{(1-y)(1+3y)}{4f_\epsilon((1+y)/2))}$ and define $g_\epsilon(t) = \gamma_\epsilon f_\epsilon(t)$. Now the curve $t + ig_\epsilon(t)$ for $t \in [-y, 1]$ defines part of an ellipse that passes through $-y$, 1 and $\frac{1}{2}(1 + y) + \frac{1}{4}i(1 - y)(1 + 3y)$ for every ϵ. As ϵ goes to zero the eccentricity of this ellipse converges to 1. It follows that the elliptical arcs converge uniformly to the arc of a parabola. Because this parabola has to go through the same three points, it must be given by the equation $t + ig(t)$, where $g(t) = -(t + y)(t - 1)$. We claim that, to obtain Equation (1), it is now sufficient to prove that

$$|t_1g(t_2) + t_2g(t_1)| < g(t_1t_2) \tag{2}$$

for all $t_1, t_2 \in [-y, y_u]$. Indeed, if this inequality holds then, because $[-y, y_u]$ is compact, there is also some $C \in (0, 1)$ such that $|t_1g(t_2) + t_2g(t_1)| < C \cdot g(t_1t_2)$. By uniform continuity the same will then hold for g_ϵ for ϵ sufficiently small and thus also for f_ϵ.

At this point, we recall that we arrived at the inequality in Equation (2) by assuming that z_1 and z_2 are both elements of a_ϵ. If we assume the other case, that is, where $z_1 \in a_\epsilon$ and $z_2 \in \overline{a_\epsilon}$, then, instead of considering $(t_1 + f_\epsilon(t_1)i) \cdot (t_2 + f_\epsilon(t_2)i)$, we consider $(t_1 + f_\epsilon(t_1)i) \cdot (t_2 - f_\epsilon(t_2)i)$. In a completely analogous way it then follows that it is sufficient to prove the following inequality

$$|-t_1g(t_2) + t_2g(t_1)| < g(t_1t_2) \tag{3}$$

for all $t_1, t_2 \in [-y, y_u]$. To prove the inequalities in both Equations (2) and (3) we will prove that

$$|t_1|g(t_2) + |t_2|g(t_1) < g(t_1t_2)$$

for all $t_1, t_2 \in [-y, y_u]$. We distinguish three cases.

- If t_1 and t_2 are both positive then

$$g(t_1t_2) - |t_1|g(t_2) - |t_2|g(t_1) = g(t_1t_2) - t_1g(t_2) - t_2g(t_1) = (1-t_1)(1-t_2)(y - t_1t_2)$$

  This is strictly positive because $t_1t_2 \leq y_u^2 < y$.

- If t_1 and t_2 have opposite signs, say $t_2 \leq 0$, then

$$g(t_1t_2) - |t_1|g(t_2) - |t_2|g(t_1) = g(t_1t_2) - t_1g(t_2) + t_2g(t_1) = (1-t_1)(1+t_2)(y + t_1t_2)$$

  which is again strictly positive because $t_1t_2 \geq -yy_u > -y$.

- If t_1 and t_2 are both strictly negative then

$$\begin{aligned} g(t_1t_2) - |t_1|g(t_2) - |t_2|g(t_1) &= g(t_1t_2) + t_1g(t_2) + t_2g(t_1) \\ &= t_1t_2(3 - t_1 - t_2 - t_1t_2) + y(1 + t_1 + t_2 - 3t_1t_2) \end{aligned}$$

  Note that $t_1t_2(3 - t_1 - t_2 - t_1t_2)$ is strictly positive and thus if $(1 + t_1 + t_2 - 3t_1t_2)$ is also positive the whole quantity is strictly positive. Otherwise we have

$$\begin{aligned} t_1t_2(3 - t_1 - t_2 - t_1t_2) + y(1 + t_1 + t_2 - 3t_1t_2) &\geq t_1t_2(3 - t_1 - t_2 - t_1t_2) + (1 + t_1 + t_2 - 3t_1t_2) \\ &= (1+t_1)(1+t_2)(1-t_1t_2) \end{aligned}$$

  which is strictly positive.

This concludes the proof. □

Corollary 4.5. *Let $x \in (-\frac{1}{2}, 0)$ and x_0 such that*

$$\mu^{-1}(\mu(x)^2) < x_0 < \mu^{-1}(\sqrt{-\mu(x)}) \tag{4}$$

Then for ϵ sufficiently small $\mu(A(x, x_0, \epsilon))$ is a multiplicative semigroup.

Proof. Lemma 4.2 states that $\mu(A(x, x_0, \epsilon)) = B(\mu(x), \mu(x_0) + o(1), \epsilon) \setminus \{1\}$. Note that $-\mu(x) \in (0, 1)$ and thus, by Lemma 4.4, it is sufficient to have

$$(-\mu(x))^2 < \mu(x_0) < \sqrt{-\mu(x)}$$

Because μ^{-1} is orientation preserving this condition is equivalent to the condition of Equation (4). □

4.4 | Forward Invariance

In this section, we prove that for the right choice of parameters the set $A(x, x_0, \epsilon)$ satisfies the third condition of Lemma 3.5.

Lemma 4.6. *Let $\lambda > 0$, $x > -1$ and $\delta < \pi/d$, then $f_\lambda(C(x, \delta)) \subseteq C(0, d \cdot \delta)$.*

Proof. The map $z \mapsto z + 1$ sends the cone $C(x, \delta)$ to $C(x + 1, \delta)$, which is a subset of $C(0, \delta) \setminus \{0\}$. Subsequently, the map $z \mapsto z^{-d}$ sends $C(0, \delta) \setminus \{0\}$ to $C(0, d \cdot \delta) \setminus \{0\} \subset C(0, d \cdot \delta)$. Finally, because λ is a positive real number, multiplication with λ sends $C(0, d \cdot \delta)$ to itself. This concludes the proof. □

Lemma 4.7. *Let $0 < \lambda < \lambda_c$. For ϵ sufficiently small the cone $C(-\frac{1}{d+1}, \epsilon)$ is strictly forward invariant for f_λ.*

Proof. For any $\delta > 0$ define the open strip

$$S_{<\delta} = \{z \in \mathbb{C} : |\Im(z)| < \delta\}$$

and let $S_{\leq\delta}$ be its closure. We define the map

$$\phi : \mathbb{C} \setminus \mathbb{R}_{\leq -1/(d+1)} \to S_{<\pi}, \; z \mapsto \log\left(z + \frac{1}{d+1}\right)$$

This is a biholomorphism with $\phi^{-1}(w) = e^w - \frac{1}{d+1}$. Note that for any $\delta \in (0, \pi)$ we have that $\phi^{-1}(S_{\leq\delta}) = C(-1/(d+1), \delta) \setminus \{-\frac{1}{d+1}\}$. Now fix $\delta \in (0, \pi/d)$, it follows from Lemma 4.6 that the map $g_\lambda := \phi \circ f_\lambda \circ \phi^{-1}$ is well-defined as a map $S_{\leq\delta} \to S_{\leq d\delta}$.

We claim that there exist constants $c \in (0, 1)$ and $\epsilon_0 \in (0, \delta)$ such that $|g'_\lambda(w)| \leq c$ for all $z \in S_{\leq\epsilon_0}$. If we assume that this claim is true, then it follows that for any $w \in S_{\leq\epsilon}$ with $0 < \epsilon \leq \epsilon_0$ we have that

$$|\mathfrak{I}(g_\lambda(w))| = |\mathfrak{I}(g_\lambda(w) - g_\lambda(\mathfrak{R}(w)))| \leq |g_\lambda(w) - g_\lambda(\mathfrak{R}(w))| \leq c \cdot |w - \mathfrak{R}(w)| \leq c \cdot \epsilon$$

Therefore, $g_\lambda(S_{\leq\epsilon}) \subseteq S_{\leq c\epsilon}$, and thus, by conjugation, $f_\lambda(C(-1/(d+1), \epsilon)) \subseteq C(-1/(d+1), c\epsilon)$. By Lemma 4.6, we also have that $f_\lambda(C(-1/(d+1), \epsilon)) \subseteq C(0, d\epsilon)$ and thus we can conclude that f_λ maps $C(-\frac{1}{d+1}, \epsilon)$ strictly into itself.

We now prove the claim. We calculate

$$g'_\lambda(w) = -\frac{\lambda d(d+1)e^w}{(\frac{d}{d+1} + e^w)((\frac{d}{d+1} + e^w)^d + \lambda(d+1))}$$

This is a rational function in e^w, say $h(e^w)$, with the property that $h(0) = h(\infty) = 0$. If we take $w \in S_{\leq\delta}$ we can let $|e^w|$ get arbitrarily close to either 0 or ∞ by adding the requirement that $|\mathfrak{R}(w)| \geq B$ for some large $B > 0$. From now on fix B such that it guarantees that $|g'_\lambda(w)| < \frac{1}{2}$ if $w \in S_{\leq\delta}$ with $|\mathfrak{R}(w)| \geq B$.

It remains to be shown that there is a neighborhood of the real interval $[-B, B]$ on which $|g'_\lambda|$ is strictly less than 1. To prove this, we will show that there is a $\alpha \in (0, 1)$ such that $|g'_\lambda(w)| \leq \alpha$ for all $w \in \mathbb{R}$. Note that $g'_\lambda(w) < 0$ for all real w. We calculate the second derivative.

$$g''_\lambda(w) = \frac{\lambda d^2(d+1)e^w \left[\left(e^w + \frac{d}{d+1}\right)^d \left(e^w - \frac{1}{d+1}\right) - \lambda\right]}{\left(\frac{d}{d+1} + e^w\right)^2 \left(\left(\frac{d}{d+1} + e^w\right)^d + \lambda(d+1)\right)^2}$$

The sign of $g''_\lambda(w)$ is the same as that of $\left(e^w + \frac{d}{d+1}\right)^d \left(e^w - \frac{1}{d+1}\right) - \lambda$. This is a monotonically increasing function in w that is negative as $w \to -\infty$ and positive as $w \to \infty$, therefore $g''_\lambda(w)$ has a unique sign change at some w_m. It follows that $g'_\lambda(w_m)$ is the global minimum of g'_λ on the real line and that w_m satisfies

$$\lambda = \left(e^{w_m} + \frac{d}{d+1}\right)^d \left(e^{w_m} - \frac{1}{d+1}\right) = \left(\phi^{-1}(w_m) + 1\right)^d \cdot \phi^{-1}(w_m)$$

Observe that it follows that $\phi^{-1}(w_m)$ is a positive fixed point of f_λ and thus w_m is a fixed point of g_λ. Because $0 < \lambda < \lambda_c$ the unique positive fixed point of f_λ is attracting, that is, $|f'_\lambda(\phi^{-1}(w_m))| < 1$; see Lemma 4.1. The derivative of a fixed point is invariant under coordinate transformation and thus $|g'(w_m)| < 1$.

We conclude that for $\alpha := |g'(w_m)|$ we have $|g'(w)| < \alpha$ for all $w \in \mathbb{R}$. Because $[-B, B]$ is compact, there is an ϵ_0 neighborhood of $[-B, B]$ such that $|g'(w)| < (1 + \alpha)/2$ for all w in this neighborhood. It follows that $|g'(w)| \leq (1 + \alpha)/2$ for all $w \in S_{\leq\epsilon_0}$. This proves the claim and therefore concludes the proof of the lemma. □

Remark 4.8. We remark here that this shows that for $0 < \lambda < \lambda_c$ and ϵ sufficiently small $C(-1/(d+1), \epsilon)$ satisfies the third condition of Lemma 3.5. Showing that it also satisfies the first is not difficult. This is sufficient to prove zero-freeness for graphs. Lemma 4.7 therefore essentially gives an alternative proof to that of [8] of the zero-freeness of a neighborhood of $(0, \lambda_c)$ for graphs. Unfortunately, $\mu(C(-1/(d+1)), \epsilon)$ is not a multiplicative semigroup for small ϵ which is why we had to define $A(x, x_0, \epsilon)$.

Lemma 4.9. *Let $0 < \lambda < \lambda_c$ and $x_0 \geq \lambda - \frac{1}{d+1}$. For ϵ sufficiently small the region $A(-\frac{1}{d+1}, x_0, \epsilon)$ is strictly forward invariant for f_λ.*

Proof. Let a_ϵ and b_ϵ be the line-segments of the boundary of $A_\epsilon = A(-\frac{1}{d+1}, x_0, \epsilon)$ in the upper-half plane with intersection I_ϵ, with a_ϵ being the bounded line segment. Let us denote the imaginary part of I_ϵ with $h(\epsilon)$, that is

$$h(\epsilon) = \mathfrak{I}(I_\epsilon) = \left(x_0 + \frac{1}{d+1}\right)\tan(\epsilon) = \left(x_0 + \frac{1}{d+1}\right)\epsilon + \mathcal{O}(\epsilon^2)$$

First, we will prove that for sufficiently small ϵ the region A_ϵ is mapped into $S_{<h(\epsilon)}$. It is sufficient to prove that the line-segments a_ϵ, b_ϵ are mapped into $S_{<h(\epsilon)}$.

The line segment b_ϵ is part of the half-line defined by $-1 + te^{\tilde{\epsilon}i}$ where $t \in [0, \infty)$. Under f_λ it is injectively mapped into a straight line segment. Therefore, $f_\lambda(b_\epsilon)$ is the straight line segment between $f_\lambda(I_\epsilon)$ and $f_\lambda(\infty) = 0$. As $S_{<h(\epsilon)}$ is convex and contains 0, it is sufficient to show that $f_\lambda(I_\epsilon) \in S_{<h(\epsilon)}$, that is, what remains is to show that $f_\lambda(a_\epsilon) \subset S_{<h(\epsilon)}$.

Let us consider the line segment a_ϵ, that is, parametrized by the curve $\gamma_\epsilon(t) = -(1-t)\frac{1}{d+1} + tI_\epsilon$, where $t \in [0, 1]$. Then

$$f_\lambda(\gamma(t)) = \lambda\left(1 - (1-t)\frac{1}{d+1} + tI_\epsilon\right)^{-d}$$
$$= \lambda\left(\frac{d}{d+1} + t\left(x_0 + \frac{1}{d+1}\right)\right)^{-d} - \lambda dt\left(x_0 + \frac{1}{d+1}\right)\left(\frac{d}{d+1} + t\left(x_0 + \frac{1}{d+1}\right)\right)^{-d-1}\epsilon i + \mathcal{O}(\epsilon^2)$$

where the $\mathcal{O}(\epsilon^2)$ is uniform over all $t \in [0, 1]$. It therefore sufficient to show that

$$\lambda dt\left(x_0 + \frac{1}{d+1}\right)\left(\frac{d}{d+1} + t\left(x_0 + \frac{1}{d+1}\right)\right)^{-d-1} < \lim_{\epsilon\to 0} h(\epsilon)/\epsilon = x_0 + \frac{1}{d+1} \tag{5}$$

for all $t \in [0, 1]$. We find

$$\frac{\lambda dt\left(x_0 + \frac{1}{d+1}\right)}{\left(\frac{d}{d+1} + t\left(x_0 + \frac{1}{d+1}\right)\right)^{d+1}} \leq \max_{\tilde{t}\geq 0} \frac{\lambda d\tilde{t}}{\left(\frac{d}{d+1} + \tilde{t}\right)^{d+1}} = \lambda\frac{d}{d+1}$$

The inequality in (5) immediately follows and thus we conclude that $f_\lambda(A_\epsilon) \subseteq S_{<h(\epsilon)}$ if ϵ is sufficiently small.

To finish the proof observe that $A_\epsilon \subseteq C(-\frac{1}{d+1}, \epsilon)$. Let T be the closure of $f_\lambda(C(-\frac{1}{d+1}, \epsilon))$, which by Lemma 4.7 is a proper subset of the interior of $C(-\frac{1}{d+1}, \epsilon)$ for ϵ sufficiently small. Therefore, if ϵ is sufficiently small, then

$$f_\lambda(A_\epsilon) \subseteq T \cap S_{<h(\epsilon)}$$

is a proper subset of the interior of A_ϵ. □

The condition in Lemma 4.9 will unfortunately not be sufficient for $d = 2$ and $d = 3$. For those two cases, we have a separate *stronger* lemma using explicit values of x_0. The proof is very similar to the proof of the previous lemma.

Lemma 4.10.

- *Let* $\lambda \in (0, 4)$. *For* ϵ *sufficiently small the region* $A(-\frac{1}{3}, 2, \epsilon)$ *is strictly forward invariant for the map* $z \mapsto \lambda/(1+z)^2$.
- *Let* $\lambda \in (0, \frac{27}{16})$. *For* ϵ *sufficiently small the region* $A(-\frac{1}{4}, 1, \epsilon)$ *is strictly forward invariant for the map* $z \mapsto \lambda/(1+z)^3$.

Proof. Define $f_{\lambda,d}(z) = \lambda/(1+z)^d$, $A_{\epsilon,2} = A(-\frac{1}{3}, 2, \epsilon)$ and $A_{\epsilon,3} = A(-\frac{1}{4}, 1, \epsilon)$. Furthermore let $\tilde{\epsilon}_2 = \tan^{-1}(\frac{7}{9}\tan(\epsilon))$ and $\tilde{\epsilon}_3 = \tan^{-1}(\frac{5}{8}\tan(\epsilon))$. By definition we have

$$A_{\epsilon,2} = C(-1/3, \epsilon) \cap C(-1, \tilde{\epsilon}_2) \quad \text{and} \quad A_{\epsilon,3} = C(-1/4, \epsilon) \cap C(-1, \tilde{\epsilon}_3)$$

It follows from Lemma 4.7 that to prove forward invariance of $A_{\epsilon,d}$ it is sufficient to prove that $f_{\lambda,d}(A_{\epsilon,d})$ is a strict subset of $C(-1, \tilde{\epsilon}_d)$ for ϵ sufficiently small. The boundary of $A_{\epsilon,d}$ consists of two finite straight line segments, $a_{\epsilon,d}$ and $\overline{a_{\epsilon,d}}$ and two infinite line segments $b_{\epsilon,d}$ and $\overline{b_{\epsilon,d}}$. These line segments intersect in $I_{\epsilon,d}$ and $\overline{I_{\epsilon,d}}$, where $I_{\epsilon,2} = 2 + \frac{7}{3}\tan(\epsilon)i$ and $I_{\epsilon,3} = 1 + \frac{5}{4}\tan(\epsilon)i$. To prove that $A_{\epsilon,d}$ gets mapped into $C(-1, \tilde{\epsilon}_d)$ it is sufficient to prove that the segments $a_{\epsilon,d}$ and $b_{\epsilon,d}$ get mapped into $C(-1, \tilde{\epsilon}_d)$. Furthermore, because $f_{\lambda,d}(b_{\epsilon,d})$ is a straight line segment from $f_{\lambda,d}(I_\epsilon)$ to 0, it is in fact sufficient to prove that $a_{\epsilon,d}$ is mapped strictly inside $C(-1, \tilde{\epsilon}_d)$ by $f_{\lambda,d}$.

We parameterize $a_{\epsilon,d}$ as $\gamma_{\epsilon,d}(t)$ for $t \in [0, 1]$ where

$$\gamma_{\epsilon,2}(t) = -\frac{1}{3}(1-t) + (2 + \frac{7}{3}\tan(\epsilon)i)t \quad \text{and} \quad \gamma_{\epsilon,3}(t) = -\frac{1}{4}(1-t) + (1 + \frac{5}{4}\tan(\epsilon)i)t$$

Write $f_{\lambda,d}(\gamma_{\epsilon,d}(t)) = g_{\epsilon,d}(t) + ih_{\epsilon,d}(t)$, where $g_{\epsilon,d}$ and $h_{\epsilon,d}$ are real valued functions. We need to show that for ϵ sufficiently small $|h_{\epsilon,d}(t)/(g_{\epsilon,d}(t)+1)| < \tan(\tilde{\epsilon}_d)$ for all $t \in [0,1]$. We can explicitly calculate[3] that

$$B_2(\lambda; t) := \lim_{\epsilon\to 0} \frac{h_{\epsilon,2}(t)}{(g_{\epsilon,2}(t)+1)\tan(\tilde{\epsilon}_2)} = -\frac{162t}{9(7t+2)+\frac{1}{\lambda}(7t+2)^3}$$

and

$$B_3(\lambda; t) := \lim_{\epsilon\to 0} \frac{h_{\epsilon,3}(t)}{(g_{\epsilon,3}(t)+1)\tan(\tilde{\epsilon}_3)} = -\frac{1536t}{64(5t+3)+\frac{1}{\lambda}(5t+3)^4}$$

Clearly $B_d(\lambda; t)$ is negative for all $t > 0$ and $|B_d(\lambda; t)|$ is increasing in λ. It therefore suffices to prove that $B_2(4; t) > -1$ and $B_3(\frac{27}{16}; t) > -1$. We calculate explicitly

$$\frac{d}{dt}B_2(4; t) = \frac{1296\left(49t^2(7t+3)-40\right)}{(7t+2)^2(7t(7t+4)+40)^2} \quad \text{and} \quad \frac{d}{dt}B_3\left(\frac{27}{16}; t\right) = \frac{7776\left(5t^2\left(25t^2+40t+18\right)-27\right)}{5(5t+3)^2(25t^3+45t^2+27t+27)^2}$$

Both these derivatives have a unique positive real zero and thus $B_2(4; t)$ and $B_3(\frac{27}{16}; t)$ have a unique global minimum. These can be calculated numerically yielding $B_2(4, t) > -0.916$ and $B_3(\frac{27}{16}, t) > -0.891$ for all $t \in [0,1]$. Because the interval $[0,1]$ is compact, it follows that the inequality $|h_{\epsilon,d}(t)/(g_{\epsilon,d}(t)+1)| < \tan(\tilde{\epsilon}_d)$ holds for all ϵ sufficiently small, which concludes the proof. □

4.5 | Proof of the Main Theorem

We are now ready to prove Theorem 1.2, which we will restate here for convenience.

Theorem 1.2. *Let $\Delta \geq 3$. There exists an open neighborhood U of the interval $(0, \lambda_c(\Delta))$ such that for any hypergraph $\mathcal{H}$ of maximum degree Δ and $\lambda \in U$ we have $Z(\mathcal{H}; \lambda) \neq 0$.*

Proof. Let $d = \Delta - 1$ and fix $\lambda_0 \in (0, \lambda_c(d+1))$. We claim that we can choose $x_0(d)$ such that λ_0 and $A_{d,\epsilon} := A(-\frac{1}{d+1}, x_0(d), \epsilon)$ satisfy the properties of Lemma 3.5. The theorem can then be concluded by Corollary 3.6.

By Lemma 4.3 the set $1 + A_{d,\epsilon}$ is log-convex for any choice of $x_0(d) \geq 0$ and $\epsilon \in (0, \pi/2)$. It follows from Corollary 4.5that $\mu(A_{d,\epsilon})$ is a multiplicative semigroup for small ϵ if

$$\frac{1}{d^2-1} < x_0(d) < \frac{1}{\sqrt{d}-1} \tag{6}$$

If we choose $x_0(2) = 2$ and $x_0(3) = 1$, we see that inequality (6) is satisfied. Moreover, it follows from Lemma 4.10 that $A_{d,\epsilon}$ is strictly forward invariant for f_{λ_0} for small ϵ in these two cases. This proves the claim for $d \in \{2, 3\}$.

Now we assume $d \geq 4$. It follows from Lemma 4.9 that $A_{d,\epsilon}$ is strictly forward invariant for f_{λ_0} for small ϵ if

$$\lambda_0 - \frac{1}{d+1} \leq x_0(d) \tag{7}$$

We can thus prove the claim if we can show that we can simultaneously satisfy inequalities (6) and (7). Because $\lambda_0 < \lambda_c(d+1)$ it is sufficient to show that for $d \geq 4$

$$\frac{d^d}{(d-1)^{d+1}} - \frac{1}{d+1} < \frac{1}{\sqrt{d}-1}$$

For $d = 4$ the inequality reads $\frac{1037}{1215} < 1$, which is true. Otherwise we observe that

$$\frac{d^d}{(d-1)^{d+1}} - \frac{1}{d+1} = \frac{d}{(d-1)^2}\left(1 + \frac{1}{d-1}\right)^{d-1} - \frac{1}{d+1} < \frac{d}{(d-1)^2}e - \frac{1}{d+1}$$

It is thus sufficient to prove that for $d \geq 5$

$$\frac{d}{(d-1)^2}e - \frac{1}{d+1} < \frac{1}{\sqrt{d}-1} \tag{8}$$

As $d \to \infty$ the left-hand side of (8) decreases as $1/d$ and the right-hand side as $1/\sqrt{d}$ and thus the inequality is certainly true for large d. Moreover, both sides are rational functions in $\sqrt{d}$ and thus they can be equal for only finitely many real values. We can accurately approximate the solutions to the resulting polynomial equation to find that the largest d for which both sides are equal is at $d \approx 4.0389$. This shows that for $d \geq 5$ inequality (8) is true, which concludes the proof of the claim. □

5 | The Large Degree Limit

For every $d \geq 1$ we let $\mathcal{U}_{d,2}$ be the maximal open zero-free region for graphs of maximum degree at most $d+1$, formally

$$\mathcal{U}_{d,2} = \mathbb{C} \setminus \overline{\{\lambda \in \mathbb{C} : \text{there exists a graph } G \text{ with } \Delta(G) \leq d+1 \text{ such that } Z(G;\lambda) = 0\}}$$

We let $\mathcal{U}_{d,\geq 2}$ denote the analogous region for hypergraphs. To conform with notation from [11], we shifted the index by one with respect to the introduction. In the previous two sections, we showed that for $d \geq 2$

$$\mathcal{U}_{d,2} \cap \mathbb{R} = \mathcal{U}_{d,\geq 2} \cap \mathbb{R} = \left(-\lambda_s(d+1), \lambda_c(d+1)\right)$$

However, we will show in Lemma 6.9 that $\mathcal{U}_{2,2} \neq \mathcal{U}_{2,\geq 2}$ and thus in general these sets are not equal. In this section we show that in the large degree limit they are in fact equal.

As d increases the sets $\mathcal{U}_{d,2}$ become arbitrarily small, for example, both $\lambda_c(d+1)$ and $\lambda_s(d+1)$ converge to 0. However, as $d \to \infty$ it is not hard to see that $d \cdot \lambda_c(d+1)$ and $d \cdot \lambda_s(d+1)$ converge to e and $1/e$, respectively. In fact, in [11] it is proved that there is a set $\mathcal{U}_\infty$ such that the sequence of rescaled regions $d \cdot \mathcal{U}_{d,2}$ converges to $\mathcal{U}_\infty$. This limit set is bounded and contains an open neighborhood of $(-1/e, e)$.

Theorem 5.1 (Theorem 1.1. in [11]). *The sets $d \cdot \mathcal{U}_{d,2}$ converge to $\mathcal{U}_\infty$ in terms of the Hausdorff distance.*

This means that for every closed $K_1 \subseteq \text{int}(\mathcal{U}_\infty)$ and open $K_2 \supseteq \overline{\mathcal{U}_\infty}$ for d sufficiently large $K_1 \subseteq d \cdot \mathcal{U}_{d,2} \subseteq K_2$. We will show that the same is true for the rescaled regions $d \cdot \mathcal{U}_{d,\geq 2}$.

To show this we define the map $E_\Lambda(Z) = \Lambda e^{-Z}$. It is shown in [11] that $\Lambda \in \mathcal{U}_\infty$ if and only if there is a compact convex set containing 0 that is forward invariant for E_Λ. We will show that, after doing the proper rescaling, this implies that there exists a set $\tilde{A}$ such that Λ/d and $\tilde{A}/d$ satisfy the conditions of Lemma 3.5 for sufficiently large d. More precisely, we prove the following.

Lemma 5.2. *Let $\Lambda_0 \in \mathbb{C}$ and $A \subseteq \mathbb{C}$ convex, compact and with $0 \in A$ such that $E_{\Lambda_0}(A) \subseteq \text{int}(A)$. Then there is a neighborhood U of Λ_0, a region $\tilde{A}$ and a $d_0 \in \mathbb{Z}_{\geq 1}$ such that for all $\Lambda \in U$ and $d \geq d_0$ the pair Λ/d and $\tilde{A}/d$ satisfy the conditions of Lemma 3.5, that is,*

- *$1 + \tilde{A}/d$ is log-convex;*
- *$\mu(\tilde{A}/d)$ is a multiplicative semi-group;*
- *$\tilde{A}/d$ is strictly forward invariant for $z \mapsto \frac{\Lambda/d}{(1+z)^d}$.*

Proof. We need to slightly alter A to define $\tilde{A}$. For $\epsilon > 0$ let A_ϵ denote the convex hull of the union of A with the closed ball B_ϵ. For ϵ sufficiently small we still have $E_{\Lambda_0}(A_\epsilon) \subseteq \text{int}(A_\epsilon)$; from now on fix such an ϵ.

We recall that $\mathbb{D}$ denotes the open unit disk. By the Riemann mapping theorem, there is a conformal bijection $h : \mathbb{D} \to \text{int}(A_\epsilon)$. Note that $h^{-1}(E_{\Lambda_0}(A_\epsilon))$ and $h^{-1}(\{0\})$ are closed in $\mathbb{D}$ and thus we can choose $\delta < 1$ such that B_δ contains them both in the interior. We let $\tilde{A} = h(B_\delta)$. Note that $0 \in \text{int}(\tilde{A})$ and $E_{\Lambda_0}(\tilde{A}) \subseteq E_{\Lambda_0}(A_\epsilon) \subseteq \text{int}(\tilde{A})$.

Observe that, because $\tilde{A}$ is compact, for d sufficiently large $1 + \tilde{A}/d$ is contained in the right half-plane and thus $\log(1 + \tilde{A}/d)$ is well defined. The interior of $\log(1 + \tilde{A}/d)$ is the image of $\mathbb{D}$ under the map $g(z) = \log(1 + h(\delta z)/d)$. It is a result from complex analysis (see e.g., [34], §2.5) that the image $g(\mathbb{D})$ of a conformal map g is convex if and only if for all $z \in \mathbb{D}$ we have $\Re(1 + zg''(z)/g'(z)) > 0$. We apply this to g:

$$\inf_{z\in\mathbb{D}} \Re\left(1 + \frac{zg''(z)}{g'(z)}\right) = \inf_{z\in\mathbb{D}} \Re\left(1 + \frac{\delta z h''(\delta z)}{h'(\delta z)} + \frac{\delta z h'(\delta z)}{h(\delta z) + d}\right) = \min_{z\in B_\delta} \Re\left(1 + \frac{zh''(z)}{h'(z)} + \frac{zh'(z)}{h(z) + d}\right)$$

Because $h(\mathbb{D}) = A_\epsilon$ is convex there is a strictly positive lower bound on $\Re(1 + zh''(z)/h'(z))$ for $z \in B_\delta$. For d sufficiently large, say $d \geq d_1$, it thus follows that the above infimum is strictly positive and thus $g(\mathbb{D}) = \log(1 + \tilde{A}/d)$ is convex.

Because $\tilde{A}$ is compact and contains 0 in its interior there are $0 < m < M$ such that $B_m \subseteq \tilde{A} \subseteq B_M$. For $0 < r < 1$, we have $B_{r/(1+r)} \subset \mu(B_r) \subset B_{r/(1-r)}$. Thus, for d sufficiently large,

$$B_{m/(d+m)} \subseteq \mu(\tilde{A}/d) \subseteq B_{M/(d-M)}$$

For $w_1, w_2 \in \tilde{A}/d$ we have

$$|w_1 w_2| \leq \left(\frac{M}{d - M}\right)^2$$

which, for say $d \geq d_2$, is less than $\frac{m}{d+m}$. We can conclude that $w_1 w_2 \in \mu(\tilde{A}/d)$ and thus $\mu(\tilde{A}/d)$ is a multiplicative semi-group for $d \geq d_2$.

Let $G_d(\Lambda, Z) = \frac{\Lambda}{(1+Z/d)^d}$. The set $\tilde{A}/d$ being strictly forward invariant for $z \mapsto \frac{\Lambda/d}{(1+z)^d}$ is equivalent to saying that $\tilde{A}$ is strictly forward invariant for $Z \mapsto G_d(\Lambda, Z)$. Let K be a compact set containing Λ_0 in its interior. The maps $(\Lambda, Z) \to G_d(\Lambda, Z)$ converge uniformly on $K \times \tilde{A}$ to $(\Lambda, Z) \mapsto \Lambda e^{-Z}$. Therefore, there is a neighborhood U of Λ_0 and a d_3 such that for $\Lambda \in U$ and $d \geq d_3$ we have that $G_d(\Lambda, \tilde{A}) \subseteq \mathrm{int}(\tilde{A})$. We conclude the proof by letting $d_0 = \max\{d_1, d_2, d_3\}$. □

This allows us to prove Theorem 1.4, which we state in the following form.

Theorem 5.3. *The sets $d \cdot \mathcal{U}_{d,\geq 2}$ converge to $\mathcal{U}_\infty$ in terms of the Hausdorff distance.*

Proof. Take a closed $K_1 \subseteq \mathrm{int}(\mathcal{U}_\infty)$ and open $K_2 \supseteq \overline{\mathcal{U}_\infty}$.

It follows from Theorem 5.1 that for d sufficiently large $d \cdot \mathcal{U}_{d,2} \subseteq K_2$. Because $\mathcal{U}_{d,\geq 2} \subseteq \mathcal{U}_{d,2}$ by definition, the same is true for $\mathcal{U}_{d,\geq 2}$.

Take a $\Lambda_0 \in K_1$. It follows from [11, Corollary 4.2 and Lemma 4.4] that there is a compact convex set containing 0 that is strictly forward invariant for E_{Λ_0}. It follows from Lemma 5.2that there is a neighborhood $U(\Lambda_0)$ of Λ_0, a region $\tilde{A}(\Lambda_0)$ and a $d_0(\Lambda_0)$ such that for $\Lambda \in U(\Lambda_0)$ and $d \geq d_0(\Lambda_0)$ the pair Λ/d and $\tilde{A}(\Lambda)/d$ satisfy the conditions of Lemma 3.5. It follows then from Corollary 3.6 that for those d we have $\Lambda/d \in \mathcal{U}_{d,\geq 2}$, that is, $U(\Lambda_0) \subseteq d \cdot \mathcal{U}_{d,\geq 2}$.

The sets $\{U(\Lambda)\}_{\Lambda\in K_1}$ form an open cover of K_1. Because $\mathcal{U}_\infty$ is bounded the set K_1 is compact and thus there is a finite $I \subseteq K_1$ such that $\{U(\Lambda)\}_{\Lambda\in I}$ is a cover of K_1. Then for $d \geq \max_{\Lambda\in I} d_0(\Lambda)$ we can conclude that $K_1 \subseteq d \cdot \mathcal{U}_{d,\geq 2}$. □

6 | The Maximal Zero-Free Disk for Bounded Degree Uniform Linear Hypertrees

In this section, we will prove Theorem 1.5. Because b, d will be more or less fixed we will drop the subscripts and write

$$f_\lambda(z) = \lambda \cdot \left[1 - \left(\frac{z}{1+z}\right)^b\right]^d \tag{9}$$

for the remainder of this section.

6.1 | Preliminaries on Complex Dynamics

In this section, we gather the required background on the theory of dynamics of rational maps. The definitions and general results from this section can be found in [35, 36]. We recall that $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ denotes the Riemann sphere.

6.1.1 | Critical Points and Holomorphic Families of Rational Maps

A point $z \in \hat{\mathbb{C}}$ is called a *critical point* of a rational map $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ if g is not locally injective around z, that is, if there does not exist a neighborhood U of z such that $g|_U$ is injective. A *holomorphic family of rational maps* parameterized by an open set $X \subseteq \hat{\mathbb{C}}$ is a holomorphic map $g : X \times \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ such that for any fixed $\lambda_0 \in X$ the map $z \mapsto g(\lambda_0, z)$ is rational. We usually denote $g(\lambda, z)$ by $g_\lambda(z)$. Note that f as defined in (9) is a holomorphic family of rational maps for any open $X \subseteq \hat{\mathbb{C}}$. We say that the critical points of a holomorphic family of rational maps *move holomorphically* around λ_0 if there is a neighborhood $U \subset X$ of λ_0 and holomorphic maps $c_1, \ldots, c_N$ from $U \to \hat{\mathbb{C}}$ such that for any $\lambda \in U$ the critical points of g_λ are $\{c_1(\lambda), \ldots, c_N(\lambda)\}$.

For any map $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ we denote by g^n the map that applies g successively n times. The set $\{g^n(z_0)\}_{n \geq 1}$ is called the *forward orbit* of z_0. We shall see that the forward orbits of the critical points play an important role in understanding the dynamic behavior of g. We first prove a lemma about the critical orbits of our map f (as defined in (9)). The lemma says that there is essentially a unique critical orbit.

Lemma 6.1. *Around any $\lambda_0 \in \mathbb{C} \setminus \{0\}$ the critical points of f_λ move holomorphically. In fact there are holomorphic functions $c_1, \ldots, c_N$ parameterizing the critical points on $\mathbb{C} \setminus \{0\}$. Moreover, for any critical point c_i*

$$f_\lambda^{n_i}(c_i(\lambda)) = \lambda$$

for some $n_i \in \{1, 2, 3\}$.

Proof. The maps $z \mapsto \lambda z$, $z \mapsto 1 - z$ and $z \mapsto z/(1+z)$ are locally injective on the whole Riemann sphere. The maps $z \mapsto z^b$ and $z \mapsto z^d$ are locally injective anywhere except for $z \in \{0, \infty\}$. Therefore, the critical points of f_λ, as long as λ is nonzero, are given by those z for which either

$$\frac{z}{1+z} \in \{0, \infty\} \quad \text{or} \quad 1 - \left(\frac{z}{1+z}\right)^b \in \{0, \infty\}$$

Since these do not depend on λ, we can conclude that the critical points do indeed move holomorphically on $\mathbb{C} \setminus \{0\}$; they are in fact constant. Moreover observe that

- if $\frac{z}{1+z} = 0$ then $f_\lambda(z) = \lambda$;
- if $1 - \left(\frac{z}{1+z}\right)^b = 0$ then $f_\lambda^2(z) = \lambda$;
- if $\frac{z}{1+z} = \infty$, equivalently if $1 - \left(\frac{z}{1+z}\right)^b = \infty$, then $f_\lambda^3(z) = \lambda$. □

6.1.2 | Normality

Let U be an open subset of $\hat{\mathbb{C}}$ and let $\mathcal{F}$ be a set of holomorphic maps $g : U \to \hat{\mathbb{C}}$. In this context, $\mathcal{F}$ is usually referred to as a family of holomorphic maps. The family $\mathcal{F}$ is called a *normal family* if every sequence $\{g_n\}_{n \geq 1} \subseteq \mathcal{F}$ has a subsequence that converges uniformly on compact subsets of U. Given a particular parameter $\lambda_0 \in U$, we say that $\mathcal{F}$ is *normal at* λ_0 if there exists an open neighborhood of λ_0 on which $\mathcal{F}$ is a normal family.

Theorem 6.2 (Montel's Theorem). *Let $\mathcal{F}$ be a family of holomorphic maps $U \mapsto \hat{\mathbb{C}}$ for some open $U \subseteq \hat{\mathbb{C}}$ and let h_1, h_2, h_3 be holomorphic maps $U \to \hat{\mathbb{C}}$ such that $h_1(\lambda)$, $h_2(\lambda)$ and $h_3(\lambda)$ are pairwise distinct for all $\lambda \in U$. If $g(\lambda) \notin \{h_1(\lambda), h_2(\lambda), h_3(\lambda)\}$ for all $g \in \mathcal{F}$ and $\lambda \in U$ then $\mathcal{F}$ is a normal family.*

Remark 6.3. This theorem is usually stated with h_1, h_2 and h_3 being constant, say 0,1 and ∞. By considering the family $\mathcal{F}' = \{\lambda \mapsto (\mu_\lambda \circ g)(\lambda) : g \in \mathcal{F}\}$, where μ_λ is a Möbius tranformation sending $(h_1(\lambda), h_2(\lambda), h_3(\lambda))$ to $(0, 1, \infty)$, we see that $\mathcal{F}$

is a normal family if and only if $\mathcal{F}'$ is a normal family. Furthermore, $\mathcal{F}$ avoids $\{h_1(\lambda), h_2(\lambda), h_3(\lambda)\}$ if and only if $\mathcal{F}'$ avoids $\{0, 1, \infty\}$. The two statements are therefore equivalent.

6.1.3 | Periodic Points

A point $z_0 \in \hat{\mathbb{C}}$ is called a *periodic fixed point* of a rational map $g : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ if there is an $n \geq 1$ such that $g^n(z_0) = z_0$. The least n for which this is the case is called the *period* of z_0. If $n = 1$, then z_0 is called a *fixed point* of g. The *multiplier* of a periodic point z_0 with period n is defined as $(g^n)'(z_0)$, that is, the derivative of f^n evaluated at z_0.[4]

Lemma 6.4. *The map f_λ has a fixed point with multiplier α if and only if $\lambda = (1 - w^b)^{-d} \cdot \frac{w}{1-w}$, where $w \neq 1$ is a solution to*

$$-bd \cdot \frac{w^b(1 - w)}{1 - w^b} = \alpha$$

Proof. Note that z is a fixed point of f_λ if and only if

$$f_\lambda(z) = z \quad \Leftrightarrow \quad \lambda = z \cdot \left[1 - \left(\frac{z}{1+z}\right)^b\right]^{-d}$$

Furthermore, this fixed point has multiplier α if and only if

$$\alpha = f_\lambda'(z) = -bd\frac{z^{b-1}}{(1+z)\big((1+z)^b - z^b\big)} \cdot f_\lambda(z) = -db\frac{\left(\frac{z}{1+z}\right)^b}{(1+z)\left(1 - \left(\frac{z}{1+z}\right)^b\right)}$$

Note that $f_\lambda(\infty) = 0$ irregardless of λ and thus ∞ is never a fixed point. Therefore, for any fixed point z, we can write $z = \frac{w}{1-w}$ for some $w \neq -1$. The result follows by making this substitution. □

A periodic fixed point with multiplier α is called either *attracting*, *indifferent*, or *repelling* according to whether $|\alpha| < 1$, $|\alpha| = 1$ or $|\alpha| > 1$, respectively. A periodic point z_0 of g_{λ_0}, where g is a holomorphic family of rational maps, is called *persistently indifferent* if there is a neighborhood U of λ_0 and a holomorphic map $w : U \to \hat{\mathbb{C}}$ such that $w(\lambda_0) = z_0$ and $w(\lambda)$ is an indifferent periodic fixed point of g_λ for every $\lambda \in U$.

Lemma 6.5. *The map f_λ has no persistently indifferent periodic points for any $\lambda \in \mathbb{C}$.*

Proof. It follows from Lemma 6.4 that there is a nonempty open set $U \subseteq \mathbb{C}$ such that f_λ has an attracting fixed point for all $\lambda \in U$. Fix any nonzero $\lambda_0 \in U$, let p denote an attracting fixed point of f_{λ_0}, and let $c_1, \ldots, c_N$ denote the critical points of f_{λ_0}. At least one of the critical orbits $\{f_{\lambda_0}^n(c_i)\}_{n\geq 1}$ converges to p; see for example, [36, Theorem 8.6]. It then follows from Lemma 6.1 that the orbit of any critical point converges to p. This means that f_{λ_0} is a hyperbolic map; see for example, ([36], §19). One property of hyperbolic maps is that every periodic orbit must be either attracting or repelling, that is, not indifferent. It is not hard to see that if f_λ were to have a persistently indifferent fixed point for some λ, then it has to have an indifferent fixed point for all but finitely many $\lambda \in \mathbb{C}$; this is proved, for example, in [27, Lemma 9]. Because we just observed that f_λ is hyperbolic for λ in an open set, we can conclude that f_λ cannot have a persistently indifferent fixed point for any λ. □

Theorem 6.6 (Part of Theorem 4.2 in [35]). *Let g be a holomorphic family of rational maps parameterized by U. Suppose there exist holomorphic maps $c_i : U \to \hat{\mathbb{C}}$ parameterizing the critical points of g. Let $\lambda_0 \in U$ and suppose that for all i the families of maps given by*

$$\mathcal{F}_i = \{\lambda \mapsto g_\lambda^n(c_i(\lambda))\}_{n\geq 1}$$

are normal at λ_0. Then there is a neighborhood $V \subseteq U$ of λ_0 such that for all $\lambda \in V$ every periodic point of g_λ is either attracting, repelling, or persistently indifferent.

6.2 | Accumulation of Roots

We will use the results gathered in the previous section to show that parameters λ for which f_λ (as defined in (9)) has an indifferent fixed point are accumulation points of zeros.

Lemma 6.7. *Suppose $\lambda_0 \in \mathbb{C}$ and $n \in \mathbb{Z}_{\geq 1}$ are such that $f^n_{\lambda_0}(\lambda_0) = -1$, then there is a $(b+1)$-uniform linear hypertree $\mathcal{T}$ with degree at most $d+1$ and $Z_{\mathcal{T}}(\lambda_0) = 0$.*

Proof. We inductively define a sequence $\{(\mathcal{T}_m, v_m)\}_{m\geq 0}$ of rooted $(b+1)$-uniform linear hypertrees whose maximum degrees are at most $d+1$. We let $\mathcal{T}_0$ consist of a single vertex v_0. For $m \geq 1$ we let $\mathcal{T}_m$ consist of the vertex v_m that is contained in d hyperedges of size $b+1$ only intersecting in v_m. For all vertices $u \neq v_m$ sharing a hyperedge with v_m, we attach a disjoint copy of $(\mathcal{T}_{m-1}, v_{m-1})$ by identifying u with v_{m-1}.

It follows from Lemma 2.4 that

$$R_{v_m}(\mathcal{T}_m; \lambda) = f^m_\lambda(\lambda)$$

for all $m \geq 0$ and $\lambda \in \mathbb{C}$. Therefore, $R_{v_n}(\mathcal{T}_n; \lambda_0) = -1$ and thus it follows from Lemma 3.1 that there is a $(b+1)$-uniform linear hypertree $\mathcal{T}$ with degree at most $d+1$ and $Z_{\mathcal{T}}(\lambda_0) = 0$ (in fact, one can take $\mathcal{T} = \mathcal{T}_n$). □

We remark that for graphs, that is, for $b = 1$, the trees defined in the proof of the lemma above are often refered to as d-ary trees of finite depth.

Lemma 6.8. *Let $\lambda_0 \in \mathbb{C} \setminus \{-1, 0\}$ and suppose f_{λ_0} has an indifferent fixed point. Then, given a neighborhood U of λ_0, there is a $\lambda_1 \in U$ and a $(b+1)$-uniform linear hypertree $\mathcal{T}$ with degree at most $d+1$ such that $Z_{\mathcal{T}}(\lambda_1) = 0$.*

Proof. Take $p_0 \in \mathbb{C}$ such that $f_{\lambda_0}(p_0) = -1$. It follows that $f^2_{\lambda_0}(p_0) = \infty$ and $f^3_{\lambda_0}(p_0) = 0$ and thus because $\lambda_0 \notin \{-1, 0, \infty\}$, it follows from Lemma 6.1 that p_0 is not a critical point of f_{λ_0}. We can therefore apply the Implicit Function Theorem to obtain a neighborhood V of λ_0 and a holomorphic map $p : V \to \hat{\mathbb{C}}$ such that $p(\lambda_0) = p_0$ and $f_\lambda(p(\lambda)) = -1$ for all $\lambda \in V$. Note that this implies that $p(\lambda) \notin \{-1, \infty\}$ for all $\lambda \in V$.

Let $U \subseteq V$ be an arbitrary neighborhood of λ_0. By Lemma 6.1, there are holomorphic functions $c_1, \ldots, c_N$ parameterizing the critical points of f_λ on U. Because f_{λ_0} has an indifferent fixed point that, by Lemma 6.5, is not persistently indifferent it follows from Theorem 6.6 that at least one of the families $\mathcal{F}_i = \{\lambda \mapsto f^n_\lambda(c_i(\lambda))\}_{n\geq 1}$ is not normal on U. It then follows from Lemma 6.1 that the family $\{\lambda \mapsto f^n_\lambda(\lambda)\}_{n\geq 1}$ is not normal on U. By Theorem 6.2, there must be a $\lambda_1 \in U$ such that $f^n_{\lambda_1}(\lambda_1) \in \{-1, \infty, p(\lambda_1)\}$. If $f^n_{\lambda_1}(\lambda_1) = \infty$ then $f^{n-1}_{\lambda_1}(\lambda_1) = -1$ and if $f^n_{\lambda_1}(\lambda_1) = p(\lambda_1)$ then $f^{n+1}_{\lambda_1}(\lambda_1) = -1$. In all three cases, we can apply Lemma 6.7 to conclude that there is a $(b+1)$-uniform linear hypertree $\mathcal{T}$ with degree at most $d+1$ such that $Z_{\mathcal{T}}(\lambda_1) = 0$. □

Using Lemma 6.4 we can calculate parameters λ for which f_λ has an indifferent fixed point. Figure 3 shows examples of such parameters, which, by the previous lemma, are accumulation points of zeros of $(b+1)$-uniform hypertrees.

Let $\mathcal{U}_{\Delta,2}$ be the maximal connected zero-free region containing 0 for graphs of maximum degree at most Δ (shifting the index back from the one used in Section 2 to the one in the introduction). In [4, Conjecture 6], it is conjectured that this region is zero-free for hypergraphs of maximum degree at most Δ as well. Using Lemma 6.8, we can show that, at least for $\Delta = 3$, this is not the case.

Corollary 6.9. *The region $\mathcal{U}_{3,2}$ is not zero-free for hypergraphs of maximum degree at most 3.*

Proof. Let $\lambda_0 = (1 - w_0^4)^{-2} \frac{w_0}{1-w_0}$, where $w_0 \approx 0.3540 + 0.5331i$ is a solution to

$$-8\frac{w_0^4(1 - w_0)}{(1 - w_0^4)} = 1$$

We find that $\lambda_0 \approx 0.0665 + 0.6015i$; see also Figure 3. By Lemmas 6.4 and 6.8, zeros of 5-uniform linear hypertrees with maximum degree at most 3 accumulate on λ_0.

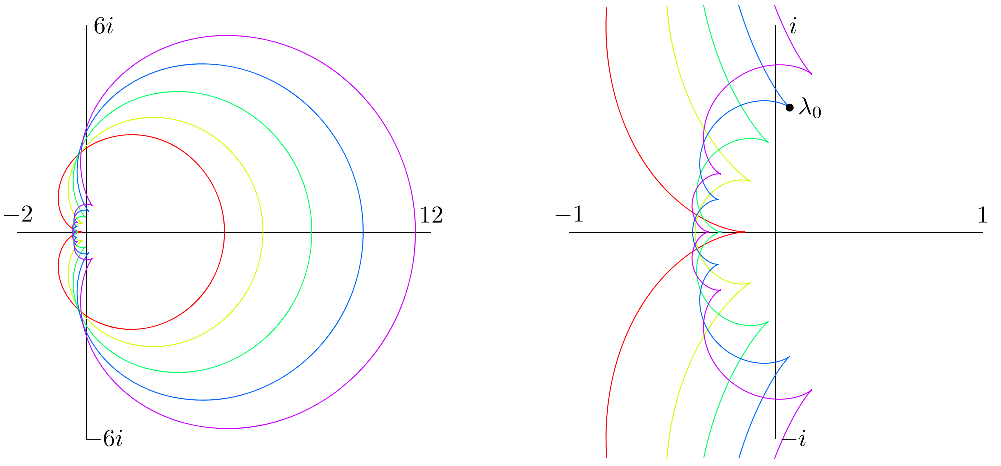

FIGURE 3 | Parameters λ for which $f_{\lambda,b}$ has an indifferent fixed point for $d = 2$ and b ranging from 1 to 5. The regions enclosed by the curves get increasingly larger.

On the other hand, it is shown in [37] that $\mathcal{U}_{d+1,2}$ contains the intersection of the disc centered at 0 of radius $\frac{7}{8}\tan(\frac{\pi}{2d})$ and the right half-plane. Because $\frac{7}{8}\tan(\frac{\pi}{4}) = 0.875$ and $|\lambda_0| \approx 0.6052$ we can conclude that $\lambda_0 \in \mathcal{U}_{3,2}$. □

6.3 | Dynamics on Disks

To prove the zero-freeness part of Theorem 1.5, we will apply Lemma 3.3. We will thus show that there is a λ with $|\lambda| = \rho_{d,b}$ such that $f_\lambda(B_R) \subseteq B_R$ for some $R > 0$. Instead of using the value of $\rho_{d,b}$, we will do this by letting λ_0 be maximal in norm for which we can apply Lemma 3.3. It will follow that there must be a λ of the same norm as λ_0 for which f_λ has a fixed point of multiplier 1. We must first prove some lemmas about disks.

Lemma 6.10. *Let $D \subset \mathbb{C}$ be a closed disk with $0 \in \mathrm{int}(D)$ whose center lies in $\mathbb{R}_{<0}$ and let z_0 be the intersection of ∂D with the negative real axis. Then*

- *for odd $n \in \mathbb{Z}_{\geq 1}$*
$$\operatorname*{argmax}_{z\in D}|z^n - 1| = \{z_0\};$$
- *for even $n \in \mathbb{Z}_{\geq 1}$ either*
$$\operatorname*{argmax}_{z\in D}|z^n - 1| = \{z_0\} \quad \text{or} \quad \operatorname*{argmax}_{z\in D}|z^n - 1| = \{z_M, \overline{z_M}\},$$
where z_M is a nonreal value in ∂D.

Proof. Assume first that n is odd. Let z_0 be the intersection of ∂D with the negative real axis. For any $r > 0$ let B_r denote the closed disk centered at 0 with radius r. For a set A let $A^n = \{z^n : z \in A\}$. Observe that $D \subseteq B_{|z_0|}$ and therefore $D^n \subseteq B^n_{|z_0|} = B_{|z_0|^n}$. The unique maximizer of $|z - 1|$ with $z \in B_{|z_0|^n}$ is $-|z_0|^n = z_0^n$. Because $D \setminus \{z_0\} \subseteq \mathrm{int}(B_{|z_0|})$ it follows that z_0 is the unique maximizer for $|z^n - 1|$ with $z \in D$.

Now assume that n is even. Let $-x$ and r be the center and the radius of D, respectively, and note that $x > r$. The disk D is the image of the closed unit disk $\overline{\mathbb{D}}$ under the map $z \mapsto rz - x$. Because n is even and $\overline{\mathbb{D}}$ is symmetric under $z \mapsto -z$ we see that D^n is the image of $\overline{\mathbb{D}}$ under the map $p(z) := (rz + x)^n$. We want to determine the values in $\overline{\mathbb{D}}$ that maximize $|p(z) - 1|$. These values have to lie on the boundary $\partial\mathbb{D}$.

We define $\gamma(t) = p(e^{\pi i t})$ for $t \in [0, 1]$, which parameterizes the image under p of the part of $\partial\mathbb{D}$ that is contained in the upper half-plane. It is sufficient to show that there is a unique $t_0 \in [0, 1)$ maximizing $|\gamma(t) - 1|$. Note that either $0 < \gamma(1) \leq 1$ or $1 \leq \gamma(1) < \gamma(0)$ so $t = 1$ is never a maximizer of $|\gamma(t) - 1|$.

The curvature of the curve γ is given by

$$\kappa(t) = \frac{1}{|p'(e^{\pi it})|}\Re\left(1 + \frac{e^{\pi it}p''(e^{\pi it})}{p'(e^{\pi it})}\right)$$

We will show that κ is strictly increasing.

We have that $p'(e^{\pi it}) = nr(re^{\pi it} + x)^{n-1}$. As t increases $re^{\pi it} + x$ travels along a circle of radius r centered at x from the positive real axis to the negative real axis. Since $x > 0$ we see that $|re^{\pi it} + x|$ is decreasing and therefore $1/|p'(e^{\pi it})|$ is increasing.

We calculate

$$1 + \frac{e^{\pi it}p''(e^{\pi it})}{p'(e^{\pi it})} = 1 + \frac{e^{\pi it}(n-1)r}{re^{\pi it} + x} = \frac{nre^{\pi it} + x}{re^{\pi it} + x}$$

The map $z \mapsto (nrz + x)/(rz + x)$ is a Möbius transformation that maps the real line to the real line. Because a Möbius transformation is conformal and sends generalized circles to generalized circles, it must map $\partial\mathbb{D}$ to a circle crossing the real line transversely at $(nr + x)/(r + x)$ and $(-nr + x)/(-r + x)$. Observe that

$$0 < \frac{nr + x}{r + x} < \frac{-nr + x}{-r + x}$$

which implies that as t increases from 0 to 1 the real part of $(nre^{\pi it} + x)/(re^{\pi it} + x)$ increases. It follows that κ is increasing.

Let C_t be the osculating circle of γ at $\gamma(t)$. This is the circle that locally approximates γ the best; see for example, [38]. A theorem by Tait and later rediscovered by Kneser [39, 40] states that if a curve has strictly increasing curvature then $t_1 > t_2$ implies that C_{t_1} strictly contains C_{t_2}. This implies that for any $t_0 \in [0, 1)$ the circle C_{t_0} strictly contains $\gamma((t_0, 1))$. Moreover, by the definition of curvature, any circle touching the curve γ at $\gamma(t_0)$ of smaller radius must have the property that there are $t > t_0$ with $\gamma(t)$ lying outside of it.

Now let $t_0 \in [0, 1)$ be the minimal t for which $|\gamma(t) - 1|$ is maximized. Let C_M be the circle with radius $|\gamma(t_0) - 1|$ centered at 1. The circle C_M touches the curve γ at $\gamma(t_0)$ and also contains the whole curve $\gamma([0, 1])$. Therefore, its radius is at least that of C_{t_0} and thus C_M contains C_{t_0}. It follows that C_M strictly contains the curve $\gamma((t_0, 1))$ and thus we conclude that t_0 is the unique maximizer of $|\gamma(t) - 1|$ for $t \in [0, 1]$. See Figure 4 for an example.

It follows that $z_M := -x - re^{\pi it_0}$ is the unique maximizer of $|z^n - 1|$ for z in the intersection of D with the closed lower half-plane. By symmetry $\overline{z_M}$ is the unique maximizer for z in the intersection of D with the closed upper half-plane. If $t_0 = 0$ then $z_M = \overline{z_M} = z_0$. □

Lemma 6.11. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a rational function and $R > 0$ for which*

- $f(B_R) \subseteq B_R$;
- *there does not exist any $\tilde{R} > 0$ such that $f(B_{\tilde{R}}) \subseteq \text{int}(B_{\tilde{R}})$;*
- *for any $z_1, z_2 \in \text{argmax}_{z\in B_R}|f(z)|$ we have $|f'(z_1)| = |f'(z_2)|$.*

Then $|f'(z)| = 1$ for all $z \in \text{argmax}_{z\in B_R}|f(z)|$.

Proof. Suppose first $|f'(z)| < 1$ for all $z \in \text{argmax}_{z\in B_R}|f(z)|$. Note that $\text{argmax}_{z\in B_R}|f(z)|$ is closed and contained in ∂B_R. It follows that there exist $\delta_1, \delta_2 > 0$ and $\eta < 1$ such that $|f'(z)| < \eta$ for all $z \in S_{\delta_1,\delta_2}$, where

$$S_{\delta_1,\delta_2} = \{r \cdot e^{i\theta} \cdot z : 1 \leq r < 1 + \delta_1, |\theta| < \delta_2 \text{ and } z \in \underset{z\in B_R}{\text{argmax}}|f(z)|\}$$

The set $\partial B_R \setminus S_{\delta_1,\delta_2}$ is a closed subset of ∂B_R whose forward image under f is contained in $\text{int}(B_R)$. Therefore, by continuity, there exists an $\epsilon \in (0, \delta_1)$ such that the forward image of $\partial B_{(1+\epsilon)R} \setminus S_{\delta_1,\delta_2}$ is contained in $\text{int}(B_R)$. Now take an element $z \in \partial B_{(1+\epsilon)R} \cap S_{\delta_1,\delta_2}$ and let $\theta_z = \arg(z)$. By construction the whole straight line segment between $Re^{i\theta_z}$ and $z = (1+\epsilon)Re^{i\theta_z}$ lies in S_{δ_1,δ_2}. Therefore

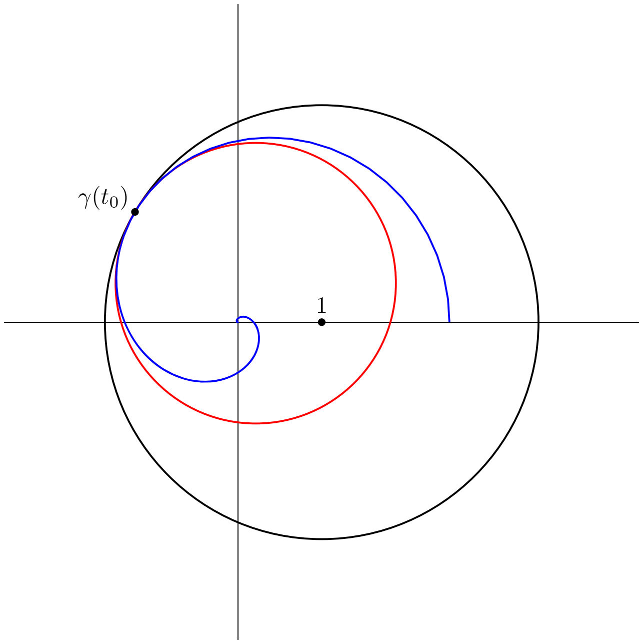

FIGURE 4 | An example accompanying the proof of Lemma 6.10. Here, $x = \frac{1}{2}, r = \frac{2}{3}$ and $n = 6$. The curve γ is drawn in blue, the osculating circle at $\gamma(t_0)$ is drawn in red and the circle touching $\gamma(t_0)$ with center 1 is drawn in black.

$$|f(z)| \leq |f(Re^{i\theta_z})| + |f(z) - f(Re^{i\theta_z})| < R + \eta|Re^{i\theta_z} - z| = (1+\eta\epsilon)R$$

Here we have used that $f(B_R) \subseteq B_R$. We see that z gets mapped into $\text{int}(B_{(1+\epsilon)R})$ by f. We can thus conclude that $f(B_{(1+\epsilon)R}) \subseteq \text{int}(B_{(1+\epsilon)R})$, which is a contradiction.

Now suppose that $|f'(z)| > 1$ for all $z \in \text{argmax}_{z\in B_R}|f(z)|$. Let $U = \{z : |f'(z)| > 1\}$, which is an open neighborhood of $\underset{z\in B_R}{\text{argmax}}|f(z)|$. Because $f(B_R) \subseteq B_R$ we can use the Schwarz–Pick theorem which states that for all $z \in \text{int}(B_R)$

$$\frac{|f'(z)|}{R^2 - |f(z)|^2} \leq \frac{1}{R^2 - |z|^2}$$

It follows that for all $z \in U \cap \text{int}(B_R)$ we have $|f(z)| < |z|$. The image of $\partial B_R \setminus U$ under f is mapped into $\text{int}(B_R)$, and thus, by continuity, there is an $\epsilon > 0$ such that $f(\partial B_{(1-\epsilon)R} \setminus U) \subseteq \text{int}(B_{(1-\epsilon)R})$. It follows that $f(B_{(1-\epsilon)R}) \subset \text{int}(B_{(1-\epsilon)R})$, which is a contradiction. □

6.4 | Proof of Theorem 1.5

We will now prove Theorem 1.5, which we restate for convenience.

Theorem 1.5. *Let $d \geq 2$, $b \geq 1$ and define the function $f_\lambda(z) = f_{\lambda,b,d}(z)$ as*

$$f_\lambda(z) = \lambda\left(1 - \left(\frac{z}{1+z}\right)^b\right)^d$$

Let $\rho_{d,b}$ be the largest $\rho \geq 0$, such that there exists a disk B_R of radius R around 0 that is mapped into itself by f_ρ. To be precise,

$$\rho_{d,b} = \max\{\rho \geq 0 \mid \text{there exists an } R > 0 \text{ such that } f_\rho(B_R) \subseteq B_R\}$$

Zero-freeness If $\mathcal{T}$ is a $(b+1)$-uniform linear hypertree with maximum degree at most $d+1$ and $\lambda \in \mathbb{C}^{V(\mathcal{T})}$ is such that $|\lambda_v| \leq \rho_{d,b}$ for all $v \in V(\mathcal{T})$, then $Z(\mathcal{T};\lambda) \neq 0$.

Maximality *There is a sequence of* $(b+1)$*-uniform linear hypertrees* $\{\mathcal{T}_n\}$ *with maximum degree at most* $d+1$ *and a sequence of parameters* $\{\lambda_n\}$ *such that* $|\lambda_n|$ *converges to* $\rho_{d,b}$ *and* $Z(\mathcal{T}_n;\lambda_n)=0$ *for all* n.

Formula

$$\rho_{d,b} = \min\left\{ \left|(1-w^b)^{-d}\frac{w}{(1-w)}\right| \,:\, w\in\mathbb{C}\text{with } -bd\frac{w^b(1-w)}{(1-w^b)} = 1 \right\}$$

Asymptotics *For fixed* b

$$\rho_{d,b} = (ebd)^{-1/b} + \mathcal{O}(d^{-2/b})$$

as $d\to\infty$.

Proof. Define

$$g(z) = \left[1-\left(\frac{z}{1+z}\right)^b\right]^d$$

so that $f_\lambda(z) = \lambda\cdot g(z)$. Consider the set

$$S = \{\lambda\in\mathbb{C} \,:\, f_\lambda(B_R)\subseteq B_R \text{ for some } R\geq 0\}$$

For any λ with $|\lambda| \leq \left(\max\left\{|g(z)| \,:\, z\in B_{1/2}\right\}\right)^{-1}$, we see that $f_\lambda(B_{1/2})\subseteq B_{1/2}$ and thus S is not empty. Moreover, S is closed and if $\lambda\in S$ then $\alpha\cdot\lambda\in S$ for every $\alpha\in\overline{\mathbb{D}}$. If $f_\lambda(B_R)\subseteq B_R$ then $\lambda = f_\lambda(0)\in B_R$ and thus $R\geq|\lambda|$. Because $f_\lambda(1)=\infty$ it follows that $|\lambda|\leq R<1$ for all $\lambda\in S$. We can thus conclude that $S = B_{\rho_{d,b}}$. Thus, zero-freeness follows from Lemma 3.3.

For the rest, let us denote $\rho=\rho_{d,b}$. Because $\{R\geq 0 \,:\, f_\rho(B_R)\subseteq B_R\}\subseteq\mathbb{R}$ is closed we can take R_M to be the largest possible radius for which $f_\rho(B_{R_M})\subseteq B_{R_M}$. Clearly, there is no $R'>0$ such that $f_\rho(B_{R'})\subseteq \mathrm{int}(B_{R'})$ because that would imply that $f_{(1+\epsilon)\rho}(B_{R'})\subseteq B_{R'}$ for some $\epsilon>0$ contradicting the maximality of ρ. Recall that we defined $\mu(z) = z/(1+z)$. We see that $\mu(B_{R_M})$ is a disk containing $\mu(0)=0$, whose center is $(\mu(R_M)+\mu(-R_M))/2 = -\frac{R_M^2}{1-R_M^2}$. This center is negative because $R_M<1$. We have

$$\underset{z\in B_{R_M}}{\operatorname{argmax}}|f_\rho(z)| = \underset{z\in B_{R_M}}{\operatorname{argmax}}|g(z)| = \underset{z\in B_{R_M}}{\operatorname{argmax}}|1-\mu(z)^b| = \mu^{-1}\left(\underset{w\in\mu(B_{R_M})}{\operatorname{argmax}}|1-w^b|\right)$$

So, by Lemma 6.10, there is either a unique $z\in B_{R_M}$ maximizing $|f_\rho(z)|$ or a conjugate pair. In both cases $|f_\rho'(z)|$ is the same for all $z\in\operatorname{argmax}_{z\in B_{R_M}}|f_\rho(z)|$.

Fix $z\in\operatorname{argmax}_{z\in B_{R_M}}|f_\rho(z)|$ then, by Lemma 6.11, $|f_\rho'(z)|=1$. By the maximum modulus principle $|z|=|f_\rho(z)|=R_M$. Let $\lambda = \rho z/f_\rho(z)$, then $f_\lambda(z)=z$ and $|f_\lambda'(z)|=1$. Thus it follows from Lemma 6.8 that for any $n\geq 1$ integer there is a $\lambda_n\in\mathbb{C}$ and a $(b+1)$-uniform linear hypertree $\mathcal{T}_n$ with degree at most $d+1$ such that $Z(\mathcal{T}_n;\lambda_n)=0$ and $|\lambda_n-\lambda|<1/n$. In particular $\lim_{n\to\infty}|\lambda_n| = |\lambda| = \rho$, which proves the maximality part.

Because z is a fixed point on the boundary of a disk that is forward invariant for f_λ, its derivative must be positive real at z. Therefore, we can conclude that $f_\lambda'(z)=1$. From Lemma 6.4 it follows that λ is of the form $\lambda=(1-w^b)^{-d}\cdot\frac{w}{1-w}$, where $w\neq 1$ is a solution to

$$-bd\cdot\frac{w^b(1-w)}{1-w^b} = 1$$

Thus, we can conclude that $|\lambda|=\rho$ is at least as large as the smallest in absolute value of such λ. If ρ were strictly larger than this minimum, then by Lemma 6.8 we would conclude that there is a $(b+1)$-uniform tree $\mathcal{T}$ of degree at most $(d+1)$ and $\lambda_1\in\mathbb{C}$ such that $|\lambda_1|<\rho$ and $Z(\mathcal{T};\lambda_1)=0$. But this contradicts the zero-freeness part. Thus ρ is equal to the minimum, which concludes the proof of the formula part.

All that remains is to prove the asymptotics. Fix b and define $h(w) = \frac{w^b(1-w)}{1-w^b}$. Consider the multivariate function $F(w,\eta) = h(\eta w)/\eta^b - 1$. We have $F(w,0) = w^b-1$, which has zeros ζ^k for $k=0,\ldots,k-1$ and $\zeta = e^{2\pi i/b}$. By the implicit function theorem, there are holomorphic functions α_k defined in a neighborhood of $\eta=0$ with $\alpha_k(\eta) = \zeta^k + \mathcal{O}(|\eta|)$ and $F(\alpha_k(\eta),\eta)=0$. The functions α_k parameterize the solutions to $F(w,\eta)=0$ for η sufficiently small.

For each $d \geq 2$ let w_d be a solution to $h(w) = -1/bd$ for which $|(1-w^b)^{-d} \cdot \frac{w}{1-w}| = \rho_{d,b}$. Moreover, let $\eta_d = (-1/bd)^{1/b}$, making an arbitrary choice of b-th root. Observe that $F(w_d/\eta_d, \eta_d) = 0$, and thus, for sufficiently large d, there is a k_d such that $w_d = \eta_d \alpha_{k_d}(\eta_d)$. It follows that

$$w_d^b = -\frac{1}{bd} + \mathcal{O}(d^{-1-1/b}) \quad \text{and} \quad |w_d| = (bd)^{-1/b} + \mathcal{O}(d^{-2/b})$$

We conclude that

$$\begin{aligned} \rho_{d,b} = \left|(1-w_d^b)^{-d} \cdot \frac{w_d}{1-w_d}\right| &= \left|1 + \frac{1}{bd} + \mathcal{O}(d^{-1-1/b})\right|^{-d} \cdot \left|w_d + \mathcal{O}(w_d^2)\right| \\ &= \left(e^{-1/b} + \mathcal{O}(d^{-1/b})\right) \cdot \left((bd)^{-1/b} + \mathcal{O}(d^{-2/b})\right) \\ &= (ebd)^{-1/b} + \mathcal{O}(d^{-2/b}) \end{aligned}$$

□

Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Endnotes

[1] A sequence of sets A_n converges to A if for every closed $K_1 \subseteq \text{int}(A)$ and open $K_2 \supseteq \overline{A}$ for n sufficiently large $K_1 \subseteq A_n \subseteq K_2$.

[2] An FPTAS for a graph polynomial Z_G is an algorithm that on an input G and $\epsilon > 0$ outputs an $\hat{Z}$ in polynomial time $|V(G)|$ and $1/\epsilon$ such that it is an ϵ-relative approximation of Z_G, that is, $|Z_G - \hat{Z}| \leq \epsilon Z_G$.

[3] Which we did with the computer-algebra system *Mathematica*.

[4] If the orbit of z_0 passes through ∞ the multiplier can be calculated by conjugating g with a chart that moves the orbit to $\mathbb{C}$.

References

1. J. I. Brown and B. Cameron, "On the stability of independence polynomials," *Electron. J. Comb* 25, no. 1 (2018): 1–46, https://doi.org/10.37236/7280.

2. M. Chudnovsky and P. Seymour, "The roots of the independence polynomial of a clawfree graph," *J. Comb. Theory, Ser. B* 97, no. 3 (2007): 350–357.

3. J. D. Leake and N. R. Ryder, "Generalizations of the matching polynomial to the multivariate independence polynomial," *Algebr. Comb* 2, no. 5 (2019): 781–802.

4. D. Galvin, G. McKinley, W. Perkins, M. Sarantis, and P. Tetali, "On the zeroes of hypergraph independence polynomials," 2022, https://doi.org/10.48550/ARXIV.2211.00464.

5. J. B. Shearer, "On a problem of spencer," *Combinatorica* 5, no. 3 (1985): 241–245, https://doi.org/10.1007/BF02579368.

6. A. D. Scott and A. D. Sokal, "The repulsive lattice gas, the independent-set polynomial, and the Lovász local lemma," *J. Stat. Phys* 118, no. 5–6 (2005): 1151–1261, https://doi.org/10.1007/s10955-004-2055-4.

7. A. D. Sokal, "A personal list of unsolved problems concerning lattice gases and antiferromagnetic Potts models," *Markov Processes Relat. Fields* 7, no. 1 (2001): 21–38.

8. H. Peters and G. Regts, "On a conjecture of sokal concerning roots of the independence polynomial," *Mich. Math. J* 68, no. 1 (2019): 33–55, https://doi.org/10.1307/mmj/1541667626.

9. D. de Boer, P. Buys, L. Guerini, H. Peters, and G. Regts, "Zeros, chaotic ratios and the computational complexity of approximating the independence polynomial," arXiv e-prints, arXiv:2104.11615 2021.

10. I. Bezáková, A. Galanis, L. A. Goldberg, and D. Štefankovič, "Inapproximability of the independent set polynomial in the complex plane," *SIAM J. Comput* 49, no. 5 (2020): STOC18-395–STOC18-448, https://doi.org/10.1137/18M1184485.

11. F. Bencs, P. Buys, and H. Peters, "The limit of the zero locus of the independence polynomial for bounded degree graphs," arXiv e-prints, page arXiv:2111.06451 2021.

12. S. Zhang, "A note on the zero-free region of hypergraph independence polynomials," arXiv:2305.17822 2023.

13. A. Barvinok, *Combinatorics and complexity of partition functions*, vol. 30 (Springer, 2016).

14. J. Liu, A. Sinclair, and P. Srivastava, "The ising partition function: Zeros and deterministic approximation," *J. Stat. Phys* 174, no. 2 (2019): 287–315.

15. V. Patel and G. Regts, "Deterministic polynomial-time approximation algorithms for partition functions and graph polynomials," *SIAM J. Comput* 46, no. 6 (2017): 1893–1919, https://doi.org/10.1137/16M1101003.

16. M. Bordewich, M. Dyer, and M. Karpinski, "Path coupling using stopping times and counting independent sets and colorings in hypergraphs," *Random Struct. Algoritm* 32, no. 3 (2008): 375–399.

17. I. Bezáková, A. Galanis, L. A. Goldberg, H. Guo, and D. Stefankovic, "Approximation via correlation decay when strong spatial mixing fails," *SIAM J. Comput* 48, no. 2 (2019): 279–349.

18. J. Hermon, A. Sly, and Y. Zhang, "Rapid mixing of hypergraph independent sets," *Random Struct. Algoritm* 54, no. 4 (2019): 730–767.

19. J. Liu and P. Lu, "FPTAS for counting monotone CNF," in *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms* (SIAM, 2014), 1531–1548.

20. J. Liu, C. Wang, Y. Yin, and Y. Yu, "Phase transitions via complex extensions of markov chains. arXiv e-prints," arXiv:2411.06857 2024.

21. D. Weitz, "Counting independent sets up to the tree threshold," in *Proceedings of the Thirty-Eighth Annual ACM Symposium on Theory of Computing* (Association for Computing Machinery, 2006), 140–149.

22. A. Sly and N. Sun, "Counting in two-spin models on d-regular graphs," *Ann. Probab* 42, no. 6 (2014): 2383–2416, https://doi.org/10.1214/13-AOP888.

23. N. Anari, K. Liu, and S. Oveis Gharan, "Spectral independence in high-dimensional expanders and applications to the hardcore model," in *Proceedings of the 2020 IEEE 61st Annual Symposium on Foundations of Computer Science (FOCS)* (IEEE Computer Society, 2020), 1319–1330, https://doi.org/10.1109/FOCS46700.2020.00125.

24. A. Blanca, P. Caputo, Z. Chen, D. Parisi, D. Štefankovič, and E. Vigoda, "On mixing of Markov chains: Coupling, spectral independence, and entropy factorization," in *Proceedings of the 2022 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)* (SIAM, 2022), 3670–3692.

25. Z. Chen, K. Liu, and E. Vigoda, "Optimal mixing of Glauber dynamics: Entropy factorization via high-dimensional expansion," in *Proceedings of the 53rd Annual ACM SIGACT Symposium on Theory of Computing* (Association for Computing Machinery, 2021), 1537–1550.

26. Z. Chen, K. Liu, and E. Vigoda, "Spectral independence via stability and applications to Holant-type problems," arXiv: 2106. 03366 2022.

27. P. Buys, "Cayley trees do not determine the maximal zero-free locus of the independence polynomial," *Mich. Math. J* 70, no. 3 (2021): 635–648, https://doi.org/10.1307/mmj/1599206419.

28. M. Trinks, "A survey on recurrence relations for the independence polynomial of hypergraphs," *Graphs Comb* 32, no. 5 (2016): 2145–2158.

29. F. Bencs, "On trees with real-rooted independence polynomial," *Discret. Math* 341, no. 12 (2018): 3321–3330, https://doi.org/10.1016/j.disc.2018.06.033.

30. J. H. Grace, "The zeros of a polynomial," *Proc. Camb. Philos. Soc* 11 (1902): 352–357.

31. G. Szegő, "Bemerkungen zu einem Satz von JH Grace über die Wurzeln algebraischer Gleichungen," *Math. Z* 13 (1922): 28–55.

32. J. L. Walsh, "On the location of the roots of certain types of polynomials," *Trans. Am. Math. Soc* 24, no. 3 (1922): 163–180.

33. P. Brändén and D. G. Wagner, "A converse to the Grace-Walsh-Szegő theorem," *Math. Proc. Camb. Philos. Soc* 147, no. 2 (2009): 447–453, https://doi.org/10.1017/S0305004109002424.

34. P. L. Duren, "Univalent functions," in *259 of Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]* (Springer-Verlag, 1983).

35. C. T. McMullen, "Complex dynamics and renormalization," in *Annals of mathematics studies*, vol. 135 (Princeton University Press, 1994).

36. J. Milnor, "Dynamics in one complex variable," in *Annals of mathematics studies*, vol. 160, 3rd ed. (Princeton University Press, 2006).

37. F. Bencs, P. Csikvári, P. Srivastava, and J. Vondrák, "On complex roots of the independence polynomial," in *Proceedings of the 2023 Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)* (SIAM, 2023), 675–699.

38. É. Ghys, S. Tabachnikov, and V. Timorin, "Osculating curves: Around the Tait-Kneser theorem," *Math. Intell* 35, no. 1 (2013): 61–66, https://doi.org/10.1007/s00283-012-9336-6.

39. A. Kneser, "Bemerkungen über die Anzahl der Extrema der Krümmung auf geschlossenen Kurven und über verwandte Fragen in einer nichteuklidischen Geometrie. Festschrift H. Weber," (1912): 170–180.

40. P. G. Tait, "Note on the circles of curvature of a plane curve," *Proc. Edinb. Math. Soc* 14 (1896): 403.