A Compressed Sensing Based Approach on Discrete Algebraic Reconstruction Technique

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***Abstract*— Discrete tomography (DT) techniques are capable of computing better results, even using less number of projections than the continuous tomography techniques. Discrete Algebraic Reconstruction Technique (DART) is an iterative reconstruction method proposed to achieve this goal by exploiting a prior knowledge on the gray levels and assuming that the scanned object is composed from a few different densities. In this paper, DART method is combined with an initial total variation minimization (TvMin) phase to ensure a better initial guess and extended with a segmentation procedure in which the threshold values are estimated from a finite set of candidates to minimize both the projection error and the total variation (TV) simultaneously. The accuracy and the robustness of the algorithm is compared with the original DART by the simulation experiments which are done under (1) limited number of projections, (2) limited view problem and (3) noisy projections conditions.**

Index Terms – **Discrete Tomography, image reconstruction, algebraic reconstruction techniques, global thresholding, compressed sensing, total variation minimization**

I. INTRODUCTION

Computed Tomography (CT) acquires several projections by scanning an object with X-rays sent from different angles. Projections are then used to obtain the 3D representation of the object, which can also be considered to be a series of 2D slices. These slice images are reconstructed from 1D projection measurements by using various image reconstruction techniques. Filtered backprojection (FBP) is the widely used analytical method that has typically lower computational cost compared to other methods. However, it requires high number of projection samples, which also means higher radiation dose. On the other hand, algebraic methods are able to reconstruct images from incomplete projections, while it takes longer time to converge. Algebraic Reconstruction Technique (ART) [1] is a type of algebraic method whose principle is Kaczmarz's method.

Discrete Tomography (DT) [2] and Compressed Sensing (CS) theory [3] are both approaches which aim image reconstruction from incomplete projections. DT deals with the reconstruction of a discrete image from its projections by limiting the range of the reconstructed image intensity function to a finite set of gray levels. Discrete Algebraic Reconstruction Technique (DART) [4] is a well-known DT algorithm and it is capable of computing better reconstructions from fewer projections, not only because of its discrete nature but also due to its ability to reduce the number of variables by focusing on the regions where the reconstruction procedure tend to fail. DART alternates between a continuous algebraic reconstruction stage and a discretization procedure. The original DART [4] exploits a-priori gray levels and uses Otsu's histogram-based threshold selection scheme [5] in order to discretize the reconstructed continuous image. However, it has also been shown that, it is possible to estimate the gray levels automatically by selecting gray levels and thresholds which minimizes a cost function on projection space [6].

Compressed Sensing (CS) theory deals with sparse (or compressible) signals, or images in our case, and states that it is possible to reconstruct them from their fewer number of projections than required according to the Nyquist - Shannon sampling criteria [7]. Therefore, the techniques which solve the CS problems aim to find the sparsest solution, corresponding to the minimization of ℓ_0- or ℓ_1-norm of the signal. Although the CT images are generally not sparse, they might have a sparse representation in another domain. Total variation minimization (TvMin) is a technique which assumes that the image is sparse in its spatial gradient domain and aims to minimize the ℓ_1-norm of the gradient magnitude of the image. It is preferred to reconstruct images, in particular the medical images due to its capability of preserving high frequency details [8].

II. METHODS

A 3D CT image is considered to be a series of 2D slices and reconstruction is applied on these slices one-by-one. Cross sectional reconstruction is said to be a transformation from 1D projection measurements to 2D images. From an algebraic point of view, this reconstruction problem can be formulated as a system of linear equations in the vector form $Ax = b$ or as the weighted sums of the pixels, i.e. the line integrals of the rays over the traversed pixels:

$$\sum_{j=1}^{n} a_{ij}x_j = b_i, \quad i = 1,2,\cdots,m \tag{1}$$

where $x \in \mathbb{R}^n$, the vector of unknowns, denotes the image and $A \in \mathbb{R}^{mxn}$ is the *projection matrix* whose entry a_{ij} corresponds to the contribution of the pixel x_j to the projection b_i where $b \in \mathbb{R}^m$.

A. The DART Algorithm

In the DART algorithm [4], the objective is to obtain discrete reconstructions in which all pixels are assigned to one of the gray levels from a pre-defined level set.

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Hence, DART combines a continuous iterative reconstruction algorithm with a discretization step, meaning that an image segmentation procedure is applied during the algorithm. In this paper, the SART algorithm, see [9] for details, will be used as the continuous reconstruction subroutine.

Here, a brief explanation on the DART algorithm is given step by step. DART starts with an initial approximation which is computed by SART, and performs the following steps iteratively:

(1) Reconstructed image is segmented, so that it has only the gray values from given level set: $x_j \in \{\xi_1, \xi_2, .., \xi_L\}$

(2) From the segmented image, *boundary pixels*, which differ from at least one of the adjacent pixels, are determined by using 8-connected neighbourhood. Then, these pixels are extended with a randomized scheme which includes a non-boundary pixel to the set of boundary pixels with $|1-p|$ probability, where p is called as *fix probability*. This new set is called as *free pixels* u^k which imply the only pixels that will be updated in the next iteration of the DART, and the remaining pixels are kept fixed.

(3) Residual sinogram r^k is computed, by subtracting the forward projection of the *fixed pixels* f^k from the projection data as $r^k = b - Af^k$.

(4) The SART algorithm is then applied on the residual sinogram to update $u^{(k)}$ and a Gaussian smoothing filter is applied on the updated $u^{(k)}$. $x^{(k+1)}$ is obtained as $x^{(k+1)} = f^{(k)} + S_{u^{(k)}} r^{(k)}$ where S denotes the SART operator, and the control is returned back to (1) as long as the termination criterion, such as the number of iterations or converge, is not met.

DART reduces the number of unknowns in the original under-determined system, since almost each succeeding iteration comes up with the reconstruction problem to be solved for only free pixels. This approach not only improves the computational cost, but also yields more accurate results.

B. The TvMin Technique

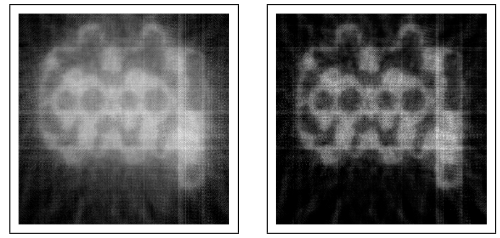
The term *total variation (TV)* implies the $\ell_1 - norm$ (or $\ell_2 - norm$ for isotropic TV model) of the image gradient and its minimization is a technique which is used to solve CS problems, in particular image reconstruction (or denoising). From compressed sensing point of view, total variation regularization algorithms recover the images by utilizing the sparsity of the gradient magnitude of the image. TV regularization can be formulated as an unconstrained optimization problem, as in Eq. (2).

$$min_{x \in \mathbb{R}^n} \sum_j \|D_j x\| + \frac{\mu}{2} \|b - Ax\|_2^2 \qquad (2)$$

where $D_j x$ is the discrete gradient of x at the pixel j and μ is the regularization parameter. Eq. (2) aims to minimize both the TV and the projection error simultaneously.

C. The Proposed TvMin+DART Algorithm Definition

In this section, we describe the proposed algorithm which aims to improve the DART algorithm by using the TvMin technique instead of ARM to obtain the initial guess and by combining the histogram-based and the projection data based approaches to select an optimal threshold value.



(a) (b)

Fig. 1: Initial reconstructions obtained with $t_0 = 3$ iterations of (a) SART and (b) TvMin.

Since it is important to start with a good initial estimate to DART, in the proposed algorithm, the initial reconstruction is computed by TvMin, as it is defined in Eq. (2), instead of SART. This approach yields a more convenient reconstruction to be used in segmentation, since the artefacts formed after SART are highly reduced while the high frequency regions are preserved, plus enhanced. Two initial reconstructions, one from SART and another from TvMin, are shown in Fig. 1.

TvMin constrained optimization problem is known having high computational cost due to its non-differentiability. However, it has already been solved in [10] efficiently and named as TV minimization scheme based on augmented Lagrangian and alternating direction algorithms (TVAL3 scheme). This solver is used to solve Eq. (2), by adding a non-negativity constraint in the proposed method, in order to obtain an easier to segment initial guess in a time efficient manner.

In this work, a thresholding scheme is also proposed to be used in the segmentation step of DART, by combining a histogram-based approach with the TV regularization problem. In the proposed threshold selection scheme, the threshold value, which minimizes both the projection error and TV, is selected among a number of *candidate thresholds*, which are computed by two-stage multilevel Otsu algorithm (TSMO) [11]. To determine the number of candidate thresholds, a *valley estimation* (VE) procedure, proposed in [12], is utilized.

To briefly explain the VE procedure, first of all, a normalized histogram binning (the number of groups is chosen as $K = 32$, as in [12]) is applied on the gray-level histogram of the continuous reconstruction and then, the groups are scanned twice. In the first scan, a probability is estimated for each group, according to its location on the histogram and its frequency. For more details about how these probabilities are estimated, the reader is referred to [12]. In the second scan, the estimated non-zero probabilities are updated to 100%, for the groups whose probabilities exceed 100% when summed with the adjacent ones and 0%, otherwise. The VE procedure is used as a pre-process to TSMO in order to estimate the number of clusters C, corresponding to the number of groups with 100% probability, and TSMO estimates C thresholds, which will then be used as candidates by the proposed algorithm. TSMO has two stages; in the first stage, the $K = 32$ normalized histogram bins are clustered into C clusters using Otsu's multi-level thresholding scheme [5], whose objective is maximizing between-class variance, and in the second stage, Otsu's bi-level thresholding is applied for each cluster. As the final stage of the threshold selection procedure, each candidate threshold $\tau \in T_{cdt}$ estimated by TSMO is tried if it minimizes Eq. (3) or not and the one which minimizes the cost, given in Eq. (3), most is used to segment the reconstructed image.

$$\tau^* = \underset{\tau \in T^C_{cdt}}{\operatorname{argmin}} \{ \sum_j \|D_j x\| + \frac{1}{2}\|b - Ax\|_2^2 \} \tag{3}$$

where τ^* is the selected optimum threshold value.

Furthermore, to prevent high fluctuations caused by the proposed segmentation step, a control operation is carried out after each threshold selection procedure. This operation checks the difference $\Delta cost$ between the current cost, given in Eq. (3), computed by using the selected threshold and the previous cost, and if this difference is greater than a predefined parameter $\varepsilon_{penalty}$, the algorithm keeps using the previous threshold value instead of the selected one. This operation can be defined as follows:

$$\tau^{(k)} = \begin{cases} \tau^*, & \Delta cost < \varepsilon_{penalty} \\ \tau^{(k-1)}, & otherwise \end{cases} \tag{4}$$

All other subsequent iterations of our algorithm are similar to the DART algorithm which has already been described in Section III.B. The pseudo code of the proposed algorithm is given as follows:

Algorithm (TvMin+DART)
$x^{(0)} \longleftarrow TVAL3(x := 0, A, b), \quad k := 0$
while (*stop criterion is not met*) **do**
begin
  Compute the histogram H^k
  Estimate the number of valleys $C \longleftarrow VE(H^k)$
  Find C candidate thresholds: $T_{cdt} \longleftarrow TSMO(H^k, C)$
  Select threshold τ_l^* which makes Eq. 2 minimum
  Segment image: $s^k \longleftarrow S(x^k, \tau^k)$
  Subdivide s^k into free pixels $U^k \subset x^k$ and fixed $F^k = x^k \setminus U^k$
  Extend U^k with the pixels in F^k with probability $|1-p|$
  Compute the residual sinogram $r^k \longleftarrow b - Af^k$
  Update the image of free pixels $u^k \longleftarrow ARM(u^k, A^k, r^k)$
  Smooth u^k and obtain $x^{k+1} \longleftarrow f^k + u^k$
end

D. The Proposed Algorithm with Gray Level Estimation

In this section, the proposed algorithm is extended to estimate the gray levels, by a formulation which exploits the differentiability of the projection error, defined as $E(x) = \sum_i^m (b_i - b_i^{'})^2$ using Euclidean distance. Since each gray level is restricted to a prescribe set after each DART iteration, one knows that $x_j^{'} \in \{\xi_1, \xi_2, .., \xi_L\}$. Hence, $E(x)$ can be rewritten as in Eq. (5) by substituting the forward projection of the image $x^{'}$ obtained from DART instead of $b_i^{'}$.

$$E(x) = \sum_i^m (b_i - \sum_{l=1}^L \xi_l Q_{il})^2 \quad where \quad Q_{il} = \sum_{j \in \Omega_l} a_{ij} \tag{5}$$

where ξ_l is the lth gray level of an L element gray level set and Ω_l is the index set of the pixels whose intensities are set to gray level ξ_l.

By differentiating Eq. (5) it with respect to the ξ_l to find the labels which minimize the error, one comes up with the following solution:

$$\begin{aligned} \frac{\partial E}{\partial \xi_l} &= \frac{\partial}{\partial \xi_l}\{\sum_{i=1}^m (b_i - \sum_{l=1}^L \xi_l Q_{il})^2\} \\ &= \frac{\partial}{\partial \xi_t}\{\sum_{i=1}^m (b_i - (\sum_{\substack{l=1 \\ l \neq t}}^L \xi_l Q_{il} + Q_{it}\xi_t))^2\} \\ &= -2\{\sum_{i=1}^m (b_i Q_{it} - \sum_{\substack{l=1 \\ l \neq t}}^L Q_{il} Q_{it} \xi_l - Q_{it}^2 \xi_t)\} \\ &= -2\sum_{i=1}^m Q_{it}(b_i - \sum_{\substack{l=1 \\ l \neq t}}^L Q_{il}\xi_l) + 2\xi_t \sum_{i=1}^m Q_{it}^2 \end{aligned} \tag{6}$$

where $t = 1, .., n$. Therefore, the gray levels can easily be estimated by using the following equation:

$$\xi_t = \frac{\sum_{i=1}^m Q_{it}(b_i - \sum_{\substack{l=1 \\ l \neq t}}^L Q_{il}\xi_l)}{\sum_{i=1}^m Q_{it}^2}. \tag{7}$$

Once Eq. (7) is computed, the previous gray level set is updated with the new labels and this procedure is repeated for each iteration, after the thresholds are determined and the image is segmented.

III. SIMULATION EXPERIMENTS

All implementations are done in MATLAB environment. AIR Tools package [13] is exploited for the simulation of the parallel beam geometry and SART algorithm. To estimate an initial guess by solving the TvMin problem, TVAL3 MATLAB solver [10] is used.

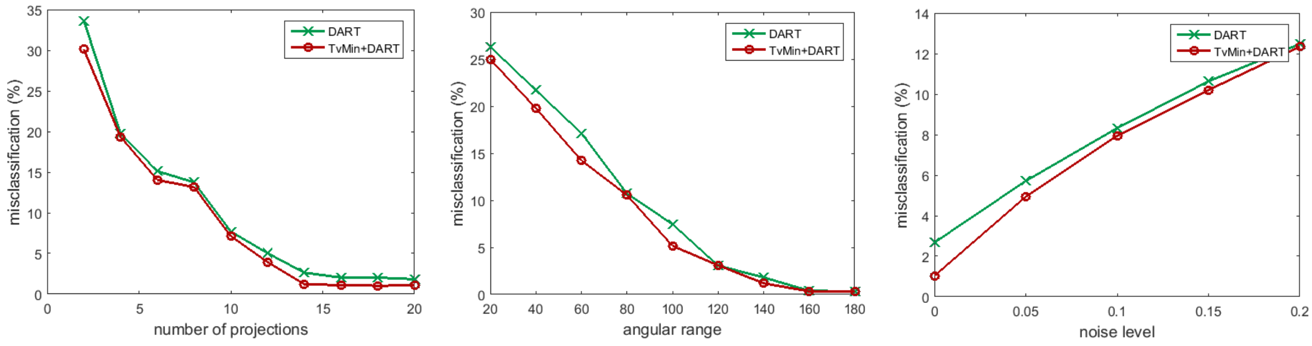
As the performance evaluation metrics, misclassification percentage, given in Eq. (8), and root means squared error (RMSE), given in Eq. (9), are used. RMSE is used to measure the quality of the image which is reconstructed by using the gray level estimation.

$$misclassification(\%) = \frac{100}{n}\sum_{j=1}^n (1 - \delta(x, x')) \tag{8}$$

$$RMSE = \Big(\frac{\sum_{j=1}^n (x_j - x'_j)^2}{n}\Big)^{1/2} \tag{9}$$

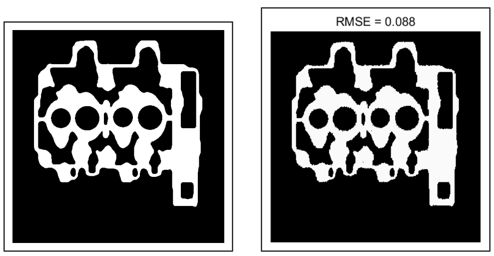
where x' denotes the reconstructed image.

Phantom image used in the experiments is same with one of the phantoms used in the paper, in which the original DART is proposed [4]. For each projection angle, 512 measurements are detected which is equal to the image dimensions. The relaxation parameter (λ) of SART and the fix probability (p) are fixed as 0.8 (as suggested in [13]), and 0.85 (as suggested in [4]), respectively. The randomized



(a) (b) (c)

Fig. 2: The misclassification percentage of DART and the proposed algorithm with respect to (a) the number of projections, (b) the angular range and (c) the noise level.



(a) (b)

Fig. 3: (a) Exact phantom and (b) reconstruction of the proposed algorithm with gray level estimation

procedure is applied both the DART and the TvMin+DART algorithms simultaneously, meaning that the free pixels, which are selected with $|1-p|$ probability, are determined once and used for both algorithms. Also, the number of the initial estimation iterations t_0 for the TvMin and the SART are both fixed to 3, like the subsequent SART iterations (t) which are also set to 3. For each cases, $k = 150$ iterations are used, which is enough to converge.

In Fig. 2a and Fig. 2b, limited number of projections and limited view experiments are considered, respectively. In the former experiment, we sampled the projections at equidistant intervals in full range $[0, \pi)$. In the later one, we gradually narrowed the range down, from $[0, \pi)$ to $[4\pi/9, 5\pi/9)$, and sampled the projections at 1 degree intervals for each range. All experiments except the one shown in Fig. 2c are done using noise-free measurements. In Fig. 2c, the robustness of the algorithms with respect to noise are compared. To simulate a noisy projection, the projection samples are polluted by adding a noise vector on them, as $\tilde{b} = b + \eta \|b\| \frac{e}{\|e\|}$ where e is a random noise vector, $\tilde{b}$ denotes the noisy projection measurements and η is the level of noise.

In Fig. 3, the result of the gray level estimation experiment is provided. Final reconstruction, shown in Fig 3b, has the labels which are very close to the exact gray levels. We started with the label set $\{0.3, 0.7\}$ while the exact gray levels are $\{0, 1\}$, after the algorithm is iterated $k = 150$ times, the final labels were estimated as $\{0, 0.97\}$. The RMSE was measured as 0.088.

IV. CONCLUSION

In this paper, we have proposed an algorithm which extends the DART by using the TvMin technique to estimate the initial guess and selecting the threshold, which is optimal in the sense of TV regularization cost function, from a finite set of candidates, which are estimated by a histogram-based thresholding scheme. Furthermore, a gray level estimation formulation has been presented and tested. The results show that the proposed algorithm improves the DART algorithm, either slightly or significantly, in terms of accuracy in almost all incomplete projection experiments.

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