Computation of the confluent hypergeometric function $M(a, b, x)$

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Abstract. The computation of the Kummer function $M(a,b,x)$ is considered in this paper. An efficient and accurate algorithm combining asymptotic expansions for large values of the parameters and power series is presented. Numerical tests show that a Matlab implementation of the algorithm allows the computation of the function with $\sim 10^{-14}$ relative accuracy in the parameter region $(a,\, b,\, x) \in (0,\, 500)\times(0,\, 500)\times(0,\, 1000)$ in double precision floating point arithmetic.

Keywords: Kummer function · Numerical computation · Matlab software.

1 Introduction

The confluent hypergeometric function $M(a,b,x)$ is given by

$$M(a,b,x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{x^n}{n!}, \tag{1}$$

where nonpositive integer values for the b parameter are not allowed. Another standard notation of this function is ${}_1F_1\left(\begin{matrix} a \\ b \end{matrix}\,;x\right)$. Confluent hypergeometric functions (also called Kummer functions) have many applications in science and engineering. These functions arise in a variety of mathematical problems, including differential equations, probability theory and statistical physics. They are also used in the analysis of complex systems, such as fluid dynamics, quantum mechanics and material science. Many mathematical properties satisfied by confluent hypergeometric functions are given in [5].

A double-precision floating-point arithmetic implementation for the computation of the other standard solution of the Kummer equation (the function $U(a,b,x)$) was recently considered in [2]. It is important to note that the behavior of the confluent hypergeometric functions $U(a,b,x)$ and $M(a,b,x)$ shows

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significant differences. For example, consider the following three-term recurrence relation (TTRR)

$$(a+k-1)f_{k+1}+(b-2a-x-2k)f_k+(a+k+1-b)f_{k+1}=0. \qquad (2)$$

The functions $u_k=\frac{\Gamma(a+k)}{\Gamma(a)}U(a+k,b,x)$ and $g_k=\frac{\Gamma(a+k)}{\Gamma(1+a+k-b)}M(a+k,b,x)$ both satisfy the same recurrence relation but while the function g_k is a dominant solution of the recurrence relation (2), the function u_k is the minimal solution of the TTRR [1]. Therefore, computational strategies for computing the confluent hypergeometric functions will differ to account for this distinct behavior.

For computing the confluent function $M(a,b,x)$ some algorithms are available in the literature (see for example [3,4]). In [3], a variety of methods are considered and compared: power series in x, continued fraction representations, asymptotic expansions for large x and in terms of incomplete gamma functions and rational approximations. In this paper we show that an efficient and accurate algorithm in double precision to compute $M(a,b,x)$ can be obtained using the asymptotic expansions for large values of the parameters described in [6] and the power series in x given in (1). Matlab implementations of the methods used in the resulting algorithm are provided.

1.1 Expansion for $M(a,b,z)$

Asymptotic expansions of the Kummer functions $M(a,b,z)$ and $U(a,b+1,z)$ in which all three positive parameters a, b, and z are allowed to be large, were given in [6]. The expansions are even valid when at least one of the parameters a, b, or z is large.

The expansion for the confluent hypergeometric function $M(a,b,z)$ is given by

$$M(a,b,z)\sim e^{z-z\mathcal{A}}\frac{\Gamma(b)}{\Gamma(a)}z^{a-b}f_0(\mu)\sum_{n=0}^{\infty}\frac{\widetilde{f}_n(\mu)}{z^n},\quad \widetilde{f}_n(\mu)=\frac{f_n(\mu)}{f_0(\mu)}, \qquad (3)$$

as $z\to\infty$. In the expansion we have that $\mathcal{A}=\mu(\tau-\ln\tau-1)-\alpha\ln(1-\mu\tau)$ where $\mu=\beta-\alpha$, $\alpha=\dfrac{a}{z}$, $\beta=\dfrac{b}{z}$ and $\tau=\dfrac{2}{\beta+1+\sqrt{(\beta+1)^2-4\mu}}$.

The first two $\widetilde{f}_n$ coefficients are $\widetilde{f}_0=1$ and

$$\widetilde{f}_1=-\frac{1}{12}\mu\tau^2\frac{(\mu^2\tau^5-13\mu^2\tau^4-\mu\tau^4+21\mu\tau^3+4\mu\tau^2-9\tau^2-2\tau-1)}{(\mu\tau^2-1)^3(\mu\tau-1)}$$

.

In numerical computations, it is convenient to write the front factor in (3) in terms of the $\Gamma^*(x)$ function, which is defined as

$$\Gamma^*(x)=\frac{\Gamma(x)}{\sqrt{2\pi/x}\,x^xe^{-x}},\ x>0. \qquad (4)$$

When x is large, this function can be very important in algorithms where the function $\Gamma(x)$ is involved because $\Gamma^*(x) = 1 + \mathcal{O}(1/x)$.

Using the $\Gamma^*(x)$ function, we have that the asymptotic expansion (3) can be written as

$$M(a,b,z) \sim e^{z(1-\mu\tau)} \frac{\Gamma^*(b)}{\Gamma^*(a)} \sqrt{\frac{a}{b}} \frac{u^b v^a}{\sqrt{\beta\mu\tau^2 - 2\mu\tau + 1}} \sum_{n=0}^{\infty} \frac{\widetilde{f}_n(\mu)}{z^n}, \tag{5}$$

where $u = \beta\tau$ and $v = \dfrac{1-\mu\tau}{\alpha\tau}$.

As discussed in [6], it is also interesting the possibility of computing scaled functions $\widetilde{M}(a,b,z)$, which are defined as

$$M(a,b,z) = e^z \frac{\Gamma(b)}{\Gamma(a)} z^{a-b} \widetilde{M}(a,b,z)\,. \tag{6}$$

The front factor $e^z \dfrac{\Gamma(b)}{\Gamma(a)} z^{a-b}$ can be used to estimate the overflow limit in numerical calculations.

2 Numerical testing and algorithm

For testing the computation of $M(a,b,x)$ we consider the recurrence relation

$$xM(a+1,b+1,x) + bM(a,b,x) = bM(a+1,b,x). \tag{7}$$

Efficient implementations in Matlab of the series given in (1) and the expansion given in (5) are implemented in the Matlab functions `Mabxseries(a,b,x)` and `Mabxlps(a,b,x)`, respectively, which are available at http://personales.unican.es/gila/Mabx.m.

The power series given in (1) is a useful method of computation for not very large values of x. This is shown in Figure 1, where two examples of the use of the power series to compute the confluent hypergeometric function $M(a,b,x)$ are shown. The blue circles indicate points where the number of terms used to compute the series is larger than 500 for an accuracy close to double precision. The green crosses correspond to points where the function values are close or above the overflow limit in double precision floating-point arithmetic. The region $(a,\, b) = (0,\, 500) \times (0,\, 500)$ has been tested with 10000 random points.

When the power series becomes inefficient, the expansion (5) provides a good alternative. Few tests of the accuracy obtained using the expansion for different values of x are shown in Figures 2 and 3. We check the recurrence relations (7) for fixed values of x in the region $(a,\, b) = (0,\, 500) \times (0,\, 500)$ with 10000 random points.

We conclude that an efficient algorithm for computing the Kummer function $M(a,b,x)$ can be obtained using the power series in the region $(a,b,z) \in (0,\, 100) \times (0,\, 100) \times (0, 20)$ and the asymptotic expansion elsewhere. It is interesting to note that the intrinsic Matlab function `hypergeom`, included in the

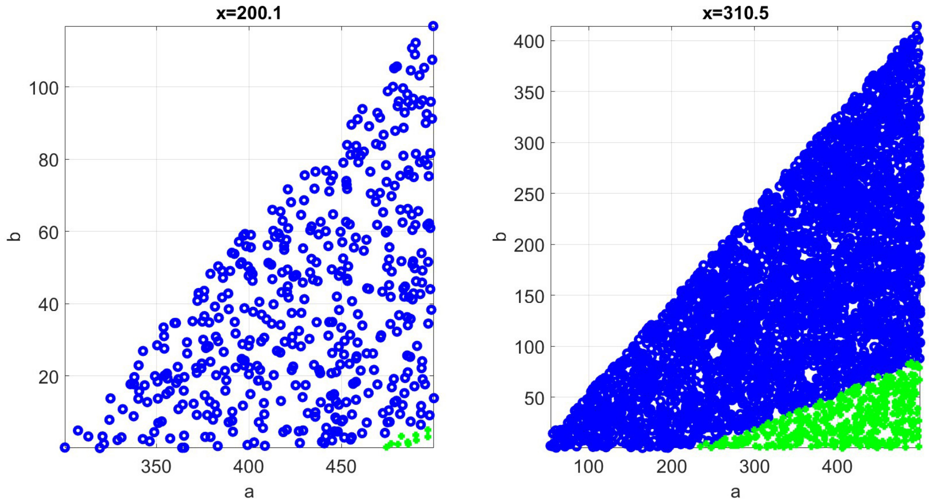

Fig. 1. Test of the power series given in (1) for two values of x. The blue circles indicate points where the number of terms used to compute the series is larger than 500 for an accuracy close to double precision. The green crosses correspond to points where the function values are close or above the overflow limit in double precision floating-point arithmetic.

symbolic toolbox but which also provides floating-point results, seems to fail for some values of the parameters. Therefore, our algorithm offers a reliable and efficient alternative to compute the confluent hypergeometric function $M(a, b, x)$.

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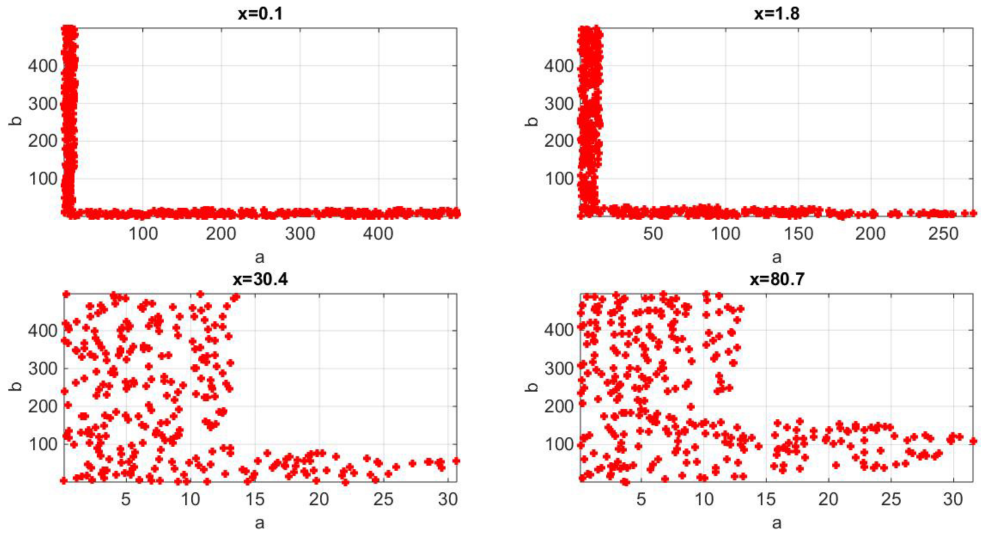

Fig. 2. Test for the expansion (5) for different values of x. The red points indicate points where the error obtained when testing the recurrence relation in (7) is greater than 10^{-12}.

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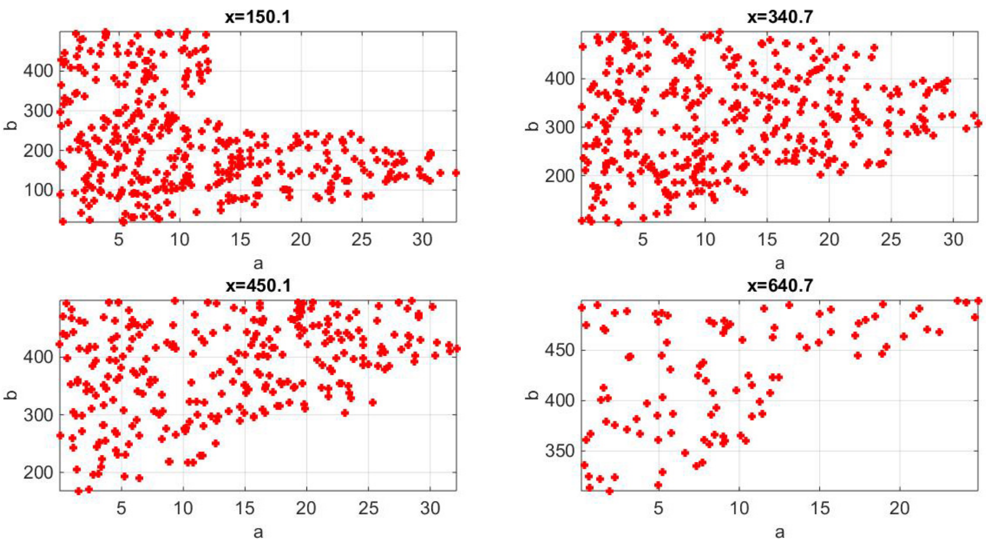

Fig. 3. Test for the expansion (5) for different values of x. The red points indicate points where the error obtained when testing the recurrence relation in (7) is greater than 10^{-12}.