Committees and Equilibria: Multiwinner Approval Voting Through the Lens of Budgeting Games

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Approval-based multiwinner voting, one of the central topics in computational social choice, addresses collective decision-making scenarios in which n voters select a committee of k candidates from a larger pool of alternatives. A fundamental aim is to ensure that the elected committee proportionately represents the preferences of the electorate. Consequently, much effort has gone into exploring various proportionality notions and developing voting rules to achieve them. A key intuition underlying many fairness axioms and voting rules is that an optimal outcome is attained when no subset of voters can improve their position by reallocating their endorsements. In this paper, we formalize this intuition by defining a new class of games, which we call *budgeting games*, where committees occur as a result of voters' decisions about how to allocate a given budget. Our primary contribution lies in introducing this new class of normal-form games and showing that key notions in multiwinner voting theory, such as priceability, the core and EJR (Extended Justified Representation) can be thought of as equilibria of budgeting games. Remarkably, our budgeting games do not just capture existing concepts, but also give rise to entirely new families of voting rules. These rules, which are guaranteed to satisfy desirable fairness axioms, are based on improving-move dynamics in the respective budgeting games, and include the well-known Method of Equal Shares. Finally, we showcase the applicability of our game-theoretic perspective by proving existence of strong equilibria in a restricted version of our budgeting games, which implies that the core in a novel special case of multiwinner elections is non-empty.

CCS Concepts: • **Theory of computation** → **Algorithmic game theory and mechanism design**; • **Computing methodologies** → **Multi-agent systems**; • **Applied computing** → *Voting / election technologies.*

Additional Key Words and Phrases: Multi-Winner Voting, Social Choice Theory

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1 Introduction

Voting is one of the oldest and most studied forms of collective decision making. One specific form of voting that has received particular attention in (computational) social choice is *approval-based multiwinner voting*, where n voters whose preferences are expressed by granting, or withholding, approval on each candidate must elect a committee of k candidates out of a larger set of available alternatives [Lackner and Skowron, 2023]. The paradigmatic example for approval-based multi-winner elections is the selection of a committee, though its usefulness extends beyond the domain usually associates with elections [Gawron and Faliszewski, 2022, Skowron et al., 2016].

One of the central requirements in multiwinner voting is that the chosen committee represents the electorate's preferences in a proportional manner [Faliszewski et al., 2017]. As a result, much recent work has been devoted to developing appropriate definitions of proportionality [Aziz et al., 2017, Brill and Peters, 2023, Peters et al., 2021, Sánchez-Fernández et al., 2017], and to the task of finding proportional committees efficiently [Aziz et al., 2018, Brill et al., 2017, Peters and Skowron, 2020], with Lackner and Skowron [2023] offering a comprehensive overview of this line of research. Fine details notwithstanding, a common intuition lies behind many of the solutions put forward so far: a good outcome is to be found by distributing control over the k seats on the final committee among the voters, in such a way that no group of voters can do better by using their 'budget of control' differently. This idea underlies important axioms, e.g., EJR, priceability and the core, as well as prominent voting rules, such as the Method of Equal Shares (MES) where it shows up explicitly.

Thus framed, this way of looking at multiwinner voting calls for a game-theoretic interpretation, in which we would see desirable outcomes of multiwinner voting scenarios occurring as equilibria in some suitably defined game. Despite existing attempts [Aziz et al., 2017, Peters et al., 2021], a more systematic game-theoretic treatment of proportionality notions in multiwinner voting is still missing. In this paper, we aim to fill this gap by introducing *budgeting games*, a class of normal-form games in which agents decide how to allocate a given budget, in a process that results in 'activating' a committee of representatives. Remarkably, many solution concepts of budgeting games turn out to correspond to central fairness notions and rules from the multiwinner voting literature: the list includes EJR, the core, priceability, and the Method of Equal Shares.

Contributions. We define *budgeting games*, a new class of normal-form games that capture the dynamics of choosing representatives in a multiwinner voting scenario. We demonstrate the usefulness of this type of game in Section 4, by establishing equivalences between solution concepts for budgeting games and fair outcomes of multiwinner voting instances described by prominent proportionality axioms. The types of game-theoretic solution concepts we leverage are linked to the absence of group deviations from agents of varying degrees of rationality. For example, the EJR axiom corresponds to what we call the *cohesive α-core*: the rule of the game, here, is that groups of agents are allowed to reallocate their budget only to unanimously approved candidates, and without relying on contributions of agents outside the group; if no group of agents can find such an improvement, Theorem 4.7 ensures we are in an equilibrium that corresponds to an EJR committee. A snapshot of the equivalences we obtain is offered in Figure 1.

In Section 5 we are able to show that certain sequences of similarly natural deviations are guaranteed to produce an outcome in the cohesive α-core, which in turn implies that the cohesive α-core is always non-empty. The family of voting rules described by such sequences turns out to include both novel methods, not yet explored, as well as, in a pleasingly reassuring twist, the familiar Method of Equal Shares [Peters et al., 2021].

Finally, in Section 6 we show the existence of strong equilibria for tree-structured budgeting games. The existence of strong equilibria also implies that the α-core of these games is non-empty, which, in turn, means that the core of the corresponding multiwinner voting instances is non-empty.

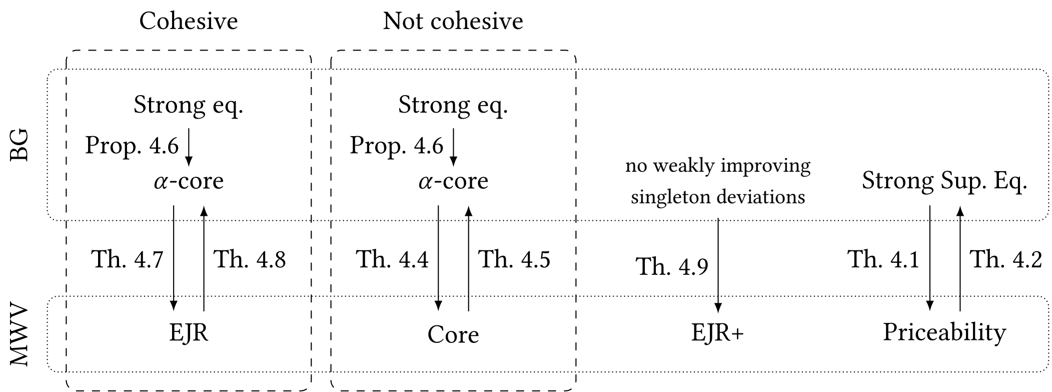

Fig. 1. Correspondences between solution concepts for BG and MWV.

Thus, this result shows the non-emptiness of the core for another restricted domain [Pierczyński and Skowron, 2022], providing a new perspective on the non-emptiness of the core, one of the major open problems in multiwinner voting.

Related work. There have been previous isolated attempts to rationalize proportionality axioms in multiwinner voting in terms of games [Aziz et al., 2017] or market equilibria [Peters et al., 2021]. However, these games have been defined in a piecemeal manner for a single axiom. The idea of characterizing proportional outcomes trough the equilibria of a game in which players coordinate their contribution to different projects has also successfully been used in divisible participatory budgeting [Fain et al., 2016] and donor coordination [Brandt et al., 2023]. However, in both settings, candidates can receive arbitrary amounts of funding and thus the resulting games look similar to our budgeting games, but are technically quite different as they have continuous utility functions, in contrast to the discrete utility functions we use. Finally, Kraiczy and Elkind [2023] use dynamics related to budgeting games to efficiently find proportional outcomes in discrete participatory budgeting, a generalization of multiwinner voting. However, these dynamics cannot be rationalized in game theoretic terms as best responses of (sets of) voters. Instead, they should be seen as local search dynamics with strong algorithmic properties.

2 Preliminaries: Multiwinner Voting

A *multiwinner voting (MWV)* instance is given by a tuple $I = \langle N, A, k, \boldsymbol{A}\rangle$, where $N = [n]$ is the *set of voters*, A is the *set of alternatives*, $k \in \mathbb{N}_{>0}$ specifies the *number of alternatives* from A to be selected, and $\boldsymbol{A} = (A_1, \dots, A_n)$ is an *approval profile*, i.e., a vector of *approval ballots* $A_i \subseteq A$ indicating the subset of alternatives agent $i \in N$ approves of.[1] The *outcome* of a MWV instance is a committee $\pi \in \mathcal{P}_k(A)$ of alternatives from A of size exactly k.[2] A set of less than k alternatives is a *partial committee*. Given a multiwinner voting instance, the goal, typically, is to find committees that satisfy desirable fairness requirements, chief among them being proportionality. We introduce below four of the most prominent proportionality notions, beginning with priceability.

A committee $\pi \in \mathcal{P}_k(A)$ is *priceable* for a MWV instance I if there exists an allowance $\alpha \in \mathbb{R}_{\geq 0}$ and a collection $(\gamma_i)_{i \in N}$ of contribution functions $\gamma_i : A \to \mathbb{R}_{\geq 0}$ such that:

(C_1) If $\gamma_i(j) > 0$, then $j \in A_i$ for all alternatives $j \in A$ and voters $i \in N$.

[1] For $n \in \mathbb{N}_{>0}$, $[n] = \{1, \dots, n\}$ is the set of the first n natural numbers (excluding 0).
[2] If A is a set, $\mathcal{P}_k(A)$ is the set of subsets of A of size exactly k.

(C_2) If $\gamma_i(j) > 0$, then $j \in \pi$ for all alternatives $j \in A$ and voters $i \in N$.
(C_3) $\sum_{j \in A} \gamma_i(j) \leq \alpha$ for all voters $i \in N$.
(C_4) $\sum_{i \in N} \gamma_i(j) = 1$ for all alternatives $j \in \pi$.
(C_5) $\sum_{i \in N : j \in A} \left(\alpha - \sum_{m \in A} \gamma_i(m)\right) \leq 1$ for all alternatives $j \in A \setminus \pi$.

Intuitively, a committee π is priceable if there exists a budget α for each voter, and a payment scheme under which all of the alternatives $j \in \pi$ receive at least a contribution of 1, and there is no alternative outside π whose supporters have an allowance left of strictly more than 1.

To define the core, we first say that, given a MWV instance I and a (partial) committee $\pi \in \mathcal{P}_k(A)$, a subset $S \subseteq N$ of voters with $|S| \geq \ell \cdot n/k$ *blocks* π if there exists a set $T \subseteq A$ of alternatives with $|T| = \ell$ such that $|A_i \cap T| > |A_i \cap \pi|$, for all $i \in S$. A (partial) committee π is in the *core* of a MWV instance I if it is not blocked. In other words, a committee π is in the core if there is no group of voters that is large enough to deserve ℓ seats and that would prefer to fill these ℓ seats by themselves. Intriguingly, it is not known, at the moment of writing, whether the core is always non-empty [Lackner and Skowron, 2023]. However, positive results can be obtained for restrictions of the core in which deviations are allowed only from voter coalitions that satisfy certain constraints.

One such constraint is cohesiveness. For $\ell \in \mathbb{N}_{>0}$, a subset $S \subseteq N$ of voters is *ℓ-cohesive* if (*i*) $|S| \geq \ell \cdot n/k$, and (*ii*) $|\bigcap_{i \in S} A_i| \geq \ell$. A (partial) committee $\pi \in \mathcal{P}_k(A)$ satisfies *extended justified representation (EJR)* if for all ℓ-cohesive groups S there is a voter $i \in S$ such that $|A_i \cap \pi| \geq \ell$. Intuitively, a subset S of voters is ℓ-cohesive if S is large enough to demand ℓ alternatives, and there are at least ℓ alternatives that all voters in S approve of. EJR has been one of the central concepts in the multiwinner voting literature, because (*i*) it can always be satisfied, and (*ii*) there exist polynomial time computable voting rules that satisfy EJR. Note that if a committee π is in the core, then π also satisfies EJR, but not the other way around [Aziz et al., 2017].

One concern regarding EJR is that larger cohesive groups do not regularly appear in real world examples [Brill and Peters, 2023]. To mitigate this concern, a new axiom has been proposed recently [Brill and Peters, 2023]. A (partial) committee $\pi \in \mathcal{P}_k(A)$ satisfies *extended justified representation plus (EJR+)* if there is no group of voters $S \subseteq N$, alternative $j \in A$ and $\ell \in \mathbb{N}_{>0}$ such that (*i*) S is large enough to deserve ℓ seats, i.e., $|S| \geq \ell \cdot n/k$, (*ii*) the voters in S jointly approve an alternative that is not in π, i.e., $(\bigcap_{i \in S} A_i) \setminus \pi \neq \emptyset$ and (*iii*) no voter in S approves of ℓ or more candidates in π, i.e., $|A_i \cap \pi| < \ell$ for all $i \in S$. In contrast to EJR, it is possible to verify whether a committee satisfies EJR+ in polynomial time and EJR+ is much more restrictive in real world instances. At the same time, the two most prominent voting rules satisfying EJR, namely Proportional Approval Voting (PAV) and the Method of Equal Shares (MES), also satisfy EJR+.

The aforementioned Method of Equal Shares (MES) is defined as follows. Given a MWV instance $I = \langle N, A, k, \boldsymbol{A} \rangle$, MES constructs a committee π iteratively, starting from the empty set. A load $\ell_i \colon 2^A \to \mathbb{R}^+$ is associated with every voter $i \in N$, and initialized as $\ell_i(\emptyset) = 0$ for all $i \in N$. Given π and a scalar $\beta \geq 0$, the contribution of voter $i \in N$ for candidate $j \in A \setminus \pi$ is defined by $\gamma_i(\pi, \beta, j) = A_i(j) \cdot \min(k/n - \ell_i(\pi), \beta)$. Given a budget allocation π, a candidate $j \in A \setminus \pi$ is said to be β-affordable, for $\beta \geq 0$, if $\sum_{i \in N} \gamma_i(\pi, \beta, j) \geq 1$. For a given round with current committee π, if no candidate is β-affordable for any β, then MES terminates. Otherwise, it selects a candidate $j \in A \setminus \pi$ that is $\beta^\star$-affordable where $\beta^\star$ is the smallest β such that one candidate is β-affordable (π is updated to $\pi \cup \{j\}$). Loads are then updated to $\ell_i(\pi \cup \{j\}) = \ell_i(\pi) + \gamma_i(\pi, \beta, j)$. A new round then starts.

Example 1. *Consider the MWV instance with voters $N = [6]$, alternatives $A = [5]$, a desired size $k = 3$ of the committee and approval sets $A_1 = \{3\}$, $A_2 = \{1, 2, 3\}$, $A_3 = \{1\}$, $A_4 = \{2\}$, $A_5 = \{1, 2, 4\}$ and $A_6 = \{4\}$ (see Figure 2). There are no 2- or 3-cohesive sets of voters, but plenty of 1-cohesive sets: $\{1, 2\}$, for instance, is a large enough coalition of agents to deserve $2 \cdot 3/6 = 1$ seats, and they agree on* *candidate* 3*. Committee* $\pi = \{3, 4, 5\}$ *manages to represent at least one voter from every* 1*-cohesive set with one candidate, and therefore satisfies EJR. It also satisfies EJR+, as both unelected candidates* 1 *and* 2 *have only three supporters and for every set of supporters* S *of size two that deserves one representative, there is at least one voter* $i \in S$ *who is already represented by one candidate. However, including candidates* 1 *and* 2 *would make voters* 2, 3, 4 *and* 5 *happier than with* π*, and these voters form a large enough group to control two seats. Thus,* π *is not in the core. Committee* π *is also not priceable: since resource* 5 *is not approved by any candidate,* 5 *is barred from receiving any contributions in a payment scheme that witnesses priceability. The different committee* $\pi' = \{1, 2, 3\}$ *is priceable: with an allowance of* 0.6*, voters* 1 *and* 2 *can buy resource* 3 *by paying* 0.6 *and* 0.4*, respectively,* $\{2, 3, 5\}$ *buy resource* 1 *by paying* 0.2*,* 0.6 *and* 0.2*, and* $\{4, 5\}$ *buy resource* 2*, contributing* 0.6 *and* 0.4*, respectively. Voter* 6 *has a budget of* 0.6 *left over but, since voter* 5 *has exhausted their allowance with different investments, resource* 4 *cannot be bought.*

3 Our New Class of Strategic Games: Budgeting Games

In this section we introduce *budgeting games*, the novel type of game central to the results of the paper. An instance of a *budgeting game (BG)* is given by a tuple $\Gamma = \langle N, M, b, \boldsymbol{X} \rangle$, where $N = [n]$ is a finite set of *players* (or *agents*) and $M = [m]$ is a finite set of *resources*. Each player $i \in N$ has a set $X_i \subseteq M$ of *relevant resources* and a *budget* $b \in \mathbb{R}_{>0}$ that i can allocate across the resources in M. We also say that the resources in $M \setminus X_i$ are *irrelevant* to i. We use $\boldsymbol{X} = (X_1, \ldots, X_n)$ to refer to the vector of relevant resource sets of the players. Note that each player i has the same budget b available; we also say that the budgets of the players are *uniform*.

A *strategy* of player $i \in N$ specifices how i allocates their budget b across the resources in M. Formally, we capture this through an *allocation function* $\lambda_i : M \longrightarrow \mathbb{R}_{\geq 0}$ which specifies for each resource $j \in M$ the amount $\lambda_i(j) \geq 0$ that i allocates on j such that the total amount player i allocates across all resources does not exceed the budget b, i.e., $\sum_{j \in M} \lambda_i(j) \leq b$, Note that i does not need to spend the entire budget and is free to contribute to resources that are irrelevant to them. Each strategy λ_i of player $i \in N$ is equivalently described by an m-dimensional vector $(\lambda_i(1), \ldots, \lambda_i(m)) \in \mathbb{R}^m_{\geq 0}$. We define the set of all strategies Λ_i of player i as

$$\Lambda_i = \left\{ \big(\lambda_i(j)\big)_{j \in M} \;\middle|\; \lambda_i(j) \geq 0 \;\; \forall j \in M, \; \sum_{j \in M} \lambda_i(j) \leq b \right\} \subset \mathbb{R}^m_{\geq 0}. \tag{1}$$

Note that the strategy set Λ_i is infinite but compact. Thus, budgeting games are infinite normal-form games with a finite set of players and each player having an infinite set of strategies.

A *strategy profile* $\boldsymbol{\lambda} = (\lambda_i)_{i \in N}$ is a vector of strategies of the players. Given an instance Γ of a BG, we define $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n$ as the set of all strategy profiles. Given a strategy profile $\boldsymbol{\lambda} \in \Lambda$, we say that a resource $j \in M$ is *active* if $\sum_{i \in N} \lambda_i(j) \geq 1$ and *inactive* otherwise. Given a strategy profile $\boldsymbol{\lambda}$, we define the *outcome* $out(\boldsymbol{\lambda})$ of $\boldsymbol{\lambda}$ as the set of all active resources in M, i.e., $out(\boldsymbol{\lambda}) = \{j \in M \mid \sum_{i \in N} \lambda_i(j) \geq 1\}$. The *utility* of player $i \in N$ is the number of active resources relevant to i, i.e., $u_i(\boldsymbol{\lambda}) = |X_i \cap out(\boldsymbol{\lambda})|$. The goal of each player $i \in N$ is to maximize their utility.

Given a strategy profile $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n)$ and a set of players $S \subseteq N$ (also referred to as a *coalition*), $\boldsymbol{\lambda}' = (\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}_{-S})$ is the strategy profile obtained from $\boldsymbol{\lambda}$ if each player $i \in S$ deviates to strategy $\lambda'_i \in \Lambda_i$, while the strategy of each player $i \in N \setminus S$ remains the same, i.e., $\lambda'_i = \lambda_i$. We call such a deviation *unilateral* if $|S| = 1$ (i.e., only a single player deviates) and *coalitional* otherwise. For a budgeting game Γ, a strategy profile $\boldsymbol{\lambda}$ is a k*-strong equilibrium* if for every coalition $S \subseteq N$ with $|S| \leq k$, there is no strategy profile $\boldsymbol{\lambda}' = (\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}_{-S})$ such that $u_i(\boldsymbol{\lambda}') > u_i(\boldsymbol{\lambda})$ for every player $i \in S$. Profile $\boldsymbol{\lambda}$ is called a *strong equilibrium* simply if it is an n-strong equilibrium. The notion of a k-strong equilibrium captures a familiar intuition (i.e., that there are no profitable deviations),

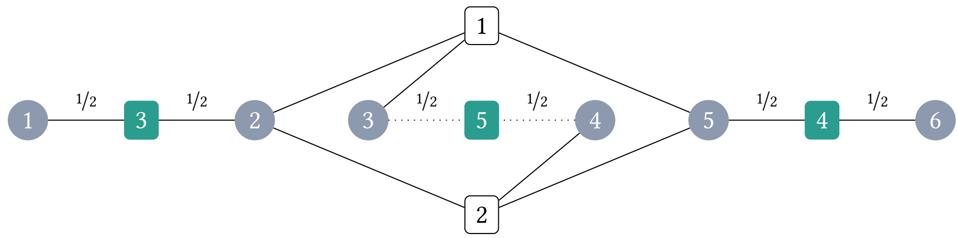

Fig. 2. Instance of budgeting game with $n = 6$ players (circle nodes) and $m = 5$ resources (square nodes). Each player i is connected to their relevant resources in X_i (solid edges), and the budget is $b = 1/2$. The edges are labeled with the allocated amounts, if any. Investment in a non-relevant resource is indicated by a dotted line. Active resources are colored in green.

under increasingly stronger conditions. Note that a k-strong equilibrium is also a $(k-1)$-strong equilibrium for $k \geq 2$, and 1-strong equilibria are more familiarly known as Nash equilibria. As is known, (pure) Nash equilibria are not guaranteed to exist for general normal-form games. However, for our budgeting games a potential function argument shows that a pure Nash equilibrium *does* always exist.

Theorem 3.1. *Every budgeting game $\Gamma = \langle N, M, b, \boldsymbol{X} \rangle$ admits a Nash equilibrium.*

Though of independent interest, this result is less relevant in the context of multi-winner voting, and we omit the proof here.

The interplay between MWV instances and budgeting games is crucial enough to warrant its own definition. If $I = \langle N, A, k, \boldsymbol{A} \rangle$ is a MWV instance, the *associated budgeting game* is $\Gamma_I = \langle N, M, k/n, \boldsymbol{X} \rangle$, where $M = A$ and $X_i = A_i$, for all $i \in N$. If $\Gamma = \langle N, M, b, \boldsymbol{X} \rangle$ is a budgeting game, with $b \in \{1/n, 2/n, \ldots, |M|/n\}$, the *associated MWV instance* is $I_\Gamma = \langle N, A, b \cdot n, \boldsymbol{A} \rangle$, where $A = M$ and the approval set of each agent $i \in N$ is $A_i = X_i$.

Example 2. *Consider a budgeting game with $N = [6]$, $M = [5]$, $b = 1/2$ and relevant resources, together with a profile $\boldsymbol{\lambda}$, depicted in Figure 2. Note that this is the budgeting game associated with the MWV instance in Example 1. The allocation $\boldsymbol{\lambda}$ activates resources 3, 4 and 5, i.e., $out(\boldsymbol{\lambda}) = \{3, 4, 5\}$, corresponding to committee $\pi = \{3, 4, 5\}$ that satisfies EJR, but is neither in the core, nor priceable. Coalition $S = \{2, 3, 4, 5\}$ can deviate to $\boldsymbol{\lambda}'_S$ in which agents 2 and 5 contribute $1/4$ to each of resources 1 and 2, 3 contributes $1/2$ to 1 and 4 contributes $1/2$ to 2. As a result, $\boldsymbol{\lambda}' = (\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}_{-S})$ is a profitable deviation for S, since $out(\boldsymbol{\lambda}') = \{1, 2\}$.*

4 Proportionality Notions as Equilibria

In this section we characterize prominent proportionality axioms as equilibria of budgeting games. One of these (the α-core) has precedent in the game-theory literature; the others arise out of the connection with multiwinner voting, and are new.

4.1 Priceability and Superset Equilibria

For priceability, the relevant equilibrium relies on a type of group deviation that results in more resources being activated. For a budgeting game Γ and a coalition $S \subseteq N$, a strategy profile $\boldsymbol{\lambda}' = (\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}_{-S})$ is a *superset deviation for* S if $out(\boldsymbol{\lambda}) \subsetneq out(\boldsymbol{\lambda}')$, and $out(\boldsymbol{\lambda}) \cap X_i \subsetneq out(\boldsymbol{\lambda}') \cap X_i$, for every $i \in S$. Note that the utility of each player in S increases as result of a superset deviation.

Definition 1 (Superset Equilibrium). *Profile $\boldsymbol{\lambda}$ is a* superset equilibrium *if there is no coalition $S \subseteq N$ of players that has a superset deviation.*

Superset equilibria can be shown to correspond to priceable committees under a mild restriction on the way agents allocate their budget. This restriction is natural: no player allocates any of their budget to resources not relevant to them.

Theorem 4.1. *For a budgeting game $\Gamma = \langle N, M, b, \boldsymbol{X} \rangle$, if $\boldsymbol{\lambda}$ is a superset equilibrium such that $\lambda_i(j) = 0$, for all $i \in N$ and $j \in M \setminus X_i$, then the committee $\pi = out(\boldsymbol{\lambda})$ is priceable for the multiwinner voting instance $I_\Gamma = \langle N, A, k, \boldsymbol{A} \rangle$, with $A = M$, $k = |\pi|$, and $\boldsymbol{A} = \boldsymbol{X}$.*

Conversely, every priceable committee can be mapped to a *subset* of (the outcome of) a superset equilibrium in a corresponding budgeting game.

Theorem 4.2. *For a MWV instance $I = \langle N, A, k, \boldsymbol{A} \rangle$ and a committee π priceable for allowance α, the budgeting game $\Gamma_I = \langle N, M, b, \boldsymbol{X} \rangle$, with $M = A$, $b = \alpha$, and $\boldsymbol{X} = \boldsymbol{A}$, has a superset equilibrium $\boldsymbol{\lambda}$ in which $\lambda_i(j) = 0$, for all $i \in N$ and $j \in M \setminus X_i$, and such that $\pi \subseteq out(\boldsymbol{\lambda})$.*

The proofs proceed by following the definitions; we omit them here. Some observations are in order, however, about the conditions under which these results hold. First, note that without restricting contributions to relevant resources only, it becomes possible to have superset equilibria whose outcomes are not priceable, i.e., Theorem 4.1 does not go through anymore.

Example 3. *Consider a budgeting game with $N = [2]$, $M = [2]$, $X_1 = X_2 = \{1\}$, and budget $b = 1/2$. Profile $\boldsymbol{\lambda}$ in which players 1 and 2 contribute $1/2$ to resource 2, which gets activated, is a superset equilibrium. However, committee $\{2\}$ in the corresponding MWV instance with $k = 1$ is not priceable: property C_1 forbids voters from investing any of their budget in alternative 2, while property C_4 requires non-negative contributions.*

Then, we remark that a strengthening of Theorem 4.2 in which every priceable committee corresponds *exactly* to a superset equilibrium is not possible.

Example 4. *Consider an MWV instance with $N = [2]$, $A = [2]$, $A_1 = \{1\}$ and $A_2 = \{2\}$. For $k = 1$, both committees $\{1\}$ and $\{2\}$ are priceable with allowance 1: in one case, voter 1 spends its allowance to buy candidate 1 while voter 2 spends nothing; the other case is symmetrical. However, in the budgeting game corresponding to the MWV instance, the output for a budget of 1 is $\{1, 2\}$: at equilibrium, with investment only in relevant resources, players 1 and 2 activate resources 1 and 2, respectively.*

The issue here is that the choice of k may require the exclusion of a perfectly affordable candidate simply because of tie-breaking, whereas in the corresponding game players are free to activate any relevant resource within the allotted budget. It is an interesting question whether setting the budget to k/n in the corresponding budgeting game, and allowing players to invest in non-relevant resources, would make it possible to simulate each priceable committee of size k.

4.2 The Core of a Multiwinner Voting Instance and the α-Core of a Budgeting Game

To characterize the core of MWV instances we can, interestingly, use the existing notion of the α-core [Aumann, 1961, Scarf, 1971]. To define it in the context of a budgeting game $\Gamma = \langle N, M, b, \boldsymbol{X} \rangle$, we first say that a coalition $S \subseteq N$ has an *α-core deviation* from a strategy profile $\boldsymbol{\lambda}$ if there is a strategy profile $\boldsymbol{\lambda}'_S$ such that $u_i(\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}'_{-S}) > u_i(\boldsymbol{\lambda})$, for every strategy profile $\boldsymbol{\lambda}'_{-S}$ and for all $i \in S$. In this case, we say that strategy profile $\boldsymbol{\lambda}$ is *α-blocked* by coalition S. In other words, deviating to $\boldsymbol{\lambda}'_S$ leads to strictly higher utility for all players in S, regardless of how players in $N \setminus S$ react.

Definition 2 (α-Core). *A strategy profile $\boldsymbol{\lambda}$ is in the α-*core *of a budgeting game Γ if there is no coalition that α-blocks it.*

The definition of the α-core can be phrased equivalently by making players outside S contribute 0 to all resources. More precisely, for a strategy profile $\boldsymbol{\lambda}$ and a coalition $S \subseteq N$, let $\boldsymbol{\lambda}^0_{-S}$ be the stategy profile in which every player in $N \setminus S$ resets their contribution to 0, i.e., $\lambda^0_i(j) = 0$, for all players $i \in N \setminus S$ and resources $j \in M$. We then have the following result.

Proposition 4.3. *For a budgeting game Γ, a coalition $S \subseteq N$ α-blocks a strategy profile $\boldsymbol{\lambda}$ with $\boldsymbol{\lambda}'_S$ if and only if for all $i \in S$ we have $u_i(\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}^0_{-S}) > u_i(\boldsymbol{\lambda})$.*

Example 5. *For the budgeting game in Example 2, profile $\boldsymbol{\lambda}$ depicted in Figure 2 is not in the α-core, as the coalition $S = \{2, 3, 4, 5\}$ α-blocks $\boldsymbol{\lambda}$ with the deviation $\boldsymbol{\lambda}'$ that activates resources* 1 *and* 2*.*

Using Proposition 4.3, we can make the relationship between the core of a multwinner voting instance and the α-core of the associated budgeting game precise.

Theorem 4.4. *If a strategy profile $\boldsymbol{\lambda}$ is in the α-core of a budgeting game $\Gamma = \langle N, M, k/n, \boldsymbol{X}\rangle$, then $out(\boldsymbol{\lambda})$ is in the core of the associated MWV instance $I_\Gamma = \langle N, A, k, \boldsymbol{A}\rangle$.*

Proof. Take a strategy profile $\boldsymbol{\lambda}$ in the α-core of Γ and assume, for the sake of contradiction, that $out(\boldsymbol{\lambda})$ is not in the core of the MWV instance I_Γ, i.e., $out(\boldsymbol{\lambda})$ is blocked. Intuitively, this means that there exists some group S of voters who would be happier with alternatives T, and S is large enough to demand $|T|$ seats on the committee: this set S can then redirect its budget to activate the resources in T, meaning $\boldsymbol{\lambda}$ could not have been in the α-core of the budgeting game Γ. More formally, there exists a set of voters $S \subseteq N$ and a set of alternatives $T \subseteq A$, such that $|S| \geq |T| \cdot n/k$ and $|A_i \cap T| > |A_i \cap out(\boldsymbol{\lambda})|$ for all $i \in S$. This implies that in the associated budgeting game the players in S together have a budget of $|S| \cdot b = |S| \cdot k/n \geq |T|$. This means, in particular, that players in S have sufficient budget to activate all resources in T. Hence there exists a profile $\boldsymbol{\lambda}'_S$ for the players in S such that $out(\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}^0_{-S}) = T$. From the fact that $out(\boldsymbol{\lambda})$ is blocked we also have that $|A_i \cap T| > |A_i \cap out(\boldsymbol{\lambda})|$, for all $i \in S$. This allows us to conclude that for all $i \in S$, we have:

$$u_i(\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}^0_{-S}) = |A_i \cap out(\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}^0_{-S})| = |A_i \cap T| > |A_i \cap out(\boldsymbol{\lambda})| = u_i(\boldsymbol{\lambda}).$$

By Proposition 4.3, this implies that S is an α-blocking coalition, which is a contradiction. □

A similar argument proves the converse.

Theorem 4.5. *If $\pi \subseteq A$ is a committee in the core of a MWV instance $I = \langle N, A, k, \boldsymbol{A}\rangle$, then any strategy profile $\boldsymbol{\lambda}$ with $out(\boldsymbol{\lambda}) = \pi$ is in the α-core of the associated budgeting game $\Gamma_I = \langle N, M, k/n, \boldsymbol{X}\rangle$.*

Proof. Take a committee π in the core of I and let $\boldsymbol{\lambda}$ be a strategy profile such that $out(\boldsymbol{\lambda}) = \pi$. Assume, for the sake of contradiction, that there exists a coalition of players $S \subseteq N$ acting as an α-blocking coalition in $\boldsymbol{\lambda}$, witnessed by the deviation $\boldsymbol{\lambda}'_S$. By Proposition 4.3, we have that $u_i(\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}^0_{-S}) > u_i(\boldsymbol{\lambda})$, for all $i \in S$. We infer that:

$$|A_i \cap out(\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}^0_{-S})| > |A_i \cap out(\boldsymbol{\lambda})| = |A_i \cap \pi|,$$

for every player $i \in S$. Since no player in $N \setminus S$ contributes any of their budget in $out(\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}^0_{-S})$, we know that $|out(\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}^0_{-S})| \leq \sum_{i \in S} b = k/n \cdot |S|$, and therefore that $|S| \geq |out(\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}^0_{-S})| \cdot n/k$. However, that means that S blocks π with $out(\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}^0_{-S})$, which contradicts the assumption that π is a committee in the core of I. □

With a budget of k/n per player, n players can always activate k resources, i.e., for every committee π there exists at least one strategy profile $\boldsymbol{\lambda}$ such that $out(\boldsymbol{\lambda}) = \pi$. Thus, Theorems 4.4 and 4.5 imply, first, that a strategy profile $\boldsymbol{\lambda}$ is in the α-core if and only if $out(\boldsymbol{\lambda})$ is in the core and, second, that the core is always non-empty if and only if for every budgeting game $\Gamma = \langle N, M, k/n, \boldsymbol{X}\rangle$, $k \in \{1, 2, \ldots, |M|\}$, there exists a profile in the α-core.

Note, however, that activation of k resources might require some players to spend their budget on irrelevant resources: if $k = 1$, activating some resource r involves each agent investing $1/n$ in r—whether the agent cares about r or not. Such behavior borders on irrationality, and it would be interesting to know if committees in the core can be rationalized with more rational profiles. It might help to focus on committees in the core that are subset minimal.

Finally, it is worth pointing out that every strong equilibrium of a budgeting game is also in the α-core, while the converse is not true, i.e., there can be profiles in the α-core that are not strong equilibria.

Proposition 4.6. *Let Γ be a budgeting game. If a strategy profile $\boldsymbol{\lambda}$ is a strong equilibrium of Γ, then $\boldsymbol{\lambda}$ is also in the α-core of Γ.*

4.3 EJR and the Cohesive α-Core

Extended Justified Representation (EJR) can be seen as a restricted version of the core. Thus, in order to find an equilibrium characterizing EJR, we restrict the type of deviation players are allowed to do compared to the α-core. Given a budgeting game Γ and a profile $\boldsymbol{\lambda}$, a *cohesive deviation* for coalition $S \subseteq N$ is a profile $\boldsymbol{\lambda}'_S$ such that if $\lambda'_i(j) > 0$, then $j \in \bigcap_{m \in S} X_m$, for all players $i \in S$. In other words, in $\boldsymbol{\lambda}'_S$ all players in S contribute only to resources that are relevant to all players in S. Profile $\boldsymbol{\lambda}'_S$ is a *cohesive α-core* deviation if, in addition to being cohesive, it also holds that $u_i(\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}'_{-S}) > u_i(\boldsymbol{\lambda})$, for all $i \in S$, and for every strategy profile $\boldsymbol{\lambda}'_{-S}$. In this case, we also say that profile $\boldsymbol{\lambda}$ is *cohesively α-blocked* by coalition $S \subseteq N$.

Definition 3 (Cohesive α-core). *A profile $\boldsymbol{\lambda}$ is in the* cohesive α-core *of a budgeting game Γ if there is no coalition that cohesively α-blocks it.*

Example 6. *For the budgeting game in Example 2, note that deviation $\boldsymbol{\lambda}'$, which makes agents in $S = \{2, 3, 4, 5\}$ better off, is not cohesive, since it involves contributions to resources that are not unanimously relevant. The lack of deviations shows that the outcome $\{3, 4, 5\}$ is in the cohesive α-core of the game.*

In Example 6 the outcome that ends up in the cohesive α-core of the budgeting game coincides with a committee that satisfies EJR in the associated MWV instance. The following results show that this is no coincidence.

Theorem 4.7. *If profile $\boldsymbol{\lambda}$ is in the cohesive α-core of budgeting game $\Gamma = \langle N, M, k/n, \boldsymbol{X}\rangle$, then $out(\boldsymbol{\lambda})$ satisfies EJR with respect to the associated MWV instance $I_\Gamma = \langle N, A, k, \boldsymbol{A}\rangle$.*

Proof. Suppose, for the sake of contradiction, that there is a profile $\boldsymbol{\lambda}$ in the cohesive α-core of Γ such that $out(\boldsymbol{\lambda})$ does not satisfy EJR. Then there exists an ℓ-cohesive group S of voters, each represented by fewer than ℓ candidates in $out(\boldsymbol{\lambda})$. In other words, there exists a set of alternatives $T \subseteq A$, with $|T| = \ell$ and $T \subseteq \bigcap_{i \in S} A_i$, such that (i) $|S| \geq \ell \cdot n/k$ and (ii) $|A_i \cap T| > |A_i \cap out(\boldsymbol{\lambda})|$ for all $i \in S$. Analogously to the proof of Theorem 4.4, it follows from statements (i) and (ii) that the players in S are an α-blocking coalition for $\boldsymbol{\lambda}$, witnessed by a profile $\boldsymbol{\lambda}'_S$ such that $out(\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}^0_{-S}) = T$. As per Proposition 4.3, we can assume without loss of generality that $\lambda_i(j) = 0$ if $j \notin T$, which, with the fact that $T \subseteq \bigcap_{i \in S} A_i$, implies that this deviation is cohesive. □

Theorem 4.8. *If committee π satisfies EJR with respect to MWV instance $I = \langle N, A, k, \boldsymbol{A}\rangle$, then any strategy profile $\boldsymbol{\lambda}$ with $out(\boldsymbol{\lambda}) = \pi$ is in the cohesive α-core of the associated budgeting game $\Gamma_I = \langle N, M, k/n, \boldsymbol{X}\rangle$.*

Proof. Take a committee π satisfying EJR and a strategy profile $\boldsymbol{\lambda}$ such that $out(\boldsymbol{\lambda}) = \pi$. Assume, for the sake of contradiction, that there is a cohesive α-blocking coalition S, witnessed by strategy profile $\boldsymbol{\lambda}'_S$. As no player in $N \setminus S$ contributes any of their budget in $out(\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}^0_{-S})$, we know that $|out(\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}^0_{-S})| \leq \sum_{i \in S} b = |S| \cdot k/n$. Moreover, $\boldsymbol{\lambda}'_S$ is cohesive so only resources j with $j \in X_i$ for all $i \in S$ can be in $out(\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}^0_{-S})$. Therefore S is $|out(\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}^0_{-S})|$-cohesive with respect to the MWV instance I. Also, because $\boldsymbol{\lambda}'_S$ is cohesive we have $|A_i \cap out(\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}^0_{-S})| = |out(\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}^0_{-S})|$. However, by Proposition 4.3, we have that $u_i(\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}^0_{-S}) > u_i(\boldsymbol{\lambda})$, for all $i \in S$. This means that:

$$\left|A_i \cap out(\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}^0_{-S})\right| = \left|out(\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}^0_{-S})\right| > |A_i \cap out(\boldsymbol{\lambda})| = |A_i \cap \pi|,$$

contradicting the assumption that π satisfies EJR. □

As for the α-core, Theorem 4.8 does not provide any guarantee that players in an EJR-satisfying profile $\boldsymbol{\lambda}$ behave sensibly, in the sense of contributing only to relevant resource. Indeed, committees of size k produced by the voting rule PAV (known to satisfy EJR but not priceability) cannot always be the outcome of a profile $\boldsymbol{\lambda}$ in which players contribute only to relevant resources (since then $out(\boldsymbol{\lambda})$ would be priceable). On the other hand, as MES produces a partial committee that always satisfies EJR and ensures that no player pays for a non-approved candidate, we conclude that every budgeting game where with budget k/n for some k has a profile in the cohesive α-core where players contribute only to relevant resources.

4.4 EJR+ and Weakly Improving Deviations

Intuitively, EJR+ is violated if there is a group S of voters such that every voter in S is represented by fewer than $k \cdot |S|/n$ candidates, and there exists a candidate jointly approved by the voters in S that is not elected. It is tempting to assume this corresponds to an equilibrium where no group of players can afford to activate an additional resource. Such a requirement, however, does not capture every violation of EJR+, as it leaves open the possibility that voters do not have enough representation because they spend more than strictly necessary for the resources they activate. Thus, in order to capture EJR+ we use deviations in which players can increase their representation either by activating a new resource, or, alternatively, by reducing the amount of money they need to spend to achieve their current level of representation. More precisely, for budgeting game Γ and profile $\boldsymbol{\lambda}$, strategy profile $\boldsymbol{\lambda}' = (\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}_{-S})$ is a *weakly improving singleton deviation* for coalition $S \subseteq N$ if $|out(\boldsymbol{\lambda}') \setminus out(\boldsymbol{\lambda})| = 1$ and for all $i \in S$ either $u_i(\boldsymbol{\lambda}') > u_i(\boldsymbol{\lambda})$, or $u_i(\boldsymbol{\lambda}') = u_i(\boldsymbol{\lambda})$ and $\sum_{j \in M} \lambda'_i(j) < \sum_{j \in M} \lambda_i(j)$.

We will show later that the outcome $out(\boldsymbol{\lambda})$ of any profile $\boldsymbol{\lambda}$ with no available weakly improving singleton deviations satisfies EJR+. Prior to this, we note that such a condition is too strong to actually capture EJR+; in fact, there exist games in which every profile allows for such a deviation.

Example 7. *Consider a budgeting game with players $N = [3]$, resources $M = [3]$, budget $b = 0.6$, and relevant resources given by $X_1 = \{1, 3\}$, $X_2 = \{1, 2\}$ and $X_3 = \{2, 3\}$. Any strategy profile in this game allows for a weakly improving singleton deviation: clearly, if no resource is activated, any set of two players can deviate by activating a resource they jointly approve. At the same time, the total budget allows for the activation of only one resource. Assume, without loss of generality, that only resource 1 is active. Then at least one of players 1 and 2 contributes more than 0.4 to 1: assume, again without loss of generality, that $\lambda_1(1) > 0.4$. Then players 1 and 3 have a weakly improving singleton deviation activating resource 3 such that $0.4 < \lambda'_1(3) < \lambda_1(1)$ and $\lambda'_3(3) = 1 - \lambda'_1(3) \leq 0.6$. But now either 1 or 3 contributes strictly more than 0.4, hence this player has a weakly improving singleton deviation together with player 2.*

This motivates restricting weakly improving singleton deviations to a well-behaved subset in which players share the cost of the newly activated resource evenly. Formally, deviation $\boldsymbol{\lambda}' = (\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}_{-S})$

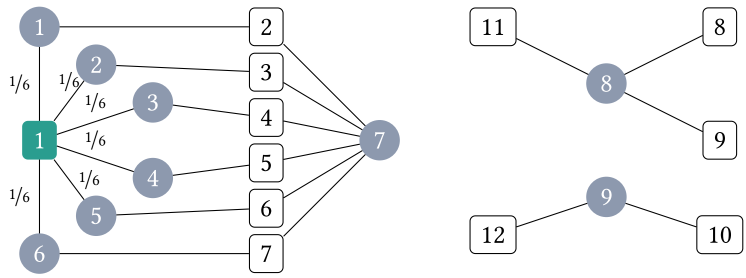

Fig. 3. Budgeting game for Example 8, with profile $\boldsymbol{\lambda}$.

is *balanced* if for all $j \in out(\boldsymbol{\lambda}') \setminus out(\boldsymbol{\lambda})$ it holds that that $\lambda'_i(j) = \lambda'_{i*}(j)$, for all players $i, i^* \in S$. Though severe, this restriction is still sufficient to guarantee EJR+.

Theorem 4.9. *For a budgeting game $\Gamma = \langle N, M, k/n, \boldsymbol{X} \rangle$, if $\boldsymbol{\lambda}$ is a profile for which no balanced, weakly improving singleton deviation is possible, then $out(\boldsymbol{\lambda})$ satisfies EJR+ with respect to the associated MWV instance $I_\Gamma = \langle N, A, k, \boldsymbol{A} \rangle$.*

Proof. Let $\boldsymbol{\lambda}$ be a strategy profile such that $out(\boldsymbol{\lambda})$ violates EJR+. Let S be a set of voters that witnesses this violation and j a candidate in $(\bigcap_{i \in S} A_i) \setminus out(\boldsymbol{\lambda})$. We claim that the corresponding set of players S has a balanced, weakly improving singleton deviation $\boldsymbol{\lambda}' = (\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}_{-S})$ activating j. If such a deviation exists we must have $\lambda'_i(j) \leq 1/|S|$. We have to show that for every player $i \in S$ either

$$b - \sum_{j \in X_i \cap out(\boldsymbol{\lambda})} \left(1 - \min\left(1, \sum_{i' \in N \setminus \{i\}} \lambda_{i'}(j)\right)\right) \geq \frac{1}{|S|},$$

i.e., i can contribute $1/|S|$ without decreasing their utility, or there exists a resource j' such that $\lambda_i(j') > 1/|S|$. Assume neither holds for player i. It follows that i contributes at most $1/|S|$ to every active resource in X_i and has less than $1/|S|$ of their budget after this. Thus we know that $1/|S| > b - (1/|S| \cdot |out(\boldsymbol{\lambda}) \cap X_i|)$, which is equivalent, after rewriting, to $|out(\boldsymbol{\lambda}) \cap X_i| \geq \lfloor k/n \cdot |S| \rfloor$. However, this contradicts the assumption that S witnesses a violation of EJR+. □

The opposite direction of Theorem 4.9 does not hold, as the absence of balanced, weakly improving singleton deviations is a stronger requirement than EJR+.

Example 8. *Consider the budgeting game depicted in Figure 3 with budget $b = 2/3$, corresponding to a MWV instance with $k = 6$. Any committee containing* 1*, e.g., $\pi = \{1, 8, \ldots, 12\}$, satisfies EJR in the associated MWV instance. Indeed, π even satisfies EJR+: every unelected candidate $j \in \{2, \ldots, 7\}$ is supported by two voters, one of which also supports candidate 1. As $\lfloor 2 \cdot 6/9 \rfloor = 1$, EJR+ is satisfied. Now, consider any strategy profile that activates (a subset of) π such as the one depicted in Figure 3. There is at least one player $i \in \{1, \ldots, 6\}$ that can contribute only $1/6$ to resource* 1*, and thus afford to activate the resource shared with player* 7 *in a balanced way, with a contribution of $1/2$ each.*

The natural question, now, is whether profiles that do not allow for balanced, weakly improving singleton deviations always exist. The answer is yes, and they can be found using MES (Proposition 5.4). This turns out to be a straightforward consequence of the game-theoretic characterization of MES in Section 5, to follow. This means that the absence of such deviations encodes a new fairness axiom, stronger than existing axioms, and always satisfied by MES!

5 Defining voting rules through improving move dynamics

We turn our attention now to finding equilibria of the type characterized in Section 4. In particular, we want to understand when sequences of acceptable deviations (such as the ones described in Section 4) can settle on an equilibrium. This is of particular interest, as it suggests a new way of looking at multiwinner voting rules: according to this view, voters can find their way to a fair outcome on their own, by repeatedly deviating from a starting profile according to pre-specified rules. The success of this approach hinges, of course, on whether sequences of such deviations are guaranteed to terminate in a stable outcome. Our first results show that for cohesive and α-core deviations such a guarantee is not possible.

Example 9. *Consider the budgeting game depicted in Figure 4, with budget $b = 1/2$. Profiles $\boldsymbol{\lambda}$, $\boldsymbol{\lambda}''$ and $\boldsymbol{\lambda}''$ form a cycle of cohesive deviations: $\boldsymbol{\lambda}'$, for instance, is obtained from $\boldsymbol{\lambda}$ by agents 2 and 3 redirecting their budget to resources 3 and 4, respectively, relevant to both.*

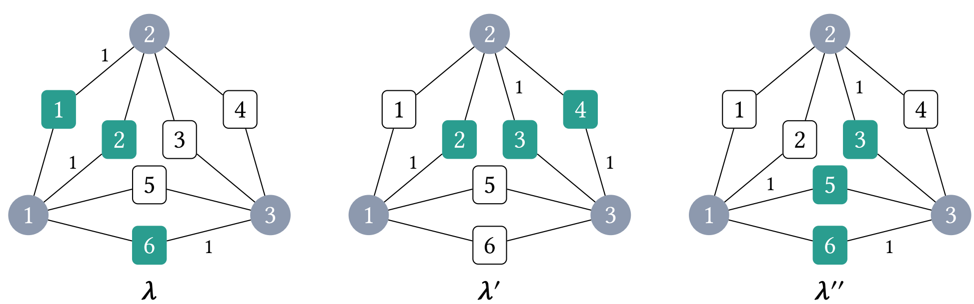

Fig. 4. A cyclic sequence of cohesive deviations ($\boldsymbol{\lambda} \longrightarrow \boldsymbol{\lambda}' \longrightarrow \boldsymbol{\lambda}'' \longrightarrow \boldsymbol{\lambda}$) for Example 9.

Though cohesive, the deviations in Example 9 are not α-core, since they depend on other players' contributions in order to be profitable. But a slightly modified version of the example creates a cycle for α-core deviations.

Example 10. *Consider the same budgeting game as in Example 9 but with additional resources 7, 8 and 9, where 7 is relevant for 1, 8 is relevant for 2 and 9 is relevant 3. The budget is $b = 3/2$. As an initial strategy profile $\boldsymbol{\lambda}$, player 2 contributes 1 to 8, player 1 contributes 1 to 7 and 1/4 to both 5 and 6, and player 3 contributes 3/4 to both 5 and 6. Players 2 and 3 can deviate to a strategy $\boldsymbol{\lambda}'$ where 2 contributes 3/4 to resources 3 and 4 while 3 contributes 1/4 to 3 and 4 and 1 to 9. This increases the utility of player 2 from 1 to 2 and the utility of player 3 from 2 to 3. Crucially, this improvement holds for any possible strategy of 1, i.e., it is an α-core deviation. It is, however, not cohesive, as 9 is not relevant to player 2. The resulting strategy profile is exactly symmetric to $\boldsymbol{\lambda}$, and now 1 and 2 can deviate in the same manner.*

This leaves cohesive α-core deviations, for which we derive a positive result: using a potential function argument, we show that every sequence of cohesive α-core deviations is finite. To this aim, we define a potential function $\Phi : \Lambda \longrightarrow \mathbb{N}_0^n$ that maps every strategy profile $\boldsymbol{\lambda} \in \Lambda$ to a non-negative integer vector $\Phi(\boldsymbol{\lambda}) = (a_1, a_2, \ldots, a_n) \in \mathbb{N}_0^n$ such that $a_1 \geq a_2 \geq \ldots \geq a_n$ (ties broken arbitrarily). As we show below, these vectors increase lexicographically with every cohesive α-core deviation.[3] Further, each entry a_i, $i \in [n]$, will be bounded. Note that the latter is crucial: despite the fact that

[3] Recall that the lexicographic ordering $\prec$ on $\mathbb{N}_0^n$ is defined as follows: for any two vectors $\boldsymbol{a}, \boldsymbol{a}' \in \mathbb{N}_0^n$, we have $\boldsymbol{a} \prec \boldsymbol{a}'$ iff there is a $k \in [n]$ such that $a_i = a'_i$ for all $i = 1, \ldots, k-1$ and $a_k < a'_k$.

our budgeting games have infinite strategy sets, this allows us to conclude that every sequence of cohesive α-core deviations must be finite.

Theorem 5.1. *Any sequence of cohesive α-core deviations is finite.*

Proof. Fix an arbitrary strategy profile $\bar{\boldsymbol{\lambda}}$ and consider a sequence of cohesive α-core deviations, starting from $\bar{\boldsymbol{\lambda}}$. Define S^k as the coalition of the k-th deviation and define $\boldsymbol{\lambda}^k = (\boldsymbol{\lambda}'_{S^k}, \boldsymbol{\lambda}^{k-1}_{-S^k})$ as the resulting strategy profile of the k-th deviation with $k > 1$, and $\boldsymbol{\lambda}^1 = (\boldsymbol{\lambda}'_{S^1}, \bar{\boldsymbol{\lambda}}_{-S^1})$ for $k = 1$. For convenience, let $\boldsymbol{a}^k = \Phi(\boldsymbol{\lambda}^k)$ for $k \geq 1$. By construction $\boldsymbol{a}^k$ has length $n = |N|$, so that for every player $i \in N$ there is a corresponding value in $\boldsymbol{a}^k$ at position $\sigma(i)$. Recall that $\boldsymbol{a}^k$ is sorted in non-increasing order, so for two players i and j, if $a^k_{\sigma(i)} > a^k_{\sigma(j)}$ then $\sigma(i) < \sigma(j)$. By the way that we will define $\boldsymbol{a}^k$, it serves as a lower bound on the utility of a player $i \in N$ after the k-th deviation, i.e., $u_i(\boldsymbol{\lambda}^k) \geq a^k_{\sigma(i)}$. We will prove that $\boldsymbol{a}^{k-1} \prec \boldsymbol{a}^k$ for any $k \geq 1$.

We begin by introducing some notation in order to formally define $\boldsymbol{a}^k$. When considering the k-th deviation, a coalition S^l with $l < k$ is *intact* if no player $i \in S^l$ was part of a coalition after the l-th deviation and before the $(k+1)$-th deviation, i.e., $i \notin \bigcup_{l<h\leq k} S^h$. Consider a coalition S^l that is still intact when the k-th deviation is made, $l < k$. Then, for any $i \in S^l$, $a^k_{\sigma(i)}$ can be interpreted as the number of resources that are relevant for all players in S^l and that the contributions $\boldsymbol{\lambda}^k_{S^l} = \boldsymbol{\lambda}'_{S^l}$ can activate without any contribution of the players in $N \backslash S^l$. It is thus a lower bound on the utility of the players in S^l for the profile $\boldsymbol{\lambda}^k$.

We say that a deviation k *breaks* deviation l, $l < k$, if the coalition S^l was intact before the k-th deviation and is no longer intact after the k-th deviation. In this case, the values $a^k_{\sigma(i)}$ are set to 0 for all $i \in S^l \backslash S^k$, which is obviously a lower bound on the utility of the players in $S^l \backslash S^k$ for the profile $\boldsymbol{\lambda}^k$. We introduce a set L^k of players whose coalition is broken due to the k-th deviation and who are not part of the k-th coalition. More formally,

$$L^k = \bigcup_{j \in S^k} L^k_j \quad \text{with} \quad L^k_j = \left\{ i \in N \backslash S^k \;\middle|\; a^{k-1}_{\sigma(i)} \neq 0, l = \underset{1\leq h<k}{\arg\max}\{i \in S^h\} \text{ and } \{i, j\} \subseteq S^l \right\},$$

where L^k_j is the set of players whose coalition is broken due to player $j \in S^k$ deviating.

We now define $\boldsymbol{a}^k$ more formally. Define $\boldsymbol{\lambda}^0$ as $\lambda^0_i(j) = 0$ for all $i \in N$ and all $j \in M$ and define $\boldsymbol{a}^0 = \boldsymbol{0}$. For the k-th deviation, $k \geq 1$, and for every player $i \in N$ we define $a^k_{\sigma(i)}$ as:

$$a^k_{\sigma(i)} = \begin{cases} |out(\boldsymbol{\lambda}'_{S^k}, \boldsymbol{\lambda}^0_{-S^k})| & \text{if } i \in S^k, \\ 0 & \text{if } i \in L^k, \\ a^{k-1}_{\sigma(i)} & \text{otherwise.} \end{cases}$$

So after the k-th deviation for an agent $i \in N$, the number $a^k_{\sigma(i)}$ denotes how profitable the last intact coalition was that i was in, or is 0 otherwise. As we consider cohesive α-core deviations, the profitability of a deviation is the same for all players in the coalition. If at some point in time, agent i decides to break the last coalition that i was in to join a new coalition, this new coalition must be more profitable for i. We continue by formally defining this. Note the following two consequences of the definitions of $\boldsymbol{a}^k$ and a cohesive α-core deviation. First, for any $i \in N$ and $k \geq 0$, it holds that $a^k_{\sigma(i)} \in \{0, 1, \ldots, \min\{|X_i|, \lfloor b \cdot n \rfloor\}\}$. Secondly, note that for any $i \in S^k$ and $k > 1$ it holds that:

$$a^k_{\sigma(i)} = u_i(\boldsymbol{\lambda}'_{S^k}, \boldsymbol{\lambda}^0_{-S^k}) > u_i(\boldsymbol{\lambda}^{k-1}) \geq a^{k-1}_{\sigma(i)} \geq 0,$$

and $a^k_{\sigma(j)} = a^k_{\sigma(h)}$ for any $j, h \in S^k$. Therefore, for any $i \in S^k$ and $k > 1$ it holds that:

$$a^k_{\sigma(i)} > \max_{j \in S^k} \left\{ a^{k-1}_{\sigma(j)} \right\}. \tag{2}$$

Note that trivially $\boldsymbol{a}^0 \prec \boldsymbol{a}^1$, as $a^1_{\sigma(i)} = |out(\boldsymbol{\lambda}'_{S^1}, \bar{\boldsymbol{\lambda}}_{-S^1})| > 0$ for all $i \in S^1$. We prove that $\boldsymbol{a}^{k-1} \prec \boldsymbol{a}^k$ for any $k \geq 2$. First, consider the case where $L^k = \emptyset$. Then it must be that $a^k_{\sigma(i)} = a^{k-1}_{\sigma(i)}$ for all $i \in N \backslash S^k$. And so by equation (2), it follows that $\boldsymbol{a}^{k-1} \prec \boldsymbol{a}^k$. Secondly, consider the case where $L^k \neq \emptyset$. Note that $L^k \cap S^k = \emptyset$ by definition. In this case $a^k_{\sigma(i)} = a^{k-1}_{\sigma(i)}$ for any $i \in N \backslash (L^k \cup S^k)$, as the intact coalitions of these players i (if any) are not affected by the k-the deviation. Note that for a player $i \in S^k$ and any $j \in L^k_i$, it holds that $a^{k-1}_{\sigma(i)} = a^{k-1}_{\sigma(j)}$, as i and j had the same profitability for the intact coalition they were in. Therefore by equation (2), for any $i \in S^k$ and any $j \in L^k \cup S^k$, it holds that $a^k_{\sigma(i)} > a^{k-1}_{\sigma(j)}$. This leads to $\boldsymbol{a}^{k-1} \prec \boldsymbol{a}^k$. Since $a^k_{\sigma(i)} \in \{0, 1, \ldots, \min\{|X_i|, \lfloor b \cdot n \rfloor\}\}$, for any $k \geq 0$ and $i \in N$, we conclude that any sequence of cohesive α-core deviations is finite. □

This result immediately gives us the tools to define a whole family of voting rules, as outlined at the beginning of the section, with the starting strategy profile and a rule for picking the next deviation (some version of cohesive α-core deviation) as parameters. Per Theorem 4.7, the outcome of such a rule is guaranteed to satisfy EJR, and hence satisfies a strong baseline proportionality constraint. Consider, for instance, the rule that starts from the empty profile (contributions uniformly set to 0), and allows any cohesive α-core deviation. Does this correspond to any existing multiwinner voting rule, e.g., MES? The example below shows that it does not.

Example 11. *Consider a budgeting game with players $N = [3]$, resources $M = [2]$ and budget $b = 3/4$. Resource 1 is relevant for players 1 and 2 and resource 2 is relevant for players 2 and 3. In the corresponding multiwinner instance, MES elects candidates 1 and 2, with voters 1 and 2 contributing 1/2 to 1, voter 2 contributing 1/4 to 2 and voter 3 contributing 3/4 to 2. However, there exists no sequence of cohesive α-core deviations starting from the empty profile that activates both resources 1 and 2. Clearly, no cohesive deviation can activate both 1 and 2 at the same time, as there is only one player for whom both resources are relevant and that player does not have sufficient budget to activate both. So assume one of the two resources is activated first, say w.l.o.g. resource 1 is activated. As player 2 does not have enough budget to activate 1 alone, we know that also player 2 is contributing to 1. However, then no cohesive deviation activating resource 2 can be profitable for player 2 under every strategy of the remaining players. Indeed, as resource 2 is not relevant for player 1, player 1 cannot be part of the deviating coalition. Moreover, player 3 will not contribute to resource 1 in any cohesive deviation. Hence, if player 1 does not contribute to 1 any longer, than the deviation is not profitable for player 2.*

Notably, all known voting rules elect both 1 and 2 in Example 11. Thus, cohesive α-core deviations on top of an empty starting profile define a different, and completely new family of voting rules, every member of which satisfies EJR out of the box. It is natural to ask how well such rules perform in practice, and which starting profiles and restrictions on the set of deviations result in the best voting rules, but we leave this to future work. Instead, we turn our attention to the question that motivated Example 11: can we characterize MES with this methodology?

5.1 Characterizing the Method of Equal Shares

MES is a prime candidate for an interpretation in terms of budgeting games, and we show now that it can be characterized using a type of *superset deviation.* Any sequence of such deviations is finite, and, by construction, rules defined in this way will be priceable.

To capture the fact that agents try to achieve the highest satisfaction for budget spent, we add the condition that deviations need to be efficient. A profitable deviation $\boldsymbol{\lambda}' = (\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}_{-S})$ for a coalition $S \subseteq N$ is *efficient* if there is no player $i \in S$, set of players S^* and profitable deviation $\boldsymbol{\lambda}^* = (\boldsymbol{\lambda}^*_{S^*}, \boldsymbol{\lambda}_{-S^*})$ such that $i \in S^*$ and:

$$\frac{\sum_{m \in M} \lambda^*_i(m) - \lambda_i(m)}{u_i(\boldsymbol{\lambda}^*) - u_i(\boldsymbol{\lambda})} < \frac{\sum_{m \in M} \lambda'_i(m) - \lambda_i(m)}{u_i(\boldsymbol{\lambda}') - u_i(\boldsymbol{\lambda})}.$$

That is, the extra money spent on a unit increase of utility for i in $\boldsymbol{\lambda}^*$ is strictly smaller than in $\boldsymbol{\lambda}'$. Note that the denominator in this definition is never 0, as the deviation has to be profitable.

Since it is always more efficient for a player if other members of the coalition contribute more, efficient deviations need to be augmented by further restrictions, in order to ensure fairness of the contributions, as well as existence. For a budgeting game Γ and a profile $\boldsymbol{\lambda}$, a superset deviation $\boldsymbol{\lambda}' = (\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}_{-S})$ for coalition $S \subseteq N$ is *p-balanced* if for every player $i \in S$ it holds that: (*i*) $\lambda'_i(j) = \lambda_i(j)$, for any resource $j \notin out(\boldsymbol{\lambda}') \setminus out(\boldsymbol{\lambda})$, and (*ii*) for any resource $j \in out(\boldsymbol{\lambda}') \setminus out(\boldsymbol{\lambda})$, we have $j \in X_i$ and either $\lambda'_i(j) = p$, or $\lambda'_i(j) < p$ and $\sum_{m \in M} \lambda'_i(m) = b$. Intuitively, in a p-balanced deviation players either pay the same amount, or, if they don't have enough funds left to do so, everything that they have left.

Finally, we also require that deviations activate only one additional resource at a time. We call a superset deviation $\boldsymbol{\lambda}' = (\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}_{-S})$ a *singleton deviation* if $|out(\boldsymbol{\lambda}') \setminus out(\boldsymbol{\lambda})| = 1$. Together, these conditions carve out the type of superset deviation we need in order to capture the dynamics of MES. Before proceeding with the main result, we add an existence sanity check.

Proposition 5.2. *For any profile $\boldsymbol{\lambda}$ with no money spent on inactive resources, if there exists a superset deviation, then there also exists an efficient p-balanced singleton deviation.*

Proof. Let $\boldsymbol{\lambda}$ be a profile for which there exists a superset deviation. It is straightforward, then, to find a p-balanced singleton deviation where only one of the resources from the superset deviation is activated.

It remains to be shown that an efficient singleton deviation exists. Consider a p-balanced singleton deviation $\boldsymbol{\lambda}^*$ with minimal p and let j be the additional resource that gets activated in $\boldsymbol{\lambda}^*$. We claim that it is also efficient. First of all, consider a player i that contributes p to j after the deviation. By definition, i has at least p money left before the deviation, hence for any p^*-balanced singleton deviation with $p^* \geq p$, i needs to contribute at least p. Now consider a player i' that contributes $q < p$ to the deviation. This player needs to contribute q to any p^*-balanced deviation with $p^* \geq q$. As $q < p \leq p^*$ for all possible p^*-balanced deviations, i' needs to contribute q for every possible deviation. Since for any singleton deviation $\boldsymbol{\lambda}'$ the increase in utility is the same and $\lambda'_i(j) - \lambda_i(j) = \lambda'_i(j)$ holds for all players i by the assumption that no money is spent on inactivate resources in $\boldsymbol{\lambda}$, this means the p-balanced deviation is efficient. □

The assumption that no money is spent on inactive resources in Proposition 5.2 is crucial. Without it there might be a voter i who already contributes p to an inactive resource j and for whom a p-balanced deviation activating j would be preferable to a p^*-balanced deviation activating another resource, even if $p^* < p$. Similarly, dropping any one of the above restrictions on superset deviations results in a loss of the existence guarantee.

It is straightforward to see that any sequence of efficient, p-balanced singleton deviations is finite. Surprisingly, any such sequence also leads to a profile in the cohesive α-core—surprising given that these deviations are much more restricted than cohesive α-core deviations.

Theorem 5.3. *For a budgeting game Γ and a superset equilibrium $\boldsymbol{\lambda}$, if $\boldsymbol{\lambda}$ can be reached from the profile $\boldsymbol{\lambda}^0$, where $\lambda^0_i(j) = 0$ for all $i \in N$, $j \in M$, through a sequence of efficient, p-balanced singleton deviations, then $\boldsymbol{\lambda}$ is in the cohesive α-core of Γ.*

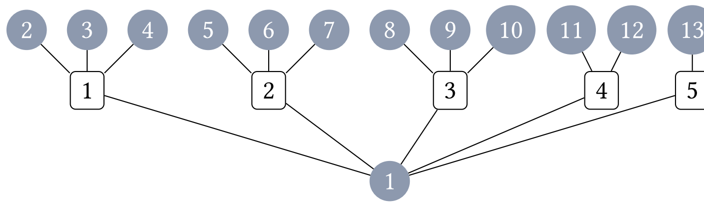

Fig. 5. Budgeting game for Example 12.

Proof. Assume there is a set of players $S \subseteq N$ that is cohesively α-blocking $\boldsymbol{\lambda}$ with $\boldsymbol{\lambda}' = (\boldsymbol{\lambda}'_S, \boldsymbol{\lambda}^0_{-S})$. Then, by definition for each $i \in S$ we have $u_i(\boldsymbol{\lambda}') > u_i(\boldsymbol{\lambda})$. Hence there is at least one $j^* \in (out(\boldsymbol{\lambda}') \setminus out(\boldsymbol{\lambda}))$. Moreover, observe that there is also a deviation $\boldsymbol{\lambda}'' = (\boldsymbol{\lambda}''_S, \boldsymbol{\lambda}^0_{-S})$ such that for all $j \in out(\boldsymbol{\lambda}')$ we have $\lambda_i(j) = 1/|S|$ and for all $j \notin out(\boldsymbol{\lambda}')$ we have $\lambda_i(j) = 0$. This deviation also witnesses that S is cohesively α-blocking and $out(\boldsymbol{\lambda}') = out(\boldsymbol{\lambda}'')$.

Now, let $(\boldsymbol{\lambda}^0, \boldsymbol{\lambda}^2, \dots, \boldsymbol{\lambda}^k)$ be the sequence of profiles generated by the sequence of efficient, p-balanced singleton deviations. Let $\boldsymbol{\lambda}^\ell$ be the first profile such that for at least one voter $i^* \in S$ we have $\sum_{j \in M} \lambda^\ell_{i^*}(j) > b - 1/|S|$. We need to make a few observations: (1) In every profile $\boldsymbol{\lambda}^t$ for $t < \ell$, every player in S has at least $1/|S|$ money left over. Therefore, the players in S have a $1/|S|$-balanced singleton deviation by funding j^*. (2) We know that $u_i(\boldsymbol{\lambda}') > u_i(\boldsymbol{\lambda}') - 1 \geq u_i(\boldsymbol{\lambda}^\ell)$. (3) As $\boldsymbol{\lambda}''$ is a possible deviation, we also know that $\sum_{j \in out(\boldsymbol{\lambda}') \setminus j^*} \lambda''_{i^*}(j) \leq b - 1/|S|$. (3) Finally, as $\boldsymbol{\lambda}^\ell$ was reached only by singleton deviations, we know that $\lambda^\ell_{i^*}(j) \neq 0$ implies $j \in X_{i^*} \cap out(\boldsymbol{\lambda}^\ell)$.

Together, these observations imply that $|X_{i^*} \cap out(\boldsymbol{\lambda}^\ell)| = u_{i^*}(\boldsymbol{\lambda}^\ell) \leq u_{i^*}(\boldsymbol{\lambda}'') - 1$, and:

$$\sum_{j \in X_{i^*} \cap out(\boldsymbol{\lambda}^\ell)} \lambda^\ell_{i^*}(j) > b - \frac{1}{|S|} \geq \sum_{j \in X_{i^*} \cap (out(\boldsymbol{\lambda}'') \setminus j^*)} \lambda''_{i^*}(j).$$

Therefore, there is at least one resource $j \in (X_{i^*} \cap out(\boldsymbol{\lambda}^\ell))$ such that $\lambda^\ell_{i^*}(j) > 1/|S|$. However, this contradicts the assumption that $\boldsymbol{\lambda}^\ell$ was reached by efficient, p-balanced singleton deviations, as a $1/|S|$-balanced singleton deviation would also have been possible for i^*. □

At this point we have stumbled upon yet another game-theoretically defined voting rule satisfying EJR. How does it relate to MES? It is easy to see that each run of MES defines a sequence of efficient, p-balanced singleton deviations, i.e., every co-winner of MES can be reached by such a sequence. The opposite, however, is not true.

Example 12. *Consider the budgeting game depicted in Figure 5 with budget $b = 7/8$. Any sequence of efficient, p-balanced singleton deviations has to start with activating the resources* 1, 2 *and* 3 *with every voter contributing* $1/4$. *However, then there are two possible p-balanced singleton deviations, either activating* 4 *or* 5. *In both cases, player* 1 *would contribute* $1/8$, *all the remaining budget. Therefore, both deviations are efficient. On the other hand,* $\{1, 2, 3, 4\}$ *is the unique winner set under MES.*

What we can thus say is that MES simulates a game where players activate resources one by one, myopically maximizing their utility per unit of money at every turn, up to some tie-breaking. This gives additional motivation to MES, as one can argue that it selects a committee that would be reached by rational, utility maximizing agents in this specific buying process. It is also possible to fully characterize MES in our model, however doing so requires restrictions that are hard to motivate from a game theoretic perspective.

Finally, this way of looking at MES shows that any committee selected by it corresponds to a profile that does not admit a balanced, weakly improving singleton deviation. This proves that the new condition introduced in Section 4.4, which is stronger than EJR+, is indeed satisfiable.

PROPOSITION 5.4. *If a superset equilibrium* $\boldsymbol{\lambda}$ *can be reached from the profile* $\boldsymbol{\lambda}^0$*, where* $\lambda_i^0(j) = 0$ *for all* $i \in N$*,* $j \in M$*, by a sequence of efficient, p-balanced singleton deviations, then* $\boldsymbol{\lambda}$ *does not admit a balanced, weakly improving singleton deviation.*

6 Strong Equilibria in Trees

Though difficult in the general case, showing non-emptiness of the core is possible for restricted classes of multiwinner voting instances relevant in practice [Pierczyński and Skowron, 2022]. We add to this literature by proving the existence of strong equilibria for a novel restricted class of budgeting games (implying non-emptiness of the core for the associated MWV instances), namely, for instances that have a tree-like underlying graph structure.

Consider an instance $\Gamma = \langle N, M, b, \boldsymbol{X} \rangle$ of a budgeting game. Define the corresponding graph $G^\Gamma = (V, E)$ of the game as $V = N \cup M$ and $E = \{(i, j) | i \in N, j \in M \text{ and } j \in X_i\}$. Then Γ has a *tree structure* if G^Γ is a tree (see Figure 6 for an example). Note that if a budgeting game Γ has a tree structure, many players can agree on one resource j, i.e., $0 \leq |\{i \in N | j \in X_i\}| \leq n$, but any subset $S \subseteq N$ of players agrees on at most one resource, i.e., $|\bigcap_{i \in S} X_i| \leq 1$. To analyse a budgeting game Γ that has a tree structure, we assign a level to each player $i \in N$ and each resource $j \in M$, denoted by L_i and L_j respectively. The root of the tree is always a player $i \in N$ with $L_i = 1$. All the relevant resources of this *root player* i are one level down, i.e., $L_j = 2$ for all $j \in X_i$. The levels in the tree alternate between levels of players and levels of resources, so for all $j \in M$ it holds that L_j is even and for all $i \in N$ it holds that L_i is uneven. A representation exists where a player $i \in N$ that is not the root player, $L_i \neq 1$, has only one resource $j \in X_i$ *above* i, i.e., $L_j = L_i - 1$, and can have many (or none) resources $h \in X_i$ *below* i, i.e., $L_h = L_i + 1$, $h \neq j$. We also define leaf players: a player $i \in N$ is a *leaf player* if the resources in X_i that are below i are not relevant to any other player. More formally, if $|X_i| = 1$ or if $\forall j \in X_i$ with $L_j = L_i + 1$, $\nexists h \in N$ such that $j \in X_h$.

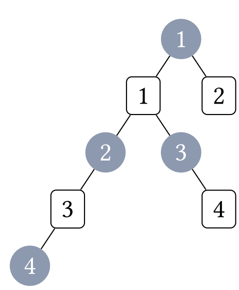

Fig. 6. Example of a budgeting game with a tree structure. Player 1 is the root of the tree and players 3 and 4 are leaf players.

For instances of a budgeting game Γ that have a tree structure, the algorithm TREE-SWEEP computes a strong equilibrium. The idea behind TREE-SWEEP is to compute the strategies of the players in a specific order, and to compute these strategies so that all players play *best response down*. We define the best response down strategy for a player i as follows:

- First, consider resources $j \in X_i$ below i ($L_j = L_i+1$) with $0 \leq \sum_{h \in N} \lambda_h(j) < 1$. If the remaining budget is sufficient, activate these resources precisely, so set $\lambda_i(j) = 1 - \sum_{h \in N} \lambda_h(j)$, in order of lowest to highest contribution required for activating (breaking ties arbitrarily).

ALGORITHM 1: TREE-SWEEP (Γ)

Input: An instance $\Gamma = \langle N, M, b, \boldsymbol{X}\rangle$ of a budgeting game.
Output: A strong equilibrium $\boldsymbol{\lambda}$ for Γ.
Initialize $\boldsymbol{\lambda}$ with $\lambda_i(j) = 0\ \forall i \in N$ and $\forall j \in M$ and define $\bar{L} = \max_{i\in N}\{L_i \mid i \text{ is not a leaf player}\}$.
for *all leaf players i* **do**
  Let player i play the best response down strategy.
end
for $l = \bar{L}, \bar{L} - 2, \ldots, 1$ **do**
  for *players i with* $L_i = l$ *that are not leaf players* **do**
    Let player i play the best response down strategy.
  end
end
return $\boldsymbol{\lambda}$

- Finally, if player i has budget left, i only contributes budget to the resource $j \in X_i$ above i ($L_j = L_i - 1$), if this is (potentially) 'useful', i.e., $\lambda_i(j) = \min\{b - \sum_{h\in N}\lambda_h(j), 1 - \sum_{h\in N}\lambda_h(j)\}$.

The algorithm initializes by computing the strategies of all the leaf players. Then, the algorithm continues by computing the strategies of all the players of level $\bar{L}$, where $\bar{L}$ is the highest level of players that contains at least one player that is not a leaf player. The algorithm continues with the next level of players until the strategy of the root player is computed.

By construction, any profile $\boldsymbol{\lambda}$ computed by TREE-SWEEP (Γ) has three useful properties: (1) every resource in $out(\boldsymbol{\lambda})$ is precisely activated, (2) if a player i spends money on an inactive resource j, then j is above i and (3) if a player i that is not the root player has budget left, then the resource above i is active. More formally:

(1) $\sum_{i\in N}\lambda_i(j) = 1$ for any $j \in out(\boldsymbol{\lambda})$;
(2) If $\lambda_i(j) > 0$ and $j \notin out(\boldsymbol{\lambda})$, then $L_j = L_i - 1$ for any $i \in N$ and $j \in M$;
(3) If $b - \sum_{h\in X_i}\lambda_i(h) > 0$, then either i is the root player or $j \in X_i$ with $L_j = L_i - 1$ is in $out(\boldsymbol{\lambda})$.

Additionally, the profile $\boldsymbol{\lambda}$ computed by TREE-SWEEP is a Nash equilibrium, which will turn out to be useful when proving that $\boldsymbol{\lambda}$ is also a strong equilibrium.

THEOREM 6.1. *For every instance of a budgeting game with a tree structure, TREE-SWEEP computes a Nash equilibrium.*

PROOF. Let Γ be an instance of a budgeting game with a tree structure and let $\boldsymbol{\lambda} = $ TREE-SWEEP(Γ). We show that no player $i \in N$ can unilaterally deviate and strictly improve their utility. This trivially holds for a player i with $X_i \cap out(\boldsymbol{\lambda}) = \emptyset$, as there is no resource $j \in X_i$ that i can activate by setting $\boldsymbol{\lambda}'_i(j) = b$. For players i with $X_i \cap out(\boldsymbol{\lambda}) \neq \emptyset$, we proof that this holds by showing that under $\boldsymbol{\lambda}$ player i activates its relevant resources in order of lowest to highest contribution required. This holds by construction for the root player i. This also holds for players $h \in N$, $h \neq i$, with the resource above them active ($j \in X_h$, $L_j = L_h - 1$ and $j \in out(\boldsymbol{\lambda})$). Consider a player $h \in N$, $h \neq i$, with the resource above them inactive, i.e., $j \in X_h$, $L_j = L_h - 1$ and $j \notin out(\boldsymbol{\lambda})$. Note that h not necessarily activates its relevant resources in order of lowest to highest contribution required. Let $k = \arg\max_{l\in X_h\cap out(\boldsymbol{\lambda})}\{\lambda_h(l)\}$. It could be that for $\lambda'_h(j)$ such that $\sum_{l\in N\setminus\{h\}}\lambda_l(j) + \lambda'_h(j) = 1$, it holds that $\lambda'_h(j) < \lambda_h(k)$. Suppose that h deviates and activates resource j precisely with $\lambda'_h(j)$ and retrieves its contribution to k. Then as h had no budget left under $\boldsymbol{\lambda}$ by (3), the budget that h has left is $\lambda_h(k) - (\lambda'_h(j) - \lambda_h(j))$, which is smaller than $\lambda_h(k)$ as $(\lambda'_h(j) - \lambda_h(j)) > 0$ because $j \notin out(\boldsymbol{\lambda})$. Note that by the best response down strategy, h cannot activate a relevant inactive resource with this budget left, as this would contradict h activating resource k under $\boldsymbol{\lambda}$. Consider

	cohesive					not cohesive				
	MWV	Exists		FIP		MWV	Exists		FIP	
Nash	—					—	✓	(Th. 3.1)	✓	
Superset	—	✓	(by def.)	✓	(by def.)	Price.	✓	(by def.)	✓	(by def.)
α-core	EJR	✓	(Th. 5.1)	✓	(Th. 5.1)	Core	?		✗	(Ex. 10)
strong eq.	—	?		✗	(Ex. 9)	—	?		✗	(Ex. 9)

Table 1. Related MWV concepts, existence and FIP (Finite Improvement Property) for the different equilibria notions that we introduced in this paper. Note that the FIP for Nash equilibria holds for budgeting games in which each player is restricted to only contribute to their relevant resources.

this $\boldsymbol{\lambda}'_h$ (with $\lambda'_h(k) = 0$ and $\lambda'_h(l) = \lambda_h(l)$ for $l \neq j, k$). Note that that the utility of h is unchanged, i.e., $u_h(\boldsymbol{\lambda}'_h, \boldsymbol{\lambda}_{-h}) = u_h(\boldsymbol{\lambda})$, and h now activates its relevant resources in order of lowest to highest contribution required. Finally, note that we only reasoned about players i contributing to relevant resources (resources in X_i). However, we can prove that a Nash equilibrium always exists for budgeting games in which each player is restricted to only contribute to their relevant resources, and that this profile also constitutes a Nash equilibrium in the original game, but omit the proof here. We therefore conclude that $\boldsymbol{\lambda}$ is a Nash equilibrium. □

Theorem 6.2. *For every instance of a budgeting game with a tree structure, Tree-Sweep computes a strong equilibrium.*

So for an instance $\Gamma = \langle N, M, k/n, \boldsymbol{X} \rangle$ of a budgeting game with a tree structure, Tree-Sweep can be used to compute a strong equilibrium $\boldsymbol{\lambda}$ = Tree-Sweep(Γ). And by Theorem 4.4 and the fact that any strong equilibrium is also in the α-core, it follows that $out(\boldsymbol{\lambda})$ is in the core of the associated MWV instance $I_\Gamma = \langle N, A, k, \boldsymbol{A} \rangle$. Significantly, this proves non-emptiness of the core for a restricted domain of MWV instances. Note that this also holds for instances $\Gamma = \langle N, M, k/n, \boldsymbol{X} \rangle$ of a budgeting game with a forest structure, i.e., G^Γ consists out of multiple trees, as Tree-Sweep can be used for each tree in order to compute a strong equilibrium.

7 Conclusion

Our paper provides the first systematic game-theoretic treatment of fairness notions in multiwinner voting. In particular, we characterize priceability, the core, EJR and MES, and find a type of equilibrium that maps activated outcomes onto committees that satisfy EJR+. We have also shown that, unlike cohesive deviations and α-core deviations, cohesive α-core deviations have the finite improvement property, and thus immediately define new classes of voting rules. Finally, we have shown how our game theoretic perspective can be used to find strong equilibria for a restricted class of budgeting games. Table 1 offers a more detailed summary of our results.

The most pressing direction for future work arising from our paper is the study of the novel class of voting rules resulting from cohesive α-core deviations. Another natural extension of our framework would be to allow resources to have different activation thresholds, which could be used to model participatory budgeting. Finally, it would be interesting to identify other graph structures for which strong equilibria exist and that capture common properties of MWV instances.

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