Christensen-Sinclair factorization via semidefinite programming ☆

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A R T I C L E   I N F O

A B S T R A C T

We show that the Christensen-Sinclair factorization theorem, when the underlying Hilbert spaces are finite dimensional, is an instance of strong duality of semidefinite programming. This gives an elementary proof of the result and also provides an efficient algorithm to compute the Christensen-Sinclair factorization.

1. Introduction

The seminal result in operator space theory of Christensen and Sinclair for matrix algebras establishes that a t-linear form $T : M_n \times \cdots \times M_n \to \mathbb{C}$ is completely contractive if and only if it can be factorized into a sequence of $*$-representations interlaced with contractions [3]. Here, M_n is the space of complex $n \times n$ matrices equipped with the operator norm of matrices when they are regarded as linear maps from $\ell_2^n(\mathbb{C})$ to $\ell_2^n(\mathbb{C})$. The completely bounded norm of T is defined in the following way. For every $m \in \mathbb{N}$, consider the map $T_m : M_{nm} \times \cdots \times M_{nm} \to M_m$ defined via

$$T_m(X_1, \ldots, X_t) = \left(\sum_{r_1, \ldots, r_{t-1} \in [m]} T((X_1)_{i,r_1}, (X_2)_{r_1,r_2}, \ldots, (X_t)_{r_{t-1},j}) \right)_{i,j \in [m]},$$

where $(X_1)_{ij}$ is the (i,j)-th block of $X_1 \in M_{nm} = M_n(M_m)$ when X_1 is regarded as a matrix with $n \times n$ blocks of size $m \times m$. Then, the completely bounded norm of T is given by

$$\|T\|_{\mathrm{cb}} = \sup\{\|T_m\| : m \in \mathbb{N}\},$$

where $\|T_m\|$ is the operator norm of T_m. Now we are ready to state the Christensen-Sinclair factorization theorem, which is the main result of [3] specified for the case where the C^* algebra is M_n.

Theorem 1.1. *(Christensen-Sinclair) Let $T : M_n \times \cdots \times M_n \to \mathbb{C}$ be a t-linear form. Then, $\|T\|_{\mathrm{cb}} \leq 1$ if and only if there exist $d \in \mathbb{N}$, unit vectors $u, v \in \mathbb{C}^d$ and matrices $A_0 \in M_{d,nd}$, $A_1, \ldots, A_{t-1} \in M_{nd,nd}$ and $A_t \in M_{nd,d}$ with operator norm at most 1 such that*

$$T(X_1, \ldots, X_t) = \langle u, A_0(X_1 \otimes \mathrm{Id}_d)A_1 \ldots A_{t-1}(X_t \otimes \mathrm{Id}_d)A_t v \rangle, \tag{1}$$

for every $X_1, \ldots, X_t \in M_n$.

To the best of our knowledge, all the proofs of Theorem 1.1 are done in more generality, considering the space of endomorphisms of some complex Hilbert space instead of M_n. Even when specified to the finite-dimensional case, they require tools from operator spaces, several applications of the Hahn-Banach theorem, going through infinite-dimensional spaces and are not constructive [3,7,2].

By contrast, we give an elementary and constructive proof that does not require the use of an infinite-dimensional separation theorem. We do this by showing that Theorem 1.1 is an instance of the strong duality of semidefinite programming. In the proof we use as a black-box a standard and elementary result due to Slater that ensures strong duality. We stress that the proof of Slater's theorem just requires a finite-dimensional separation

theorem, but no application of the infinite-dimensional Hahn-Banach theorem (for a proof see [10]). Assuming Slater's theorem, our argument just requires simple notions of linear algebra.

Semidefinite programming is an extension of linear programming that includes a bigger family of problems and can still be efficiently solved up to arbitrary precision (see [5] for an introduction to semidefinite programming). To be more precise, let H_N be the space of Hermitian matrices of M_N and let H_N^+ be the cone of positive semidefinite matrices. A collection of matrices $C, B_1, \ldots, B_L \in H_N$ and a vector $b \in \mathbb{R}^L$ define a *primal semidefinite program (P)* and a *dual semidefinite program (D)*, which in their *canonical form* are given by

$$(P) \quad \inf \quad \langle C, Y \rangle \qquad\qquad (D) \quad \sup \quad \langle b, y \rangle \qquad\qquad (2)$$
$$\text{s.t.} \quad Y \in H_N^+ \qquad\qquad\qquad \text{s.t.} \quad y \in \mathbb{R}^L$$
$$\mathcal{B}(Y) = b \qquad\qquad\qquad\qquad C - \mathcal{B}^*(y) \in H_N^+,$$

where $\mathcal{B} : H_N \to \mathbb{R}^L$ is given by $\mathcal{B}(Y) := (\langle B_1, Y \rangle, \ldots, \langle B_L, Y \rangle)$, $\mathcal{B}^*(y) = \sum_{i \in [L]} y_i B_i$ and $\langle B, Y \rangle = \mathrm{Tr}(BY)$. It is always satisfied that the optimal value of (P) is at least the optimal value of (D), what is known as *weak duality*. In addition, under some mild assumptions provided by Slater's theorem (see Theorem 2.2 below), both values are equal, what is known as *strong duality*.[1] Note that if all matrices $C, B_1, \ldots, B_L$ were diagonal, (P) and (D) would be linear programs. Indeed, in that case the value of (P) would not change if we further impose that Y is diagonal, which makes (P) a linear program. Also, the constraint $C - \mathcal{B}^*(y) \in H_N^+$ is equivalent to saying that the diagonal entries of $C - \mathcal{B}^*(y)$ are non-negative, so (D) is also a linear program.

It will be convenient to introduce the representation norm of a t-linear form $T : M_n \times \cdots \times M_n \to \mathbb{C}$, which is given by

$$\|T\|_{\mathrm{rep}} = \inf \quad w$$
$$\text{s.t.} \quad T(X_1, \ldots, X_t) = \langle u, A_0(X_1 \otimes \mathrm{Id}_d)A_1 \ldots A_{t-1}(X_t \otimes \mathrm{Id}_d)A_t v \rangle,$$
$$\forall X_1, \ldots, X_t \in M_n,$$
$$d \in \mathbb{N}, \ u, v \in \mathbb{C}^d, \ \|u\|_2^2 = \|v\|_2^2 = w,$$
$$A_0 \in M_{d,nd}, \ A_1, \ldots, A_{t-1} \in M_{nd,nd}, \ A_t \in M_{nd,d} \text{ contractions.}$$

$\|\cdot\|_{\mathrm{rep}}$ is a norm, as shown in [3]. Now, we are ready to state our main result.

Theorem 1.2. *Given a t-linear form $T : M_n \times \cdots \times M_n \to \mathbb{C}$, there is a pair of semidefinite programs (P_{CS}) and (D_{CS}) such that*

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[1] Usually semidefinite programs are phrased in terms of real symmetric matrices, but the weak and strong dualities hold when one substitutes symmetric matrices by Hermitian ones, as they are general properties of conic programs (see for instance [12]).

i) (P_{CS}) *optimal value equals* $\|T\|_{\mathrm{rep}}$,
ii) (D_{CS}) *optimal value equals* $\|T\|_{\mathrm{cb}}$,
iii) (D_{CS}) *is the dual of* (P_{CS}) *and their optimal values are equal.*

Theorem 1.2 has three important consequences. The first one is already clear from the statement, and the other two will become clear later (see Remark 2.1). These consequences are:

a) Theorem 1.2 implies Theorem 1.1;
b) (P_{CS}) and (D_{CS}) have $O(\mathrm{poly}(n)^t)$ variables, so the known algorithms to approximate semidefinite programs can be used to efficiently compute the completely bounded norm;
c) from the solution returned by these algorithms one can extract a description of the vectors and matrices appearing in a factorization as in Eq. (1).

1.1. Some remarks

We note that Theorem 1.2 works when we substitute M_n by any subspace of M_n, i.e., when we consider an operator space that inherits its structure from M_n, and the algorithmic consequences hold as well. By this we mean that if V is a normed subspace of M_n, then, for every $m \in \mathbb{N}$, $V \otimes M_m$ inherits a norm as a subspace of M_{mn}, and that norm defines a notion of completely bounded norm for t-linear maps $T : V \times \cdots \times V \to \mathbb{C}$. We also remark that our proof of Theorem 1.2 can be extended to the case where M_n is substituted by the space of endomorphisms of a separable complex space. However, in that case one requires Hahn-Banach to prove strong duality and the algorithmic consequences would be lost. It is not clear to us how to extend the proof to the non-separable case, as our technique heavily relies on the existence of a countable orthonormal basis.

1.2. Related work

An analogous result for the linear case ($t = 1$) was proven by Watrous [10,11]. Gribling and Laurent showed that the completely bounded norm of a form $T : \ell_\infty^n(\mathbb{C}) \times \cdots \times \ell_\infty^n(\mathbb{C}) \to \mathbb{C}$ can be computed as a semidefinite program, so item (b) above, for the special case of $\ell_\infty^n(\mathbb{C})$, also follows from their work [4]. The initial motivation of our work comes from the applications of the Christensen-Sinclair theorem to quantum information. Arunachalam, Briët and Palazuelos used this result to give a characterization of quantum query complexity, showing that every quantum query algorithm is a completely contractive form, and vice versa [1]. However, prior to this work, given a completely contractive form there was no way of obtaining the corresponding algorithm, which in our context is equivalent to finding the matrices that define a factorization as in Eq. (1). Item (c) fills this gap.

Our work is not the first example of an elementary proof of a celebrated result in operator algebras via ideas of theoretical computer science. Previously, Regev and Vidick, who, after finding applications of the operator space Grothendieck theorem in quantum computing [6,9], gave an elementary proof of that result using ideas of theoretical computer science [8]. We expect this interplay to be fruitful in the future, potentially leading to the proof of new results in operator algebras.

2. Proof of Theorem 1.2

We divide the proof in 3 parts. In the first, we introduce (P_{CS}) and prove Theorem 1.2 (*i*), in the second we introduce (D_{CS}) and prove Theorem 1.2 (*ii*), and in the third we show that (P_{CS}) and (D_{CS}) are semidefinite programs and prove Theorem 1.2 (*iii*). Before diving into the proof, we introduce some notation.

We use $[n]$ to denote $\{1, \ldots, n\}$. Sometimes we will consider multi-indices in $([n] \times [n])^s$ and refer to them as $\mathbf{I}$, where $I_1, \ldots, I_s \in [n] \times [n]$. $M_{m,n}$ is the space of $m \times n$ complex matrices and $M_n = M_{n,n}$. $\{E_{i,j}\}_{i,j \in [n]}$ is the canonical basis of M_n, where the (i', j')-th entry of $E_{i,j}$ is given by $\delta_{i,i'} \delta_{j,j'}$. Given a matrix $X \in M_n$ and $I = (i, j) \in [n] \times [n]$, we say that X_I is the coordinate of X in the former basis, corresponding to $E_{i,j}$. Sometimes we will consider $X \in M_{mn}$ and $I \in [n] \times [n]$, in which case $X(I)$ will be I-th block when we regard X as $n \times n$ block-matrix with blocks of size $m \times m$. Given s matrices $X_1, \ldots, X_s \in M_n$, we denote by $\mathbf{X}$ the matrix-vector defined by $(X_1, \ldots, X_s)$. Given $\mathbf{I} \in ([n] \times [n])^s$ and $\mathbf{X} \in (M_{mn})^s$, $\mathbf{X}(\mathbf{I})$ is defined as the matrix product given by $X_1(I_1) \ldots X_t(I_t) \in M_m$. The norm of vectors of $\mathbb{C}^d$ is the Euclidean norm, and the norm of matrices is the operator norm when considered as linear maps from $\ell_2(\mathbb{C})$ to $\ell_2(\mathbb{C})$. We say that a matrix is a contraction if its norm is at most 1. Given a vector $\alpha \in \mathbb{C}^n$, $\mathrm{Diag}(\alpha)$ is the diagonal matrix of M_n whose diagonal is α. Id_d is the identity matrix of M_d. Given $z \in \mathbb{C}$, $\Re z \in \mathbb{R}$ is its real part and $\Im z \in \mathbb{R}$ its imaginary part.

We will often identify a t-form $T : M_n \times \cdots \times M_n \to \mathbb{C}$ with its tensor of coefficients $(T_{\mathbf{I}})_{\mathbf{I}} \in (\mathbb{C}^{n \times n})^t$ defined via $T_{\mathbf{I}} = T(E_{I_1}, \ldots, E_{I_t})$, so $T(\mathbf{X}) = \sum_{\mathbf{I} \in ([n] \times [n])^t} T_{\mathbf{I}} X(\mathbf{I})$, where $\mathbf{X} \in (M_n)^t$. One can also write $T_m : M_{nm} \times \cdots \times M_{nm} \to \mathbb{C}$ in terms of these coefficients, namely, $T_m(\mathbf{X}) = \sum_{\mathbf{i} \in ([n] \times [n])^t} T_{\mathbf{I}} X(\mathbf{I})$; where, given $I \in [n] \times [n]$ and $X \in M_{nm}$, $\mathbf{X}(I)$ is the I-th $m \times m$-dimensional block of X when regarded as a block-matrix with $n \times n$ blocks. In particular, the completely bounded norm of T can be written as

$$\|T\|_{\mathrm{cb}} = \sup \left| \sum_{\mathbf{I} \in ([n] \times [n])^t} T_{\mathbf{I}} \langle u, X_1(I_1) \ldots X_t(I_t) v \rangle \right|, \tag{3}$$

where the supremum runs over all $m \in \mathbb{N}$, all contractions $X_1, \ldots X_t \in M_{nm}$ and all unit vectors $u, v \in \mathbb{C}^m$.

We recall the reader that for $Y \in M_N$, $Y \in H_N^+$ if and only if there exist vectors v_i such that $Y_{(i,j)} = \langle v_i, v_j \rangle$ for every $i, j \in [n]$. In that case, we say that Y is the Gram matrix of $\{v_i\}_{i \in [n]}$.

2.1. The primal semidefinite program

In this section, we introduce (P_{CS}) and prove Theorem 1.2 (i). Before doing that we give some intuition of why $\|T\|_{\mathrm{rep}}$ can be formulated as a semidefinite program. Assume that T factors as in Eq. (1) with vectors of u and v of square norm w. Then, we consider the following block structure for the contractions A_s:

$$
A_0 = (A_0(1) \quad \ldots \quad A_0(n)),
$$

$$
A_s = \begin{pmatrix} A_s(1,1) & \ldots & A_s(1,n) \\ \vdots & \ddots & \vdots \\ A_s(n,1) & \ldots & A_s(n,n) \end{pmatrix}, \; A_t = \begin{pmatrix} A_t(1) \\ \ldots \\ A_t(n) \end{pmatrix}. \tag{4}
$$

We define the following vectors,

$$
v_i = A_t(i)v, \text{ for } i \in [n], \tag{5}
$$

$$
v_{\mathbf{I}} = A_{t-s}(I_1)\ldots A_{t-1}(I_s)A_t(I_{s+1})v, \text{ for } \mathbf{I} \in ([n] \times [n])^s \times [n], \; s \in [t-1], \tag{6}
$$

$$
v_{\mathbf{I}} = A_0(I_1)A_1(I_2)\ldots A_t(I_{t+1})v, \text{ for } \mathbf{I} \in [n] \times ([n] \times [n])^{t-1} \times [n]. \tag{7}
$$

We note that $T_{\mathbf{I}} = \langle u, v_{\mathbf{I}} \rangle$, where we understand $\mathbf{I}$ as an element of $([n] \times [n])^t$ in the left-hand-side and as an element of $[n] \times ([n] \times [n])^{t-1} \times [n]$ in the right-hand side. Hence, $T_{\mathbf{I}}$ is encoded in the entries of $Y =\mathrm{Gram}\{u, v_{\mathbf{I}}\}$ (which corresponds to (9) below). In addition, the fact that A_i are contractions can be encoded in the entries of this Gram matrix (which gives rise to Eqs. (11) to (13) below). With these intuitions, we are ready to state (P_{CS}):

$$
\inf \quad w \tag{P_{CS}}
$$

$$
\text{s.t.} \quad w \geq 0, \; Y \succeq 0, \tag{8}
$$

$$
\Re Y_{0,\mathbf{I}} = \Re T_{\mathbf{I}}, \; \Im Y_{0,\mathbf{I}} = \Im T_{\mathbf{I}}, \; \mathbf{I} \in ([n] \times [n])^t, \tag{9}
$$

$$
Y_{0,0} = w, \tag{10}
$$

$$
\sum_{i \in [n]} Y_{i,i} \leq w, \tag{11}
$$

$$
\sum_{i \in [n]} \left(Y_{(i,j)\mathbf{J},(i,j')\mathbf{J}'} \right)_{j,j' \in [n], \mathbf{J}, \mathbf{J}' \in ([n] \times [n])^{s-1} \times [n]}
$$

$$
\preceq \oplus_{k \in [n]} (Y_{\mathbf{J},\mathbf{J}'})_{\mathbf{J}, \mathbf{J}' \in ([n] \times [n])^{s-1} \times [n]}, \; s \in [t-1], \tag{12}
$$

$$
(Y_{\mathbf{J},\mathbf{J}'})_{\mathbf{J},\mathbf{J}' \in ([n] \times [n])^t} \preceq \oplus_{k \in [n]} (Y_{\mathbf{J},\mathbf{J}'})_{\mathbf{J}, \mathbf{J}' \in ([n] \times [n])^{t-1} \times [n]}, \tag{13}
$$

where $Y \in M_D$ and $D = 1 + n + n^3 + \cdots + n^{2t-3} + n^{2t-1} + n^{2t}$. The rows and columns of Y are labeled by the elements of $\{0\} \cup [n] \cup [n]^3 \cup \ldots [n]^{2t-3} \cup [n]^{2t-1} \cup [n]^{2t}$.

Proof of Theorem 1.2 (i). Note that Eq. (9) ensures that $(Y_{0,\mathbf{I}})_{\mathbf{I}}$ equals $(T_{\mathbf{I}})_{\mathbf{I}}$ for every $\mathbf{I} \in ([n] \times [n])^t$. Thus, it suffices to show that a multilinear form R satisfies $\|R\|_{\mathrm{rep}} \leq w$ if and only if there is a matrix Y that satisfies Eqs. (10) to (13) and $Y_{0,\mathbf{I}} = R_{\mathbf{I}}$.

Assume first that $(y_{0,\mathbf{I}})_{\mathbf{I}}$ factors as in Eq. (1) for some vectors with $\|u\|^2 = \|v\|^2 = w$. Then, consider the block structure for the contractions A_s given in Eq. (4), and define the vectors $\{u, v_{\mathbf{I}} : \mathbf{I} \in [n] \cup_{s \in [t-1]} ([n] \times [n])^s \times [n] \cup [n] \times ([n] \times [n])^{t-1} \times [n]\}$ as in Eqs. (5) to (7). Then, $y_{0,\mathbf{I}} = \langle u, v_{\mathbf{I}} \rangle$, for every $\mathbf{I} \in ([n] \times [n])^t$. This way, if we consider the positive semidefinite matrix

$$Y := \mathrm{Gram}\{u, v_{\mathbf{I}} : \mathbf{I} \in [n] \cup_{s \in [t-1]} ([n] \times [n])^s \times [n] \cup [n] \times ([n] \times [n])^{t-1} \times [n]\},$$

and we label the rows and columns corresponding to u with 0 and the ones corresponding to $v_{\mathbf{I}}$ with $\mathbf{I}$, we have that $y_{0,\mathbf{I}} = Y_{0,\mathbf{I}}$ for every $\mathbf{I} \in ([n] \times [n])^t$. Eq. (10) follows from the fact that $\|u\|^2 = w$. From the fact that A_t is a contraction, Eq. (11) follows:

$$\sum_{i \in [n]} Y_{i,i} = \sum_{i \in [n]} \langle v_i, v_i \rangle = \left\langle v, \sum_{i \in [n]} A(i)^\dagger A(i) v \right\rangle = \left\langle v, A_t^\dagger A_t v \right\rangle \leq \langle v, v \rangle = w.$$

From the fact that A_s are contractions for $s \in [t-1]$ Eq. (12) follows. Indeed, let $\lambda \in \mathbb{C}^{n \times (n \times n)^{s-1} \times n}$. Then,

$$\left\langle \lambda, \sum_{i \in [n]} (Y_{(i,j)\mathbf{J},(i,j')\mathbf{J}'})_{j,j' \in [n], \mathbf{J}, \mathbf{J}' \in ([n] \times [n])^{s-1} \times [n]} \lambda \right\rangle$$

$$= \sum_{i \in [n], j,j' \in [n], \mathbf{J}, \mathbf{J}' \in ([n] \times [n])^{s-1} \times [n]} \bar{\lambda}_{j\mathbf{J}} \left\langle v_{(i,j)\mathbf{J}}, v_{(i,j')\mathbf{J}'} \right\rangle \lambda_{j'\mathbf{J}'}$$

$$= \sum_{j,j' \in [n], \mathbf{J}, \mathbf{J}' \in ([n] \times [n])^{s-1} \times [n]} \bar{\lambda}_{j\mathbf{J}} \left\langle v_{\mathbf{J}}, \left(\sum_{i \in [n]} A_{t-s}^\dagger(j,i) A_{t-s}(i,j') \right) v_{\mathbf{J}'} \right\rangle \lambda_{j'\mathbf{J}'}$$

$$= \underbrace{\sum_{j,j' \in [n], \mathbf{J}, \mathbf{J}' \in ([n] \times [n])^{s-1} \times [n]} \bar{\lambda}_{j\mathbf{J}} \left\langle v_{\mathbf{J}}, (A_{t-s}^\dagger A_{t-s})(j,j') v_{\mathbf{J}'} \right\rangle \lambda_{j'\mathbf{J}'}}_{(*)},$$

where in the second equality we have used that $A_{t-s}(i,j)^\dagger = A_{t-s}^\dagger(j,i)$. Now, if we define $w_{\mathbf{J}} = (\lambda_{1\mathbf{J}} v_{\mathbf{J}}, \ldots, \lambda_{n\mathbf{J}} v_{\mathbf{J}})$, it follows that

$$(*) = \sum_{\mathbf{J},\mathbf{J}'} \langle w_{\mathbf{J}}, A_{t-s}^\dagger A_{t-s} w_{\mathbf{J}'} \rangle = \left\langle \left(\sum_{\mathbf{J}} w_{\mathbf{J}} \right), A_{t-s}^\dagger A_{t-s} \left(\sum_{\mathbf{J}'} w_{\mathbf{J}'} \right) \right\rangle.$$

Hence, as $A_{t-s}^\dagger A_{t-s} \preceq \mathrm{Id}$, it is satisfied that

$$(*) \leq \left\langle \left(\sum_{\mathbf{J}} w_{\mathbf{J}} \right), \left(\sum_{\mathbf{J}'} w_{\mathbf{J}'} \right) \right\rangle = \sum_{j \in [n], \mathbf{J}, \mathbf{J}' \in ([n] \times [n])^{s-1} \times [n]} \lambda_{j\mathbf{J}}^* \langle v_{\mathbf{J}}, v_{\mathbf{J}'} \rangle \lambda_{j\mathbf{J}'}$$

$$= \langle \lambda, \oplus_{k \in [n]} (Y_{\mathbf{J}, \mathbf{J}'})_{\mathbf{J}, \mathbf{J}' \in [n] \times [n])^{s-1} \times [n]} \lambda \rangle,$$

as desired. The fact that A_0 is a contraction implies Eq. (13), and this can be shown similarly to how we showed that Eq. (12) holds.

Now, assume that there exists $Y \succeq 0$, satisfying equations Eqs. (10) to (13) and $y_{0,\mathbf{I}} = Y_{0,\mathbf{I}}$. Consider $d \in \mathbb{N}$ and vectors $\{u, v_{\mathbf{I}} : \mathbf{I} \in [n] \cup_{s \in [t-1]} ([n] \times [n])^s \times [n] \cup [n] \times ([n] \times [n])^{t-1} \times [n]\} \in \mathbb{C}^d$ such that

$$Y = \text{Gram}\{u, v_{\mathbf{I}} : \mathbf{I} \in [n] \cup_{s \in [t-1]} ([n] \times [n])^s \times [n] \cup [n] \times ([n] \times [n])^{t-1} \times [n]\},$$

where again we label by 0 the rows and columns corresponding to u and by $\mathbf{I}$ the ones corresponding to $v_{\mathbf{I}}$. Eq. (10) implies that $\|u\|^2 = w$. We define A_t through its blocks. Let $v \in \mathbb{C}^d$ be a vector with $\|v\|^2 = w$. $A_t(i) \in M_d$ is defined as the matrix that maps v to v_i and is extended by 0 to the orthogonal complement of $\text{span}\{v\}$. This way, A_t is a contraction, because

$$\|A_t\|^2 = \frac{\langle v, A_t^\dagger A_t v \rangle}{w} = \frac{1}{w} \sum_{i \in [n]} \langle v, A_t(i)^\dagger A_t(i) v \rangle = \frac{1}{w} \sum_{i \in [n]} \langle v_i, v_i \rangle = \frac{1}{w} \sum_{i \in [n]} Y_{i,i} \leq 1,$$

where in the inequality we have used Eq. (11). The definition of A_{t-s} for $s \in [t-1]$ is slightly more complicated. Given $(i, j) \in [n] \times [n]$, the block $A_{t-s}(i, j)$ is defined as linear map on $\text{span}\{v_{\mathbf{I}} : \mathbf{I} \in ([n] \times [n])^{s-1} \times [n]\}$ by

$$A_{t-s}(i, j) v_{\mathbf{J}} = v_{(i,j)\mathbf{J}}$$

and extended by 0 to the orthogonal complement. First, we have to check that this a good definition, namely that for every $\lambda \in \mathbb{C}^{n^{2(s-1)+1}}$

$$\sum_{\mathbf{J} \in ([n] \times [n])^{(s-1)} \times [n]} \lambda_{\mathbf{J}} v_{\mathbf{J}} = 0 \implies \sum_{\mathbf{J} \in ([n] \times [n])^{(s-1)} \times [n]} \lambda_{\mathbf{J}} v_{(i,j)\mathbf{J}} = 0.$$

Indeed, we can prove something stronger. For any $\lambda \in \mathbb{C}^{n^{2(s-1)+1}}$, we define $\tilde{\lambda} \in \mathbb{C}^{n^{1+2(s-1)+1}}$ by $\tilde{\lambda}_{j'\mathbf{J}} := \delta_{j,j'} \lambda_{\mathbf{J}}$, where j is the second index in the pair (i, j) that indexes the block $A_{t-s}(i, j)$. Hence,

$$\left\langle \sum_{\mathbf{J} \in ([n] \times [n])^{s-1} \times [n]} \lambda_{\mathbf{J}} v_{(i,j)\mathbf{J}}, \sum_{\mathbf{J}' \in ([n] \times [n])^{s-1} \times [n]} \lambda_{\mathbf{J}'} v_{(i,j)\mathbf{J}'} \right\rangle$$

$$= \langle \lambda, (Y_{(i,j)\mathbf{J},(i,j)\mathbf{J}'})_{\mathbf{J}, \mathbf{J}' \in ([n] \times [n])^{s-1} \times [n]} \lambda \rangle$$

$$= \langle \tilde{\lambda}, (Y_{(i,j')\mathbf{J},(i,j'')\mathbf{J}'})_{j',j'' \in [n], \mathbf{J}, \mathbf{J}' \in ([n] \times [n])^{s-1} \times [n]}, \tilde{\lambda} \rangle$$

$$\leq \left\langle \tilde{\lambda}, \sum_{i \in [n]} \left(Y_{(i,j')\mathbf{J},(i,j'')\mathbf{J}'} \right)_{j',j'' \in [n], \mathbf{J}, \mathbf{J}' \in ([n] \times [n])^{s-1} \times [n]} \tilde{\lambda} \right\rangle$$

$$\leq \left\langle \tilde{\lambda}, \oplus_{k \in [n]} (Y_{\mathbf{J},\mathbf{J}'})_{\mathbf{J},\mathbf{J}' \in ([n] \times [n])^{s-1} \times [n]} \tilde{\lambda} \right\rangle$$

$$= \left\langle \lambda, (Y_{\mathbf{J},\mathbf{J}'})_{\mathbf{J},\mathbf{J}' \in ([n] \times [n])^{s-1} \times [n]} \lambda \right\rangle$$

$$= \left\langle \sum_{\mathbf{J} \in ([n] \times [n])^{s-1} \times [n]} \lambda_{\mathbf{J}} v_{\mathbf{J}}, \sum_{\mathbf{J}' \in ([n] \times [n])^{s-1} \times [n]} \lambda_{\mathbf{J}'} v_{\mathbf{J}'} \right\rangle,$$

where in the first inequality we have used that

$$\left(Y_{(i,j')\mathbf{J},(i,j'')\mathbf{J}'} \right)_{j',j'' \in [n], \mathbf{J}, \mathbf{J}' \in ([n] \times [n])^{s-1} \times [n]} \succeq 0$$

for every $i \in [n]$, and in the second inequality we have used (12). Now, we have to check that A_{t-s} is a contraction. By the definition of A_{t-s}, we just have to check that for every $\lambda \in \mathbb{C}^{n^{1+2(s-1)+1}}$,

$$\lambda v := \begin{pmatrix} \sum_{\mathbf{J} \in ([n] \times [n])^{s-1} \times [n]} \lambda_{1\mathbf{J}} v_{\mathbf{J}} \\ \vdots \\ \sum_{\mathbf{J} \in ([n] \times [n])^{s-1} \times [n]} \lambda_{n\mathbf{J}} v_{\mathbf{J}} \end{pmatrix}$$

is mapped to a vector with smaller or equal norm. Indeed,

$$\langle A_{t-s} \lambda v, A_{t-s} \lambda v \rangle = \sum_{i,j,j' \in [n], \mathbf{J}, \mathbf{J}' \in [n]^{2(s-1)+1}} \bar{\lambda}_{j\mathbf{J}} \left\langle v_{(i,j)\mathbf{J}}, v_{(i,j')\mathbf{J}'} \right\rangle \lambda_{j'\mathbf{J}'}$$

$$= \left\langle \lambda, \sum_{i \in [n]} \left(Y_{(i,j)\mathbf{J},(i,j')\mathbf{J}'} \right)_{j,j' \in [n], \mathbf{J}, \mathbf{J}' \in ([n] \times [n])^{s-1} \times [n]} \lambda \right\rangle$$

$$\leq \left\langle \lambda, \oplus_{k \in [n]} (Y_{\mathbf{J},\mathbf{J}'})_{\mathbf{J},\mathbf{J}' \in ([n] \times [n])^{s-1} \times [n]} \lambda \right\rangle$$

$$= \langle \lambda v, \lambda v \rangle,$$

where in the inequality we have used Eq. (12). Finally, we define A_0 through its blocks. $A_0(i) \in M_d$ is defined by $A_0(i) v_{\mathbf{J}} = v_{i\mathbf{J}}$ and extended by 0 to the orthogonal complement of span$\{v_{\mathbf{J}} : \mathbf{J} \in [n] \times ([n] \times [n])^{t-1}\}$. Using Eq. (13), we can check that these blocks are well-defined and that A_0 is a contraction using a similar argument to the one that we have just used to verify the same properties of A_{t-s}.

It just remains to show that the form

$$R(X_1, \ldots, X_t) = \sum_{\mathbf{I} \in ([n] \times [n])^t} Y_{0,\mathbf{I}} X_1(I_1) \ldots X_t(I_t)$$

satisfies

$$R(X_1, \ldots, X_t) = \langle u, A_0(X_1 \otimes \mathrm{Id}_d)A_1 \ldots A_{t-1}(X_t \otimes \mathrm{Id}_d)A_t v \rangle. \tag{14}$$

Eq. (14) holds if and only if it holds for a basis of M_n. We verify it for the canonical basis $\{E_I\}_{I \in [n] \times [n]}$, where the (i,j)-th entry of E_I is 1 if $I = (i,j)$ and 0 otherwise. On the one hand, by definition, we have that

$$R(E_{I_1}, \ldots, E_{I_t}) = Y_{0,\mathbf{I}} = \langle u, A(\mathbf{I})v \rangle.$$

On the other hand, a simple calculation shows that

$$A_0(E_{I_1} \otimes \mathrm{Id}_d)A_1 \ldots A_{t-1}(E_{I_t} \otimes \mathrm{Id}_d)A_t = A_1(I_1) \ldots A_t(I_t),$$

so $\langle u, A_0(E_{I_1} \otimes \mathrm{Id}_d)A_1 \ldots A_{t-1}(E_{I_t} \otimes \mathrm{Id}_d)A_t v \rangle = \langle u, A(\mathbf{I})v \rangle$, as desired. $\quad\square$

Remark 2.1. (P_{CS}) has $O(\mathrm{poly}(n)^t)$ variables, so item (b) holds. Item (c) can be inferred from the proof of Theorem 1.2 (i), where a recipe to extract a factorization as in Eq. (1) for $(Y_{0,\mathbf{I}})_\mathbf{I}$ satisfying Eqs. (10) to (13) is given.

2.2. The dual semidefinite program

In this section, we introduce (D_{CS}) and prove Theorem 1.2 (ii). (D_{CS}) is given by:

$$\sup \qquad \sum_{\mathbf{I} \in ([n] \times [n])^t} \Re(T_\mathbf{I})\Re(y_{0,\mathbf{I}}) + \Im(T_\mathbf{I})\Im(y_{0,\mathbf{I}}) \tag{D_{CS}}$$

$$\text{s.t.} \qquad y_0, y_0' \geq 0, \ (y_{\mathbf{J},\mathbf{J}'})_{\mathbf{J},\mathbf{J}' \in ([n] \times [n])^s} \succeq 0, \ \text{for } s \in [t], \tag{15}$$

$$y_0 + y_0' \leq 1, \tag{16}$$

$$\mathrm{Diag}(y_0, \ldots, y_0) \succeq \sum_{k \in [n]} (y_{ki,kj})_{i,j \in [n]}, \tag{17}$$

$$\oplus_{k \in [n]} (y_{\mathbf{J},\mathbf{J}'})_{\mathbf{J},\mathbf{J}' \in ([n] \times [n])^s}$$
$$\succeq \sum_{i \in [n]} (y_{(ij)\mathbf{J},(ij')\mathbf{J}'})_{j,j' \in [n], \mathbf{J},\mathbf{J}' \in ([n] \times [n])^s}, \tag{18}$$

for $s \in [t-1]$,

$$\begin{pmatrix} y_0' & \cdots & (y_{0,\mathbf{J}})_{\mathbf{J} \in ([n] \times [n])^t}/2 & \cdots \\ \vdots & & & \\ \dfrac{(\bar{y}_{0,\mathbf{I}})_{\mathbf{J} \in ([n] \times [n])^t}}{2} & & (y_{\mathbf{J},\mathbf{J}'})_{\mathbf{J},\mathbf{J}' \in ([n] \times [n])^t} & \\ \vdots & & & \end{pmatrix} \succeq 0. \tag{19}$$

Before diving into the proof, we give some intuition of why the optimal value of (D_{CS}) is $\|T\|_{\mathrm{cb}}$. One should note that Eq. (19) means that the variables $y_{0,\mathbf{I}}$ can be written as the

$\langle u, v_\mathbf{I} \rangle$ for some vectors u, $v_\mathbf{I}$. Then, roughly speaking, Eqs. (17) and (18) encode that $v_\mathbf{I}$ equal $X_1(I_1) \ldots X_t(I_t)v$ for some contractions $X_1, \ldots, X_t$ and a vector v, and Eq. (16) encodes that u and v are bounded vectors. From there, one can relate the optimal value of Eq. (D_{CS}) to the formula of $\|T\|_{\mathrm{cb}}$ given in Eq. (3).

Proof of Theorem 1.2 (ii). First, we note that Eq. (15) means that there exist $d \in \mathbb{N}$ and vectors $\{u, v, v_\mathbf{I} : \mathbf{I} \in ([n] \times [n])^s, \ s \in [t]\} \subset \mathbb{C}^m$ such that $y'_0 = \langle u, u \rangle$, $y_0 = \langle v, v \rangle$, and $y_{\mathbf{I}, \mathbf{I}'} = \langle v_\mathbf{I}, v_{\mathbf{I}'} \rangle$ for every $\mathbf{I} \in ([n] \times [n])^s$ and $s \in [t]$. Then, Eq. (17) means that $\langle u, u \rangle + \langle v, v \rangle \leq 1$ and Eq. (19) means that $y_{0, \mathbf{I}} = 2\langle u, v_\mathbf{I} \rangle$ for every $\mathbf{I} \in ([n] \times [n])^t$. Thus, we can rewrite Eq. (D_{CS}) as

$$\sup \qquad 2\Re \Big(\sum_{\mathbf{I} \in ([n] \times [n])^t} T_\mathbf{I} \overline{\langle u, v_\mathbf{I} \rangle} \Big), \tag{20}$$

$$\text{s.t.} \qquad m \in \mathbb{N}, \ u, v, v_\mathbf{I} \in \mathbb{C}^m, \ \mathbf{I} \in ([n] \times [n])^s, \ s \in [t],$$

$$\langle u, u \rangle + \langle v, v \rangle \leq 1,$$

$$\mathrm{Diag}(\langle v, v \rangle, \ldots, \langle v, v \rangle) \succeq \sum_{k \in [n]} (\langle v_{ki}, v_{kj} \rangle)_{i,j \in [n]}, \tag{21}$$

$$\oplus_{k \in [n]} \big(\langle v_\mathbf{J}, v_{\mathbf{J}'} \rangle \big)_{\mathbf{J}, \mathbf{J}' \in ([n] \times [n])^s}$$

$$\succeq \sum_{i \in [n]} \big(\langle v_{(ij)\mathbf{J}}, v_{(ij')\mathbf{J}'} \rangle \big)_{j,j' \in [n], \mathbf{J}, \mathbf{J}' \in ([n] \times [n])^s}, \tag{22}$$

$$\text{for } s \in [t-1].$$

Next, we will show that Eqs. (21) and (22) are equivalent to the existence of contractions $X_1, \ldots, X_t \in M_{nm}$ such that

$$v_\mathbf{I} = X_{t-s+1}(I_1) \ldots X_t(I_s)v, \tag{23}$$

for every $\mathbf{I} \in ([n] \times [n])^s$ and every $s \in [t]$, where $X_s(I_s) \in M_m$ is the I_s-th block of X_s when regarded as a block-matrix with $n \times n$ blocks. Indeed, assume that Eqs. (21) and (22) hold. Then, we define the I-th block of X_{t-s} by

$$X_{t-s}(I)v_\mathbf{J} := v_{I\mathbf{J}}$$

for every $\mathbf{J} \in ([n] \times [n])^s$ and extend it by 0 on the orthogonal complement of $\mathrm{span}\{v_\mathbf{J} : \mathbf{J} \in ([n] \times [n])^s\}$. Before proving that X_s are contractions, we have to check that $X_s(I)$ are well-defined as linear maps. Namely, that for every $\lambda \in \mathbb{C}^{n^{2s}}$ we have

$$\sum_{\mathbf{J} \in ([n] \times [n])^s} \lambda_\mathbf{J} v_\mathbf{J} = 0 \implies \sum_{\mathbf{J} \in ([n] \times [n])^s} \lambda_\mathbf{J} v_{I\mathbf{J}} = 0.$$

In fact, we can prove something stronger. Let $\lambda \in \mathbb{C}^{n^{2s}}$, and define $\tilde{\lambda} \in \mathbb{C}^{n^{1+2s}}$ by $\tilde{\lambda}_{j'\mathbf{J}} = \delta_{j,j'} \lambda_\mathbf{J}$, where $I = (i, j)$. Then

$$\left\langle \sum_{\mathbf{J}\in([n]\times[n])^s} \lambda_{\mathbf{J}} v_{I\mathbf{J}}, \sum_{\mathbf{J}'\in([n]\times[n])^s} \lambda_{\mathbf{J}'} v_{I\mathbf{J}'} \right\rangle$$

$$= \left\langle \lambda, \left(\langle v_{I\mathbf{J}}, v_{I\mathbf{J}'} \rangle \right)_{\mathbf{J},\mathbf{J}'\in([n]\times[n])^s} \lambda \right\rangle$$

$$= \left\langle \tilde{\lambda}, \left(\langle v_{(i,j)\mathbf{J}}, v_{(i,j')\mathbf{J}'} \rangle \right)_{j,j'\in[n],\mathbf{J},\mathbf{J}'\in([n]\times[n])^s} \tilde{\lambda} \right\rangle$$

$$\leq \left\langle \tilde{\lambda}, \sum_{i'\in[n]} \left(\langle v_{(i',j)\mathbf{J}}, v_{(i',j')\mathbf{J}'} \rangle \right)_{j,j'\in[n],\mathbf{J},\mathbf{J}'\in([n]\times[n])^s} \tilde{\lambda} \right\rangle$$

$$\leq \left\langle \lambda, \left(\langle v_{\mathbf{J}}, v_{\mathbf{J}'} \rangle \right)_{\mathbf{J},\mathbf{J}'\in([n]\times[n])^s} \lambda \right\rangle$$

$$= \left\langle \sum_{\mathbf{J}\in([n]\times[n])^s} \lambda_{\mathbf{J}} v_{\mathbf{J}}, \sum_{\mathbf{J}'\in([n]\times[n])^s} \lambda_{\mathbf{J}'} v_{\mathbf{J}'} \right\rangle,$$

where in the first inequality we have used that

$$\left(\langle v_{(i,j)\mathbf{J}}, v_{(i,j')\mathbf{J}'} \rangle \right)_{j,j'\in[n],\mathbf{J},\mathbf{J}'\in([n]\times[n])^s} \succeq 0$$

for every $i \in [n]$, and in the second inequality we have used Eq. (22) (or Eq. (21) if $s = 0$). Now we shall prove that X_s are contractions. By their definition, we only have to check that for every $\lambda \in \mathbb{C}^{n^{1+2s}}$ the vector

$$\lambda v := \begin{pmatrix} \sum_{\mathbf{J}\in([n]\times[n])^s} \lambda_{1\mathbf{J}} v_{\mathbf{J}} \\ \vdots \\ \sum_{\mathbf{J}\in([n]\times[n])^s} \lambda_{n\mathbf{J}} v_{\mathbf{J}} \end{pmatrix}$$

is mapped through X_{t-s} to a vector of smaller or equal norm. That is true because

$$\langle X_s \lambda v, X_s \lambda v \rangle = \sum_{i,j,j'\in[n],\mathbf{J},\mathbf{J}'\in([n]\times[n])^s} \bar{\lambda}_{j\mathbf{J}} \langle v_{(i,j)\mathbf{J}}, v_{(i,j')\mathbf{J}'} \rangle \lambda_{j'\mathbf{J}}$$

$$= \left\langle \lambda, \sum_{i\in[n]} \left(\langle v_{(ij)\mathbf{J}}, v_{(ij')\mathbf{J}'} \rangle \right)_{j,j'\in[n],\mathbf{J},\mathbf{J}'\in([n]\times[n])^s} \lambda \right\rangle$$

$$\leq \left\langle \lambda, \begin{pmatrix} \left(\langle v_{\mathbf{J}}, v_{\mathbf{J}'} \rangle \right)_{\mathbf{J},\mathbf{J}'\in([n]\times[n])^s} & & \\ & \ddots & \\ & & \left(\langle v_{\mathbf{J}}, v_{\mathbf{J}'} \rangle \right)_{\mathbf{J},\mathbf{J}'\in([n]\times[n])^s} \end{pmatrix} \lambda \right\rangle$$

$$= \langle \lambda v, \lambda v \rangle,$$

where in the inequality we have used Eq. (22) (or Eq. (21) in the case of $s = 0$).

On the other hand, if Eq. (23) holds, it is a routine check showing that Eqs. (21) and (22) hold. Putting everything together, we can rewrite (20) as

$$\sup \qquad 2\Re\Big(\sum_{\mathbf{I}\in([n]\times[n])^t} T_{\mathbf{I}}\overline{R_{\mathbf{I}}}\Big),$$

$$\text{s.t.} \qquad R\in(\mathbb{C}^{n\times n})^t,\ m\in\mathbb{N},\ u,v\in\mathbb{C}^m, X_s\in M_m \text{ contractions for } s\in[t],$$

$$\langle u,u\rangle+\langle v,v\rangle\leq 1,$$

$$R_{\mathbf{I}}=\langle u, X_1(I_1)\ldots X_t(I_t)v\rangle,\ \text{for } \mathbf{I}\in([n]\times[n])^t.$$

Note that R satisfies the constraints above if and only if $\overline{R}$ does it, so we can change $\overline{R}$ by R in the supremum above. Additionally, by multiplying the v of an optimal solution of this optimization program by an appropriate complex phase we can change the real part by the module, so we can express (D_{CS}) as

$$\sup \qquad 2\Big|\sum_{\mathbf{I}\in([n]\times[n])^t} T_{\mathbf{I}}R_{\mathbf{I}}\Big|, \qquad\qquad (24)$$

$$\text{s.t.} \qquad R\in(\mathbb{C}^{n\times n})^t,\ m\in\mathbb{N},\ u,v\in\mathbb{C}^m, X_s\in M_m \text{ contractions for } s\in[t],$$

$$\langle u,u\rangle+\langle v,v\rangle\leq 1,$$

$$R_{\mathbf{I}}=\langle u, X_1(I_1)\ldots X_t(I_t)v\rangle,\ \text{for } \mathbf{I}\in([n]\times[n])^t.$$

We finally claim that the above optimization problem is equivalent to

$$\sup \qquad 2\Big|\sum_{\mathbf{I}\in([n]\times[n])^t} T_{\mathbf{I}}R_{\mathbf{I}}\Big|, \qquad\qquad (25)$$

$$\text{s.t.} \qquad R\in(\mathbb{C}^{n\times n})^t,\ m\in\mathbb{N},\ u,v\in\mathbb{C}^m, X_s\in M_m \text{ contractions for } s\in[t],$$

$$\langle u,u\rangle,\langle v,v\rangle\leq 1/2,$$

$$R_{\mathbf{I}}=\langle u, X_1(I_1)\ldots X_t(I_t)v\rangle,\ \text{for } \mathbf{I}\in([n]\times[n])^t.$$

We first note that the value of Eq. (24) is greater or equal than the one of Eq. (25), because the feasible region is larger in the case of Eq. (24). On the other hand, if one picks a feasible instance (u,v,X) of Eq. (24), one can define the instance $(\tilde{u},\tilde{v},X)$ by

$$\tilde{u}=\frac{u\sqrt{\|u\|^2+\|v\|^2}}{\sqrt{2}\|u\|},\ \tilde{v}=\frac{v\sqrt{\|u\|^2+\|v\|^2}}{\sqrt{2}\|v\|},$$

which is feasible for Eq. (25) and attains a value greater or equal than (u,v,X), because

$$\Big|\sum T_{\mathbf{I}}\langle\tilde{u}, X_1(I_1)\ldots X_t(I_t)\tilde{v}\rangle\Big| = \frac{\|u\|^2+\|v\|^2}{2\|u\|\|v\|}\Big|\sum T_{\mathbf{I}}\langle u, X_1(I_1)\ldots X_t(I_t)v\rangle\Big|$$

$$\geq \Big|\sum T_{\mathbf{I}}\langle u, X_1(I_1)\ldots X_t(I_t)v\rangle\Big|.$$

Now, the result follows from the fact that the optimal value of Eq. (25) is $\|T\|_{\mathrm{cb}}$ as in Eq. (3). $\square$

2.3. Strong duality

Finally, we prove Theorem 1.2 (iii). Before that, we formally state the condition that ensures strong duality (for a reference see [5, Theorem 2]).

Theorem 2.2 *(Slater's theorem). Let* (P) *and* (D) *be a primal-dual pair of semidefinite programs, as in Eq.* (2). *Assume that* (P) *is feasible and there exists a strictly positive instance for* (D), *i.e., there exists* $y \in \mathbb{R}^L$ *such that* $C - \mathcal{B}^*(y)$ *is strictly positive. Then the optimal values of* (P) *and* (D) *are equal.*[2]

Proof of Theorem 1.1 (iii). First, we show that (P_{CS}) can be expressed as in the canonical form of (P) in Eq. (2). To do that we introduce the slack matrix variable Z and write (P_{CS}) as

$$\inf \quad w \qquad\qquad\qquad (\widetilde{P}_{CS})$$

$$\text{s.t.} \quad X := \begin{pmatrix} w & 0 & 0 \\ 0 & Y & 0 \\ 0 & 0 & Z \end{pmatrix} \succeq 0$$

$$\Re Y_{0,\mathbf{I}} = \Re T_{\mathbf{I}}, \ \Im Y_{0,\mathbf{I}} = \Im T_{\mathbf{I}}, \ \mathbf{I} \in ([n] \times [n])^t, \tag{26}$$

$$Y_{0,0} = w, \tag{27}$$

$$\sum_{i \in [n]} Y_{i,i} - w = Z_{0,0}, \tag{28}$$

$$\sum_{i \in [n]} (Y_{(i,j)\mathbf{J},(i,j')\mathbf{J}'})_{j,j' \in [n], \mathbf{J},\mathbf{J}' \in ([n]\times[n])^{s-1} \times [n]}$$

$$- \oplus_{k \in [n]} (Y_{\mathbf{J},\mathbf{J}'})_{\mathbf{J},\mathbf{J}' \in ([n]\times[n])^{s-1} \times [n]} \tag{29}$$

$$= (Z_{j\mathbf{J},j'\mathbf{J}'})_{j,j' \in [n], \mathbf{J},\mathbf{J}' \in ([n]\times[n])^{s-1} \times [n]}, \ s \in [t-1],$$

$$(Y_{\mathbf{J},\mathbf{J}'})_{\mathbf{J},\mathbf{J}' \in ([n]\times[n])^t} - \oplus_{k \in [n]} (Y_{\mathbf{J},\mathbf{J}'})_{\mathbf{J},\mathbf{J}' \in ([n]\times[n])^{t-1} \times [n]}$$

$$= (Z_{\mathbf{J},\mathbf{J}'})_{\mathbf{J},\mathbf{J}' \in ([n]\times[n])^t}. \tag{30}$$

One can regard X as a positive semidefinite matrix with some entries set to 0, which can be imposed via linear constraints. Additionally, note that the objective function w is a linear function of the entries of X, and so are the restrictions Eqs. (26) to (30). Hence, (P_{CS}) has the form of (P) in Eq. (2).

Second, we show that (D_{CS}) can be expressed as in the canonical form of (D) in Eq. (2). We can rewrite (D_{CS}) as

$$\sup \quad \sum_{\mathbf{I} \in ([n]\times[n])^t} \Re(T_{\mathbf{I}}) r_{\mathbf{I}} - \Im(T_{\mathbf{I}}) c_{\mathbf{I}} \qquad\qquad (\widetilde{D}_{CS})$$

---

[2] We recall that an optimization problem is feasible if there exists an instance satisfying its constraints.

s.t. $y_0, y_0', r_{\mathbf{I}}, c_{\mathbf{I}}, y_{\mathbf{I},\mathbf{I}'} \in ([n] \times [n])^s, \ s \in [t]$

$$y_0 \geq 0, \ y_0' \geq 0, \ \sum_{\mathbf{J},\mathbf{J}' \in ([n] \times [n])^s} y_{\mathbf{J},\mathbf{J}'} E_{\mathbf{J},\mathbf{J}'} \succeq 0, \ \text{for } s \in [t], \tag{31}$$

$$y_0 + y_0' \leq 1, \tag{32}$$

$$y_0 \, \mathrm{Id}_n \succeq \sum_{i,j,k \in [n]} y_{ki,kj} E_{i,j}, \tag{33}$$

$$\sum_{\mathbf{J},\mathbf{J}' \in ([n] \times [n])^s} y_{\mathbf{J},\mathbf{J}'} \oplus_{k \in [n]} E_{\mathbf{J},\mathbf{J}'}$$

$$\succeq \sum_{i,j,j' \in [n], \mathbf{J},\mathbf{J}' \in ([n] \times [n])^s} y_{(ij)\mathbf{J},(ij')\mathbf{J}'} E_{j\mathbf{J},j'\mathbf{J}'}, \tag{34}$$

for $s \in [t-1]$,

$$y_0' E_{0,0} + \sum_{\mathbf{J} \in ([n] \times [n])^t} r_{\mathbf{J}} \frac{E_{0,\mathbf{J}} + E_{\mathbf{J},0}}{2} + c_{\mathbf{J}} \frac{i E_{0,\mathbf{J}} - i E_{\mathbf{J},0}}{2}$$

$$7 + \sum_{\mathbf{I},\mathbf{I}' \in ([n] \times [n])^t} y_{\mathbf{I},\mathbf{I}'} E_{\mathbf{I},\mathbf{I}'} \succeq 0. \tag{35}$$

Thus, we have written (D_{CS}) as an optimization problem $(\widetilde{D}_{CS})$ on the variables $y_0, y_0', r_{\mathbf{I}}, c_{\mathbf{I}}, y_{\mathbf{I},\mathbf{I}'}$. Moreover, the objective function is a linear combination of these variables. Also, the constraints are positive semidefinite constraints on matrices that are linear combinations of other matrices, where the coefficients of these linear combinations are $y_0, y_0', r_{\mathbf{I}}, c_{\mathbf{I}}, y_{\mathbf{I},\mathbf{I}'}$. Putting everything together, it follows that (D_{CS}) is of the form of (D) in Eq. (2).

Third, we show that (D_{CS}) is the dual of (P_{CS}). Equivalently, we prove that $(\widetilde{D}_{CS})$ is the dual of $(\widetilde{P}_{CS})$. To take the dual of a primal semidefinite program such as $(\widetilde{P}_{CS})$ it is convenient to assign a dual variable to every linear constraint. We assign $r_{\mathbf{I}}$ and $c_{\mathbf{I}}$ to the constraints in Eq. (26), y_0' to Eq. (27), y_0 to Eq. (28), and $y_{\mathbf{I},\mathbf{I}}$ to Eqs. (29) and (30). In addition, one should note that every variable in the primal corresponds to a restriction in the dual. With this in mind, from the definition of the dual given in Eq. (2), it follows that $(\widetilde{D}_{CS})$ is the dual of $(\widetilde{P}_{CS})$, and that the constraints of Eq. (31) correspond to variable Z in $(\widetilde{P}_{CS})$, Eq. (32) to variable w, and Eqs. (33) to (35) to variable Y.

Finally, we show that the conditions of Theorem 2.2 are satisfied by $(\widetilde{P}_{CS})$ and $(\widetilde{D}_{CS})$, which implies that their values are equal. $(\widetilde{P}_{CS})$ is feasible, as every T factors as in Eq. (1) for some u, v with sufficiently large norm (if this was not true, $\|T\|_{\mathrm{cb}}$ would not be a norm). In addition, we claim that the following parameters define a strictly positive feasible instance for $(\widetilde{D}_{CS})$

$$y_0 = y_0' = \frac{1}{3},$$

$$y_{\mathbf{I},\mathbf{J}} = \frac{\delta_{\mathbf{I},\mathbf{J}}}{3(n+1)^s}, \text{ for } \mathbf{I},\mathbf{J} \in ([n] \otimes [n])^s, \ s \in [t],$$

$$r_{\mathbf{I}} = c_{\mathbf{I}} = 0, \text{ for } \mathbf{I} \in ([n] \otimes [n])^s, \ s \in [t].$$

Indeed, with these parameters Eqs. (31) to (35) read as follows:

$$\frac{1}{3} \geq 0, \ \mathrm{Id} \succ 0$$

$$\frac{1}{3} + \frac{1}{3} \leq 1,$$

$$\frac{1}{3} \mathrm{Id}_n \succ \frac{n}{3(n+1)} \mathrm{Id}_n,$$

$$\frac{1}{3(n+1)^s} \mathrm{Id}_n \succ \frac{n}{3(n+1)^{s+1}} \mathrm{Id}_n, \text{ for } s \in [t-1],$$

$$\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3(n+1)^t} \mathrm{Id}_{n^t} \end{pmatrix} \succ 0,$$

and these identities are true because $1 > n/(n+1)$. □

Declaration of competing interest

None declared.

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Data availability

No data was used for the research described in the article.

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