Optimal compilation of parametrised quantum circuits

John van de Wetering[1], Richie Yeung[2,3], Tuomas Laakkonen[3], and Aleks Kissinger[2]

[1]University of Amsterdam
[2]University of Oxford
[3]Quantinuum
October 31st 2023

Parametrised quantum circuits contain phase gates whose phase is determined by a classical algorithm prior to running the circuit on a quantum device. Such circuits are used in variational algorithms like QAOA and VQE. In order for these algorithms to be as efficient as possible it is important that we use the fewest number of parameters. We show that, while the general problem of minimising the number of parameters is NP-hard, when we restrict to circuits that are Clifford apart from parametrised phase gates and where each parameter is used just once, we *can* efficiently find the optimal parameter count. We show that when parameter transformations are required to be sufficiently well-behaved that the only rewrites that reduce parameters correspond to simple 'fusions'. Using this we find that a previous circuit optimisation strategy by some of the authors [Kissinger, van de Wetering. PRA (2019)] finds the optimal number of parameters. Our proof uses the ZX-calculus. We also prove that the standard rewrite rules of the ZX-calculus suffice to prove any equality between parametrised Clifford circuits.

1 Introduction

A quantum circuit is built out of small unitary gates that together make it possible to perform an arbitrary quantum computation. In a *parametrised* quantum circuit, we allow certain quantum gates to be specified by a classical parameter that is determined before running the circuit on a quantum device. Usually parametrised gates are either phase gates or controlled-phase gates, and the parameter, a real number, specifies the phase to be applied. Parametrised quantum circuits are an increasingly important construction for quantum algorithms, especially for near-term applications. For instance, variational algorithms such as QAOA [13] and VQE [24] use a feedback loop between a classical side and a quantum side where the parameters are updated by a classical optimisation procedure, based on measurement outcomes of the quantum device. Each training step of a typical optimisation procedure involves estimating the gradient of the cost function with respect to each parameter (e.g. using the parameter-shift rule [28]). In order to be as efficient as possible with our resources we should hence make sure that we are not

John van de Wetering: john@vdwetering.name, http://vdwetering.name
Richie Yeung: richie.yeung@cs.ox.ac.uk
Tuomas Laakkonen: tuomas.laakkonen@quantinuum.com
Aleks Kissinger: aleks.kissinger@cs.ox.ac.uk

using superfluous parameters. We then would like some classical optimisation algorithm for parametrised quantum circuits that reduces the circuit to a form that has the minimal number of parameters, while still being able to express the same set of unitaries.

Parameter optimisation is also relevant in the field of measurement-based quantum computation (MBQC) [26, 27]. In most literature on MBQC the measurement patterns have to be deterministic regardless of the chosen measurement angles [6, 8, 12]. We can hence view this as a computation parametrised by those measurement angles. In this setting minimising the number of parameters corresponds to minimising the number of measurements needed [6], while still preserving deterministic realisability of the pattern.

There are many results in general circuit optimisation, that try to simplify the presentation of quantum circuits in terms of total number or gates or circuit depth, but also in trying to reduce the count of a specific gate, like the number of CNOTs or T gates. Several of these techniques only include special-case behaviour for phase gates that are Clifford, like the S or Z gate, but not for other types of phases [1, 11, 18, 22, 34]. This means that these techniques also apply to parametrised phase gates, as we can then just treat these as 'black-box' non-Clifford phase gates.

There are only a small number of rewrites known that we can do with such black-box phase gates. Most of these correspond to a 'fusing' of two parameters. The simplest case is when we have two parametrised phase gates next to each other on the same qubit:

$$- \boxed{R_Z(\alpha_1)} - \boxed{R_Z(\alpha_2)} - \quad = \quad - \boxed{R_Z(\alpha_1 + \alpha_2)} - \tag{1}$$

We see that the two different parameters α_1 and α_2 are combined into one gate, so that we can build the same set of unitaries using a single parameter α' by making the identification $\alpha' := \alpha_1 + \alpha_2$. More complicated versions of the same basic idea can be made by using the sum-of-paths approach [2], which allows us to fuse phases that apply to the same parity of qubits [1], or by compiling the circuit into a series of Pauli exponentials and exploiting the Pauli commutation relations [18, 34]. We will refer to these techniques collectively as *phase folding*.

There are also more complicated rewrites that can in principle be performed on parametrised phase gates, for instance by exploiting the different Euler angle decompositions of a single-qubit unitary:

$$- \boxed{R_Z(\alpha_1)} - \boxed{R_X(\alpha_2)} - \boxed{R_Z(\alpha_3)} - \quad = \quad - \boxed{R_X(\alpha_1')} - \boxed{R_Z(\alpha_2')} - \boxed{R_X(\alpha_3')} - \tag{2}$$

Here the parameters on the right depend on those on the left by some trigonometric relations, and are in particular discontinuous as a function of $(\alpha_1, \alpha_2, \alpha_3)$. Hence, when the use-case is for instance QAOA, this might not be desirable to apply, as it transforms the parameter space in pathological ways.

All this then raises a number of questions:

- What is the right notion of equivalence of parametrised quantum circuits?
- What are the possible rewrites we can do to transform parametrised quantum circuits while preserving equivalence?
- Is there an efficient algorithm to find an equivalent parametrised quantum circuit that uses the minimal number of parameters?

In this work, we answer each of these questions in the case of parametrised quantum circuits without repeated parameters. Regarding the first point, we show that a broad set of parameter transformations (analytic functions on the unit circle) actually already forces the relations between parameters in equivalent quantum circuits to be given by simple additive relations that correspond to just 'fusions' of parameters. We answer the second point by finding that the Clifford rewrite rules of the ZX-calculus [4] suffice to prove any equality between parametrised Clifford circuits, under the condition that each parameter occurs uniquely.

Our main result is an answer to the third question: we find that an existing optimisation approach, previous work of some of the authors [18], finds the optimal number of parameters, under the condition that every parameter in the circuit occurs on a unique gate. More formally we show the following:

Theorem 1.1. Given a parametrised circuit which consists of Clifford gates and parametrised Z phase gates each of which is parametrised by a unique parameter, we can efficiently find an equivalent parametrised circuit with an optimal number of parameters. Furthermore, these new parameters correspond to sums and differences of the original parameters, and the algorithm for finding the circuit is that described in [18].

Note that if we drop the requirement here on the non-parametrised gates being Clifford that the problem likely no longer has an efficient solution: we show that when Clifford+T gates are allowed, parameter optimisation is NP-hard. We conjecture that the same is true when parameters are allowed to be used multiple times on different gates, and in fact we show this is the case for optimising *post-selected* quantum circuits with repeated parameters.

2 Parametrised circuits

We will consider a parametrised quantum circuit to be a quantum circuit consisting of gates from a discrete set, together with *parametrised phase gates* $Z[\alpha]$. Here $\alpha \in \mathbb{R}$ is a classical parameter that has to be determined prior to running the quantum circuit. We consider such a parameter to be a complete unknown which can be in any value in $\mathbb{R}$. A parametrised quantum circuit C depending on a vector of values $\vec{\alpha} \in \mathbb{R}^k$ can then be viewed as a map from parameter space $\mathbb{R}^k$ to the space of unitaries $\mathcal{U}$ as $C : \mathbb{R}^k \to \mathcal{U}$. A specific instantiation of C, which we will write as $C[\vec{\alpha}]$, is then just a regular quantum circuit where the phases have been filled in.

In order to consider optimising parametrised quantum circuits, we first need to discuss what it means for parametrised circuits to be equal.

Definition 2.1. We say a circuit C_1 with parameter space $\mathbb{R}^k$ *reduces* to C_2 with parameter space $\mathbb{R}^l$ when there exists a function $f : \mathbb{R}^k \to \mathbb{R}^l$ such that $C_1[\vec{\alpha}] = \lambda(\vec{\alpha})C_2[f(\vec{\alpha})]$ for all $\vec{\alpha} \in \mathbb{R}^k$ where $\lambda : \mathbb{R}^k \to \mathbb{C} \setminus \{0\}$ is a function representing a global scalar $\lambda(\vec{\alpha})$ that may depend on the parameters.

The notion of reduction will be sufficient for us, but a natural stronger property to consider would be *equivalence*, which would require a reduction to exist in both directions, essentially stating that both circuits are equally expressive.

We can then define the main problem we will study in this paper:

Definition 2.2. Parameter optimisation: Given a parametrised quantum circuit C_1 with parameter space $\mathbb{R}^k$, find a parametrised quantum circuit C_2 with parameter space $\mathbb{R}^l$ that it reduces to, such that l is *minimal*.

Without any further restriction on C_1, C_2 and f this problem is likely to be hard, and so we don't expect an efficient solution. In particular, if the discrete gate set we are allowed to use is approximately universal, then parameter optimisation is likely to be hard. Consider for instance the following result regarding parametrised circuits with Clifford+T gates.

Proposition 2.3. Parameter optimisation when the circuits are allowed to contain Clifford+T gates is NP-hard.

Proof. We use an argument similar to [32]. When we allow T gates in our circuit we can construct classical oracles $U_f |\vec{x}, y\rangle = |\vec{x}, f(\vec{x}) \oplus y\rangle$ for a Boolean function $f : \{0,1\}^n \to \{0,1\}$. Using parametrised phase gates $Z(\alpha)$ and $Z(\beta)$ we can then build a circuit implementing the diagonal unitary $|\vec{x}, y\rangle \mapsto e^{i\alpha y + i\beta(f(\vec{x})\oplus y)} |\vec{x}, y\rangle$, by first applying $Z(\alpha)$ on $|y\rangle$, then computing with U_f, applying $Z(\beta)$, and uncomputing U_f. In this case it is clear that the parameters α and β can be fused if and only if f is never satisfiable or always satisfiable. Hence, an oracle for finding the minimal number of parameters for Clifford+T circuits with parametrised phase gates would allow us to solve the NP-complete SAT problem with a little bit of post-processing. □

For this reason we will restrict to just Clifford gates as our allowed discrete gates. But even with just Clifford gates and parametrised phase gates we need to be careful to not make the problem too hard. For instance, if we were to allow the mapping between the parameter space and the concrete phase parameters that get used in the phase gates to be non-identity, then we also quickly run into problems. Consider for instance two parametrised phase gates on the same qubit, which depend on α via $Z(f_1(\alpha))$ and $Z(f_2(\alpha))$. Then if we take $f_1(\alpha) = \alpha$ and $f_2(\alpha) = -\alpha + \frac{\pi}{4}$, then these two phase gates fuse (Eq. (1)) and we get a $Z(\frac{\pi}{4}) = T$ gate, so that optimising such a circuit is again NP-hard.

Hence, we should work with circuits consisting of Clifford gates and parametrised phase gates $Z(\alpha)$ where α exactly corresponds to the value of our parameter space. In this setting we are still allowed to reuse parameters into multiple phase gates in the circuit. This allows us to for instance construct parametrised controlled-phase gates. We suspect however that this problem is also already hard.

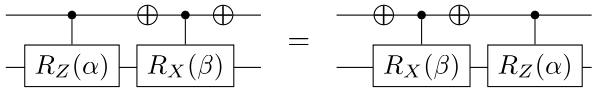
> **Conjecture**: The parameter optimisation problem for Clifford circuits with parametrised phase gates where parameters are allowed to be used in multiple gates is NP-hard.

While we have not managed to prove this, we have proven this result in the setting of post-selected quantum circuits:

Proposition 2.4. Parameter optimisation for post-selected Clifford quantum circuits with repeated parametrised phase gates is NP-hard.

The proof can be found in Appendix A. While it might seem obvious that a problem involving post-selected circuits would be hard, we point out that our main result about optimally minimising parameter count for Clifford+parameter circuits where the parameters occur uniquely *also* applies to post-selected circuits (see Remark 5.5). Hence, the optimisation problem is efficiently solvable for post-selected circuits when each parameter occurs uniquely, but it is NP-hard when the parameters are allowed to repeat.

While our proof of hardness does not extend to unitary circuits, we still expect the problem to be hard. We note for instance that for unitary circuits with repeated parametrised phase gadgets there are additional rewrites of the circuits we should consider that do not play a role in the case where every parameter is used once, such as:



(3)

I.e. this says that a controlled Z-phase gate commutes with an anti-controlled X-phase gate. This rewrite rule was needed to prove completeness for universal ZX-diagrams in [16], and hence seems to be a 'harder' rule then the ones we need. This rewrite rule is not covered by our optimality result, since the ZX-diagram of a controlled-phase gate requires the parameter to be used three times [9].

This discussion on the hardness of the problem then brings us then to the types of circuits we will consider in this paper.

Definition 2.5. A *parametrised Clifford circuit* (PCC) is a parametrised circuit consisting of Clifford gates and $Z[\alpha]$ parametrised phase gates where each parameter is used in at most one gate.

One way to interpret this condition on each parameter occurring uniquely is that all the parametrised phase gates are truly 'black boxes': we know nothing about their values, not even whether two of the parametrised phase gates are implementing the same phase. In any case it is clear that any circuit with repeated parameters reduces to one where the parameters occur uniquely: for every repeat occurrence we instead introduce a fresh parameter. Note that this setting of unique parameters is also a natural setting to consider when talking about determinism in measurement-based quantum computing; see Section 6.

We will also restrict the types of functions f that may appear in a reduction of a parametrised Clifford circuit. Instead of treating the parameters as real numbers, it makes sense to treat them as phases modulo 2π, and hence as elements of the unit circle S^1. It then additionally makes sense to require the parameter map $f : (S^1)^k \to (S^1)^l$ to be continuous: many use-cases of parametrised quantum circuits require us to take derivatives and slightly change the parameters bit by bit. A reduction then shouldn't break these properties. In fact, we will go beyond that and assume that the parameter map is *smooth*: infinitely differentiable. Since there also exist pathological smooth functions, we will make our final assumption, which is that parameter maps are *analytic*, meaning they can be written as convergent infinite power series. Since the details of this are a bit technical to formalise, we present them in Appendix B. We just present here the main conclusion.

Proposition 2.6. Any parameter map $\Phi : \mathbb{R}^k \to \mathbb{R}^l$ consisting of analytic functions on the unit circle (as formalised in Appendix B) is equal to $\Phi(\vec{\alpha}) = M\vec{\alpha} + \vec{c} \bmod 2\pi$ where M is a matrix of integers and $\vec{c} \in \mathbb{R}^l$ is some constant.

Hence, we see that what a priori seems like quite a general class of transformations actually reduces to something quite simple.

Definition 2.7. We say a PCC C_1 *affinely reduces* to C_2 when for all $\vec{\alpha}$ we have $C_1[\vec{\alpha}] = C_2[M\vec{\alpha} + \vec{c}]$ for some integer matrix M and constant $\vec{c}$.

Note that when we have an affine reduction $C_1[\vec{\alpha}] = C_2[M\vec{\alpha} + \vec{c}]$, that a parameter α_j gets copied to all the parametrised gates in C_2 where the jth column of M is non-zero. Hence, such an affine reduction can break the condition on having the parameter only appear in one place. For this reason we define a stricter version of reduction.

Definition 2.8. We say an affine reduction is *parsimonious* when the matrix M appearing in the reduction has at most 1 non-zero element in each column.

Definition 2.9. The **affine parameter optimisation problem for Clifford circuits** is to find, given a parametrised Clifford circuit C_1, a parametrised Clifford circuit C_2, such that C_1 parsimoniously affinely reduces to C_2 and C_2 has a minimal number of parameters amongst those circuits that C_1 parsimonious affinely reduces to.

The main result of this paper is then the following theorem.

Theorem 2.10. There is an efficient algorithm that constructively solves the affine parameter optimisation problem for Clifford circuits.

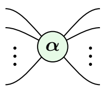
This algorithm is the T-count optimisation algorithm of [18]. In that paper it was already noted that their algorithm can be used to reduce the number of parameters in a circuit. Our results show that it is in fact *optimal* in doing so (given suitable restrictions on the type of reductions that are allowed as discussed above).

Note that the assumption of the reduction being parsimonious seems quite strong. We suspect however that this condition is not needed, and that for any parametrised Clifford circuit D_1 for which there is an affine reduction to D_2, that there is then also a parsimonious affine reduction to D_2. We have however only managed to prove this in certain special cases (see Section 5.1).

3 The ZX-calculus

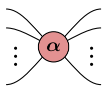
The algorithm that implements the optimal parameter reduction strategy uses the ZX-calculus [9, 10], and our proof is also most naturally expressed in the ZX-calculus. We will hence give a brief introduction to the ZX-calculus and the rewrite strategy of [18]. For an in-depth review, see [31].

ZX-diagrams are built out of two types of generators, called *spiders*. We have *Z-spiders*:



$$= \; |0\cdots0\rangle\langle0\cdots0| + e^{i\alpha}\,|1\cdots1\rangle\langle1\cdots1|\,. \quad (4)$$

And *X-spiders*:



$$= \; |+\cdots+\rangle\langle+\cdots+| + e^{i\alpha}\,|-\cdots-\rangle\langle-\cdots-| \quad (5)$$

The phase $\alpha \in \mathbb{R}$ can be any real number and we take it modulo 2π. When $\alpha = 0$ we don't write it on the spider. Spiders can have any number of inputs, including zero, and any number of outputs, also including zero. In particular, the zero-input zero-output spider corresponds to just a number:  $= 1 + e^{i\alpha}$. Another relevant special case is the 1-input 1-output X-spider with a π phase, which is the Pauli X (the NOT gate).

We can compose spiders together either vertically, corresponding to taking the tensor products of their linear maps, and horizontally, corresponding to regular composition of linear maps. Such a composition of spiders is called a *ZX-diagram*. When the phases on the spiders are allowed to be arbitrary, ZX-diagrams are *universal*, meaning that they can represent any linear map from $\mathbb{C}^{2^n}$ to $\mathbb{C}^{2^m}$. If instead we restrict the phases to be multiples of $\frac{\pi}{4}$, then the diagrams are universal for linear maps we can construct from Clifford+T circuits including ancilla preparations and postselections. If we further restrict to multiples

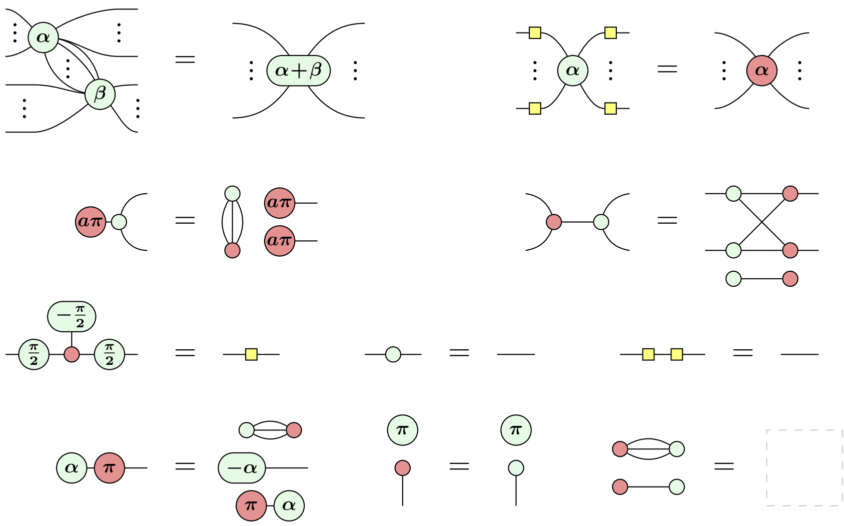


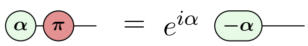
Figure 1: A complete set of rules for the Clifford fragment of the ZX-calculus. Here $a \in \{0, 1\}$ is a Boolean variable, and the equations involving α or β hold for any $\alpha, \beta \in \mathbb{R}$, not just the Clifford angles.

of $\frac{\pi}{2}$, we get Clifford circuits, stabiliser state preparations, and stabiliser projections. In particular, we say a spider is *Clifford* when α is a multiple of $\frac{\pi}{2}$.

The usefulness of ZX-diagrams comes from the set of rewrite rules that we can apply to them that preserve their semantics as linear maps. First of all, the interpretation of the diagram only depends on the connectivity, and we can hence deform the diagram in the plane as we wish. This means we can treat the diagrams as undirected graphs. Second, there is a set of local rewrite rules that preserve the semantics; see Figure 1. These rules suffice to prove any true equality between Clifford diagrams, and hence are *complete* for the Clifford fragment [4]. Note however that these rules do not prove all equalities between general diagrams where the phases are not restricted to multiples of $\frac{\pi}{2}$.

Here we drew a yellow square for the *Hadamard gate*, for which we can derive the equation on the left in Figure 1 as its definition.

Note that most of these rules only apply when the spiders have specific phases, except for *spider fusion* (top left), *colour change* (top right) and *π-pushing* (bottom left), which are sound for any value $\alpha, \beta \in \mathbb{R}$. In particular π-pushing will be relevant to us, so let us note that we can also directly write the scalar diagrams as numbers to present this rule as:


(6)

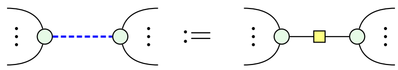
This just says that $X(|0\rangle + e^{i\alpha}|1\rangle) = e^{i\alpha}(|0\rangle + e^{-i\alpha}|1\rangle)$.

Because we can treat ZX-diagrams as undirected graphs and the rewrite rules apply regardless of which wires are inputs or outputs, we will in this paper often work with *states* for convenience: diagrams that have no inputs. This corresponds to using the Choi-Jamiołkowski isomorphism, and in the diagram is represented by bending the input wires to be outputs instead. When we introduce parametrised diagrams in Section 4 we will usually denote the parameters as being connected to inputs, so that it is a 'parametrised state'.

3.1 Local complementation and pivoting

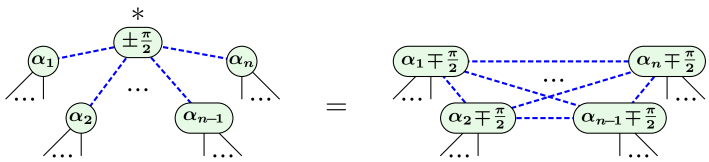
For this paper it is important that we can bring Clifford diagrams to particular pseudo-normal forms, so we will review some useful rewriting tools that we need to reduce to these normal forms.

First, instead of working with arbitrary ZX-diagrams, it is often helpful to restrict to *graph-like* ZX-diagrams (which can be done without loss of generality). These are diagrams where all spiders are Z-spiders, and every spider is 'maximally fused', meaning that the only connections left are via Hadamard gates [11]. Parallel edges and self-loops can be removed, so that we are left with a simple graph where the vertices are the Z-spiders and the edges correspond to Hadamards, which we will refer to as *Hadamard edges* from now on. Each vertex is additionally labeled by the phase of the spider. For clarity we will write Hadamard edges as blue-dotted wires:



(7)

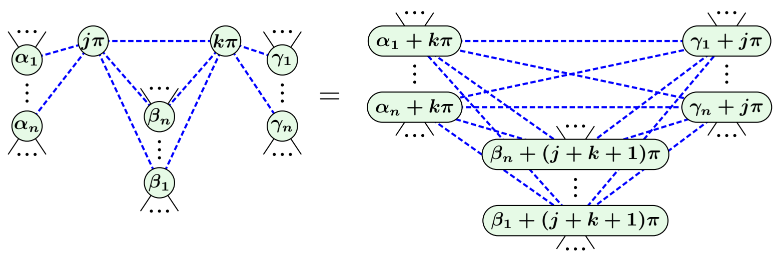
The spiders that are connected to an input or output wire of the diagram we will refer to as *boundary* spiders, while the spiders that are only connected to other spiders we will call *internal* spiders. The rewrite strategy of [11] describes how to remove all the internal Clifford spiders. First, the *local complementation* rewrite rule removes any internal spider with a phase of $\pm\frac{\pi}{2}$:



(8)

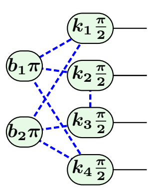
Note that this equation is actually only accurate up to some scalar value, depending on the number of neighbours n and the phase $\pm\frac{\pi}{2}$, but as this value is representable by a scalar Clifford diagram it will not be important to us.

The second equation is called the *pivot* rule, and it can remove any pair of connected internal spiders that both have a 0 or π phase:



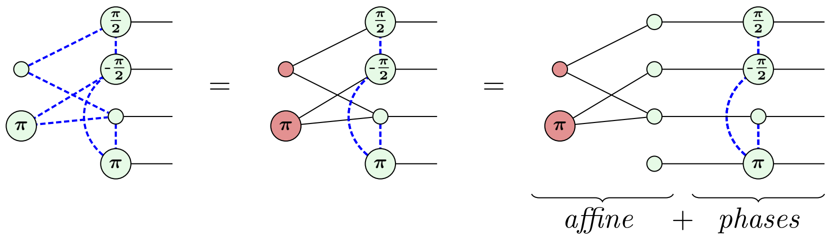
(9)

Hence, if we started with a Clifford diagram, one where all the phases are multiples of $\frac{\pi}{2}$, then the only internal spiders left, after removing those with a $\pm\frac{\pi}{2}$ phase with Eq. (8) and the remaining ones with a 0 or π phase that are connected, are those that have a 0 or π phase and are not connected to any other internal spider. Assuming for simplicity that the diagram only has outputs and no inputs, we can write such a diagram in two layers, with first the internal spiders, and then the boundary spiders:

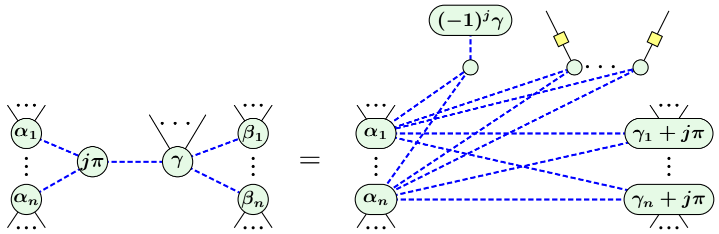


(10)

We say such a diagram is in *affine with phases* form, or AP form. This is because we can see such a diagram as encoding an affine subspace, together with a phase polynomial [20, 25]:

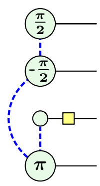


Using an additional rewrite we can also get rid of the final internal spiders, at the cost of introducing Hadamard edges on the output wires. We call this rewrite rule a *boundary pivot*, as it applies a pivot between an internal spider and a boundary spider:



(11)

This introduces a new element to the diagram that we call a *phase gadget*, that we will have more to say about in the next section. But for now note that if $\gamma = \pm\frac{\pi}{2}$, that we can remove it with a local complementation, after which its neighbour will also have a $\pm\frac{\pi}{2}$ phase so that it can also be removed. If instead γ is 0 or π then we can remove both of these spiders with a regular pivot. We then see that in both cases we can again remove both introduced spiders. Hence, by repeating the boundary pivot we can get rid of the final internal spiders. Hence, when we start with a Clifford diagram, we can efficiently reduce to a pseudo-normal-form where there are no internal spiders, but there may be Hadamard gates on the output wires:

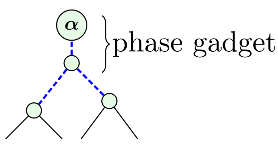


(12)

This is a graph state with local Cliffords (GSLC). But note that these local Cliffords are from a small set. They are either a power of an S gate (a $k\frac{\pi}{2}$ phase), or a Hadamard gate, or a Z gate (a π phase) followed by a Hadamard.

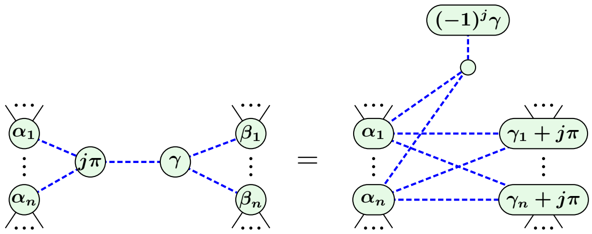
3.2 Optimising non-Clifford diagrams

The above rewrite strategy can also be applied when we have non-Clifford phases in the diagram. In that case local complementation still removes all internal spiders with a $\pm\frac{\pi}{2}$ phase, and pivoting still removes all connected internal spiders with a 0 or π phase. If an internal $j\pi$ phased spider is connected to some boundary spider with a Clifford phase then we can also get rid of it with a boundary pivot as described above. However, now we can have internal 0 or π spiders that are connected only to other spiders with a non-Clifford phase. In that case we can't remove the spider, but we can make it part of a *phase gadget*:



(13)

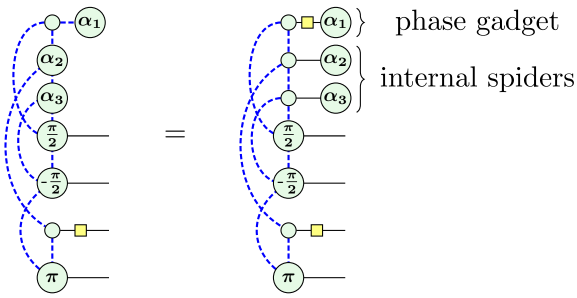
Namely, if an internal $j\pi$-phased spider is connected to a non-Clifford spider, we apply a *gadget pivot*:



(14)

If instead it is only connected to boundary non-Clifford spiders, then we apply the boundary pivot of Eq. (11), but now we don't remove the phase gadget. Note that in both cases the phase gadget ends up connected to what the $j\pi$ spider was connected to originally. As we assumed that this spider was only connected to non-Clifford spiders, we see that when we repeat this procedure that phase gadgets never end up connected to each other.

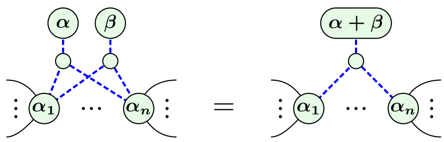
After we have done these operations whereever we can, we see that the only internal spiders have a non-Clifford phase or are part of a phase gadget with a non-Clifford phase. We can then write this new diagram also as a GSLC, but now where some of the outputs are plugged with a non-Clifford phase:



(15)

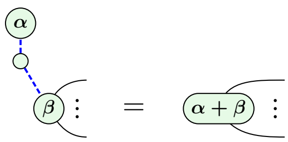
Hence, we see that the phase gadgets correspond to an output where there is a Hadamard and an effect corresponding to the phase, while for internal non-Clifford spiders we can view them as appearing as effects on output wires where there is no Hadamard in between.

Note that none of these steps so far actually remove any of the non-Clifford phases. To do that, we use *gadget fusion* [18]:



(16)

This applies whenever two phase gadgets have exactly the same set of neighbours, and it fuses them into one phase gadget. The other rule that combines non-Clifford phases applies instead to a phase gadget that has exactly one neighbour:



(17)

Additionally, any internal spider or phase gadget that has no neighbours only contributes a scalar to the overall diagram and hence can also be removed. Note that these fusions of non-Clifford phases can result in a spider that has a Clifford phase. For instance, two $\frac{\pi}{4}$ phases combine into a $\frac{\pi}{2}$ phase. In that case we can us the previously described rewrites to remove this newly acquired Clifford spider.

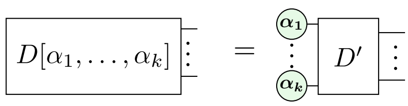
The set of rewrites we have described here and the rewrite strategy it implies was proposed in [18]. To use it to optimise diagrams, in [18] they used the technique of *phase teleportation* to let it inform when phases in the original circuit were allowed to be fused. Later, in [6] it was shown that a quantum circuit can also be directly extracted from the diagram using techniques from measurement-based quantum computing. This procedure preserves the number of non-Clifford phases in the diagram.

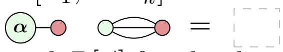
As it will be important for us later, let us describe the type of pseudo-normal-form the rewriting procedure from [18] gives: we get a graph state where on each output we have either a $k\frac{\pi}{2}$ phase, or a $Z^a H$ unitary where $a \in \{0, 1\}$, and some of the outputs can be plugged by a non-Clifford phase. Those plugged spiders with a Hadamard on them are phase gadgets and have at least 2 neighbours. Each phase gadget additionally has a unique set of neighbours (since if two gadgets had the same neighbours, we would have fused them).

4 Parametrised circuits in the ZX-calculus

In order to work with parametrised circuits in the ZX-calculus we will define the notion of a parametrised diagram.

Definition 4.1. A *parametrised diagram with k parameters D* is a ZX-diagram that is entirely Clifford, apart from the phases $\vec{\alpha} = (\alpha_1, \ldots, \alpha_k)$ that are each present *at most once* in the diagram. That is, for each choice of $\alpha_j \in \mathbb{R}$ we have



where the equality is on-the-nose (so with accurate scalars), and D' is a Clifford diagram. We wrote the diagram here as a state (no input wires) without loss of generality. Note that if some phase α_j is actually not present in $D[\alpha_1, \ldots \alpha_k]$ that we can still find a Clifford diagram D' such that the above holds, as . We will write D for the ZX-diagram with 'abstract' parameters, and $D[\vec{\alpha}]$ for the diagram instantiated with a particular choice of values $\alpha_1, \ldots \alpha_k$.

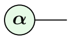
The condition here that each phase only appears at most once is important, as it means that on the input wires of D', each has a phase we can choose independently. If some phase α_j appeared multiple times in D, then to 'extract' it out of the Clifford portion would mean that we have a number of input wires that are 'correlated'. As a result we would only have information about the symmetric subspace of these wires.

Remark 4.2. Scalar factors will be important in this paper. Every equality written here will hence be meant to be *on the nose*, meaning that they represent equal linear diagrams, even with the correct global phase.

As we will want to consider reductions between parametrised diagrams, it will be helpful to first note a couple of properties regarding equalities between parametrised diagrams and what the ZX-calculus can prove about these.

4.1 Completeness of the Clifford+parameters fragment

We can straightforwardly prove that the ZX-calculus is complete for parametrised Clifford diagrams.

Definition 4.3. Write $|+_\alpha\rangle =$  $= |0\rangle + e^{i\alpha}|1\rangle$.

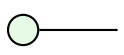
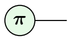
Note that in this definition we use unnormalised states. Hence $|+_0\rangle = \sqrt{2}\,|+\rangle$.

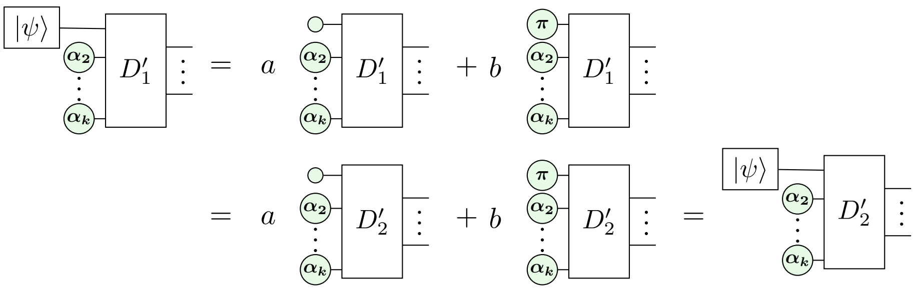
Lemma 4.4. Let $\alpha, \beta \in \mathbb{R}$ be two different phases (modulo 2π). Then $|+_\alpha\rangle$ and $|+_\beta\rangle$ form a basis. In particular, writing $|+_\gamma\rangle = a\,|+_\alpha\rangle + b\,|+_\beta\rangle$ we have $a + b = 1$ and

$$a \;=\; \frac{e^{i\gamma} - e^{i\beta}}{e^{i\alpha} - e^{i\beta}} \qquad\qquad b \;=\; \frac{e^{i\alpha} - e^{i\gamma}}{e^{i\alpha} - e^{i\beta}} \tag{18}$$

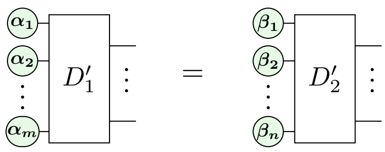
In the special case of $\alpha = 0$ and $\beta = \pi$ we have $a - b = e^{i\gamma}$.

Proposition 4.5. The rules of Figure 1 are complete for parametrised Clifford diagrams: Let D_1 and D_2 be two parametrised diagrams with the same number of parameters such that $D_1[\vec{\alpha}] = D_2[\vec{\alpha}]$ for every choice of $\alpha_1, \ldots, \alpha_k$. Then $D_1' = D_2'$ and $D_1[\vec{\alpha}]$ and $D_2[\vec{\alpha}]$ can be uniformly rewritten into each other using the ZX-calculus Clifford rewrite rules.

Proof. Let $|\psi\rangle = a\,|+_0\rangle + b\,|+_\pi\rangle$ be an arbitrary (not necessarily normalised) qubit state. Note that we can write $|\psi\rangle = a$  $+\, b$ . Hence:



As these two linear map agree on whatever state we put in, they must be equal:



We can repeat this procedure for all the other α_j to conclude that $D_1' = D_2'$. As these are both Clifford diagrams and they represent the same linear map, by Clifford completeness there is a sequence of Clifford rewrites that transforms D_1' into D_2'. When we plug in some α_j phases into these diagrams, this set of rewrites will still be applicable. Hence, there is a set of Clifford rewrites transforming $D_1[\vec{\alpha}]$ into $D_2[\vec{\alpha}]$. $\square$

Remark 4.6. In this result we didn't have to use the fact that these diagrams were equal for any choice of α here. It suffices for the diagrams to agree on just two different values of α_j for each j. This is because $\{|+_\alpha\rangle, |+_\beta\rangle\}$ forms a basis of the qubit state space if $\alpha \neq \beta$. This is a special case of the possibility to verify parametrised equations using a finite number of cases of [21]. In particular, if we had repeated parameters, then if a parameter was repeated k times, we would have to verify equality of the diagrams at $k+1$ different values of the parameter. This is because the symmetric subspace of k qubits is $(k+1)$-dimensional.

Remark 4.7. Note that these completeness results require each parameter to appear freely in the diagram, and hence that we can't have repeated parameters. There are equations involving repeated parameters that cannot be proven by the Clifford rewrite rules, such as supplementarity [23]. In [17], it is shown that when we allow $\frac{\pi}{4}$ phases, so that the diagrams correspond to the Clifford+T fragment, that we can then prove any equation involving repeated parameters (what they call 'linear diagrams with constants in $\frac{\pi}{4}$'), if we use the extended complete axiomatisation for Clifford+T ZX-diagrams.

In Proposition 4.5 we had to assume that the diagrams were equal on the nose. But when we are dealing with unitary parametrised circuits, we only care about the exact unitary up to global phase. A stronger result would hence be to be able to rewrite diagrams that are only equal up to a global phase, where this phase may depend on the value of the parameters. We hence care about the situation where $D_1[\vec{\alpha}] = \lambda(\vec{\alpha})D_2[\vec{\alpha}]$ for all $\vec{\alpha}$ and $\lambda(\vec{\alpha})$ is a phase depending on the value of α. We will work in a slightly more general setting then that and assume that $\lambda : \mathbb{R}^n \to \mathbb{C}^n$ is just any function mapping phases into a scalar value (which is also allowed to be zero a priori). We can then prove an analogous result to Proposition 4.5, but to do that we will need to make an assumption on the parameters.

Definition 4.8. Let D be a parametrised diagram with parameters $\alpha, \vec{\alpha}$. We say the parameter α is *trivial*, if there is a choice of Clifford angles $\vec{\alpha} \in \mathbb{Z}[\frac{\pi}{2}]^n$ and two distinct choices of value β and γ of α such that the diagrams $D[\beta, \vec{\alpha}]$ and $D[\gamma, \vec{\alpha}]$ are proportional to each other. Otherwise we call the parameter *non-trivial*.

This definition is motivated by the following result.

Lemma 4.9. Suppose α is a trivial parameter in a parametrised diagram $D[\alpha, \vec{\alpha}]$. Then $D[\beta, \vec{\alpha}]$ is proportional to $D[\gamma, \vec{\alpha}]$ for every choice of β, γ where $\vec{\alpha}$ is some Clifford choice of the other parameters.

Proof. Let β and γ be two choices for the trivial parameter α such that the associated diagrams are proportional to each other. Write $D[\beta, \vec{\alpha}] = \lambda D[\gamma, \vec{\alpha}]$ where λ denotes the proportionality scalar. Let δ be any other phase and let a and b be the values such that $|+_\delta\rangle = a\,|+_\beta\rangle + b\,|+_\gamma\rangle$. These values exist because $|+_\beta\rangle$ and $|+_\gamma\rangle$ form a basis. Then $D[\delta, \vec{\alpha}] = aD[\beta, \vec{\alpha}] + bD[\gamma, \vec{\alpha}] = (a\lambda + b)D[\gamma, \vec{\alpha}]$. Hence, the diagram for δ is proportional to that of γ. □

This lemma says that if a diagrammatic parameter does not change the diagram for two choices of the parameter, then it does not change the diagram for *any other* choice of that parameter as well. Note that the converse says that if a parameter is *non-trivial*, then that must mean that for any choice of the other parameters, changing the parameter must result in a different linear map. We primarily care about diagrams coming from a quantum circuit, in which case every parameter is necessarily non-trivial.

Proposition 4.10. Let D be a parametrised diagram that comes from a quantum circuit which contains parametrised phase gates. Then every parameter is non-trivial.

Proof. Pick some parameter α. As D comes from a quantum circuit we can then write it as $D = U_2 \circ Z_1[\alpha] \circ U_1$ for some unitaries U_1 and U_2 (built out of the gates appearing before and behind the $Z[\alpha]$ gate), and where we write $Z_1[\alpha]$ as shorthand for the $Z[\alpha]$ gate appearing on the first qubit. Note that while U_1 and U_2 can depend on the other parameters, they do not depend on α as the parameter only appears in one gate. Let $D[\alpha]$ and $D[\alpha']$ be the diagram instantiated with the specific choices of α and α'. We then claim that $D[\alpha] \circ D[\alpha']^\dagger$ is not equal to the identity. To see this note that $D[\alpha] \circ D[\alpha']^\dagger = U_1 \circ Z_1[\alpha - \alpha'] \circ U_1^\dagger$. Hence, if this were equal to the identity, then

$$\mathrm{id} \;=\; U_1^\dagger \circ \mathrm{id} \circ U_1 \;=\; U_1^\dagger \circ U_1 \circ Z_1[\alpha - \alpha'] \circ U_1^\dagger U_1 \;=\; Z_1[\alpha - \alpha'].$$

But we know that this is not the identity when $\alpha \neq \alpha'$. □

In Appendix C we prove the following result that shows that when two diagrams are equal up to some scalar that may vary depending on the parameters, that actually this scalar does *not* depend on the parameters and is just some constant.

Proposition 4.11. Let D_1 and D_2 be (not necessarily Clifford) parametrised diagrams with the same number of parameters and suppose $D_1[\vec{\alpha}] = \lambda(\vec{\alpha})D_2[\vec{\alpha}]$ for some non-zero scalar function $\lambda(\vec{\alpha}) \in \mathbb{C}$, and suppose that all the parameters of D_1 are non-trivial. Then $D_1' = \lambda' D_2'$ for some constant scalar λ'.

Using this, it is then straightforward to get the completeness result for parametrised Clifford diagrams, allowing equality to vary per parameter.

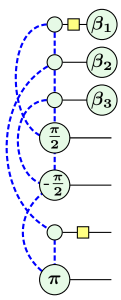
Theorem 4.12. Let D_1 and D_2 be two parametrised diagrams with the same number of parameters and where all the parameters of D_1 are non-trivial such that $D_1[\vec{\alpha}] = \lambda(\vec{\alpha})D_2[\vec{\alpha}]$ for all $\vec{\alpha}$ for some scalar function $\lambda(\vec{\alpha}) \in \mathbb{C}$. Then $D_1' = C \otimes D_2'$ for some Clifford scalar C and we can uniformly rewrite $D_1[\vec{\alpha}]$ into $C \otimes D_2[\vec{\alpha}]$ using Clifford rewrites.

4.2 Optimising parametrised Clifford circuits

The optimisation strategy for diagrams containing non-Clifford phases of Section 3.2 applies equally well to parametrised Clifford diagrams. We just treat spiders that contain a phase depending on a parameter as a non-Clifford. The local complementation rewrite Eq. (8) can add a $\pm\frac{\pi}{2}$ phase to a spider containing a parameter, while the pivot rewrite rule Eq. (9) can add a π phase. The boundary pivot (11) and gadget pivot (14) can flip the parameter α to $-\alpha$, and finally the gadget fusion (16) and neighbour fusion (17) can add together the phases in two spiders so that we can end up with expressions like $\alpha + \beta$ on a single spider.

We see then that if we started with parameters $\alpha_1, \ldots, \alpha_k$ that we then end up with spiders that can have an expression like $k\frac{\pi}{2} + (-1)^{a_1}\alpha_{j_1} + \cdots + (-1)^{a_l}\alpha_{j_l}$ for some different indices $j_1, \ldots, j_l$. We will then label this expression by a new parameter name β. Note that since none of the rewrite rules copy the phase of a non-Clifford spider that each parameter appears uniquely on some spider, and hence the new parameters $\beta_1, \ldots, \beta_{k'}$ we get are all independent. In particular, the new parameters $\vec{\beta}$ are given by an affine transformation $\vec{\beta} = P\vec{\alpha} + \vec{c}$, where $\vec{c} = (b_1\frac{\pi}{2}, \ldots, b_{k'}\frac{\pi}{2})$ is a vector of Clifford phases, P is a matrix only containing entries in $\{0, 1, -1\}$, and each column of P contains a single non-zero element (corresponding to each α_j appearing in a unique location).

We are then left with a diagram that is a 'plugged' GSLC, where the only things plugged are our new parameters β_j:



(19)

It was established in [6] that if our starting diagram is a circuit with non-Clifford phase gates, that then this rewrite strategy produces a diagram that can efficiently be transformed back into a circuit with the same number of non-Clifford phases as in the optimised diagram. The paper [6] assumed these were unknown non-Clifford phases, but the same argument continues to hold for parametrised phases, as the circuit extraction algorithm is agnostic to the specific non-Clifford values. Note that the properties of the graph described in Section 3.2 carry over, and that in particular every parameter that is connected via a Hadamard to a spider has at least two neighbours, and that its neighbourhood does not match that of any other parameter connected via a Hadamard. This is important for the proof of optimality in the next section.

The sequence of rewrites that constitutes this algorithm is not unique. There are usually many choices of where to apply the local complementations and pivots that remove spiders, and these different choices lead to different final diagrams. However, as the results of the next section will show, regardless of the choice of rewrites, the algorithm finds the minimal number of parameters.

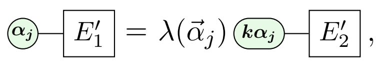
5 Proving optimality

We will now show that the algorithm for minimising parameter counts is optimal, in the sense that it is the best possible among the parsimonious affine reductions.

First, we show that if the parameters are non-trivial, then any parsimonious affine reduction can't multiply a phase α to something like 2α: only α or $-\alpha$ is allowed.

Lemma 5.1. Let $D_1[\vec{\alpha}] = \lambda(\vec{\alpha})D_2[P\vec{\alpha} + \vec{c}]$ be a parsimonious affine reduction where $P \in \mathbb{Z}^{m\times n}, \vec{c} \in \mathbb{R}^m$, and the parameters of D_2 are non-trivial. Then the coefficients of P are either 1, -1, or 0.

Proof. First suppose α_j in D_1 is trivial. In that case the non-trivial parameters of D_2 cannot depend on it, so the coefficients of the j-th column of P must be 0. Now suppose α_j is non-trivial. By the parsimonious property of P, it is mapped to a unique parameter β_i in D_2. Write $\vec{\alpha}_j$ for the indicator vector $(\vec{\alpha}_j)_l = \delta_{jl}\alpha_j$ where we set every other parameter to zero, and define $\vec{\beta}_i$ similarly. Define the one-parameter diagrams $E_1[\alpha_j] := D_1[\vec{\alpha}_j]$ and $E_2[\beta_i] := D_2[\vec{\beta}_i + \vec{c}]$. Note that α_j and β_i are non-trivial parameters in E_1 and E_2. Now let $k = P_{ij}$ be the coefficient corresponding to the mapping of α_j to β_i. Then

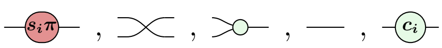


If $k \notin \{1, -1, 0\}$ then note that

$$E_1[2\pi/k] = \lambda' E_2[k2\pi/k] = \lambda' E_2[2\pi] = \lambda' E_2[0] = \lambda' E_1[0]$$

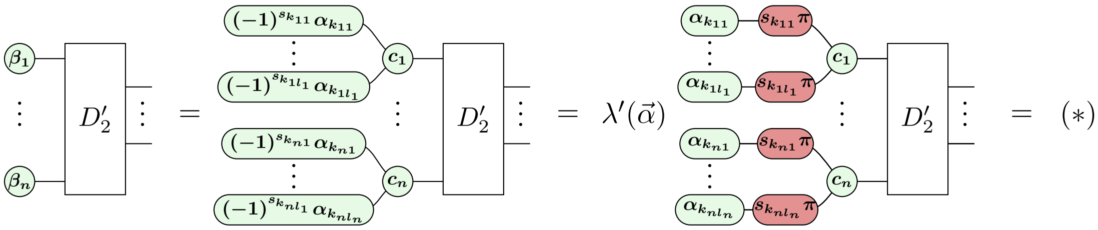
for some constant λ'. Hence, $E_1[2\pi/k] \propto E_1[0]$, so that α_j must be trivial in E_1 and hence in D_1, which is a contradiction. We conclude that the coefficients of the j-th column are either 1, -1, or 0. □

Now that we know that such a reduction is particularly simple, we can construct a ZX-diagram that implements it.

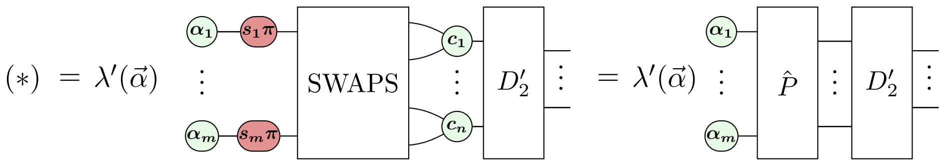
Lemma 5.2. Let $D_1[\vec{\alpha}] = \lambda(\vec{\alpha})D_2[P\vec{\alpha} + \vec{c}]$ be a parsimonious affine reduction where all the parameters of D_1 and D_2 are non-trivial. Then there is a linear map $\hat{P} \in \mathbb{C}^{2^m\times 2^n}$ constructed using  such that $\hat{P} \bigotimes_{i=1}^{n} |+_{\alpha_i}\rangle \propto \bigotimes_{j=1}^{m} \left|+_{\beta_j}\right\rangle$ where $\vec{\beta} = P\vec{\alpha} + \vec{c}$. In addition $D_1' = \lambda' D_2' \circ \hat{P}$ for some constant scalar λ'.

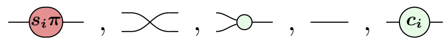
Proof. Write $\vec{\beta} = P\vec{\alpha} + \vec{c}$. By Lemma 5.1, P only contains elements from $\{1, 0, -1\}$, and there is at most one non-zero element in every column. Because all the parameters of D_1

are non-trivial, a column cannot be all zero, so that each column contains exactly 1 non-zero element, either 1 or -1. Let s_i be 1 if the non-zero element in the ith row is -1, and otherwise let s_1 be zero. Then for each j we have $\beta_j = (-1)^{s_{kj1}}\alpha_{k_{j1}} + \cdots + (-1)^{s_{kjl_j}}\alpha_{k_{jl_j}} + c_j$ where l_j is the number of non-zero elements in the jth row of P and the $k_{j1}, \ldots, k_{jl_j}$ enumerate those positions. Hence:



Here $\lambda'(\vec{\alpha}) = \prod e^{is_i\alpha_i}$ is the phase that is needed for the rewrite $X\,|+_\alpha\rangle = e^{i\alpha}\,|+_{-\alpha}\rangle$. Note that for each i there is a unique k_{jm} that is equal to i, so that α_i occurs uniquely in the equation above. Hence, by using some swaps we can write:



Here $\hat{P}$ is defined as the combination of elements from  that makes this equation work. We now have

$$D_1[\vec{\alpha}] \;=\; \lambda(\vec{\alpha})D_2[\beta] \;=\; \lambda(\vec{\alpha})\lambda'(\vec{\alpha})(D_2 \circ \hat{P})[\vec{\alpha}].$$

We see then that Proposition 4.11 applies, and that hence $\lambda(\alpha)\lambda'(\alpha)$ must be some constant scalar λ', and that $D'_1 = \lambda' D'_2 \circ \hat{P}$. □

We are now almost ready to prove the main result. We just need the following lemma about stabilisers of weight 2 of graph states with local Cliffords.

Lemma 5.3. Let $|\psi\rangle = (C_1 \otimes \ldots \otimes C_n)|G\rangle$ be a state expressed as a graph state $|G\rangle$ for graph G followed by local Cliffords C_i. Assume that for a pair of distinct qubits $u, v \in \{1, \ldots, n\}$, $Z_uZ_v|\psi\rangle = |\psi\rangle$ and $C_u, C_v \in \{I, H\}$. It must be the case that either:

(i) u and v are connected in G, have no other neighbours, and $C_u \otimes C_v$ is $I \otimes H$ or $H \otimes I$

(ii) u and v are not connected in G, have identical neighbours, and $C_u \otimes C_v = H \otimes H$

Proof. The layer of local Cliffords $C_1 \otimes \ldots \otimes C_n$ preserves the weight of stabilisers, so any stabiliser of the form Z_uZ_v of $|\psi\rangle$ must come from a weight-2 stabiliser P_uP_v of $|G\rangle$. The latter are generated by stabilisers of the form $S(w) := X_w \prod_{w' \in \text{nhd}(w)} Z_{w'}$ for all vertices w in G. Any product of k distinct generators has weight $\geq k$, hence a weight-2 stabiliser S for $|G\rangle$ must be the product of 1 or 2 generators. If S is a generator itself, it must be of the form X_uZ_v or Z_uX_v, either of which will be transformed into Z_uZ_v by local Cliffords when $C_u \otimes C_v = H \otimes I$ or $I \otimes H$, respectively. Hence u and v satisfy condition (i).

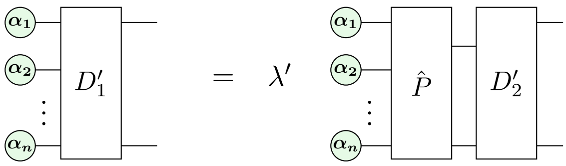
If S is the product of 2 distinct generators, then $S = S(u)S(v)$ and $\{u\} \cup \text{nhd}(u) = \{v\} \cup \text{nhd}(v)$, otherwise S would have support at some qubit $\notin \{u, v\}$. If u and v are connected, then $S = Y_uY_v$, which cannot be translated into Z_uZ_v via local Cliffords in the set $\{I, H\}$. Therefore u and v must not be connected, $\text{nhd}(u) = \text{nhd}(v)$, and $S = X_uX_v$. This gives a stabiliser Z_uZ_v for $|\psi\rangle$ when $C_u \otimes C_v = H \otimes H$. Hence u and v satisfy condition (ii). □

We can now prove the main theorem.

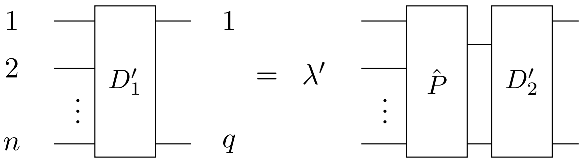
Theorem 5.4. The parameter optimisation algorithm of [18] described in Section 4.2 is optimal: the circuits it produces do not parsimoniously affinely reduce to any other diagram with fewer parameters.

Proof. Suppose we start with some parametrised Clifford circuit $C[\vec{\alpha}']$. Then all its parameters are non-trivial by Proposition 4.10. Let $D_1[\vec{\alpha}]$ be the parametrised Clifford diagram produced by applying the simplification strategy of Section 4.2 to C. Hence, C parsimoniously affinely reduces to D_1. We need to show that D_1 has the minimal number of parameters amongst the parsimonious affine reductions.

Suppose now that there exists a parametrised diagram $D_2[\vec{\beta}]$ such that $D_1[\vec{\alpha}] = \lambda(\vec{\alpha})D_2[P\vec{\alpha} + \vec{c}]$ and that the length of $\vec{\beta}$ is less than that of $\vec{\alpha}$. Because all the parameters of D_1 are non-trivial and P is parsimonious, P must map the parameters only to parameters of D_2 that act non-trivially in the subspace of the image. We may hence without loss of generality assume that all the parameters of D_2 are non-trivial. By Lemma 5.2 we then have:

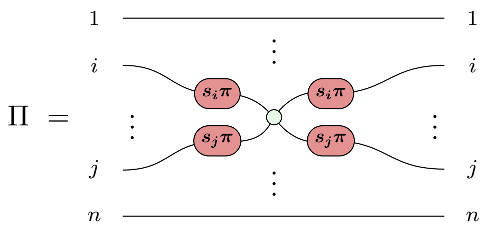


Here λ' is some constant and P is built from the generators , , , , . Since this equality holds for all $\vec{\alpha}$ we have:



(20)

Because $D_2[\vec{\beta}]$ has fewer parameters than $D_1[\vec{\alpha}]$, $\hat{P}$ should contain at least one Z fusion . Let α_i and α_j be two parameters that fuse. Then define the projector Π by the projection operation along these parameters:



Note that by definition $\hat{P}\circ\Pi = \hat{P}$. Hence, using Eq. (20) we see that $D_1'\circ\Pi = \lambda' D_2'\circ\hat{P}\circ\Pi = \lambda' D_2' \circ \hat{P} = D_1'$, so that Π is a stabiliser of D_1'.

The projector Π is exactly the projector onto the $+1$-eigenspace of the Pauli $(-1)^{s_i+s_j}Z_iZ_j$. Hence D_1' has a weight 2 stabiliser. Applying $X_i^{s_i+s_j}$ before D_1' then transforms this stabiliser into Z_iZ_j. Since D_1' arose from the simplification algorithm, we know it is a GSLC diagram. Furthermore, the indices i and j correspond to where a parameter is plugged in, which means that the local Clifford there is either an identity or a Hadamard. Now, applying Lemma 5.3 we conclude that the vertices corresponding to the indices i and j are either connected to each other and not to anything else, or that they are not connected to each other and have an identical set of neighbours, with both their local Cliffords being a Hadamard. In the first case there are choices of the parameters that would send the diagram to zero, which is not possible as the parameters are non-trivial. But in the second case the indices i and j would correspond to phase gadgets with identical neighbourhoods, in which case the algorithm would have already fused them.

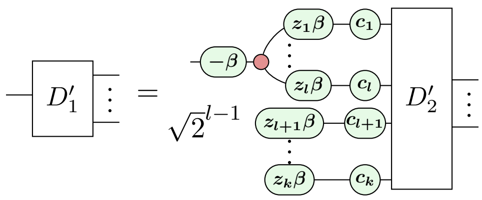
We are hence lead to a contradiction, which means that D_1 does not reduce to a diagram D_2 that has fewer parameters. □

Remark 5.5. In this proof we only needed the fact that C is a circuit to argue that its parameters are non-trivial, so that this optimality result applies to any type of diagram that has non-trivial parameters. In addition, we didn't make any assumptions on the hypothetical diagram D_2 other than that it has fewer parameters. This means that even allowing, for instance, post-selection does not allow one to reduce the number of parameters further.

5.1 General affine reductions

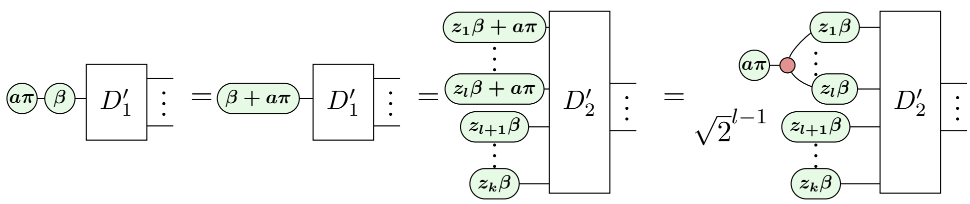
We needed to assume the affine reductions were parsimonious in order to prove optimality. In this section we will prove some properties of general affine reductions where parameters are allowed to be copied to multiple places. In particular, we will see that if parameters of D_1 are non-trivial and are copied to at most 3 places in D_2, then there is a different parsimonious reduction from D_1 to D_2 (where parameters are not used more than once).

Lemma 5.6. Let $z_1, \ldots, z_k \in \mathbb{Z}$ be integers and $c_1, \ldots, c_k \in \mathbb{R}$ some constants, and suppose that $D_1[\alpha] = D_2[z_1\alpha + c_1, \ldots, z_k\alpha + c_k]$. Suppose furthermore without loss of generality that the first $z_1, \ldots, z_l$ are odd and $z_{l+1}, \ldots, z_k$ are even. Then $\forall \beta \in \mathbb{R}$ we have:



Proof. Define $\tilde{D}_2$ as D_2 with the constant phases $\vec{c} \in \mathbb{R}^k$ added to the wires corresponding to the parameters. I.e. $\tilde{D}_2[\vec{\beta}] := D_2[\vec{\beta} + \vec{c}]$. Then we can write $D_1[\alpha] = \tilde{D}_2[z_1\alpha, \ldots, z_k\alpha]$. Hence, without loss of generality we may assume that $\vec{c} = 0$ (at the cost of making D_2 a Clifford diagram with potential non-Clifford phases on its parameter wires).

Now let $\beta \in \mathbb{R}$. Then for $a \in \{0, 1\}$ we have $D_1[\beta + a\pi] = D_2[z_1\beta + a\pi, \ldots, z_l\beta + a\pi, z_{l+1}\beta, \ldots, z_k\beta]$. Now we can do the following rewrite:



(21)

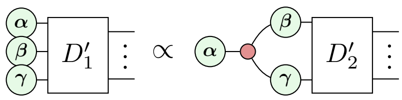
So now on the left-hand side and on the right-hand side we have a single input of $|+_{a\pi}\rangle$. Since these form a basis we conclude that this must hold for any input, and hence we can leave that wire open. Applying a $Z(-\beta)$ to both sides then gives us the desired equation. □

This lemma already shows that something weird goes on when we have a copied parameter: the diagram is invariant under a continuous group of phases. Note furthermore that this shows that any non-parsimonious reduction is equivalent to a parsimonious one where we precompose the diagram with a small number of spiders.

Proposition 5.7. Let $D_1[\vec{\alpha}] = D_2[M\vec{\alpha} + \vec{c}]$ for some integer matrix M and constants $\vec{c}$. Then $D_1[\vec{\alpha}] = (D_2' \circ E)[\vec{\alpha}]$ where E is built out Z-spiders with phases from $\vec{c}$ and fan-out X-spiders.

Proof. Do the same construction as in Lemma 5.6 iteratively where we fix every parameter except one, to extract it into E and set $\beta = 0$ in all cases. □

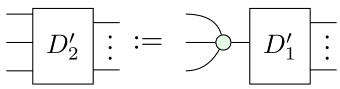
This does not imply that non-parsimonious reductions can be ignored. One could for instance have the non-parsimonious reduction $D_1[\alpha, \beta, \gamma] = D_2[\alpha + \beta, \alpha + \gamma]$. The construction above then gives:



(22)

So while the construction of Proposition 5.7 gives us a parsimonious reduction between D_1' and $D_2' \circ E$, the diagram $D_2' \circ E$ might have more parameters than the original non-parsimonious one. In addition, if there are non-Clifford constants $\vec{c}$, then these get incorporated in E, so that the reduction is no longer to a Clifford diagram.

We suspect however that every non-parsimonious reduction is 'pathological'. For instance, that the only parameters that can get copied are those that are trivial anyway, or otherwise that the reduction function can be simplified to a parsimonious one. For example, suppose we have:

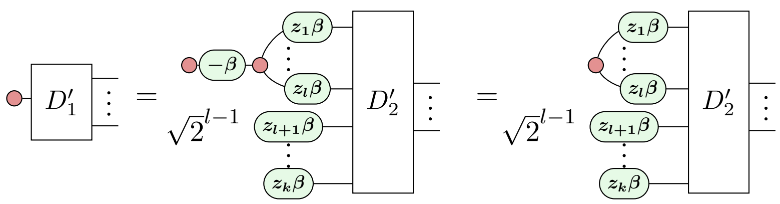


Then we of course have $D_1[\alpha] = D_2[\alpha, \alpha, -\alpha]$ by spider fusion. But we also have the parsimonious reduction $D_1[\alpha] = D_2[\alpha, 0, 0]$, and so the copying of the parameters is not 'necessary' in this case. We can see from Lemma 5.6 that one case where we can get rid of the copying of the parameter, is when only one of the z_j is odd, with the rest being even: the X-spider is then an identity and can be removed to get a straightforward parsimonious reduction.

We can also derive some stronger consequences from the equation we found in Lemma 5.6:

Lemma 5.8. Let $z_1, \ldots, z_k \in \mathbb{Z}$ be integers and $c_1, \ldots, c_k \in \mathbb{R}$ some constants, and suppose that $D_1[\alpha] = D_2[z_1\alpha + c_1, \ldots, z_k\alpha + c_k]$. Let furthermore $|A_w\rangle := \sum_{\vec{x}\,;\,\vec{z}\cdot\vec{x}=w} |\vec{x}\rangle$ be a superposition of all the basis states $|\vec{x}\rangle$ with equal *weight* $\vec{z} \cdot \vec{x}$. Then $D_2' |A_w\rangle = 0$ if $w \notin \{0, 1\}$.

Proof. Start with the equation from Lemma 5.6, and without loss of generality assume all the c_i are zero. We see that only the right-hand side contains a β and that this equation must hold for for all choices of β. Now, let's plug in $|0\rangle$:



(23)

The input to D_2' as a state is equal to

$$|\psi_\beta\rangle \;=\; \sum_{\substack{\vec{x}\in\mathbb{F}_2^k \\ \vec{z}\cdot\vec{x}\equiv 0 \bmod 2}} e^{i\beta\vec{z}\cdot\vec{x}}\,|\vec{x}\rangle \;=\; \sum_{w \text{ even}} e^{i\beta w} \sum_{\substack{\vec{x}\in\mathbb{F}_2^k \\ \vec{x}\cdot\vec{z}=w}} |\vec{x}\rangle\,.$$

This is because the X-spider forces the first $x_1,\ldots,x_l$ to have even parity, and since all the $z_1,\ldots,z_l$ are odd, this is the same as $z_1x_1+\cdots+z_lx_l$ being even. Since $z_{l+1},\ldots,z_k$ are all even, we can also just add the z_jx_j terms to this expression for $j>l$ without changing the parity. Hence $\vec{z}\cdot\vec{x}$ must be even. We will call this the *weight* and we set $w=\vec{z}\cdot\vec{x}$. In the latter expression we grouped together all the terms with the same weight. Note that because some of the z_j can be negative, that the weight can also be negative. We can hence write $|\psi_\beta\rangle=\sum_{w \text{ even}} e^{i\beta w}\,|A_w\rangle$ where $|A_w\rangle := \sum_{\vec{x}\,;\,\vec{z}\cdot\vec{x}=w}|\vec{x}\rangle$. Since D_2' maps every $|\psi_\beta\rangle$ to the same thing, it must map $|\psi_\beta\rangle-|\psi_{\beta'}\rangle$ to zero, for any $\beta,\beta'\in\mathbb{R}$. Let's call the kernel of D_2' K. Then $|\psi_\beta\rangle-|\psi_{\beta'}\rangle\in K$. Note that:

$$|\psi_\beta\rangle-|\psi_{\beta'}\rangle \;=\; \sum_{\substack{w\neq 0 \\ w \text{ even}}} (e^{iw\beta}-e^{i\beta' w})\,|A_w\rangle \;=\; \sum_{\substack{w>0 \\ w \text{ even}}} \Big[(e^{iw\beta}-e^{i\beta' w})\,|A_w\rangle+(e^{-iw\beta}-e^{-i\beta' w})\,|A_{-w}\rangle\Big]\,.$$

In particular if we set $\beta'=-\beta$ respectively $\beta'=-\beta+\pi$ we get

$$\sum_{\substack{w>0 \\ w \text{ even}}} 2i\sin w\beta(|A_w\rangle-|A_{-w}\rangle) \quad\text{and}\quad \sum_{\substack{w>0 \\ w \text{ even}}} 2\cos w\beta(|A_w\rangle+|A_{-w}\rangle)$$

These vectors are part of K regardless of the value of $\beta\in\mathbb{R}$. Since sine and cosine mostly take irrational values, and the angle doubling formulas are not linear, we can only have all these vectors part of K if each of the $|A_w\rangle-|A_{-w}\rangle$ and $|A_w\rangle+|A_{-w}\rangle$ is part of K themselves. Hence $|A_w\rangle\in K$ if $w\neq 0$ for even w.

If in Eq. (23) we instead plugged in $|1\rangle$ we would have gotten an input state that is a combination of the *odd* weight bit string superpositions. To be precise, the input state for a given β would be

$$|\psi_\beta\rangle \;=\; e^{-i\beta}\sum_{w \text{ odd}} e^{iw\beta}\,|A_w\rangle \;=\; \sum_{w \text{ odd}} e^{i(w-1)\beta}\,|A_w\rangle\,.$$

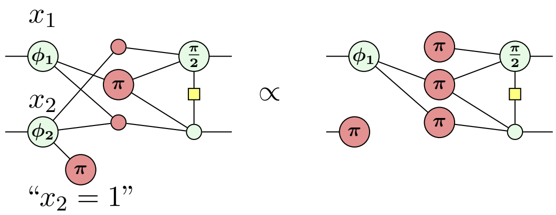
The global $e^{-i\beta}$ factor comes from inputting a $|1\rangle$ which absorbs the first $-\beta$ phase, introducing a $e^{-i\beta}$. Now as before $|\psi_\beta\rangle-|\psi_{\beta'}\rangle$ is in the kernel K of D_2'. As a result all the odd $|A_w\rangle\in K$ when $w\neq 1$. For $w=1$, the term $e^{i(w-1)\beta}=1$ is constant and so does not appear in the expression $|\psi_\beta\rangle-|\psi_{\beta'}\rangle$.

We see then that $D_2'\,|A_w\rangle=0$ if $w\notin\{0,1\}$. □

This property is useful for ruling out certain possibilities for a non-parsimonious reduction. Note for instance that there is a unique maximum weight w_m which we get when we set $x_j=1$ iff $z_j\geq 0$. Hence $|A_{w_m}\rangle=|\vec{x}\rangle$ is just a single bit string. We then know that $D_2'\,|\vec{x}\rangle=0$. If we assume that the constants $\vec{c}$ are zero, then D_2' is a Clifford diagram. This then has a normal form where we can represent the computational basis states it is non-zero on as an affine subspace: $D_2' := \sum_{\substack{\vec{x},\vec{y} \\ A(\vec{x},\vec{y})=(\vec{b},\vec{d})}} e^{i\frac{\pi}{2}\phi(\vec{x},\vec{y})}\,|\vec{y}\rangle\langle\vec{x}|$. The fact that a $\vec{x}$ is not part of that subspace implies there must be a certain equation in A that witnesses that. That equation cannot involve any variables from $\vec{y}$, since otherwise we could find a $\vec{y}$ so that $(\vec{x},\vec{y})$ satisfies the equation.

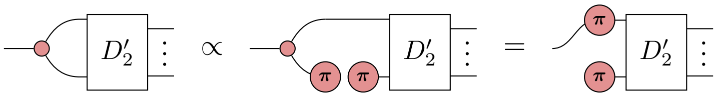
To make the following more clear, let's pick a particular example. Suppose we had $D_1[\alpha]=D_2[z_1\alpha,z_2\alpha]$ where $z_1>0$ and $z_2<0$ and where both are odd, so that $\vec{x}=(1,0)$

is the unique state giving maximum weight, and hence $D_2' |10\rangle = 0$. Then there must an equation defining the affine subspace of D_2 involving the two variables x_1 and x_2 for which $(1, 0)$ is not a solution. Examples could be $x_1 = 0$ or $x_2 = 1$. In that case the qubit wire leading to the corresponding parameter can be disconnected from the rest of the diagram:



(24)

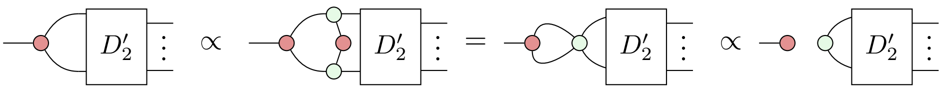
We can then use this to simplify the expression of Lemma 5.6:



(25)

This hence leads to $D_1[\alpha] = D_2[-\alpha, 0]$, which is parsimonious.

We could have also had an equation involving both variables, like $x_1 \oplus x_2 = 0$, for which $(1, 0)$ is also not a solution. In that case we can also simplify the expression coming from Lemma 5.6:



(26)

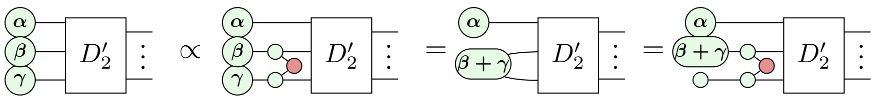
So in this case we have $D_1[\alpha] = D_1[0]$, so that the parameter is trivial.

This in fact covers all the possibilities for a parameter that appears twice, so we see that either we can reduce it to a parsimonious case, or the parameter is necessarily trivial. Note that we didn't need the fact that the constants $\vec{c}$ were Clifford, as we only used information about the affine subspace, and not the phases, so that this holds for any affine reduction were variables appear at most twice. If we have more than one parameter, we can repeat this argumentation for each parameter independently. We have then proven the following:

Proposition 5.9. Let $D_1[\vec{\alpha}] = D_2[M\vec{\alpha} + \vec{c}]$ be an affine reduction where every column of M contains at most two non-zero elements, and suppose that all parameters $\vec{\alpha}$ are non-trivial. Then there exists a parsimonious reduction $D_1[\vec{\alpha}] = D_2[M'\vec{\alpha} + \vec{c}]$.

We can do a similar argument for when we have three repetitions of a variable, although the method breaks down when we have four repetitions.

When we have three variables we also get an equation restricting the affine subspace of D_2. If this involves just one variable, the reduction can be restricted to one where the variable only occurs twice using a version of Eq. (25), so that the previous case applies and we are done. If the equation involves two variables, then we can use a version of Eq. (26) to reduce it to the parsimonious case as well:



(27)

In this particular case then $D_2[\alpha, \beta, \gamma] = D_2[\alpha, \beta + \gamma, 0]$, but in general there might also be a minus sign. In any case we can reduce D_2 to the two parameter case, so that we can repeat the argument for the two parameter reductions, and we get a parsimonious reduction.

Now if the restricting equation involves all three variables we need to make a case distinction. If none or only one of the z_j is odd, then by Lemma 5.6 we already know the reduction can be made parsimonious. If all three z_j are odd, then the three-qubit version of Eq. (26) applies, and the parameter in D_1 is trivial. The remaining case is then when two of the z_j are odd. In that case, the equivalent of Eq. (26) would toggle the connectivity of the X-spider, so that it is equivalent to the X-spider only being connected to just 1 of the spiders, which would again make it parsimonious.

Proposition 5.10. Let $D_1[\vec{\alpha}] = D_2[M\vec{\alpha} + \vec{c}]$ be an affine reduction where every column of M contains at most *three* non-zero elements, and suppose that all parameters $\vec{\alpha}$ are non-trivial. Then there exists a parsimonious reduction $D_1[\vec{\alpha}] = D_2[M'\vec{\alpha} + \vec{c}]$.

For any number of variables repetitions, if the restricting equation contains 1 or 2 variables, we reduce to a case with fewer repetitions. For four repetitions the only problematic cases are then when there are exactly 2 odd z_j, and the restricting equation contains 3 variables such that this does not contain all odd z_j, or the restricting equation contains all 4 variables. We conjecture that using some other consequences of Lemma 5.8 we should also be able to prove that when we have this situation that we can still construct a parsimonious reduction.

6 Measurement-based quantum computing

We have so far only talked about parameters in the context of parametrised circuits, but they are also a useful concept in measurement-based quantum computing (MBQC). The most well-studied form of MBQC is the *one-way model* [26, 27]. In this model you start with a graph state and you do measurements in one of the three principal planes of the Bloch sphere. Subsequent measurement angles may depend on previous measurement outcomes, which is needed in order to correct for 'wrong' measurement outcomes and to get a deterministic outcome. In general it might not be possible to pick a measurement order and set of corrections that results in a deterministic outcome. In order to ensure this we need to require certain combinatorial properties of the underlying graph state that ensure there is a *flow* for the errors to be pushed along. In particular, a measurement pattern is *uniformly*, *strongly* and *stepwise* deterministic if and only if the underlying graph state has a *generalised flow* (gflow) [8].

A pattern is uniformly deterministic if it is deterministic for every choice of measurement angle for the chosen measurement planes. It is strongly deterministic if every measurement outcome happens with equal probability, and it is stepwise deterministic if after every single measurement the set of corrections to make it deterministic is known again. A generalised flow is a property that an *open* graph (a graph with a specified set of inputs and outputs) can have which has to do with the neighbourhoods of the vertices.

The most general known flow condition is known as *Pauli flow*. In this setting, qubits are either measured as some Pauli, or they are measured in an arbitrary angle in a given plane of the Bloch sphere. The uniform determinism condition is then weakened to only apply to those qubits measured in the arbitrary angle. When represented as a ZX-diagram, such patterns correspond to parametrised Clifford diagrams [30]. In [6] and [30] it is shown how to rewrite measurement patterns with gflow or Pauli flow in order to remove certain measured qubits while preserving the flow conditions. On the level of ZX-diagrams these rewrites correspond precisely to the strategy of [18] that we have shown to be optimal when it comes to parameter optimisation.

We hence also have the following result.

Theorem 6.1. The simplification strategy for measurement patterns with gflow of [6] produces a measurement pattern that has a minimal number of parametrised measurements amongst those patterns that it parsimoniously affinely reduces to.

7 Conclusion and discussion

We showed that we can find the minimal number of parameters of a Clifford circuit with parametrised phase gates under the condition that each parameter occurs uniquely and that the notion of reduction is restricted to analytic maps that don't clone any parameter to multiple places. The algorithm that finds the minimal parameter count is the one described in our earlier work [18]. Our result also applies to measurement patterns with gflow and shows that the technique of [6] produces a measurement pattern with the minimal number of non-Clifford measurements (under the condition that we treat these non-Clifford angles as 'black box parameters').

Note that the ZX-calculus-based approach of [18] was primarily interested in optimising T-count, and as an approach it always seemed to match the T-count of the circuit-based approach based on fusing Pauli exponentials of [34]. Evidence that these methods should result in the same non-Clifford count was further given in [30]. Hence, it seems likely that the approach of [34] should also give optimal parameter counts. That means that writing a circuit as a series of Pauli exponentials and merging those exponentials of the same Pauli when there are only commuting Paulis in between them is essentially the best possible rewrite strategy when our goal is minimising the number of non-Clifford components of the circuit and we can't use any specific knowledge on the angles of rotation involved (barring the use of discontinuous parameter changing rules like Eq. (2)).

Our results demonstrate that any circuit optimisation strategy that wants to do better at removing non-Clifford phases needs to use specific knowledge of the phases involved. This is indeed the case in T-count optimisation wherein relations to Reed-Muller decoding [3] and symmetric 3-tensor factorisation [15] are used. In particular, in [3] it was shown that for diagonal circuits built out of CNOT gates and phase gates, there are only non-trivial identities when the phases involved are dyadic rational multiples of π.

There are several possible generalisations of our result that could be considered. We have already shown that generalising the discrete gate set from Cliffords to Clifford+T results in a parameter optimisation problem that is NP-hard, but we left open the question of the hardness of other possible generalisations. In particular, we have only established the hardness of optimisation when parameters are allowed to be reused in the setting of post-selected quantum circuits. We don't know if the problem remains hard in the unitary setting. This is particularly relevant, because many ansätze for variational circuits involve controlled-phase gates [14, 29] that get decomposed into repeated parametrised phase gates. We have also left open the hardness when we relax the notion of reduction between parametrised circuits to include parameter transformations that are discontinuous in parameter space, like the Euler angle transformation of Eq. (2). This rule suffices to make the ZX-calculus complete for arbitrary linear maps [33] so it stands to reason that rewriting when this rule is allowed becomes hard. Lastly, our optimality result applies to *parsimonious* reductions: those that do not clone the parameter. We've showed in Section 5.1 that when a parameter is used at most 3 times in a reduction, that we can also find an equivalent parsimonious reduction. We suspect that this should hold for any non-parsimonious reduction, and hence that our optimality result should hold even when allowing any affine reduction.

References

[1] M. Amy, D. Maslov, M. Mosca, and M. Roetteler. A meet-in-the-middle algorithm for fast synthesis of depth-optimal quantum circuits. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, 32(6):818–830, 6 2013.

[2] Matthew Amy. Towards large-scale functional verification of universal quantum circuits. In Peter Selinger and Giulio Chiribella, editors, *Proceedings of the 15th International Conference on Quantum Physics and Logic, Halifax, Canada, 3-7th June 2018*, volume 287 of *Electronic Proceedings in Theoretical Computer Science*, pages 1–21. Open Publishing Association, 2019.

[3] Matthew Amy and Michele Mosca. T-count optimization and Reed-Muller codes. *Transactions on Information Theory*, 2019.

[4] Miriam Backens. The ZX-calculus is complete for stabilizer quantum mechanics. *New Journal of Physics*, 16(9):093021, 2014.

[5] Miriam Backens and Aleks Kissinger. ZH: A complete graphical calculus for quantum computations involving classical non-linearity. In Peter Selinger and Giulio Chiribella, editors, *Proceedings of the 15th International Conference on Quantum Physics and Logic, Halifax, Canada, 3-7th June 2018*, volume 287 of *Electronic Proceedings in Theoretical Computer Science*, pages 18–34. Open Publishing Association, 2019.

[6] Miriam Backens, Hector Miller-Bakewell, Giovanni de Felice, Leo Lobski, and John van de Wetering. There and back again: A circuit extraction tale. Preprint, 2020.

[7] Miriam Backens, Hector Miller-Bakewell, Aleks Kissinger, and John van de Wetering. Completeness of the ZH-calculus. *Forthcoming*, 2020.

[8] Daniel E. Browne, Elham Kashefi, Mehdi Mhalla, and Simon Perdrix. Generalized flow and determinism in measurement-based quantum computation. *New Journal of Physics*, 9(8):250, 2007.

[9] Bob Coecke and Ross Duncan. Interacting quantum observables. In *Proceedings of the 37th International Colloquium on Automata, Languages and Programming (ICALP)*, Lecture Notes in Computer Science, 2008.

[10] Bob Coecke and Ross Duncan. Interacting quantum observables: categorical algebra and diagrammatics. *New Journal of Physics*, 13:043016, 2011.

[11] Ross Duncan, Aleks Kissinger, Simon Perdrix, and John van de Wetering. Graph-theoretic Simplification of Quantum Circuits with the ZX-calculus. *Quantum*, 4:279, 6 2020.

[12] Ross Duncan and Simon Perdrix. Rewriting Measurement-Based Quantum Computations with Generalised Flow. In *Proceedings of ICALP*, Lecture Notes in Computer Science, pages 285–296. Springer, 2010.

[13] Edward Farhi, Jeffrey Goldstone, and Sam Gutmann. A quantum approximate optimization algorithm. *arXiv preprint arXiv:1411.4028*, 2014.

[14] Vojtěch Havlíček, Antonio D Córcoles, Kristan Temme, Aram W Harrow, Abhinav Kandala, Jerry M Chow, and Jay M Gambetta. Supervised learning with quantum-enhanced feature spaces. *Nature*, 567(7747):209–212, 2019.

[15] Luke E Heyfron and Earl T Campbell. An efficient quantum compiler that reduces T count. *Quantum Science and Technology*, 4(015004), 2018.

[16] Emmanuel Jeandel, Simon Perdrix, and Renaud Vilmart. A Complete Axiomatisation of the ZX-Calculus for Clifford+T Quantum Mechanics. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '18, pages 559–568, New York, NY, USA, 2018. ACM.

[17] Emmanuel Jeandel, Simon Perdrix, and Renaud Vilmart. Diagrammatic Reasoning Beyond Clifford+T Quantum Mechanics. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*, LICS '18, pages 569–578, New York, NY, USA, 2018. ACM.

[18] Aleks Kissinger and John van de Wetering. Reducing the number of non-Clifford gates in quantum circuits. *Physical Review A*, 102:022406, 8 2020.

[19] Stach Kuijpers, John van de Wetering, and Aleks Kissinger. Graphical fourier theory and the cost of quantum addition. Preprint, 2019.

[20] Tommy McElvanney and Miriam Backens. Complete flow-preserving rewrite rules for MBQC patterns with Pauli measurements. *arXiv preprint arXiv:2205.02009*, 2022.

[21] Hector Miller-Bakewell. Finite Verification of Infinite Families of Diagram Equations. In Bob Coecke and Matthew Leifer, editors, *Proceedings 16th International Conference on Quantum Physics and Logic, Chapman University, Orange, CA, USA., 10-14 June 2019*, volume 318 of *Electronic Proceedings in Theoretical Computer Science*, pages 27–52. Open Publishing Association, 2020.

[22] Yunseong Nam, Neil J Ross, Yuan Su, Andrew M Childs, and Dmitri Maslov. Automated optimization of large quantum circuits with continuous parameters. *npj Quantum Information*, 4(1):23, 2018.

[23] Simon Perdrix and Quanlong Wang. Supplementarity is necessary for quantum diagram reasoning. In *41st International Symposium on Mathematical Foundations of Computer Science (MFCS 2016)*, volume 58 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 76:1–76:14, Krakow, Poland, 2016.

[24] Alberto Peruzzo, Jarrod McClean, Peter Shadbolt, Man-Hong Yung, Xiao-Qi Zhou, Peter J Love, Alán Aspuru-Guzik, and Jeremy L O'brien. A variational eigenvalue solver on a photonic quantum processor. *Nature communications*, 5:4213, 2014.

[25] Boldizsár Poór, Robert I. Booth, Titouan Carette, John van de Wetering, and Lia Yeh. The Qupit Stabiliser ZX-travaganza: Simplified Axioms, Normal Forms and Graph-Theoretic Simplification. In Shane Mansfield, Benoit Valîron, and Vladimir Zamdzhiev, editors, Proceedings of the Twentieth International Conference on *Quantum Physics and Logic,* Paris, France, 17-21st July 2023, volume 384 of *Electronic Proceedings in Theoretical Computer Science*, pages 220–264. Open Publishing Association, 2023.

[26] Robert Raussendorf and Hans J. Briegel. A One-Way Quantum Computer. *Physical Review Letters*, 86:5188–5191, 5 2001.

[27] Robert Raussendorf, Dan E. Browne, and Hans J. Briegel. Measurement-based quantum computation on cluster states. *Physical Review A*, 68(2):22312, 2003.

[28] Maria Schuld, Ville Bergholm, Christian Gogolin, Josh Izaac, and Nathan Killoran. Evaluating analytic gradients on quantum hardware. *Physical Review A*, 99(3):032331, 2019.

[29] Sukin Sim, Peter D Johnson, and Alán Aspuru-Guzik. Expressibility and entangling capability of parameterized quantum circuits for hybrid quantum-classical algorithms. *Advanced Quantum Technologies*, 2(12):1900070, 2019.

[30] Will Simmons. Relating Measurement Patterns to Circuits via Pauli Flow. In Chris Heunen and Miriam Backens, editors, *Proceedings 18th International Conference on Quantum Physics and Logic, Gdansk, Poland, and online, 7-11 June 2021*, volume 343 of *Electronic Proceedings in Theoretical Computer Science*, pages 50–101. Open Publishing Association, 2021.

[31] John van de Wetering. ZX-calculus for the working quantum computer scientist. *arXiv preprint arXiv:2012.13966*, 2020.

[32] John van de Wetering and Matt Amy. Optimising T-count is NP-hard. *arXiv preprint arXiv:2310.05958*, 2023.
[33] Renaud Vilmart. A Near-Minimal Axiomatisation of ZX-Calculus for Pure Qubit Quantum Mechanics. In *2019 34th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS)*, pages 1–10, 2019.
[34] Fang Zhang and Jianxin Chen. Optimizing T gates in Clifford+T circuit as $\pi/4$ rotations around Paulis. Preprint, 2019.

A Hardness of optimisation with repeated parameters

We have not yet determined the hardness of optimisation of Clifford circuits with repeated parameters. The argument of Proposition 2.3 does not work, since we cannot 'reliably' produce a Toffoli gate using just Clifford gates and parametrised phase gates: if all the parameters are set to Clifford angles, then the whole circuit is Clifford and hence certainly cannot implement a Toffoli. It turns out that it is however possible to construct a Toffoli for any other choice of parameter using repeated parametrised phase gates if we allow the circuit to be *post-selected*.

In order to define the reduction problem for post-selected quantum circuits we slightly relax Definition 2.1 to allow for a zero scalar.

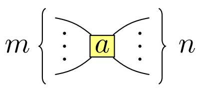
Definition A.1. We say a parametrised post-selected circuit C_1 with parameter space $\mathbb{R}^k$ *reduces* to post-selected circuit C_2 with parameter space $\mathbb{R}^l$ when there exists a function $f : \mathbb{R}^k \to \mathbb{R}^l$ such that $C_1[\vec{\alpha}] = \lambda(\vec{\alpha}) C_2[f(\vec{\alpha})]$ for all $\vec{\alpha} \in \mathbb{R}^k$ where $\lambda : \mathbb{R}^k \to \mathbb{C}$ is a function representing a global scalar $\lambda(\vec{\alpha})$ that may depend on the parameters and is allowed to be zero. We define an affine reduction as before by restricting f to be affine.

Definition A.2. (Affine) Parameter optimisation for post-selected circuits: Given a parametrised post-selected quantum circuit C_1 with parameter space $\mathbb{R}^k$, find a parametrised quantum circuit C_2 with parameter space $\mathbb{R}^l$ that it (affinely) reduces to, such that l is *minimal*.

In this appendix we will prove the following result.

Proposition A.3. Boolean satisfiability reduces to affine parameter optimisation for post-selected circuits.

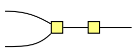
In order to prove this it will be helpful to use the *ZH-calculus* [5, 7]. In particular we define a new type of generator, the *H-box*:



$$= \sum_{\vec{x},\vec{y}} a^{x_1 \cdots x_m y_1 \cdots y_n} |\vec{y}\rangle\langle\vec{x}| \tag{28}$$

This is a matrix that has a 1 in every entry except for the bottom-right corner (where $x_1 = \cdots = x_m = y_1 = \cdots = y_n = 1$) where there is an a. In particular, the 1-input 1-output H-spider with $a = -1$ is a rescaled Hadamard. For this reason, when $a = -1$ we don't write the label in the H-box.

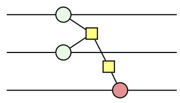
The reason we are interested in H-boxes, is because they allow us to easily represent the AND operation $\text{AND}\,|x, y\rangle = |xy\rangle$:



$= 2$

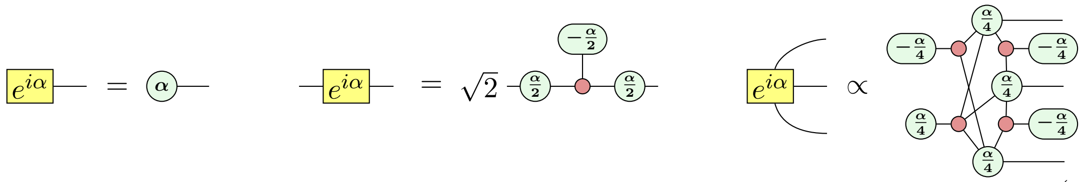
(29)

Hence, we can use H-boxes to construct the Toffoli:



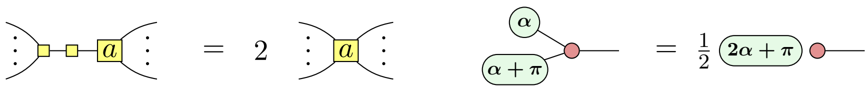
(30)

In order to relate H-boxes to parametrised phase gates we need to decompose it into the generators of the ZX-calculus. How to do this is described in [19]:



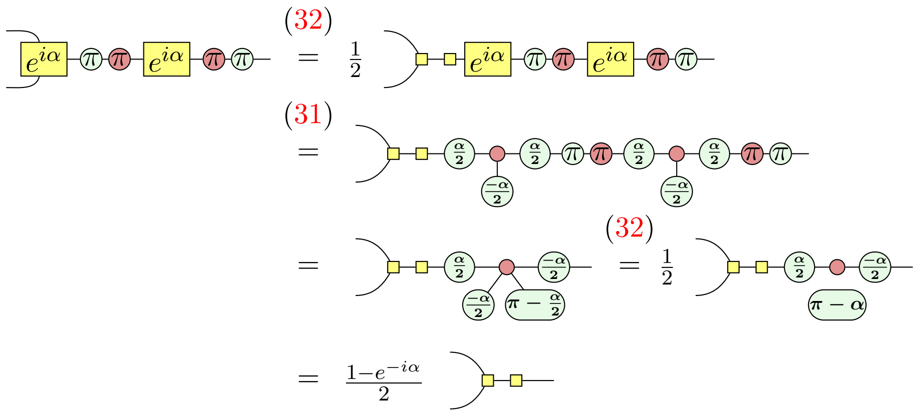
(31)

To decompose an n-ary H-box with an $e^{i\alpha}$ phase we require $2^n - 1$ phase gadgets with $\pm\alpha/2^{n-1}$ phases. Now to show how to implement a Toffoli using parametrised phase gates we will need to additional rewrites, *H-box fusion* and *supplementarity*:



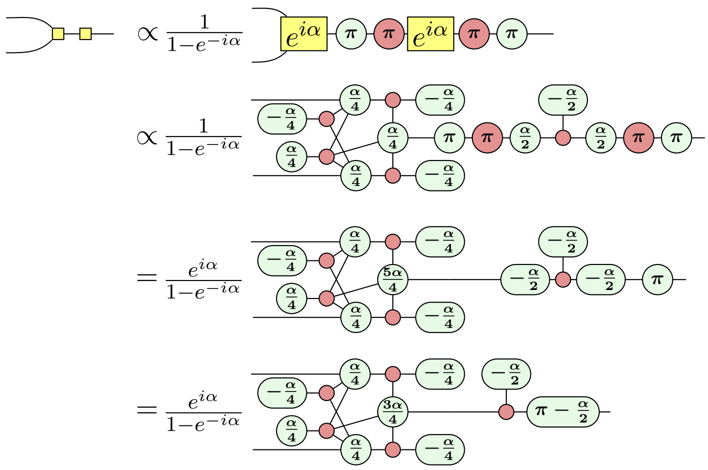
(32)

We then show the following:



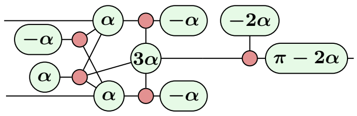
(33)

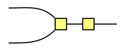
From this we conclude that:



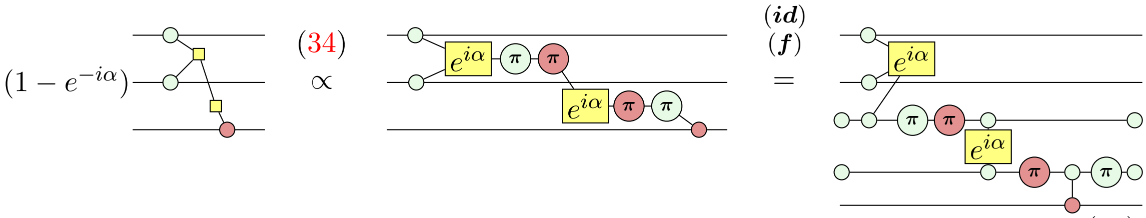
(34)

This equation holds whenever $\alpha \neq 0$ (since otherwise the scalar factor in Eq. (33) would be zero). Since we want to work with integer multiples of α we can replace α by 4α in this equation (because the equation must hold regardless of the value of α this substitution is valid):



$\propto e^{-i4\alpha}(1 - e^{-i4\alpha})$  (35)

In order to make the connection to post-selected quantum circuits more clear we can use this construction to build a Toffoli as a post-selected circuit using controlled-phase gates:



(36)

Since we can build a Toffoli (up to a scalar that depends on the parameters), we can construct for every Boolean function $f : \{0,1\}^n \to \{0,1\}$ the map $U_f[\alpha]$ acting as $U_f[\alpha] |\vec{x}, y\rangle = \lambda(\vec{\alpha}) |\vec{x}, y \oplus f(\vec{x})\rangle$ for some scalar function λ which goes to zero when $\alpha = k\frac{\pi}{2}$. Note that we only need the single parameter α as we can construct each Toffoli using the same repeated parameter. In particular, if the construction of U_f involves k Toffolis, then the scalar is $\lambda(\alpha) = (1 - e^{i4\alpha})^k$. Now with U_f in hand, we can build the same circuit as in Proposition 2.3 using two additional parameters β and γ to get the diagonal map $|\vec{x}, y\rangle \mapsto \lambda'(\alpha) e^{i\beta y + i\gamma(f(\vec{x}) \oplus y)} |\vec{x}, y\rangle$. Let's call the post-selected circuit implementing this map $C_f[\alpha, \beta, \gamma]$.

Note that if f is not satisfiable that then β and γ can fuse, meaning C_f reduces to a circuit $C'[\delta]$ implementing $C'[\delta] |\vec{x}, y\rangle = e^{i\delta y} |\vec{x}, y\rangle$. The reduction is $C_f[\alpha, \beta, \gamma] = \lambda'(\alpha) C'[\beta + \gamma]$. Hence, the optimal parameter count of C_f is 1 (because it can't be zero). If instead f is always satisfiable, then we can also show that it reduces to a circuit with parameter count 1. When f is satisfiable, but not always satisfiable then we can show that β and γ have separate effects on at least two different input states to C_f so that the optimal parameter count is at least 2.

We can now state our reduction from SAT to parameter optimisation with post-selected quantum circuits. For a given Boolean formula f, construct the post-selected parametrised circuit C_f. Then determine its optimal parameter count. If this is more than 1, then f is satisfiable. If it is exactly 1, then it is either not satisfiable or always satisfiable. Test the value $f(0 \cdots 0)$. If this is 1 then f is satisfiable, and if it is not, then it is not satisfiable everywhere and hence must not be satisfiable at all.

B Affine functions are all you need

We will here formalise what we mean by the parameter maps consisting of analytic functions on the unit circle. We require a parameter map $\Phi = (\Phi_1, \ldots, \Phi_l)$ to consist of functions $\Phi_j(\vec{\alpha}) = 2\pi f((\vec{\alpha} \bmod 2\pi)/(2\pi)) \bmod 2\pi$ for some smooth f. Here we are dividing and multiplying by 2π to make the following arguments more convenient. We additionally take the modulo 2π inside of the argument of f so that we can view f as a function $f : [0,1]^n \to \mathbb{R}$. To make Φ_j a proper map of the unit circle we need $f(1, x_2, \ldots, x_n) - f(0, x_2, \ldots, x_n) = k$ for some integer $k \in \mathbb{Z}$ and arbitrary $x_j \in [0,1]$, and similarly for the other arguments (note that this k cannot depend on the choice of $x_2, \ldots x_n$ due to continuity of f). In addition, for f to be smooth as a function on the unit circle we also need $f'(0, x_2, \ldots, x_n) = f'(1, x_2, \ldots, x_n)$, so that there is no discontinuity in the derivative at the boundary. The same holds for all higher derivatives as well. We assume that f is in each argument *analytic*, meaning that it is infinitely differentiable, and given by a power series. Concretely, writing $g(x) := f(x, x_2, \ldots, x_n)$ for some fixed choice of $x_2, \ldots, x_n$, we will assume that

$$g(x) \ := \ \sum_{m=0}^{\infty} a_m (x-b)^m$$

for some choice of $b, a_m \in \mathbb{R}$, such that this infinite sum absolutely converges within an open interval I containing $[-1, 1]$.

Our discussion above then shows that we should assume that $g(1) - g(0) = k \in \mathbb{Z}$, and $g^{(j)}(0) = g^{(j)}(1)$, where $g^{(j)}$ denotes the jth derivative of g.

Lemma B.1. Let g be an analytic function satisfying the conditions $g(1) = g(0) + k$ for some $k \in \mathbb{Z}$ and $g^{(j)}(0) = g^{(j)}(1)$ for all j. The g is equal to $g(x) = kx + c$ for some constant $c \in \mathbb{R}$.

Proof. Define $h(x) := g(x-1) + k$. Note first that h is still an analytic function, and furthermore that $h(1) = g(0) + k = g(1)$ and $h^{(j)}(1) = g^{(j)}(0) = g^{(j)}(1)$. Hence g and h agree on all derivatives and hence must be equal. As a result $g(x) = g(x-1) + k$, and hence also $g^{(j)}(x) = g^{(j)}(x-1)$.

Now let's use this equality and evaluate g and its derivatives at the base point b of the series as well as in $b-1$. We of course have $g(b) = a_0$. But also $g(b) = g(b-1) + k = a_0 + k + \sum_{j=1}^{\infty} a_j(-1)^j$. Hence, $\sum_{j=1}^{\infty} a_j(-1)^j = -k$. We do the same for the first derivative: $a_1 = g^{(1)}(b) = g^{(1)}(b-1) = \sum_{j=1}^{\infty} j a_j (-1)^j - 1$. After some rewriting we get $\sum_{j=2}^{\infty} j a_j (-1)^j = 0$. In general, doing this with the lth derivative, we get the equation

$$\sum_{j=l+1}^{\infty} j!/(j-l)! a_j (-1)^j \ = \ 0.$$

This then gives us an infinite set of linear equations for the a_j in upper triangular form:

$$\begin{array}{rclllll} -k & = & -a_1 & +a_2 & -a_3 & +a_4 & \ldots \\ 0 & = & & +2a_2 & -3a_3 & +4a_4 & \ldots \\ 0 & = & & & -6a_3 & +12a_4 & \ldots \\ & \vdots & & & & & \end{array} \tag{37}$$

Using standard induction arguments one can show that this has a unique solution given by $a_1 = k$, and $a_j = 0$ for $j > 1$. Hence $g(x) = a_0 + k(x-b) = kx - ba_0$ where $ba_0 = g(0)$. $\square$

Recalling that we defined $g(x) := f(x, x_2, \ldots, x_n)$ we see with the above lemma that $f(x, x_2, \ldots, x_n) = kx + c$, where the c is necessarily $f(0, x_2, \ldots, x_n)$. We can do the exact same argument for the argument x_2 as well to arrive at $f(x, x_2, \ldots, x_n) = k_2 x_2 + f(x, 0, x_3, \ldots, x_n) = kx + k_2 x_2 + f(0, 0, x_3, \ldots, n)$. Hence, repeating the argument for each parameter we see that $f(x_1, \ldots, x_n) = k_1 x_1 + \cdots k_n x_n + c$ where $c = f(0, \ldots, 0)$. Note that we can write this as $f(\vec{x}) = \vec{k} \cdot \vec{x} + c$ where $\cdot$ is the dot product and $\vec{k} = (k_1, \ldots, k_n)$.

Now $\Phi = (\Phi_1, \ldots, \Phi_l)$ where $\Phi_j(\vec{\alpha}) = 2\pi f_j((\vec{\alpha} \bmod 2\pi)/(2\pi)) \bmod 2\pi = 2\pi(\vec{k}_j \cdot \frac{\vec{\alpha}}{2\pi} + c_j \bmod 2\pi) = (\vec{k}_j \cdot \alpha + c_j) \bmod 2\pi$. Writing $\vec{c} = (c_1, \ldots, c_l)$ and $M = (\vec{k}_1 \ \cdots \ \vec{k}_l)$ we see that we then indeed get $\Phi(\vec{\alpha}) = M\vec{\alpha} + \vec{c}$.

C Proof of Parametrised completeness

In this Appendix we will set out to prove Theorem 4.12.

We will need the following facts:

Lemma C.1. Let S be a scalar Clifford diagram. Then S is equal to $\sqrt{2}^n e^{ik\frac{\pi}{4}}$ for some integers $n, k \in \mathbb{Z}$.

Lemma C.2. Let D be a Clifford diagram. Consider now the diagrams D_+ and D_- we get by plugging in a $|+\rangle$, respectively a $|-\rangle$ state into one designated input or output wires of D. Then if $D \neq 0$ exactly one of the following is true:

- $\|D_+\| = \|D_-\|$.
- $\|D_+\| = 0$.
- $\|D_-\| = 0$.

Proof. Without loss of generality we can take D to be a state, which is hence (proportional to) a stabiliser state. Then plugging in $|+\rangle$ or $|-\rangle$ into one of the wires corresponds to a post-selection on an X measurement. It is well known that a Pauli measurement on a stabiliser state is either deterministic or unbiased, corresponding to the scenarios described above. □

We will now first study in-depth the behaviour of trivial parameters, and diagrams which just have a single parameter, before moving to the general case.

Definition C.3. We say a parametrised diagram D is *singular* when there exists a choice of parameters $\vec{\alpha}$ such that $D[\vec{\alpha}] = 0$.

Lemma C.4. Let D be a singular parametrised diagram with just a single parameter. Then that parameter is trivial. Furthermore, if there is more than one value of the parameter for which D is zero, then $D = 0$. Otherwise the unique value α of the parameter such that $D[\alpha] = 0$ is Clifford: $\alpha = k\frac{\pi}{2}$.

Proof. Suppose that for some phase α we have $D[\alpha] = 0$, and let β and γ be any other phases. Write $|+_\gamma\rangle = a\,|+_\alpha\rangle + b\,|+_\beta\rangle$. Then $D[\gamma] = aD[\alpha] + bD[\beta] = bD[\beta]$, so that the parameter is indeed trivial. Now, if there were a second value of the parameter for which D is zero, then we could have taken β to be equal to that value, which would have shown $D[\gamma] = 0$. Since γ is arbitrary this indeed shows $D = 0$.

So suppose that α is the unique value for which $D[\alpha] = 0$. Suppose that $\alpha \neq 0$ and $\alpha \neq \pi$. We will show that then $\alpha = \pm\frac{\pi}{2}$. Now with the above argument, taking $\gamma := \pi$ and $\beta := 0$ we calculate $D[\pi] = bD[0]$. Since we know that $D[\pi] \neq 0$ and $D[0] \neq 0$, Lemma C.2 shows that necessarily $|b| = 1$. But Lemma 4.4 also gives us $b = \frac{e^{i\alpha}+1}{e^{i\alpha}-1}$. Hence we must have $|e^{i\alpha} + 1| = |e^{i\alpha} - 1|$. Converting the expressions in sine and cosine shows that this only holds when $\alpha = \pm\frac{\pi}{2}$. □

Proposition C.5. Let $D \neq 0$ be a parametrised diagram with a single parameter α and suppose α is trivial. Then exactly one of the following is true:

- $D[0] = 0$ and $D[\alpha] = \frac{1}{2}(1 - e^{i\alpha})D[\pi]$.
- $D[\pi] = 0$ and $D[\alpha] = \frac{1}{2}(1 + e^{i\alpha})D[0]$.
- $D[\frac{\pi}{2}] = 0$ and $D[\alpha] = \frac{1}{2}(1 + e^{i(\alpha+\frac{\pi}{2})})D[0]$.

- $D[-\frac{\pi}{2}] = 0$ and $D[\alpha] = \frac{1}{2}(1 + e^{i(\alpha - \frac{\pi}{2})})D[0]$.
- $D[\pi] = D[0]$ and $D[\alpha] = D[0]$.
- $D[\pi] = -D[0]$ and $D[\alpha] = e^{i\alpha}D[0]$.

In particular, if D is non-singular, then $D[\pi] = \pm D[0]$.

Proof. If $D[0] = 0$, then using $|+_\alpha\rangle = a\,|+_0\rangle + b\,|+_\pi\rangle$, we get $D[\alpha] = bD[\pi]$. Using Lemma 4.4 we see that $b = \frac{1}{2}(1 - e^{i\alpha})$. The other 3 cases where D is singular are similarly calculated.

So suppose D is not singular. By Lemma C.2 we must then have $\|D[\pi]\| = \|D[0]\|$ and hence $D[\pi] = e^{i\phi}D[0]$. Now because $e^{i\phi}$ must be expressible as a Clifford phase, we must have $\phi = k\frac{\pi}{4}$. But actually, since ϕ is a relative phase that only gets applied when π is inserted in the diagram, and not when 0 is, we must have $\phi = k\frac{\pi}{2}$, since it otherwise would correspond to a T gate. It is straightforwardly verified using the previously used technique, that if $\phi = \pm\frac{\pi}{2}$ that then $D[\mp\frac{\pi}{2}] = 0$. Hence, as D is non-singular, we must have $\phi = 0$ or $\phi = \pi$, which correspond to the two remaining cases. □

Proposition C.6. Let D_1 and D_2 be parametrised diagrams with a single parameter. Suppose that $D_1[\alpha] = \lambda(\alpha)D_2[\alpha]$ for some scalar function $\lambda : \mathbb{R} \to \mathbb{C}$ which is non-zero for all α. If α is trivial in D_1, then α is trivial in D_2 and there exists some scalar Clifford diagram S such that either $D_1' = S \otimes D_2'$ or $D_1' = S \otimes (D_2' \circ X(\pi))$. In either case we can rewrite, up to a scalar, $D_1[\alpha]$ into $D_2[\alpha]$, using the Clifford rewrite rules.

Proof. We make some case distinctions. First, if $D_1[\alpha] = 0$ for all α, then the same is true for $D_2[\alpha]$ and we are done. So assume $D_1 \neq 0$. By Proposition C.5 there are then 6 cases to check. The first 4 are very similar, so we only do the first of those. Suppose $D_1[0] = 0$. Then because $\lambda(0) \neq 0$ by assumption, we must also have $D_2[0] = 0$. Hence, by the first case in Proposition C.5 we have

$$\lambda(\alpha)D_2[\alpha] = D_1[\alpha] = \frac{1}{2}(1 - e^{i\alpha})D_1[\pi] = \lambda(\pi)\frac{1}{2}(1 - e^{i\alpha})D_2[\pi] = \lambda(\pi)D_2[\alpha].$$

Hence, $\lambda(\alpha) = \lambda(\pi)$ for all $\alpha \neq 0$. Since $D_2[0] = 0$ we might as well take $\lambda(0) = \lambda(\pi)$. Let S denote the scalar Clifford diagram denoting $\lambda(\pi)$. Then $D_1 = S \otimes D_2$.

The other cases where one of the diagrams is zero for one of the values of the parameter are handled similarly. Hence, assume that $D_1[\alpha] \neq 0$ for all α. Then also $D_2[\alpha] \neq 0$. We must then have $D_1[\pi] = \pm D_1[0]$, and $D_2[\pi] = \pm D_2[0]$. Again, we make some case distinctions.

Suppose $D_1[\pi] = D_1[0]$ and $D_2[\pi] = D_2[0]$. Then $D_1[\alpha] = D_1[0]$ and $D_2[\alpha] = D_2[0]$, so that $D_1[\alpha] = D_1[0] = \lambda(0)D_2[0] = \lambda(0)D_2[\alpha]$, and we are done. If instead $D_1[\pi] = -D_1[0]$ and $D_2[\pi] = -D_2[0]$, then $D_1[\alpha] = e^{i\alpha}D_1[0]$ and $D_2[\alpha] = e^{i\alpha}D_2[0]$, so that we can again verify that $D_1[\alpha] = \lambda(0)D_2[\alpha]$ and we are done.

Hence, assume that $D_1[\pi] = D_1[0]$, while $D_2[\pi] = -D_2[0]$ (the other remaining case is handled analogously). Using that $D_2[\alpha] = e^{i\alpha}D_2[0]$ we then calculate

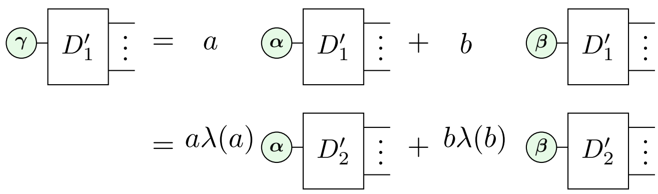
$$\lambda(\alpha)e^{i\alpha}D_2[0] = \lambda(\alpha)D_2[\alpha] = D_1[\alpha] = D_1[0] = \lambda(0)D_2[0],$$

so that $\lambda(\alpha) = e^{-i\alpha}\lambda(0)$. Letting $\overline{D}_2' = D_2' \circ X(\pi)$ we note that $\overline{D}_2[\alpha] = e^{i\alpha}D_2[-\alpha] = D_2[0]$. Hence, $D_1[\alpha] = e^{-i\alpha}\lambda(0)D_2[\alpha] = \lambda(0)D_2[0] = \lambda(0)\overline{D}_2[\alpha]$. We see then that ZX proves that $D_1[\alpha]$ and $S \otimes \overline{D}_2[\alpha]$ are equal, for $S = \lambda(0)$. □

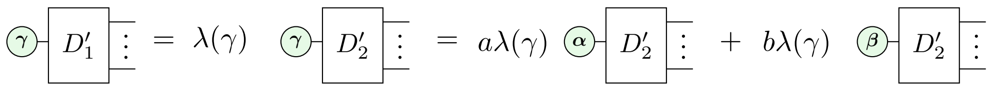
Lemma C.7. Let $D_1[\alpha]$ and $D_2[\alpha]$ be (not necessarily Clifford) parametrised diagrams with just one non-trivial parameter α. Suppose there is some scalar function $\lambda : \mathbb{R} \to \mathbb{C}$ such that $D_1[\alpha] = \lambda(\alpha) D_2[\alpha]$, then $\lambda(\alpha) = \lambda'$ is constant and we have $D_1' = \lambda' D_2'$.

Proof. Pick some α and β distinct from one another. By non-triviality of the parameter we know that $D_1[\alpha]$ is not proportional to $D_1[\beta]$ for some choice of α and β. This then also implies that $D_2[\alpha]$ is not proportional to $D_2[\beta]$. Let γ be any other phase, i.e. $\gamma \neq \alpha$ and $\gamma \neq \beta$.

Let a and b be the constants such that $|+_\gamma\rangle = a\,|+_\alpha\rangle + b\,|+_\beta\rangle$. These a and b exist because $|+_\alpha\rangle$ and $|+_\beta\rangle$ form a basis. Note that necessarily $a \neq 0$ and $b \neq 0$, as $\gamma \neq \alpha, \beta$. Then:



But on the other hand:



Hence, we must have:

$$a\lambda(\alpha)D_2[\alpha] + b\lambda(\beta)D_2[\beta] \;=\; a\lambda(\gamma)D_2[\alpha] + b\lambda(\gamma)D_2[\beta].$$

Now bring matching terms to each side:

$$a(\lambda(\alpha) - \lambda(\gamma))D_2[\alpha] = -b(\lambda(\beta) - \lambda(\gamma))D_2[\beta].$$

Since by assumption $D_2[\alpha]$ and $D_2[\beta]$ are not proportional to each other, we must have that each side is equal to 0. As we furthermore know that $a, b \neq 0$, we must then have $\lambda(\alpha) - \lambda(\gamma) = 0 = \lambda(\beta) - \lambda(\gamma)$. Hence: $\lambda(\alpha) = \lambda(\gamma) = \lambda(\beta)$. Since γ was an arbitrary phase not equal to α or β we see hence that $\lambda(\gamma) = \lambda'$ is a constant. But then $D_1[\alpha] = \lambda' D_2[\alpha]$ for all α, and hence the diagrams must be equal when we leave the wire open: $D_1' = \lambda' D_2'$. □

As we also noted in Remark 4.6, to use Proposition 4.5 we only really need to know that the two parametrised maps agree on two values. The proof of Lemma C.7 furthermore required a third value in order to get the phases to be equal. We hence note the following corollary.

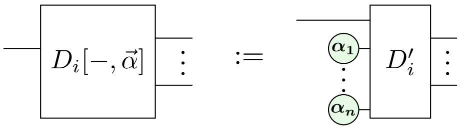
Corollary C.8. Let $D_1[\alpha]$ and $D_2[\alpha]$ be parametrised diagrams with just one parameter α that is furthermore non-trivial, and suppose that $D_1[k\frac{\pi}{2}] = \lambda(k) D_2[k\frac{\pi}{2}]$ for some scalar function $\lambda : \mathbb{Z} \to \mathbb{C}$ and all $k \in \mathbb{Z}$. Then $D_1' = \lambda' D_2'$ for some constant scalar λ'.

Proposition (Restatement of Prop. 4.11)**.** Let D_1 and D_2 be (not necessarily Clifford) parametrised diagrams with the same number of parameters and where all the parameters of D_1 are non-trivial. Then if $D_1[\vec{\alpha}] = \lambda(\vec{\alpha}) D_2[\vec{\alpha}]$ for all $\vec{\alpha}$, then $D_1' = \lambda' D_2'$ for some constant scalar λ'.

Proof. For convenience we will write $D_1[\alpha, \vec{\alpha}]$ for the set of parameters in D_1. For an integer string $\vec{k} \in \{0,1\}^n$ let $\vec{\alpha}^{\vec{k}} := (k_1\frac{\pi}{2}, \ldots, k_n\frac{\pi}{2})$. We then write $D_i^{\vec{k}} := D_i[-, \vec{\alpha}^{\vec{k}}]$ for the parametrised diagrams $D_1^{\vec{k}}$ and $D_2^{\vec{k}}$ which each have a single non-trivial parameter α.

Note that $D_1^{\vec{k}}[\alpha] = \lambda(\alpha, \vec{\alpha}^{\vec{k}}) D_2^{\vec{k}}[\alpha]$ for every α. Hence, by setting $\lambda^{\vec{k}}(\alpha) := \lambda(\alpha, \vec{\alpha}^{\vec{k}})$ we satisfy the conditions of Lemma C.7 so that we must have $(D_1^{\vec{k}})' = \lambda_{\vec{k}}(D_2^{\vec{k}})'$, where $\lambda_{\vec{k}}$ is a scalar.

Now consider the diagrams $D_i[-, \vec{\alpha}]$ for $i = 1, 2$, which are like $D_i[\alpha, \vec{\alpha}]$, but with the wire where α is plugged in left open:



Consider now a shorter integer string $\vec{k}' \in \{0,1\}^{n-1}$ and the one-parameter diagrams $E_i^{\vec{k}'}[\beta] := D_i[-, \beta, \vec{\alpha}^{\vec{k}'}]$. Then $E_1^{\vec{k}'}[k\frac{\pi}{2}] = \lambda_{k,\vec{k}'} E_2^{\vec{k}'}[k\frac{\pi}{2}]$ so that this satisfies the assumptions of Corollary C.8 (with $\lambda(k) := \lambda_{k,\vec{k}'}$). We hence note that $\lambda_{k,\vec{k}'}$ must be independent of k, so let's call the scalar just $\lambda_{\vec{k}'}$. Additionally, we must then have $(E_1^{\vec{k}'})' = \lambda_{\vec{k}'}(E_2^{\vec{k}'})'$.

We can now repeat the same argument until we have gone through all the parameters. We then conclude that $D_1' = \lambda' D_2'$ for some constant scalar λ'. □

Theorem (Restatement of Theorem 4.12)**.** Let D_1 and D_2 be two parametrised diagrams with the same number of parameters and where all the parameters of D_1 are non-trivial. Then if $D_1[\vec{\alpha}] = \lambda(\vec{\alpha}) D_2[\vec{\alpha}]$ for all $\vec{\alpha}$, then $D_1' = C \otimes D_2'$ for some Clifford scalar C and we can uniformly rewrite $D_1[\vec{\alpha}]$ into $C \otimes D_2[\vec{\alpha}]$ using Clifford rewrites.

Proof. By Proposition 4.11 we may assume that λ is in fact a constant λ'. Now if we put $\vec{\alpha} = (0 \cdots 0)$ then both $D_1[0 \cdots 0]$ and $D_2[0 \cdots 0]$ are entirely Clifford diagrams. But as we also have $D_1[0 \cdots 0] = \lambda' D_2[0 \cdots 0]$ we see that λ' must be a scalar that is representable as a Clifford diagram. Denote the Clifford diagram representing λ' by C. Then we see that $D_1[\alpha] = C \otimes D_2[\alpha]$ for all α. Hence by Proposition 4.5 we see that $D_1' = C \otimes D_2'$ and that these can be rewritten into one another by the Clifford rewrite rules.

□

Remark C.9. This proof currently requires a strong form of non-triviality, which is that for any choice of $\vec{\alpha}$, $D[\alpha, \vec{\alpha}]$ is not proportional to $D[\alpha', \vec{\alpha}]$, for any choice of $\alpha \neq \alpha'$. Or in the contra-positive form: we already call a parameter trivial when there is at least one choice of $\vec{\alpha}$ such that there exist $\alpha \neq \alpha'$ with $D[\alpha, \vec{\alpha}]$ proportional to $D[\alpha', \vec{\alpha}]$. We conjecture that we can generalise the proof to only requiring that $D[\alpha, \vec{\alpha}] \neq 0$ for any choice of parameter.