Grothendieck inequalities characterize converses to the polynomial method

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A surprising 'converse to the polynomial method' of Aaronson et al. (CCC'16) shows that any bounded quadratic polynomial can be computed exactly in expectation by a 1-query algorithm up to a universal multiplicative factor related to the famous Grothendieck constant. Here we show that such a result does not generalize to quartic polynomials and 2-query algorithms, even when we allow for additive approximations. We also show that the additive approximation implied by their result is tight for bounded bilinear forms, which gives a new characterization of the Grothendieck constant in terms of 1-query quantum algorithms. Along the way we provide reformulations of the completely bounded norm of a form, and its dual norm.

1 Introduction

Quantum query complexity is one of the few models of computation in which the strengths and weaknesses of quantum computers can be rigorously studied with currently-available techniques (see e.g., [Amb18, Aar21] for recent surveys). On the one hand, many of quantum computing's best-known algorithms, such as for unstructured search [Gro96], period finding (the core of Shor's algorithm for integer factoring) [Sho97] and element distinctness [Amb07], are most naturally described in the query model. On the other hand, the model admits powerful lower-bound techniques.

For a (possibly partial) Boolean function $f : D \to \{-1, 1\}$ defined on a set $D \subseteq \{-1, 1\}^n$, the celebrated *polynomial method* of Beals, Buhrman, Cleve, Mosca and de Wolf [BBC+01] gives a lower bound on the quantum query complexity of f, denoted $Q(f)$, in terms of the minimal degree of an approximating polynomial for f, or approximate degree, $\widetilde{\deg}(f)$. The method relies on the basic fact that for any t-query quantum algorithm $\mathcal{A}$ that takes an n-bit

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input and returns a sign, there is a real n-variable polynomial p of degree at most $2t$ such that $p(x) = \mathbb{E}[\mathcal{A}(x)]$ for every x. Here, the expectation is taken with respect to the randomness in the measurement done by $\mathcal{A}$.[1] Using this method, many well-known quantum algorithms were proved to be optimal in terms of query complexity (see e.g., [BKT20] and references therein).

Since polynomials are simpler objects than quantum query algorithms, it is of interest to know how well approximate degree approximates quantum query complexity. There are total functions f that satisfy $Q(f) \geq \widetilde{\deg}(f)^c$ for some absolute constant $c > 1$ [Amb06, ABDK16]; the second reference gives an exponent $c = 4 - o(1)$, which was shown to be optimal in [ABDK16]. For partial functions it was recently shown that this separation can even be exponential [AB23]. These separations rule out a direct converse to the polynomial method, whereby a given bounded degree-$2t$ polynomial p can be computed by a t-query quantum algorithm $\mathcal{A}$. However, since these results concern functions whose approximate degree grows with n, they leave room for the possibility that such an $\mathcal{A}$ approximates p with some error that depends on t.

We will say that a polynomial p is *bounded* if its restriction to the Boolean hypercube takes values in the interval $[-1, 1]$ and that $\mathcal{A}$ *approximates* p if for some constant *additive* error parameter $\varepsilon < 1$, we have that $|p(x) - \mathbb{E}[\mathcal{A}(x)]| \leq \varepsilon$ for every x. Note that an additive error of 1 can trivially be achieved with a uniformly random coin flip. For a function $p : \{-1, 1\}^n \to \mathbb{R}$ and positive integer t, we denote the smallest additive error that a t-query quantum algorithm can achieve by

$$\mathcal{E}(p, t) := \inf \{\varepsilon \geq 0 \mid \exists\ t\text{-query quantum algorithm } \mathcal{A} \text{ with } |p(x) - \mathbb{E}[\mathcal{A}(x)]| \leq \varepsilon \quad \forall x \in \{-1, 1\}^n\}. \tag{1}$$

For bounded polynomials of degree at most 2, a "multiplicative converse" to the polynomial method was proved in [AAI⁺16], showing that up to an absolute constant scaling, quadratic polynomials can indeed be computed by 1-query quantum algorithms.

Theorem 1.1 (Quadratic multiplicative converse [AAI⁺16])**.** *There exists an absolute constant $C \in (0, 1]$ such that $\mathcal{E}(Cp, 1) = 0$ for every bounded polynomial p of degree at most 2.*

This result directly implies the following additive version.

Corollary 1.2 (Quadratic additive converse)**.** *There exists an absolute constant $\varepsilon \in (0, 1)$ such that the following holds. For every bounded polynomial p of degree at most 2, we have $\mathcal{E}(p, 1) \leq \varepsilon$. In particular, one can take $\varepsilon = 1 - C$ for the constant C appearing in Theorem 1.1.*

In light of the polynomial method, Corollary 1.2 shows that one-query quantum algorithms are roughly equivalent to bounded quadratic polynomials. The authors of [AAI⁺16] asked whether this result generalizes to higher degrees. Two ways to interpret this question are that for any k, any bounded degree-$2k$ polynomial p satisfies:

(a) Multiplicative converse: $\mathcal{E}(Cp, k) = 0$ for some $C = C(k) > 0$, or;

(b) Additive converse: $\mathcal{E}(p, k) \leq \varepsilon$ for some $\varepsilon = \varepsilon(k) < 1$.

[1] We identify quantum query algorithms with the (random) functions giving their outputs.

The dependence on the degree k in these options is necessary due to the known separations between bounded-error quantum query complexity and approximate degree. Option (a), the higher-degree version of Theorem 1.1, was ruled out in [ABP19].

Theorem 1.3. *For any $C > 0$, there exist an $n \in \mathbb{N}$ and a bounded quartic n-variable polynomial p such that no two-query quantum algorithm $\mathcal{A}$ satisfies $\mathbb{E}[\mathcal{A}(x)] = Cp(x)$ for every $x \in \{-1,1\}^n$.*

Note that Option (a) with C implies Option (b) with $1-C$, but Theorem 1.3 does not rule out Option (a).

1.1 Our contributions.

Our first contribution concerns an error in the original proof of Theorem 1.3, which was based on a probabilistic example. Here, we show that Theorem 1.3 holds as stated, both by considering a slightly modified probabilistic example and by giving a completely explicit example. More importantly, we prove a stronger result that subsumes Theorem 1.3: we rule out the possibility of Option (b).

Theorem 1.4. *There is no constant $\varepsilon \in (0,1)$ such that for every bounded polynomial p of degree at most 4, we have $\mathcal{E}(p,2) \leq \varepsilon$.*

In the context of quantum query complexity of Boolean functions, this rules out arguably the most natural way to *upper* bound $Q(f)$ in terms of $\widetilde{\deg}(f)$: First, ε-approximate f by a degree-$2t$ polynomial p, then ε'-approximate p with a t-query quantum algorithm $\mathcal{A}$, with $\varepsilon+\varepsilon' < 1$, and finally boost the success probability of $\mathcal{A}$ so that it approximates f, for instance by taking the majority of independent runs of $\mathcal{A}$. Corollary 1.2 gives the only exceptional case where this is possible in general.

Our second contribution concerns 1-query quantum algorithms. Theorem 1.1 was proved using a surprising application of Grothendieck's theorem from Banach space theory [Gro53] (see also Section 1). For bounded bilinear forms, $p(x,y) = x^{\mathsf{T}}Ay$ given by a matrix $A \in \mathbb{R}^{n\times n}$, the result holds with $1/C$ equal to the Grothendieck constant K_G (see [ABP19, Section 5] for a short proof). Determining the precise value of K_G is a notorious open problem posed in [Gro53]; the best-known lower and upper bounds place it in the interval $(1.676, 1.782)$ [Dav84, Ree91, BMMN13]. The general form of Theorem 1.1 follows from decoupling techniques. It is not hard to show that $1/K_G$ is the optimal constant in the bilinear case for the multiplicative setting of Theorem 1.1. Here, we also show that the additive approximation implied by the multiplicative setting is optimal.

Theorem 1.5. *The worst-case minimum error for one-query quantum algorithms satisfies*

$$\sup_p \mathcal{E}(p,1) = 1 - \frac{1}{K_G},$$

where the supremum is taken over the set of bounded bilinear forms.

This complements another well-known characterization of K_G in terms of the largest-possible Bell-inequality violations in two-player XOR games [Tsi80].

1.2 Our main technical result

Both Theorems 1.3 and 1.4 are in fact corollaries of our main result (Theorem 3.1 below), which gives a formula for $\mathcal{E}(p,t)$ when p is a block-multilinear form. Block-multilinear forms already played an important role in other works related to quantum query complexity [OZ15, AAI+16, BSdW22], theoretical computer science [KN07, Lov10, KM13] and in the polarization theory of functional analysis [BH31, Har72].

The formula characterizes $\mathcal{E}(p,t)$ in terms of a ratio of norms appearing naturally in Grothendieck's theorem for bilinear forms. A polynomial p is a bilinear form if it can be expressed as $p(x,y) = x^\mathsf{T} Ay$ for a matrix $A \in \mathbb{R}^{n\times n}$. A central role is played by two norms associated to A (and hence p). The first is the ℓ_∞-norm of A when identified with the map $(x,y) \mapsto p(x,y)$,

$$\|A\|_\infty = \max_{x,y\in\{-1,1\}^n} x^\mathsf{T} Ay.$$

The second is its completely bounded norm, which may be given by

$$\|A\|_{\mathrm{cb}} = \sup_{d\in\mathbb{N}, u,v:[n]\to S^{d-1}} \sum_{i,j=1}^{n} A_{i,j}\langle u(i), v(j)\rangle, \tag{2}$$

where S^{d-1} denotes the Euclidean unit sphere of $\mathbb{R}^d$. The celebrated Grothendieck theorem asserts that these norms are equivalent up to a constant factor.

Theorem 1.6 (Grothendieck's theorem)**.** *There exists a constant $K < \infty$ such that for any $n \in \mathbb{N}$ and $A \in \mathbb{R}^{n\times n}$, we have*

$$\|A\|_\infty \le \|A\|_{\mathrm{cb}} \le K\|A\|_\infty. \tag{3}$$

The non-trivial part of this result is the second inequality in (3) and the smallest K for which it holds is the above-mentioned Grothendieck constant K_G. Our characterization of $\mathcal{E}(p,1)$ involves the *dual norms* of the ℓ_∞-bilinear norm and the completely bounded norm (c.f. Section 2.1). The dual formulation of (the second inequality in) Grothendieck's theorem asserts that, for any matrix A,

$$\|A\|_{\infty,*} \le K_G\|A\|_{\mathrm{cb},*}. \tag{4}$$

Similar norms can be defined for block-multilinear forms of higher degree. Endowing the space of polynomials with the standard inner product of the coefficient vectors in the monomial basis, our formula for $\mathcal{E}(p,t)$ is as follows.

Theorem 1.7 (Informal version of Theorem 3.1)**.** *For a block-multilinear form p of degree $2t$, we have*

$$\mathcal{E}(p,t) = \sup_q \frac{\langle p,q\rangle - \|q\|_{\mathrm{cb},*}}{\|q\|_{\infty,*}}.$$

where the supremum runs over all block-multilinear forms q of degree $2t$.

The proof of Theorem 1.7 uses a characterization of quantum query algorithms in terms of completely bounded polynomials [ABP19]. Recently, this characterization was also used to make progress on the problem to determine "the need for structure in quantum speed-ups" in works by Bansal, Sinha and de Wolf [BSdW22] and by the second author [Esc24]. In addition, it led to a new exact SDP-based formulation for quantum query complexity, due to Laurent and the third author [GL19].

Theorems 1.4 and 1.5 follow from Theorem 1.7 by taking suprema over particular sequences of bounded degree-$2t$ block-multilinear forms. From Theorem 1.7 it follows that

$$\sup_p \mathcal{E}(p,t) = \sup_q \left[\left(\sup_p \frac{\langle p, q\rangle}{\|q\|_{\infty,*}} \right) - \frac{\|q\|_{\mathrm{cb},*}}{\|q\|_{\infty,*}} \right] = 1 - \inf_q \frac{\|q\|_{\mathrm{cb},*}}{\|q\|_{\infty,*}}. \tag{5}$$

Now, Theorem 1.5 follows from Eq. (5) and the dual version of Grothendieck's inequality (Eq. (4)). Similarly, Theorem 1.4 is proven by using Eq. (5) and constructing a family of degree-4 polynomials $(p_n)_n$ that witnesses the failure of Grothendieck inequality. By this we mean that $(p_n)_n$ exhibit the separation

$$\frac{\|p_n\|_{\mathrm{cb}}}{\|p_n\|_{\infty}} \to \infty. \tag{6}$$

By duality this implies that there is a sequence $(r_n)_n$ with $\|r_n\|_{\mathrm{cb},*}/\|r_n\|_{\infty,*} \to 0$, which alongside Eq. (5) implies that $\sup_p \mathcal{E}(p,2) = 1$, as desired.

1.3 Organization

The rest of the paper is organized as follows. Section 2 compiles the necessary preliminaries, in Section 3 we prove our main technical result, in Section 4 we show the norm separations displayed in Eq. (6), and in Section 5 we prove Theorems 1.4 and 1.5. In Section 6 we pose an open question, whose positive answer would strenghten our main result and establish a clean link between quantum query algorithms and XOR games. Finally, in Section 7 we phrase the dual norms as (efficiently solvable) convex optimization programs and use these formulations to give an alternative proof of our main result. (Strictly speaking, Section 7 is not necessary to understand the proofs of Theorems 1.4, 1.5 and 1.7.)

2 Preliminaries

2.1 Notation

For $n \in \mathbb{N}$, write $[n] := \{1, \ldots, n\}$. We endow $\mathbb{R}^d$ with the standard inner product $\langle x, y\rangle = \sum_{i\in[d]} x_i y_i$ and write the resulting norm as $\|x\| = \sqrt{\langle x, x\rangle}$. Denote the set of unit vectors of $\mathbb{R}^d$ by S^{d-1}.

We endow the space of matrices $\mathbb{R}^{d\times d}$ with the standard operator norm. We write $M(d) = \mathbb{R}^{d\times d}$ and let $B_{M(d)}$ denote the unit ball in $M(d)$ with respect to the operator norm (i.e., the set of contractions).

Given a normed vector space $(V, \|\ \|)$ with $V \subseteq \mathbb{R}^d$, the dual norm of an element $v \in V$ is given by

$$\|v\|_* = \sup\{|\langle v, w\rangle| \mid w \in V,\ \|w\| \le 1\}.$$

2.2 Polynomials

As usual we let $\mathbb{R}[x_1, \ldots, x_n]$ be the ring of n-variate polynomials with real coefficients, whose elements we write as

$$p(x) = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha x^\alpha, \tag{7}$$

where $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and $c_\alpha \in \mathbb{R}$. We define the support of p by

$$\mathrm{supp}(p) = \{\alpha \in \mathbb{Z}^n_{\geq 0} \mid c_\alpha \neq 0\}.$$

For $\alpha \in \mathbb{Z}^n_{\geq 0}$, write $|\alpha| = \alpha_1 + \cdots + \alpha_n$, which is the degree of the monomial x^α. A form of degree d is a homogeneous polynomial of degree d, i.e., a polynomial whose support consists of α for which $|\alpha| = d$. Denote by $\mathbb{R}[x_1, \ldots, x_n]_{=d}$ the space of forms of degree d. For p as in Eq. (7), define its homogeneous degree-d part by

$$p_{=d}(x) = \sum_{|\alpha|=d} c_\alpha x^\alpha.$$

We endow $\mathbb{R}[x_1, \ldots, x_n]$ with the inner product given by

$$\langle p, q\rangle = \sum_{\alpha \in \mathbb{Z}^n_{\geq 0}} c_\alpha c'_\alpha,$$

where c_α and c'_α are the coefficients of p and q, respectively.

Multilinear polynomials (those that are affine in every variable) will play an important role in this work, as they can be identified with the real-valued functions defined on the Boolean hypercube.

We recall the definition of $\|\cdot\|_1$ and $\|\cdot\|_\infty$, which are seminorms of polynomials in $\mathbb{R}[x_1, \ldots, x_n]$, and norms on the space of multilinear polynomials.

$$\|p\|_\infty := \sup_{x \in \{-1,1\}^n} |p(x)|,$$
$$\|p\|_1 := \mathbb{E}_{x \in \{-1,1\}^n} |p(x)|,$$

where the expectation is taken with respect to the uniform probability measure.

2.3 Tensors, norms and quantum query complexity

Toward defining the completely bounded norm, we restrict our attention to forms only. For every $p \in \mathbb{R}[x_1, \ldots, x_n]_{=t}$ there are many t-tensors $T \in \mathbb{R}^{n \times \cdots \times n}$ such that $T(x) = p(x)$ for every $x \in \mathbb{R}^n$, where

$$T(x) := \sum_{\mathbf{i} \in [n]^t} T_\mathbf{i} x(\mathbf{i}),$$

where $x(\mathbf{i}) := x_{i_1} x_{i_2} \cdots x_{i_t}$. We let $\mathcal{S}_n$ be the symmetric group on n elements. These tensors only have to satisfy

$$\sum_{\mathbf{i} \in \mathcal{I}_\alpha} T_\mathbf{i} = c_\alpha \quad \forall \alpha \in \mathbb{Z}^n_{\geq 0}, \tag{8}$$

where $\mathcal{I}_\alpha$ is the set of elements $\mathbf{i}$ of $[n]^t$ such that the element $m \in [n]$ occurs α_m times in $\mathbf{i}$. Each of these tensors gives a way of evaluating p in matrices, namely for every matrix-valued map $A : [n] \to M(d)$,

$$T(A) := \sum_{\mathbf{i}\in[n]^t} T_{\mathbf{i}} A(\mathbf{i}),$$

where $A(\mathbf{i}) := A(i_1)A(i_2)\dots A(i_t)$. It is also important to consider the unique symmetric t-tensor $T_p \in \mathbb{R}^{n\times\cdots\times n}$ such that

$$p(x) = T_p(x).$$

For a tensor $T = (T_{\mathbf{i}})_{\mathbf{i}\in[n]^t}$ and a permutation $\sigma \in \mathcal{S}_t$ we define $T \circ \sigma := (T_{\sigma(\mathbf{i})})_{\mathbf{i}\in[n]^t}$, where $\sigma(\mathbf{i}) := ((\sigma(i_1), \dots, \sigma(i_d))$. We say that T symmetric if $T = T \circ \sigma$ for all $\sigma \in \mathcal{S}_t$. The entries of this T_p are given by

$$(T_p)_{\mathbf{i}} = \frac{c_{e_{i_1}+\cdots+e_{i_t}}}{\tau(i_1,\dots,i_t)} \text{ for } \mathbf{i} \in [n]^t, \tag{9}$$

where $\tau(i_1, \dots, i_t)$ is the number of distinct permutations of the sequence $(i_1, \dots, i_t)$ and $\{e_i\}$ is an orthonormal basis of $\mathbb{R}^n$. The completely bounded norm of a tensor T is given by[2]

$$\|T\|_{\mathrm{cb}} = \sup\Big\{\Big\| \sum_{\mathbf{i}\in[n]^t} T_{\mathbf{i}} A_1(i_1)\dots A_t(i_t)\Big\| \ \Big|\ A_s\colon [n] \to B_{M(d)},\ s \in [t],\ d \in \mathbb{N}\Big\}. \tag{10}$$

It is not hard to show that for a 2-tensor (matrix) T this definition agrees with the one given in the introduction in Eq. (2).

The completely bounded norm of a form p is given by

$$\|p\|_{\mathrm{cb}} = \inf \sum_{\sigma\in\mathcal{S}_t} \|T^\sigma\|_{\mathrm{cb}}, \tag{11}$$

where the infimum runs over all the families of t-tensors $\{T_\sigma\}_{\sigma\in S_t}$ such that $\sum_{\sigma\in\mathcal{S}_t} T^\sigma \circ \sigma = T_p$. In Section 3.1 we provide easier expressions for both $\|T\|_{\mathrm{cb}}$ (showing that a single contraction-valued map $A\colon [n] \to B_{M(d)}$ suffices) and $\|p\|_{\mathrm{cb}}$. To prove our main technical result, Theorem 3.1, which expresses $\mathcal{E}(p, t)$ in the language of completely bounded polynomials, we combine these simplified expressions with the following characterization of quantum query algorithms, proved in [ABP19].

Theorem 2.1 (Quantum query algorithms are completely bounded forms)**.** *Let $p : \{-1, 1\}^n \to [-1, 1]$ and let $t \in \mathbb{N}$. Then,*

$$\begin{aligned} \mathcal{E}(p,t) = \inf \quad & \|p - q\|_\infty \\ \text{s.t.} \quad & h \in \mathbb{R}[x_1,\dots,x_{n+1}]_{=2t} \textit{ with } \|h\|_{\mathrm{cb}} \le 1 \\ & q : \{-1,1\}^n \to \mathbb{R},\ \text{with } q(x) = h(x,1)\ \forall\ x \in \{-1,1\}^n. \end{aligned}$$

[2]This is the completely bounded norm of T when regarded as an element of $\ell_n^1 \otimes_h \cdots \otimes_h \ell_n^1$, where h stands for the Haagerup tensor product, which determines a well-studied tensor norm. See for instance [Pau03, Chapter 17].

2.4 Block-multilinear forms

Our main result Theorem 3.1 is stated for a special kind of polynomials, which are the block-multilinear forms.

Definition 2.2. Let $\mathcal{P} = \{I_1, \dots, I_t\}$ be a partition of $[n]$ into t (pairwise disjoint) non-empty subsets. Define the set of *block-multilinear polynomials with respect to* $\mathcal{P}$ to be the linear subspace

$$V_{\mathcal{P}} = \mathrm{span}\{x_{i_1} \cdots x_{i_t} \mid i_1 \in I_1, \dots, i_t \in I_t\}.$$

We also work with the larger space of polynomials spanned by monomials where in the above we replace linearity by odd degree.

Definition 2.3. For a family $\mathcal{Q} \subseteq 2^{[n]}$ of pairwise disjoint subsets, let $W_{\mathcal{Q}} \subseteq \mathbb{R}[x_1, \dots, x_n]$ be the subspace of polynomials spanned by monomials x^α with $\alpha \in \mathbb{Z}^n_{\geq 0}$ satisfying

$$\sum_{i \in I} \alpha_i \equiv 1 \bmod 2 \ \forall I \in \mathcal{Q}. \tag{12}$$

We use $\Pi_{\mathcal{Q}} : \mathbb{R}[x_1, \dots, x_n] \to W_{\mathcal{Q}}$ to refer to the orthogonal projector onto $W_{\mathcal{Q}}$.

Remark 2.4. Given a partition $\mathcal{P}$ of $[n]$, we have $V_{\mathcal{P}} \subset W_{\mathcal{P}}$. In particular, $V_{\mathcal{P}}$ consists of precisely the multilinear polynomials in $W_{\mathcal{P}}$.

Although the projector $\Pi_{\mathcal{Q}}$ onto $W_{\mathcal{Q}}$ is properly defined on the space of polynomials, we will slightly abuse notation and let it act on a t-tensor $T \in \mathbb{R}^{n \times \cdots \times n}$ as follows. Define $\mathcal{I}_{\mathcal{Q}} \subseteq [n]^t$ to be the set of t-tuples that contain an odd number of elements from each set $I \in \mathcal{Q}$. Then, we let $\Pi_{\mathcal{Q}} T$ be the tensor given by

$$(\Pi_{\mathcal{Q}} T)_{\mathbf{i}} := \begin{cases} T_{\mathbf{i}} & \text{if } \mathbf{i} \in \mathcal{I}_{\mathcal{Q}}, \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

It is not hard to see that if p is a polynomial satisfying $T(x) = p(x)$ for every $x \in \{-1,1\}^n$, then $\Pi_{\mathcal{Q}} T(x) = \Pi_{\mathcal{Q}} p(x)$ for every $x \in \{-1,1\}^n$.

We note that all the norms and seminorms we have mentioned are norms on the space $V_{\mathcal{P}}$ for any partition $\mathcal{P}$ of $[n]$. Hence, we can take the dual of these norms with respect to this subspace, so from now on $\|p\|_{\infty,*}$ and $\|p\|_{\mathrm{cb},*}$ will be the dual of $\|p\|_\infty$ and $\|p\|_{\mathrm{cb}}$ of $V_{\mathcal{P}}$, respectively. By contrast, when we write $\|R\|_{\mathrm{cb},*}$ for some t-tensor $\mathbb{R}^{n \times \cdots \times n}$ we refer to the dual norm of the completely bounded norm of R with respect to the whole space of t-tensors.

We stress that $\|\cdot\|_{\infty,*}$ need not be equal to $\|\cdot\|_1$. This is because we are taking the dual norms with respect to $V_{\mathcal{P}}$ and not with respect to the space of all multilinear maps, in which case the dual norm would be $\|p\|_1$. The following example shows that $\|p\|_{\infty,*} \neq \|p\|_1$ in general.

Example 2.5. Consider $n = 3$, $t = 1$, $p = (x_1 + x_2 + x_3)/3$ and $\mathcal{P} = \{[3]\}$, so $V_{\mathcal{P}}$ is the set of linear polynomials. Then, $\|p\|_1 > 1/3$, but $\|p\|_{\infty,*} \leq 1/3$. Indeed, as $|p(x)| \geq 1/3$ for every $x \in \{-1,1\}^3$ and $|p(x)| > 1/3$ for some $x \in \{-1,1\}^3$, we have that $\|p\|_1 > 1/3$. Note that if q is linear, then $\|\hat{q}\|_1 = \|q\|_\infty$, where $\hat{q}$ is the Fourier transform of q. Hence

$$\|p\|_{\infty,*} = \sup_{q \in V_{\mathcal{P}}, \|q\|_\infty \leq 1} \langle p, q \rangle = \sup_{q \in V_{\mathcal{P}}, \|\hat{q}\|_1 \leq 1} \langle \hat{p}, \hat{q} \rangle \leq \sup_{\|\hat{q}\|_1 \leq 1} \|\hat{p}\|_\infty \|\hat{q}\|_1 = \frac{1}{3},$$

where in second equality we used Parseval's identity (see [O'D14, Chapter 1] for an introduction to Fourier analysis in the Boolean hypercube).

3 $\mathcal{E}(p,t)$ for block-multilinear forms

In this section we formally state and prove our main result:

Theorem 3.1. *Let $\mathcal{P}$ be a partition of $[n]$ in $2t$ subsets and $p \in V_{\mathcal{P}}$. Then,*

$$\mathcal{E}(p,t) = \sup\left\{\langle p, r\rangle - \|r\|_{\mathrm{cb},*} \mid r \in V_{\mathcal{P}},\ \|r\|_{\infty,*} \le 1\right\}.$$

For the proof, we use more convenient expressions for the completely bounded norms and the fact that the projector Π_Q is contractive under several norms.

3.1 Reformulation of the completely bounded norm of a form

In this section we derive alternative expressions for the completely bounded norms that facilitate our proof of Theorem 3.1. These expressions may also be useful in future applications of the of characterization quantum query algorithms in terms of completely bounded forms. After this section, we will implicitly use the formulas of Propositions 3.2 and 3.3 as definitions for $\|p\|_{\mathrm{cb}}$ and $\|T\|_{\mathrm{cb}}$, respectively.

First, we provide a simpler expression (compared to Eq. (11)) for $\|p\|_{\mathrm{cb}}$.

Proposition 3.2. *Let $p \in \mathbb{R}[x_1,\dots,x_n]_{=t}$ be a form of degree t. Then,*

$$\|p\|_{\mathrm{cb}} = \inf\Big\{\|T\|_{\mathrm{cb}} \mid p(x) = T(x)\ \forall x \in \mathbb{R}^n\Big\}.$$

Proof: We begin by noting that for any t-tensor T and permutation $\sigma \in \mathcal{S}_t$, we have

$$(T\circ\sigma)(x) = \sum_{\mathbf{i}\in[n]^t} T_{\sigma(\mathbf{i})} x(\mathbf{i}) = \sum_{\mathbf{i}\in[n]^t} T_{\sigma(\mathbf{i})} x(\sigma(\mathbf{i})) = T(x). \tag{14}$$

We first show that $\|p\|_{\mathrm{cb}} \le \inf\{\|T\|_{\mathrm{cb}} \mid p(x) = T(x)\}$. By restricting in (11) to decompositions where $T^\sigma = T/t!$ for all σ, we have

$$\begin{aligned}\|p\|_{\mathrm{cb}} &= \inf\Big\{\sum_{\sigma\in\mathcal{S}_t} \|T^\sigma\|_{\mathrm{cb}} \mid T_p = \sum_\sigma T^\sigma\circ\sigma\Big\}\\ &\le \inf\Big\{\|T\|_{\mathrm{cb}} \mid T_p = \frac{1}{t!}\sum_\sigma T\circ\sigma\Big\}.\end{aligned}$$

Hence, it suffices to show that $T_p = \frac{1}{t!}\sum_\sigma T\circ\sigma$ if and only if $p(x) = T(x)$ for all $x \in \mathbb{R}^n$. For the 'only if' implication, assume $T_p = \frac{1}{t!}\sum_\sigma T\circ\sigma$, then for any $x \in \mathbb{R}^n$ we have that

$$p(x) = T_p(x) = \frac{1}{t!}\sum_\sigma T\circ\sigma(x) = \frac{1}{t!}\sum_\sigma T(x) = T(x),$$

where in the second equality we have used (14). For the reverse implication, assume that $p(x) = T(x)$ for all $x \in \mathbb{R}^n$. Then, using (14), we obtain

$$\frac{1}{t!}\sum_\sigma T\circ\sigma(x) = p(x)$$

for all $x \in \mathbb{R}^n$ and therefore $T_p = \frac{1}{t!}\sum_\sigma T \circ \sigma$ since T_p is the unique symmetric t-tensor with $T_p(x) = p(x)$ for all $x \in \mathbb{R}^n$. This concludes the proof of the first inequality.

Now for the other inequality, $\|p\|_{\mathrm{cb}} \geq \inf\{\|T\|_{\mathrm{cb}} \mid p(x) = T(x)\}$, let T^σ ($\sigma \in \mathcal{S}_t$) be such that $T_p = \sum_\sigma T^\sigma \circ \sigma$ and define $T = \sum_\sigma T^\sigma$. Then we have

$$p(x) = T_p(x) = \sum_\sigma T^\sigma \circ \sigma(x) = \sum_\sigma T^\sigma(x) = T(x),$$

and $\|T\|_{\mathrm{cb}} \leq \sum_\sigma \|T^\sigma\|_{\mathrm{cb}}$ by the triangle inequality. □

Second, we show that the contraction-valued maps A_s in the definition of $\|T\|_{\mathrm{cb}}$ (see Equation (10)) can be taken to be the same. This result can be understood as the fact that the polarization constant of completely bounded multilinear maps is 1.[3]

Proposition 3.3. *Let $T \in \mathbb{R}^{n\times\cdots\times n}$ be a t-tensor. Then,*

$$\|T\|_{\mathrm{cb}} = \sup\left\{\|T(A)\| \mid A\colon [n] \to B_{M(d)},\ d \in \mathbb{N}\right\}.$$

Proof: Let $|||T|||$ be the expression in the right-hand side of the statement. Note that it is the same as the expression of $\|T\|_{\mathrm{cb}}$, but now the contraction-valued maps $A_1, \dots, A_t$ are all equal. This shows that $|||T||| \leq \|T\|_{\mathrm{cb}}$. To prove the other inequality, let $A_1, \dots, A_t\colon [n] \to B_{M(d)}$ and $u, v \in S^{d-1}$. Now, define the contraction-valued map A by $A(i) := \sum_{s\in[t]} e_s e_{s+1}^\mathsf{T} \otimes A_s(i)$ for $i \in [n]$, and define the unit vectors $u' := e_1 \otimes u$ and $v' := e_{t+1} \otimes v$. They satisfy

$$\langle u, A_1(i_1)\dots A_t(i_t)v\rangle = \langle u', A(\mathbf{i})v'\rangle \quad \text{for all } \mathbf{i} \in [n]^t,$$

so in particular

$$\sum_{\mathbf{i}\in[n]^t} T_\mathbf{i}\langle u, A_1(i_1)\dots A_t(i_t)v\rangle = \sum_{\mathbf{i}\in[n]^t} T_\mathbf{i}\langle u', A(\mathbf{i})v'\rangle.$$

Taking the supremum over all maps A_s and u, v shows that $\|T\|_{\mathrm{cb}} \leq |||T|||$, which concludes the proof. □

3.2 Contractivity of the projector Π_Q.

A key element of the proof of Theorem 3.1 is that can restrict the infimum in Theorem 2.1 to the space of polynomials $W_\mathcal{Q}$ given in Definition 2.3. To do that, we prove that the orthogonal projector onto this space, $\Pi_\mathcal{Q}$ is contractive in several norms. This will follow from the fact that $\Pi_\mathcal{Q}$ has a particularly nice structure in the form of an averaging operator. Let $\mathcal{Q}$ be a family of disjoint subsets of $[n]$. For each $I \in \mathcal{Q}$ let z_I be a random variable that takes the values -1 and 1 with probability 1/2 and let $z = (z_I)_{I\in\mathcal{Q}}$. For a bit string $x \in \{-1,1\}^n$, we define the random variable $x \cdot z \in \{-1,1\}^n$ as

$$(x\cdot z)(i) := \begin{cases} x_i z_I & \text{if } i \in I \text{ for some } I \in \mathcal{Q}, \\ x_i & \text{otherwise.} \end{cases}$$

[3] In Banach space theory a t-linear map $T : X \times \cdots \times X \to Y$ determines a homogeneous degree-t polynomial $P : X \to Y : A \to T(A, \dots, A)$. The operator norms of T and P are equivalent if T is symmetric: $\|T\| \leq \|P\| \leq K(t)\|T\|$, where $K(t)$ is the polarization constant of degree t. For a survey on the topic see [MMFPSS22, Section 5.1].

For a matrix-valued map $A : [n] \to M(d)$ we define the random variable $A \cdot z$ in an analogous way.

Proposition 3.4. *For any $p \in \mathbb{R}[x_1, \dots, x_n]$ and $x \in \mathbb{R}^n$, we have that*

$$\Pi_{\mathcal{Q}} p(x) = \mathbb{E}_z \Big[p(x \cdot z) \prod_{I \in \mathcal{Q}} z_I \Big].$$

Similarly, for any t-tensor $T \in \mathbb{R}^{n \times \cdots \times n}$, positive integer d and matrix-valued map $A\colon [n] \to M(d)$, we have that

$$\Pi_{\mathcal{Q}} T(A) = \mathbb{E}_z \left[T(A \cdot z) \prod_{I \in \mathcal{Q}} z_I \right].$$

Proof: By linearity, it suffices to prove the equality for monomials. Let $\alpha \in \mathbb{Z}^n_{\geq 0}$. Then we have

$$(x \cdot z)^\alpha \prod_{I \in \mathcal{Q}} z_I = x^\alpha \prod_{I \in \mathcal{Q}} z_I^{1 + \sum_{i \in I} \alpha_i}.$$

It follows that

$$\mathbb{E}_z \Big[(x \cdot z)^\alpha \prod_{I \in \mathcal{Q}} z_I \Big] = \begin{cases} x^\alpha & \text{if } 1 + \sum_{i \in I} \alpha_i = 0 \bmod 2 \ \forall I \in \mathcal{Q}, \\ 0 & \text{otherwise.} \end{cases}$$

It remains to observe that this is precisely the projection of x^α on $W_{\mathcal{Q}}$. The statement for tensors follows analogously. □

Finally, we prove that $\Pi_{\mathcal{Q}}$ is contractive with respect to the relevant norms.

Lemma 3.5. *Let $\mathcal{Q}$ be a family of disjoint subsets of $[n]$ and $p \in \mathbb{R}[x_1, \dots, x_n]$ and let* $\text{norm} \in \{\text{cb}, \infty, 1\}$ *where for the* cb*-norm we moreover require p to be homogeneous. Then*

$$\|\Pi_{\mathcal{Q}} p\|_{\text{norm}} \leq \|p\|_{\text{norm}}.$$

Proof: First, we consider the $\|\cdot\|_\infty$ norm. For every $x \in \{-1, 1\}^n$, we have that $x \cdot z \in \{-1, 1\}^n$, so

$$|\Pi_{\mathcal{Q}} p(x)| \leq \mathbb{E}_z |p(x \cdot z) \prod_{I \in \mathcal{Q}} z_I| = \mathbb{E}_z |p(x \cdot z)| \leq \mathbb{E}_z \|p\|_\infty = \|p\|_\infty,$$

where in the first inequality we used Proposition 3.4 and the triangle inequality.

Second, we consider $\|\cdot\|_{\text{cb}}$. Arguing as in the $\|\cdot\|_\infty$ case and using Proposition 3.3, it follows that for any t-tensor $T \in \mathbb{R}^{n \times \cdots \times n}$ we have that $\|\Pi_{\mathcal{Q}} T\|_{\text{cb}} \leq \|T\|_{\text{cb}}$. Given that $\Pi_{\mathcal{Q}} p(x) = \Pi_{\mathcal{Q}} T(x)$ if $p(x) = T(x)$, it follows that

$$\|\Pi_{\mathcal{Q}} p\|_{\text{cb}} \leq \|\Pi_{\mathcal{Q}} T\|_{\text{cb}} \leq \|T\|_{\text{cb}}$$

for every t-tensor $T \in \mathbb{R}^{n \times \cdots \times n}$ such that $T(x) = p(x)$. Taking the infimum over all those T we arrive at $\|\Pi_{\mathcal{Q}} p\|_{\text{cb}} \leq \|p\|_{\text{cb}}$.

Finally, for $\|\cdot\|_1$ we have

$$\|\Pi_{\mathcal{Q}} p\|_1 = \mathbb{E}_x |\mathbb{E}_z p(x \cdot z) \prod_{I \in \mathcal{Q}} z_I| \leq \mathbb{E}_x \mathbb{E}_z |p(x \cdot z)| = \mathbb{E}_z \mathbb{E}_x |p(x)| = \|p\|_1,$$

where in the first equality we have used Proposition 3.4 and in the third we have used the fact that the uniform measure is invariant under multiplication by $z \in \{-1, 1\}^n$. □

3.3 Putting everything together

We are now ready to prove Theorem 3.1. To this end, we start from the expression given in Theorem 2.1 for $\mathcal{E}(p,t)$ and let $h \in \mathbb{R}[x_1,\dots,x_{n+1}]_{=2t}$ with $\|h\|_{\mathrm{cb}} \le 1$ and let $q : \{-1,1\}^n \to \mathbb{R}$ be defined by $q(x) = h(x,1)$ for every $x \in \{-1,1\}^n$.

We first show that we can project q (and h) onto $W_{\mathcal{P}}$ and obtain a feasible solution whose objective value is at least as good as q. Since $\mathcal{P}$ is a partition of $[n]$, it defines a family of disjoint subsets of $[n+1]$, so by Lemma 3.5, we have $\|\Pi_{\mathcal{P}} h\|_{\mathrm{cb}} \le \|h\|_{\mathrm{cb}} \le 1$. Since the degree of h is at most $2t$, the polynomial $\Pi_{\mathcal{P}} h$ has degree at most $2t$. This shows that each monomial in its support contains exactly one variable from each of the $2t$ sets in $\mathcal{P}$. We can therefore observe that $\Pi_{\mathcal{P}} h$ does not depend on x_{n+1}. Since $h(x,1) = q(x)$, we have $\Pi_{\mathcal{P}} h(x,1) = \Pi_{\mathcal{P}} q(x)$ and therefore $\Pi_{\mathcal{P}} q \in V_{\mathcal{P}}$. We then use Proposition 3.2 to show that $\|\Pi_{\mathcal{P}} q\|_{\mathrm{cb}} \le 1$. Indeed, applying $\Pi_{\mathcal{P}}$ to a $2t$-tensor $T \in \mathbb{R}^{(n+1)\times\cdots(n+1)}$ that certifies $\|h\|_{\mathrm{cb}} \le 1$ results in a tensor $\Pi_{\mathcal{P}} T$ that satisfies $\Pi_{\mathcal{P}} T(\mathbf{i}) = 0$ whenever $\mathbf{i}$ contains an index equal to $n+1$. So, $\Pi_{\mathcal{P}} T(x,1) = \Pi_{\mathcal{P}} q(x)$ for every $x \in \{-1,1\}^n$ and thus $\Pi_{\mathcal{P}} T$, viewed as a $2t$-tensor in $\mathbb{R}^{n\times\cdots\times n}$, certifies $\|\Pi_{\mathcal{P}} q\|_{\mathrm{cb}} \le 1$. For the objective value of $\Pi_{\mathcal{P}} q$ we finally observe that

$$\|p - \Pi_{\mathcal{P}} q\|_\infty = \|\Pi_{\mathcal{P}}(p-q)\|_\infty \le \|p-q\|_\infty,$$

where we used that $p \in V_{\mathcal{P}}$ in the equality and Lemma 3.5 in the inequality. This shows that

$$\mathcal{E}(p,t) \ge \inf\{\|p-q\|_\infty \mid q \in V_{\mathcal{P}} \text{ with } \|q\|_{\mathrm{cb}} \le 1\}.$$

To show that the above inequality is in fact an equality it suffices to observe that given a polynomial $q \in V_{\mathcal{P}}$, we can define $h \in \mathbb{R}[x_1,\dots,x_{n+1}]$ as $h(x,x_{n+1}) = q(x)$ and then we have $\|h\|_{\mathrm{cb}} \le \|q\|_{\mathrm{cb}}$.

Finally, in the above reformulation of $\mathcal{E}(p,t)$, we can express $\|p-q\|_\infty$ in terms of its dual norm and obtain

$$\begin{aligned}\mathcal{E}(p,t) = \inf_q \sup_r \quad & \langle p-q, r\rangle \\ \text{s.t.} \quad & q \in V_{\mathcal{P}} \text{ with } \|q\|_{\mathrm{cb}} \le 1, \\ & r \in V_{\mathcal{P}} \text{ with } \|r\|_{\infty,*} \le 1.\end{aligned}$$

Finally, we need the von Neumann's minimax theorem (see [Nik54] for a proof).

Theorem 3.6 (Minimax)**.** *Let X and Y convex compact sets. Let $f : X \times Y \to \mathbb{R}$ such that f is concave in the first variable and convex in the second. Then,*

$$\sup_{x\in X} \inf_{y\in Y} f(x,y) = \inf_{y\in Y} \sup_{x\in X} f(x,y).$$

The desired result then follows by exchanging the infimum and supremum, which we are allowed to do by Theorem 3.6.

4 Separations between infinity and completely bounded norms

In this section we show that the completely bounded norm of a degree 4 bounded polynomial can be unbounded. In other words, we prove the following Theorem.

Theorem 4.1. *There is a sequence $p_n \in \mathbb{R}[x_1, \dots, x_n]_{=4}$ such that*

$$\frac{\|p_n\|_{\mathrm{cb}}}{\|p_n\|_\infty} \to \infty.$$

To prove Theorem 4.1 we first provide a framework to lower bound the completely bounded norm inspired on a technique due to Varopoulos [Var74].[4] Second, we construct two sequences of bounded polynomials, one random and one explicit, that fit in that framework and have unbounded completely bounded norm.

4.1 Lower bounding the completely bounded norm

We will first talk about general cubic forms, that is polynomials given by:

$$p(x) = \sum_{S \in \binom{[n]}{3}} c_S \prod_{i \in S} x_i, \tag{15}$$

where the c_S are some real coefficients. We will lower bound its completely bounded norm. Then, we will extend this lower bound to an associated quartic form, given by $x_0 p(x)$. For $i \in [n]$, define the ith *slice* of p to be the symmetric matrix $M_i \in \mathbb{R}^{n \times n}$ with (j,k)-coefficient equal to $c_{\{i,j,k\}}$ if i, j, k are pairwise distinct and 0 otherwise. Then, define

$$\Delta(p) = \max_{i \in [n]} \|M_i\|.$$

Lemma 4.2 (tri-linear Varopoulos decomposition)**.** *Let p be an n-variate multilinear cubic form as in* (15)*. Then, for some $d \in \mathbb{N}$, there exists $A : [n] \to B_{M(d)}$ and orthogonal vectors $u, v \in S^{d-1}$ such that $[A(j), A(i)] = 0$, and*

$$A(i)^2 = 0 \tag{16}$$
$$\langle u, A(i)v \rangle = 0 \tag{17}$$
$$\langle u, A(i)A(j)v \rangle = 0 \tag{18}$$
$$\langle u, A(i)A(j)A(k)v \rangle = \frac{c_{\{i,j,k\}}}{\Delta(p)} \tag{19}$$

for all pairwise distinct $i, j, k \in [n]$.

Proof: For each $i \in [n]$, define M_i as above. Define $W_i = \Delta(p)^{-1} M_i$ and note that W_i has operator norm at most 1. For each $i \in [n]$, define the $(2n+2) \times (2n+2)$ block matrix

$$A(i) = \left[\begin{array}{c|c|c|c} & & & \\ \hline e_i & & & \\ \hline & W_i^\intercal & & \\ \hline & & e_i^\intercal & \end{array}\right], \tag{20}$$

[4] We use the same construction as the one proposed by Varopoulos, but we apply it to multilinear polynomials, which gives it the extra property displayed in Eq. (16)

where the first and last rows and columns have size 1, the second and third have size n and where the empty blocks are filled with zeros. Define $u = e_{2n+2}$ and $v = e_1$. Eq. (17) is true because the bottom left corner of $A(i)$ is 0. From Eq. (20) follows that

$$A(i)A(j) = \left[\begin{array}{c|c|c|c} & & & \\ \hline & & & \\ \hline W_i^\intercal e_j & & & \\ \hline & e_i^\intercal W_j^\intercal & & \end{array}\right], \quad A(i)A(j)A(k) = \left[\begin{array}{c|c|c|c} & & & \\ \hline & & & \\ \hline & & & \\ \hline e_i W_j^\intercal e_k & & & \end{array}\right],$$

so Eqs. (18) and (19) follow by looking at the bottom left corner of these matrices. Finally, the property that $A(i)^2 = 0$ follows from the fact that

$$A(i)^2 = \left[\begin{array}{c|c|c|c} & & & \\ \hline & & & \\ \hline W_i^\intercal e_i & & & \\ \hline & e_i^\intercal W_i^\intercal & & \end{array}\right]$$

and that the ith row and ith column of M_i (and hence W_i) are zero. □

Corollary 4.3. *Let p be an n-variate multilinear cubic form as in* (15). *Suppose that an $(n+2)$-variate quartic form $h \in \mathbb{R}[x_0, x_1, \ldots, x_n, z]$ satisfies $h(x, 1) = x_0 p(x_1, \ldots, x_n)$ for every $x \in \{-1, 1\}^{n+1}$. Then,*

$$\|h\|_{\mathrm{cb}} \geq \frac{\|p\|_2^2}{\Delta(p)}.$$

Proof: From the orthonormality of the characters, it follows that h and $x_0 p$ have equal coefficients for each quartic multilinear monomial in the variables $x_0, \ldots, x_n$, which are c_S for $x_0 \chi_S$ with $S \in \binom{[n]}{3}$ and 0 otherwise. Let $A : [n] \to B_{M(d)}$ and $u, v \in S^d$ be as in Lemma 4.2, and extend A by $A(0) = I, A(n+1) = 0$. Commutativity and properties (16)–(18) imply that if a quartic monomial expression $A((i,j,k,l))$ with $i, j, k, l \in \{0, \ldots, n+1\}$ has repeated indices or an index equal to $n+1$, then $\langle u, A((i,j,k,l))v \rangle = 0$. With this, it follows that, for every T_h such that $T_h(x) = h(x)$, we have

$$\|T_h\|_{\mathrm{cb}} \geq \sum_{\mathbf{i} \in (\{0\} \cup [n+1])^4} T_{\mathbf{i}} \Big\langle u, A(\mathbf{i})v \Big\rangle = \sum_{S \in \binom{[n]}{3}} c_S \Big\langle u,\, A(0) \prod_{i \in S} A(i) v \Big\rangle. \tag{21}$$

Finally, if we use that $A(0) = I$, property (19) and Parseval's identity, we obtain the desired result:

$$\|h\|_{\mathrm{cb}} = \inf \|T_h\|_{\mathrm{cb}} \geq \sum_{S \in \binom{[n]}{3}} c_S \langle u, \prod_{i \in S} A(i) v \rangle = \Delta(p)^{-1} \sum_{S \in \binom{[n]}{3}} c_S^2 = \frac{\|p\|_2^2}{\Delta(p)}.$$

□

4.2 A separation based on a random example

We begin by defining a random cubic form as in (15) where the coefficients c_S are chosen to be independent uniformly distributed random signs. Parseval's identity then gives $\|p\|_2^2 = \binom{n}{3}$. We now use a standard random-matrix inequality to upper bound $\Delta(p)$ (see [Tao12, Corollary 2.3.6] for a proof).

Lemma 4.4. *There exist absolute constants $C, c \in (0, \infty)$ such that the following holds. Let n be a positive integer and let M be a random $n \times n$ symmetric random matrix such that for $j \geq i$, the entries M_{ij} are independent random variables with mean zero and absolute value at most 1. Then, for any $\tau \geq C$, we have*

$$\Pr\big[\|M\| > \tau\sqrt{n}\big] \leq Ce^{-c\tau n}.$$

Applying Lemma 4.4 to the slices M_i and the union bound then imply that $\Delta(p) \leq C\sqrt{n}$ with probability $1 - \exp(-Cn)$. By Hoeffding's inequality [BLM13, Theorem 2.8] and the union bound, we have that $\|p\|_\infty \leq Cn^2$ with probability $1 - \exp(-Cn)$. Rescaling p then gives that there exists a bounded multilinear cubic form such that $\|p\|_2^2/\Delta(p) \geq C\sqrt{n}$. Now Theorem 4.1 follows from Corollary 4.3.

4.3 A construction based on an explicit example

We also give an explicit construction using techniques from [BP19], which were used there to disprove a conjecture on a tri-linear version of Grothendieck's theorem. We do not exactly use the construction from that paper because it involves complex functions. Instead, we will use the Möbius function (defined below), which is real valued and has the desired properties.

The construction uses some notions from additive combinatorics. For a function $f : \mathbb{Z}_n \to [-1, 1]$ (on the cyclic group of order n), define the 3-linear form

$$p(x_1, x_2, x_3) = \sum_{a,b\in\mathbb{Z}_n} x_{1,a}x_{2,a+b}x_{3,a+2b}f(a + 3b).$$

where $x_1, x_2, x_3 \in \{-1, 1\}^n$ and the sums of a and b are done in $\mathbb{Z}_n$.

We begin by upper bounding $\Delta(p)$. The polynomial p has $3n$ slices, $M_{i,a} \in \mathbb{R}^{[3]\times\mathbb{Z}_n}$ for each $i \in [3]$ and $a \in \mathbb{Z}_n$, which we view as 3×3 block-matrices with blocks indexed by $\mathbb{Z}_n$. The slice $M_{1,a}$ is supported only on the $(2, 3)$ and $(3, 2)$ blocks, which are each others' transposes. On its $(2, 3)$ block it has value $f(a+3b)$ on coordinate $(a+b, a+2b)$ for each b. In particular, this matrix has at most one nonzero entry in each row and column. It follows that a relabeling of the rows turns $M_{1,a}$ into a diagonal matrix with diagonal entries in $[-1, 1]$, and therefore $\|M_{1,a}\| \leq 1$. Similarly, we get that $\|M_{i,a}\| \leq 1$ for $i = 2, 3$. Hence,

$$\Delta(p) \leq 1. \tag{22}$$

for any choice of f.

Now we will choose a specific f for which we will be able to upper bound $\|p\|_\infty$ and lower bound $\|p\|_2^2$. Identify $\mathbb{Z}_n$ with $\{0, 1 \ldots, n-1\}$ in the standard way. We choose f to be the Möbius function restricted to this interval. That is, set $f(0) = 0$ and for $a > 0$, set

$$f(a) = \begin{cases} 1 & \text{if } a \text{ is square-free with an even number of prime factors} \\ -1 & \text{if } a \text{ is square-free with an odd number of prime factors} \\ 0 & \text{otherwise.} \end{cases}$$

The infinity norm of p can be upper bounded in terms of the Gowers 3-uniformity norm of f. This norm plays a central role in additive combinatorics and is defined by

$$\|f\|_{U^3} = \Big(\mathbb{E}_{a,b_1,b_2,b_3\in\mathbb{Z}_n} \prod_{c\in\{0,1\}^3} f(a + c_1b_1 + c_2b_2 + c_3b_3)\Big)^{\frac{1}{8}}.$$

The proof of the announced bound can be found in [Gre07, Proposition 1.11].

Lemma 4.5 (generalized von Neumann inequality)**.** *Suppose that n is coprime to 6. Then, for any $f : \mathbb{Z}_n \to [-1, 1]$, we have that*

$$\|p\|_\infty \leq n^2 \|f\|_{U^3}.$$

A recent result by Tao and Teräväinen [TT23] given an upper bound to the Gowers 3-uniformity norm of the Möbius function.

Theorem 4.6. *Let $f : \mathbb{Z}_n \to \mathbb{R}$ be the Möbius function. Then,*

$$\|f\|_{U^3} \leq \frac{1}{(\log \log n)^C}.$$

for some constant $C > 0$.

Combining Lemma 4.5 and Theorem 4.6 it follows that

$$\|p\|_\infty \leq \frac{n^2}{(\log \log n)^C} \tag{23}$$

for some constant $C > 0$.

To lower bound $\|p\|_2^2$ we begin using Parseval's identity, which implies that

$$\|p\|_2^2 = n \sum_{a \in \mathbb{Z}_n} f(a)^2. \tag{24}$$

Given that $|f(a)|^2$ is 1 if a is square-free and 0 otherwise, we can use a classical result of number theory to lower bound $\|p\|_2^2$ (see [HW+79, page 269] for a proof).

Proposition 4.7. *There are $\frac{6}{\pi^2} n - O(\sqrt{n})$ natural numbers between 1 and n that are square-free.*

From Eq. (24) and Proposition 4.7 follows that

$$\|p\|_2^2 = \frac{6}{\pi^2} n^2 - O(\sqrt{n^3}). \tag{25}$$

Finally, we substitute p by $p/(n^2/(\log \log n)^C)$, and it follows from Eqs. (22), (23) and (25) that p is bounded and

$$\frac{\|p\|_2^2}{\Delta(p)} \geq \frac{6}{\pi^2} (\log \log n)^C - o(1).$$

Again, Theorem 4.1 now follows from Corollary 4.3.

Remark 4.8. The *jointly completely bounded norm* of p is given by

$$\|p\|_{\text{jcb}} = \sup_{d \in \mathbb{N}} \| \sum_{a,b \in \mathbb{Z}_n} A(1, a) A(2, a + b) A(3, a + 2b) f(a + 3b) \|,$$

where the supremum is taken over maps $A : [3] \times [n] \to \mathbb{C}^{d \times d}$ such that $\|A(i, a)\| \leq 1$ and $[A(i, a), A(j, b)] = [A(i, a), A(j, b)^*] = 0$ for all $i \neq j$ and $a, b \in \mathbb{Z}_n$. This norm can also be stated in terms of tensor products and the supremum is attained by observable-valued maps. As such, this norm appears naturally in the context of non-local games. It was shown in [BBB+19] that Proposition 4.5 also holds for the jointly completely bounded norm, that is $\|p\|_{\text{jcb}} \leq n^2 \|f\|_{U^3}$. The proof of Corollary 4.3 easily implies that $\|p\|_{\text{cb}} \geq \|p\|_2^2 / \Delta(p)$. This was used in [BP19] to prove that the jcb and cb norms are inequivalent.

5 Grothendieck inequalities characterize converses to the polynomial method

In this section we show, as a corollary of our main result Theorem 3.1, that Grothendieck inequalities characterize converses to the polynomial method . By this we mean that: i) for 1-query algorithms an additive converse is possible and moreover this converse characterizes K_G; and ii) for 2-query algorithms no additive converse is possible, because Grothendieck's inequality fails for 3-linear forms.

5.1 Characterizing K_G with 1-query quantum algorithms

Here we prove Theorem 1.5. Before doing that, we should prove Corollary 5.2, which states that the completely bounded norm of a bilinear form p regarded as polynomial is equal to its completely bounded norm as matrix A. This is not obvious from Eq. (11), because for a t-tensor T and permutation $\sigma \in \mathcal{S}_t$, it is in general not true that $\|T\|_{\mathrm{cb}} = \|T \circ \sigma\|_{\mathrm{cb}}$ (see for instance [ABP19]). It is not hard to show however, that when T is a matrix we have $\|T^{\mathsf{T}}\|_{\mathrm{cb}} = \|T\|_{\mathrm{cb}}$. This gives the following reformulation of the completely bounded norm of forms of degree 2.

Proposition 5.1. *Let $p \in \mathbb{R}[x_1, \dots, x_n]_{=2}$ be a quadratic form and let T_p be the unique symmetric matrix associated to p via (9). Then $\|p\|_{\mathrm{cb}} = \|T_p\|_{\mathrm{cb}}$.*

Proof: Let $T \in \mathbb{R}^{n\times n}$ be a matrix. First, we have

$$\begin{aligned} \|T^{\mathsf{T}}\|_{\mathrm{cb}} &= \sup\Big\{ \| \sum_{i,j} T_{j,i} A(i)B(j) \| \ \mid \ A, B \colon [n] \to B_{M(d)} \Big\} \qquad (26) \\ &= \sup\Big\{ \| \sum_{i,j} T_{j,i} B(j)^{\mathsf{T}} A(i)^{\mathsf{T}} \| \ \mid \ A, B \colon [n] \to B_{M(d)} \Big\} \\ &= \|T\|_{\mathrm{cb}}, \end{aligned}$$

where we use (twice) that for any matrix M we have $\|M\| = \|M^{\mathsf{T}}\|$.

Let $T \in \mathbb{R}^{n\times n}$ be a matrix with $p(x) = T(x)$. Then, $T_p = (T + T^{\mathsf{T}})/2$, so from the above and the triangle inequality it follows that

$$\|T_p\|_{\mathrm{cb}} = \frac{1}{2}\|T + T^{\mathsf{T}}\|_{\mathrm{cb}} \leq \frac{1}{2}(\|T\|_{\mathrm{cb}} + \|T^{\mathsf{T}}\|_{\mathrm{cb}}) = \|T\|_{\mathrm{cb}}.$$

Using Proposition 3.2 we conclude that $\|p\|_{\mathrm{cb}} = \|T_p\|_{\mathrm{cb}}$. □

Considering bilinear forms gives the following corollary.

Corollary 5.2. *Let $p : \{-1, 1\}^n \times \{-1, 1\}^n \to \mathbb{R}$ be a bilinear form and let $A \in \mathbb{R}^{n\times n}$ be such that $p(x, y) = x^{\mathsf{T}} A y$ for all $x, y \in \mathbb{R}^n$. Then, $\|A\|_{\mathrm{cb}} = \|p\|_{\mathrm{cb}}$.*

Proof: By Proposition 5.1 we have $\|p\|_{\mathrm{cb}} = \|T_p\|_{\mathrm{cb}}$. Now observe that $T_p = \frac{1}{2}\begin{pmatrix} 0 & A \\ A^{\mathsf{T}} & 0 \end{pmatrix}$ and hence

$$\begin{aligned} \|T_p\|_{\mathrm{cb}} &\leq \frac{1}{2}\left(\Big\| \begin{pmatrix} 0 & A \\ 0 & 0 \end{pmatrix} \Big\|_{\mathrm{cb}} + \Big\| \begin{pmatrix} 0 & 0 \\ A^{\mathsf{T}} & 0 \end{pmatrix} \Big\|_{\mathrm{cb}} \right) \\ &\leq \frac{1}{2}\left(\|A\|_{\mathrm{cb}} + \|A^{\mathsf{T}}\|_{\mathrm{cb}} \right) \\ &= \|A\|_{\mathrm{cb}}, \end{aligned}$$

where the last equality uses (26).

Conversely, let $B, C\colon [n] \to B_{M(d)}$. Note that

$$\|\sum_{i,j\in[n]} A_{i,j}B(i)C(j)\| = \sup_{u,v}\langle u, \sum_{i,j\in[n]} A_{i,j}B(i)C(j)v\rangle$$

where the supremum is taken over possibly distinct unit vectors u and v. We argue that we can take $u = v$ without loss of generality. Indeed, let U be a unitary matrix for which $Uu = v$, then

$$\|\sum_{i,j\in[n]} A_{i,j}B(i)C(j)U\| = \|\sum_{i,j\in[n]} A_{i,j}B(i)C(j)\| = \langle u, \sum_{i,j\in[n]} A_{i,j}B(i)C(j)Uu\rangle.$$

As we are optimizing over all $B, C\colon [n] \to B_{M(d)}$, replacing each $C(j)$ by $C(j)U$, we may assume that

$$\|\sum_{i,j\in[n]} A_{i,j}B(i)C(j)\| = \Big|\sum_{i,j\in[n]} A_{i,j}\langle u, B(i)C(j)u\rangle\Big|,$$

for a single unit vector u. For every $i \in [n]$, define the following contractions:

$$Q_1(i) = B(i) = Q_2^{\mathsf{T}}(i), \qquad Q_2(n+i) = C(i) = Q_1^{\mathsf{T}}(n+i).$$

Then,

$$\begin{aligned}\|T_p\|_{\mathrm{cb}} &\geq \Big\|\sum_{i,j\in[n]} \frac{1}{2}(A_{i,j}Q_1(i)Q_2(n+j) + A_{j,i}Q_1(n+i)Q_2(j))\Big\| \\ &= \Big|\sum_{i,j\in[n]} A_{i,j}\langle u, \frac{B(i)C(j) + C^{\mathsf{T}}(j)B^{\mathsf{T}}(i)}{2}u\rangle\Big| \\ &= \Big|\sum_{i,j\in[n]} A_{i,j}\langle u, B(i)C(j)u\rangle\Big|\end{aligned}$$

Taking the supremum over $B, C\colon [n] \to B_{M(d)}$ shows that $\|T_p\|_{\mathrm{cb}} \geq \|A\|_{\mathrm{cb}}$. □

We recall that it was shown in [AAI+16] that for every bilinear form there exists a 1-query quantum algorithm that makes additive error at most $1 - 1/K_G$. It thus remains to show the lower bound.

Theorem 1.5. *The worst-case minimum error for one-query quantum algorithms satisfies*

$$\sup_p \mathcal{E}(p, 1) = 1 - \frac{1}{K_G},$$

where the supremum is taken over the set of bounded bilinear forms.

Proof: Theorem 3.1 shows the following:

$$\begin{aligned}\sup_{p\in\mathcal{BB}} \mathcal{E}(p, 1) &= \sup_{\|p\|_\infty\leq 1}\ \sup_{\|r\|_{\infty,*}\leq 1} \langle p, r\rangle - \|r\|_{\mathrm{cb},*} &(27)\\ &= \sup_{\|r\|_{\infty,*}\leq 1} \|r\|_{\infty,*} - \|r\|_{\mathrm{cb},*} &(28)\\ &= \sup_{\|r\|_{\infty,*}=1} 1 - \|r\|_{\mathrm{cb},*}.\end{aligned}$$

It thus remains to show that for bilinear forms $\|r\|_{\infty,*} \leq K_G \|r\|_{\mathrm{cb},*}$. We do so starting from Grothendieck's theorem for matrices. It states that for $A \in \mathbb{R}^{n\times n}$ we have $\|A\|_{\mathrm{cb}} \leq K_G\|A\|_\infty$. Each bilinear form $q : \{-1,1\}^n \times \{-1,1\}^n \to \mathbb{R}$ uniquely corresponds to a matrix $A \in \mathbb{R}^{n\times n}$ such that $q(x,y) = x^\mathsf{T} Ay$. Moreover, for such q and A one has $\|q\|_\infty = \|A\|_\infty$ (immediate) and in Corollary 5.2 we showed $\|q\|_{\mathrm{cb}} = \|A\|_{\mathrm{cb}}$, so $\|q\|_{\mathrm{cb}} \leq K_G\|q\|_\infty$. A duality argument then concludes the proof:

$$\|r\|_{\infty,*} = \sup_{\|q\|_\infty \leq 1} \langle r, q\rangle \leq \sup_{\|q\|_{\mathrm{cb}} \leq K_G} \langle r, q\rangle = K_G\|r\|_{\mathrm{cb},*}.$$

□

Remark 5.3. If in Theorem 1.5 we restrict the supremum to bilinear forms on $n+n$ variables, for a fixed n, then we obtain a characterization of $K_G(n)$ instead of K_G. Here, $K_G(n) = \sup \|A\|_{\mathrm{cb}}/\|A\|_\infty$, where the supremum is taken over all non-zero $n \times n$ real matrices.

5.2 No converse for the polynomial method

In this section we show that there is no additive nor multiplicative converse for polynomials of degree 4 and 2-query algorithms. In other words, we will prove Theorems 1.3 and 1.4. Before doing that, we explain what was the error in the proof of Theorem 1.3 given in [ABP19].

Their proof arrives at the equation

$$\sum_{\alpha,\beta\in\{0,1,2,3,4\}^n:|\alpha|+|\beta|=4} d'_{\alpha,\beta} x^\alpha = C \sum_{\alpha\in\{0,1\}^n:|\alpha|=4} d_\alpha x^\alpha \quad \forall\, x \in \{-1,1\}^n, \tag{29}$$

where $d'_{\alpha,\beta}$, d_α and C are some real numbers, x^α stands for $\prod_{i=1}^n x_i^{\alpha_i}$ and $|\alpha|$ for $\sum_{i=1}^n \alpha_i$. It follows from the orthogonality of the characters that $d'_{\alpha,0} = Cd_\alpha$ for all $\alpha \in \{0,1\}^n$ such that $|\alpha| = 4$. What is used, however, is that $d'_{\alpha,0} = Cd_\alpha$ for all $\alpha \in \{0,1,2,3,4\}^n$ such that $|\alpha| = 4$, which is not true in general. For instance if $n = 1$, $C = 1$ and $d'_{(2,0),(0,2)} = -d'_{(0,0),(4,0)} = 1$ and the rest of the coefficients set to 0, then (29) becomes $x^2 - 1 = 0,\ \forall\, x \in \{-1,1\}$.

We now prove that there is no additive converse, from which the non-multiplicative converse result quickly follows.

Theorem 1.4. *There is no constant $\varepsilon \in (0,1)$ such that for every bounded polynomial p of degree at most 4, we have $\mathcal{E}(p,2) \leq \varepsilon$.*

Proof: For any partition $\mathcal{P}$ of $\{0\} \cup [3n]$ in $2t$ subsets, Theorem 3.1 shows that

$$\begin{aligned}
\sup_{p\in V_\mathcal{P},\|p\|_\infty\leq 1} \mathcal{E}(p,t) &= \sup_{p\in V_\mathcal{P},\|p\|_\infty\leq 1}\ \sup_{r\in V_\mathcal{P},\|r\|_{\infty,*}\leq 1} \langle p, r\rangle - \|r\|_{\mathrm{cb},*} \\
&= \sup_{r\in V_\mathcal{P},\|r\|_{\infty,*}\leq 1} \|r\|_{\infty,*} - \|r\|_{\mathrm{cb},*} \\
&= \sup_{r\in V_\mathcal{P},\|r\|_{\infty,*}= 1} 1 - \|r\|_{\mathrm{cb},*}.
\end{aligned}$$

Consider now the case $t = 2$ and the partition $\mathcal{P}_n = \{\{0\}, \{1,\dots,n\}, \{n+1,\dots,2n\}, \{2n+1,\dots,3n\}\}$ of $\{0\} \cup [3n]$. In Theorem 4.1 a sequence of forms $p_n \in V_{\mathcal{P}_n}$ was constructed with the property that

$$\frac{\|p_n\|_{\mathrm{cb}}}{\|p_n\|_\infty} \to \infty. \tag{30}$$

Hence, by a duality argument we get that there is a sequence $r_n \in V_{\mathcal{P}_n}$ such that $\|r_n\|_{\mathrm{cb},*}/\|r_n\|_{\infty,*} \to 0$. Indeed, suppose towards a contradiction that there is a $K > 0$ such that for every $n \in \mathbb{N}$ and every $r \in V_{\mathcal{P}_n}$ we have that $\|r\|_{\mathrm{cb},*} \geq K\|r\|_{\infty,*}$. Then,

$$\|p\|_{\mathrm{cb}} = \sup_{\|r\|_{\mathrm{cb},*}\leq 1} \langle r, p\rangle \leq \frac{1}{K} \sup_{\|r\|_{\infty,*}\leq 1} \langle r, p\rangle = \frac{1}{K}\|p\|_{\infty},$$

which contradicts Eq. (30). The sequence r_n shows that

$$\sup_{p\in V_{\mathcal{P}_n}, \|p\|_\infty \leq 1, n\in\mathbb{N}} \mathcal{E}(p, 2) = 1,$$

which implies the stated result. □

Theorem 1.3. *For any $C > 0$, there exist an $n \in \mathbb{N}$ and a bounded quartic n-variable polynomial p such that no two-query quantum algorithm $\mathcal{A}$ satisfies $\mathbb{E}[\mathcal{A}(x)] = Cp(x)$ for every $x \in \{-1, 1\}^n$.*

Proof: First note that we can assume $C \leq 1$, because $|\mathbb{E}[\mathcal{A}(x)]| \leq 1$ for any algorithm $\mathcal{A}$ and any $x \in \{-1, 1\}^n$. Assume that there exists $0 < C \leq 1$ such that for every bounded p of degree 4 there is a 2-query algorithm $\mathcal{A}$ with $\mathbb{E}[\mathcal{A}(x)] = p(x)$ for every $x \in \{-1, 1\}^n$. We claim that that $\mathcal{A}$ approximates p up to an additive error $1 - 1/C$, which contradicts Theorem 1.4. Indeed,

$$|p(x) - \mathbb{E}[\mathcal{A}(x)]| = |p(x)(1 - C)| \leq 1 - C.$$

□

6 An open question

Let $\mathcal{P}$ be a partition of $[n]$ in $2t$ subsets and let $p \in V_{\mathcal{P}}$ with $\|p\|_\infty \leq 1$. From the characterization of quantum t-query algorithms of [ABP19] we know that there is a quantum query algorithm $\mathcal{A}$ that outputs $p/\|p\|_{\mathrm{cb}}$ on expectation. In particular,

$$|\mathbb{E}[\mathcal{A}(x)] - p(x)| = \left|\frac{p(x)}{\|p\|_{\mathrm{cb}}} - p(x)\right| \leq \|p\|_\infty\Big(1 - \frac{1}{\|p\|_{\mathrm{cb}}}\Big).$$

As a consequence, one has that

$$\mathcal{E}(p, t) \leq \|p\|_\infty\Big(1 - \frac{1}{\|p\|_{\mathrm{cb}}}\Big). \tag{31}$$

Our Theorem 3.1 implies that both sides of Equation (31) are equal when you take the supremum over all $p \in V_{\mathcal{P}}$. We wonder if that is true for every $p \in V_{\mathcal{P}}$.

Question 6.1. *Let $\mathcal{P}$ be a partition of $[n]$ in $2t$ subsets and let $p \in V_{\mathcal{P}}$. Is it true that*

$$\mathcal{E}(p, t) = \|p\|_\infty\Big(1 - \frac{1}{\|p\|_{\mathrm{cb}}}\Big)?$$

A positive answer to this question would strengthen our main technical contribution, Theorem 3.1. In addition, if we focus on the case $t = 1$, it would imply that the method proposed in [AAI+16] to give an algorithm that computes $p/\|p\|_{\mathrm{cb}}$ with 1 query, provides the best 1 query approximation for every p (here the best means the one that minimizes $\mathcal{E}(p, 1)$). For $t \geq 1$ it was showed in [ABP19] that $p/\|p\|_{\mathrm{cb}}$ is the output of a t query algorithm, which would also be the best possible t query approximation for p.

Finally, for the case $t = 1$, it would imply a clean link between the biases of two player XOR games and quantum query algorithms. Indeed, given a matrix $A \in \mathbb{R}^{n\times n}$ it both defines a bounded bilinear form $p_A(x, y) = x^{\mathsf{T}} Ay$ and a two player XOR game G_A, where the referee asks the pair of questions (i, j) with probability

$$\pi(i, j) = \frac{|A_{i,j}|}{\sum_{i,j\in[n]} |A_{i,j}|}$$

and the payoff is given by

$$\mu(i, j, a, b) = \frac{1 + ab \cdot \mathrm{sgn}[A_{i,j}]}{2}.$$

Corollary 5.2 states that

$$\|p_A\|_\infty = \|A\|_\infty \text{ and } \|p_A\|_{\mathrm{cb}} = \|A\|_{\mathrm{cb}},$$

while Tsirelson's work [Tsi80] implies that the classical and quantum biases of G_A are

$$\beta(G_A) = \|A\|_\infty \text{ and } \beta^*(G_A) = \|A\|_{\mathrm{cb}}.$$

Thus, a positive answer to Question 6.1 would imply that

$$\mathcal{E}(p_A, 1) = \beta(G_A)\Big(1 - \frac{1}{\beta^*(G_A)}\Big).$$

7 An approach via duality

In this section we give an alternative proof of our main result from the dual picture. We stress that our main result gives a formula for $\mathcal{E}(p, t)$ in terms of certain dual norms, but it is not clear yet how to compute them. Along the way to the alternative proof of our main result, we will give formulations of the dual norms as efficiently solvable convex optimization problems. In particular, we will show that $\|p\|_{\mathrm{cb},*}$ can be computed using a semidefinite program and $\|p\|_{\infty,*}$ via a linear program. The semidefinite programming formulation of $\|p\|_{\mathrm{cb},*}$ essentially follows from a semidefinite programming formulation for the completely bounded norm of a tensor that was provided in [GL19]. However, proving this, and using it to provide an alternative proof of Theorem 2.1, requires us to further develop some of the theory of completely bounded norms of polynomials.

7.1 How to compute the dual norms

First, we give a convenient formula for the dual of the completely bounded norm of a tensor with respect to the inner product given by

$$\langle T, R\rangle = \sum_{\mathbf{i}\in[n]^t} T_{\mathbf{i}} R_{\mathbf{i}}.$$

To state it in an compact way, we introduce the following subset of t-tensors in $\mathbb{R}^{n\times\cdots\times n}$:

$$K(n,t) := \{\langle u, A(\mathbf{i})v\rangle_{\mathbf{i}\in[n]^t} \mid d\in\mathbb{N},\ u,v\in S^{d-1},\ A\colon [n]\to B_{M(d)}\}. \tag{32}$$

Proposition 7.1. *Let $R\in\mathbb{R}^{n\times\cdots\times n}$ be a t-tensor. Then,*

$$\|R\|_{\mathrm{cb},*} = \inf\{w>0 \mid R\in wK(n,t)\}. \tag{33}$$

Remark 7.2. Note that Proposition 7.1 states that $\|R\|_{\mathrm{cb},*}$ is the Minkowski norm defined by $K(n,t)$, so $K(n,t)$ is the unit ball of $(\mathbb{R}^{n\times\cdots\times n}, \|\cdot\|_{\mathrm{cb},*})$.

Remark 7.3. Proposition 7.1 allows us to compute $\|R\|_{\mathrm{cb},*}$ as a SDP. Indeed, it follows from Eq. (33) that

$$\begin{aligned}\|R\|_{\mathrm{cb},*} = \quad & \inf && w\\ & \text{s.t.} && w\in\mathbb{R}_{>0},\, d\in\mathbb{N},\, u,v\in wS^{d-1},\ A\colon[n]\to B_{M(d)},\\ & && R=\langle u, A(\mathbf{i})v\rangle_{\mathbf{i}\in[n]^t},\end{aligned}$$

and in [GL19] it was shown (implicitly) that the constraints can be written as linear equations on the entries of a positive semidefinite matrix.

Proof of Proposition 7.1: Let $|||R|||$ be the expression in the right-hand side of Eq. (33). First, we show that $|||\cdot|||$ is a norm. We should check that $|||R|||$ is well-defined, i.e., that every tensor can be decomposed as $\langle u, A(\mathbf{i})v\rangle$. We observe that the standard basis elements for the space of t-tensors are contained in $K(n,t)$. Indeed, let $u := e_1$, $v := e_{t+1}$ and $A(i) = \sum_{s\in\mathbf{i}^{-1}(i)} e_s e_{s+1}^{\mathsf{T}}$ then

$$\langle u, A(\mathbf{j})v\rangle = \begin{cases}1 & \text{if } \mathbf{j}=\mathbf{i},\\ 0 & \text{otherwise.}\end{cases}$$

To conclude that $|||R|||$ is well-defined it then suffices to observe that the set of scalar multiples of elements in $K(n,t)$ is closed under addition. Indeed, if

$$R_{\mathbf{i}} = \langle u, A(\mathbf{i})v\rangle \text{ and } \tilde{R}_{\mathbf{i}} = \langle \tilde{u}, \tilde{A}(\mathbf{i})\tilde{v}\rangle,$$

for some $u,v,\tilde{u},\tilde{v}\in\mathbb{R}^d$ with $\|u\|^2=\|v\|^2=w$, $\|\tilde{u}\|^2=\|\tilde{v}\|^2=\tilde{w}$ and maps $A,\tilde{A}\colon[n]\to B_{M(d)}$, then

$$R_{\mathbf{i}} + \tilde{R}_{\mathbf{i}} = \langle \hat{u}, \hat{A}(\mathbf{i})\hat{v}\rangle,$$

where $\hat{u},\hat{v}\in\mathbb{R}^{2d}$ are the vectors with $\|\hat{u}\|^2=\|\hat{v}\|^2=w+\tilde{w}$ defined by $\hat{u}:=u\oplus\tilde{u}$, $v:=v\oplus\tilde{v}$ and the map $\hat{A}\colon[n]\to B_{M(2d)}$ is defined via

$$\hat{A}(i) = \begin{pmatrix} A(i) & 0\\ 0 & \tilde{A}(i)\end{pmatrix}.$$

This construction also shows that $|||\cdot|||$ satisfies the triangle inequality. It is also clear that $|||\cdot|||$ is homogeneous and that $|||R|||=0$ if and only if $R=0$, so $|||\cdot|||$ is a norm.

Finally, note that the completely bounded norm of a t-tensor $R\in\mathbb{R}^{n\times\cdots\times n}$ is given by

$$\|T\|_{\mathrm{cb}} = \sup\Big\{\big|\sum_{\mathbf{i}\in[n]^t} T_{\mathbf{i}}\langle u, A(\mathbf{i})v\rangle\big| \mid d\in\mathbb{N},\ u,v\in S^{d-1},\ A\colon[n]\to B_{M(d)}\Big\},$$

so $\|T\|_{\text{cb}} = |||T|||_*$, and by the fact that the dual of the dual norm is the primal norm for finite-dimensional normed spaces, we conclude that

$$|||\cdot||| = |||\cdot|||_{**} = \|\cdot\|_{\text{cb},*}.$$

□

Proposition 7.4. *Let $\mathcal{P}$ be a partition of $[n]$ in t subsets and $p \in V_{\mathcal{P}}$. Then,*

$$\|p\|_{\infty,*} = \inf\{\|r\|_1 \;|\; r : \{-1,1\}^n \to \mathbb{R},\, r \in W_{\mathcal{P}},\, r_{=t} = p\}. \tag{34}$$

Remark 7.5. Eq. (34) can be phrased as a linear program, so it provides an efficient way of computing $\|p\|_{\infty,*}$.

Proof: We prove the statement in a few steps:

$$\begin{aligned}\|p\|_{\infty,*} &= \sup\left\{\langle p,q\rangle \;|\; q \in W_{\mathcal{P}},\, \|q\|_\infty \leq 1\right\}\\ &= \sup\left\{\langle p,q\rangle \;|\; q \in \mathbb{R}[x_1,\dots,x_n]_{=t},\, \|q\|_\infty \leq 1\right\}\\ &= \inf\left\{\|r\|_1 \;|\; r : \{-1,1\}^n \to \mathbb{R},\, r_{=t} = p\right\}\\ &= \inf\left\{\|r\|_1 \;|\; r : \{-1,1\}^n \to \mathbb{R},\, r \in W_p,\, r_{=t} = p\right\},\end{aligned}$$

where the first equality is the definition, in the second equality we have used Lemma 3.5 with $\|\cdot\|_\infty$ to remove the condition $q \in W_{\mathcal{P}}$, the third equality follows from Lagrange duality (cf. [BV04, Sec. 5.1.6]), and the fourth from Lemma 3.5 for $\|\cdot\|_1$. □

We have already seen in Example 2.5 that $\|p\|_{\infty,*} \neq \|p\|_1$ in general, because we are taking the dual norm with respect to $V_{\mathcal{P}}$. For completeness we give an alternative proof of the separation using Proposition 7.4.

Example 7.6. The upper bound $\|p\|_{\infty,*} \leq 1/3$ of Example 2.5 follows from Proposition 7.4 by considering the multilinear map $r(x) = (x_1 + x_2 + x_3 + x_1x_2x_3)/3$ that belongs to $W_{\mathcal{P}}$, satisfies $r_{=1}(x) = p(x) = (x_1 + x_2 + x_3)/3$ and $\|r\|_1 = 1/3$.

Proposition 7.7. *Let $\mathcal{P}$ be a partition of $[n]$ in t subsets and let $p \in V_{\mathcal{P}}$. Then,*

$$\|p\|_{\text{cb},*} = t!\|T_p\|_{\text{cb},*}. \tag{35}$$

Proof: By duality and definition of $\|\cdot\|_{\text{cb}}$, we have that

$$\begin{aligned}\|p\|_{\text{cb},*} &= \sup\{\sum_{\alpha\in\mathbb{Z}^n_{\geq 0}} c_\alpha c'_\alpha : q \in V_{\mathcal{P}},\, \|q\|_{\text{cb}} \leq 1\}\\ &= \sup\{\sum_{\alpha\in\mathbb{Z}^n_{\geq 0}} c_\alpha \sum_{\mathbf{i}\in\mathcal{I}_\alpha} T_{\mathbf{i}} : q \in V_{\mathcal{P}},\, T(x) = q(x),\, \|T\|_{\text{cb}} \leq 1\},\end{aligned}$$

where c_α and c'_α are the coefficients of p and q, respectively. Now, let $R \in \mathbb{R}^{n\times\cdots\times n}$ be a t-tensor such that $R(x) = p(x)$ for every $x \in \mathbb{R}^n$. Then we have, using Eq. (8), that

$$\|p\|_{\text{cb},*} = \sup\{\sum_{\alpha\in\mathbb{Z}^n_{\geq 0}} \sum_{\mathbf{j}\in\mathcal{I}_\alpha} R_{\mathbf{j}} \sum_{\mathbf{i}\in\mathcal{I}_\alpha} T_{\mathbf{i}} : q \in V_{\mathcal{P}},\, T(x) = q(x),\, \|T\|_{\text{cb}} \leq 1\}.$$

In particular, if we choose R to be T_p, then we have

$$\begin{aligned}\|p\|_{\mathrm{cb},*} &= t! \sup\{\sum_{\alpha\in\mathbb{Z}^n_{\geq 0}}\sum_{\mathbf{i}\in\mathcal{I}_\alpha}(T_p)_{\mathbf{i}}T_{\mathbf{i}} : q\in V_{\mathcal{P}},\, T(x)=q(x),\, \|T\|_{\mathrm{cb}}\leq 1\}\\ &= t!\sup\{\langle T_p, T\rangle : q\in V_{\mathcal{P}},\, T(x)=q(x),\, \|T\|_{\mathrm{cb}}\leq 1\}.\end{aligned}$$

We now show that the expression on the right equals $t!$ times $\|T_p\|_{\mathrm{cb},*}$, which we recall can be written as

$$\|T_p\|_{\mathrm{cb},*} = \sup\{\langle T_p, T\rangle : \|T\|_{\mathrm{cb}}\leq 1\}. \tag{36}$$

By inclusion of the feasible region we have that $\|p\|_{\mathrm{cb},*}\leq t!\|T_p\|_{\mathrm{cb},*}$. For the other inequality, let $T\in\mathbb{R}^{n\times\cdots\times n}$ be a t-tensor and consider $\Pi_{\mathcal{P}}T$ as in Eq. (13). By Proposition 3.4 we have $\|\Pi_{\mathcal{P}}T\|_{\mathrm{cb}}\leq\|T\|_{\mathrm{cb}}\leq 1$. Also note that the polynomial $\Pi_{\mathcal{P}}T(x)$ belongs to $V_{\mathcal{P}}$, because $(\Pi_{\mathcal{P}}T)_{\mathbf{i}}=0$ unless $\mathbf{i}$ contains exactly one index from each set in the partition $\mathcal{P}$. It remains to observe that $(\Pi_{\mathcal{P}}T)_{\mathbf{i}}=T_{\mathbf{i}}$ for all indices $\mathbf{i}\in[n]^t$ for which $(T_p)_{\mathbf{i}}\neq 0$ and therefore

$$\langle T_p, T\rangle = \langle T_p, \Pi_{\mathcal{P}}T\rangle.$$

This shows that $\|p\|_{\mathrm{cb},*}\geq t!\|T_p\|_{\mathrm{cb},*}$. □

7.2 Alternative proof of the main result via semidefinite programming

First of all, we will state Theorem 7.8 (which corresponds, after some reformulation, to equation (20) of [GL19]), that gives an optimization problem equivalent to the dual of the SDP $\mathcal{E}(p,t)$. Before that, we introduce the following notation. Given $\mathbf{i}\in[n+1]^{2t}$, $\alpha(\mathbf{i})\in\{0,1\}^n$ is defined as

$$(\alpha(\mathbf{i}))_m := \begin{cases} 1 & \text{if } m\in[n] \text{ and } m \text{ occurs an odd number of times in } \mathbf{i},\\ 0 & \text{otherwise.}\end{cases}$$

Theorem 7.8 ([GL19])**.** *Let $p:\{-1,1\}^n\to\mathbb{R}$ and $t\in\mathbb{N}$. Then,*

$$\begin{aligned}\mathcal{E}(p,t) = \sup\ & (\langle p,r\rangle - w)/\|r\|_1 \\ \text{s.t. } & r:\{-1,1\}^n\to\mathbb{R},\ d\in\mathbb{R}\\ & A_s\colon[n+1]\to B_{M(d)} \text{ for all } s\in[2t]\\ & u,v\in\mathbb{R}^d,\ w=\|u\|^2=\|v\|^2\\ & c_{\alpha(\mathbf{i})} = \langle u, A_1(i_1)\dots A_{2t}(i_{2t})v\rangle \text{ for all } \mathbf{i}\in[n+1]^{2t},\end{aligned} \tag{37}$$

where c_α are the coefficients of r.[5]

Second, we show that in the case of p belonging to $W_{\mathcal{P}}$, we can restrict r to belong to $W_{\mathcal{P}}$. As before, we also show that a single contraction-valued map A suffices.[6]

[5]Following [GL19] one obtains Theorem 7.8, but with the A_s being unitary-valued maps. Every contraction-valued map can be turned into an equivalent unitary-valued map by block-encoding contractions into the top-left corner of unitaries.

[6]In fact, all of the contraction-valued maps A_s can be taken to be the same regardless of whether p belongs to $W_{\mathcal{P}}$ or not, similarly to what is done in Proposition 3.3.

Lemma 7.9. *Let $\mathcal{P}$ be a partition of $[n]$ in $2t$ subsets and let $p \in W_{\mathcal{P}}$. Then,*

$$\begin{aligned}\mathcal{E}(p,t) = \sup\ & (\langle p, r\rangle - w)/\|r\|_1 \\ \text{s.t. } & r : \{-1,1\}^n \to \mathbb{R},\ r \in W_{\mathcal{P}},\ d \in \mathbb{R} \\ & A\colon [n+1] \to B_{M(d)}, \\ & u, v \in \mathbb{R}^d,\ w = \|u\|^2 = \|v\|^2 \\ & c_{\alpha(\mathbf{i})} = \langle u, A(\mathbf{i})v\rangle \text{ for all } \mathbf{i} \in [n+1]^{2t}\ ,\end{aligned} \tag{38}$$

where c_α are the coefficients of r.

Proof: Let $\mathcal{E}^*(p,t)$ be the expression in the right-hand side of Eq. (38). By inclusion of the feasible region, $\mathcal{E}(p,t) \geq \mathcal{E}^*(p,t)$. To prove the other inequality, consider a feasible instance (u, v, w, A_s, r) for the SDP (37). Consider the contraction-valued map $\tilde{A}(i) := \sum_{s\in[2t]} e_s e_{s+1}^{\mathsf{T}} \otimes \hat{A}_s(i)$ for $i \in [n]$, where $\hat{A}_s(i)$ is a $d2^{2t} \times d2^{2t}$ matrix defined by

$$\hat{A}_s(i) = \bigoplus_{z\in\{-1,1\}^{2t}} (A_s \cdot z)(i).$$

We also define $\tilde{A}(n+1) := 0$ and the vectors $\tilde{u} = e_1 \otimes \hat{u}$ and $\tilde{v} = e_{2t+1} \otimes \hat{v}$, where $\hat{u}$ is the $2^t d$-dimensional vector defined as the (normalized) direct sum of 2^t copies of u, i.e.,

$$\hat{u} = \frac{1}{\sqrt{2^{2t}}} \bigoplus_{z\in\{-1,1\}^{2t}} u,$$

and the same for $\hat{v}$, but with an appropriate sign in each of the copies

$$\hat{v} = \frac{1}{\sqrt{2^{2t}}} \bigoplus_{z\in\{-1,1\}^{2t}} v \prod_{I\in\mathcal{P}} z_I.$$

This way, $(\tilde{u}, \tilde{v}, w, \tilde{A}, \Pi_{\mathcal{P}} r)$ is a feasible instance of $\mathcal{E}^*(p,t)$. Indeed, if $\mathbf{i}$ takes the value $n+1$ at least once or has any repeated indices, then $\alpha(\mathbf{i})_1 + \cdots + \alpha(\mathbf{i})_n < 2t$, so $(\Pi_{\mathcal{P}} r)_{\alpha(\mathbf{i})} = 0$ because $\Pi_{\mathcal{P}} r \in W_{\mathcal{P}}$, and $\langle \tilde{u}, \tilde{A}(\mathbf{i})\tilde{v}\rangle = 0$ by construction. If $\mathbf{i} \in [n]^{2t}$ and has no repeated indices, we get that

$$\langle \tilde{u}, \tilde{A}(\mathbf{i})\tilde{v}\rangle = \frac{1}{2^{2t}} \sum_{z\in\{-1,1\}^{2t}} \langle u, A_1(i_1)\ldots A_{2t}(i_{2t})v\rangle \prod_{I\in\mathcal{P}} z_I^{1+\sum_{j\in I}\alpha(\mathbf{i})_j}.$$

Now, reasoning as in Proposition 3.4, we get that

$$\langle \tilde{u}, \tilde{A}(\mathbf{i})\tilde{v}\rangle = \begin{cases} \langle u, A_1(i_1)\ldots A_{2t}(i_{2t})v\rangle & \text{if } \mathbf{i} \text{ takes one value in each } I \in \mathcal{P}, \\ 0 & \text{otherwise,}\end{cases}$$

so putting everything together we get that

$$\tilde{c}_{\alpha(\mathbf{i})} = \langle \tilde{u}, \tilde{A}(\mathbf{i})\tilde{v}\rangle,$$

where $\tilde{c}_\alpha$ are the coefficients of $\Pi_{\mathcal{P}} r$. Since $\tilde{A}$ is contraction-valued and both

$$\|\tilde{u}\|^2 = \frac{1}{2^{2t}} \sum_{z\in\{-1,1\}^{2t}} \|u\|^2 = \|u\|^2 = w$$

and $\|\tilde{v}\|^2 = w$, we conclude that $(\tilde{u}, \tilde{v}, w, \tilde{A}, \Pi_{\mathcal{P}} r)$ is a feasible instance of Eq. (38).

Finally, the value of $(\tilde{u}, \tilde{v}, w, \tilde{A}, \Pi_{\mathcal{P}} r)$ is at least as large as the one of (u, v, w, A, r):

$$\frac{\langle p, \Pi_{\mathcal{P}} r\rangle - w}{\|\Pi_{\mathcal{P}} r\|_1} = \frac{\langle p, r\rangle - w}{\|\Pi_{\mathcal{P}} r\|_1} \geq \frac{\langle p, r\rangle - w}{\|r\|_1},$$

where in the equality we have used that that p belongs to $W_{\mathcal{P}}$ and in the inequality we have used Lemma 3.5. □

Now, we are ready to prove Theorem 3.1, again.

Proof of Theorem 3.1: First note that given a feasible instance (u, v, w, A, r) for Eq. (38) we clearly have that $r_{=2t} \in V_{\mathcal{P}}$. We show that also $w \geq \|r_{=2t}\|_{\mathrm{cb},*}$. By Propositions 7.1 and 7.7, this requires us to show that $t!$ times the unique symmetric $2t$-tensor $T_{r_{=2t}}$ associated to $r_{=2t}$ belongs to $wK(n,t)$. To do so, we show that $t!T_{r_{=2t}} = (\langle u, A(\mathbf{i})v\rangle)_{\mathbf{i}\in[n]^{2t}}$. Let $\mathbf{i} \in [n]^{2t}$. If $\mathbf{i}$ has repeated indices then $(T_{r_{=2t}})_{\mathbf{i}} = 0$ because $r_{=2t}$ is multilinear, and also $\langle u, A(\mathbf{i})v\rangle = 0$, because r is a feasible instance of Eq. (38). If $\mathbf{i}$ does not have repeated indices, then $t!(T_{r_{=2t}})_{\mathbf{i}} = c_{\alpha(\mathbf{i})}$, and also $\langle u, A(\mathbf{i})v\rangle = c_{\alpha(\mathbf{i})}$ because r is a feasible solution for Eq. (38).

On the other hand, given $r \in W_{\mathcal{P}}$ there is an instance (u, v, w, A, r) with $w = \|r_{=2t}\|_{\mathrm{cb},*}$. Indeed, by Proposition 7.7 there is a map $A\colon [n] \to B_{M(d)}$ and vectors u, v whose norm squared is $\|r_{=2t}\|_{\mathrm{cb},*}$ such that $c_\alpha(r_{=2t}) = \langle u, A(\mathbf{i})v\rangle$ for every $\mathbf{i} \in \mathcal{I}_\alpha$ and every $\alpha \in \mathbb{Z}^n_{\geq 0}$. Note that in order to have a feasible instance for Eq. (38) we need to satisfy its last condition, and with these contractions we can only satisfy it for α such that $\alpha_1 + \cdots + \alpha_n = 2t$. To satisfy it for every α with $\alpha_1 + \cdots + \alpha_n \leq 2t$, we just have to change the contractions and the vectors by $\hat{A}(i) := \sum_{s\in[2t]} e_s e_{s+1}^{\mathsf{T}} \otimes A_s(i)$ and $\hat{u} := e_1 \otimes u$ and $\hat{v} := e_{t+1} \otimes v$ and define the extra contraction as $\hat{A}(n+1) = 0$. This, way $(\hat{u}, \hat{v}, \|r_{=2t}\|_{\mathrm{cb},*}, \hat{A}, r)$ is a feasible instance. To sum up, so far we have proved that

$$\begin{aligned} \mathcal{E}(p,t) = \sup\ & (\langle p, r\rangle - \|r_{=2t}\|_{\mathrm{cb},*})/\|r\|_1 \\ \text{s.t. } & r : \{-1,1\}^n \to \mathbb{R},\ r \in W_{\mathcal{P}}. \end{aligned}$$

We finally reformulate the above in terms of $r_{=2t}$ using the following two observations. Since $p \in V_{\mathcal{P}}$ we have $\langle p, r\rangle = \langle p, r_{=2t}\rangle$. Moreover, by Proposition 7.4, we have $\|r_{=2t}\|_{\infty,*} = \inf\{\|\tilde{r}\|_1 \mid \tilde{r} : \{-1,1\}^n \to \mathbb{R}, \tilde{r} \in W_{\mathcal{P}}, \tilde{r}_{=2t} = r_{=2t}\}$. Hence,

$$\begin{aligned} \mathcal{E}(p,t) = \sup\ & (\langle p, r_{=2t}\rangle - \|r_{=2t}\|_{\mathrm{cb},*})/\|r_{=2t}\|_{\infty,*} \\ \text{s.t. } & r_{=2t} \in V_{\mathcal{P}}, \end{aligned}$$

which concludes the proof. □

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