



Linear Programming with Unitary-Equivariant Constraints

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Abstract: Unitary equivariance is a natural symmetry that occurs in many contexts in physics and mathematics. Optimization problems with such symmetry can often be formulated as semidefinite programs for a d^{p+q} -dimensional matrix variable that commutes with $U^{\otimes p} \otimes \bar{U}^{\otimes q}$, for all $U \in U(d)$. Solving such problems naively can be prohibitively expensive even if $p + q$ is small but the local dimension d is large. We show that, under additional symmetry assumptions, this problem reduces to a linear program that can be solved in time that does not scale in d , and we provide a general framework to execute this reduction under different types of symmetries. The key ingredient of our method is a compact parametrization of the solution space by linear combinations of walled Brauer algebra diagrams. This parametrization requires the idempotents of a Gelfand–Tsetlin basis, which we obtain by adapting a general method inspired by the Okounkov–Vershik approach. To illustrate potential applications of our framework, we use several examples from quantum information: deciding the principal eigenvalue of a quantum state, quantum majority vote, asymmetric cloning and transformation of a black-box unitary. We also outline a possible route for extending our method to general unitary-equivariant semidefinite programs.

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It is only slightly overstating the case to say that physics is the study of symmetry.
Philip W. Anderson [[And72](#)]

1. Introduction

Symmetry plays a fundamental role in physics, particularly in quantum mechanics and particle physics, where thanks to Noether’s theorem symmetry is responsible for conservation laws. At the same time, symmetry is also the focus of a major area of mathematics—group theory, whose applications range from physics to biology, chemistry and the arts. “Symmetry argument” is a common problem solving technique both in physics and mathematics: observing the symmetries inherent to a problem often allows simplifying the problem. In the absence of symmetry, difficult problems can often be made more tractable by assuming some form of symmetry. Generally speaking, the more symmetry a problem has the more tractable it is.

1.1. Schur–Weyl duality. The unitary group of symmetries plays a special role in the context of quantum mechanics and quantum information, and, for systems consisting of many identical parts, the unitary symmetry becomes intertwined with permutational symmetry. This intimate connection between unitary and permutation groups acting on several identical quantum systems is expressed via the so-called *Schur–Weyl duality*. In its simplest form, it states that the two-qubit *singlet state* $|\psi^-\rangle := (|01\rangle - |10\rangle)/\sqrt{2}$ is the unique (up to a global phase) state that is invariant under identical local unitary rotations: $(U \otimes U)|\psi^-\rangle \sim |\psi^-\rangle$ for any $U \in U(2)$, as well as the unique anti-symmetric state: $\text{SWAP}|\psi^-\rangle = -|\psi^-\rangle$. A similar dual characterization in terms of unitary and permutational symmetries applies not only to $|\psi^-\rangle$ but also its orthogonal complement, allowing to decompose the whole two-qubit space $\mathbb{C}^2 \otimes \mathbb{C}^2$ into invariant subspaces. This duality extends also to $(\mathbb{C}^d)^{\otimes p}$ where each of the p systems has dimension d .

1.2. Applications of Schur–Weyl duality in quantum computing. A natural situation where Schur–Weyl duality arises in quantum computing is when dealing with many copies of some quantum state ρ . The total state $\rho^{\otimes p}$ is then invariant under permutations and transforms in a straightforward way under simultaneous unitary basis change on each of the p systems. This scenario is very common in quantum information theory where Schur–Weyl duality has become an important tool [Har05, Bot16]. Its algorithmic manifestations, weak Schur sampling [CHW07] and quantum Schur transform [Har05, BCH06, KS18, Kro19], play a similarly important role in quantum algorithm design [Wri16]. Specific quantum algorithmic tasks where Schur–Weyl duality is used include quantum spectrum [KW01a] and entropy [Ach20] estimation, quantum spectrum testing [OW15], state tomography [Key06, Haa17, OW16, OW17], and quantum majority vote [Buh22].

1.3. Group-covariant quantum channels. Transformations that preserve symmetry are common in quantum computing. A quantum channel Φ is called *G-covariant*, for some group G , if there exist two unitary representations of G , ϕ_{in} and ϕ_{out} , such that $\Phi(\phi_{\text{in}}(g) \rho \phi_{\text{in}}(g)^\dagger) = \phi_{\text{out}}(g) \Phi(\rho) \phi_{\text{out}}(g)^\dagger$, for any group element $g \in G$ and state ρ . The structure of group-covariant quantum channels can be much simpler than the structure of general channels [MSD17]. For example, while perfect universal programming of general quantum channels is impossible [NC97], covariant channels can be programmed [GBW21], even in infinite dimensions [GW21].

1.4. Group covariance in different contexts. Group covariance is important in many contexts. Let us briefly illustrate two that are less obvious: quantum error correction and machine learning.

1.4.1. Group covariance in quantum error correction Group covariance is particularly important in the context of quantum error correction and fault tolerance. Many quantum error correcting codes are covariant with respect to the Clifford group and thus allow for simple or so-called “transversal” implementation of Clifford gates. An even higher degree of symmetry, namely possessing a universal set of transversal gates, would be ideal for devising schemes that can manipulate encoded quantum data. However, codes with such continuous symmetries are ruled out by the well-known Eastin–Knill theorem [EK09]. The interplay between continuous symmetries and quantum error correction has received revived attention in the context of holography and quantum gravity where an approximate version of the Eastin–Knill theorem was recently established [Fai20]. Group-covariant quantum codes with continuous symmetries are also closely related to the notion of a quantum reference frame [Hay21, Yan22].

1.4.2. Equivariance in machine learning A special case of group covariance is *equivariance*, which means that the representations ϕ_{in} and ϕ_{out} are either identical or related to each other in some simple way. Intuitively, equivariance says that applying some transformation on the input is equivalent to applying the same transformation on the output. This is a natural condition that occurs in many contexts. For example, in machine learning, the structure of neural networks should respect the symmetries of the problem at hand, such as translations and rotations when dealing with images. Such group-equivariant neural networks can have substantially increased expressive capacity without the need to increase their number of parameters [Coh21, CW16, Bro21]. In particular, unitary-equivariant neural networks that capture symmetries of many-body quantum systems have recently found applications to quantum chemistry [Qia22]. More generally, coordinate-independent convolutional networks on Riemannian manifolds require equivariance under local gauge transformations [Wei21].

In quantum machine learning, group-equivariant convolutional quantum circuits have been proposed for speeding up learning of quantum states [Zhe21]. In general, equivariant gatesets can be used to exploit symmetry in variational quantum algorithms [Mey23]. A general framework for group-invariant and equivariant quantum machine learning was recently outlined in [Lar22].

1.5. Local unitary equivariance. A particularly natural special case of group covariance is *local unitary equivariance* which corresponds to the case when Φ is a quantum channel from p to q systems, each of dimension d , the symmetry group G is the full unitary group $U(d)$, and the two unitary representations are given by $\phi_p(U) := U^{\otimes p}$ and $\phi_q(U) := U^{\otimes q}$. In other words, applying the same unitary $U \in U(d)$ on each of the p input systems of Φ is equivalent to applying U on each of the q output systems. This is concisely captured by the condition that the Choi matrix X^Φ of Φ obeys the symmetry $[X^\Phi, U^{\otimes p} \otimes \bar{U}^{\otimes q}] = 0$, for all $U \in U(d)$.

The classic Schur–Weyl duality mentioned in Sect. 1.1 admits various generalizations [Ber12, MS14, Ben96, Ben94, Dot08]. In particular, the above setting is sometimes referred to as *mixed Schur–Weyl duality* since X^Φ can be thought of as a “mixed” $(p+q)$ -tensor with two types of indices [Hal96, Nik07]. The regular Schur–Weyl duality is then

the special case when either $p = 0$ or $q = 0$, namely when there are either no input systems (such as in state preparation) or no output systems (such as in quantum measurement). We are interested in the general case of arbitrary p, q and local dimension d .

The most common scenario is when either $p = 1$ or $q = 1$. Such symmetry naturally occurs in many quantum tasks with a single input or a single output system, such as asymmetric quantum cloning [Cer00,NPR21], port-based teleportation [IH08,Moz18,Stu17,Led22,Chr21,SMK22,SS21], quantum majority vote [Buh22], or $U(d)$ -covariant quantum error correction [KL21,KL22]. It also occurs in situations that involve a partial transpose on a single system, such as in entanglement detection [EW01,COS18,Hub21].

Symmetries with general values of $p > 1$ and $q > 1$ correspond to scenarios with multiple input and output systems [Key02, Section 7], such as quantum state purification [KW01b] and cloning [Sca05,Fan14], multi-port-based teleportation [Kop21,Stu22,MSK21]. Such symmetries also occur in situations that involve the partial transpose on several systems, such as in extendability problem [JV13,JSZ22], entanglement detection [BCS20,BSH21], or universality of qudit gate sets [SMZ22,DS22,SS23]. Universality of quantum circuits with two-local $U(d)$ -equivariant gates has recently been considered in [Mar22,HLM21,MLH22] from the perspective of conservation laws. Finally, this class of symmetries is of independent interest also in high-energy physics [KR07,Can11] and the study of quantum spin systems [Rya21,BRR22].

1.6. Optimization subject to local unitary equivariance. Taking problem’s symmetry into account is a good idea for almost any problem, including optimization problems, since this can significantly reduce the number of parameters. In particular, this is the case in semidefinite optimization [Bac12,RMB21]. Given the wide range of problems in quantum information with local unitary equivariance symmetry, the main focus of our work is on linear and semidefinite optimization under a local (p, q) -unitary equivariance constraint. Typically this means optimizing over unitary-equivariant quantum channels or other tensors with this symmetry.

An early example of using $U \otimes \bar{U}$ symmetry to reduce a semidefinite optimization problem to a linear one is the work of Rains [Rai01] on entanglement distillation under completely positive-partial-transpose preserving operations. He characterizes the optimal distillation fidelity by a semidefinite program and then exploits the symmetry

$$(U \otimes \bar{U}) \sum_i |i\rangle \otimes |i\rangle = \sum_i |i\rangle \otimes |i\rangle \tag{1}$$

of the canonical maximally entangled state to reduce this to a linear program (see Example 3.9 for more details). This work has inspired a long sequence of results [APE03,LM15,Wan18,WW20]. However, generalizing them to PPT-extendible channels [HSW23] requires taking advantage of more complicated symmetries.

The closely related $U \otimes U$ symmetry appears in the so-called *quantum max cut* problem that has recently received significant attention [AGM20,Hwa21,PT21, KP22, PT22, Lee22, Kin22]. This problem is concerned with approximating the ground state and ground energy of a local Hamiltonian on a graph whose vertices are qubits and edges are assigned the projector $|\psi^-\rangle\langle\psi^-|$ onto the singlet state $|\psi^-\rangle := (|01\rangle - |10\rangle)/\sqrt{2}$ (see Sect. 1.1). The $U \otimes U$ symmetry of $|\psi^-\rangle$ is one of the basic cases captured by our framework (see Example 3.2), while the general case of $U^{\otimes p}$ is captured by Schur–Weyl duality (see Sect. 3.1).

Other instances of semidefinite optimization problems with $U^{\otimes p} \otimes \bar{U}^{\otimes q}$ symmetry appearing in quantum computing are quantum majority vote and basis-independent evaluation of Boolean functions [Buh22], black-box transformations of quantum gates [Qui19b, Qui19a, QE22, YSM23, YSM22, Ebl22], asymmetric cloning [NPR21], and entanglement witnesses [Hub22].

Previously each of these problems has been approached individually and by ad hoc methods that work only for restricted choices of p, q, d , such as $p = 1, q = 1$, or $d = 2$. Our goal is to provide a general framework for solving unitary-equivariant semidefinite optimization problems for a wide range of values of p, q, d . More specifically, our aim is to answer the following two questions:

- (1) *How to efficiently eliminate the irrelevant degrees of freedom from a $U(d)$ -equivariant optimization problem?*
- (2) *Can a $U(d)$ -equivariant optimization problem be solved in time that does not scale in d ?*

We provide answers to these questions using representation-theoretic and diagrammatic methods.

1.7. Summary of our main result. Consider the following general semidefinite program (SDP) for a Hermitian matrix variable X :

$$\begin{aligned}
 & \max_X \quad \text{Tr}(CX) \\
 & \text{s.t.} \quad \text{Tr}(A_k X) \leq b_k, \quad \forall k \in [m_1], \\
 & \quad \quad \text{Tr}_{S_k}(X) = D_k, \quad \forall k \in [m_2], \\
 & \quad \quad [X, U^{\otimes p} \otimes \bar{U}^{\otimes q}] = 0, \quad \forall U \in U(\mathbb{C}^d), \\
 & \quad \quad X \geq 0,
 \end{aligned} \tag{2}$$

where C, A_k, D_k are fixed Hermitian matrices, $b_k \in \mathbb{R}$ are fixed scalars, m_1 and m_2 denote the number of constraints that involve full trace and partial trace, and $S_k \subseteq [p+q]$ denote subsets of systems that are traced out.

Note that all matrices in eq. (2) are of size $d^{p+q} \times d^{p+q}$. For this problem to have an efficient description, we assume that C, A_k, D_k are s -sparse, i.e., can be specified as a linear combination of at most s walled Brauer algebra diagrams (see Sect. 3.2) and elementary rank-1 matrices $|i\rangle\langle j|$ where $i, j \in [d]^{p+q}$. Our main result, Theorem 6.4, provides an efficient way of converting the above semidefinite program to an equivalent linear program (LP) when the matrix variable X is subject to one of the following additional symmetries: $S_p \times S_q$ symmetry, walled Brauer algebra symmetry, or Gelfand–Tsetlin symmetry (see Sect. 6.1 for more details).

Theorem (Informal). *Assuming one of the above additional symmetries on X , the SDP (2) can be converted to an equivalent LP with $n \leq N$ variables and $m_1 + m_2 N + n$ constraints where $N := (p + q)!$.*

Our approach has the advantage that d can be arbitrary¹ and the computational resources are tied only to the value of $p + q$. For example, let $(p, q) = (2, 3)$ be small constants and $d = 1000$ be very large. In this regime, naively solving the above SDP is

¹ In fact, d can even be symbolic!

impossible since it has $d^{2(p+q)} = 1000^{10} = 10^{30}$ scalar variables. However, the complexity of our method scales only in the parameter $N = (p + q)!$, which in this case is $(2 + 3)! = 5! = 120$.

Our method can provide an advantage also when d is small. For example, if $d = 2$ then $d^{2(p+q)} = 2^{2(2+3)} = 2^{10} = 1024$ variables are needed naively while only 42 suffice with our method (see Table 7 in Appendix E).

Even though our method provides a significant improvement in terms of d , its complexity still scales as $(p + q)!$, so in practice we can only deal with relatively small values of p and q . However, numerical solutions to small problem instances are still valuable since they may reveal structures that can help to tackle larger instances. For example, numerical insights can lead to a refined ansatz with fewer parameters that scales up more easily. Good ansatz, even if suboptimal, can still be used to obtain numerical bounds. If the ansatz is sufficiently simple, it might even be amenable to analytic methods. In this way, our method can potentially be used to bootstrap from small problem instances to much larger ones.

We have implemented our method and made our code available [GO22]. To illustrate potential applications of our method, in Sect. 7 we provide several examples from quantum information: deciding the principal eigenvalue of a quantum state, quantum majority vote, asymmetric cloning and transformation of a black-box unitary operation. We have implemented the calculations for these examples as *Wolfram Mathematica* notebooks that are also available [GO22].

1.8. Intuition. The main idea behind our result is as follows. While the naive semidefinite program (2) has $d^{2(p+q)}$ scalar variables, the unitary equivariance condition alone reduces this down to $\dim(\mathcal{A}_{p,q}^d)$, where $\mathcal{A}_{p,q}^d$ is the matrix algebra of partially transposed permutations. This observation already gives us a d -independent upper bound on the number of parameters since $\dim(\mathcal{A}_{p,q}^d) \leq (p + q)!$. While generally this bound is loose for small d , it saturates when $d \geq p + q$.

This reduction in the number of parameters happens thanks to a generalized or “mixed” Schur–Weyl duality (see Sect. 3). Together with Schur’s lemma, this duality implies that any unitary-equivariant matrix variable X can be written as

$$X \cong \bigoplus_{\lambda \in \text{Irr}(\mathcal{A}_{p,q}^d)} [X_\lambda \otimes I_{m_\lambda}] \tag{3}$$

in some basis, where the size of each block X_λ is independent of d . The main difficulty then lies in obtaining an ansatz that captures all relevant degrees of freedom for such X , in a way that does not scale in the local dimension d . In particular, we cannot afford to apply the “mixed” Schur transform that implements the basis change in eq. (3) since the underlying space has dimension d^{p+q} .

For simplicity, in this work, we only consider the special case when each X_λ is diagonal. This assumption is justified when X is subject to some additional symmetry (see Sect. 6.1 for more details). For example, under certain symmetry X is diagonal in the so-called Gelfand–Tsetlin basis and can thus be written as a linear combination of primitive idempotents of the partially transposed permutation algebra $\mathcal{A}_{p,q}^d$. Since this algebra is closely related to the walled Brauer algebra $\mathcal{B}_{p,q}^d$, which is multiplicity-free, we can adapt a general framework of [DLS18] to compute its idempotents. A crucial ingredient in this computation are so-called Jucys–Murphy elements. In Theorem 5.6, which is our main technical contribution, we show that these elements can be lifted

from $\mathcal{B}_{p,q}^d$ to $\mathcal{A}_{p,q}^d$. This allows us to perform the entire computation within the walled Brauer algebra $\mathcal{B}_{p,q}^d$. In contrast to $\mathcal{A}_{p,q}^d$, $\mathcal{B}_{p,q}^d$ is diagrammatic and hence we do not need to manipulate any matrices of size d^{p+q} . This is precisely why the complexity of our approach does not scale in d . In particular, we do not require explicit knowledge of the mixed Schur transform.

1.9. Open problems. Our work raises several interesting open questions.

- (1) Derive an explicit basis in which our ansatz for X is diagonal under the $S_p \times S_q$ symmetry when $d = 2$ or $\min(p, q) = 2$.
- (2) We believe that our method can be extended to solve general unitary-equivariant SDPs. This requires removing the additional symmetry assumption that reduces such SDP to an LP. This could be done along the lines outlined in Appendix D, but it remains to formally verify this.
- (3) What is the role of representation theory in our approach? Could there be a more direct way of doing this without the use of representation theory?
- (4) Characterize the kernel of the map $\psi_{p,q}^d$ defined in eq. (41) that embeds the walled Brauer algebra $\mathcal{B}_{p,q}^d$ into the partially-transposed permutation matrix algebra $\mathcal{A}_{p,q}^d$. This would allow removing the technical restriction (118) in Theorem 6.4 that d must be sufficiently large. Moreover, this could also provide additional speedups for small d since all calculations could be performed using a linearly independent diagram basis of size much smaller than $(p + q)!$. One possible method for computing $\ker \psi_{p,q}^d$ is sketched in Remark 6.3.
- (5) Given a solution X of a unitary-equivariant SDP that describes a Choi matrix, we would like to find an efficient quantum circuit that implements the corresponding quantum channel. Note that [Buh22] achieves this when $d = 2$, $p = 2k + 1$ and $q = 1$. An efficient implementation of the mixed Schur transform $U_{\text{Sch}(p,q)}$ (see Remark 3.10) for general values of p, q, d would be a useful subroutine for achieving the general case.
- (6) It should be possible to treat the local dimension d symbolically and deduce the asymptotic scaling of the solution as $d \rightarrow \infty$.
- (7) The applications we provide in Sect. 7 are only for illustrative purposes. We expect that one should be able to go much further by bearing the full weight of our method. For example, concerning the application in Sect. 7.4, we would like to find, for any given d , how many copies of a black-box unitary U are needed to implement $f(U)$ exactly and deterministically via a sequential superchannel.
- (8) Our approach is an instance of a general philosophy outlined in [Bac12] for solving SDPs with $*$ -matrix algebra symmetries. Other instances of this setting are also useful in quantum information [GNW21] and hence worthwhile investigating.

1.10. Outline. Our paper is structured as follows. In Sect. 2, we introduce the main preliminary concepts, such as quantum channels and their symmetries, and semidefinite programming. In Sect. 3, we discuss Schur–Weyl duality and its generalization to mixed tensors. Here we also introduce walled Brauer algebras and their matrix analogs — partially transposed permutation algebras—that play a central role in mixed Schur–Weyl duality. In Sect. 4, we review a general construction of primitive central idempotents for any multiplicity-free family of algebras due to [DLS18]. In Sect. 5, we specialize this construction to walled Brauer algebras and adapt it to their matrix analogues (which

can be of independent interest). Finally, in Sect. 6 we derive our main result—a general meta-algorithm for reducing unitary-equivariant semidefinite programs to significantly smaller linear programs. This reduction is possible under certain natural permutation symmetry assumptions. We conclude in Sect. 7 by providing several applications of our framework to quantum computing.

2. Preliminaries

Throughout the paper, we fix a local dimension $d \geq 2$ and let $V := \mathbb{C}^d$. For any integers $p, q \geq 0$, let $V^p := V^{\otimes p}$ denote the p -fold tensor product of V , and let $V^{p,q} := V^{\otimes p} \otimes V^{*\otimes q}$ denote the *mixed* tensor product where V^* is the dual space of V . While $V^* \cong V = \text{span}_{\mathbb{C}}\{|1\rangle, \dots, |d\rangle\}$, we make the distinction to emphasize that V^* is subject to the *dual* action (see below).

Let $\text{End}(V)$ denote the set of all complex $d \times d$ matrices. We denote by $U(V)$ all *unitary* matrices on V , i.e., all $U \in \text{End}(V)$ such that $U^\dagger U = I_V$, where $U^\dagger := \bar{U}^T$ denotes the conjugate transpose of U and I_V is the identity matrix on V . The *dual action* of $U \in U(d)$ on V^* is given by the entry-wise complex conjugate \bar{U} .

We denote by $D(V)$ the set of all *density matrices* on V , i.e., all matrices $\rho \in \text{End}(V)$ such that $\rho \geq 0$ (ρ is *positive semidefinite*) and $\text{Tr}(\rho) := \sum_{i=1}^d \langle i | \rho | i \rangle = 1$.

A *representation* of a group G on a complex vector space V is a homomorphism $R: G \rightarrow U(V)$, i.e., $R(gh) = R(g)R(h)$ for all $g, h \in G$. The representation is *faithful* if R is an injection, i.e., different group elements are represented by different matrices. A representation R is *reducible*² if there exists a basis change $U \in U(V)$ and two representations R_1 and R_2 of dimension at least one such that $UR(g)U^\dagger = R_1(g) \oplus R_2(g)$ for all $g \in G$. Otherwise R is called *irreducible*.

An *algebra* is a vector space equipped with a bilinear product. If G is a group, its corresponding *group algebra* $\mathbb{C}G := \text{span}_{\mathbb{C}}\{|g\rangle : g \in G\}$ is an algebra that extends the group operation of G by linearity: $|g\rangle \cdot |h\rangle = |gh\rangle$, for all $g, h \in G$. If V is a vector space, a *matrix algebra* on V is a linear subspace of $\text{End}(V)$ that is closed under matrix multiplication. The *centralizer* of a matrix algebra \mathcal{A} in $\text{End}(V)$ is the set of all matrices acting on V that commute with \mathcal{A} :

$$\text{End}_{\mathcal{A}}(V) := \{B \in \text{End}(V) : [A, B] = 0 \text{ for every } A \in \mathcal{A}\}, \tag{4}$$

where $[A, B] := AB - BA$ denotes the *commutator* of A and B . For more background on algebras and their representations see Sect. 4.1.

For any integer $n \geq 1$, we let $[n] := \{1, \dots, n\}$. We write $\lambda \vdash p$ to mean that λ is a *partition* of an integer $p \geq 0$, i.e., $\lambda = (\lambda_1, \dots, \lambda_k)$ is a tuple of k integers, for some $k \geq 1$, such that $\lambda_1 \geq \dots \geq \lambda_k \geq 0$ and $\lambda_1 + \dots + \lambda_k = p$. We also think of λ as a *Young diagram*, i.e., a collection of p square cells arranged in k rows with λ_i of them in the i -th row. For example,



$$\tag{5}$$

² Technically such representation is called *decomposable* and the notion of irreducibility is weaker (this distinction will become important when we consider representations of algebras instead of groups, see Sect. 4.1). However, in the context of group representations the two notions are equivalent since all representations we consider are unitary.

depicts the partition $(5, 3, 1)$. We denote by $\text{len}(\lambda) := k$ the *length* of λ , i.e., the length of the first column (or the number of rows). We denote by $\text{size}(\lambda) := p$ the *size* of $\lambda \vdash p$, i.e., the number of cells in the Young diagram. The *content* of cell $(i, j) \in \lambda$ is $j - i$, where i denotes the row and j denotes the column of the corresponding cell. The *content* of partition λ is the total content of all its cells: $\text{cont}(\lambda) := \sum_{(i,j) \in \lambda} (j - i)$, where the sum runs over all cells in the diagram.

2.1. Quantum channels. Let $V_{\text{in}} := \mathbb{C}^{d_{\text{in}}}$ and $V_{\text{out}} := \mathbb{C}^{d_{\text{out}}}$ be finite-dimensional complex Euclidean spaces. A *quantum channel* $\Phi: \text{End}(V_{\text{in}}) \rightarrow \text{End}(V_{\text{out}})$ is a completely positive and trace-preserving linear map. *Complete positivity* means that, for any reference space V_{ref} and state $\rho \in \text{D}(V_{\text{in}} \otimes V_{\text{ref}})$, we have $(\Phi \otimes I_{\text{ref}})(\rho) \succeq 0$ where I_{ref} denotes the identity channel on V_{ref} . *Trace preservation* means that $\text{Tr}(\Phi(\rho)) = \text{Tr}(\rho)$, for all $\rho \in \text{D}(V_{\text{in}})$.

A convenient way of representing a quantum channel is by its *Choi matrix* $X^\Phi \in \text{End}(V_{\text{in}} \otimes V_{\text{out}})$ defined as

$$X^\Phi := \sum_{i,j=1}^{d_{\text{in}}} |i\rangle\langle j| \otimes \Phi(|i\rangle\langle j|) \tag{6}$$

where $\{|1\rangle, \dots, |d_{\text{in}}\rangle\}$ is an orthonormal basis for V_{in} . The action of Φ on $\rho \in \text{D}(V_{\text{in}})$ can be recovered from its Choi matrix X^Φ as follows:

$$\Phi(\rho) = \text{Tr}_{V_{\text{in}}}[X^\Phi(\rho^T \otimes I_{\text{out}})]. \tag{7}$$

A given matrix $X \in \text{End}(V_{\text{in}} \otimes V_{\text{out}})$ describes a quantum channel if and only if

$$X \succeq 0, \quad \text{Tr}_{V_{\text{out}}}(X) = I_{\text{in}}. \tag{8}$$

For more background on quantum information theory see [Wat18].

2.2. Unitary equivariance. Our main motivating problem is the optimization of a linear function over unitary-equivariant quantum channels. Recall that $V := \mathbb{C}^d$ for some $d \geq 2$ and $V^p := V^{\otimes p}$.

Definition 2.1. We say that $\Phi: \text{End}(V^p) \rightarrow \text{End}(V^q)$ is a $p \rightarrow q$ *channel*. Such channel is *locally U(V)-equivariant* or simply *unitary-equivariant* if

$$\Phi(U^{\otimes p} \rho U^{\dagger \otimes p}) = U^{\otimes q} \Phi(\rho) U^{\dagger \otimes q} \tag{9}$$

for every $U \in \text{U}(V)$ and $\rho \in \text{D}(V^p)$.

To optimize over such channels, we need to understand their structure or, equivalently, the structure of the associated Choi matrices. The following is a well-known characterization of the unitary equivariance of Φ in terms of its Choi matrix.

Proposition 2.2. Let $X^\Phi \in \text{End}(V^{p,q})$ be the Choi matrix of a $p \rightarrow q$ channel Φ . Then Φ is unitary-equivariant if and only if

$$[X^\Phi, U^{\otimes p} \otimes \bar{U}^{\otimes q}] = 0, \quad \forall U \in \text{U}(V). \tag{10}$$

Note that the last q registers of X^Φ are subject to the dual action of $\text{U}(V)$.

2.3. *Semidefinite and linear programming.* *Semidefinite programming* is an important subfield of optimization [WSV12] that has numerous applications in quantum information theory [ST21, Wat18]. A typical formulation of a *semidefinite program* (SDP) has the form [WSV12, Section 1.1]:

$$\begin{aligned} \max_X \quad & \text{Tr}(C^\top X) \\ \text{s.t.} \quad & \text{Tr}(A_i^\top X) = b_i, \quad \forall i \in [m], \\ & X \succeq 0, \end{aligned} \tag{11}$$

where X is a Hermitian matrix variable, C and A_i are constant Hermitian matrices, and b_i are real constants. A special case of SDPs are *linear programs* (LPs) which correspond to the case when all matrices involved are diagonal. Any LP can be formulated in the standard form

$$\begin{aligned} \max_x \quad & c^\top x \\ \text{s.t.} \quad & a_i^\top x = b_i, \quad \forall i \in [m], \\ & x \geq 0, \end{aligned} \tag{12}$$

where x is a real vector variable, c and a_i are constant real vectors, and b_i are real constants. In practice LPs are much faster to solve than SDPs. Therefore being able to reduce a given SDP to an LP under additional symmetry assumptions is often desirable.

2.4. *Motivating problem.* Consider the following motivating semidefinite optimization problem. Fix a quantum state $\rho \in \mathcal{D}(V^p)$ and an arbitrary Hermitian matrix $H \in \text{End}(V^q)$, and assume we want to find a unitary-equivariant $p \rightarrow q$ channel Φ that maximizes the linear function $\text{Tr}[\Phi(\rho)H]$. For example, if $H = |\psi\rangle\langle\psi|$ for some pure state $|\psi\rangle \in V^q$ then $\text{Tr}[\Phi(\rho)H] = \langle\psi|\Phi(\rho)|\psi\rangle$ is the fidelity between the output state $\Phi(\rho)$ and the desired target state $|\psi\rangle$.

According to eq. (7), $\text{Tr}[\Phi(\rho)H] = \text{Tr}[X^\Phi(\rho^\top \otimes H)]$, so the constraints on X^Φ from eq. (8) give us the following SDP:

$$\max_{X^\Phi \in \text{End}(V^{p,q})} \text{Tr}[X^\Phi(\rho^\top \otimes H)], \quad \text{Tr}_{V^q}(X^\Phi) = I_{V^p}, \quad X^\Phi \succeq 0. \tag{13}$$

This problem has a trivial solution—a channel that ignores its input ρ and prepares the principal eigenvector of H as output. To make the problem non-trivial, consider instead a collection of n pairs (ρ_i, H_i) , with the goal of maximizing the smallest value of $\text{Tr}[\Phi(\rho_i)H_i]$, $i = 1, \dots, n$. This is captured by the following SDP:

$$\max_{\substack{X^\Phi \in \text{End}(V^{p,q}) \\ c \in \mathbb{R}}} c, \quad \forall i: \text{Tr}[X^\Phi(\rho_i^\top \otimes H_i)] \geq c, \quad \text{Tr}_{V^q}(X^\Phi) = I_{V^p}, \quad X^\Phi \succeq 0. \tag{14}$$

This problem no longer admits a trivial solution where Φ ignores the input state.

Motivated by applications to problems mentioned in Sect. 1.6, we would like to incorporate the unitary-equivariance constraint (9) on the channel Φ in the SDPs (13) and (14). Note that using Proposition 2.2 in a naive way would result in an optimization problem with an uncountable number of linear constraints and a matrix variable of dimension d^{p+q} that scales badly in d even for constant p and q . Our main contribution

is an efficient method that can deal with both of these issues simultaneously. Under additional symmetry assumptions, it reduces the above SDPs to finite linear programs whose size does not scale in d (we sketch in Appendix D a possible way to remove the symmetry assumption).

2.5. Optimization under unitary equivariance. Symmetry is a powerful tool for simplifying almost any type of problem, including problems in semidefinite optimization [Bac12,RMB21]. Our goal is to investigate semidefinite optimization for a matrix variable subject to a unitary equivariance constraint. To simplify the problem even further, we assume one of several additional symmetries that reduce the semidefinite program to a linear program.

A naive way of imposing unitary equivariance on Φ in our motivating problem in eq. (14) is by including eq. (10) as an extra linear constraint. However, this technically constitutes an uncountably infinite set of constraints. To get around this issue, we could instead demand that

$$\int_{U \in \text{U}(d)} (U^{\otimes p} \otimes \bar{U}^{\otimes q}) X^\Phi (U^{\otimes p} \otimes \bar{U}^{\otimes q})^\dagger dU = X^\Phi \tag{15}$$

where dU denotes the Haar measure on $\text{U}(d)$. This integral can in principle be evaluated using Weingarten calculus [CMN21,CS06], producing a single linear constraint on X^Φ . The resulting SDP has finite size and can be supplied to a standard solver.

However, there is another serious issue that can prevent the SDP from being solvable in practice. Namely, X^Φ is a matrix of size $d^{p+q} \times d^{p+q}$. Since each matrix entry of X^Φ is represented by a separate scalar variable in the SDP, the total number of variables is $d^{2(p+q)}$, which is prohibitive even for moderate values of d .

Motivated by this issue, our main goal is to understand whether optimization problems with a $\text{U}(d)$ -equivariance constraint can be solved in time that does not scale in d . In this work, we focus on linear programming as a special case of semidefinite programming and answer the above question in the affirmative (see Theorem 6.4 for our main result).

In the context of unitary-equivariant channels, linear programs occur naturally when additional symmetries are imposed on Φ . Indeed, appropriately chosen symmetries guarantee that the Choi matrix X^Φ is diagonal in a certain basis, allowing the semidefinite constraint $X^\Phi \geq 0$ to be replaced by scalar inequalities.

2.6. Additional symmetries. One natural example of additional $p \rightarrow q$ channel symmetries is invariance under permutations of the p input and q output systems, where each type of system is permuted separately.

The symmetric group S_p on p elements acts naturally on p qudits by permuting them. This can be captured by a representation $\psi_p^d: S_p \rightarrow \text{End}(V^p)$ defined on simple tensors $|i_1\rangle \otimes \dots \otimes |i_p\rangle$ with $i_1, \dots, i_p \in \{1, \dots, d\}$ as

$$\psi_p^d(\pi)(|i_1\rangle \otimes \dots \otimes |i_p\rangle) := |i_{\pi^{-1}(1)}\rangle \otimes \dots \otimes |i_{\pi^{-1}(p)}\rangle, \tag{16}$$

for all $\pi \in S_p$, and extended linearly to all vectors in V^p .

Definition 2.3. A $p \rightarrow q$ channel Φ is *input-symmetric* if $\Phi(\psi_p^d(\pi) \rho \psi_p^d(\pi)^\dagger) = \Phi(\rho)$, for every $\rho \in \text{D}(V^p)$ and $\pi \in S_p$. Similarly, Φ is *output-symmetric* if $\Phi(\rho) =$

$\psi_q^d(\sigma) \Phi(\rho) \psi_q^d(\sigma)^\dagger$, for every $\rho \in D(V^p)$ and $\sigma \in S_q$. We call Φ *symmetric* if it is both input- and output-symmetric:

$$\Phi(\psi_p^d(\pi) \rho \psi_p^d(\pi)^\dagger) = \psi_q^d(\sigma) \Phi(\rho) \psi_q^d(\sigma)^\dagger. \tag{17}$$

if for every $\rho \in D(V^p)$ and every pair of permutations $(\pi, \sigma) \in S_p \times S_q$.

Similar to Proposition 2.2, this symmetry of Φ can also be expressed in terms of its Choi matrix X^Φ .

Proposition 2.4. *A $p \rightarrow q$ channel Φ is symmetric if and only if its Choi matrix X^Φ satisfies*

$$[X^\Phi, \psi_p^d(\pi) \otimes \psi_q^d(\sigma)] = 0, \quad \forall (\pi, \sigma) \in S_p \times S_q. \tag{18}$$

In Sect. 6.1 we consider two additional types of symmetries and discuss when a unitary-equivariant SDP reduces to an LP under such symmetries.

The structure of unitary-equivariant quantum channels can be described using a generalization of Schur–Weyl duality to mixed tensor products.

3. Mixed Schur–Weyl Duality

To understand the interplay between unitary equivariance and permutational or other symmetries, it is very useful to know about the relationship between their representations. Indeed, the irreducible representations of $U(d)$ and S_p on the p -fold tensor product space $V^p = (\mathbb{C}^d)^{\otimes p}$ are intimately related because the actions of $U^{\otimes p}$, $U \in U(d)$ and $\psi_p^d(\pi)$, $\pi \in S_p$ on this space mutually commute. This relationship, known as *Schur–Weyl duality*, is an important tool in quantum information theory, see [Har05, Chapter 6] and [Bot16]. We will need a generalization of this duality to the *mixed* tensor product space $V^{p,q} = V^p \otimes V^{*q}$.

3.1. Schur–Weyl duality. Let us first review the classical Schur–Weyl duality on V^p where $V = \mathbb{C}^d$. For more background, see [Har05, Section 5.3].

A natural way for $U(d)$ to act on p qudits is by applying an identical unitary transformation on each of them. This is captured by a representation $\phi_p^d: U(d) \rightarrow \text{End}(V^p)$ defined as

$$\phi_p^d(U) := U^{\otimes p}, \tag{19}$$

for all $U \in U(d)$. Let us denote the algebra generated by the image of $U(d)$ under ϕ_p^d by

$$\mathcal{U}_p^d := \text{span}_{\mathbb{C}}\{\phi_p^d(U) : U \in U(d)\}. \tag{20}$$

Let $\mathbb{C}S_p := \text{span}_{\mathbb{C}} S_p$ denote the group algebra of the symmetric group. We extend the representation $\psi_p^d: S_p \rightarrow \text{End}(V^p)$ in eq. (16) from S_p to $\mathbb{C}S_p$ by linearity and denote the resulting algebra of p -qudit permutations by

$$\mathcal{A}_p^d := \psi_p^d(\mathbb{C}S_p). \tag{21}$$

Recall from eq. (4) that $\text{End}_{\mathcal{A}_p^d}(V^p)$ denotes the centralizer of \mathcal{A}_p^d in $\text{End}(V^p)$, i.e., all matrices in $\text{End}(V^p)$ that commute with \mathcal{A}_p^d . One way to state the Schur–Weyl duality is that \mathcal{U}_p^d is the centralizer of \mathcal{A}_p^d , and vice versa.

Theorem 3.1 (Schur–Weyl duality). *The algebra \mathcal{U}_p^d is the centraliser algebra of \mathcal{A}_p^d in $\text{End}(V^p)$ and vice versa, i.e.,*

$$\mathcal{U}_p^d = \text{End}_{\mathcal{A}_p^d}(V^p), \quad \mathcal{A}_p^d = \text{End}_{\mathcal{U}_p^d}(V^p). \tag{22}$$

Moreover, when $d \geq p$ the representation ψ_p^d is faithful, i.e., $\mathcal{A}_p^d \cong \mathbb{C}S_p$.

One of the simplest instances of this duality is in the case of two qubits.

Example 3.2. ($p = 2$ and $d = 2$) Since $S_2 = \{(), (12)\}$, the algebra \mathcal{A}_2^2 is generated by the identity matrix and SWAP:

$$\mathcal{A}_2^2 := \left\{ x \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + y \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : x, y \in \mathbb{C} \right\}, \tag{23}$$

whereas \mathcal{U}_2^2 is generated by tensor squares of unitary matrices:

$$\mathcal{U}_2^2 := \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{U}(2) \right\}. \tag{24}$$

Using the two-qubit Schur transform [Har05, p. 121]

$$U := \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & -1 & 0 \\ \sqrt{2} & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2} \end{pmatrix}, \tag{25}$$

we can simultaneously block-diagonalize both algebras:

$$U\mathcal{A}_2^2U^\top = \left\{ \left(\begin{array}{c|ccc} x-y & 0 & 0 & 0 \\ \hline 0 & x+y & 0 & 0 \\ 0 & 0 & x+y & 0 \\ 0 & 0 & 0 & x+y \end{array} \right) : x, y \in \mathbb{C} \right\}, \tag{26}$$

$$U\mathcal{U}_2^2U^\top = \text{span}_{\mathbb{C}} \left\{ \left(\begin{array}{c|ccc} ad-bc & 0 & 0 & 0 \\ \hline 0 & a^2 & \sqrt{2}ab & b^2 \\ 0 & \sqrt{2}ac & ad+bc & \sqrt{2}bd \\ 0 & c^2 & \sqrt{2}cd & d^2 \end{array} \right) : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{U}(2) \right\}. \tag{27}$$

These algebras centralize each other since

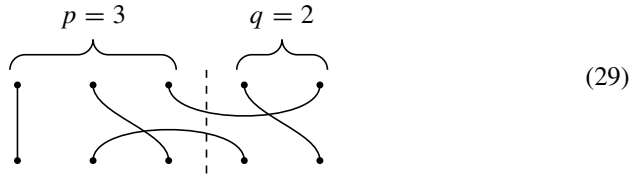
$$U\mathcal{A}_2^2U^\top = \mathbb{C} \oplus \mathbb{C}I_3, \quad U\mathcal{U}_2^2U^\top = \mathbb{C} \oplus \text{End}(\mathbb{C}^3), \tag{28}$$

where the second equality follows from Burnside’s theorem [LR04] since the 3×3 block in eq. (27) is irreducible.

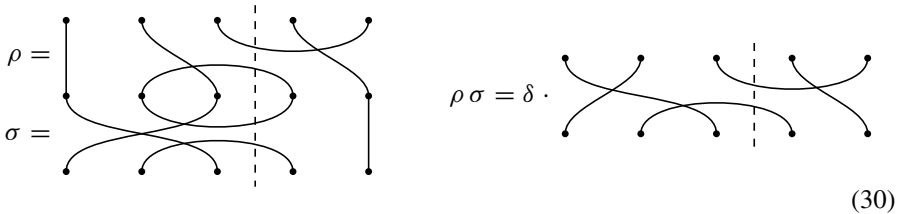
Example 3.3 (Unfaithfulness of ψ_3^d). It is important to observe that ψ_p^d has a non-trivial kernel when the local dimension d is smaller than p . For example, ψ_3^d ($\sum_{\sigma \in S_3} \text{sign}(\sigma)\sigma$) vanishes when $d = 2$ and does not vanish when $d \geq 3$. For this reason we will often have to make a distinction between small and large local dimensions d .

3.2. *Walled Brauer algebra.* To analyze the Choi matrix of unitary-equivariant quantum channel, we need a generalization of Schur–Weyl duality for the space $V^{p,q}$ of mixed tensor products. This generalization requires the notion of walled Brauer algebras [Tur89, Koi89, Ben94, Ben96, Nik07, Bul20], a restricted version of Brauer algebras (see [Koe08] for a survey on Brauer and other diagram algebras).

Let $p, q \geq 0$ and $\delta \in \mathbb{C}$. The *walled Brauer algebra* $\mathcal{B}_{p,q}^\delta$ consists of formal complex linear combinations of diagrams, where each diagram has two rows of $p + q$ nodes each, with a vertical “wall” between the first p and the last q nodes. These nodes are connected up in pairs, with each pair subject to the following restriction: if both nodes are in the same row, they must be on different sides of the wall, while if they are in different rows, they must be on the same side of the wall. For example, a diagram in $\mathcal{B}_{3,2}^\delta$ may look like this:

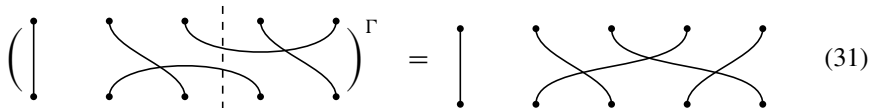


Multiplication in $\mathcal{B}_{p,q}^\delta$ corresponds to concatenation of diagrams, with the bottom row of the first diagram identified with the top row of the second diagram. If some loops appear in this process, one should erase them and multiply the resulting diagram with the scalar $\delta^{\#\text{loops}}$:



The group algebra of the permutation group $S_p \times S_q$ forms a subalgebra of the walled Brauer algebra $\mathcal{B}_{p,q}^\delta$ consisting only of those diagrams where no edge goes across the wall. In fact, the two algebras are isomorphic when $p = 0$ or $q = 0$, i.e., $\mathbb{C}S_p \cong \mathcal{B}_{p,0}^\delta \cong \mathcal{B}_{0,p}^\delta$ for any value of δ . The walled Brauer algebra $\mathcal{B}_{p,q}^\delta$ itself is a subalgebra of the full Brauer algebra \mathcal{B}_{p+q}^δ that is defined similarly but without the wall and with no restrictions on which pairs of nodes can be connected. This algebra was originally introduced by Brauer [Bra37] for studying Schur–Weyl-like dualities of orthogonal and symplectic groups.

For any diagram $\sigma \in \mathcal{B}_{p,q}^\delta$, the *partial transpose* σ^Γ is obtained by exchanging the last q nodes of both rows (i.e., the nodes on the right-hand side of the wall):



Note that $\sigma \in \mathcal{B}_{p,q}^\delta$ is a walled Brauer algebra diagram if and only if $\sigma^\Gamma \in S_{p+q}$. In particular, $\dim(\mathcal{B}_{p,q}^\delta) = (p + q)!$ since $(\mathcal{B}_{p,q}^\delta)^\Gamma = \mathbb{C}S_{p+q}$ as vector spaces.

The walled Brauer algebra $\mathcal{B}_{p,q}^\delta$ is generated by *transpositions* σ_i that swap the i -th and $(i + 1)$ -th node of the two rows, where $i \in \{1, \dots, p + q - 1\} \setminus \{p\}$, and a *contraction*

$\bar{\sigma}_p$ between the p -th and $(p + 1)$ -th node within each row. For example, $\mathcal{B}_{2,2}^\delta$ is generated by

$$\begin{aligned}
 \sigma_1 &= \begin{array}{c} \text{---} \cdot \quad \cdot \text{---} \\ \diagdown \quad \diagup \\ \cdot \quad \cdot \\ \diagup \quad \diagdown \\ \text{---} \cdot \quad \cdot \text{---} \end{array} \quad \begin{array}{c} | \\ | \\ | \\ | \end{array} \quad \begin{array}{c} | \\ | \\ | \\ | \end{array} \\
 \bar{\sigma}_2 &= \begin{array}{c} | \\ | \\ | \\ | \end{array} \quad \begin{array}{c} \text{---} \cdot \quad \cdot \text{---} \\ \text{---} \cdot \quad \cdot \text{---} \end{array} \quad \begin{array}{c} | \\ | \\ | \\ | \end{array} \\
 \sigma_3 &= \begin{array}{c} | \\ | \\ | \\ | \end{array} \quad \begin{array}{c} | \\ | \\ | \\ | \end{array} \quad \begin{array}{c} \text{---} \cdot \quad \cdot \text{---} \\ \diagdown \quad \diagup \\ \cdot \quad \cdot \\ \diagup \quad \diagdown \\ \text{---} \cdot \quad \cdot \text{---} \end{array}
 \end{aligned} \tag{32}$$

For completeness, below is an algebraic definition of the walled Brauer algebra $\mathcal{B}_{p,q}^\delta$ in terms of relations obeyed by the transpositions σ_i and contraction $\bar{\sigma}_p$. One can easily verify that the corresponding diagrams obey these relations, while the completeness of these relations is less obvious [Nik07].

Definition 3.4 ([BS12]). Let $p, q \geq 0$ and $\delta \in \mathbb{C}$. The walled Brauer algebra $\mathcal{B}_{p,q}^\delta$ is a finite associative algebra over \mathbb{C} with generators $\bar{\sigma}_p$ and σ_i , with $i \in \{1, \dots, p + q - 1\} \setminus \{p\}$, and the following relations between them:

$$\sigma_i^2 = 1, \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}, \quad \sigma_i \sigma_j = \sigma_j \sigma_i \quad (|i - j| > 1), \tag{33}$$

$$\bar{\sigma}_p^2 = \delta \bar{\sigma}_p, \quad \bar{\sigma}_p \sigma_{p \pm 1} \bar{\sigma}_p = \bar{\sigma}_p, \quad \bar{\sigma}_p \sigma_i = \sigma_i \bar{\sigma}_p \quad (i \neq p \pm 1), \tag{34}$$

$$\bar{\sigma}_p \sigma_{p+1} \sigma_{p-1} \bar{\sigma}_p \sigma_{p-1} = \bar{\sigma}_p \sigma_{p+1} \sigma_{p-1} \bar{\sigma}_p \sigma_{p+1}, \tag{35}$$

$$\sigma_{p-1} \bar{\sigma}_p \sigma_{p+1} \sigma_{p-1} \bar{\sigma}_p = \sigma_{p+1} \bar{\sigma}_p \sigma_{p+1} \sigma_{p-1} \bar{\sigma}_p. \tag{36}$$

Walled Brauer algebras have a natural notion of trace and partial trace. The trace $\text{Tr}: \mathcal{B}_{p,q}^\delta \rightarrow \mathbb{C}$ of a walled Brauer algebra diagram σ is defined as

$$\text{Tr}(\sigma) := \delta^{\text{loops}(\sigma)}, \tag{37}$$

where $\text{loops}(\sigma)$ denotes the number of loops formed by connecting all nodes in the top row of σ to the corresponding nodes in the bottom row. This definition is extended to the whole of $\mathcal{B}_{p,q}^\delta$ by linearity. For any subset $S \subseteq [p + q]$, the corresponding partial trace $\text{Tr}_S: \mathcal{B}_{p,q}^\delta \rightarrow \mathcal{B}_{p',q'}^\delta$ is defined similarly, except we connect only those pairs of nodes in σ that are indicated by S :

$$\text{Tr}_S(\sigma) := \delta^{\text{loops}_S(\sigma)} \sigma', \tag{38}$$

where $\text{loops}_S(\sigma)$ denotes the number of loops formed in this way and $\sigma' \in \mathcal{B}_{p',q'}^\delta$ denotes the smaller diagram left after erasing the loops. Note that $p + q = p' + q' + |S|$ where $p' := p - |S \cap [p]|$ and $q' := q - |(S - p) \cap [q]|$.

Example 3.5. The trace of a diagram in $\mathcal{B}_{4,1}^\delta$:

$$\text{Tr} \left(\begin{array}{c} | \\ | \\ | \\ | \end{array} \begin{array}{c} \diagdown \quad \diagup \\ \cdot \quad \cdot \\ \diagup \quad \diagdown \end{array} \begin{array}{c} \cdot \quad \cdot \\ \text{---} \cdot \quad \cdot \text{---} \end{array} \right) = \begin{array}{c} | \\ | \\ | \\ | \end{array} \left(\begin{array}{c} \diagdown \quad \diagup \\ \cdot \quad \cdot \\ \diagup \quad \diagdown \end{array} \right) \left(\begin{array}{c} \cdot \quad \cdot \\ \text{---} \cdot \quad \cdot \text{---} \end{array} \right) = \delta^3. \tag{39}$$

The partial trace of the same diagram over $S = \{2, 3, 4\}$:

$$\text{Tr}_S \left(\begin{array}{c} 1 \\ | \\ | \\ | \end{array} \begin{array}{c} 2 \\ \diagdown \quad \diagup \\ \cdot \quad \cdot \\ \diagup \quad \diagdown \end{array} \begin{array}{c} 3 \\ \cdot \quad \cdot \\ \text{---} \cdot \quad \cdot \text{---} \end{array} \begin{array}{c} 4 \\ \cdot \quad \cdot \\ \text{---} \cdot \quad \cdot \text{---} \end{array} \begin{array}{c} 5 \\ | \\ | \\ | \end{array} \right) = \begin{array}{c} | \\ | \\ | \\ | \end{array} \left(\begin{array}{c} \diagdown \quad \diagup \\ \cdot \quad \cdot \\ \diagup \quad \diagdown \end{array} \right) \left(\begin{array}{c} \cdot \quad \cdot \\ \text{---} \cdot \quad \cdot \text{---} \end{array} \right) = \delta \cdot \begin{array}{c} | \\ | \\ | \\ | \end{array}. \tag{40}$$

3.3. *Matrix algebra of partially transposed permutations.* One of the central ideas of our approach is to represent matrices by diagrams and to use graphical methods to perform linear algebra operations such as matrix multiplication and partial traces. For this purpose we consider a matrix representation of the walled Brauer algebra $\mathcal{B}_{p,q}^d$ by extending eq. (16) from a representation of S_p to a representation $\psi_{p,q}^d : \mathcal{B}_{p,q}^d \rightarrow \text{End}(V^{p,q})$ of $\mathcal{B}_{p,q}^d$. Denoting the nodes in the first row of a diagram $\sigma \in \mathcal{B}_{p,q}^d$ by $1, \dots, p+q$ and in the second row by $\underline{1}, \dots, \underline{p+q}$, the action of $\psi_{p,q}^d(\sigma)$ on the standard basis tensors of $V^{p,q}$ is given by

$$\psi_{p,q}^d(\sigma)(|i_1\rangle \otimes \dots \otimes |i_{p+q}\rangle) = \sum_{1 \leq i_1, \dots, i_{p+q} \leq d} \sigma_{i_1, \dots, i_{p+q}}^{i_1, \dots, i_{p+q}} |i_1\rangle \otimes \dots \otimes |i_{p+q}\rangle, \quad (41)$$

for all $i_1, \dots, i_{p+q} \in \{1, \dots, d\}$, where the coefficients are given by

$$\sigma_{i_1, \dots, i_{p+q}}^{i_1, \dots, i_{p+q}} := \begin{cases} 1 & \text{if } i_r = i_s \text{ for all connected pairs of vertices} \\ & r, s \in \{1, \dots, p+q, \underline{1}, \dots, \underline{p+q}\} \text{ of } \sigma, \\ 0 & \text{otherwise.} \end{cases} \quad (42)$$

Equivalently,

$$\sigma_{i_1, \dots, i_{p+q}}^{i_1, \dots, i_{p+q}} = \prod_{(r,s) \in \sigma} \delta_{i_r, i_s}, \quad (43)$$

where the product is over all pairs (r, s) of nodes that are connected in σ .

Example 3.6. According to eqs. (41) and (43),

$$\begin{aligned} \langle x_1, \dots, x_5 | \psi_{3,2}^d \left(\begin{array}{c} \text{Diagram} \end{array} \right) | y_1, \dots, y_5 \rangle &= \begin{array}{c} x_1 \ x_2 \ x_3 \ x_4 \ x_5 \\ \begin{array}{c} \text{Diagram} \end{array} \\ y_1 \ y_2 \ y_3 \ y_4 \ y_5 \end{array} \quad (44) \\ &= \delta_{x_1, y_1} \delta_{x_2, y_3} \delta_{x_3, x_5} \delta_{x_4, y_5} \delta_{y_2, y_4} \quad (45) \end{aligned}$$

for any choice of $x_1, \dots, x_5 \in [d]$ and $y_1, \dots, y_5 \in [d]$.

A crucial fact about the matrix $\psi_{p,q}^d(\sigma)$ representing a diagram σ is that its partial traces are related to those of the diagram. Namely, for any $\sigma \in \mathcal{B}_{p,q}^d$ and $S \subseteq [p+q]$,

$$\text{Tr}(\psi_{p,q}^d(\sigma)) = \text{Tr}(\sigma), \quad \text{Tr}_S(\psi_{p,q}^d(\sigma)) = \psi_{p',q'}^d(\text{Tr}_S(\sigma)), \quad (46)$$

where the diagrammatic traces $\text{Tr}(\sigma)$ and $\text{Tr}_S(\sigma)$ are defined in eqs. (37) and (38), respectively. We formally establish these two identities in Appendix A.

Similar to eq. (21), we denote the image of $\mathcal{B}_{p,q}^d$ under $\psi_{p,q}^d$ by

$$\mathcal{A}_{p,q}^d := \psi_{p,q}^d(\mathcal{B}_{p,q}^d). \quad (47)$$

This is known as *matrix algebra of partially transposed permutations* as it is generated by permutation matrices on $p+q$ qudit registers, partially transposed on the last q registers. Recall that the *partial transpose* $\Gamma : \text{End}(V^{p+q}) \rightarrow \text{End}(V^{p+q})$ is defined as

$(M \otimes N)^\Gamma := M \otimes N^\Gamma$, for all $M \in \text{End}(V^p)$ and $N \in \text{End}(V^q)$. This operation can be used to relate the maps $\psi_{p,q}^d$ and ψ_{p+q}^d defined in eqs. (16) and (41), respectively:

$$\psi_{p,q}^d(\sigma) = \left(\psi_{p+q}^d(\sigma^\Gamma)\right)^\Gamma, \tag{48}$$

where σ^Γ denotes the partial transpose of the diagram $\sigma \in \mathcal{B}_{p,q}^d$, see eq. (31). Hence, as vector spaces, the algebras $\mathcal{A}_{p,q}^d$ and \mathcal{A}_{p+q}^d are related as follows:

$$\mathcal{A}_{p,q}^d = \left(\mathcal{A}_{p+q}^d\right)^\Gamma. \tag{49}$$

However, because of different product operations, the two algebras are generally not isomorphic.

Example 3.7 (Transposition versus contraction). When $p = 2$ and $q = 0$, the only non-trivial element of $\mathcal{B}_{2,0}^d$ is the transposition (12). Its matrix version acts as

$$\psi_{2,0}^d \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right) : |i\rangle|j\rangle \mapsto |j\rangle|i\rangle, \tag{50}$$

for all $i, j \in \{1, \dots, d\}$, since the diagram is encoded by $\sigma_{k,l}^{i,j} = \delta_{i,l}\delta_{j,k}$. More generally, any diagram that represents a permutation (i.e., has no edges across the wall) simply translates into the corresponding permutation of the tensor factors.

When $p = q = 1$, the only non-trivial element of $\mathcal{B}_{1,1}^d$ is the contraction of 1 and 2. The corresponding matrix acts as

$$\psi_{1,1}^d \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \vdots \end{array} \right) : |i\rangle|j\rangle \mapsto \delta_{i,j} \sum_{k=1}^d |k\rangle|k\rangle, \tag{51}$$

for all $i, j \in \{1, \dots, d\}$, since $\sigma_{k,l}^{i,j} = \delta_{i,j}\delta_{k,l}$ in this case. In particular, when $d = 2$,

$$\psi_{2,0}^2 \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \psi_{1,1}^2 \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \\ \vdots \end{array} \right) = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \psi_{2,0}^2 \left(\begin{array}{c} \diagup \quad \diagdown \\ \diagdown \quad \diagup \end{array} \right)^\Gamma, \tag{52}$$

which are known as the SWAP operator and the un-normalized projector onto the canonical maximally entangled state.

3.4. Mixed Schur–Weyl duality. Mixed Schur–Weyl duality is concerned with the action of $U(d)$ and $\mathcal{B}_{p,q}^d$ on the mixed tensor product space $V^{p,q} = V^{\otimes p} \otimes V^{*\otimes q}$ where $p, q \geq 0$ and $V = \mathbb{C}^d$. The $q = 0$ case is equivalent to the usual Schur–Weyl duality discussed in Sect. 3.1 while the $p = 0$ case is isomorphic to it.

As a generalization of eq. (19), consider the natural representation $\phi_{p,q}^d : U(d) \rightarrow \text{End}(V^{p,q})$ of $U(d)$ defined as

$$\phi_{p,q}^d(U) := U^{\otimes p} \otimes \bar{U}^{\otimes q} \tag{53}$$

for all $U \in \text{U}(d)$, where the entry-wise complex conjugate \bar{U} is also known as the *dual*³ of the *defining representation*. Similar to eq. (20), let

$$\mathcal{U}_{p,q}^d := \text{span}_{\mathbb{C}}\{\phi_{p,q}^d(U) : U \in \text{U}(d)\}. \tag{54}$$


We are particularly interested in the matrix algebra $\text{End}_{\mathcal{U}_{p,q}^d}(V^{p,q})$, i.e., the centralizer of the natural $\text{U}(d)$ action on $V^{p,q}$, which captures the unitary equivariance condition. Indeed, recall from Proposition 2.2 that a $p \rightarrow q$ channel Φ is unitary-equivariant if and only if $X^\Phi \in \text{End}_{\mathcal{U}_{p,q}^d}(V^{p,q})$. The following result generalizes Theorem 3.1 to the mixed tensor product space $V^{p,q}$ and says that $\text{End}_{\mathcal{U}_{p,q}^d}(V^{p,q})$ is equal to the matrix algebra $\mathcal{A}_{p,q}^d$ of partially transposed permutations (see Sect. 3.3) and, when $d \geq p + q$, isomorphic to the walled Brauer algebra $\mathcal{B}_{p,q}^d$.

Theorem 3.8 (Mixed Schur–Weyl duality [Koi89, Ben94]). *The algebra $\mathcal{U}_{p,q}^d$ is the centraliser algebra of $\mathcal{A}_{p,q}^d$ in $\text{End}(V^{p,q})$ and vice versa, i.e.,*

$$\mathcal{U}_{p,q}^d = \text{End}_{\mathcal{A}_{p,q}^d}(V^{p,q}), \mathcal{A}_{p,q}^d = \text{End}_{\mathcal{U}_{p,q}^d}(V^{p,q}). \tag{55}$$

Moreover, when $d \geq p + q$ the representation $\psi_{p,q}^d$ is faithful, i.e., $\mathcal{A}_{p,q}^d \cong \mathcal{B}_{p,q}^d$.

The simplest non-trivial instance of this duality is illustrated in Example 3.9 below. It is similar to Example 3.2, except here the second system of $\mathcal{U}_{1,1}^2$ and $\mathcal{A}_{1,1}^2$ is subject to the dual action of $\text{U}(d)$ and the partial transpose, respectively.

Example 3.9. ($p = q = 1$ and $d = 2$) The algebra $\mathcal{A}_{1,1}^2$ is generated by the identity matrix and the matrix $\psi_{1,1}^d$ () given in eq. (52):

$$\mathcal{A}_{1,1}^2 := \left\{ x \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + y \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} : x, y \in \mathbb{C} \right\}. \tag{56}$$

Note from eq. (23) that $\mathcal{A}_{1,1}^2 = (\mathcal{A}_{2,0}^2)^\Gamma$. We can describe $\mathcal{U}_{1,1}^2$ by introducing the *dual representation*: for any invertible matrix M , let $M^* := (M^{-1})^\Gamma$. Note that $M^* = \bar{M}$ when M is unitary and, for a general M , the entries of M^* are rational functions in the entries of M . In particular,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \frac{1}{ad - bc} \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \tag{57}$$

for any $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$. The algebra $\mathcal{U}_{1,1}^2$ is then given by

$$\mathcal{U}_{1,1}^2 := \text{span}_{\mathbb{C}} \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}^* : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{U}(2) \right\}. \tag{58}$$

³ More generally, the dual of the defining representation is $M \mapsto (M^{-1})^\Gamma$ where $M \in \text{GL}(d, \mathbb{C})$. However, when M is unitary, $(M^{-1})^\Gamma = (M^\dagger)^\Gamma = \bar{M}$.

Using the dual two-qubit Schur transform [Buh22]

$$U := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & -\sqrt{2} & 0 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 0 & \sqrt{2} & 0 \end{pmatrix}, \tag{59}$$

we can simultaneously block-diagonalize both algebras:

$$U\mathcal{A}_{1,1}^2 U^\top = \left\{ \left(\begin{array}{c|ccc} x+2y & 0 & 0 & 0 \\ \hline 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \end{array} \right) : x, y \in \mathbb{C} \right\}, \tag{60}$$

$$U\mathcal{U}_{1,1}^2 U^\top = \text{span}_{\mathbb{C}} \left\{ \frac{1}{ad-bc} \begin{pmatrix} ad-bc & 0 & 0 & 0 \\ \hline 0 & a^2 & \sqrt{2}ab & b^2 \\ 0 & \sqrt{2}ac & ad+bc & \sqrt{2}bd \\ 0 & c^2 & \sqrt{2}cd & d^2 \end{pmatrix} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{U}(2) \right\}. \tag{61}$$

These algebras centralize each other since

$$U\mathcal{A}_{1,1}^2 U^\top = \mathbb{C} \oplus \mathbb{C}I_3, \quad U\mathcal{U}_{1,1}^2 U^\top = \mathbb{C} \oplus \text{End}(\mathbb{C}^3). \tag{62}$$

While this appears identical to eq. (28), the decompositions of $\mathcal{A}_{p,q}^d$ and $\mathcal{U}_{p,q}^d$ are generally very different from those of \mathcal{A}_{p+q}^d and \mathcal{U}_{p+q}^d , see Sect. 5.4.

Remark 3.10. Theorem 3.8 can be seen as an instance of the Double Centralizer Theorem, see [Eti11, Theorem 4.54]. Indeed, we can rephrase it as follows: as a representation of $\mathcal{A}_{p,q}^d \times \mathcal{U}_{p,q}^d$, the space $V^{p,q}$ decomposes as

$$V^{p,q} \cong \bigoplus_{\lambda \in \text{Irr}(\mathcal{A}_{p,q}^d)} V^\lambda \otimes U^\lambda, \tag{63}$$

where V^λ are simple modules of $\mathcal{A}_{p,q}^d$ and U^λ are simple modules of $\mathcal{U}_{p,q}^d$. It follows from Proposition 2.2 that there exists a *mixed Schur transform* $U_{\text{Sch}(p,q)} \in \hat{\text{U}}(V^{p,q})$ that block-diagonalizes any unitary-equivariant Choi matrix X^Φ as follows:

$$U_{\text{Sch}(p,q)} X^\Phi U_{\text{Sch}(p,q)}^\dagger = \bigoplus_{\lambda \in \text{Irr}(\mathcal{A}_{p,q}^d)} [X_\lambda^\Phi \otimes I_{m_\lambda}], \tag{64}$$

where the operator X_λ^Φ acts on a $\mathcal{A}_{p,q}^d$ -register of dimension d_λ , and the identity matrix I_{m_λ} acts on a $\mathcal{U}_{p,q}^d$ -register of dimension m_λ . When $q = 0$, $U_{\text{Sch}(p,q)}$ reduces to the usual Schur transform considered in [Har05].

The distinction between $\mathcal{A}_{p,q}^d$ and $\mathcal{B}_{p,q}^d$ is crucial if one is interested in small dimensions $d < p + q$ since the two algebras are not isomorphic in this case. For example, the algebra $\mathcal{A}_{p,q}^d$ is always semisimple because $\mathcal{U}_{p,q}^d$ is known to be semisimple from the representation theory of Lie groups [Eti11, Theorem 4.66], so its commutant $\mathcal{A}_{p,q}^d$ must also be semisimple by the Double Centralizer Theorem [Eti11, Theorem 4.54]. However, $\mathcal{B}_{p,q}^d$ is not semisimple for integer $d < p + q - 1$ [Cox08].

Previously the algebra $\mathcal{A}_{p,q}^d$ was studied in [Koi89, Ben94] in the context of mixed Schur–Weyl duality. For $q = 1$, it was explicitly studied in [ZKW07, SHM13, MHS14, MSH18] motivated by applications in quantum information. Some aspects of it were also studied for general q in [Stu22, MSK21].

One of our main technical results is the determination of primitive central idempotents of the matrix algebras $\mathcal{A}_{p,q}^d$, see Sect. 5, which will play a central role in our results in Sect. 6 on linear programming with unitary-equivariant constraints. In the following section we summarize a general method for constructing primitive central idempotents of multiplicity-free families of algebras due to [DLS18].

4. Review of the [DLS18] Construction

Inspired by the Okounkov–Vershik approach [OV96, VO05] to the representation theory of the symmetric groups and their algebras, the authors of [DLS18] present a general algorithm for finding the primitive central idempotents of any multiplicity-free family of algebras. In this section, we introduce the necessary background on algebras and their representations, and then summarize the [DLS18] algorithm. We will apply this algorithm in Sect. 5 to the algebra $\mathcal{A}_{p,q}^d$ of partially transposed permutation matrices defined in eq. (47). The idempotents of $\mathcal{A}_{p,q}^d$ are the main ingredient of our optimization framework in Sect. 6.

4.1. Background on algebras and their modules. An algebra over a field is a vector space over this field, equipped with a bilinear product. A canonical example of an algebra is the full matrix algebra $\text{End}(V)$, i.e., the set of all linear operators acting on some vector space V . As part of our definition, we assume that all algebras are *complex, finite-dimensional, associative, and unital*. Namely, the vector space underlying an algebra \mathcal{A} is complex and finite-dimensional, the product operation in \mathcal{A} is associative, and there is a *unit* element $1 \in \mathcal{A}$ such that $1a = a1 = a$ for any $a \in \mathcal{A}$. Similar to Cayley’s theorem for groups, any algebra is isomorphic to a subalgebra of the full matrix algebra $\text{End}(\mathbb{C}^{\dim \mathcal{A}})$. Indeed, the action of \mathcal{A} on any basis of \mathcal{A} produces a matrix algebra that is analogous to the left-regular representation of a group. Due to the Artin–Wedderburn Theorem, an algebra over \mathbb{C} is *semisimple* if it is isomorphic to a direct sum of full matrix algebras over \mathbb{C} . If \mathcal{A} and \mathcal{B} are algebras, $\varphi: \mathcal{A} \rightarrow \mathcal{B}$ is an *algebra embedding* if φ is an injective homomorphism. The embedding is *unity-preserving* if $\varphi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$. We write $\mathcal{A} \hookrightarrow \mathcal{B}$ to mean that such embedding exists (in such case one can intuitively think of \mathcal{A} as a subalgebra of \mathcal{B}). If $\mathcal{B} \subseteq \mathcal{A}$ is a subalgebra then we denote by $\mathcal{Z}_{\mathcal{B}}(\mathcal{A})$ the *centralizer* of \mathcal{B} in \mathcal{A} :

$$\mathcal{Z}_{\mathcal{B}}(\mathcal{A}) := \{a \in \mathcal{A} : ab = ba \text{ for every } b \in \mathcal{B}\}. \tag{65}$$

If $\mathcal{B} = \mathcal{A}$ in the above definition then $\mathcal{Z}(\mathcal{A}) := \mathcal{Z}_{\mathcal{A}}(\mathcal{A})$ is known as the *center* of the algebra \mathcal{A} . Note that $\text{End}_{\mathcal{B}}(V)$ defined in eq. (4) is a special case of $\mathcal{Z}_{\mathcal{B}}(\mathcal{A})$ where $\mathcal{A} = \text{End}(V)$.

If an algebra \mathcal{A} acts on a complex vector space V , we call V an \mathcal{A} -*module*⁴ (in the context of groups it is analogous to the notion of a group representation). A *submodule* of an \mathcal{A} -module V is a subspace W of V such that $aw \in W$ for all $w \in W$ and $a \in \mathcal{A}$.

⁴ Module is generally defined as a ring acting on an abelian group. Note that any unital, associative algebra is a ring and any vector space under addition is an abelian group.

(Note that W is an \mathcal{A} -module in its own right.) If an \mathcal{A} -module V has submodules W_1 and W_2 such that $V = W_1 \oplus W_2$ as a vector space then we say that V is the direct sum of W_1 and W_2 . A module V is *indecomposable* if it is not the direct sum of two non-zero submodules, and is *decomposable* otherwise. A module V is *simple* if V has no submodules except V and 0 (in the context of groups, simple modules are known as irreducible representations). We denote the set of all non-isomorphic simple modules of \mathcal{A} by $\text{Irr}(\mathcal{A})$. Given a subalgebra \mathcal{A} of an algebra \mathcal{B} and an \mathcal{B} -module V we can consider its *restriction* to the subalgebra \mathcal{A} . We denote such \mathcal{A} -module V by $\text{Res}_{\mathcal{A}}^{\mathcal{B}} V$. See [DK12, Cox12] for more background on finite-dimensional algebras and their modules.

4.2. Multiplicity-free families of algebras. A particularly nice situation is when all restrictions of an algebra are multiplicity-free. This is formalized in the following definition which is motivated by the canonical example of the sequence $\mathbb{C}S_0 \hookrightarrow \mathbb{C}S_1 \hookrightarrow \dots \hookrightarrow \mathbb{C}S_n$ of symmetric group algebras considered in [OV96, VO05].

Definition 4.1 (*Definition 1.1 in [DLS18]*). A family $\mathcal{A}_0, \dots, \mathcal{A}_n$ of finite-dimensional semisimple.⁵ algebras over \mathbb{C} is *multiplicity-free* if the following axioms hold:

- (a) $\mathcal{A}_0 \cong \mathbb{C}$.
- (b) For each k , there is a unity-preserving algebra embedding $\mathcal{A}_k \hookrightarrow \mathcal{A}_{k+1}$.
- (c) The restriction of a simple \mathcal{A}_k -module to \mathcal{A}_{k-1} is isomorphic to a direct sum of pairwise non-isomorphic simple \mathcal{A}_{k-1} -modules. We say that in that case the restriction from \mathcal{A}_k to \mathcal{A}_{k-1} is *multiplicity-free*.

4.3. Bratteli diagram. Given a multiplicity-free family of algebras, one can create a graph that shows how different simple modules of these algebras restrict to their subalgebras. We denote by V^λ a simple \mathcal{A} -module corresponding to $\lambda \in \text{Irr}(\mathcal{A})$. The dimension of this module is denoted by $d_\lambda := \dim V^\lambda$. We can represent the multiplicity-free restrictions among \mathcal{A}_k by a directed acyclic graph known as Bratteli diagram [Bra72].

Definition 4.2. Let $\mathcal{A}_0, \dots, \mathcal{A}_n$ be a multiplicity-free family of algebras. Its *Bratteli diagram* is a directed acyclic graph whose vertices are the isomorphism classes $\bigsqcup_{k=0}^n \text{Irr}(\mathcal{A}_k)$ of simple \mathcal{A}_k -modules. There is an edge $\lambda \rightarrow \mu$ from vertex $\lambda \in \text{Irr}(\mathcal{A}_k)$ to vertex $\mu \in \text{Irr}(\mathcal{A}_{k+1})$ if and only if V^λ is isomorphic to a direct summand of $\text{Res}_{\mathcal{A}_k}^{\mathcal{A}_{k+1}} V^\mu$, the restriction of V^μ to the subalgebra \mathcal{A}_k . We call $\text{Irr}(\mathcal{A}_k)$ the *k-th level* of the Bratteli diagram. We denote the unique vertex in $\text{Irr}(\mathcal{A}_0)$ by \varnothing^6 and call it the *root*, while $\text{Irr}(\mathcal{A}_n)$ are called *leaves*.

An example of a Bratteli diagram for a multiplicity-free family of semisimple symmetric group algebras is shown in Fig. 1.

4.4. Primitive central idempotents. Idempotents of an algebra play an important role in its structure.

⁵ In [DLS18] the algebras are also required to be *split*. However, since we consider only algebras over \mathbb{C} , which is an algebraically closed field, in our setting all algebras are automatically split.

⁶ We will later use $(\varnothing, \varnothing)$ for the root of the Bratteli diagram of the walled Brauer algebra because the irreducible representations of this algebra are labeled by pairs of Young diagrams.

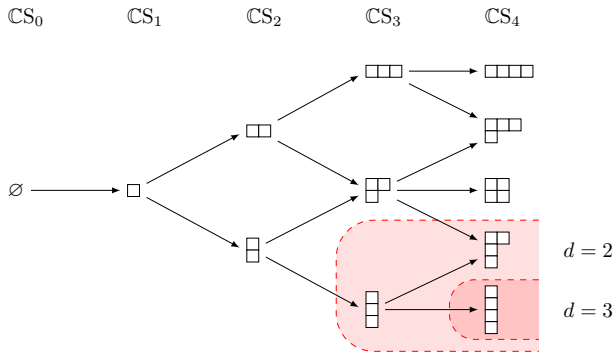


Fig. 1. Bratteli diagram for the symmetric group algebras $\mathbb{C}S_0 \hookrightarrow \mathbb{C}S_1 \hookrightarrow \mathbb{C}S_2 \hookrightarrow \mathbb{C}S_3 \hookrightarrow \mathbb{C}S_4$, also known as Young’s lattice. The Bratteli diagram for the permutation matrix algebras $\mathcal{A}_0^d \hookrightarrow \mathcal{A}_1^d \hookrightarrow \mathcal{A}_2^d \hookrightarrow \mathcal{A}_3^d \hookrightarrow \mathcal{A}_4^d$ defined in eq. (21) is the same when $d \geq 4$. When $d = 2$ or $d = 3$, vertices with Young diagrams containing more than d rows are removed

Definition 4.3. An *idempotent* e in the algebra \mathcal{A} is an element with the property $e^2 = e$. Two idempotents $a, b \in \mathcal{A}$ are said to be *orthogonal* if $ab = ba = 0$. A *central idempotent* $e \in \mathcal{A}$, is an idempotent that commutes with every element $a \in \mathcal{A}$, i.e., $ea = ae$.

Certain types of idempotents are extremely useful for studying the representation theory of a given algebra. We define them as follows.

Definition 4.4. A *primitive idempotent* is an idempotent that cannot be written as a sum of two nonzero orthogonal idempotents. A *primitive central idempotent* is a central idempotent that cannot be written as a sum of two nonzero orthogonal central idempotents.

If algebra \mathcal{A} is semisimple, we have the isomorphism

$$\mathcal{A} = \bigoplus_{\lambda \in \text{Irr}(\mathcal{A})} \varepsilon(\lambda)\mathcal{A} \cong \bigoplus_{\lambda \in \text{Irr}(\mathcal{A})} \text{End}(V^\lambda) \tag{66}$$

where $\varepsilon(\lambda)$ are the *primitive central idempotents* of \mathcal{A} and the first direct sum should be understood as a decomposition of the left-regular representation of \mathcal{A} . This isomorphism is very useful since it allows to think of an abstract semisimple algebra \mathcal{A} as an algebra of block-diagonal matrices, a perspective that we will repeatedly use. Note from eq. (66) that $\dim \mathcal{A} = \sum_{\lambda \in \text{Irr}(\mathcal{A})} d_\lambda^2$, which is analogous to the dimension formula for irreducible representations of groups. The primitive central idempotents $\varepsilon(\lambda)$ are in one-to-one correspondence with simple \mathcal{A} -modules labeled by $\lambda \in \text{Irr}(\mathcal{A})$ and provide a resolution of the unit element of the algebra \mathcal{A} : $\sum_{\lambda \in \text{Irr}(\mathcal{A})} \varepsilon(\lambda) = 1$.

4.5. Gelfand–Tsetlin basis and subalgebra. Let $\mathcal{A}_0, \dots, \mathcal{A}_n$ be a multiplicity-free family of algebras (see Definition 4.1). We denote by $\text{Paths}(\lambda)$ the set of all paths in the Bratteli diagram starting from the root $\emptyset \in \text{Irr}(\mathcal{A}_0)$ and terminating at the leaf $\lambda \in \text{Irr}(\mathcal{A}_n)$. Any $T \in \text{Paths}(\lambda)$ has the form

$$T = \lambda_0 \rightarrow \lambda_1 \rightarrow \dots \rightarrow \lambda_n \tag{67}$$

where $\lambda_0 = \emptyset$ and $\lambda_n = \lambda$. Let $T[k] := \lambda_k$ denote the k -th vertex on the path T . We also define $\text{Paths}(n) := \bigcup_{\lambda \in \text{Irr}(\mathcal{A}_n)} \text{Paths}(\lambda)$ and say that $T \in \text{Paths}(n)$ is a path of length n .

We can now define a certain basis for the direct sum $\bigoplus_{\lambda \in \text{Irr}(\mathcal{A}_n)} V^\lambda$ of all simple \mathcal{A}_n -modules, where each element of the basis corresponds to a path in the Bratteli diagram. This basis can be obtained by choosing any leaf $\lambda \in \text{Irr}(\mathcal{A}_n)$ and considering the restriction $\text{Res}_{\mathcal{A}_{n-1}}^{\mathcal{A}_n} V^\lambda$ of the corresponding simple \mathcal{A}_n -module V^λ to \mathcal{A}_{n-1} , which according to Definition 4.1 is multiplicity-free. This restriction can then be iterated further along any path in the Bratteli diagram towards the root \emptyset that corresponds to the one-dimensional algebra $\mathcal{A}_0 \cong \mathbb{C}$. Doing this along all $\text{Paths}(\lambda)$ between \emptyset and λ results in a decomposition of the chosen simple \mathcal{A}_n -module V^λ into one-dimensional simple \mathcal{A}_0 -modules. Repeating this procedure for all leaves $\lambda \in \text{Irr}(\mathcal{A}_n)$ produces the Gelfand–Tsetlin basis of $\bigoplus_{\lambda \in \text{Irr}(\mathcal{A}_n)} V^\lambda$:

$$\{ |T\rangle : T \in \text{Paths}(n) \}. \tag{68}$$

These vectors are labeled by elements of $\text{Paths}(n)$ since each sequence of restrictions corresponds to some leaf-root path in the Bratteli diagram.

To find the Gelfand–Tsetlin basis explicitly, we can look at the maximal commutative subalgebras of \mathcal{A}_k . Due to eq. (66) one can think of them as subalgebras of diagonal matrices, carrying the information about the projectors onto the Gelfand–Tsetlin basis. This motivates the following definition.

Definition 4.5. For each $k \in [n]$, the corresponding Gelfand–Tsetlin subalgebra is

$$\mathcal{X}_k := \langle \mathcal{Z}(\mathcal{A}_1), \dots, \mathcal{Z}(\mathcal{A}_k) \rangle \subseteq \mathcal{A}_k \tag{69}$$

where $\mathcal{Z}(\mathcal{A}_i)$ denotes the center of the subalgebra \mathcal{A}_i .

Note that $\mathcal{X}_1 \subseteq \dots \subseteq \mathcal{X}_n$. The Gelfand–Tsetlin subalgebra \mathcal{X}_k is a maximal commutative subalgebra of \mathcal{A}_k , see Proposition 1.1 of [OV96, VO05]. We will later find a particular set of generators for \mathcal{X}_k that act nicely on the Gelfand–Tsetlin basis, which will help us to construct the primitive central idempotents of \mathcal{A}_n .

For each path $T = \lambda_0 \rightarrow \lambda_1 \rightarrow \dots \rightarrow \lambda_n \in \text{Paths}(n)$ in the Bratteli diagram, set

$$\varepsilon_T := \varepsilon(\lambda_1)\varepsilon(\lambda_2) \cdots \varepsilon(\lambda_n) \tag{70}$$

where $\varepsilon(\lambda_i)$ are the primitive central idempotents of \mathcal{A}_i , see eq. (66). Note that ε_T is an element of the Gelfand–Tsetlin subalgebra \mathcal{X}_n since $\varepsilon(\lambda_i) \in \mathcal{Z}(\mathcal{A}_i)$ for each i .

Proposition 4.6 (Proposition 1.6 and Corollary 1.7 [DLS18]). *The collection $\{\varepsilon_T : T \in \text{Paths}(n)\}$ is a family of orthogonal primitive idempotents in \mathcal{A}_n that sums to the identity 1 and is a basis for the Gelfand–Tsetlin subalgebra \mathcal{X}_n . Moreover, the primitive central idempotents of \mathcal{A}_n are given by*

$$\varepsilon(\lambda) = \sum_{T \in \text{Paths}(\lambda)} \varepsilon_T. \tag{71}$$

Using the isomorphism in eq. (66), the primitive idempotents ε_T correspond to the projectors $|T\rangle\langle T|$ onto the Gelfand–Tsetlin basis vectors $|T\rangle \in \bigoplus_{\lambda \in \text{Irr}(\mathcal{A}_n)} V^\lambda$.

4.6. *Jucys–Murphy elements.* In this section, we define a certain nice set of elements of the algebra \mathcal{A}_k that generate the Gelfand–Tsetlin subalgebra \mathcal{X}_k . They are commonly known as Jucys–Murphy elements.

Definition 4.7 (*Definition 3.1 in [DLS18]*). Let $\mathcal{A}_0, \dots, \mathcal{A}_n$ be a multiplicity-free family of algebras and let $\mathcal{X}_1, \dots, \mathcal{X}_n$ be their Gelfand–Tsetlin subalgebras. Let J_1, \dots, J_n be a sequence of elements in \mathcal{A}_n such that $J_k \in \mathcal{X}_k$ for each $k \in [n]$. This sequence is

- (a) *additively central* if $J_1 + \dots + J_k \in \mathcal{Z}(\mathcal{A}_k)$ for all $k \in [n]$,
- (b) *separating* if $\mathcal{X}_k = \langle J_1, \dots, J_k \rangle$ for all $k \in [n]$.

It is a *Jucys–Murphy sequence* if it is both additively central and separating.

Since $J_1, \dots, J_n \in \mathcal{X}_n$ and $\{\varepsilon_T : T \in \text{Paths}(n)\}$ is a basis of \mathcal{X}_n due to Proposition 4.6, we can expand each J_k as a linear combination of ε_T .

Definition 4.8. For a given sequence J_1, \dots, J_n with $J_k \in \mathcal{X}_k$, we define scalars $c_T(1), \dots, c_T(n) \in \mathbb{C}$ such that for all $k \in [n]$:

$$J_k = \sum_{T \in \text{Paths}(n)} c_T(k) \varepsilon_T. \tag{72}$$

Note that under the isomorphism in eq. (66) $\{c_T(k) : T \in \text{Paths}(n)\}$ are the eigenvalues of J_k . An important observation regarding the $c_T(k)$ is that the value of $c_T(k)$ does not depend on the whole path T but only on the vertices $T[k]$ and $T[k - 1]$, see Lemma 3.9 in [DLS18] which is a consequence of the property (a) in Definition 4.7. This means that the number $c_T(k)$ can be assigned to the edge $T[k - 1] \rightarrow T[k]$ in the Bratteli diagram and we can equivalently write

$$c_{T[k-1] \rightarrow T[k]} := c_T(k). \tag{73}$$

The concepts introduced above are well-established for the group algebra $\mathbb{C}S_n$, as summarized in the following example.

Example 4.9. ($\mathbb{C}S_n$) The Bratteli diagram for $\mathbb{C}S_0 \hookrightarrow \mathbb{C}S_1 \hookrightarrow \dots \hookrightarrow \mathbb{C}S_n$ is known as *Young lattice*, see Fig. 1. A path T in this Bratteli diagram can also be viewed as a *standard Young tableau*. Jucys–Murphy elements of $\mathbb{C}S_n$ [Juc74, Mur81] are

$$J_k := \begin{cases} 0 & \text{if } k = 1, \\ \sum_{i=1}^{k-1} \sigma_{i,k} & \text{if } 2 \leq k \leq n, \end{cases} \tag{74}$$

where $\sigma_{i,k}$ is the transposition of elements i and k . Moreover,

$$c_T(k) := j - i, \tag{75}$$

where i and j are the coordinates of the cell (i, j) occupied by k in the standard Young tableau T . The number $j - i$ is also known as the *content* of cell (i, j) in a Young diagram. The *content* of a Young diagram λ is defined as the total content of all its cells:

$$\text{cont}(\lambda) := \sum_{(i,j) \in \lambda} (j - i) \tag{76}$$

where the sum runs over all cells in the diagram λ .

4.7. *DLS algorithm for computing primitive idempotents.* We have all ingredients to state the [DLS18] algorithm for computing primitive central and canonical primitive pairwise orthogonal idempotents of any multiplicity-free family $\mathcal{A}_0, \dots, \mathcal{A}_n$ of semisimple finite-dimensional algebras.

Following [DLS18], we assign to each edge $\lambda \rightarrow \mu$ between levels $k - 1$ and k of the Bratteli diagram an *interpolating polynomial* $P_{\lambda \rightarrow \mu}$ of $x \in \mathcal{A}_k$ defined as

$$P_{\lambda \rightarrow \mu}(x) := \prod_{\tilde{\mu} : \lambda \rightarrow \tilde{\mu} \neq \mu} \frac{x - c_{\lambda \rightarrow \tilde{\mu}}}{c_{\lambda \rightarrow \mu} - c_{\lambda \rightarrow \tilde{\mu}}}, \tag{77}$$

where the product is over all edges $\lambda \rightarrow \tilde{\mu}$ (other than $\lambda \rightarrow \mu$) outgoing from the vertex λ . According to their main result [DLS18, Theorem 3.11], the primitive central idempotents of \mathcal{A}_k can be computed recursively for any $k \in [n]$ and $\mu \in \text{Irr}(\mathcal{A}_k)$ as follows:

$$\varepsilon(\mu) = \sum_{\lambda : \lambda \rightarrow \mu} P_{\lambda \rightarrow \mu}(J_k) \varepsilon(\lambda), \tag{78}$$

where the sum is over all edges $\lambda \rightarrow \mu$ incoming into μ and J_1, \dots, J_n is a Jucys–Murphy sequence for the algebras $\mathcal{A}_0, \dots, \mathcal{A}_n$. The base case of the recursion is $\varepsilon(\emptyset) = 1$. According to [DLS18, Theorem 3.8], the canonical primitive idempotents corresponding to the Gelfand–Tsetlin basis can be found by substituting eq. (77) into eq. (70):

$$\varepsilon_T = \prod_{k=1}^n P_{\lambda_{k-1} \rightarrow \lambda_k}(J_k) = \prod_{k=1}^n \prod_{\mu : \lambda_{k-1} \rightarrow \mu \neq \lambda_k} \frac{J_k - c_{\lambda_{k-1} \rightarrow \mu}}{c_{\lambda_{k-1} \rightarrow \lambda_k} - c_{\lambda_{k-1} \rightarrow \mu}} \tag{79}$$

where $T = \lambda_0 \rightarrow \lambda_1 \rightarrow \dots \rightarrow \lambda_n$ is a path in the Bratteli diagram.

Using these formulas requires the following data about the family $\mathcal{A}_0, \dots, \mathcal{A}_n$:

- (1) the Bratteli diagram of $\mathcal{A}_0, \dots, \mathcal{A}_n$,
- (2) a Jucys–Murphy sequence J_1, \dots, J_n for $\mathcal{A}_0, \dots, \mathcal{A}_n$,
- (3) the scalars $c_T(k)$ for all $k \in [n]$ and paths $T \in \text{Paths}(n)$ in the Bratteli diagram.

In the following section we discuss how this information can be obtained for the family of partially transposed permutation matrix algebras $\mathcal{A}_{p,q}^d$ using the same known data for walled Brauer algebras $\mathcal{B}_{p,q}^d$.

5. Adapting [DLS18] to the Matrix Algebras $\mathcal{A}_{p,q}^d$

In this section, we provide the necessary ingredients for applying the [DLS18] framework to the partially transposed permutation matrix algebras $\mathcal{A}_{p,q}^d$. Our main technical contribution is Theorem 5.6 which shows that Jucys–Murphy elements of partially transposed permutation matrix algebras can be obtained from Jucys–Murphy elements of walled Brauer algebras, even when the corresponding walled Brauer algebras are not semisimple.

Consider the following multiplicity-free family of walled Brauer algebras:

$$\mathbb{C} \cong \mathcal{B}_{0,0}^d \hookrightarrow \mathcal{B}_{1,0}^d \hookrightarrow \dots \hookrightarrow \mathcal{B}_{p,0}^d \hookrightarrow \mathcal{B}_{p,1}^d \hookrightarrow \dots \hookrightarrow \mathcal{B}_{p,q}^d, \tag{80}$$

where the embeddings correspond to adding on the right of the diagram an extra pair of nodes that are connected with a vertical line. For the sake of brevity, let us denote this family by $\mathcal{B} := (\mathcal{B}_0, \dots, \mathcal{B}_{p+q})$ where

$$\mathcal{B}_k := \begin{cases} \mathbb{C} & \text{if } k = 0, \\ \mathcal{B}_{k,0}^d & \text{if } 1 \leq k \leq p, \\ \mathcal{B}_{p,k-p}^d & \text{if } p + 1 \leq k \leq p + q. \end{cases} \tag{81}$$

Similarly, let $\mathcal{A} := (\mathcal{A}_0, \dots, \mathcal{A}_{p+q})$ where, for every $k \in \{0, \dots, p + q\}$,

$$\mathcal{A}_k := \psi_{p,q}^d(\mathcal{B}_k) \subseteq \text{End}(V^{p,q}) \tag{82}$$

is the corresponding partially transposed permutation matrix algebra and $\psi_{p,q}^d$ is the map from eq. (41). Note that \mathcal{A}_{k-1} is the subalgebra of \mathcal{A}_k that consists of all matrices of the form $M \otimes I_d$ for some M .

For every $k \in [p + q]$, let

$$\mathcal{X}_k^{\mathcal{B}} := \langle \mathcal{Z}(\mathcal{B}_1), \dots, \mathcal{Z}(\mathcal{B}_k) \rangle, \quad \mathcal{X}_k^{\mathcal{A}} := \langle \mathcal{Z}(\mathcal{A}_1), \dots, \mathcal{Z}(\mathcal{A}_k) \rangle \tag{83}$$

denote the Gelfand–Tsetlin subalgebras of \mathcal{B} and \mathcal{A} , respectively, see Definition 4.5. Sometimes we abuse the notation and refer to \mathcal{A} (or \mathcal{B}) as the Bratteli diagram of the corresponding algebra family.

5.1. Bratteli diagram for walled Brauer algebras. The simple modules of the walled Brauer algebra $\mathcal{B}_{p,q}^\delta$ are labeled by pairs of Young diagrams (λ^l, λ^r) where λ^l and λ^r are partitions of $p - k$ and $q - k$ for some $0 \leq k \leq \min(p, q)$:

$$\text{Irr}(\mathcal{B}_{p,q}^\delta) = \{ \lambda = (\lambda^l, \lambda^r) : 0 \leq k \leq \min(p, q), \lambda^l \vdash p - k, \lambda^r \vdash q - k \}. \tag{84}$$

The Bratteli diagram (see Definition 4.2) for the family \mathcal{B} of walled Brauer algebras is defined as follows [BO20]. The only vertex at level $k = 0$ is the root (\emptyset, \emptyset) . For any $k \in [p + q]$, the vertices at level k are given by $\text{Irr}(\mathcal{B}_k)$, see eqs. (81) and (84). For any pair of adjacent levels $k - 1$ and k where $k \in [p + q]$, an edge $\lambda \rightarrow \mu$ between $\lambda \in \text{Irr}(\mathcal{B}_{k-1})$ and $\mu \in \text{Irr}(\mathcal{B}_k)$ is present if and only if

- (1) $k \leq p$ and the diagram μ is obtained from λ by adding one cell to the diagram λ^l ,
- (2) $k > p$ and μ is obtained from λ by either adding a cell to λ^r or removing a cell from λ^l .

For example, the Bratteli diagram for the multiplicity-free family ending with $\mathcal{B}_{2,2}^\delta$ is given in Fig. 2.

By construction, the number of paths from the root vertex to any leaf λ in the Bratteli diagram is equal to the dimension $d_\lambda = \dim(V^\lambda)$ of the corresponding simple module of $\mathcal{B}_{p,q}^\delta$.

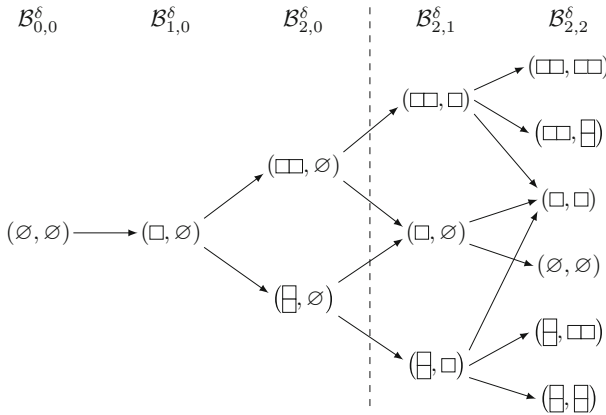


Fig. 2. Bratteli diagram associated to the multiplicity-free family $\mathbb{C} \cong \mathcal{B}_{0,0}^\delta \hookrightarrow \mathcal{B}_{1,0}^\delta \hookrightarrow \mathcal{B}_{2,0}^\delta \hookrightarrow \mathcal{B}_{2,1}^\delta \hookrightarrow \mathcal{B}_{2,2}^\delta$ of walled Brauer algebras when they are semisimple

5.2. Jucys–Murphy elements for walled Brauer algebras. If $\delta \in \mathbb{C}$ is such that the walled Brauer algebras $\mathcal{B}_{p,q}^\delta$ are semisimple, their Jucys–Murphy elements are given by [BS12, SS15, JK20] (cf. Example 4.9)

$$J_k^\mathcal{B} := \begin{cases} 0 & \text{if } k = 1, \\ \sum_{i=1}^{k-1} \sigma_{i,k} & \text{if } 2 \leq k \leq p, \\ \sum_{i=p+1}^{k-1} \sigma_{i,k} - \sum_{i=1}^p \bar{\sigma}_{i,k} + \delta & \text{if } p + 1 \leq k \leq p + q, \end{cases} \tag{85}$$

where $\sigma_{i,k}$ is the transposition of elements i and k , and $\bar{\sigma}_{i,k}$ is the corresponding contraction. When the walled Brauer algebra is not semisimple, we still define $J_k^\mathcal{B}$ via the above formula.

5.3. Content vectors for walled Brauer algebras. Let $T \in \text{Paths}(p + q)$ be an arbitrary root-leaf path in the Bratteli diagram of $\mathcal{B}_{p,q}^\delta$ and let $T[k - 1] \rightarrow T[k]$ where $k \in [p + q]$ denote an edge on this path. Recall from eq. (84) that each vertex $T[k]$ of T is labeled by some bipartition (λ^l, λ^r) . The number $c_{T[k-1] \rightarrow T[k]}$ introduced in Definitions 4.8 and 73 that corresponds to this edge is calculated via the following rule [BO20, JK20]:

- (1) if $1 \leq k \leq p$ and $T[k]$ is obtained from $T[k - 1]$ by adding a cell (i, j) to the first diagram in the bipartition $T[k - 1]$ then

$$c_{T[k-1] \rightarrow T[k]} = j - i,$$

- (2) if $p + 1 \leq k \leq p + q$ and $T[k]$ is obtained from $T[k - 1]$ by removing a cell (i, j) from the first diagram in the bipartition $T[k - 1]$ then

$$c_{T[k-1] \rightarrow T[k]} = i - j,$$

- (3) if $p + 1 \leq k \leq p + q$ and $T[k]$ is obtained from $T[k - 1]$ by adding a cell (i, j) to the second diagram in the bipartition $T[k - 1]$ then

$$c_{T[k-1] \rightarrow T[k]} = j - i + \delta.$$

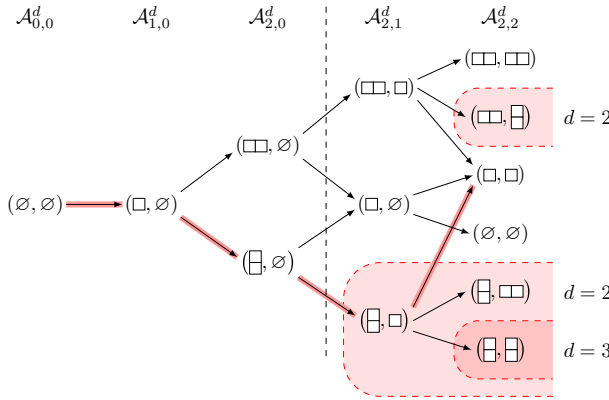


Fig. 3. Bratteli diagram associated to the multiplicity-free family $\mathbb{C} \cong \mathcal{A}_{0,0}^d \hookrightarrow \mathcal{A}_{1,0}^d \hookrightarrow \mathcal{A}_{2,0}^d \hookrightarrow \mathcal{A}_{2,1}^d \hookrightarrow \mathcal{A}_{2,2}^d$ of partially transposed permutation matrix algebras for different values of the local dimension d . When $d \geq 4$, this diagram coincides with that of the walled Brauer algebras (see Fig. 2). However, for small values of d (i.e., $d = 2$ and $d = 3$) the diagram has to be modified by removing the designated vertices. Note that removing the vertex (\square, \square) when $d = 2$ eliminates the highlighted path from the root (\emptyset, \emptyset) to the leaf (\square, \square) , which decreases the dimension of the corresponding simple $\mathcal{A}_{2,2}^d$ -module $V^{\square, \square}$ by one, i.e., $\dim(V^{\square, \square}) = 4$ if $d > 2$ while $\dim(V^{\square, \square}) = 3$ if $d = 2$

Since these formulas are very similar to the definition of the *content* $j - i$ of a cell (i, j) in a Young tableau, see eq. (75), we refer to $(c_T(1), \dots, c_T(p + q))$ as the *content vector* of the path $T \in \text{Paths}(p + q)$ where $c_T(k) = c_{T[k-1] \rightarrow T[k]}$.

5.4. Adapting the Bratteli diagram from $\mathcal{B}_{p,q}^\delta$ to $\mathcal{A}_{p,q}^d$. According to [Ben94, Theorem 1.11], the Bratteli diagram for the family of partially transposed permutation matrix algebras \mathcal{A} can be obtained from the Bratteli diagram for semisimple walled Brauer algebras \mathcal{B} by removing all vertices (λ^l, λ^r) that violate the condition $\text{len}(\lambda^l) + \text{len}(\lambda^r) \leq d$, where $\text{len}(\mu)$ denotes the length of the first column of the Young diagram μ . For example, Fig. 3 shows how Fig. 2 should be adapted for small values of d .

Note that along with the removed vertices we also remove their incident edges. Depending on the local dimension d we may need to remove some vertices that are not leaves of the diagram, which in turn can decrease the number of root-leave paths, thus affecting the dimension of V^λ (see Fig. 3). In particular, $d_\lambda = \dim(V^\lambda)$ of the simple \mathcal{A}_n -module V^λ generally depends on the local dimension d .

5.5. Adapting JM elements and content vectors from $\mathcal{B}_{p,q}^\delta$ to $\mathcal{A}_{p,q}^d$. The Jucys–Murphy elements for semisimple walled Brauer algebras \mathcal{B} , given in eq. (85), can be used in the DLS algorithm in Sect. 4.7 to find the primitive central idempotents $\varepsilon^{\mathcal{B}}(\lambda)$ and canonical primitive idempotents $\varepsilon_T^{\mathcal{B}}$ of $\mathcal{B}_{p,q}^\delta$. In this section, we show how this procedure can be adapted to the partially transposed permutation matrix algebras $\mathcal{A}_{p,q}^d$. To construct the primitive central idempotents $\varepsilon^{\mathcal{A}}(\lambda)$ and canonical primitive idempotents $\varepsilon_T^{\mathcal{A}}$ of $\mathcal{A}_{p,q}^d$, we can use the modified Bratteli diagram from Sect. 5.4, and it only remains to adapt the Jucys–Murphy elements and content vectors from $\mathcal{B}_{p,q}^\delta$ to $\mathcal{A}_{p,q}^d$. Below we show that

we can use lifted versions of Jucys–Murphy elements of $\mathcal{B}_{p,q}^\delta$ and their content vectors for that purpose.

Throughout this section we set $\delta := d$ and, for every $k \in [p + q]$, let

$$J_k^A := \psi_{p,q}^d(J_k^B) \in \mathcal{A}_{p,q}^d, \tag{86}$$

where J_k^B are the Jucys–Murphy elements of \mathcal{B} given in eq. (85) and $\psi_{p,q}^d$ is the map from eq. (41). To establish that J_1^A, \dots, J_{p+q}^A is a Jucys–Murphy sequence for $\mathcal{A}_{p,q}^d$, we need to show that it is both additively central and separating (see Definition 4.7).

Lemma 5.1. *The sequence J_1^A, \dots, J_{p+q}^A is additively central in $\mathcal{A}_{p,q}^d$.*

Proof. For any $k \in [p + q]$, one can verify that the Jucys–Murphy elements J_k^B defined in eq. (85) satisfy $J_k^B \in \mathcal{X}_k^B$ and $J_1^B + \dots + J_k^B \in \mathcal{Z}(\mathcal{B}_k)$ [JK20]. Since $\psi_{p,q}^d$ is a homomorphism, $\psi_{p,q}^d(\mathcal{Z}(\mathcal{B}_k)) \subseteq \mathcal{Z}(\mathcal{A}_k)$ and hence $\psi_{p,q}^d(\mathcal{X}_k^B) \subseteq \mathcal{X}_k^A$. Therefore $J_k^A = \psi_{p,q}^d(J_k^B) \in \psi_{p,q}^d(\mathcal{X}_k^B) \subseteq \mathcal{X}_k^A$ and the sequence J_1^A, \dots, J_{p+q}^A is additively central in $\mathcal{A}_{p,q}^d$. \square

Lemma 5.2. *For any $T \in \text{Paths}(p + q)$ in the Bratteli diagram of $\mathcal{A}_{p,q}^d$,*

$$(J_1^A + \dots + J_{p+q}^A) \varepsilon_T^A = (\text{cont}(\lambda^l) + \text{cont}(\lambda^r) + d \cdot \text{size}(\lambda^r)) \varepsilon_T^A \tag{87}$$

where ε_T^A is the corresponding canonical primitive idempotent of $\mathcal{A}_{p,q}^d$, $\lambda = (\lambda^l, \lambda^r) = T[p + q]$ is the last vertex of the path T , $\text{cont}(\lambda)$ is the total content of all cells of the Young diagram λ , see eq. (76), and $\text{size}(\lambda)$ is the number of cells in λ .

Proof. See Appendix C for proof. Our proof is reminiscent of [BS12, Lemma 2.3], which is a similar statement for the walled Brauer algebras. \square

Corollary 5.3. *For any $k \in [p + q]$ and $T \in \text{Paths}(p + q)$, $J_k^A \varepsilon_T^A = c_T(k) \varepsilon_T^A$ where $c_T(k) = c_{T[k-1] \rightarrow T[k]}$ is the notion of content for the walled Brauer algebra $\mathcal{B}_{p,q}^d$, see Sect. 5.3.*

Proof. Let $T^k := \lambda_0 \rightarrow \lambda_1 \rightarrow \dots \rightarrow \lambda_k$ denote the first k edges of the path T . Recall from eq. (70) that the canonical primitive idempotents of \mathcal{A} are given by

$$\varepsilon_{T^k}^A = \varepsilon^A(\lambda_1) \varepsilon^A(\lambda_2) \dots \varepsilon^A(\lambda_k). \tag{88}$$

Each value of k effectively corresponds to truncating the Bratteli diagram to a certain p and q . Thus Lemma 5.2 with appropriate p and q allows us to compute the eigenvalue of $J_1^A + \dots + J_k^A$ for two consecutive $\varepsilon_{T^k}^A$:

$$(J_1^A + \dots + J_k^A) \varepsilon_{T^k}^A = (\text{cont}(\lambda_k^l) + \text{cont}(\lambda_k^r) + d \cdot \text{size}(\lambda_k^r)) \varepsilon_{T^k}^A, \tag{89}$$

$$(J_1^A + \dots + J_{k-1}^A) \varepsilon_{T^{k-1}}^A = (\text{cont}(\lambda_{k-1}^l) + \text{cont}(\lambda_{k-1}^r) + d \cdot \text{size}(\lambda_{k-1}^r)) \varepsilon_{T^{k-1}}^A. \tag{90}$$

Multiplying both equations with the primitive central idempotents of \mathcal{A} ranging from $\varepsilon^A(\lambda_k)$ to $\varepsilon^A(\lambda_{p+q})$ transforms the subscripts T^k and T^{k-1} into $T^{p+q} = T$:

$$(J_1^A + \dots + J_k^A) \varepsilon_T^A = (\text{cont}(\lambda_k^l) + \text{cont}(\lambda_k^r) + d \cdot \text{size}(\lambda_k^r)) \varepsilon_T^A, \tag{91}$$

$$(J_1^A + \dots + J_{k-1}^A) \varepsilon_T^A = (\text{cont}(\lambda_{k-1}^l) + \text{cont}(\lambda_{k-1}^r) + d \cdot \text{size}(\lambda_{k-1}^r)) \varepsilon_T^A. \tag{92}$$

Subtracting these two equations we get

$$J_k^A \varepsilon_T^A = \left(\text{cont}(\lambda_k^l) - \text{cont}(\lambda_{k-1}^l) + \text{cont}(\lambda_k^r) - \text{cont}(\lambda_{k-1}^r) + d \cdot (\text{size}(\lambda_k^r) - \text{size}(\lambda_{k-1}^r)) \right) \varepsilon_T^A. \tag{93}$$

If $k \leq p$ then $\lambda_k^r = \lambda_{k-1}^r = \emptyset$ and $\text{cont}(\lambda_k^l) - \text{cont}(\lambda_{k-1}^l) = j - i$, where (i, j) is the location where adding a cell to the Young diagram λ_{k-1}^l transforms it into λ_k^l . If $k > p$ then there are two cases. If $\lambda_k^r = \lambda_{k-1}^r$ then $\text{cont}(\lambda_k^l) - \text{cont}(\lambda_{k-1}^l) = i - j$ because the cell (i, j) is removed from λ_{k-1}^l . If $\lambda_k^l = \lambda_{k-1}^l$ then $\text{cont}(\lambda_k^l) - \text{cont}(\lambda_{k-1}^l) = j - i$ and $\text{size}(\lambda_k^r) - \text{size}(\lambda_{k-1}^r) = 1$, where (i, j) is the location where adding a cell to the Young diagram λ_{k-1}^r transforms it into λ_k^r . In either case,

$$J_k^A \varepsilon_T^A = c_T(k) \varepsilon_T^A, \tag{94}$$

where $c_T(k) = c_{T[k-1] \rightarrow T[k]}$ is exactly the notion of content for the walled Brauer algebra $\mathcal{B}_{p,q}^d$ as defined in Sect. 5.3. □

Lemma 5.4. *If $S, T \in \text{Paths}(p + q)$ are two paths in the Bratteli diagram of $\mathcal{A}_{p,q}^d$ then $S = T$ if and only if $c_S = c_T$.*

Proof. The forward direction is obvious. For the reverse implication, assume that $S \neq T$ and $c_S = c_T$. We consider two cases depending on the first location $k \in [p + q]$ where the two paths differ, i.e., $S[k] \neq T[k]$ while $S[k - 1] = T[k - 1] =: (\lambda^l, \lambda^r)$.

If $k \leq p$, we can only add cells to the Young diagram λ^l , so let (i, j) and (i', j') denote the two possible locations. It follows from Corollary 5.3 that $c_T(k) = j - i$ and $c_S(k) = j' - i'$, and hence $j - i = j' - i'$. Since any Young diagram has at most one location on any diagonal where a new cell can be added, $(i, j) = (i', j')$ and therefore $S[k] = T[k]$.

If $k > p$, the condition $c_T(k) = c_S(k)$ can be satisfied only if we assume (without loss of generality) that $c_T(k) = i^l - j^l$ and $c_S(k) = j^r - i^r + d$, implying that $i^l + i^r = j^l + j^r + d$ for the removed cell (i^l, j^l) of λ^l and the added cell (i^r, j^r) of λ^r . In particular, $i^l + i^r = j^l + j^r + d \geq d + 2$ since $j^l > 0$ and $j^r > 0$. On the other hand, the total length of the two diagrams satisfies $\text{len}(\lambda^l) + \text{len}(\lambda^r) \leq d$, which implies that $i^l + i^r \leq d$, a contradiction. □

Corollary 5.5. *The sequence J_1^A, \dots, J_{p+q}^A is separating in \mathcal{A} .*

Proof. This follows from Lemma 5.4 and Proposition 3.5 of [DLS18]. □

Theorem 5.6. *J_1^A, \dots, J_{p+q}^A is a Jucys–Murphy sequence for \mathcal{A} .*

Proof. This follows from Lemma 5.1 and corollary 5.5. □

Theorem 5.6 establishes that the operators J_k^A , which were obtained in eq. (86) by lifting the Jucys–Murphy sequence J_k^B of the walled Brauer algebras $\mathcal{B}_{p,q}^d$ to the matrix algebras $\mathcal{A}_{p,q}^d$, are indeed Jucys–Murphy elements of $\mathcal{A}_{p,q}^d$. Furthermore, Corollary 5.3 shows that the content vectors of the two families of algebras agree.

5.6. *Primitive central and canonical primitive idempotents of $\mathcal{A}_{p,q}^d$.* We have now established the three ingredients required to apply the [DLS18] algorithm to the partially transposed matrix algebras $\mathcal{A}_{p,q}^d$:

- (1) the Bratteli diagram for \mathcal{A} is obtained by truncating the Bratteli diagram of \mathcal{B} as discussed in Sect. 5.4,
- (2) the Jucys–Murphy elements of \mathcal{A} are obtained by applying $\psi_{p,q}^d$ to the Jucys–Murphy elements of \mathcal{B} given in eq. (85),
- (3) the content vectors of \mathcal{A} agree with those of \mathcal{B} and are given in Sect. 5.3.

This allows us to use the algorithm described in Sect. 4.7 to compute the primitive central idempotents and canonical primitive idempotents of \mathcal{A} .

A major advantage of our approach is that the entire computation can be performed by employing linear combinations of diagrams instead of actual matrices (i.e., staying within the diagrammatic walled Brauer algebra $\mathcal{B}_{p,q}^d$ rather than working in the matrix algebra $\mathcal{A}_{p,q}^d$). This results in a diagrammatic representation of an idempotent of $\mathcal{A}_{p,q}^d$ as a preimage of the actual idempotent under $\psi_{p,q}^d$. More explicitly, the primitive central idempotents of $\mathcal{A}_{p,q}^d$ can be computed iteratively as

$$\varepsilon^{\mathcal{A}}(\mu) := \sum_{\lambda: \lambda \rightarrow \mu} \psi_{p,q}^d \left(P_{\lambda \rightarrow \mu} (J_k^{\mathcal{B}}) \right) \varepsilon^{\mathcal{A}}(\lambda), \tag{95}$$

where the polynomials $P_{\lambda \rightarrow \mu}$ from eq. (77) are evaluated for the Bratteli diagram of the family \mathcal{A} described in Sect. 5.4. Similarly to eq. (79), canonical primitive idempotents $\varepsilon_{\mathbb{T}}$ are adapted to the matrix algebras \mathcal{A} as follows:

$$\varepsilon_{\mathbb{T}}^{\mathcal{A}} = \psi_{p,q}^d \left(\prod_{k=1}^{p+q} \prod_{\mu: \lambda_{k-1} \rightarrow \mu \neq \lambda_k} \frac{J_k^{\mathcal{B}} - c_{\lambda_{k-1} \rightarrow \mu}}{c_{\lambda_{k-1} \rightarrow \lambda_k} - c_{\lambda_{k-1} \rightarrow \mu}} \right), \tag{96}$$

where the second product runs over edges in the Bratteli diagram of the family \mathcal{A} .

This representation of idempotents is more compact compared to the naive one when d is large, and easily amenable to further fast diagrammatic calculations. This allows to significantly lower the computational complexity of various tasks within the partially transposed permutation matrix algebra, as illustrated in the next section for a certain class of optimization problems.

6. Reducing Unitary-Equivariant SDPs to LPs

In this section we derive our main result—a pre-processing algorithm for LP solvers which accepts a sparse SDP with a $U(d)$ -equivariant constraint as input. Our algorithm also requests one of several additional symmetries (see Sect. 6.1) that guarantee that the provided SDP reduces to an LP. While the input problem has a compact representation due to all involved matrices being sparse, naively solving it might be impossible in practice due to a prohibitively large d (the SDP matrix variable has dimension d^{p+q}). Our algorithm converts the implicit input LP to an explicit smaller LP whose naive representation has size that no longer depends on d . Although it may generally not be sparse, this LP is much smaller and can thus be further supplied as input to any standard LP solver.

In Sect. 6.1 we list the additional symmetries our algorithm requires, in Sect. 6.2 we specify the input format of our algorithm, and in Sect. 6.3 we state and prove our main result.

6.1. *Types of symmetries.* To achieve a reduction from SDP to LP, the SDP matrix variable X needs some further symmetry in addition to unitary equivariance. For example, one option is the $S_p \times S_q$ permutational symmetry which is natural in the context of $p \rightarrow q$ quantum channels, see Lemma 2.4. We show in Sect. 6.1.3 that an SDP with such symmetry reduces to an LP when $\min(p, q) \leq 2$. The full list of possible symmetries we consider for the SDP variable X is as follows.

Definition 6.1. A matrix $X \in \text{End}(V^{p,q})$ possesses

- the $S_p \times S_q$ permutational symmetry if for every $\sigma \in \mathbb{C}(S_p \times S_q) \subset \mathcal{B}_{p,q}^d$

$$[X, \psi_{p,q}^d(\sigma)] = 0, \tag{97}$$

or equivalently $X \in \mathcal{Z}_{\psi_{p,q}^d(\mathbb{C}(S_p \times S_q))}(\mathcal{A}_{p,q}^d)$;

- the walled Brauer algebra symmetry if for every $\sigma \in \mathcal{B}_{p,q}^d$, see Sect. 3.2,

$$[X, \psi_{p,q}^d(\sigma)] = 0, \tag{98}$$

or equivalently $X \in \mathcal{Z}(\mathcal{A}_{p,q}^d)$;

- the Gelfand–Tsetlin symmetry if for every $A \in \mathcal{X}_{p+q}^A$, see eq. (83),

$$[X, A] = 0, \tag{99}$$

or equivalently $X \in \mathcal{X}_{p+q}^A$ since \mathcal{X}_{p+q}^A is the maximal commutative subalgebra of $\mathcal{A}_{p,q}^d$ (see Sect. 4.5).

The last two symmetries have an intuitive interpretation in Schur basis. Recall from Remark 3.10 that there exists a mixed Schur transform $U_{\text{Sch}(p,q)} \in U(V^{p,q})$ that block-diagonalizes any unitary-equivariant X :

$$U_{\text{Sch}(p,q)} X U_{\text{Sch}(p,q)}^\dagger = \bigoplus_{\lambda \in \text{Irr}(\mathcal{A}_{p,q}^d)} [X_\lambda \otimes I_{m_\lambda}]. \tag{100}$$

We will assume that $U_{\text{Sch}(p,q)}$ is adapted to the Gelfand–Tsetlin basis, meaning that each X_λ is expressed in this basis. Each symmetry from Definition 6.1 results in some simplification of the matrix variable X since each block X_λ assumes a special form. We discuss this in more detail in the following sections.

6.1.1. *Full walled Brauer algebra symmetry* The maximal possible symmetry that can be assumed is the full walled Brauer algebra symmetry, which means that each X_λ in eq. (100) is proportional to the identity matrix, i.e.,

$$X_\lambda = v_\lambda I_{d_\lambda} \tag{101}$$

for some scalar $v_\lambda \in \mathbb{R}$. In this case, the semidefinite constraint $X_\lambda \geq 0$ reduces to $v_\lambda \geq 0$, thus simplifying the problem from an SDP to an LP with variables $\{v_\lambda : \lambda \in \text{Irr}(\mathcal{A}_{p,q}^d)\}$. The total number of variables in the LP is

$$n_{p,q}^d(\text{Brauer}) := |\text{Irr}(\mathcal{A}_{p,q}^d)| = \sum_{k=0}^{\min(p,q)} f_{p-k,q-k}^d, \tag{102}$$

where $f_{p-k,q-k}^d$ is the number of pairs (λ^l, λ^r) of Young diagrams such that $\lambda^l \vdash p - k$, $\lambda^r \vdash q - k$ and $\text{len}(\lambda^l) + \text{len}(\lambda^r) \leq d$. In this case, we can write X as a linear combination of the primitive central idempotents from eq. (95):

$$X = \sum_{\lambda \in \text{Irr}(\mathcal{A}_{p,q}^d)} v_\lambda \varepsilon^{\mathcal{A}}(\lambda). \tag{103}$$

6.1.2. Gelfand–Tsetlin symmetry A minimal symmetry that can be assumed is the *Gelfand–Tsetlin symmetry*, which means that each X_λ is diagonal in the Gelfand–Tsetlin basis, i.e.,

$$X_\lambda = \sum_{i=1}^{d_\lambda} v_{\lambda,i} |i\rangle \langle i| \tag{104}$$

for some scalars $v_{\lambda,i} \in \mathbb{R}$ that become the variables of the LP. The $X \geq 0$ constraint then reduces to $v_{\lambda,i} \geq 0$ for all $\lambda \in \text{Irr}(\mathcal{A}_{p,q}^d)$ and $i \in [d_\lambda]$. The total number of LP variables in this case is

$$n_{p,q}^d(\text{Gelfand–Tsetlin}) := \sum_{\lambda \in \text{Irr}(\mathcal{A}_{p,q}^d)} d_\lambda, \tag{105}$$

where $d_\lambda = \dim(V^\lambda)$ is the dimension of the corresponding simple module of $\mathcal{A}_{p,q}^d$. In this case, we can write X as a linear combination of the canonical primitive idempotents from eq. (96):

$$X = \sum_{T \in \text{Paths}(\mathcal{A}_{p,q}^d)} v_T \varepsilon_T^{\mathcal{A}}. \tag{106}$$

6.1.3. $S_p \times S_q$ permutational symmetry A somewhat intermediate symmetry that can be assumed is the *$S_p \times S_q$ permutational symmetry*. This symmetry allows us to simplify X_λ to

$$X_\lambda \cong \bigoplus_{\substack{\mu \vdash p, \text{len}(\mu) \leq d \\ \nu \vdash q, \text{len}(\nu) \leq d}} (I_\mu \otimes I_\nu) \otimes \tilde{X}_{\mu,\nu}^\lambda, \tag{107}$$

where $\tilde{X}_{\mu,\nu}^\lambda$ is a Hermitian matrix of dimension $m_{\mu,\nu}^\lambda(d)$, see eq. (200) in Appendix B, and a priori we do not know the basis on the right-hand side.

There are two cases when this symmetry leads to a reduction from SDP to LP. We show in Lemmas B.1 and B.2 that $m_{\mu,\nu}^\lambda(d) \in \{0, 1\}$ when $\min(p, q) \leq 2$ or $d = 2$, meaning that the corresponding term in eq. (107) either drops out or $\tilde{X}_{\mu,\nu}^\lambda$ becomes a scalar $\tilde{x}_{\mu,\nu}^\lambda \in \mathbb{R}$:

$$X_\lambda \cong \bigoplus_{\substack{\mu \vdash p, \text{len}(\mu) \leq d \\ \nu \vdash q, \text{len}(\nu) \leq d \\ m_{\mu,\nu}^\lambda(d) = 1}} \tilde{x}_{\mu,\nu}^\lambda (I_\mu \otimes I_\nu). \tag{108}$$

We show in Proposition 6.2 that when $\min(p, q) = 1$ each block X_λ becomes diagonal specifically in the Gelfand–Tsetlin basis. In contrast to eq. (104), some of the diagonal

entries of X_λ must be equal in this case. In the following proposition we assume $q = 1$. The argument is completely analogous when $p = 1$ since one just has to use a different sequence \mathcal{A} of algebras $\mathbb{C} \hookrightarrow \mathcal{A}_{0,1}^d \hookrightarrow \dots \hookrightarrow \mathcal{A}_{0,q}^d \hookrightarrow \mathcal{A}_{1,q}^d$ when constructing the Bratteli diagram and the corresponding Gelfand–Tsetlin basis.

Proposition 6.2. *Fix $d \geq 2$ and $p \geq 1$, and let $X \in \text{End}(V^{p,1})$ be a Hermitian matrix with unitary equivariance and $S_p \times S_1$ symmetry:*

$$[X, U^{\otimes p} \otimes \bar{U}] = 0, \quad \forall U \in U(V), \tag{109}$$

$$[X, \psi_p^d(\pi) \otimes I_d] = 0, \quad \forall \pi \in S_p. \tag{110}$$

Then X can be written as

$$X = \sum_{T \in \text{Paths}(\mathcal{A}_{p,1}^d)} v_{T[p],T[p+1]} \varepsilon_T^A, \tag{111}$$

where $v_{T[p],T[p+1]} \in \mathbb{R}$ depends only on the last edge of path T .

Proof. Let us first understand in what way the $S_p \times S_1$ symmetry is special among general $S_p \times S_q$ symmetries. Since the group S_1 is trivial, $\mathbb{C}(S_p \otimes S_1) \cong \mathbb{C}S_p$. Moreover, $\psi_{p,1}^d(\pi \times e) = \psi_p^d(\pi) \otimes I_d$ for any $\pi \in S_p$, where $e \in S_1$ denotes the identity permutation. Hence

$$\psi_{p,1}^d(\mathbb{C}(S_p \otimes S_1)) = \psi_p^d(\mathbb{C}S_p) \otimes I_d = \mathcal{A}_{p,0}^d \hookrightarrow \mathcal{A}_{p,1}^d, \tag{112}$$

so the algebra $\mathcal{A}_{p,0}^d$ generated by the $S_p \times S_1$ symmetry appears in the multiplicity-free family $\mathbb{C} \hookrightarrow \mathcal{A}_{1,0}^d \hookrightarrow \dots \hookrightarrow \mathcal{A}_{p,0}^d \hookrightarrow \mathcal{A}_{p,1}^d$.

We can use this observation to get a better grip on the matrix X . Thanks to the mixed Schur–Weyl duality (Theorem 3.8), eq. (109) implies that $X \in \mathcal{A}_{p,1}^d$. Also, notice from eq. (112) that eq. (110) is equivalent to $[X, \mathcal{A}_{p,0}^d] = 0$. By combining these two observations we see that $X \in \mathcal{Z}_{\mathcal{A}_{p,0}^d}(\mathcal{A}_{p,1}^d)$.

Writing any $A \in \mathcal{A}_{p,1}^d$ in Schur basis we get

$$U_{\text{Sch}(p,1)} A U_{\text{Sch}(p,1)}^\dagger = \bigoplus_{\lambda \in \text{Irr}(\mathcal{A}_{p,1}^d)} [A_\lambda \otimes I_{m_\lambda}], \tag{113}$$

where the blocks A_λ are expressed in the Gelfand–Tsetlin basis. If we instead take $A \in \mathcal{A}_{p,0}^d$ then, thanks to how the Gelfand–Tsetlin basis is recursively constructed from paths in the Bratteli diagram, the A_λ in eq. (113) are block-diagonal. That is, $A_\lambda = \bigoplus_\mu A_{\lambda,\mu}$ (there are no multiplicities here since the embedding $\mathcal{A}_{p,0}^d \hookrightarrow \mathcal{A}_{p,1}^d$ is part of a multiplicity-free family).

Since $[X, A] = 0$ for all $A \in \mathcal{A}_{p,0}^d$, the blocks X_λ of X are of the form $X_\lambda = \bigoplus_\mu c_{\lambda,\mu} I_{\lambda,\mu}$ for some $c_{\lambda,\mu} \in \mathbb{R}$. In particular, they are diagonal in the Gelfand–Tsetlin basis, so $X = \sum_{T \in \text{Paths}(\mathcal{A}_{p,1}^d)} v_T \varepsilon_T^A$ for some $v_T \in \mathbb{R}$ for each path T . Moreover, all variables v_T that correspond to paths T that go through a fixed vertex at level p of the Bratteli diagram have the same value (this vertex labels a simple $\mathcal{A}_{p,0}^d$ -module). Formally this means that $v_T = v_S$ for every $T, S \in \text{Paths}(\mathcal{A}_{p,1}^d)$ such that the paths T and S in the Bratteli diagram of $\mathcal{A}_{p,1}^d$ share the same last edge, i.e., $(T[p], T[p+1]) = (S[p], S[p+1])$. In particular, if $(T[p], T[p+1]) = (\mu, \lambda)$ then $v_T = c_{\lambda,\mu}$. \square

Table 1. Summary of symmetries from Definition 6.1. For each symmetry, we provide pointers to the corresponding ansatz for X , the number of variables $n_{p,q}^d$ in this ansatz, and tables for numerical values of $n_{p,q}^d$. Note that the ansatz (111) of X in case of the $S_p \times S_q$ symmetry is valid only when $q = 1$; a similar formula can also be obtained for $p = 1$ as explained just before Proposition 6.2

Symmetry	Ansatz for X	Formula for $n_{p,q}^d$	Values of $n_{p,q}^d$
Full walled Brauer algebra	eq. 103	eq. 102	Table 4
$S_p \times S_q$	eq. 111	eq. 114	Table 5
Gelfand–Tsetlin	eq. 106	eq. 105	Table 6
Only unitary equivariance	eq. 100	$\dim(\mathcal{A}_{p,q}^d)$	Table 7

The number of variables in this case is

$$n_{p,q}^d(S_p \times S_q) := \sum_{\substack{\lambda \in \text{Irr}(\mathcal{A}_{p,q}^d) \\ \mu \vdash p, \text{len}(\mu) \leq d \\ \nu \vdash q, \text{len}(\nu) \leq d}} (m_{\mu,\nu}^\lambda(d))^2. \tag{114}$$

This number can be easily calculated numerically using the results of Appendix B. The results of this calculation for small d, p, q can be found in Appendix E.

6.1.4. Summary The chosen symmetry, together with unitary equivariance, guarantees that X can be expressed as a linear combination of idempotents of the algebra $\mathcal{A}_{p,q}^d$, see eqs. (106), (103) and (111):

$$X = \sum_{i=1}^n v_i \varepsilon_i^A, \tag{115}$$

where the number of terms n is given by eqs. (102), (105) and (114), respectively (see Table 1 for a summary). Since we effectively know the basis in which X is diagonal, the SDP reduces to an LP. In particular, the scalars $v_1, \dots, v_n \in \mathbb{R}$ in eq. (115) will be the variables of the output LP produced by our algorithm. The number of variables n varies dramatically depending on the chosen symmetry type, see Appendix E.

6.2. Input specification. Our algorithm accepts a sparse SDP composed of the following objects:

- (1) $p, q \geq 0$ – the number of input and output systems,
- (2) $d \geq 2$ – the local dimension of $V^{p,q} = (\mathbb{C}^d)^{\otimes p} \otimes (\mathbb{C}^d)^{* \otimes q}$,
- (3) $X \geq 0$ – a Hermitian matrix variable acting on $V^{p,q}$ that has to obey the $U(d)$ -equivariance condition in eq. (10),
- (4) (in)equality constraints that involve constant sparse matrices,
- (5) a desired additional type of symmetry (see Sect. 6.1 for possible options) which guarantees that the problem reduces to an LP.

Our algorithm outputs an explicit LP, equivalent to the input one, whose size does not depend on d and which can be further fed as an input to a standard LP solver. We are concerned only with this pre-processing step and its complexity.

The input to our algorithm is an SDP in the following form:

$$\begin{aligned}
 & \max_X \quad \text{Tr}(CX) \\
 & \text{s.t.} \quad \text{Tr}(A_k X) \leq b_k, \quad \forall k \in [m_1], \\
 & \quad \quad \text{Tr}_{S_k}(X) = D_k, \quad \forall k \in [m_2], \\
 & \quad \quad [X, U^{\otimes p} \otimes \bar{U}^{\otimes q}] = 0, \quad \forall U \in \text{U}(\mathbb{C}^d), \\
 & \quad \quad X \succeq 0,
 \end{aligned} \tag{116}$$

where m_1 and m_2 denote the number of constraints that involve full trace and partial trace, respectively. Recall from eq. (10) that the penultimate condition is equivalent to unitary-equivariance of the superoperator associated to X .

The Hermitian matrices C and A_k in eq. (116) are constant and s -sparse, meaning that they can be written as a linear combination of at most s terms, where each term is either $\psi_{p,q}^d(\sigma)$ for some diagram σ of the walled Brauer algebra $\mathcal{B}_{p,q}^d$ or an elementary standard basis matrix whose all entries are 0 and only one entry is 1, i.e., $|i\rangle\langle j|$ for some $i, j \in [d]^{p+q}$. Each D_k is a Hermitian linear combination of at most s diagrams from the walled Brauer algebra \mathcal{B}_{p_k,q_k}^d obtained by removing the nodes $S_k \subseteq [p+q]$ from $\mathcal{B}_{p,q}^d$. The remaining number of nodes on each side of the wall of \mathcal{B}_{p_k,q_k}^d is $p_k := p - |S_k \cap [p]|$ and $q_k := q - |(S_k - p) \cap [q]|$. The total size of the input SDP in terms of the number of scalars needed to specify the matrices C, A_k, D_k is⁷

$$(1 + m_1 + m_2)s. \tag{117}$$

The SDP may contain additional scalar variables that need not obey the unitary equivariance condition. Such variables do not require any pre-processing by our algorithm, so they do not incur additional costs in our setting.

If the input SDP contains partial trace constraints, our algorithm has the following technical restriction: we require the local dimension d to be sufficiently large, namely

$$d \geq p + q - \min_k |S_k|. \tag{118}$$

In particular, if X is a Choi matrix of a $p \rightarrow q$ channel and we include the partial trace constraint $\text{Tr}_{V^q}(X) = I_{V^p}$ to capture trace preservation, we require that $d \geq p$. Due to this we cannot apply our formalism to the setting of [Buh22] where $d = 2$ and p is large. To remove this restriction, one would have to know the kernel of the map $\psi_{p,q}^d$ in order to correctly process the partial trace constraints in the SDP (see Example 3.3 for an instance where the kernel is non-trivial).

Remark 6.3. As outlined in Appendix D, one way to obtain the kernel of $\psi_{p,q}^d$ is by using the primitive idempotents ε_T^A to compute the blocks of $\psi_{p,q}^d(\sum_{j=1}^{(p+q)!} b_j \sigma_j)$ where b_j are symbolic variables. Equating these blocks to zero produces linear equations in b_j that reveal the linear dependencies among the matrices $\psi_{p,q}^d(\sigma_j)$. We can store this information in a database and use it to reduce the complexity of multiplication in the diagrammatic algebra $(\psi_{p,q}^d)^{-1}(\mathcal{A}_{p,q}^d)$, i.e., the preimage of $\mathcal{A}_{p,q}^d$ under $\psi_{p,q}^d$. This can lead to an improved complexity in our main result Theorem 6.4 since the complexity parameter N can be lowered from $(p+q)!$ to $\dim(\mathcal{A}_{p,q}^d)$.

⁷ For simplicity, we do not count the m_2 additional parameters needed to specify the scalars b_k and the additional information needed to specify the subsets S_k in eq. (116).

6.3. *Main result.* The input to our algorithm is an SDP of the form (116) which involves a unitary-equivariant constraint and has the following parameters:

- p and q – number of input and output systems,
- d – local dimension of each system,
- s – sparsity of matrices C, A_k, D_k ,
- m_1 – number of inequality constraints,
- m_2 – number of equality constraints with partial trace.

We assume that p and q are small constants while d may generally be large. The complexity of our algorithm will scale in

$$N := (p + q)! \tag{119}$$

but not d . This is in contrast to the naive approach of solving an SDP with a matrix variable X of dimension d^{p+q} . While the naive approach quickly becomes impractical as d grows, our method does not suffer from this problem. In fact, it even offers performance improvements for small d such as $d = 2$. Our algorithm requires assuming that X has one of the symmetries listed in Sect. 6.1, which guarantees that the SDP reduces to an LP.

The following is a formal statement of our main result.

Theorem 6.4. *Any SDP of the form (116), where X has one of the symmetries listed in Definition 6.1, can be converted to an equivalent LP with n variables and $m_1 + m_2N + n$ constraints. The number of variables $n := n_{p,q}^d(\text{sym})$, which depends on p, q, d and the chosen symmetry sym (see Table 1), can always be bounded as*

$$n_{p,q}^d(\text{sym}) \leq \dim(\mathcal{A}_{p,q}^d) \leq (p + q)! = N. \tag{120}$$

The algorithm consists of two parts:

- (1) an input-independent pre-computation that needs to be done only once for each set of parameters p, q, d , and whose complexity does not scale in d ,
- (2) and SDP-to-LP conversion that takes time

$$(1 + m_1 + m_2)s \cdot nN \tag{121}$$

where $(1 + m_1 + m_2)s$ is the size of the input SDP.

If $d \geq p + q$, the run-time of the pre-computation does not scale in d , while for small $d < p + q$ additional speedup is gained. If the SDP contains partial trace constraints, i.e., $m_2 > 0$, we require that $d \geq p + q - \min_k |S_k|$.

Remark 6.5. For the sake of simplicity we will ignore various details in our analysis. In particular, we will assume that the following operations take constant time: multiplying two walled Brauer algebra diagrams, contracting a diagram with a rank-1 matrix, or computing the (partial) trace of a diagram. In reality the complexity of these operations scales with $p + q$, which is small compared to our yardstick $N = (p + q)!$. Similarly, we will ignore the fact that the input size scales as $2(p + q) \log_2 d$ when the SDP contains rank-1 matrices. Finally, we will also ignore the fact that storing the value $d^{\text{loops}(\sigma)}$ requires $(p + q) \log_2 d$ bits.

Proof. Our algorithm consists of two parts: (1) pre-computation of a database of $\mathcal{A}_{p,q}^d$ idempotents and (2) processing the input SDP to an LP.

The pre-computation of a database of $\mathcal{A}_{p,q}^d$ idempotents can be done upfront since it depends only on the parameters p, q, d but not the input SDP. The type of idempotents needed depends on the specified symmetry type sym (see Sect. 6.1). In either case, they can be computed diagrammatically using the DLS algorithm from Sect. 5.6. It produces a list of preimages $\varepsilon_1, \dots, \varepsilon_n \in \mathcal{B}_{p,q}^d$ of $\mathcal{A}_{p,q}^d$ idempotents, with each ε_i expressed as a linear combination of walled Brauer algebra diagrams σ_j ⁸:

$$\varepsilon_i = \sum_{j=1}^{(p+q)!} \alpha_{ij} \sigma_j. \tag{122}$$

The resulting $n \times (p+q)!$ coefficient matrix α is the output of the pre-computation step. By construction, its entries are rational.

The second part of the algorithm requires a $U(d)$ -equivariant SDP as input and reduces it to an explicit LP whose size is d -independent. The main idea of this algorithm is that we can evaluate all traces appearing in eq. (116) diagrammatically without ever explicitly computing any of the $d^{p+q} \times d^{p+q}$ matrices involved (see Appendix A). Let us discuss this step in more detail.

Due to unitary equivariance and the additional symmetry sym , we can express the matrix variable X as a linear combination of idempotents $\psi_{p,q}^d(\varepsilon_i)$ with unknown coefficients $v_i \in \mathbb{R}$ as in eq. (115):

$$X = \sum_{i=1}^n v_i \psi_{p,q}^d(\varepsilon_i). \tag{123}$$

These coefficients will be the variables of the output LP. The number $n = n_{p,q}^d(sym)$ of variables v_i and idempotent preimages ε_i depends on the type of symmetry (see Table 1 in Sect. 6.1). Since $\psi_{p,q}^d(\varepsilon_i) \geq 0$ and these idempotents are mutually orthogonal for different i , the positive semidefinite constraint $X \geq 0$ reduces to

$$v_i \geq 0, \quad \forall i = 1, \dots, n. \tag{124}$$

For the target function and each of the constraints in eq. (116), we can evaluate the corresponding trace via diagram contraction. The main idea is to expand X as a linear combination of $\psi_{p,q}^d(\sigma_j)$ using eqs. (122) and (123):

$$X = \sum_{i=1}^n v_i \sum_{j=1}^{(p+q)!} \alpha_{ij} \psi_{p,q}^d(\sigma_j). \tag{125}$$

Since the constant matrices C and A_k in eq. (116) are s -sparse, they are already provided to us as linear combinations of diagrams and elementary rank-1 matrices. Using these expansions together with eq. (125), we can diagrammatically evaluate all traces in eq. (116) by linearity (see Appendix A for more details).

⁸ By knowing the kernel of $\psi_{p,q}^d$, we can express each ε_i more economically as $\varepsilon_i = \sum_{j=1}^m \alpha_{ij} \sigma_j$ where $m = \dim(\mathcal{A}_{p,q}^d) \leq (p+q)!$.

In particular, the objective function can be written in terms of the LP variables v_i as follows:

$$\text{Tr}(CX) = \sum_{i=1}^n v_i \sum_{j=1}^{(p+q)!} \alpha_{ij} \text{Tr}(C\psi_{p,q}^d(\sigma_j)) = \sum_{i=1}^n v_i c_i = c^\top v \quad (126)$$

where $c \in \mathbb{R}^n$ is a vector with entries

$$c_i := \sum_{j=1}^{(p+q)!} \alpha_{ij} \text{Tr}(C\psi_{p,q}^d(\sigma_j)). \quad (127)$$

Since C is s -sparse, we can use Proposition A.1 in Appendix A to evaluate the trace. The total time it takes to compute the vector c is

$$\#i \cdot \#j \cdot s = nNs. \quad (128)$$

Similarly, the k -th inequality constraint can be expressed as

$$\text{Tr}(A_k X) = a_k^\top v \leq b_k \quad (129)$$

where each $a_k \in \mathbb{R}^n$ is a vector with entries

$$(a_k)_i := \sum_{j=1}^{(p+q)!} \alpha_{ij} \text{Tr}(A_k \psi_{p,q}^d(\sigma_j)). \quad (130)$$

The total time it takes to compute the tensor a is

$$\#k \cdot \#i \cdot \#j \cdot s = m_1 nNs. \quad (131)$$

Next, let us fix k and deal with the k -th partial trace equality constraint. First, we expand the partial trace $\text{Tr}_{S_k}(X)$ by linearity using eq. (125):

$$\text{Tr}_{S_k}(X) = \sum_{i=1}^n v_i \sum_{j=1}^{(p+q)!} \alpha_{ij} \text{Tr}_{S_k}(\psi_{p,q}^d(\sigma_j)). \quad (132)$$

If $\sigma_j^{S_k}$ denotes the diagram σ_j with pairs of nodes in the set S_k that are opposite to each other contracted, then the matrix corresponding to σ_j has partial trace

$$\text{Tr}_{S_k}(\psi_{p,q}^d(\sigma_j)) = d^{\text{loops}_{S_k}(\sigma_j)} \psi_{p_k, q_k}^d(\sigma_j^{S_k}) \quad (133)$$

where $\text{loops}_{S_k}(\sigma_j)$ is the number of loops formed and (p_k, q_k) is the remaining number of systems on each side of the wall, see Proposition A.2 in Appendix A. Substituting this into eq. (132),

$$\text{Tr}_{S_k}(X) = \sum_{i=1}^n v_i \sum_{j=1}^{(p+q)!} \alpha_{ij} d^{\text{loops}_{S_k}(\sigma_j)} \psi_{p_k, q_k}^d(\sigma_j^{S_k}). \quad (134)$$

We need to compare this to D_k and derive a set of linear constraints.

Since D_k is a linear combination of diagrams,

$$D_k = \sum_{l=1}^{(p_k+q_k)!} e_{kl} \psi_{p_k, q_k}^d(\rho_l) \tag{135}$$

for some coefficients $e_{kl} \in \mathbb{R}$, where $\rho_l \in \mathcal{B}_{p_k, q_k}^d$ ranges over all walled Brauer algebra diagrams on $p_k + q_k = p + q - |S_k|$ nodes. Since $\sigma_j^{S_k}$ is a diagram in \mathcal{B}_{p_k, q_k}^d ,

$$\psi_{p_k, q_k}^d(\sigma_j^{S_k}) = \sum_{l=1}^{(p_k+q_k)!} \delta(\sigma_j^{S_k}, \rho_l) \psi_{p_k, q_k}^d(\rho_l) \tag{136}$$

where δ denotes the Kronecker delta function. Substituting this in eq. (134),

$$\text{Tr}_{S_k}(X) = \sum_{l=1}^{(p_k+q_k)!} \sum_{i=1}^n v_i \sum_{j=1}^{(p+q)!} \alpha_{ij} d^{\text{loops}_{S_k}(\sigma_j)} \delta(\sigma_j^{S_k}, \rho_l) \psi_{p_k, q_k}^d(\rho_l). \tag{137}$$

Because of the assumption (118) that $d \geq p + q - \min_k |S_k|$, the representation ψ_{p_k, q_k}^d is faithful due to Theorem 3.8, and hence the matrices $\psi_{p_k, q_k}^d(\rho_l)$ are linearly independent. Comparing the coefficients at $\psi_{p_k, q_k}^d(\rho_l)$ in eqs. (135) and (137), we conclude that

$$\sum_{i=1}^n v_i \sum_{j=1}^{(p+q)!} \alpha_{ij} d^{\text{loops}_{S_k}(\sigma_j)} \delta(\sigma_j^{S_k}, \rho_l) = e_{kl}. \tag{138}$$

In other words, for every $1 \leq l \leq (p_k + q_k)!$ we get a linear constraint

$$\sum_{i=1}^n v_i (d_{kl})_i = d_{kl}^\top v = e_{kl} \tag{139}$$

where each $d_{kl} \in \mathbb{R}^n$ is a vector with entries

$$(d_{kl})_i := \sum_{j=1}^{(p+q)!} \alpha_{ij} d^{\text{loops}_{S_k}(\sigma_j)} \delta(\sigma_j^{S_k}, \rho_l). \tag{140}$$

To compute all entries d_{kli} of the above tensor, we can fix k and i and then evaluate the sum over j . For each j , we determine l such that $\sigma_j^{S_k} = \rho_l$ and add the contribution $\alpha_{ij} d^{\text{loops}_{S_k}(\sigma_j)}$ to the corresponding entry d_{kli} . The total time it takes to perform this computation is

$$\#k \cdot \#i \cdot \#j = m_2 \cdot n \cdot (p + q)! = m_2 n N. \tag{141}$$

Combining everything together, the output of our algorithm is the following LP:

$$\begin{aligned} \max_v \quad & c^\top v \\ \text{s.t.} \quad & a_k^\top v \leq b_k, \quad \forall k \in [m_1], \\ & d_{kl}^\top v = e_{kl}, \quad \forall k \in [m_2], l \in [(p_k + q_k)!], \\ & v_i \geq 0, \quad \forall i \in [n]. \end{aligned} \tag{142}$$

It has $n \leq N$ variables v_i and

$$m_1 + \sum_{k=1}^{m_2} (p_k + q_k)! + n \leq m_1 + m_2 N + n \tag{143}$$

constraints. The total number of scalar constants needed to specify the tensors c, a, d appearing⁹ in this LP is

$$(1 + m_1 + m_2 N)n. \tag{144}$$

The total amount of time it takes to compute these tensors is obtained by adding together eqs. (128), (131) and (141):

$$nNs + m_1 nNs + m_2 nN \leq (1 + m_1 + m_2)nNs. \tag{145}$$

This completes the description and complexity analysis of our algorithm. □

Remark 6.6. Our proof did not use the assumption that the input matrices D_k are s -sparse. This assumption only helps to keep the input SDP more compact and makes it easier to compare its size to that of the output LP. The size of the problem description grows by a factor of nN during the conversion.

Remark 6.7. Our framework can be straightforwardly generalized from SDPs of the form (116) to the following slightly more general form:

$$\begin{aligned} \max_{X, x_1, \dots, x_M} \quad & c_1 x_1 + \dots + c_M x_M \\ \text{s.t.} \quad & \text{Tr}(A_k X) \leq x_1 a_{k1} + \dots + x_M a_{kM} + b_k, \quad \forall k \in [m_1], \\ & \text{Tr}_{S_k}(\tilde{A}_k X) = D_k, \quad \forall k \in [m_2], \\ & [X, U^{\otimes p} \otimes \tilde{U}^{\otimes q}] = 0, \quad \forall U \in \text{U}(\mathbb{C}^d), \\ & X \geq 0, \end{aligned} \tag{146}$$

where A_k are constant s -sparse matrices that are provided as a linear combination of diagrams and elementary rank-1 matrices, and \tilde{A}_k, D_k are linear combinations of at most s walled Brauer diagrams on registers that are left after tracing out systems S_k . We use this more general form of SDPs in Sect. 7.2.

7. Applications

In this section we discuss several applications of our framework, focusing on four natural unitary-equivariant problems in quantum information theory: deciding the principal eigenvalue of a quantum state, quantum majority vote, asymmetric cloning, and transformation of a black-box unitary operation (they are inspired by [KW01a, Buh22, NPR21], and [QE22], respectively). These are only meant as toy examples that illustrate how our framework can be easily applied to a variety of problems, hence we do not attempt to derive full analytical solutions. Similarly, this list of applications is by no means exhaustive. Other potential applications (within quantum information) include entanglement

⁹ The tensor e defined in eq. (135) is already given to us as part of the input.

witness construction [BSH21], and quantum error-correction [KL22] and machine learning [QE22,Zhe21]. We leave exploring these other applications in more depth to future work.

The four main applications mentioned above are discussed in separate sections below. For each application we provide a separate *Wolfram Mathematica* notebook [GO22] that performs the required calculations and produces the resulting plots.

7.1. Deciding the principal eigenvalue. This application is inspired by the problem of estimating the spectrum of a given quantum state [KW01a]. Let ρ be a d -dimensional quantum state that is picked from some unitary-invariant measure. Given $\rho^{\otimes n}$ and a threshold value $c \in [1/d, 1]$, the problem is to decide whether $\lambda_{\max} < c$ or $\lambda_{\max} \geq c$, where λ_{\max} is the principal eigenvalue of ρ .

For concreteness, we assume that ρ is produced by choosing a uniformly random pure state in $\mathbb{C}^d \otimes \mathbb{C}^k$ and discarding the ancillary k -dimensional system. This guarantees that ρ has a unitary-invariant measure. Moreover, the eigenvalues $\lambda = (\lambda_1, \dots, \lambda_d)$ of such ρ have the same probability density as those of a normalized Wishart matrix [Nec07, Proposition 4]:

$$\mu_{d,k}(\lambda_1, \dots, \lambda_d) := \frac{1}{\sqrt{d}} C_{d,k} V(\lambda)^2 \prod_{i=1}^d \lambda_i^{k-d} \tag{147}$$

where $C_{d,k}$ is a normalization constant and $V(\lambda)$ is the Vandermonde determinant:

$$C_{d,k} := \frac{\Gamma(dk)}{\prod_{j=0}^{d-1} \Gamma(d+1-j)\Gamma(k-j)}, \quad V(\lambda) := \prod_{1 \leq i < j \leq d} (\lambda_i - \lambda_j). \tag{148}$$

The $1/\sqrt{d}$ factor in eq. (147) accounts for the fact that (unlike in [Nec07]) we treat all $\lambda_1, \dots, \lambda_d$ as independent variables. The density $\mu_{d,k}$ is normalized to 1 on the standard probability simplex

$$\Delta_{d-1} := \{(\lambda_1, \dots, \lambda_d) : \lambda_1 + \dots + \lambda_d = 1, \lambda_1, \dots, \lambda_d \geq 0\}. \tag{149}$$

Any strategy for this problem can be described by a two-outcome measurement with operators $P, Q \geq 0$ such that $P + Q = I_{d^n}$, where P and Q correspond to outcomes $\lambda_{\max} < c$ and $\lambda_{\max} \geq c$, respectively. The optimal probability of distinguishing the two cases correctly is

$$p_{d,k}^n(c) = \min_{U \in U(d)} \max_{\substack{P, Q \geq 0 \\ P+Q=I}} \tag{150}$$

$$\left(\int_{\lambda \in \Delta_{d-1}} \mu_{d,k}(\lambda) \text{Tr} \left[\rho(\lambda, U)^{\otimes n} \left(\delta \left(\frac{1}{d} \leq \lambda_{\max} < c \right) P + \delta(c \leq \lambda_{\max} \leq 1) Q \right) \right] \right),$$

where $\rho(\lambda, U) := U \text{diag}(\lambda) U^\dagger$ and δ denotes the indicator function for the corresponding subregion of Δ_{d-1} .

To simplify this expression, we focus on the trace. Using the cyclic property, we can move the unitary dependence from ρ onto P and Q :

$$\text{Tr} \left[\text{diag}(\lambda)^{\otimes n} \left(\delta \left(\frac{1}{d} \leq \lambda_{\max} < c \right) U^{\dagger \otimes n} P U^{\otimes n} + \delta(c \leq \lambda_{\max} \leq 1) U^{\dagger \otimes n} Q U^{\otimes n} \right) \right]. \tag{151}$$

Then, by twirling over $U(d)$, we can turn the worst case probability into the average case and thus remove the minimization over U in eq. (150) altogether. Hence, without loss of generality $P, Q \in \mathcal{A}_{n,0}^d$ in an optimal strategy, i.e., they can be written as linear combinations of n -qudit permutations, see eq. (21). Moreover, since $\rho(\lambda, U)^{\otimes n}$ is invariant under qudit permutations, we can also twirl P and Q over S_n . Hence, we can write P as a non-negative linear combination of primitive central idempotents of $\mathcal{A}_{n,0}^d$, see eq. (115), and set $Q = I_{d^n} - P$.

With these simplifications, we can evaluate the trace in eq. (151) diagrammatically, giving us a polynomial in the eigenvalues λ_i . Plugging this back into eq. (150) allows us to evaluate the integral over λ . The resulting expression depends only on the decomposition of P into idempotents. This reduces the problem from an SDP to an LP, where we only need to optimize the weights in the decomposition of P . The following example provides an explicit formula for $p_{d,k}^n(c)$, for a specific combination of parameters, obtained using this procedure.

Example 7.1. ($n = 3, d = 2, k = 2$) An exact formula for the success probability as a function of the threshold value c in this case is given by

$$p_{2,2}^3(c) := \begin{cases} 2(1 - c)(4c^2 - 2c + 1) & \text{if } c \in [1/2, c_1], \\ \frac{7}{5} - \frac{6}{5}c(16c^4 - 40c^3 + 40c^2 - 20c + 5) & \text{if } c \in [c_1, c_2], \\ (2c - 1)^3 & \text{if } c \in [c_2, 1], \end{cases} \quad (152)$$

where $c_1 \approx 0.821569391$ and $c_2 \approx 0.913830846$ are roots of the polynomials $96x^5 - 240x^4 + 200x^3 - 60x^2 + 3$ and $24x^5 - 60x^4 + 70x^3 - 45x^2 + 15x - 3$, respectively. A plot of the function $p_{2,2}^3(c)$ is shown in Fig. 4. Note that $p_{2,2}^3(1/2) = p_{2,2}^3(1) = 1$ since the problem of deciding the largest eigenvalue becomes trivial for extreme values of c . The success probability $p_{2,2}^3(c)$ is always at least $p_{2,2}^3(c_2) \approx 0.566968020$ and never below the trivial $n = 1$ lower bound

$$p_{2,2}^1(c) := \max\{2(1 - c)(4c^2 - 2c + 1), (2c - 1)^3\} \quad (153)$$

whose minimum is $1/2$ at $c = \frac{1}{2} + \frac{1}{2^{4/3}}$. Using the same procedure we also obtained explicit expressions for $p_{2,2}^n(c)$ with $n = 1, \dots, 8$. Their plots are shown in Fig. 5.

7.2. Quantum majority vote. This application of our framework is inspired by the work of [Buh22] on optimal unitary-equivariant quantum channels for evaluating permutation-equivariant and symmetric Boolean functions.

As usual, let $p, q \geq 0$ denote the number of inputs and outputs, and let $d \geq 2$ denote their dimension. We are interested in functions of the form $f : [d]^p \rightarrow [d]^q$, or more generally in multi-valued functions or relations $f \subseteq [d]^p \times [d]^q$.

We call f *equivariant* with respect to the symmetric group S_d on the alphabet $[d]$ if $f(\pi \cdot x) = \pi \cdot f(x)$ for every $\pi \in S_d$, where $\pi \cdot (x_1, \dots, x_n) := (\pi(x_1), \dots, \pi(x_n))$ for all $x_1, \dots, x_n \in [d]$ (this is a classical analogue of Definition 2.1). We say that f is *symmetric* if its inputs and outputs have $S_p \times S_q$ permutation symmetry, i.e., if $f(\pi \circ x) = \sigma \circ f(x)$ for every $(\pi, \sigma) \in S_p \times S_q$, where $\pi \circ x := (x_{\pi^{-1}(1)}, \dots, x_{\pi^{-1}(p)})$ and $\sigma \circ f(x)$ is defined similarly (this is a classical analogue of Definition 2.3).

The most important example of an equivariant and symmetric function is the *majority vote* function $f : [d]^p \rightarrow [d]$ that outputs the most frequently occurring input symbol.

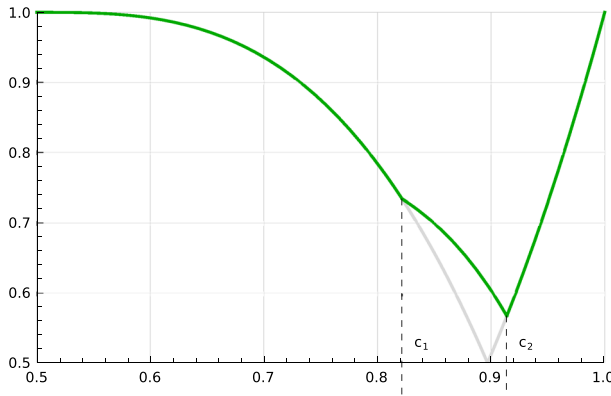


Fig. 4. Plot of the success probability $p_{2,2}^3(c)$ from eq. (152) as a function of $c \in [1/2, 1]$. The gray curves represent the trivial lower bound from eq. (153) obtained by setting $n = 1$

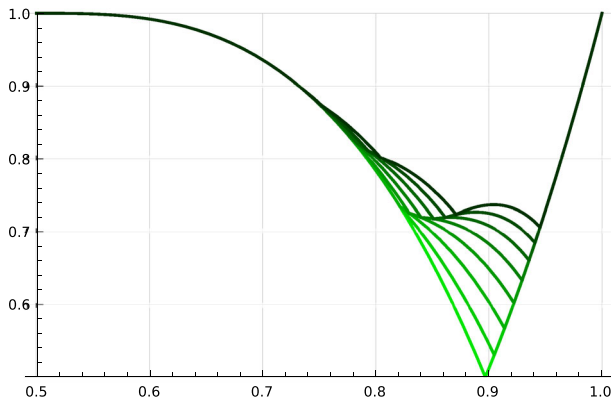


Fig. 5. Plots of $p_{2,2}^n(c)$ for $n = 1, \dots, 8$ (darker lines correspond to larger values of n). As n gets larger, the curves move upwards and the number of their segments increases

Since generally this symbol may not be unique, we prefer to think of $f \subseteq [d]^p \times [d]$ as a relation. One can also consider generalizations with $q \geq 1$ where the q most popular symbols must be output in any order.

A natural quantum generalization of an equivariant function f is a unitary-equivariant map $\Psi_f : \text{End}(V^p) \rightarrow \text{End}(V^q)$ such that $\Psi_f(|x\rangle\langle x|) = |f(x)\rangle\langle f(x)|$ for all $x \in [d]^p$. In case of a multi-valued function f , the ideal quantum map can be taken as $\Psi_f(|x\rangle\langle x|) = \Pi_{f(x)}$, where $\Pi_{f(x)} := \sum_{y \in f(x)} |y\rangle\langle y|$ is the rank- $|f(x)|$ orthogonal standard basis projector on all valid output states. Note that the ideal map Ψ_f may not be a quantum channel in general.

Given a classical S_d -equivariant and $S_p \times S_q$ symmetric relation $f \subseteq [d]^p \times [d]^q$, we would like to find a unitary-equivariant $p \rightarrow q$ quantum channel Φ_f that approximates the ideal functionality. Namely, one that maximizes the worst-case fidelity

$$\min_{x \in [d]^p} \text{Tr}(\Phi_f(|x\rangle\langle x|) \Pi_{f(x)}). \tag{154}$$

We can formulate this as an SDP for computing the worst-case fidelity $F \in \mathbb{R}$ of an optimal quantum channel represented by its Choi matrix $X \in \text{End}(V^{p,q})$:

$$\begin{aligned}
 & \max_{X, F} F \\
 & \text{s.t. } \text{Tr} \left(X(|x\rangle\langle x| \otimes \Pi_{f(x)}) \right) \geq F, \quad \forall x \in [d]^p, \\
 & \quad \text{Tr}_{V^q}(X) = I_{V^p}, \\
 & \quad [X, U^{\otimes p} \otimes \bar{U}^{\otimes q}] = 0, \quad \forall U \in \text{U}(\mathbb{C}^d), \\
 & \quad X \geq 0.
 \end{aligned} \tag{155}$$

As a generalization of [Buh22], we consider the majority relation on any alphabet $[d]$ with $d \geq 3$. For simplicity, we restrict to the case of $p = 3$ inputs (and $q = 1$ outputs). In this case the majority relation for any $d \geq 3$ is fully defined by

$$\begin{aligned}
 & 111 \mapsto 1, \\
 & 112 \mapsto 1, \\
 & 123 \mapsto 1, 2, 3,
 \end{aligned} \tag{156}$$

which are extended to the whole domain using symmetry and equivariance. These rules cover three distinct cases: when all three inputs are equal, when one of them is different, and when all three are different. Since there is no clear majority in the last case, the relation can output any of the three symbols.

For quantum majority vote with $p = 3$, $q = 1$, and $d \geq 3$ the symmetries of the Choi matrix X of the optimal quantum channel Φ_f allow us to reduce the SDP (155) to an LP using the ansatz (115) for X :

$$X = \sum_{i=1}^{n(d)} v_i \varepsilon_{T_i}^A, \tag{157}$$

where ε_{T_i} are primitive idempotents that correspond to distinct root-leaf paths T_i in the Bratteli diagram of $\mathcal{A}_{3,1}^d$, and $n(d) := n_{3,1}^d$ (Gelfand–Tsetlin) is the total number of such paths, see eq. (105). Based on eq. (156), which defines the majority relation on three symbols, the resulting LP for any $d \geq 3$ has the following form:

$$\begin{aligned}
 & \max_{F, v_1, \dots, v_{n(d)}} F \\
 & \text{s.t. } \sum_{i=1}^{n(d)} v_i \langle 111, 1 | \varepsilon_{T_i}^A | 111, 1 \rangle \geq F, \\
 & \quad \sum_{i=1}^{n(d)} v_i \langle 112, 1 | \varepsilon_{T_i}^A | 112, 1 \rangle \geq F, \\
 & \quad \sum_{i=1}^{n(d)} v_i \sum_{y=1}^3 \langle 123, y | \varepsilon_{T_i}^A | 123, y \rangle \geq F, \\
 & \quad \sum_{i=1}^{n(d)} v_i \text{Tr}_{V^{\text{out}}} \varepsilon_{T_i}^A = I_{V^{\text{in}}},
 \end{aligned}$$

$$v_i \geq 0, \quad \forall i \in [n(d)],$$

$$v_i = v_j \text{ whenever } T_i \text{ and } T_j \text{ share the same last edge.} \quad (158)$$

The last condition is a consequence of Proposition 6.2 and ensures $S_p \times S_1$ symmetry of the majority relation. Enforcing this symmetry effectively decreases the number of variables in the LP (158) from $n(d)$, which corresponds to the Gelfand–Tsetlin symmetry, to the smaller value of $n_{3,1}^d(S_3 \times S_1)$ (see eq. (114)) that corresponds to the $S_3 \times S_1$ symmetry. Explicit values of these numbers can be found in Tables 5 and 6 in Appendix E.

The LP (158) can be solved exactly and the optimal fidelity turns out to be $F = 8/9$ for all $d \geq 2$, thus extending the $d = 2$ result of [Buh22].

7.3. Asymmetric cloning. In this section, we provide an example of how our approach based on primitive idempotents can be used to solve a general unitary-equivariant SDP that does not reduce to an LP. This example is based on the problem of asymmetric cloning from [NPR21]. The problem is to find a $1 \rightarrow q$ quantum channel $\Phi: \text{End}(V) \rightarrow \text{End}(V^q)$ whose marginals $\Phi_i := \text{Tr}_{[q]\setminus\{i\}} \circ \Phi$ satisfy

$$\Phi_i(\rho) = p_i \rho + (1 - p_i) \frac{I}{d} \quad (159)$$

for all $i \in [q]$ and states $\rho \in D(V)$ where $V = \mathbb{C}^d$. We consider the case $q = 3$ with $d = 2$ and $d = 3$, and plot the set of triples (p_1, p_2, p_3) in eq. (159) that are physically realizable. Note that this set is invariant under permutations of p_i .

According to [NPR21], the Choi matrix X^Φ of the channel Φ is a linear combination of partially transposed permutation matrices, i.e., $X^\Phi \in \mathcal{A}_{1,q}^d$. We can use the primitive idempotents from eq. (96) to construct the blocks X_λ^Φ in Remark 64 without explicitly computing the Schur transform $U_{\text{Sch}(1,q)}$, see Appendix D for more details. The resulting blocks are then subject to positive semidefinite and trace constraints. In this way we can formulate the question of physical realizability of the channel Φ as a semidefinite feasibility problem.

Two examples of SDPs resulting from this procedure are given below. They characterize asymmetric $1 \rightarrow 3$ cloning in dimensions $d = 2$ and $d = 3$. Here we denote for brevity $X^i := X_{\lambda_i}^\Phi$ for every $\lambda_i \in \text{Irr}(\mathcal{A}_{p,q}^d)$.

Example 7.2. ($q = 3$ and $d = 2$) Positive semidefinite constraints:

$$(X_{1,1}^1) \succeq 0, \quad \begin{pmatrix} X_{1,1}^2 & X_{1,2}^2 \\ X_{2,1}^2 & X_{2,2}^2 \end{pmatrix} \succeq 0, \quad \begin{pmatrix} X_{1,1}^3 & X_{1,2}^3 & X_{1,3}^3 \\ X_{2,1}^3 & X_{2,2}^3 & X_{2,3}^3 \\ X_{3,1}^3 & X_{3,2}^3 & X_{3,3}^3 \end{pmatrix} \succeq 0. \quad (160)$$

Trace constraint:

$$5X_{1,1}^1 + X_{1,1}^2 + X_{2,2}^2 + 3X_{1,1}^3 + 3X_{2,2}^3 + 3X_{3,3}^3 = 2. \quad (161)$$

Expressions for realizable triples (p_1, p_2, p_3) :

$$p_1 = \frac{1}{3} \left(3X_{1,1}^3 + X_{1,1}^2 + 5X_{1,1}^1 + 3X_{2,2}^3 + 9X_{3,3}^3 + 3X_{2,2}^2 - 3 \right), \tag{162}$$

$$p_2 = \frac{1}{6} \left(6X_{1,1}^3 + 5X_{1,1}^2 + 10X_{1,1}^1 + 15X_{2,2}^3 + 3\sqrt{3}X_{2,3}^3 + 3\sqrt{3}X_{3,2}^3 + 9X_{3,3}^3 + \sqrt{3}X_{1,2}^2 + \sqrt{3}X_{2,1}^2 + 3X_{2,2}^2 - 6 \right), \tag{163}$$

$$p_3 = \frac{1}{6} \left(14X_{1,1}^3 + 5X_{1,1}^2 + 10X_{1,1}^1 + 2\sqrt{2}X_{1,2}^3 + 2\sqrt{2}X_{2,1}^3 + 7X_{2,2}^3 + 9X_{3,3}^3 + 2\sqrt{6}X_{1,3}^3 + 2\sqrt{6}X_{3,1}^3 + \sqrt{3}X_{2,3}^3 + \sqrt{3}X_{3,2}^3 - \sqrt{3}X_{1,2}^2 - \sqrt{3}X_{2,1}^2 + 3X_{2,2}^2 - 6 \right). \tag{164}$$

The feasible region for (p_1, p_2, p_3) is shown in Fig. 6.

Example 7.3. ($q = 3$ and $d = 3$) Positive semidefinite constraints: $(X_{1,1}^1) \geq 0$,

$$\begin{pmatrix} X_{1,1}^2 & X_{1,2}^2 \\ X_{2,1}^2 & X_{2,2}^2 \end{pmatrix} \geq 0, \quad \begin{pmatrix} X_{1,1}^3 & X_{1,2}^3 & X_{1,3}^3 \\ X_{2,1}^3 & X_{2,2}^3 & X_{2,3}^3 \\ X_{3,1}^3 & X_{3,2}^3 & X_{3,3}^3 \end{pmatrix} \geq 0, \quad \begin{pmatrix} X_{1,1}^4 & X_{1,2}^4 & X_{1,3}^4 \\ X_{2,1}^4 & X_{2,2}^4 & X_{2,3}^4 \\ X_{3,1}^4 & X_{3,2}^4 & X_{3,3}^4 \end{pmatrix} \geq 0. \tag{165}$$

Trace constraint:

$$X_{1,1}^4 + 2X_{1,1}^3 + 5X_{1,1}^2 + 8X_{1,1}^1 + X_{2,2}^4 + X_{3,3}^4 + 2X_{2,2}^3 + 2X_{3,3}^3 + 5X_{2,2}^2 = 1. \tag{166}$$

Expressions for realizable triples (p_1, p_2, p_3) :

$$p_1 = \frac{1}{8} \left(3X_{1,1}^4 + 6X_{1,1}^3 + 15X_{1,1}^2 + 24X_{1,1}^1 + 3X_{2,2}^4 + 12X_{3,3}^4 + 6X_{2,2}^3 + 24X_{3,3}^3 + 15X_{2,2}^2 - 4 \right), \tag{167}$$

$$p_2 = \frac{1}{8} \left(3X_{1,1}^4 + 6X_{1,1}^3 + 15X_{1,1}^2 + 24X_{1,1}^1 + 11X_{2,2}^4 + 2\sqrt{2}X_{2,3}^4 + 2\sqrt{2}X_{3,2}^4 + 4X_{3,3}^4 + 22X_{2,2}^3 + 4\sqrt{2}X_{2,3}^3 + 4\sqrt{2}X_{3,2}^3 + 8X_{3,3}^3 + 15X_{2,2}^2 - 4 \right), \tag{168}$$

$$p_3 = \frac{1}{8} \left(9X_{1,1}^4 + 21X_{1,1}^3 + 15X_{1,1}^2 + 24X_{1,1}^1 + 8X_{3,3}^3 + 15X_{2,2}^2 + 5X_{2,2}^4 + 7X_{2,2}^3 + 4X_{3,3}^4 + 2\sqrt{3}X_{1,2}^4 + 2\sqrt{3}X_{2,1}^4 - 4 - \sqrt{2}X_{2,3}^4 - \sqrt{2}X_{3,2}^4 - \sqrt{6}X_{3,1}^4 - \sqrt{6}X_{1,3}^4 + \sqrt{15}X_{1,2}^3 + \sqrt{15}X_{2,1}^3 + \sqrt{30}X_{3,1}^3 + \sqrt{30}X_{1,3}^3 + \sqrt{2}X_{2,3}^3 + \sqrt{2}X_{3,2}^3 \right). \tag{169}$$

The feasible region for (p_1, p_2, p_3) is shown in Fig. 7.

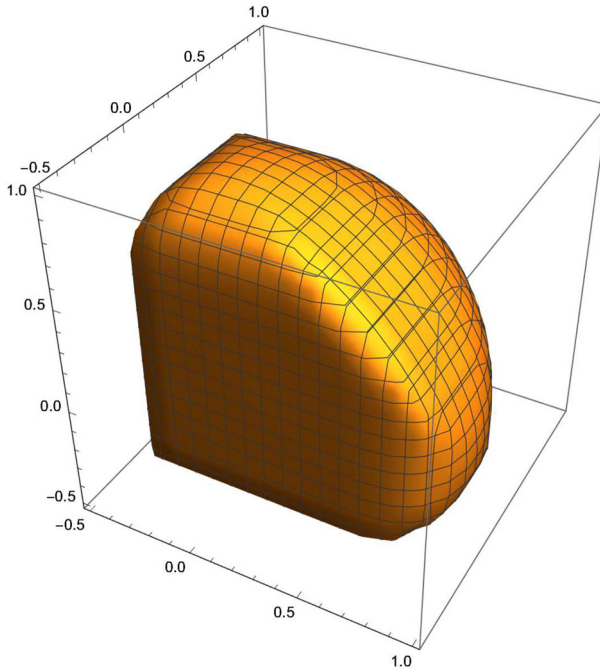


Fig. 6. Plot of the feasible region for (p_1, p_2, p_3) in the asymmetric cloning SDP for $q = 3$ and $d = 2$

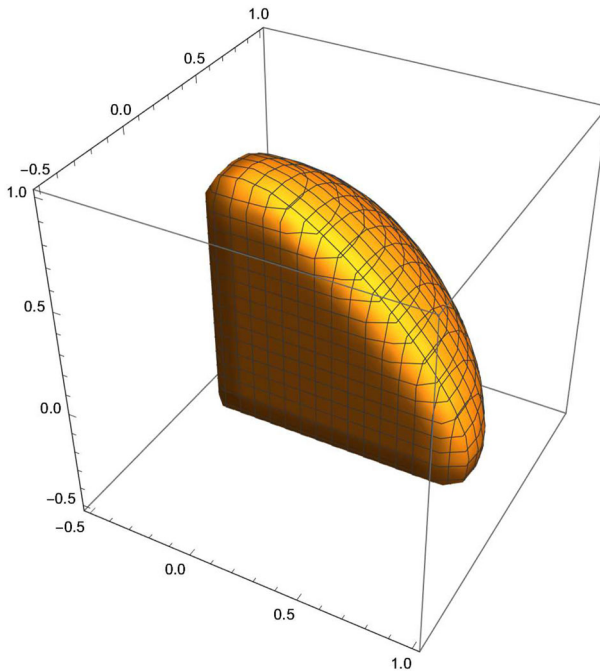


Fig. 7. Plot of the feasible region for (p_1, p_2, p_3) in the asymmetric cloning SDP for $q = 3$ and $d = 3$

7.4. Transformation of a black-box unitary operation. Following the work of [QE22] and others [Qui19b, Qui19a, YSM23, YSM22, Ebl22], we present another application of our method—transforming a black-box unitary operation. Consider the following general problem: given n copies of an unknown d -dimensional unitary U , the task is to find a universal protocol that implements $f(U)$, where f is some function of U . This protocol can be either *deterministic* or *probabilistic*, depending on whether it always succeeds or not, and either *exact* or *non-exact*, depending on the channel fidelity between the ideal channel and the one implemented by the protocol. In particular, we focus on the deterministic case where $f(U) = U^T$. Our main result in this section is the existence of an exact and deterministic protocol which transforms 4 copies of a black-box single-qubit unitary U into $f(U) = U^T$. This result is similar to the recent work [YSM22], which proves the same claim for the function $f(U) = U^{-1}$. For more detailed background on this topic we refer the reader to [QE22, YSM22]; here we only introduce the necessary ingredients needed to describe our approach.

Following previous work, we use the formalism of quantum superchannels. We search for a deterministic sequential protocol accomplishing our task by expressing it as a quantum sequential superchannel [QE22]. A *quantum superchannel* is a linear map $\mathcal{C}: \bigotimes_{i=1}^n (\text{End}(\mathcal{I}_i) \rightarrow \text{End}(\mathcal{O}_i)) \rightarrow (\text{End}(\mathcal{P}) \rightarrow \text{End}(\mathcal{F}))$ that transforms n quantum channels into a new quantum channel. Here the spaces $\mathcal{I}_i = \mathcal{O}_i = \mathcal{P} = \mathcal{F} = \mathbb{C}^d$ correspond to the inputs \mathcal{I}_i and outputs \mathcal{O}_i of the i -th copy of the channel $\mathcal{U}(\rho) := U\rho U^\dagger$ associated with the unknown input unitary U , and \mathcal{P} and \mathcal{F} are the input and output spaces of the desired output channel $\mathcal{U}_f(\rho) := f(U)\rho f(U)^\dagger$ that represents the target unitary $f(U)$. Let $\mathcal{I}^n := \bigotimes_{i=1}^n \mathcal{I}_i$ and $\mathcal{O}^n := \bigotimes_{i=1}^n \mathcal{O}_i$. A *quantum sequential superchannel* \mathcal{C} (also known as a *quantum comb*) is a quantum superchannel with certain additional constraints on its Choi matrix $C \in \text{End}(\mathcal{P} \otimes \mathcal{I}^n \otimes \mathcal{O}^n \otimes \mathcal{F})$ [CDP08]:

$$C \geq 0, \tag{170}$$

$$\text{Tr} C = 1, \tag{171}$$

$$\text{Tr}_{\mathcal{I}_i} C_i = C_{i-1} \otimes I_{\mathcal{O}_{i-1}}, \quad \forall i \in [n + 1], \tag{172}$$

where $C_{n+1} := C$, $\mathcal{I}_{n+1} := \mathcal{F}$, $\mathcal{O}_0 := \mathcal{P}$ and $C_{i-1} := \frac{1}{d} \text{Tr}_{\mathcal{I}_i, \mathcal{O}_{i-1}} C_i$.

Finding a deterministic sequential superchannel \mathcal{C} which implements the operation $\mathcal{C}(\mathcal{U}^{\otimes n}) = \mathcal{U}^T$ with highest possible average channel fidelity is equivalent to solving the following SDP for the Choi matrix C of \mathcal{C} [QE22]:

$$\begin{aligned} \max_C \quad & \text{Tr}(C\Omega_{n,d}) \\ \text{s.t.} \quad & C \text{ satisfies (170)–(172),} \end{aligned} \tag{173}$$

where $\Omega_{n,d}$ is given by

$$\Omega_{n,d} := \frac{1}{d^2} \sum_{\substack{\lambda \in \text{Irr}(\mathcal{A}_{n,1}^d) \\ i,j \in [d_\lambda]}} \frac{(E_{ij}^\lambda)_{\mathcal{O}^n \mathcal{P}} \otimes (E_{ij}^\lambda)_{\mathcal{I}^n \mathcal{F}}}{m_\lambda} \tag{174}$$

and $E_{ij}^\lambda := \psi_{n,1}^d(\varepsilon_{ij}^\lambda)$, where ε_{ij}^λ is the same as ε_{ij} for a given $\lambda \in \text{Irr}(\mathcal{A}_{n,1}^d)$ in Appendix D where $(p, q) = (n, 1)$. Notice that $\Omega_{n,d}$ has the mixed unitary symmetry:

$$[\Omega_{n,d}, U_{\mathcal{O}^n}^{\otimes n} \otimes \bar{U}_{\mathcal{P}} \otimes V_{\mathcal{I}^n}^{\otimes n} \otimes \bar{V}_{\mathcal{F}}] = 0, \quad \forall U, V \in \text{U}(\mathbb{C}^d). \tag{175}$$

Table 2. Optimal values of the SDP (173). The column $f(U)$ indicates the task, for which we want to find a deterministic sequential superchannel \mathcal{C} . In both tasks we reproduce the results of [QE22] for $d = 2, n \leq 3$ and $d = 3, n \leq 2$. We also reproduce the results for $f(U) = U^{-1}$ [YSM22] when $d = 2, n = 4$ and $d = 3, n = 3$ and $d = 3, n = 4$. Finally, we obtain new results for the unitary transposition task for $d = 2, n = 4$ and $d = 3, n = 3$ and $d = 3, n = 4$

$f(U)$	d	n			
		1	2	3	4
U^\top	2	0.500000	0.750000	0.933013	1.000000
	3	0.222222	0.407407	0.626597	0.799250
U^{-1}	2	0.500000	0.750000	0.933013	1.000000
	3	0.222222	0.333333	0.444444	0.555556

Therefore without loss of generality the optimal solution of the SDP (173) also has the same symmetry:

$$[C, U_{\mathcal{O}^n}^{\otimes n} \otimes \bar{U}_{\mathcal{P}} \otimes V_{\mathcal{I}^n}^{\otimes n} \otimes \bar{V}_{\mathcal{F}}] = 0, \quad \forall U, V \in U(\mathbb{C}^d), \tag{176}$$

which allows us to use the following ansatz for C :

$$C = \sum_{\lambda, \mu \in \text{Irr}(\mathcal{A}_{n,1}^d)} \sum_{i,j=1}^{d_\lambda} \sum_{k,l=1}^{d_\mu} c_{ijkl}^{\lambda\mu} (E_{ij}^\lambda)_{\mathcal{O}^n \mathcal{P}} \otimes (E_{kl}^\mu)_{\mathcal{I}^n \mathcal{F}}. \tag{177}$$

Both matrices C and $\Omega_{n,d}$ can be thought of as linear combinations of tensor products of two walled Brauer algebra diagrams, so our techniques from Sect. 6 can be used to rewrite the SDP (173) explicitly in terms of the variables $c_{ijkl}^{\lambda\mu}$ and then numerically solve it.

The trace constraint (171) involves a linear combination of diagrams and hence is easy to evaluate using eq. (37). The semidefinite constraint (170) becomes

$$C \geq 0 \quad \Leftrightarrow \quad [c_{ijkl}^{\lambda\mu}]_{(ik),(jl)} \geq 0, \quad \forall \lambda, \mu \in \text{Irr}(\mathcal{A}_{n,1}^d), \tag{178}$$

where we think of $[c_{ijkl}^{\lambda\mu}]_{(ik),(jl)} \in \text{End}(\mathbb{C}^{d_\lambda} \otimes \mathbb{C}^{d_\mu})$ as matrices. The only tricky constraint is (172). The i -th constraint says that a certain linear combination of diagrams from the algebra $\mathcal{B}_{i-1,1}^d \otimes \mathcal{B}_{i-1,0}^d$ equals to 0 under the map $\psi_{i-1,1}^d \otimes \psi_{i-1,0}^d$, i.e., it belongs to the kernel of this map. One way of dealing with this type of constraint is to map each diagram on both sides of the tensor product to all of its irreducible blocks. This can be done using the method outlined in Appendix D.

We can also solve a similar problem for the function $f(U) = U^{-1}$ and verify the results obtained in [YSM22]. In that case, the SDP (173) has the same form, except that the matrices C and $\Omega_{n,d}$ possess a different symmetry: they commute with $U_{\mathcal{O}^n \mathcal{P}}^{\otimes n+1} \otimes V_{\mathcal{I}^n \mathcal{F}}^{\otimes n+1}$ for every $U, V \in U(\mathbb{C}^d)$. The elementary matrices $E_{ij}^\lambda := \psi_{n+1,0}^d(\varepsilon_{ij}^\lambda)$ are labeled by irreducible representations $\lambda \in \text{Irr}(\mathcal{A}_{n+1,0}^d)$ of the symmetric group algebra $\mathbb{C}S_{n+1}$. This situation corresponds to the $(p, q) = (n + 1, 0)$ case of our formalism.

Our numerical results are summarized in Table 2, and our *Wolfram Mathematica* code for obtaining these values and the corresponding optimal Choi matrices C can be found on GitHub [GO22].

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Data Availability The supporting code can be found on GitHub [GO22].

Declarations

Conflict of interest The authors state that there is no Conflict of interest.

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Appendix A. Lifting Traces From $\mathcal{B}_{p,q}^d$ to $\mathcal{A}_{p,q}^d$

Representing the elements of the matrix algebra $\mathcal{A}_{p,q}^d$ as preimages of diagrams under $\psi_{p,q}^d$ is particularly useful when computing traces and partial traces of $\psi_{p,q}^d(\sigma)$ for any $\sigma \in \mathcal{B}_{p,q}^d$. The following two propositions relate the matrix traces of $\psi_{p,q}^d$ to the diagrammatic traces of σ defined in eqs. (37) and (38).

Proposition A.1. For any $\sigma \in \mathcal{B}_{p,q}^d$,

$$\text{Tr}(\psi_{p,q}^d(\sigma)) = \text{Tr}(\sigma). \tag{179}$$

In particular, when σ is a single diagram,

$$\text{Tr}(\psi_{p,q}^d(\sigma)) = d^{\text{loops}(\sigma)} \tag{180}$$

where $\text{loops}(\sigma)$ is the number of loops created by connecting all pairs of opposite vertices of σ .

Proof. To establish eq. (179), it suffices to consider only the case when σ is a single diagram since the general case follows by linearity. Using eqs. (41) and (43),

$$\text{Tr}(\psi_{p,q}^d(\sigma)) = \sum_{i_1, \dots, i_{p+q} \in [d]} (\langle i_1 | \otimes \dots \otimes \langle i_{p+q} | \psi_{p,q}^d(\sigma) (|i_1 \rangle \otimes \dots \otimes |i_{p+q} \rangle) \tag{181}$$

$$= \sum_{\substack{i_1, \dots, i_{p+q} \in [d] \\ \underline{i}_1, \dots, \underline{i}_{p+q} \in [d]}} \sigma_{i_1, \dots, i_{p+q}}^{i_1, \dots, i_{p+q}} \prod_{k \in [p+q]} \delta_{i_k, \underline{i}_k} \tag{182}$$

$$= \sum_{\substack{i_1, \dots, i_{p+q} \in [d] \\ \underline{i}_1, \dots, \underline{i}_{p+q} \in [d]}} \prod_{(r,s) \in \sigma} \delta_{i_r, i_s} \prod_{k \in [p+q]} \delta_{i_k, \underline{i}_k} \tag{183}$$

$$= d^{\text{loops}(\sigma)}. \tag{184}$$

The last equality follows by partitioning the product of delta functions into closed loops and observing that all indices on a given loop must have the same value. The final value agrees with the diagrammatic definition of $\text{Tr}(\sigma)$ in eq. (37). \square

The following generalization allows to graphically compute the partial trace $\text{Tr}_S(\psi_{p,q}^d(\sigma))$ for any subset of systems $S \subseteq [p+q]$.

Proposition A.2. For any $\sigma \in \mathcal{B}_{p,q}^d$ and subset $S \subseteq [p+q]$,

$$\text{Tr}_S(\psi_{p,q}^d(\sigma)) = \psi_{p',q'}^d(\text{Tr}_S(\sigma)) \tag{185}$$

where $\text{Tr}_S(\sigma)$ is defined in eq. (38) and p' and q' denote the number of leftover systems on both sides of the wall. In particular, when σ is a single diagram,

$$\text{Tr}_S(\psi_{p,q}^d(\sigma)) = d^{\text{loops}_S(\sigma)} \psi_{p',q'}^d(\sigma^S) \tag{186}$$

where σ^S denotes the diagram σ with opposite pairs of nodes that belong to S contracted, and $\text{loops}_S(\sigma)$ is the number of loops formed in this process.

Proof. By linearity, it suffices to establish the result for any diagram $\sigma \in \mathcal{B}_{p,q}^d$. Note from eq. (41) that

$$\psi_{p,q}^d(\sigma) = \sum_{\substack{l_1, \dots, l_{p+q} \in [d] \\ l_1, \dots, l_{p+q} \in [d]}} \sigma_{l_1, \dots, l_{p+q}}^{l_1, \dots, l_{p+q}} |l_1\rangle \langle l_1| \otimes \dots \otimes |l_{p+q}\rangle \langle l_{p+q}| = \sum_{l, \underline{l} \in [d]^{p+q}} \sigma_l^{\underline{l}} |l\rangle \langle l| \tag{187}$$

where $l = (l_1, \dots, l_{p+q})$ and $\underline{l} = (l_1, \dots, l_{p+q})$. Letting $\bar{S} := [p+q] \setminus S$, we can generalize eq. (181) as follows:

$$\text{Tr}_S(\psi_{p,q}^d(\sigma)) = \sum_{i \in [d]^S} (\langle i|_S \otimes I_{\bar{S}}) \psi_{p,q}^d(\sigma) (|i\rangle_S \otimes I_{\bar{S}}) \tag{188}$$

$$= \sum_{i \in [d]^S} (\langle i|_S \otimes I_{\bar{S}}) \left(\sum_{l, \underline{l} \in [d]^{p+q}} \sigma_l^{\underline{l}} |l\rangle \langle l| \right) (|i\rangle_S \otimes I_{\bar{S}}) \tag{189}$$

$$= \sum_{i \in [d]^S} \sum_{l, \underline{l} \in [d]^{p+q}} \sigma_l^{\underline{l}} \left(\prod_{k \in S} \delta_{i_k, l_k} \delta_{l_k, i_k} \right) \left(\bigotimes_{k \in \bar{S}} |l_k\rangle \langle l_k| \right) \tag{190}$$

$$= \sum_{l, \underline{l} \in [d]^{p+q}} \sigma_l^{\underline{l}} \left(\prod_{k \in S} \delta_{l_k, l_k} \right) \left(\bigotimes_{k \in \bar{S}} |l_k\rangle \langle l_k| \right), \tag{191}$$

where we eliminated the sum over i in the last equality by noting that

$$\sum_{i_k \in [d]} \delta_{i_k, l_k} \delta_{l_k, i_k} = \delta_{l_k, l_k} \tag{192}$$

for all $k \in S$. Substituting the definition of $\sigma_l^{\underline{l}}$ from eq. (43) we get

$$\text{Tr}_S(\psi_{p,q}^d(\sigma)) = \sum_{l, \underline{l} \in [d]^{p+q}} \left(\prod_{(r,s) \in \sigma} \delta_{r, l_s} \right) \left(\prod_{k \in S} \delta_{l_k, l_k} \right) \left(\bigotimes_{k \in \bar{S}} |l_k\rangle \langle l_k| \right) \tag{193}$$

$$= d^{\text{loops}_S(\sigma)} \sum_{l', \underline{l}' \in [d]^{\bar{S}}} \left(\prod_{(u,v) \in \sigma^S} \delta_{l'_u, l'_v} \right) |l'\rangle \langle l'|, \tag{194}$$

where the second equality is obtained by using a generalization of eq. (192) to contract the chains of delta functions along all loops and paths. This collapses the sum from

$[d]^{p+q}$ to $[d]^{\bar{S}}$ and reduces the product to run over edges (u, v) in the remaining smaller diagram σ^S . Using eqs. (43) and (187) in reverse,

$$\text{Tr}_S(\psi_{p,q}^d(\sigma)) = d^{\text{loops}_S(\sigma)} \sum_{l', l'' \in [d]^{\bar{S}}} (\sigma^S)_{l'}^{l''} \langle l' | \langle l'' | \quad (195)$$

$$= d^{\text{loops}_S(\sigma)} \psi_{p',q'}^d(\sigma^S) \quad (196)$$

$$= \psi_{p',q'}^d(\text{Tr}_S(\sigma)), \quad (197)$$

where we used eq. (38) that defines the diagrammatic partial trace $\text{Tr}_S(\sigma)$. □

Finally, note from eq. (187) that we can also easily evaluate the trace of $\psi_{p,q}^d(\sigma)$ with any elementary rank-1 standard basis matrix $|i\rangle\langle j|$ for any $i, j \in [d]^{p+q}$:

$$\text{Tr}(\psi_{p,q}^d(\sigma) |i\rangle\langle j|) = \sigma_j^i, \quad (198)$$

where σ_j^i is given in eq. (43). The partial trace $\text{Tr}_S(\psi_{p,q}^d(\sigma) |i\rangle\langle j|)$ can be evaluated similarly.

Appendix B. Restriction to $S_p \times S_q$ Permutational Symmetry

In this appendix we describe how the simple module V^λ of $\mathcal{A}_{p,q}^d$ restricts to the algebra $\psi_{p,q}^d(\mathbb{C}(S_p \times S_q))$. This question was first answered in [Kin70, Kin71]. It also follows from [Koi89, Proposition 2.2, Corollary 2.3.1] and [Hal96, Theorem 1.7] that a general formula for this restriction is

$$\text{Res}_{S_p \times S_q}^{\mathcal{A}_{p,q}^d} V^\lambda \cong \bigoplus_{\substack{\mu \vdash p, \text{len}(\mu) \leq d \\ \nu \vdash q, \text{len}(\nu) \leq d}} (S^\mu \otimes S^\nu)^{\oplus m_{\mu,\nu}^\lambda(d)} \quad (199)$$

where S^μ and S^ν are simple modules of $\mathbb{C}S_p$ and $\mathbb{C}S_q$, respectively. For $\lambda = (\lambda^l, \lambda^r) \in \text{Irr}(\mathcal{A}_{p,q}^d)$ with $\text{len}(\lambda^l) + \text{len}(\lambda^r) \leq d$ and $\mu \vdash p, \nu \vdash q$ with $\text{len}(\mu), \text{len}(\nu) \leq d$ the multiplicity $m_{\mu,\nu}^\lambda(d)$ is given by the following formula:

$$m_{\mu,\nu}^\lambda(d) := \sum_{\substack{\tilde{\lambda}: f(\tilde{\lambda}, d) = \lambda \\ \text{len}(\tilde{\lambda}^l) \leq d \\ \text{len}(\tilde{\lambda}^r) \leq d}} g(\tilde{\lambda}, d) \sum_{\gamma \vdash k(\tilde{\lambda})} c_{\gamma \tilde{\lambda}^l}^\mu c_{\gamma \tilde{\lambda}^r}^\nu \quad (200)$$

where $\tilde{\lambda} = (\tilde{\lambda}^l, \tilde{\lambda}^r)$ is a pair of partitions $\tilde{\lambda}^l \vdash p - k(\tilde{\lambda})$ and $\tilde{\lambda}^r \vdash q - k(\tilde{\lambda})$ for some integer $k(\tilde{\lambda}) \geq 0$, and $c_{\gamma \tilde{\lambda}^l}^\mu, c_{\gamma \tilde{\lambda}^r}^\nu$ are the Littlewood–Richardson coefficients. The numbers $g(\tilde{\lambda}, d) \in \{-1, 0, 1\}$ and the bipartitions $f(\tilde{\lambda}, d)$ are defined as follows. If $\text{len}(\tilde{\lambda}^l) + \text{len}(\tilde{\lambda}^r) \leq d$ then $f(\tilde{\lambda}, d) := \tilde{\lambda}$ and $g(\tilde{\lambda}, d) := 1$. If $\text{len}(\tilde{\lambda}^l) + \text{len}(\tilde{\lambda}^r) > d$ then $f(\tilde{\lambda}, d)$ and $g(\tilde{\lambda}, d)$ are obtained via the following procedure (here μ' denotes the transpose of the Young diagram μ):

- If $d - \tilde{\lambda}_i^{r'} - \tilde{\lambda}_1^{r'} + i = \tilde{\lambda}_j^{l'} - \tilde{\lambda}_1^{l'} - j + 1$ for some $i \in [\tilde{\lambda}_1^{l'}]$ and $j \in [\tilde{\lambda}_1^{r'}]$, then $g(\tilde{\lambda}, d) := 0$ and $f(\tilde{\lambda}, d)$ is left undefined.

- Otherwise, sort the (distinct) numbers

$$(d - \tilde{\lambda}_i^{r'} - \tilde{\lambda}_1^r + i : i = \tilde{\lambda}_1^r, \dots, 1) \cup (\tilde{\lambda}_j^{l'} - \tilde{\lambda}_1^r - j + 1 : j = 1, \dots, \tilde{\lambda}_1^l) \quad (201)$$

in decreasing order and denote the resulting list by $(k_1, \dots, k_{\tilde{\lambda}_1^r + \tilde{\lambda}_1^l})$. Denote the permutation that achieves this sorting by π and let $g(\tilde{\lambda}, d) := \text{sgn}(\pi)$. The bipartition $f(\tilde{\lambda}, d) = (f(\tilde{\lambda}, d)^l, f(\tilde{\lambda}, d)^r)$ is then defined via its transpose as follows:

$$f(\tilde{\lambda}, d)^{l'} := (d - k_i - i + 1 : i = \tilde{\lambda}_1^r, \dots, 1), \quad (202)$$

$$f(\tilde{\lambda}, d)^{r'} := (k_{\tilde{\lambda}_1^r + j} + \tilde{\lambda}_1^r + j - 1 : j = 1, \dots, \tilde{\lambda}_1^l). \quad (203)$$

Using a more geometric understanding of this procedure from [Kin71, eq. (2.18)] we arrive at the following multiplicity-free results.

Lemma B.1. *If $\min(p, q) \leq 2$ then the multiplicity $m_{\mu, \nu}^\lambda(d)$ defined in eq. (200) is either 0 or 1 for any valid λ, μ, ν, d .*

Proof. Fix a valid combination of λ, μ, ν, d , i.e., $\mu \vdash p, \nu \vdash q$ with $\text{len}(\mu), \text{len}(\nu) \leq d$ and $\lambda = (\lambda^l, \lambda^r)$ with $p - \text{size}(\lambda^l) = q - \text{size}(\lambda^r) \geq 0$ and $\text{len}(\lambda^l) + \text{len}(\lambda^r) \leq d$. Assume that $\tilde{\lambda} = (\tilde{\lambda}^l, \tilde{\lambda}^r)$ is a pair of partitions $\tilde{\lambda}^l \vdash p - k(\tilde{\lambda})$ and $\tilde{\lambda}^r \vdash q - k(\tilde{\lambda})$ for some integer $k(\tilde{\lambda}) \geq 0$. Without loss of generality it is enough to consider only the $q = 1$ and $q = 2$ cases.

Case $q = 1$. Assume that $\text{len}(\tilde{\lambda}^l) \leq d$ and $\text{len}(\tilde{\lambda}^r) = 1$. Then according to [Kin71] we have that

$$g(\tilde{\lambda}, d) = \begin{cases} 1 & \text{if } \text{len}(\tilde{\lambda}^l) + \text{len}(\tilde{\lambda}^r) \leq d, \\ 0 & \text{if } \text{len}(\tilde{\lambda}^l) + \text{len}(\tilde{\lambda}^r) = d + 1. \end{cases} \quad (204)$$

The set $\{\tilde{\lambda} : f(\tilde{\lambda}, d) = \lambda\}$ contains at most two elements corresponding to $\text{len}(\tilde{\lambda}^l) + \text{len}(\tilde{\lambda}^r) \leq d$ and $\text{len}(\tilde{\lambda}^l) + \text{len}(\tilde{\lambda}^r) = d + 1$. Moreover, there is only one term in the sum $\sum_{\gamma \vdash k(\tilde{\lambda})} c_{\gamma \tilde{\lambda}^l}^\mu c_{\gamma \tilde{\lambda}^r}^\nu$ in eq. (200) corresponding to either the empty partition $\gamma = \emptyset$ or $\gamma = (1)$. Each of these Littlewood–Richardson coefficients is either zero or one due to the Pieri rule [Mac98], which is a special case of the Littlewood–Richardson rule [BZ88, RS98, Gas98, Ste02]. Therefore $m_{\mu, \nu}^\lambda(d) \in \{0, 1\}$.

Case $q = 2$. Assume that $\text{len}(\tilde{\lambda}^l) \leq d$ and $\text{len}(\tilde{\lambda}^r) \leq 2$. Analogously, from [Kin71] it follows that

$$g(\tilde{\lambda}, d) = \begin{cases} 1 & \text{if } \text{len}(\tilde{\lambda}^l) + \text{len}(\tilde{\lambda}^r) \leq d, \\ -1 & \text{if } \text{len}(\tilde{\lambda}^l) = d \text{ and } \text{len}(\tilde{\lambda}^r) = 2 \text{ and } \tilde{\lambda}_1^{l'} > \tilde{\lambda}_2^{l'}, \\ 0 & \text{otherwise.} \end{cases} \quad (205)$$

There are three possible cases depending on the value of $k(\tilde{\lambda}) \in \{0, 1, 2\}$:

- If $k(\tilde{\lambda}) = 0$ then the only term in the sum $\sum_{\gamma \vdash k(\tilde{\lambda})} c_{\gamma \tilde{\lambda}^l}^\mu c_{\gamma \tilde{\lambda}^r}^\nu$ corresponds to $\gamma = \emptyset$ and each Littlewood–Richardson coefficient is either zero or one, which implies that $\sum_{\gamma \vdash k(\tilde{\lambda})} c_{\gamma \tilde{\lambda}^l}^\mu c_{\gamma \tilde{\lambda}^r}^\nu \in \{0, 1\}$.
- If $k(\tilde{\lambda}) = 1$ then $\tilde{\lambda}^r = (1)$ and the only term in the sum $\sum_{\gamma \vdash k(\tilde{\lambda})} c_{\gamma \tilde{\lambda}^l}^\mu c_{\gamma \tilde{\lambda}^r}^\nu$ corresponds to $\gamma = (1)$, which means that $\sum_{\gamma \vdash k(\tilde{\lambda})} c_{\gamma \tilde{\lambda}^l}^\mu c_{\gamma \tilde{\lambda}^r}^\nu \in \{0, 1\}$.

- If $k(\tilde{\lambda}) = 2$ then $\tilde{\lambda}^r = \emptyset$ and therefore

$$\sum_{\gamma \vdash k(\tilde{\lambda})} c_{\gamma \tilde{\lambda}^l}^\mu c_{\gamma \tilde{\lambda}^r}^\nu = \sum_{\gamma \vdash k(\tilde{\lambda})} c_{\gamma \tilde{\lambda}^l}^\mu \delta_{\nu, \gamma} = c_{\nu \tilde{\lambda}^l}^\mu \in \{0, 1\}, \tag{206}$$

where the conclusion follows from $\nu \vdash 2$ and the Pieri rule.

Moreover, for any valid λ there is either exactly one $\tilde{\lambda}$ for which $f(\tilde{\lambda}, d) = \lambda$ (namely $\tilde{\lambda} = \lambda$) or exactly two different $\tilde{\lambda}_1, \tilde{\lambda}_2$ with $g(\tilde{\lambda}_1, d) = -1$ and $g(\tilde{\lambda}_2, d) = 1$. Therefore $m_{\mu, \nu}^\lambda(d) \in \{0, 1\}$. \square

Lemma B.2. *If $d = 2$ then the multiplicity $m_{\mu, \nu}^\lambda(d)$ defined in eq. (200) is either 0 or 1 for any valid λ, μ, ν .*

Proof. Fix a valid combination of λ, μ, ν , i.e., $\mu \vdash p, \nu \vdash q$ with $\text{len}(\mu), \text{len}(\nu) \leq 2$ and $\lambda = (\lambda^l, \lambda^r)$ with $k := p - \text{size}(\lambda^l) = q - \text{size}(\lambda^r) \geq 0$ and $\text{len}(\lambda^l) + \text{len}(\lambda^r) \leq d$. Assume that $\tilde{\lambda} = (\tilde{\lambda}^l, \tilde{\lambda}^r)$ is a pair of partitions $\tilde{\lambda}^l \vdash p - k(\tilde{\lambda})$ and $\tilde{\lambda}^r \vdash q - k(\tilde{\lambda})$ for some integer $k(\tilde{\lambda}) \geq 0$ with $\text{len}(\tilde{\lambda}^l) \leq 2$ and $\text{len}(\tilde{\lambda}^r) \leq 2$. Again, we use a result of [Kin71] stating that for $d = 2$,

$$g(\tilde{\lambda}, 2) = \begin{cases} 1 & \text{if } \text{len}(\tilde{\lambda}^l) + \text{len}(\tilde{\lambda}^r) \leq 2, \\ -1 & \text{if } \text{len}(\tilde{\lambda}^l) = \text{len}(\tilde{\lambda}^r) = 2 \text{ and } \tilde{\lambda}_1^{l'} > \tilde{\lambda}_2^{l'} \text{ and } \tilde{\lambda}_1^{r'} > \tilde{\lambda}_2^{r'}, \\ 0 & \text{otherwise.} \end{cases} \tag{207}$$

We need to consider only two different cases.

Case $k = \min(p, q)$. Without loss of generality assume $k = q$, i.e., $\lambda^r = \emptyset$. Then according to eq. (207) the first sum in eq. (200) contains only one term corresponding to $\tilde{\lambda} = \lambda$:

$$m_{\mu, \nu}^\lambda(2) = \sum_{\gamma \vdash k} c_{\gamma \lambda^l}^\mu c_{\gamma \emptyset}^\nu = \sum_{\gamma \vdash k} c_{\gamma \lambda^l}^\mu \delta_{\nu, \gamma} = c_{\nu \lambda^l}^\mu. \tag{208}$$

Since $\text{len}(\mu), \text{len}(\nu), \text{len}(\lambda^l) \leq 2$, it follows from the Littlewood–Richardson rule that $m_{\mu, \nu}^\lambda(2) = c_{\nu \lambda^l}^\mu \in \{0, 1\}$.

Case $k < \min(p, q)$. In this case λ^l, λ^r are non-empty Young diagrams with only one row, i.e., $\text{len}(\lambda^l) = \text{len}(\lambda^r) = 1$. According to eq. (207) there are now two $\tilde{\lambda}$ such that $f(\tilde{\lambda}, 2) = \lambda$ and $g(\tilde{\lambda}, 2) \neq 0$, namely $\tilde{\lambda} = \lambda$ which corresponds to $g(\tilde{\lambda}, 2) = 1$ and $\tilde{\lambda} = (\tilde{\lambda}^l, \tilde{\lambda}^r) = ((\lambda_1^l, 1), (\lambda_1^r, 1))$ which corresponds to $g(\tilde{\lambda}, 2) = -1$. Therefore

$$m_{\mu, \nu}^\lambda(2) = \sum_{\gamma \vdash k} c_{\gamma \lambda^l}^\mu c_{\gamma \lambda^r}^\nu - \sum_{\gamma \vdash k-1} c_{\gamma (\lambda_1^l, 1)}^\mu c_{\gamma (\lambda_1^r, 1)}^\nu. \tag{209}$$

Since $\text{len}(\lambda^l) = \text{len}(\lambda^r) = 1$ and $\text{len}(\mu), \text{len}(\nu) \leq 2$ we can use the Pieri rule to deduce $c_{\gamma \lambda^l}^\mu, c_{\gamma \lambda^r}^\nu \in \{0, 1\}$ and calculate

$$\begin{aligned} c_{\gamma \lambda^l}^\mu &\neq 0 \text{ iff } \gamma \vdash k \text{ and } \text{len}(\gamma) \leq 2 \text{ and } \mu_2 \leq \gamma_1 \leq \mu_1 \text{ and } \gamma_2 \leq \mu_2, \\ c_{\gamma \lambda^r}^\nu &\neq 0 \text{ iff } \gamma \vdash k \text{ and } \text{len}(\gamma) \leq 2 \text{ and } \nu_2 \leq \gamma_1 \leq \nu_1 \text{ and } \gamma_2 \leq \nu_2. \end{aligned}$$

Therefore the first sum in eq. (209) becomes

$$\begin{aligned} & |\{(\gamma_1, \gamma_2) \vdash k : \max(\mu_2, v_2) \leq \gamma_1 \leq \min(\mu_1, v_1), \gamma_2 \leq \min(\mu_2, v_2)\}| \\ &= \left| \left\{ \gamma_2 \in \mathbb{Z} : \max(0, k - \mu_1, k - v_1) \leq \gamma_2 \leq \min\left(\left\lfloor \frac{k}{2} \right\rfloor, k - \mu_2, k - v_2, \mu_2, v_2\right) \right\} \right|. \end{aligned} \tag{210}$$

It can be rewritten as

$$\begin{aligned} \sum_{\gamma \vdash k} c_{\gamma \lambda^l}^\mu c_{\gamma \lambda^r}^v &= \max\left(\min\left(\left\lfloor \frac{k}{2} \right\rfloor, k - \mu_2, k - v_2, \mu_2, v_2\right) \right. \\ &\quad \left. - \max(0, k - \mu_1, k - v_1) + 1, 0\right). \end{aligned} \tag{211}$$

Similarly, the Littlewood–Richardson rule implies that $c_{\gamma(\lambda_{1,1}^l),1}^\mu, c_{\gamma(\lambda_{1,1}^r),1}^v \in \{0, 1\}$. Furthermore,

$$\begin{aligned} c_{\gamma(\lambda_{1,1}^l),1}^\mu &\neq 0 \text{ iff } \gamma \vdash k - 1 \text{ and } \text{len}(\gamma) \leq 2 \text{ and } \mu_2 \leq \gamma_1 + 1 \leq \mu_1 \text{ and } \gamma_2 \leq \mu_2 - 1, \\ c_{\gamma(\lambda_{1,1}^r),1}^v &\neq 0 \text{ iff } \gamma \vdash k - 1 \text{ and } \text{len}(\gamma) \leq 2 \text{ and } v_2 \leq \gamma_1 + 1 \leq v_1 \text{ and } \gamma_2 \leq v_2 - 1. \end{aligned}$$

The second sum in eq. (209) now becomes

$$\begin{aligned} & |\{(\gamma_1, \gamma_2) \vdash k - 1 : \max(\mu_2, v_2) \leq \gamma_1 + 1 \leq \min(\mu_1, v_1), \gamma_2 \leq \min(\mu_2, v_2) - 1\}| \\ &= \left| \left\{ \gamma_2 \in \mathbb{Z} : \max(0, k - \mu_1, k - v_1) \leq \gamma_2 \leq \min\left(\left\lfloor \frac{k-1}{2} \right\rfloor, k - \mu_2, k - v_2, \mu_2 - 1, v_2 - 1\right) \right\} \right|. \end{aligned} \tag{212}$$

It can be rewritten as

$$\begin{aligned} \sum_{\gamma \vdash k-1} c_{\gamma(\lambda_{1,1}^l),1}^\mu c_{\gamma(\lambda_{1,1}^r),1}^v &= \max\left(\min\left(\left\lfloor \frac{k-1}{2} \right\rfloor, k - \mu_2, k - v_2, \mu_2 - 1, v_2 - 1\right) \right. \\ &\quad \left. - \max(0, k - \mu_1, k - v_1) + 1, 0\right). \end{aligned} \tag{213}$$

From eqs. (209), (211) and (213) we clearly see that $m_{\mu,v}^\lambda(2) \in \{0, 1\}$. □

Appendix C. Proof of Lemma 5.2

Lemma 5.2. *For any $T \in \text{Paths}(p + q)$ in the Bratteli diagram of $\mathcal{A}_{p,q}^d$,*

$$(J_1^A + \dots + J_{p+q}^A) \varepsilon_T^A = (\text{cont}(\lambda^l) + \text{cont}(\lambda^r) + d \cdot \text{size}(\lambda^r)) \varepsilon_T^A \tag{87}$$

where ε_T^A is the corresponding canonical primitive idempotent of $\mathcal{A}_{p,q}^d$, $\lambda = (\lambda^l, \lambda^r) = T[p + q]$ is the last vertex of the path T , $\text{cont}(\lambda)$ is the total content of all cells of the Young diagram λ , see eq. (76), and $\text{size}(\lambda)$ is the number of cells in λ .

Proof. We will use the correspondence between a path T and a pair (τ, L) of a tableau $\tau = (\tau^l, \tau^r)$ of shape (λ^l, λ^r) and a tuple L of pairs of numbers from the set $[p+q]$, see Theorem 1.11 of [Ben94]. It follows from Lemma 5.1 that $J_1^A + \dots + J_{p+q}^A \in \mathcal{Z}(\mathcal{A}_{p+q})$. Therefore it is enough to consider how this sum acts on any vector in the isotypic component $V^\lambda \otimes U^\lambda$ corresponding to the simple module labeled by $\lambda = (\lambda^l, \lambda^r) \in \text{Irr}(\mathcal{A}_{p,q}^d)$ in the mixed Schur–Weyl duality, see eq. (63). In particular, we can take a maximal vector

$$|t_{\tau,L}\rangle := \psi(y_\tau \bar{\sigma}_L) |\beta_{\tau,L}\rangle \in V^{p,q} \tag{214}$$

from [Ben94, Definition 2.4], where $y_\tau \in \mathbb{C}(S_p \times S_q) \subseteq \mathcal{B}_{p,q}^d$ is the *Young symmetrizer* for the tableau τ , $\bar{\sigma}_L \in \mathcal{B}_{p,q}^d$ is a diagram that contracts all pairs in L , and $\psi := \psi_{p,q}^d$ is defined in eq. (41). The Young symmetrizer y_τ is defined as $y_\tau := y_{\tau^l} y_{\tau^r}$, the product of standard Young symmetrizers y_{τ^l} and y_{τ^r} , see [Ben94, Eq. 2.2]. The Young symmetrizers y_{τ^l} and y_{τ^r} are products of *column* and *row symmetrizers* (the terms which correspond to *column* and *row groups* in [Ben94, Eq. 2.2]). The standard basis vector $|\beta_{\tau,L}\rangle \in V^{p,q}$ is defined as follows:

$$|\beta_{\tau,L}\rangle := |u_1\rangle \otimes \dots \otimes |u_p\rangle \otimes |u_{p+1}\rangle \otimes \dots \otimes |u_{p+q}\rangle \tag{215}$$

where each $u_i \in [d]$ and we distinguish two cases: if $1 \leq i \leq p$ then

$$u_i := \begin{cases} j & \text{if } i \text{ belongs to the } j\text{-th row of } \tau^l, \\ 1 & \text{if } i \in L, \end{cases} \tag{216}$$

while if $p+1 \leq i \leq p+q$ then

$$u_i := \begin{cases} d-j+1 & \text{if } i \text{ belongs to the } j\text{-th row of } \tau^r, \\ 1 & \text{if } i \in L. \end{cases} \tag{217}$$

Recall that $J_1^A + \dots + J_{p+q}^A \in \mathcal{Z}(\mathcal{A}_{p+q})$, so

$$\left(J_1^A + \dots + J_{p+q}^A \right) |t_{\tau,L}\rangle = \psi(y_\tau \bar{\sigma}_L) \left(J_1^A + \dots + J_{p+q}^A \right) |\beta_{\tau,L}\rangle. \tag{218}$$

Since $J_k^A := \psi(J_k^B)$ and ψ is a homomorphism, we can use the definition of J_k^B from eq. (85) to write the right-hand side more explicitly:

$$\psi \left(y_\tau \bar{\sigma}_L \left(\sum_{\substack{1 \leq i < j \leq p \text{ or} \\ p+1 \leq i < j \leq p+q}} \sigma_{i,j} - \sum_{\substack{1 \leq i \leq p \\ p+1 \leq j \leq p+q}} \bar{\sigma}_{i,j} + d \cdot q \right) \right) |\beta_{\tau,L}\rangle. \tag{219}$$

To simplify this, we need to evaluate all expressions of the form $\psi(y_\tau \bar{\sigma}_L \sigma_{i,j}) |\beta_{\tau,L}\rangle$ and $\psi(y_\tau \bar{\sigma}_L \bar{\sigma}_{i,j}) |\beta_{\tau,L}\rangle$, where we distinguish between transpositions $\sigma_{i,j}$ and contractions $\bar{\sigma}_{i,j}$ which can be located in three possible positions relative to L :

- (1) $i \notin L, j \notin L$,
- (2) $i \in L, j \notin L$ or $i \notin L, j \in L$,
- (3) $i \in L, j \in L$.

We now proceed to consider each of these six cases separately (we will write (I) and (\bar{I}) to distinguish between the cases with $\sigma_{i,j}$ and $\bar{\sigma}_{i,j}$).

Case (I) : $\sigma_{i,j}$ with $i \notin L, j \notin L$. There are two sub-cases:

- (a) If i and j are in the same row of either τ^l or τ^r , the row symmetrizer of y_τ does not change the resulting vector, i.e., $\psi(y_\tau \sigma_{i,j})|\beta_{\tau,L}\rangle = \psi(y_\tau)|\beta_{\tau,L}\rangle$. Since $\bar{\sigma}_L$ and y_τ commute,

$$\psi(y_\tau \bar{\sigma}_L \sigma_{i,j})|\beta_{\tau,L}\rangle = |t_{\tau,L}\rangle. \tag{220}$$

- (b) If i and j are in different rows of either τ^l or τ^r , we denote the corresponding row numbers by r_i and r_j . In this case the row symmetrizer of y_τ acting on $\psi(\sigma_{i,j})|\beta_{\tau,L}\rangle$ produces a product of two “ W states” at positions defined by the rows r_i and r_j of $\tau^{l/r}$ with the numbers i and j swapped. The column antisymmetrizer of y_τ will then kill most terms, leaving only the terms with basis vectors $|r_i\rangle, |r_j\rangle$ in positions within the same column. There are $\lambda_{\max\{r_i,r_j\}}^{l/r}$ such terms, where $\lambda_{\max\{r_i,r_j\}}^{l/r}$ is the size of the smallest of the two rows r_i, r_j within the corresponding left or right tableaux $\tau^{l/r}$. After this operation the resulting vector acquires a minus sign and a different factor compared to $\psi(y_\tau)|\beta_{\tau,L}\rangle$:

$$\psi(y_\tau \sigma_{i,j})|\beta_{\tau,L}\rangle = -\frac{\psi(y_\tau)|\beta_{\tau,L}\rangle}{\lambda_{\max\{r_i,r_j\}}^{l/r}}. \tag{221}$$

Again, since $\bar{\sigma}_L$ and y_τ commute,

$$\psi(y_\tau \bar{\sigma}_L \sigma_{i,j})|\beta_{\tau,L}\rangle = -\frac{1}{\lambda_{\max\{r_i,r_j\}}^{l/r}} |t_{\tau,L}\rangle. \tag{222}$$

Case (\bar{I}) : $\bar{\sigma}_{i,j}$ with $i \notin L, j \notin L$. In this case $\psi(\bar{\sigma}_{i,j})|\beta_{\tau,L}\rangle$ can only be non-zero when $u_i = u_j$, which is equivalent to $r_i = d - r_j + 1$. Thus $r_i + r_j = d + 1$. But since $r_i + r_j \leq \text{len}(\lambda^l) + \text{len}(\lambda^r) \leq d$,

$$\psi(y_\tau \bar{\sigma}_L \bar{\sigma}_{i,j})|\beta_{\tau,L}\rangle = 0. \tag{223}$$

Case (2) : $\sigma_{i,j}$ with $i \in L, j \notin L$ or $i \notin L, j \in L$. Without loss of generality we can assume that $i \in L$ and $j \notin L$. There are two sub-cases:

- (a) If $u_j = 1$ then $\psi(\bar{\sigma}_L \sigma_{i,j})|\beta_{\tau,L}\rangle = \psi(\bar{\sigma}_L)|\beta_{\tau,L}\rangle$. This happens when $j \in \tau^l$ and $r_j = 1$ or $j \in \tau^r$ and $r_j = d$. Therefore

$$\psi(y_\tau \bar{\sigma}_L \sigma_{i,j})|\beta_{\tau,L}\rangle = |t_{\tau,L}\rangle. \tag{224}$$

- (b) If $u_j \neq 1$ then $\psi(\bar{\sigma}_L \sigma_{i,j})|\beta_{\tau,L}\rangle = 0$ because $\psi(\bar{\sigma}_L)$ would annihilate the vector $\psi(\sigma_{i,j})|\beta_{\tau,L}\rangle$. Therefore

$$\psi(y_\tau \bar{\sigma}_L \sigma_{i,j})|\beta_{\tau,L}\rangle = 0. \tag{225}$$

Case $(\bar{2})$: $\bar{\sigma}_{i,j}$ with $i \in L, j \notin L$ or $i \notin L, j \in L$. Assume again that $i \in L$ and $j \notin L$. There are two sub-cases:

(a) If $u_j = 1$ then

$$\psi(\bar{\sigma}_L \bar{\sigma}_{i,j})|\beta_{\tau,L}\rangle = \psi(\bar{\sigma}_L) \left(|\beta_{\tau,L}\rangle + \sum_{k=2}^d (\cdots \otimes |k\rangle \otimes \cdots \otimes |k\rangle \otimes \cdots) \right), \tag{226}$$

where the basis vectors labeled by k are in positions i and j . But since $j \notin L$ all vectors $\cdots \otimes |k\rangle \otimes \cdots \otimes |k\rangle \otimes \cdots$ for $k \geq 2$ are annihilated by $\bar{\sigma}_L$. Therefore in this case

$$\psi(y_\tau \bar{\sigma}_L \bar{\sigma}_{i,j})|\beta_{\tau,L}\rangle = |t_{\tau,L}\rangle. \tag{227}$$

(b) If $u_j \neq 1$ then $\psi(\bar{\sigma}_{i,j})|\beta_{\tau,L}\rangle = 0$ and therefore

$$\psi(y_\tau \bar{\sigma}_L \bar{\sigma}_{i,j})|\beta_{\tau,L}\rangle = 0. \tag{228}$$

Case (3): $\sigma_{i,j}$ with $i \in L, j \in L$. Since i and j must belong together either to τ^l or τ^r , they cannot belong simultaneously to one pair of L . Since $u_i = u_j = 1$, it follows that $\psi(\sigma_{i,j})|\beta_{\tau,L}\rangle = |\beta_{\tau,L}\rangle$ and therefore

$$\psi(y_\tau \bar{\sigma}_L \sigma_{i,j})|\beta_{\tau,L}\rangle = |t_{\tau,L}\rangle. \tag{229}$$

Case ($\bar{3}$): $\bar{\sigma}_{i,j}$ with $i \in L, j \in L$. In contrast to the previous case, two sub-cases can occur:

(a) If i and j belong to the same pair in L then

$$\psi(\bar{\sigma}_L \bar{\sigma}_{i,j})|\beta_{\tau,L}\rangle = \psi(\bar{\sigma}_L) \left(\sum_{k=1}^d (\cdots \otimes |k\rangle \otimes \cdots \otimes |k\rangle \otimes \cdots) \right) = d \cdot \psi(\bar{\sigma}_L)|\beta_{\tau,L}\rangle, \tag{230}$$

where the basis vectors labeled by k are in positions i and j . Therefore

$$\psi(y_\tau \bar{\sigma}_L \bar{\sigma}_{i,j})|\beta_{\tau,L}\rangle = d \cdot |t_{\tau,L}\rangle. \tag{231}$$

(b) If i and j belong to different pairs in L then similarly to the previous case we can write

$$\psi(\bar{\sigma}_L \bar{\sigma}_{i,j})|\beta_{\tau,L}\rangle = \psi(\bar{\sigma}_L) \left(\sum_{k=1}^d (\cdots \otimes |k\rangle \otimes \cdots \otimes |k\rangle \otimes \cdots) \right), \tag{232}$$

where the basis vectors labeled by k are in positions i and j . But now since the positions i and j are not contracted with each other by $\psi(\bar{\sigma}_L)$, we will not acquire a factor of d :

$$\psi(y_\tau \bar{\sigma}_L \bar{\sigma}_{i,j})|\beta_{\tau,L}\rangle = |t_{\tau,L}\rangle. \tag{233}$$

Collecting everything together and separating the sums in Lemma 219 according to the cases above, we arrive at the following expression:

$$\begin{aligned}
 (J_1^A + \dots + J_{p+q}^A) |t_{\tau, L}\rangle &= \psi \left(y_{\tau} \bar{\sigma}_L \left(\sum_{\substack{1 \leq i < j \leq p \text{ or} \\ p+1 \leq i < j \leq p+q}} \sigma_{i,j} - \sum_{\substack{1 \leq i \leq p \\ p+1 \leq j \leq p+q}} \bar{\sigma}_{i,j} + d \cdot q \right) \right) |\beta_{\tau, L}\rangle \\
 &= \sum_{\substack{1 \leq i < j \leq p \text{ or} \\ p+1 \leq i < j \leq p+q \\ \text{case (1a)}}} |t_{\tau, L}\rangle + \sum_{\substack{1 \leq i < j \leq p \text{ or} \\ p+1 \leq i < j \leq p+q \\ \text{case (2a)}}} |t_{\tau, L}\rangle + \sum_{\substack{1 \leq i < j \leq p \text{ or} \\ p+1 \leq i < j \leq p+q \\ \text{case (3)}}} |t_{\tau, L}\rangle + d \cdot q \tag{234} \\
 &- \sum_{\substack{1 \leq i < j \leq p \text{ or} \\ p+1 \leq i < j \leq p+q \\ \text{case (1b)}}} \frac{|t_{\tau, L}\rangle}{\lambda_{\max\{r_i, r_j\}}^{1/r}} - \sum_{\substack{1 \leq i \leq p \\ p+1 \leq j \leq p+q \\ \text{case(2a)}}} |t_{\tau, L}\rangle - \sum_{\substack{1 \leq i \leq p \\ p+1 \leq j \leq p+q \\ \text{case (3a)}}} d \cdot |t_{\tau, L}\rangle \\
 &- \sum_{\substack{1 \leq i \leq p \\ p+1 \leq j \leq p+q \\ \text{case (3b)}}} |t_{\tau, L}\rangle.
 \end{aligned}$$

To simplify this we need to do some counting. Counting all possible pairs within each row of τ gives us

$$\sum_{\substack{1 \leq i < j \leq p \text{ or} \\ p+1 \leq i < j \leq p+q \\ \text{case (1a)}}} |t_{\tau, L}\rangle = \left(\sum_{i=1}^{\text{len}(\lambda^l)} \binom{\lambda_i^l}{2} + \sum_{i=1}^{\text{len}(\lambda^r)} \binom{\lambda_i^r}{2} \right) |t_{\tau, L}\rangle. \tag{235}$$

Pairing i and j across different rows gives us:

$$\sum_{\substack{1 \leq i < j \leq p \text{ or} \\ p+1 \leq i < j \leq p+q \\ \text{case (1b)}}} \frac{|t_{\tau, L}\rangle}{\lambda_{\max\{r_i, r_j\}}^{1/r}} = \left(\sum_{i=1}^{\text{len}(\lambda^l)} \lambda_i^l (i-1) + \sum_{i=1}^{\text{len}(\lambda^r)} \lambda_i^r (i-1) \right) |t_{\tau, L}\rangle. \tag{236}$$

Note that for a single diagram λ it is true that

$$\sum_{i=1}^{\text{len}(\lambda)} \binom{\lambda_i}{2} - \sum_{i=1}^{\text{len}(\lambda)} \lambda_i (i-1) = \text{cont}(\lambda), \tag{237}$$

therefore

$$\sum_{\substack{1 \leq i < j \leq p \text{ or} \\ p+1 \leq i < j \leq p+q \\ \text{case (1a)}}} |t_{\tau, L}\rangle - \sum_{\substack{1 \leq i < j \leq p \text{ or} \\ p+1 \leq i < j \leq p+q \\ \text{case (1b)}}} \frac{|t_{\tau, L}\rangle}{\lambda_{\max\{r_i, r_j\}}^{1/r}} = \left(\text{cont}(\lambda^l) + \text{cont}(\lambda^r) \right) |t_{\tau, L}\rangle. \tag{238}$$

Next, simple combinatorics gives us

$$\sum_{\substack{1 \leq i < j \leq p \text{ or} \\ p+1 \leq i < j \leq p+q \\ \text{case (2a)}}} |t_{\tau, L}\rangle = \sum_{\substack{1 \leq i \leq p \\ p+1 \leq j \leq p+q \\ \text{case (2a)}}} |t_{\tau, L}\rangle = \left(\lambda_1^l \cdot |L| + \lambda_d^r \cdot |L| \right) |t_{\tau, L}\rangle, \quad (239)$$

$$\sum_{\substack{1 \leq i < j \leq p \text{ or} \\ p+1 \leq i < j \leq p+q \\ \text{case (3)}}} |t_{\tau, L}\rangle = \sum_{\substack{1 \leq i \leq p \\ p+1 \leq j \leq p+q \\ \text{case (3b)}}} |t_{\tau, L}\rangle = 2 \cdot \binom{|L|}{2} |t_{\tau, L}\rangle, \quad (240)$$

so the corresponding sums cancel each other. Finally,

$$\sum_{\substack{1 \leq i \leq p \\ p+1 \leq j \leq p+q \\ \text{case (3a)}}} d \cdot |t_{\tau, L}\rangle = d \cdot |L| \cdot |t_{\tau, L}\rangle. \quad (241)$$

Using $q - |L| = \text{size}(\lambda^r)$ and combining everything together gives us the desired result:

$$\left(J_1^A + \dots + J_{p+q}^A \right) |t_{\tau, L}\rangle = \left(\text{cont}(\lambda^l) + \text{cont}(\lambda^r) + d \cdot \text{size}(\lambda^r) \right) |t_{\tau, L}\rangle. \quad (242)$$

Since $J_1^A + \dots + J_{p+q}^A \in \mathcal{Z}(\mathcal{A}_{p+q})$ and $\varepsilon_T^A |t_{\tau, L}\rangle = |t_{\tau, L}\rangle$, we can draw the same conclusion for ε_T^A :

$$\left(J_1^A + \dots + J_{p+q}^A \right) \varepsilon_T^A = \left(\text{cont}(\lambda^l) + \text{cont}(\lambda^r) + d \cdot \text{size}(\lambda^r) \right) \varepsilon_T^A \quad (243)$$

which completes the proof. □

Appendix D. Computing the Blocks of $\mathcal{A}_{p,q}^d$ in the Gelfand–Tsetlin Basis

Here we propose an algorithm for computing the blocks of an arbitrary $\mathcal{A}_{p,q}^d$ algebra element in the Gelfand–Tsetlin basis. In line with our philosophy, the algorithm is fully diagrammatic, namely, all computation takes place in $\mathcal{B}_{p,q}^d$ instead of $\mathcal{A}_{p,q}^d$. Here we only sketch the reasoning behind the algorithm, and we leave it to future work to establish its correctness formally. This paves one possible route for removing the additional symmetry assumption in Theorem 6.4 and thus extending our framework from linear to general semidefinite unitary-equivariant programs.

The natural $*$ -algebra structure of $\mathcal{B}_{p,q}^d$ is the important ingredient in this section. Consider a random hermitian (with respect to the natural $*$ -algebra structure) element of $\mathcal{B}_{p,q}^d$ given as a linear combination of diagrams with random real coefficients b_i :

$$B := \sum_{i=1}^{(p+q)!} b_i \sigma_i. \quad (244)$$

Throughout this section we fix $\lambda \in \text{Irr}(\mathcal{A}_{p,q}^d)$ and choose an arbitrary ordering $i \in [d_\lambda]$ of all paths in $\text{Paths}(\lambda)$. Let B_{ij} denote the (i, j) -th entry of block λ when the matrix $\psi_{p,q}^d(B) \in \mathcal{A}_{p,q}^d$ is expressed in the Gelfand–Tsetlin basis. That is, let $B_{ij} := (A_\lambda)_{ij}$

for all $i, j \in [d_\lambda]$, where A_λ is a matrix of size $d_\lambda \times d_\lambda$ that appears in the first register of the decomposition (64):

$$U_{\text{Sch}(p,q)} \psi_{p,q}^d(B) U_{\text{Sch}(p,q)}^\dagger = \bigoplus_{\lambda \in \text{Irr}(\mathcal{A}_{p,q}^d)} [A_\lambda \otimes I_{m_\lambda}]. \tag{245}$$

Note from eq. (244) that B_{ij} is a linear combination of the variables b_k . Our goal is to determine B_{ij} for all choices of $\lambda \in \text{Irr}(\mathcal{A}_{p,q}^d)$ and $i, j \in [d_\lambda]$.

Let $T = \lambda_0 \rightarrow \dots \rightarrow \lambda_{p+q}$ be the i -th path in the Bratteli diagram of $\mathcal{B}_{p,q}^d$ that goes from the root to the leaf λ . Let us denote the preimage of ε_T^A under $\psi_{p,q}^d$ by

$$\varepsilon_i := \prod_{k=1}^{p+q} \prod_{\mu: \lambda_{k-1} \rightarrow \mu \neq \lambda_k} \frac{J_k^{\mathcal{B}} - c_{\lambda_{k-1} \rightarrow \mu}}{c_{\lambda_{k-1} \rightarrow \lambda_k} - c_{\lambda_{k-1} \rightarrow \mu}} \in \mathcal{B}_{p,q}^d, \tag{246}$$

where, in contrast to eq. (96), the second product runs over edges in the Bratteli diagram of the family \mathcal{B} instead of \mathcal{A} . By construction, the block λ of ε_i is equal to $|i\rangle\langle i|$ while all other blocks vanish:

$$U_{\text{Sch}(p,q)} \psi_{p,q}^d(\varepsilon_i) U_{\text{Sch}(p,q)}^\dagger = \bigoplus_{\mu \in \text{Irr}(\mathcal{A}_{p,q}^d)} [\delta_{\lambda,\mu} |i\rangle\langle i| \otimes I_{m_\mu}]. \tag{247}$$

Since $\psi(\varepsilon_i B \varepsilon_i) = B_{ii} \cdot \psi(\varepsilon_i)$, knowing ε_i allows us to diagrammatically extract B_{ii} by computing

$$B_{ii} = \frac{\text{Tr}(B \varepsilon_i)}{\text{Tr}(\varepsilon_i)}, \tag{248}$$

where $\text{Tr}(\varepsilon_i) = m_\lambda$ for every $i \in [d_\lambda]$, see Remark 64.

To extract the off-diagonal entries B_{ij} , we would like to have operators ε_{ij} that are analogous to ε_i but instead of $|i\rangle\langle i|$ have $|i\rangle\langle j|$, for any $i, j \in [d_\lambda]$, in block λ of eq. (247). While we do not have an expression for ε_{ij} , knowing ε_i and ε_j is enough to diagrammatically extract B_{ij} . This can be done via the following algorithm:

- (1) For every $i \in [d_\lambda]$, diagrammatically compute $\varepsilon_1 B \varepsilon_i$ and $\varepsilon_i B \varepsilon_1$. Since $\psi(\varepsilon_1 B \varepsilon_i) \cdot \psi(\varepsilon_i B \varepsilon_1) = B_{1i} B_{i1} \cdot \psi(\varepsilon_1)$ we can diagrammatically compute

$$B_{1i} B_{i1} = \frac{\text{Tr}((\varepsilon_1 B \varepsilon_i) \cdot (\varepsilon_i B \varepsilon_1))}{\text{Tr}(\varepsilon_1)}. \tag{249}$$

- (2) Since the element B is hermitian with real coefficients, $B_{1i} = B_{i1}$ as real numbers. From eq. (249) we can set

$$B_{1i} = B_{i1} := \sqrt{\frac{\text{Tr}((\varepsilon_1 B \varepsilon_i) \cdot (\varepsilon_i B \varepsilon_1))}{\text{Tr}(\varepsilon_1)}} \tag{250}$$

- (3) For every $i \in [d_\lambda]$ we set

$$\varepsilon_{1i} := \frac{\varepsilon_1 B \varepsilon_i}{B_{1i}}, \quad \varepsilon_{i1} := \frac{\varepsilon_i B \varepsilon_1}{B_{i1}}. \tag{251}$$

(4) Once we know all of the ε_{1i} and ε_{i1} , we can diagrammatically compute

$$B_{ij} = \frac{\text{Tr}(\varepsilon_{1i} B \varepsilon_{j1})}{\text{Tr}(\varepsilon_1)} \tag{252}$$

for every $i, j \in [d_\lambda]$. We can also compute $\varepsilon_{ij} = \varepsilon_{i1} \varepsilon_{1j}$ for every $i, j \in [d_\lambda]$.

Since the multiplication of two arbitrary linear combinations of diagrams has complexity $O(\dim(\mathcal{B}_{p,q}^d)^2)$, the complexity $O(\dim(\mathcal{B}_{p,q}^d)^2 \dim(\mathcal{A}_{p,q}^d))$ of the above algorithm does not depend on d asymptotically since

$$O(((p+q)!)^2 \dim(\mathcal{A}_{p,q}^d)) \leq O(((p+q)!)^3), \tag{253}$$

where we used $\dim(\mathcal{A}_{p,q}^d) \leq \dim(\mathcal{B}_{p,q}^d) = (p+q)!$ (that is quite crude bound for small d).

Appendix E. Numerical Values for the Number of Variables n

See Tables 3, 4, 5, 6, 7.

Table 3. The logarithm of the number of variables $\log_{10}(d^{2(p+q)})$ in a naive implementation of the SDP (116)

$p+q$	2	3	4	5	6	7	8	9	10	
d	2	1.20	1.81	2.41	3.01	3.61	4.21	4.82	5.42	6.02
	3	1.91	2.86	3.82	4.77	5.73	6.68	7.63	8.59	9.54
	4	2.41	3.61	4.82	6.02	7.22	8.43	9.63	10.84	12.04
	5	2.80	4.19	5.59	6.99	8.39	9.79	11.18	12.58	13.98
	6	3.11	4.67	6.23	7.78	9.34	10.89	12.45	14.01	15.56
	7	3.38	5.07	6.76	8.45	10.14	11.83	13.52	15.21	16.90
	8	3.61	5.42	7.22	9.03	10.84	12.64	14.45	16.26	18.06
	9	3.82	5.73	7.63	9.54	11.45	13.36	15.27	17.18	19.08
	10	4.00	6.00	8.00	10.00	12.00	14.00	16.00	18.00	20.00

Table 4. The number of variables $n_{p,q}^d$ in a unitary-equivariant LP with full walled Brauer algebra symmetry, see eq. (102)

p	1	1	1	2	1	2	1	2	3	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	5
q	1	2	3	2	4	3	5	4	3	6	5	4	4	7	6	5	4	8	7	6	5	4	3	2	1	5
d	2	2	3	3	3	4	4	4	4	4	4	5	5	5	5	5	5	5	5	5	5	5	5	6	6	6
	3	3	4	5	6	6	7	8	8	9	10	10	10	11	12	12	13	13	14	15	15	15	15	17	17	18
	4	4	5	6	7	8	10	12	12	13	15	16	17	21	21	21	23	21	25	27	28	27	28	32	34	36
	5	6	7	8	9	11	14	14	14	16	19	21	21	27	28	31	28	35	39	41	36	46	50	56	54	
	6	7	8	9	11	14	15	15	17	21	23	24	31	33	37	32	41	47	50	43	57	63	72	70		
	7	8	9	10	11	12	13	14	15	16	17	18	22	24	25	33	35	39	45	52	56	47	63	71	82	80
	8	9	10	11	12	13	14	15	16	17	18	22	24	25	33	35	40	47	54	58	50	67	76	88	86	
	9	10	11	12	13	14	15	16	17	18	22	24	25	33	35	40	47	54	58	50	67	76	88	90	88	
	10	11	12	13	14	15	16	17	18	22	24	25	33	35	40	47	54	58	50	67	76	88	90	91	89	

Table 7. The number of variables $n_{p,q}^d = \dim(\mathcal{A}_{p,q}^d)$ in a unitary-equivariant SDP with no additional symmetry

$p + q$	2	3	4	5	6	7	8	9	10
d	2	5	14	42	132	429	1,430	4,862	16,796
	3	6	23	103	513	2,761	15,767	94,359	586,590
	4		24	119	694	4,582	33,324	261,808	2,190,688
	5			120	719	5003	39429	344837	3,291,590
	6				720	5,039	40,270	361,302	3,587,916
	7					5040	40,319	362,815	3,626,197
	8						40,320	3,62,879	3,628,718
	9							362,880	3,628,799
	10								3,628,800

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