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INFINITE-HORIZON FUK–NAGAEV INEQUALITIES

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Abstract

We develop explicit bounds for the tail of the distribution of the all-time supremum of a random walk with negative drift, where the increments have a truncated heavy-tailed distribution. As an application, we consider a ruin problem in the presence of reinsurance.

Keywords: Random walk; heavy tails; concentration bounds; reinsurance

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1. Introduction and main results

Let X, X_i, $i \geq 1$, be a real-valued independent and identically distributed (i.i.d.) sequence with $\mathbb{E}[X] < 0$, $\mu_\beta = \mathbb{E}[|X|^\beta] < \infty$ for some $\beta > 1$, and $\mathbb{E}[\mathrm{e}^{sX}] = \infty$ for $s > 0$, so that X is heavy-tailed. Define, for $y > 0$, $S_0(y) = 0$, $S_n(y) = \sum_{i=1}^n \min\{X_i, y\}$, $n \geq 1$, and

$$M(y) = \sup_{n \geq 0} S_n(y). \tag{1}$$

We derive explicit bounds for $\mathbb{P}(M(y) > x)$, motivated by our desire to understand the impact of truncation on the behavior of $M(y)$. Truncated heavy tails naturally occur in risk theory [4], where large claims can be reinsured. Other applications can be found in queueing theory, in which large jobs may get terminated [14], and scale-free random graphs, where the degree of a vertex cannot be larger than the size of the graph [15, 20].

Asymptotic properties of $\mathbb{P}(S_n(y) > x)$ where x and y go to ∞ at a rate comparable to n have been analyzed in [7, 8]. Asymptotics for the finite-time ruin probability for various reinsurance contracts have been derived in [2, 5, 9, 10, 19].

For $M(y)$, such results seem unavailable. If y is fixed, Cramér–Lundberg theory [3] implies that

$$\mathbb{P}(M(y) > x) \leq \mathrm{e}^{-\gamma(y)x}, \tag{2}$$

with $\gamma(y)$ the unique strictly positive solution of the Cramér–Lundberg equation

$$\mathbb{E}\Big[\mathrm{e}^{s\min\{X,y\}}\Big] = 1, \tag{3}$$

so that $\gamma(y) = \sup\Big\{s \geq 0 \colon \mathbb{E}\Big[\mathrm{e}^{s\min\{X,y\}}\Big] \leq 1\Big\}$. Since X is heavy-tailed, $\gamma(y) \to 0$ as $y \to \infty$. In [4], the authors derive asymptotic expansions for $\gamma(y)$ in the slightly different setting of a Lévy

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process with truncated jumps; their results correspond to the case where $\mathbb{P}(X > x)$ is either regularly varying or of the form $\mathrm{e}^{-\sqrt{x}}$.

We mainly focus on non-asymptotic estimates. In particular, we aim to derive upper bounds for $\mathbb{P}(M(y) > x)$. Such bounds can be valuable, as convergence rates of asymptotic estimates can be slow for heavy-tailed random variables [17]. Our first main result is the following theorem.

Theorem 1. *If* $\mathbb{E}[X] < 0$ *and* $\mu_\beta < \infty$ *for some* $\beta > 1$*, there exists a* $y_\beta < \infty$ *such that*

$$\mathbb{P}(M(y) > x) \leq y^{-(\beta-1)x/y}, \quad x > 0, \; y > y_\beta. \tag{4}$$

An explicit expression for y_β is given in (16). We will argue that the bound (4) is convenient for applications when the tail of X has a power law. If $\mu_\beta < \infty$ for all $\beta > 1$, (4) is not the tightest possible upper bound. Our next result provides an alternative bound which, while less generally applicable, is sharper when $\mathbb{P}(X > x)$ is of the form $\exp\{-(\log x)^\xi\}$, $\xi > 1$ (which includes log-normal tails) or $\exp\{-x^\xi\}$, $\xi \in (0, 1)$ (which includes Weibull tails). Define $q(x) = -\log \mathbb{P}(X > x)$.

Theorem 2. *Assume that* $\mathbb{E}[X] < 0$, $\mu_2 < \infty$*, and suppose that there exists a* $\kappa \in (0, 1)$ *and* $y_\kappa > 0$ *such that* $(\log y)^{1+\kappa} \leq q(y) \leq y^{1-\kappa}$ *for* $y \geq y_\kappa$*, and that q(y) is concave. Then, for every* $\eta < 1$*, there exists a* $y^*_\eta < \infty$ *such that*

$$\mathbb{P}(M(y) > x) \leq \left(\frac{y\mathbb{P}(X > y)}{\eta|\mathbb{E}[X]|}\right)^{x/y}, \quad x > 0, \; y > y^*_\eta. \tag{5}$$

An explicit expression for y^*_η is given in (27).

Theorems 1 and 2 complement similar bounds for $\mathbb{P}(S_n(y) > x)$, which are known as Fuk–Nagaev inequalities after [12]. The approach to proving such inequalities, surveyed in [18], has been modified in [6] and [11] to investigate $\mathbb{P}(\max_{k\leq n} S_k(y) > x)$; our results do not seem to follow from the estimates in [6, 11]. Related bounds for $M(\infty)$ have been derived in [16], which are shown to be sharp (for large x and/or $\mathbb{E}[X]$ small) for distributions with power-law tails.

A key step in the approach in [18] and the other cited works includes using the Chernoff bound for $\mathbb{P}(S_n(y) > x)$ and deriving convenient upper bounds for $\mathbb{E}\big[\mathrm{e}^{s\min\{X,y\}}\big]$. Our proof ideas are similar, but we rely on (2) and (3) instead, and derive lower bounds for $\gamma(y)$. We have tried to strike a balance between the accuracy of such lower bounds and the user-friendliness (in terms of explicitness) of the final expressions (4) and (5). This is partly made possible by restricting the range of y over which these bounds apply. It is possible to extend this range, at the expense of making the bound less sharp for larger values of y, using the explicit forms of y_β and y^*_η given in (16) and (27). For more details, we refer to Sections 2 and 3. For a general framework that can be applied to develop sharp estimates for $\mathbb{E}\big[\mathrm{e}^{s\min\{X,y\}}\big]$, we refer to [13].

To illustrate the applicability and accuracy of our bounds, we consider a variation of the Sparre Andersen risk model where claims exceeding the value y are reinsured. We analyze the infinite-time ruin probability $P_h(x)$ if the initial capital is x, and y is chosen as $y = hx$, where h is a fixed constant and $x \to \infty$, using Theorem 1. Finite-time ruin versions of this reinsurance problem have been analyzed in [5, 19] using sample-path large-deviation principles for heavy tails. Extending such a result to the infinite-horizon case requires an additional interchange of limit argument. In Section 4, we illustrate how our bounds can help in this regard.

The result in Section 4 shows that if $y = hx$, our bounds almost predict the correct asymptotic behavior, except for certain round-offs. Informally, if claim sizes have a tail of the form

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$x^{-\alpha}$, ruin occurs due to the consequence of $\lceil 1/h \rceil$ large claims, leading to a ruin probability that behaves like $x^{-(\alpha-1)\lceil 1/h \rceil}$, while Theorem 1 would suggest a behavior of $x^{-(\alpha-1)/h}$.

The remainder of this article is organized as follows. In Section 2, we give a proof of Theorem 1. The proof of Theorem 2 is given in Section 3. The application to a risk model is given in Section 4. The appendix presents the results from [13] that are needed in the main body of the paper.

2. Proof of Theorem 1

Proof. We use the following notation: $x^+ = \max\{x, 0\}$, $F(x) = \mathbb{P}(X \le x)$, $\bar{F}(x) = 1 - F(x)$. Indicator functions are written as $\mathbf{1}(\cdot)$. We write $\mu_\beta^+ = \mathbb{E}[(X^+)^\beta]$, $\mu_\beta^- = \mathbb{E}[((-X)^+)^\beta]$, and we drop β from these expressions if $\beta = 1$, for example $\mu^- = \mathbb{E}[(-X)^+]$. We write $\mu_\beta = \mathbb{E}[|X|^\beta]$. Observe that $\mu_\beta = \mu_\beta^+ + \mu_\beta^-$.

In view of the inequality (2), (4) holds if we can show that there exists a y_β such that $\gamma(y) \ge s(y) = ((\beta - 1)\log y)/y$ for $y > y_\beta$. For this, using (3) for $\gamma(y)$, it is sufficient to show that $\mathbb{E}\big[\mathrm{e}^{s(y)\min\{X,y\}}\big] \le 1$. We therefore derive an appropriate bound for $\mathbb{E}\big[\mathrm{e}^{s\min\{X,y\}}\big]$. Let $\delta = \min\{2, \beta\}$. We have, for $s > 0$,

$$\mathbb{E}\big[\mathrm{e}^{s\min\{X,y\}}\big] = \int_{-\infty}^{0} \mathrm{e}^{sx}\, \mathrm{d}F(x) + \int_{0}^{y} \mathrm{e}^{sx}\, \mathrm{d}G(x), \tag{6}$$

with $G(x) = \mathbb{P}(\min\{X, y\} \le x)$. We estimate

$$\begin{aligned}\int_{-\infty}^{0} \mathrm{e}^{sx}\, \mathrm{d}F(x) &= \int_{-\infty}^{0} (\mathrm{e}^{sx} - 1 - sx)\, \mathrm{d}F(x) + \int_{-\infty}^{0} (1 + sx)\, \mathrm{d}F(x) \\ &= \int_{-\infty}^{0} \frac{\mathrm{e}^{sx} - 1 - sx}{(-sx)^\delta} s^\delta (-x)^\delta\, \mathrm{d}F(x) + F(0) - s\mu^- \\ &\le s^\delta MG_\delta \mu_\delta^- + F(0) - s\mu^-, \end{aligned} \tag{7}$$

where (see the Appendix)

$$MG_\delta = \max_{u \ge 0} \frac{\mathrm{e}^{-u} - 1 + u}{u^\delta} \le \left(\frac{\delta - 1}{\delta}\right)^{\delta - 1} (2 - \delta)^{2-\delta} \le \frac{1}{\delta}. \tag{8}$$

Next, we bound the integral $\int_0^y \mathrm{e}^{sx}\, \mathrm{d}G(x)$ using arguments similar to those in the proof of [18, Lemma 1.4]. Observe that

$$\begin{aligned}\int_{0}^{y} \mathrm{e}^{sx}\, \mathrm{d}G(x) &= \int_{0}^{y} (\mathrm{e}^{sx} - 1 - sx)\, \mathrm{d}G(x) + \int_{0}^{y} (1 + sx)\, \mathrm{d}G(x) \\ &\le \int_{0}^{y} (\mathrm{e}^{sx} - 1 - sx)\, \mathrm{d}G(x) + \bar{F}(0) + s\mu^+. \end{aligned} \tag{9}$$

To bound the remaining integral on the right-hand side of (9), we distinguish between the cases $0 \le s \le \beta/y$ and $s > \beta/y$. In the former case, we have

$$\int_{0}^{y} (\mathrm{e}^{sx} - 1 - sx)\, \mathrm{d}G(x) \le \int_{0}^{\beta/s} (\mathrm{e}^{sx} - 1 - sx)\, \mathrm{d}G(x).$$

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Recalling that $\delta = \min\{2, \beta\}$, we have, for $0 \le x \le \beta/s$,

$$0 \le \frac{e^{sx} - 1 - sx}{(sx)^\delta} = e^{sx} \frac{1 - (1 + sx)e^{-sx}}{(sx)^\delta} \le e^{sx} ME_\delta \le e^\beta ME_\delta,$$

where, see the Appendix,

$$ME_\delta = \max_{v \ge 0} \frac{1 - (1 + v)e^{-v}}{v^\delta} \le \frac{(2 - \delta)^{2-\delta}}{4 - \delta} \le \frac{1}{1 + \delta/2}. \tag{10}$$

Hence, for the case $0 \le s \le \beta/y$, we have

$$\int_0^y (e^{sx} - 1 - sx)\, dG(x) \le e^\beta ME_\delta \int_0^{\beta/s} (sx)^\delta\, dG(x) \le e^\beta s^\delta ME_\delta \mu_\delta^+. \tag{11}$$

If $s > \beta/y$ we have, from (11),

$$\int_0^y (e^{sx} - 1 - sx)\, dG(x) \le e^\beta s^\delta ME_\delta \mu_\delta^+ + \int_{\beta/s}^y (e^{sx} - 1 - sx)\, dG(x).$$

Now, observe that

$$\frac{d}{dx}\left[\frac{e^{sx} - 1 - sx}{x^\beta}\right] = x^{-\beta-1}(e^{sx}(sx - \beta) + (\beta - 1)sx + \beta) > 0$$

for $x \ge \beta/s$, and so

$$\int_{\beta/s}^y (e^{sx} - 1 - sx)\, dG(x) = \int_{\beta/s}^y \frac{e^{sx} - 1 - sx}{x^\beta} x^\beta\, dG(x) \le \frac{e^{sy} - 1 - sy}{y^\beta} \mu_\beta^+.$$

We conclude that, when $s \ge \beta/y$,

$$\int_0^y (e^{sx} - 1 - sx)\, dG(x) \le e^\beta s^\delta ME_\delta \mu_\delta^+ + \frac{e^{sy} - 1 - sy}{y^\beta} \mu_\beta^+. \tag{12}$$

Evidently – see (11) – the bound (12) also holds when $0 \le s \le \beta/y$. Returning to (6) and using (7), (9), (11), and (12), we get

$$\mathbb{E}\big[e^{s \min\{X,y\}}\big] \le s^\delta MG_\delta \mu_\delta^- + F(0) - s\mu^- + \bar{F}(0) + s\mu^+ + e^\beta s^\delta ME_\delta \mu_\delta^+ + \frac{e^{sy} - 1 - sy}{y^\beta} \mu_\beta^+.$$

Using that $F(0) + \bar{F}(0) = 1$ and $\mu^+ - \mu^- = \mathbb{E}[X]$, we see that

$$\begin{aligned}\mathbb{E}\big[e^{s \min\{X,y\}}\big] &\le 1 + s\mathbb{E}[X] + s^\delta MG_\delta \mu_\delta^- + e^\beta s^\delta ME_\delta \mu_\delta^+ + \frac{e^{sy} - 1 - sy}{y^\beta} \mu_\beta^+ \\ &= 1 + s\left(\mathbb{E}[X] + s^{\delta-1} MG_\delta \mu_\delta^- + e^\beta s^{\delta-1} ME_\delta \mu_\delta^+ + \frac{e^{sy} - 1 - sy}{sy^\beta} \mu_\beta^+\right).\end{aligned}$$

We consider this with $s = s(y) = (1/y)(\beta - 1)\log y$, so that $e^{sy} = y^{\beta-1}$. For $y > 1$ we therefore obtain

$$0 \le \frac{e^{sy} - 1 - sy}{sy^\beta} = \frac{y^{\beta-1} - 1 - sy}{y^{\beta-1}(\beta - 1)\log y} \le \frac{1}{(\beta - 1)\log y}.$$

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Concluding, we get, for $y > 1$ and $s = (1/y)(\beta - 1)\log y$,

$$\mathbb{E}\big[\mathrm{e}^{s\min\{X,y\}}\big] \le 1 + s\left(\mathbb{E}[X] + s^{\delta-1}(MG_\delta\mu_\delta^- + \mathrm{e}^\beta ME_\delta\mu_\delta^+) + \frac{\mu_\beta^+}{(\beta-1)\log y}\right). \tag{13}$$

The quantity in parentheses on the right-hand side is negative for large enough y, since $\mathbb{E}[X] < 0$. To quantify this statement, write

$$K = MG_\delta\mu_\delta^- + \mathrm{e}^\beta ME_\delta\mu_\delta^+, \qquad L = \mu_\beta^+, \tag{14}$$

so that the relevant quantity in (13) takes the form

$$Q = \mathbb{E}[X] + \left(Ks^{\delta-1} + \frac{L}{(\beta-1)\log y}\right) = \mathbb{E}[X] + \frac{L + (\beta-1)Ks^{\delta-1}\log y}{(\beta-1)\log y}. \tag{15}$$

With $y > 1$ and $s = (1/y)(\beta - 1)\log y$, observe that

$$(\beta-1)s^{\delta-1}\log y = (\beta-1)^\delta \frac{(\log y)^\delta}{y^{\delta-1}} \le C(\beta-1)^\delta,$$

where

$$C = \max_{y>1} \frac{(\log y)^\delta}{y^{\delta-1}} = \left(\max_{y>1} \frac{\log y}{y^{(\delta-1)/\delta}}\right)^\delta = \left(\frac{\delta}{(\delta-1)\mathrm{e}}\right)^\delta,$$

with the maximizing y given by $\exp\{\delta/(\delta-1)\}$. Thus,

$$(\beta-1)s^{\delta-1}\log y \le \left(\frac{\beta-1}{\delta-1}\frac{\delta}{\mathrm{e}}\right)^\delta.$$

Consequently, the quantity Q in (15) is negative when

$$y > y_\beta = \exp\left\{\frac{L + K\{[(\beta-1)/(\delta-1)](\delta/\mathrm{e})\}^\delta}{(\beta-1)|\mathbb{E}[X]|}\right\}. \tag{16}$$

Here, K and L are given in (14) where MG_δ and ME_δ can be bounded using (8) and (10).

Thus, for $y > y_\beta$, it follows that $\gamma(y) \ge (1/y)(\beta - 1)\log y$. Inserting this bound into (2) completes the proof of (4). □

Remark 1. Upon inspection of the proof, at the expense of increasing the threshold y_β we could have chosen $s = s(y) = ((\beta - 1)\log y + \log\log y + \xi)/y$ with $\mathrm{e}^\xi < (\beta - 1)|\mathbb{E}[X]|/\mu_\beta^+$, which would lead to a slightly sharper bound of the form

$$\mathbb{P}(M(y) > x) \le y^{-(\beta-1)x/y}(\log y)^{-x/y}\mathrm{e}^{-\xi x/y}, \quad x > 0,\ y > y_\beta.$$

Remark 2. If we require an upper bound valid for all y, we can simply use $\mathbb{P}(M(y) > x) \le (y/y_\beta)^{-(\beta-1)x/y}$.

3. Proof of Theorem 2

Proof. As in the proof of Theorem 1, we bound $\mathbb{E}\big[\mathrm{e}^{s\min\{X,y\}}\big]$, now using the assumptions made in the formulation of Theorem 2. For a fixed $\eta \in (0, 1)$, we then make a particular choice for $s = s(y)$, and we construct a y_η^* such that $\mathbb{E}\big[\mathrm{e}^{s\min\{X,y\}}\big] \le 1$ for $y \ge y_\eta^*$ and $s = s(y)$.

By Cramér–Lundberg theory, this implies $\gamma(y) \geq s(y)$, and inserting this inequality into (2), we obtain (5).

Let $s > 0$ and $y > 1/s$. We have

$$\begin{aligned}\mathbb{E}\big[\mathrm{e}^{s\min\{X,y\}}\big] &= \int_{-\infty}^{y} \mathrm{e}^{sx}\,\mathrm{d}F(x) + \mathrm{e}^{sy}\bar{F}(y) \\ &= \int_{-\infty}^{0} \mathrm{e}^{sx}\,\mathrm{d}F(x) + \bar{F}(0) + \int_{0}^{1/s} s\mathrm{e}^{sx}\bar{F}(x)\,\mathrm{d}x + \int_{1/s}^{y} s\mathrm{e}^{sx}\bar{F}(x)\,\mathrm{d}x. \end{aligned} \tag{17}$$

We bound the three integrals in (17). For the first integral we can use (7), where we observe that $\delta = \min\{2, \beta\} = 2$ since $\mu_2 < \infty$, and that $MG_2 = 1/2$. Thus, we have

$$\int_{-\infty}^{0} \mathrm{e}^{sx}\,\mathrm{d}F(x) \leq \frac{1}{2}s^2\mu_2^- + F(0) - s\mu^-. \tag{18}$$

To bound the second integral in (17), we use the inequality $\mathrm{e}^z \leq 1 + z\mathrm{e}^z$ with $z = sx$ and the inequalities $\int_0^u \bar{F}(x)\,\mathrm{d}x \leq \mu^+$, $\int_0^u 2x\bar{F}(x)\,\mathrm{d}x \leq \mu_2^+$, and $u \geq 0$ to obtain

$$\begin{aligned}\int_{0}^{1/s} s\mathrm{e}^{sx}\bar{F}(x)\,\mathrm{d}x &\leq \int_{0}^{1/s} s(1 + sx\mathrm{e}^{sx})\bar{F}(x)\,\mathrm{d}x \\ &\leq s\int_{0}^{1/s} \bar{F}(x)\,\mathrm{d}x + \frac{s^2}{2}\int_{0}^{1/s} 2x\mathrm{e}\bar{F}(x)\,\mathrm{d}x \leq s\mu^+ + s^2\frac{\mathrm{e}}{2}\mu_2^+. \end{aligned} \tag{19}$$

To bound the third integral in (17), note that $sx - q(x)$ is convex, so that

$$sx - q(x) \leq \max\{1 - q(1/s), sy - q(y)\}, \quad x \in [1/s, y].$$

Consequently,

$$\int_{1/s}^{y} s\mathrm{e}^{sx}\bar{F}(x)\,\mathrm{d}x = \int_{1/s}^{y} s\mathrm{e}^{sx-q(x)}\,\mathrm{d}x \leq sy\max\{\mathrm{e}^{1-q(1/s)}, \mathrm{e}^{sy-q(y)}\}. \tag{20}$$

From (18), (19), and (20), we then see that

$$\begin{aligned}\mathbb{E}\big[\mathrm{e}^{s\min\{X,y\}}\big] &\leq \left(\frac{1}{2}s^2\mu_2^- + F(0) - s\mu^-\right) + \bar{F}(0) \\ &\quad + \left(s\mu^+ + s^2\frac{\mathrm{e}}{2}\mu_2^+\right) + sy\max\{\mathrm{e}^{1-q(1/s)}, \mathrm{e}^{sy-q(y)}\} \\ &= 1 + s\left(\mathbb{E}[X] + s\frac{\mathrm{e}}{2}\mu_2^+ + \frac{1}{2}s\mu_2^- + y\max\{\mathrm{e}^{1-q(1/s)}, \mathrm{e}^{sy-q(y)}\}\right) \\ &\leq 1 + s\left(\mathbb{E}[X] + s\frac{\mathrm{e}}{2}\mu_2 + y\max\{\mathrm{e}^{1-q(1/s)}, \mathrm{e}^{sy-q(y)}\}\right).\end{aligned}$$

In the previous display, we used that $\mu^+ - \mu^- = \mathbb{E}[X]$, $F(0) + \bar{F}(0) = 1$, and $\mu_2^+ + \mu_2^- = \mu_2$. We now want to choose $s(y)$ such

$$Q = \mathbb{E}[X] + s\frac{\mathrm{e}}{2}\mu_2 + y\max\{\mathrm{e}^{1-q(1/s)}, \mathrm{e}^{sy-q(y)}\} \leq 0 \tag{21}$$

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when y is sufficiently large. We set

$$s = s(y) := \frac{1}{y}(q(y) - \log y + r), \qquad r = \log(|\mathbb{E}[X]|\eta), \tag{22}$$

with $\eta \in (0, 1)$ fixed. Then, we have

$$Q = \mathbb{E}[X] + s\frac{\mathrm{e}}{2}\mu_2 + \max\{y\mathrm{e}^{1-q(1/s)}, \mathrm{e}^r\}. \tag{23}$$

We now invoke the assumption made in Theorem 2 that there exists a $\kappa \in (0, 1)$ and $y_\kappa > 0$ such that

$$(\log y)^{1+\kappa} \le q(y) \le y^{1-\kappa}, \quad y \ge y_\kappa. \tag{24}$$

We have

$$y \exp\{-q(1/s(y))\} \le y \exp\{-q(y^\kappa)\} \tag{25}$$

when $y \ge \max\{\mathrm{e}^r, y_\kappa\}$. Indeed, by the monotonicity of q, the inequality in (25) is equivalent to

$$\frac{1}{s(y)} = \frac{y}{q(y) - \log y + r} \ge y^\kappa,$$

and this follows from the second inequality in (24) and $y \ge \mathrm{e}^r$. Furthermore, from the first inequality in (24), we have

$$q(y^\kappa) \ge (\log(y^\kappa))^{1+\kappa} = (\kappa \log y)^{1+\kappa}, \quad y \ge y_\kappa^{1/\kappa}. \tag{26}$$

Now, define $y_r = \max\left\{\sup\{y\colon\, y \exp[-(\kappa \ln y)^{\kappa+1}] \ge \mathrm{e}^{r-1}\}, \mathrm{e}^r, y_\kappa^{1/\kappa}\right\}$. Then, by (25), (26), and the definition of y_r, for $y \ge y_r$, $\max\{y\mathrm{e}^{1-q(1/s)}, \mathrm{e}^r\} \le \max\{y \exp[1 - (\kappa \ln y)^{\kappa+1}], \mathrm{e}^r\} \le \mathrm{e}^r$. Therefore, Q in (21) and (23) is bounded by

$$Q \le \mathbb{E}[X] + \frac{1}{2}\mathrm{e}s\mu_2 + \mathrm{e}^r \le \mathbb{E}[X] + \frac{1}{2}\mathrm{e}\mu_2 y^{-\kappa} + \mathrm{e}^r, \quad y \ge y_r,$$

where the latter inequality follows from the definition of $s = s(y)$ in (22) and the second inequality in (24). Since $\mathrm{e}^r = \eta|\mathbb{E}[X]|$, we conclude that $Q \le 0$ when $y \ge y_\eta^*$, where

$$y_\eta^* = \max\left\{y_r, \left(\frac{\mathrm{e}\mu_2/2}{|\mathbb{E}[X]|(1-\eta)}\right)^{1/\kappa}\right\}. \tag{27}$$

We conclude that $s(y) \le \gamma(y)$ for $y > y_\eta^*$. The proof of Theorem 2 is now completed by combining the inequalities $s(y) \le \gamma(y)$ and (2), and using the definition of $s(y)$ in (22). □

Remark 3. The value of y_η^* has not been optimized. In particular cases, the structure of $q(y)$ may be exploited to get sharper estimates, especially if $q(y)$ is of the form $(\log y)^{1+\kappa}$ or $y^{1-\kappa}$.

Remark 4. Our assumptions on $q(y)$ may be slightly generalized. However, we conjecture that (5) does not hold when the distribution of X is too close to the exponential distribution, in particular if $q(y) = y/(\log y)^\zeta$, $\zeta \in (0, 1)$.

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4. Application to a reinsurance risk model

Consider an insurance problem where premiums come in at rate c, claim sizes B, B_i, $i \geq 1$, are i.i.d., and the time between claims is governed by a renewal process, with A, A_i, $i \geq 1$, inter-renewal times which are independent of the claim sizes. A reinsurer covers claims exceeding the value hx, with x the initial capital and $h \in (0, \infty)$ a fixed constant. The ruin probability $P_h(x)$ is

$$P_h(x) = \mathbb{P}\left(\sup_{n\geq 0} \sum_{i=1}^{n} [B_i \mathbf{1}(B_i \leq hx) - cA_i] > x \right).$$

We assume $c\mathbb{E}[A] > \mathbb{E}[B]$ and that c is independent of x, so that $P_h(x) \to 0$. We wish to understand the impact of h on the convergence rate of $P_h(x)$ and aim to illustrate the applicability of Theorem 1. To simplify our presentation, we assume $A \equiv 1/c$.

Proposition 1. *Let c be a constant such that $c\mathbb{E}[A] > \mathbb{E}[B]$ and let $A \equiv 1/c$. Furthermore, let $\mathbb{P}(B > x) = L(x)x^{-\alpha}$ with $\alpha > 1$, and $L(ux)/L(x) \to 1$ for $u > 0$ as $x \to \infty$. If $h \in (0, \infty)$ is such that $1/h$ is not an integer, there exists a constant $C_h \in (0, \infty)$ such that*

$$P_h(x) = (L(x)x^{-(\alpha-1)})^{\lceil 1/h \rceil} C_h(1 + o(1)).$$

The intuition behind this result is that ruin is caused by $\lceil 1/h \rceil$ large claims, similar to what has been established in finite-horizon problems [1, 9, 10]. To prove Proposition 1, we reduce the problem to analyzing the ruin probability over a large but finite time horizon of length Tx with $T \in (0, \infty)$. To this end, we use the bounds

$$\begin{aligned} P_h(x) &\geq \mathbb{P}\left(\sup_{n\leq Tx} \sum_{i=1}^{n} [B_i - 1] > x; \sup_{n\leq Tx} B_i \leq hx \right) =: P_{h,T}(x), \\ P_h(x) &\leq \mathbb{P}\left(\sup_{n\leq Tx} \sum_{i=1}^{n} [B_i \mathbf{1}(B_i \leq hx) - 1] > x \right) + \mathbb{P}\left(\sup_{n\geq Tx} \sum_{i=1}^{n} [B_i \mathbf{1}(B_i \leq hx) - 1] > x \right). \end{aligned} \tag{28}$$

Proof. Since $\mathbb{P}(B\mathbf{1}(B \leq hx) > u) \leq \mathbb{P}(B > u \mid B \leq hx)$, $u \in (0, hx)$, we can apply a stochastic monotonicity argument to upper bound the first term on the right-hand side of (28) by

$$\mathbb{P}\left(\sup_{n\leq Tx} \sum_{i=1}^{n} [B_i - 1] > x \mid \sup_{i\leq Tx} B_i \leq hx \right) = \frac{P_{h,T}(x)}{\mathbb{P}\left(\sup_{i\leq Tx} B_i \leq hx \right)}. \tag{29}$$

To bound the second term on the right-hand side of (28), define $X_i = B_i - 1$, and note that $B_i \mathbf{1}(B_i \leq hx) - 1 \leq \min\{X_i, hx\}$ to obtain

$$\mathbb{P}\left(\sup_{n\geq Tx} \sum_{i=1}^{n} [B_i \mathbf{1}(B_i \leq hx) - 1] > x \right) \leq \mathbb{P}\left(\sup_{n\geq Tx} \sum_{i=1}^{n} \min\{X_i, hx\} > x \right).$$

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This expression can be bounded further by observing that

$$
\begin{aligned}
&\mathbb{P}\biggl(\sup_{n\geq Tx}\sum_{i=1}^{n}[B_i\mathbf{1}(B_i\leq hx)-1]>x\biggr)\\
&\leq \mathbb{P}\biggl(\sup_{n\geq Tx}\sum_{i=1}^{n}\min\{X_i,hx\}>x\biggr)\\
&\leq \mathbb{P}\biggl(\sup_{n\geq Tx}\sum_{i=1}^{n}\min\{X_i,hx\}>x,\ \sum_{i=1}^{Tx}\min\{X_i,hx\}\leq\frac{Tx}{2}\biggr)+\mathbb{P}\biggl(\sum_{i=1}^{Tx}\min\{X_i,hx\}>\frac{Tx}{2}\biggr)\\
&\leq \mathbb{P}\biggl(\sup_{n\geq Tx}\sum_{i=Tx+1}^{n}\min\{X_i,hx\}>\frac{Tx}{2}\biggr)+\mathbb{P}\biggl(\sup_{n\geq 0}\sum_{i=1}^{n}\min\{X_i,hx\}>\frac{Tx}{2}\biggr)\\
&=2\mathbb{P}\biggl(M(hx)>\frac{Tx}{2}\biggr),
\end{aligned}
\tag{30}
$$

with $M(hx)$ being the protagonist of this paper, defined in (1), with $y=hx$. Substituting (29) and (30) into (28), we obtain

$$
P_h(x)\leq\frac{P_{h,T}(x)}{\mathbb{P}\bigl(\sup_{i\leq Tx}B_i\leq hx\bigr)}+2\mathbb{P}\biggl(M(hx)>\frac{Tx}{2}\biggr). \tag{31}
$$

Since $\mathbb{E}[B]<\infty$, $\mathbb{P}\bigl(\sup_{i\leq Tx}B_i\leq hx\bigr)=\mathbb{P}(B_1\leq hx)^{\lfloor Tx\rfloor}\to 1$. Consequently, it suffices to establish the asymptotic behavior of $P_{h,T}(x)$ and to show that $\mathbb{P}(M(hx)>Tx/2)=o(P_{h,T}(x))$ for a suitably chosen T. In [19, Section 5.1], it is shown that, given h such that $1/h$ is not an integer, there exists a constant $C_{h,T}$ such that

$$
P_{h,T}(x)\sim C_{h,T}(L(x)x^{-(\alpha-1)})^{\lceil 1/h\rceil}(1+o(1)). \tag{32}
$$

The constant $C_{h,T}$ can be expressed as

$$
C_{h,T}=\int_{\substack{\mathbf{u}\in[0,T]^k\\ \mathbf{x}\in(0,\infty)^k}}\alpha^k\prod_i x_i^{-1-\alpha}\mathbf{1}\biggl(\sup_{t\in[0,T]}\sum_{i=1}^{k}x_i\mathbf{1}(u_i\geq t)-t\geq 1;\sup_{i\leq k}x_i\leq h\biggr)\,\mathrm{d}\mathbf{u}\,\mathrm{d}\mathbf{x},
$$

with $k=\lceil 1/h\rceil$, $\mathbf{u}=(u_1,\ldots,u_k)$, $\mathbf{x}=(x_1,\ldots,x_k)$, $\mathrm{d}\mathbf{u}=\mathrm{d}u_1\cdots\mathrm{d}u_k$, and $\mathrm{d}\mathbf{x}=\mathrm{d}x_1\cdots\mathrm{d}x_k$. From this representation, it follows that $C_{h,T}\in(0,\infty)$ if $1/h$ is not an integer. In addition, $C_{h,T}$ is constant in T for $T\geq kh$. Next, using (4), we see that, for every $\beta\in(1,\alpha)$ and x large enough, $\mathbb{P}(M(hx)>Tx/2)\leq(hx)^{-(\beta-1)T/(2h)}$, which, using (32), is $o(P_{h,T}(x))$ for $T>((\alpha-1)/(\beta-1))h\lceil 1/h\rceil$. This concludes the proof of Proposition 1 by setting $C_h=C_{h,kh}$. □

Remark 5. To derive tail asymptotics for the ruin probability if $\mathbb{P}(X>x)$ is of Weibull type, we expect that it is possible to apply the sample-path large-deviations results in [5] to derive tail asymptotics for $P_{h,T}(x)$, and combine them with the bound (31) and Theorem 2; a fully worked-out argument would require careful investigation of a quasi-variational problem, extending the analysis in [5, Section 4], which is beyond the scope of this paper.

Remark 6. In this section, we took A constant. The case of non-deterministic A may be dealt with by using bounds of the form

$$
\sup_{n\geq 0}\sum_{i=1}^{n}[B_i\mathbf{1}(B_i\leq hx)-cA_i]\leq\sup_{n\geq 0}\sum_{i=1}^{n}[B_i\mathbf{1}(B_i\leq hx)-c']+\sup_{n\geq 0}\sum_{i=1}^{n}[c'-cA_i],
$$

with $c' \in (\mathbb{E}[B_i], c\mathbb{E}[A_i])$. The second term on the right-hand side of this inequality is independent of the first term. It has a moment-generating function that is finite in a neighborhood of the origin and therefore does not impact the asymptotic behavior of $P_h(x)$.

Remark 7. The assumption that $1/h$ is non-integer cannot easily be removed. If $1/h$ is an integer, the probabilistic framework in [19] is inconclusive in determining whether ruin is caused by $1/h$ or $1/h + 1$ large claims. Recent partial results for random walks (without taking a supremum) in this direction in [8] suggest that this problem is highly non-trivial.

Remark 8. If $h > 1$, the number k of large claims required for ruin equals 1. It can be shown that when $h \to \infty$, the constant C_h converges to the limiting pre-constant appearing in the classical result of the asymptotic form of the ruin probability $P_\infty(x)$, cf. [3].

Appendix A. Bounding Taylor approximation errors

In the proof of Theorem 1, the following quantities appear:

$$ME_\delta = \max_{s \geq 0} \frac{1 - (1+s)\mathrm{e}^{-s}}{s^\delta}, \qquad \delta \in [0, 2], \tag{33}$$

$$MG_\delta = \max_{u \geq 0} \frac{\mathrm{e}^{-u} - 1 + u}{u^\delta}, \qquad \delta \in [1, 2]. \tag{34}$$

We analyze these quantities using the machinery developed in [13], which contains a framework to bound Taylor polynomial approximation errors for the exponential function in the presence of a power weight function. For the case that the degree n of the Taylor polynomial equals 1, this leads to consideration of quantities $ME_{1,\delta}$ and $MG_{1,\delta}$ that agree with (33) and (34), from which the degree $n = 1$ has been omitted for brevity. In this appendix, we summarize the results of [13] for the case $n = 1$ that are relevant to this paper.

We have $ME_0 = 1$, $ME_2 = \frac{1}{2}$, $MG_1 = 1$, $MG_2 = \frac{1}{2}$. Furthermore, ME_δ is a continuous, log-convex function of $\delta \in [0, 2]$, and MG_δ is a continuous, log-convex function of $\delta \in [1, 2]$. For $\delta \in (0, 2)$, the maximum ME_δ of

$$E_\delta(s) = \frac{1 - (1+s)\mathrm{e}^{-s}}{s^\delta} \tag{35}$$

over $s \geq 0$ is assumed by the unique positive solution $s = s(\delta)$ of the equation

$$\mathrm{e}^s = 1 + s + \frac{1}{\delta} s^2. \tag{36}$$

For $\delta \in (0, 1)$, the maximum MG_δ of

$$G_\delta(u) = \frac{\mathrm{e}^{-u} - 1 + u}{u^\delta} \tag{37}$$

over $u \geq 0$ is assumed by the unique positive solution $u = u(\delta)$ of the equation

$$\mathrm{e}^{-u} = 1 - \frac{\delta u}{u + \delta}. \tag{38}$$

An upper bound for ME_δ, $\delta \in [0, 2]$, follows from [13, Proposition 2] in the form $F_2(2 - \delta)$, with

$$F_2(s) = \frac{1}{\delta} \frac{s^{2-\delta}}{1 + s + (1/\delta)s^2}, \qquad s \geq 0.$$

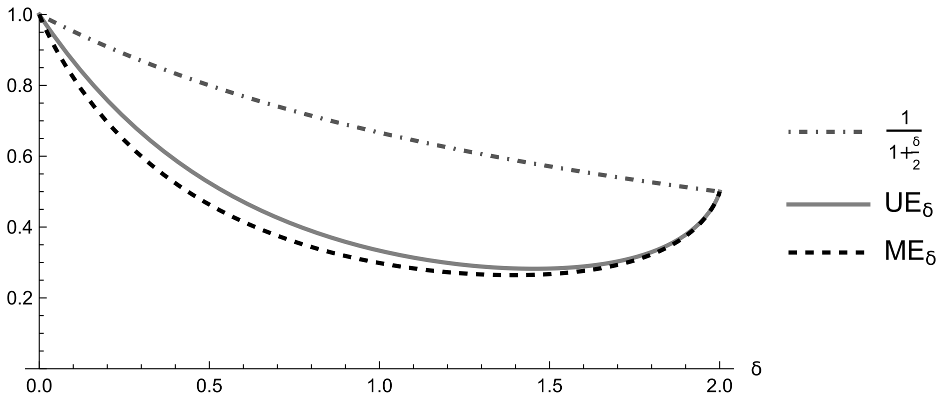

FIGURE 1. ME_δ, UE_δ, and $1/(1+\delta/2)$ as functions of $\delta \in (0, 2)$.

Thus, we have

$$ME_\delta \le \frac{(2-\delta)^{2-\delta}}{4-\delta} =: UE_\delta, \quad \delta \in [0, 2]. \tag{39}$$

An upper bound for MG_2, $\delta \in [1, 2]$, is found using the approach given at the end of [13, Section 2.2] in the form

$$H\left(\frac{(2-\delta)\delta}{\delta-1}\right), \qquad H(u) = \frac{u^{2-\delta}}{u+\delta}, \quad u \ge 0.$$

Thus, we have

$$MG_\delta \le \left(\frac{\delta-1}{\delta}\right)^{\delta-1}(2-\delta)^{2-\delta} =: UG_\delta, \quad \delta \in [1, 2]. \tag{40}$$

The bounds in (39) and (40) are relatively sharp; see Figures 1 and 2. Less sharp, but simpler (and still effective) bounds are

$$ME_\delta \le \frac{1}{1+\delta/2}, \quad \delta \in [0, 2]; \qquad MG_\delta \le \frac{1}{\delta}, \quad \delta \in [1, 2]. \tag{41}$$

The bound for MG_δ also appears in the proof of [18, Lemma 1.4]. We show for completeness that the bounds in (41) follow from the two bounds in (39) and (40), respectively.

The first bound in (41) follows via (39) from the inequality

$$f_E(\delta) := \ln(1+\delta/2) + (2-\delta)\ln(2-\delta) - \ln(4-\delta) \le 0, \quad \delta \in [0, 2]. \tag{42}$$

To establish (42), we observe that the function $f_E(\delta)$ is continuous in $\delta \in [0, 2]$ and that $f_E(0) = 0 = f_E(2)$. Furthermore, a simple computation yields

$$f_E''(\delta) = \frac{-1}{(2+\delta)^2} + \frac{1}{2-\delta} + \frac{1}{(4-\delta)^2} > -\frac{1}{4} + \frac{1}{2} + \frac{1}{16} > 0, \quad \delta \in (0, 2).$$

Hence, $f_E(\delta)$ is convex in $\delta \in (0, 2)$, and so we get (42) from $f_E(0) = 0 = f_E(2)$ and the continuity of $f_E(\delta)$, $\delta \in [0, 2]$.

The second bound in (41) follows via (40) from the inequality

$$f_G(\delta) := (2-\delta)\ln\delta + (\delta-1)\ln(\delta-1) + (2-\delta)\ln(2-\delta) \le 0, \quad \delta \in [1, 2]. \tag{43}$$

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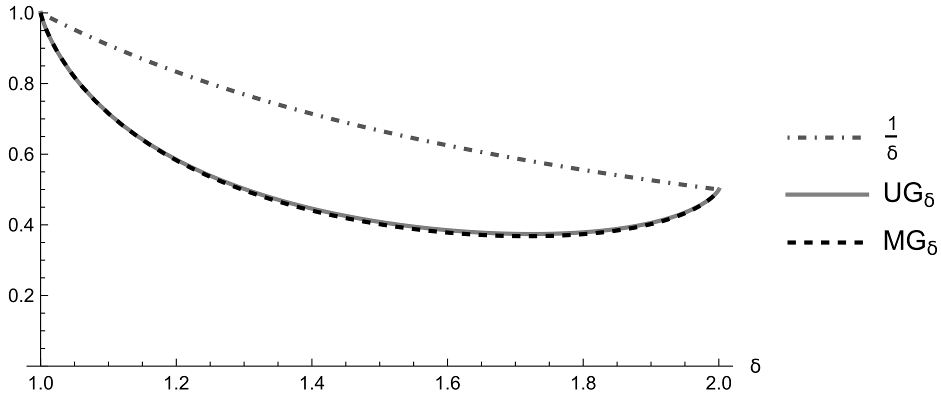

FIGURE 2. MG_δ, UG_δ, and $1/\delta$ as functions of $\delta \in (1, 2)$.

To establish (43), we observe that the function $f_G(\delta)$ is continuous in $\delta \in [1, 2]$ and that $f_G(1) = 0 = f_G(2)$. Furthermore, a simple computation yields

$$f_G''(\delta) = -\frac{2}{\delta^2} - \frac{1}{\delta} + \frac{1}{\delta - 1} + \frac{1}{2 - \delta}, \quad \delta \in (1, 2).$$

Consequently,

$$\delta^2(\delta - 1)(2 - \delta)f_G''(\delta) = -(\delta - 1)(2 - \delta)(2 + \delta) + \delta^2 > -\frac{1}{4} \cdot 4 + 1^2 = 0, \quad \delta \in (1, 2).$$

Hence, $f_G(\delta)$ is convex in $\delta \in (1, 2)$, and therefore we get (43) from $f_G(1) = 0 = f_G(2)$ and the continuity of $f_G(\delta)$, $\delta \in [1, 2]$.

Figure 1 shows a plot of ME_δ, together with plots of the upper bounds UE_δ in (39) and $(1 + \delta/2)^{-1}$ in (41), as functions of $\delta \in [0, 2]$. The value of ME_δ is obtained as $E_\delta(s(\delta))$, where E_δ is given in (35) and the unique solution $s = s(\delta)$ of the equation in (36) is computed using Newton's method (starting value: $s^{(0)} = \ln(2/\delta) + (2 - \delta)$, see [13, (37)]).

Figure 2 shows a plot of MG_δ, together with plots of the upper bounds UG_δ in (40) and δ^{-1} in (41), as functions of $\delta \in [1, 2]$. The value of MG_δ is obtained as $G_\delta(u(\delta))$, where G_δ is given in (37) and the unique solution $u = u(\delta)$ of the equation in (38) is computed using Newton's method (starting value: $u^{(0)} = (2 - \delta)\delta/(\delta - 1) - \ln(\delta - 1)$, see [13, (275)]).

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