Counting base phi representations

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Abstract

In a base phi representation a natural number is written as a sum of powers of the golden mean φ . There are many ways to do this. How many? Even if the number of powers of φ is finite, then any number has infinitely many base phi representations. By not allowing an expansion to end with the digits 0,1,1, the number of expansions becomes finite, a solution proposed by Ron Knott. Our first result is a recursion to compute this number of expansions. This recursion is closely related to the recursion given by Neville Robbins to compute the number of Fibonacci representations of a number, also known as Fibonacci partitions. We propose another way to obtain finitely many expansions, which we call the natural base phi expansions. We prove that these are closely connected to the Fibonacci partitions.

Keywords: Base phi; Lucas numbers; Fibonacci numbers; Fibonacci partitions

1 Introduction

A natural number N is written in base phi if N has the form

$$N = \sum_{i=-\infty}^{\infty} a_i \varphi^i,$$

where $a_i = 0$ or $a_i = 1$, and where $\varphi := (1 + \sqrt{5})/2$ is the golden mean.

There are infinitely many ways to do this. When the number of powers of φ is finite we write these representations (also called expansions) as

$$\alpha(N) = a_L a_{L-1} \dots a_1 a_0 \cdot a_{-1} a_{-2} \dots a_{R+1} a_R.$$

Infinitely many expansions can be generated in a rather trivial way from expansions with just a few powers of φ using the replacement $100 \to 011$ at the end of the expansion.

So we use Knott's truncation rule from [11]:

$$a_{R+2}a_{R+1}a_R \neq 011. (1)$$

Let $\operatorname{Tot}^{\kappa}(N)$ be the number of base phi expansions of the number N satisfying Equation (1):

$$\text{Tot}^{\kappa} = 0, 1, 1, 2, 3, 3, 5, 5, 5, 8, 8, 8, 5, 10, 13, 12, 12, 13, 10, 7, 15, 18, 21, 16, 20, 20, 16, 21, 18, 15, 7, 17, ...$$

In 1957 George Bergman ([1]) proposed restrictions on the digits a_i which entail that the representation becomes unique and finite. This is generally accepted as *the* representation of the natural numbers in base phi. A natural number N is written in the Bergman representation if N has the form

$$N = \sum_{i=-\infty}^{\infty} d_i \varphi^i,$$

¹In OEIS ([16]): A289749 Number of ways not ending in 011 to write n in base phi.

with digits $d_i = 0$ or $d_i = 1$, and where $d_{i+1}d_i = 11$ is not allowed. We write these representations as

$$\beta(N) = d_L d_{L-1} \dots d_1 d_0 \cdot d_{-1} d_{-2} \dots d_{R+1} d_R.$$

A natural number N is written in base Fibonacci if N has the form

$$N = \sum_{i=2}^{\infty} c_i F_i,$$

where $c_i = 0$ or $c_i = 1$, and $(F_i)_{i \ge 0} = 0, 1, 1, 2, 3, \dots$ are the Fibonacci numbers.

Let $Tot^{FIB}(N)$ be the total number of Fibonacci expansions of the number N. Then

$$Tot^{FIB} = 1, 1, 1, 2, 1, 2, 2, 1, 3, 2, 2, 3, 1, 3, 3, 2, 4, 2, 3, 3, 1, 4, 3, 3, 5, \dots^{2}$$

This sequence has received a lot of attention, see e.g., the papers [9], [8], [4], [5], [17], [2], [14], and [3].

In 1952 the paper [12] proposed restrictions on the digits c_i which entail that the representation becomes unique. This is known as the Zeckendorf expansion of the natural numbers after the paper [15].

A natural number N is written in the Zeckendorf representation if N has the form

$$N = \sum_{i=2}^{\infty} e_i F_i,$$

with digits $e_i = 0$ or $e_i = 1$, and where $e_{i+1}e_i = 11$ is not allowed.

The Fibonacci representation and the base phi representation are closely related. We make a list.

Property	Fibonacci	Base phi
	$F_n: n \geq 2$	φ^n : n integer
Fundamental recursion	$F_{n+1} = F_n + F_{n-1}$	$\varphi^{n+1} = \varphi^n + \varphi^{n-1}$
Golden mean flip	$100 \rightarrow 011$	$100 \rightarrow 011$
Unique expansion	Zeckendorf	Bergman
Condition on the digits	no 11	no 11
Fundamental intervals	$[F_n, F_{n+1} - 1]$	$[L_{2n}, L_{2n+1}], [L_{2n+1} + 1, L_{2n+2} - 1]$
Examples $F_5 = 5$, $L_4 = 7$	$[5,7] = [2\ 2\ 1]$	[7,11] = [58885]
Examples $F_6 = 8$, $L_5 = 11$	[8,12] = [32231]	[12, 17] = [10 13 12 12 13 10]

Here the L_n are the Lucas numbers defined by $L_0 = 2$, $L_1 = 1$ and $L_{n+1} = L_n + L_{n-1}$ for $n \ge 1$. The intervals $\Lambda_{2n} = [L_{2n}, L_{2n+1}]$, $\Lambda_{2n+1} = [L_{2n+1} + 1, L_{2n+2} - 1]$ are called the even and odd Lucas intervals.

Replacing the digits 100 in an expansion by 011 will be called a golden mean flip. Our Theorem 2.1 shows that any base phi expansion can be obtained from the Bergman expansion by a finite number of such golden mean flips. There is a special case which needs attention, which we illustrate with an example. Let N=4. Then $\beta(4)=101\cdot01$. Applying the golden mean flip at the right gives the expansion $101\cdot0011$, which is not an allowed expansion. However, if we apply a second golden mean flip we can obtain $100\cdot1111$, which is an allowed expansion. We call this operation a double golden mean flip.

In Section 2 we determine a formula for $\text{Tot}^{\kappa}(N)$. In Section 3 we give simple formula's for $N = F_n$, and for $N = L_n$. In Section 4 we introduce a new way to count expansions, by defining *natural expansions*, and give a formula for $\text{Tot}^{\nu}(N)$, the number of natural base phi expansions of N. We moreover show that $(\text{Tot}^{\nu}(N))$ is a subsequence of the sequence of total numbers of Fibonacci representations. Section 5 gives important information on the different behaviour of phi expansions on the odd and the even Lucas intervals.

²In OEIS ([16]): A000119 Number of representations of n as a sum of distinct Fibonacci numbers.

2 A recursive formula for the number of Knott expansions

In this section we determine a formula for $\operatorname{Tot}^{\kappa}(N)$ for each natural number N.

Because of the fundamental recursion $\varphi^{n+1} = \varphi^n + \varphi^{n-1}$, one can transform a base phi expansion $\alpha(N) = a_L a_{L-1} \dots a_1 a_0 \cdot a_{-1} a_{-2} \dots a_{R+1} a_R$ of N with $a_{i+1} a_i a_{i-1} = 100$ to another base phi expansion of N, by the map

$$T_i: a_{i+1}a_ia_{i-1} \to [a_{i+1}-1][a_i+1][a_{i-1}+1],$$

where $R-1 \le i \le L-1$. This is a more detailed definition of the golden mean flip. In the definition we put of course $a_{R-1} = a_{R-2} = 0$.

The map T_i has an inverse denoted U_i for $R-1 \leq i \leq L$ given by

$$U_i: a_{i+1}a_ia_{i-1} \to [a_{i+1}+1][a_i-1][a_{i-1}-1],$$

as soon as $a_{i+1}a_ia_{i-1} = 011$. We call this map the reverse golden mean flip. Example: $U_1(110.01) = 1000.01$.

Theorem 2.1 Any finite base phi expansion $\alpha(N)$ with digits 0 and 1 of a natural number N can be obtained from the Bergman expansion $\beta(N)$ of N by a finite number of applications of the golden mean flip.

Proof: We prove this by showing that any expansion of N will be mapped to its Bergman expansion by a finite number of applications of the reverse golden mean flip. Let $\alpha(N) = a_L a_{L-1} \dots a_1 a_0 \cdot a_{-1} a_{-2} \dots a_{R+1} a_R$ be an expansion of N with digits 0 and 1. When 11 does not occur in $\alpha(N)$, then $\alpha(N) = \beta(N)$, and there is nothing to do. Otherwise, let $m := \max\{i : a_i a_{i-1} = 11\}$. First, suppose $m \le L - 2$. Then by the definition of m, we have $a_{i+1} = 0$. So for the two possibilities $a_{i+2} = 0$ and $a_{i+2} = 1$

$$U_i(\dots 0a_{i+1}a_ia_{i-1}\dots) = U_i(0011) = 0100$$
, and $U_i(\dots 1a_{i+1}a_ia_{i-1}\dots) = U_i(1011) = 1100$.

Note that in the first case the total number of 11 occurring in the expansion of N has decreased by 1, and in the second case it remained constant. However, in the second case the m of $U_i(\alpha(N))$ has increased by 2. If we keep iterating the reverse golden mean flip on the left most occurrence of 11, then either 0011 will occur, or if not, then $\alpha(N) = 1101...$ This is the case m = L, where there is a decrease in the number of 11, since $U_L(1101...) = 10001...$ Conclusion: in all cases the number of 11 will decrease by at least 1 after a finite number of applications of the reverse golden mean flip. So after a finite number of applications of the reverse golden mean flip we reach an expansion with no occurrences of 11. By definition, this is the Bergman expansion.

The case m=L has already been considered above, the case m=L-1 corresponds to $\alpha(N)=011\ldots$, where an application of the reverse golden mean flip leads also to a decrease in the number of 11.

Example 2.2 Let N = 5, with $\beta(5) = 1000 \cdot 1001$:

$$10\underline{1}\cdot1111 \to \underline{1}10\cdot0111 \to 1000\cdot0\underline{1}11 \to 1000\cdot1001$$
, with maps U_0, U_2, U_{-2} .

Our proof for Tot^{κ} resembles the work of Neville Robbins [17] on Fibonacci representations, but we have to incorporate the double golden mean flip defined in the Introduction. It then appears that the two recursions for Fibonacci representations and golden mean (Knott) representations are the same, but that there is a difference in the initial conditions.

The emphasis will be on the manipulation of 0-1-words, not on numbers. Let $\beta(N) = d_L d_{L-1} \dots d_1 d_0 \cdot d_{-1} d_{-2} \dots d_{R+1} d_R$. By removing the radix point, we obtain a 0-1-word $B(N) := d_L d_{L-1} \dots d_1 d_0 d_{-1} d_{-2} \dots d_{R+1} d_R$. Let us denote $r(B(N)) := \operatorname{Tot}^{\kappa}(N)$.

More generally, r(w) is the number of words satisfying the Knott condition that can be obtained from a word w by golden mean flips. Note that in general the representations that we obtain are not representations of a natural number—not for any choice of the radix point. An example is w = 100001, which represents $\varphi^5 + 1$. Nevertheless, these words represent numbers $a + b\varphi$ with non-negative natural numbers a an b in the ring $\mathbb{Z}(\varphi)$.

For example w = 100001 represents $5\varphi + 4$. This is the justification for continuing with the terminology of representations.

Here are two basic examples.

$$r(10^s) = \frac{1}{2}s + 1$$
 s even (2)

$$r(10^s) = \frac{1}{2}(s+1)$$
 s odd (3)

This follows easily by making golden mean flips from left to right.

Suppose the Bergman representation $\beta(N)$ of a number N contains n+1 ones. Then we can write for some numbers s_1, s_2, \ldots, s_n

$$B(N) = 10^{s_n} \dots 10^{s_2} 10^{s_1} 1.$$

We start with the case n = 2, so

$$B(N) = 10^{s_2} 10^{s_1} 1.$$

Let us call $I_2 := 10^{s_2}$ the initial segment of B(N), and $T_1 := 10^{s_1} 1$ the terminal segment of B(N).

We want to deduce $r(B(N)) = r(I_2T_1)$ from the number of representations $r(I_2)$ and $r(T_1)$. There are two cases to consider.

Type 1: Arbitrary combinations of representations of I_2 and T_1 .

Type 2: Arbitrary combinations of representations of I_2 and T_1 plus an 'overlap' combination.

Type 1 typically occurs if s_2 is even. For example for the case $s_2 = 4$, we have the three representations 10000, 01100, 01011. Note that in general these representations always end in 00 or 11.

So for Type 1 one has simply

$$r(B(N)) = r(I_2T_1) = r(I_2)r(T_1). (4)$$

But for s_2 odd, for example $s_2 = 5$, 100000, 011000, 010110 are the three representations of I_2 . Note that in general these representations always end in 00 or 10.

So if a representation 0v of T_1 starts with a 0, then the representation 0101100v generates an 'overlap' representation 0101011v via the golden mean flip.

Obviously it is true in general that an I_2 word with s_2 odd will have exactly one representation that ends in 10. Also important: there is no representation that ends in 01. Therefore, if $r^{(i)}(T_1)$ denotes the number of representations of T_1 starting with i for i = 0, 1, then we obtain for Type 2:

$$r(B(N)) = r(I_2T_1) = r(I_2)r(T_1) + r^{(0)}(T_1).$$
(5)

It thus follows from Equation (5), the trivial equation $r^{(0)}(T_1) + r^{(1)}(T_1) = r(T_1)$, and the fact that the segment $T_1 = 10^{s_1}$ 1 has just one representation that starts with a 1, that

$$r(B(N)) = r(T_1)[r(I_2) + 1] - r^{(1)}(T_1) = r(T_1)[r(I_2) + 1] - 1.$$
(6)

We continue with the case n = 3, so

$$B(N) = 10^{s_3} 10^{s_2} 10^{s_1} 1.$$

Now $I_3 := 10^{s_3}$ is the *initial segment*, and $T_2 := 10^{s_2}10^{s_1}1$ the *terminal segment*. As before there are two cases to consider to compute $r(B(N)) = r(I_3T_2)$.

Type 1: Arbitrary combinations of representations of I_3 and T_2 .

Type 2: Arbitrary combinations of representations of I_3 and I_2 plus an 'overlap' combination.

For Type 1 one has simply

$$r(B(N)) = r(I_3T_2) = r(I_3)r(T_2). (7)$$

For Type 2 one has:

$$r(B(N)) = r(I_3T_2) = r(I_3)r(T_2) + r^{(0)}(T_2).$$
(8)

Next, we split $T_2 = I_2T_1$, where $I_2 =: 10^{s_2}$. Then we have, since I_2 has just one representation that starts with a 1, that $r^{(1)}(T_2) = r(T_1)$. It thus follows from Equation(8) and $r^{(0)}(T_2) + r^{(1)}(T_2) = r(T_2)$ that

$$r(B(N)) = r(I_3)r(T_2) + r(T_2) - r^{(1)}(T_2) = r(T_2)[r(I_3) + 1] - r^{(1)}(T_2) = r(T_2)[r(I_3) + 1] - r(T_1).$$
 (9)

In the same way one proves that for any $k=1,\ldots n-1$ the following formula holds for s_{k+1} odd, where T_{k+1} is split as $T_{k+1}=I_{k+1}T_k$.

$$r(T_{k+1}) = r(T_k)[r(I_{k+1}) + 1] - r(T_{k-1}).$$
(10)

Defining $r_n := r(B(N))$, $r_k := r(T_k)$ for k = 1, ..., n-1 and $r_0 = 1$ (cf. Equation (6)), we have obtained a recursion that computes r(B(N)).

Theorem 2.3 For a natural number N let the Bergman expansion of N have n+1 digits 1. Suppose $\beta(N) = 10^{s_n} \dots 10^{s_1} 1$. Let $\text{Tot}^{\kappa}(N) = r_n$ be the number of Knott representations of N. Define the initial conditions: $r_0 = 1$ and $r_1 = \frac{1}{2}s_1 + 1$ if s_1 is even, $r_1 = \frac{1}{2}(s_1 + 1) + 1$ if s_1 is odd. Then for $n \geq 2$:

$$r_n = \begin{cases} \left[\frac{1}{2}s_n + 1\right] r_{n-1} & \text{if } s_n \text{ is even} \\ \left[\frac{1}{2}(s_n + 1) + 1\right] r_{n-1} - r_{n-2} & \text{if } s_n \text{ is odd} \end{cases}$$

The initial condition for r_1 is different from the Fibonacci case: if s_1 is odd, then the base phi expansion has an extra representation that is generated by the "double golden mean flip" (see Section 5).

3 Expansions of the Fibonacci numbers and the Lucas numbers

Let $(F_n) = 0, 1, 1, 2, 3, 5, ...$ be the Fibonacci numbers. We will determine the number of Knott representations of these numbers. Then we first have to find a formula for the Bergman expansions of the Fibonacci numbers. Recall that L(N) is the left most position of a 1 in $\beta(N)$, and that B(N) is $\beta(N)$ without the radix point in the expansion. In the following proposition and its proof the simple 1-to-1 correspondence between the pair (L(N), B(N)) and the Bergman expansion $\beta(N)$ plays an essential role.

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Proposition 3.1 a) For n \ge 3 one has L(F_n) = n - 1.
b) If n \ge 3 is odd, then B(F_n) = (1000)^p 1001, with p = (n - 3)/2.
c) If n \ge 4 is even, then B(F_n) = (1000)^p 10001, with p = (n - 4)/2.
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Proof: This will, of course, be proved by induction. It is simple to check that $\beta(F_3) = \beta(2) = 10.01$, $\beta(F_4) = \beta(3) = 100.01$, $\beta(F_5) = \beta(5) = 1000.1001$. So the statements hold for n = 3, 4, 5. We start the induction at n = 6. Since $F_6 = F_4 + F_5$, we have

$$\beta(F_4) = 100.01$$

$$\beta(F_5) = 1000.1001$$

$$\beta(F_4) + \beta(F_5) = 1100.1101$$

$$\beta(F_4) + \beta(F_5) = 10001.0001.$$

Here we applied the reverse golden mean flip twice in the last step. Since the last expansion does not have any 11, we must have $\beta(F_6) = 10001.0001$, and $L(F_6) = 5$. Next we show what happens at n = 7.

$$\beta(F_5) = 1000 \cdot 1001$$

$$\beta(F_6) = 10001 \cdot 0001$$

$$\beta(F_5) + \beta(F_6) = 11001 \cdot 1002$$

$$\beta(F_5) + \beta(F_6) = 100010 \cdot 001001.$$

Here we applied the reverse golden mean flip twice, and a shifted version of $\beta(2) = 10.01$ in the last step. Since the last expansion does not have any 11, we must have $\beta(F_7) = 100010.001001$, and $L(F_7) = 6$.

These addition schemes clearly generalize to $\beta(F_{n-2})+\beta(F_{n-1})$ with n-2 even, respectively odd, finishing the induction proof.

Theorem 3.2 For all $n \ge 1$ one has $\operatorname{Tot}^{\kappa}(F_n) = F_n$.

Proof: It is easily checked that the proposition holds for n=1 and n=2. So let $n\geq 3$. According to Proposition 3.1, the number of ones in $\beta(F_n)$ is p+2, with p+2=(n+1)/2 if n is odd, and p+2=n/2 if n is even. Also, $\beta(F_n)=10^{s_{p+1}}\dots 10^{s_k}\dots 10^{s_1}$ 1, with $s_k=3$ for $k=2,\dots,p+1$, and $s_1=2$ for n odd, $s_1=3$ for n even.

We apply Theorem 2.3. This yields that $\text{Tot}^{\kappa}(F_n) = r_{p+1}$, the number of Knott representations of the Bergman representation of F_n satisfies

$$r_{p+1} = 3r_p - r_{p-1}.$$

Here the initial conditions are $r_0 = 1$, $r_1 = s_1/2 + 1 = 2$ for n even, and $r_1 = (s_1 + 1)/2 + 1 = 3$ for n odd. Amusingly, the same recurrence relation holds for the subsequences of even and odd Fibonacci numbers:

$$F_{n+1} = F_n + F_{n-1} = 2F_{n-1} + F_{n-2} = 3F_{n-1} - F_{n-1} + F_{n-2} = 3F_{n-1} - F_{n-3}.$$

$$(11)$$

(I) Suppose n=2m+1 is odd. Then p=m-1, so $\operatorname{Tot}^{\kappa}(F_{2m+1})=r_m$.

We claim that $r_m = F_{2m+1}$ for all $m \ge 0$.

For m = 0, we have $r_0 = 1 = F_1$, and for m = 1 we have $r_1 = 2 = F_3$.

For $m \geq 2$,

$$r_m = 3r_{m-1} - r_{m-2} = 3F_{2m-1} - F_{2m-3} = F_{2m+1},$$

by the induction hypothesis and Equation (11).

(II) Suppose n = 2m + 2 is even. Then p = m - 1, so $\operatorname{Tot}^{\kappa}(F_{2m+2}) = r_m$. We claim that $r_m = F_{2m+2}$ for all $m \ge 0$.

For m = 0, we have $r_0 = 1 = F_2$, and for m = 1 we have $r_1 = 3 = F_4$.

For $m \geq 2$,

$$r_m = 3r_{m-1} - r_{m-2} = 3F_{2m} - F_{2m-2} = F_{2m+2}$$

by the induction hypothesis and Equation (11).

Combining (I) and (II) yields the conclusion: $\operatorname{Tot}^{\kappa}(F_n) = F_n$ for all $n \geq 1$.

At the Fibonacci numbers the total number of expansions is very large, but here we show that it is very small at the Lucas numbers (L_n) .

Theorem 3.3 For all $n \ge 1$ one has $\operatorname{Tot}^{\kappa}(L_{2n}) = \operatorname{Tot}^{\kappa}(L_{2n+1}) = 2n+1$.

Proof: The Lucas numbers have simple representations: $\beta(L_{2n}) = 10^{2n} \cdot 0^{2n-1}1$, $\beta(L_{2n+1}) = 1(01)^n \cdot (01)^n$. This can be shown using the golden mean flip as in Theorem 2.1.

So the representation of L_{2n} has only two ones. It follows therefore from Theorem 2.3 that $\text{Tot}^{\kappa}(L_{2n}) = r_1 = (s_1 + 1)/2 + 1 = 2n + 1$, since $s_1 = 4n - 1$ is odd.

The representation of L_{2n+1} has 2n+1 ones, and each s_k of the blocks 10^{s_k} is equal to 1, which is odd. It follows therefore from Theorem 2.3 that $\operatorname{Tot}^{\kappa}(L_{2n+1}) = r_n = 2r_{n-1} - r_{n-2}$. And indeed, induction gives that $r_n = 2(2n-1) - (2n-3) = 2n+1$.

4 Natural base phi expansions

A consequence of the application of the double golden mean shift is that length of the negative part of the Knott expansions may take two different values.

To obtain what we will call the *natural* expansions, let us delete all expansions that have a length of the negative part that is not equal to the length of the negative part of the Bergman expansion.

For example in the case N=4 Knott proposes the three expansions $101\cdot01, 100\cdot1111$ and 11.1111. However, there is only one natural expansion: the Bergman expansion $101\cdot01$.

Let $Tot^{\nu}(N)$ denote the number of natural base phi expansions. Then we have the following

$$(\operatorname{Tot}^{\nu}(N)) = 1, 1, 2, 2, 1, 5, 5, 4, 5, 4, 3, 1, 10, 13, 12, 12, 13, 10, 6, 11, 12, \dots$$

instead of

$$(\operatorname{Tot}^{\kappa}(N)) = 1, 1, 2, 3, 3, 5, 5, 5, 8, 8, 8, 5, 10, 13, 12, 12, 13, 10, 7, 15, 18, \dots$$

The number of natural base phi expansions can be determined in a way that is very similar to the Knott expansion case.

Theorem 4.1 For a natural number N let the Bergman expansion of N have n+1 digits 1. Suppose $\beta(N)=10^{s_n}\dots 10^{s_1} 1$. Let $\operatorname{Tot}^{\nu}(N)=r_n$ be the number of natural base phi representations of N. Define the initial conditions: $r_0=1$ and $r_1=\frac{1}{2}s_1+1$ if s_1 is even, $r_1=\frac{1}{2}(s_1+1)$ if s_1 is odd. Then for $n\geq 2$:

$$r_n = \begin{cases} \left[\frac{1}{2} s_n + 1 \right] r_{n-1} & \text{if } s_n \text{ is even} \\ \left[\frac{1}{2} (s_n + 1) + 1 \right] r_{n-1} - r_{n-2} & \text{if } s_n \text{ is odd} \end{cases}$$

Proof: This follows directly from Theorem 2.3 and its proof. The only difference between the process of generating all Knott expansions and all natural expansions is the double golden mean flip, which is performed in the Knott expansion at the segment $10^{s_1}1$, and only when s_1 is odd. So $\text{Tot}^{\nu}(N) = r_n$ satisfies the same recursion as $\text{Tot}^{\text{FIB}}(N)$, except that $r_1 = \frac{1}{2}(s_1+1)+1$ has to be replaced by $r_1 = \frac{1}{2}(s_1+1)$ in the case that s_1 is odd.

We will determine the total number of natural expansions of the Fibonacci numbers. First we present a lemma that emphasizes the inter-connection between the Fibonacci and the Lucas numbers. Recall the even and odd Lucas intervals $\Lambda_{2n} = [L_{2n}, L_{2n+1}], \Lambda_{2n+1} = [L_{2n+1} + 1, L_{2n+2} - 1]$ (cf. [6]).

Lemma 4.2 For all n = 1, 2, ... one has $F_{2n+2} \in \Lambda_{2n}$, $F_{2n+3} \in \Lambda_{2n+1}$.

Proof: By induction. For n = 1 we have $F_3 = 4 \in \Lambda_2 = [3, 4]$, and $F_5 = 4 \in \Lambda_3 = [5, 6]$. For n = 2 we have $F_6 = 8 \in \Lambda_4 = [7, 11]$, and $F_7 = 13 \in \Lambda_5 = [12, 17]$.

Suppose the statement of the lemma has been proved for F_{2n+1} and F_{2n+2} . So we know

$$F_{2n+1} \in [L_{2n-1} + 1, L_{2n} - 1] = \Lambda_{2n-1}$$

 $F_{2n+2} \in [L_{2n}, L_{2n+1}] = \Lambda_{2n}$

Adding the numbers in these two equations vertically, we obtain

$$F_{2n+3} \in [L_{2n+1} + 1, L_{2n+2} - 1] = \Lambda_{2n+1}.$$

We can then write

$$F_{2n+2} \in [L_{2n}, L_{2n+1}] = \Lambda_{2n}$$

 $F_{2n+3} \in [L_{2n+1} + 1, L_{2n+2} - 1] = \Lambda_{2n+1}.$

This time, adding gives

$$F_{2n+4} \in [L_{2n+2} + 1, L_{2n+3} - 1].$$

Since $F_{2n+4} \neq L_{2n+2}$, this implies that $F_{2n+4} \in \Lambda_{2n+2}$.

Theorem 4.3 For all n = 0, 1, 2, ... one has $Tot^{\nu}(F_{2n+2}) = F_{2n+1}$ and $Tot^{\nu}(F_{2n+3}) = F_{2n+3}$.

Proof: We use the result from Proposition 5.1, which gives that for all N from Λ_{2n+1} if $\beta(N) = ...10^{s_1}1$, then s_1 is even. So for all N from Λ_{2n+1} we have that the total number of natural expansions is equal to the total number of Knott expansions. In particular we obtain from Lemma 4.2, using Theorem 3.2, that

$$\operatorname{Tot}^{\nu}(F_{2n+3}) = \operatorname{Tot}^{\kappa}(F_{2n+3}) = F_{2n+3}.$$

From Proposition 3.1, part c) we have that $B(F_{2n+2}) = (1000)^p 10001$ with p = (2n+2-4)/2 = n-1. Therefore r_n satisfies the recurrence relation $r_n = 3r_{n-1} - r_{n-2}$, with $r_1 = \frac{1}{2}(3+1) = 2 = F_3$. This is the recurrence relation for the Fibonacci numbers with odd indices, cf. Equation (11). Therefore $r_n = F_{2n+1}$. \square

There is a direct connection between the total number of natural expansions and the total number of Fibonacci expansions.

Theorem 4.4 For every N > 3 let $\beta(N) = d_{L(N)} \dots d_{R(N)}$ be the Bergman expansion of N. Then

$$\operatorname{Tot}^{\nu}(N) = \operatorname{Tot}^{\operatorname{FIB}}(F_{-R(N)+2} N).$$

Proof: Suppose that $\beta(N) = d_L \dots d_R$, so $N = \sum_L^R d_i \varphi^i$. Multiply by φ^{-R+2} :

$$\varphi^{-R+2}N = \sum_{i=R}^{L} d_i \varphi^{i-R+2} = \sum_{j=2}^{L-R+2} d_{j+R-2} \varphi^j = \sum_{j=2}^{L-R+2} e_j \varphi^j$$

where we substituted j = i - R + 2, and defined $e_j := j + R - 2$.

Next we use the well known equation $\varphi^j = F_j \varphi + F_{j-1}$:

$$[F_{-R+2}\varphi + F_{-R+1}]N = \sum_{j=2}^{L-R+2} e_j [F_j\varphi + F_{j-1}].$$

This implies that

$$F_{-R+2}N = \sum_{j=2}^{L-R+2} e_j F_j.$$

We conclude that the number $F_{-R+2}N$ has a Zeckendorf expansion given by the sum on the right side.

But the manipulations above can be made for any 0-1-word of length L-R+1, so the golden mean flips of $d_L \dots d_R$ are in 1-to-1 correspondence with golden mean flips of $e_2 \dots e_{L-R+2}$. This implies that $\operatorname{Tot}^{\nu}(N) = \operatorname{Tot}^{\operatorname{FIB}}(F_{-R(N)+2}N)$.

Example 1 The Bergman expansion of 4 is 101.01, and $F_4 = 3$. So $\text{Tot}^{\nu}(4) = \text{Tot}^{\text{FIB}}(12) = 1$.

Example 2 The Bergman expansion of 14 is 100100·001001, and $F_8 = 21$. So $\text{Tot}^{\nu}(14) = \text{Tot}^{\text{FIB}}(294) = 12$.

Example 3 Consider the Lucas numbers. From $L_{2n} = \varphi^{2n} + \varphi^{-2n}$, and $L_{2n+1} = L_{2n} + L_{2n-1}$:

 $\beta(L_{2n}) = 10^{2n} \cdot 0^{2n-1}1, \quad \beta(L_{2n+1}) = 1(01)^n \cdot (01)^n.$

We read off: $R(L_{2n}) = -2n$, $R(L_{2n+1}) = -2n$.

It is also clear that $\operatorname{Tot}^{\nu}(L_{2n})=2n$, and $\operatorname{Tot}^{\nu}(L_{2n+1})=1$.

So Theorem 4.4 gives the total number of Fibonacci representations of $F_{2n+2}L_{2n}$ and $F_{2n+2}L_{2n+1}$:

 $\operatorname{Tot^{FIB}}(F_{2n+2}L_{2n}) = 2n, \operatorname{Tot^{FIB}}(F_{2n+2}L_{2n+1}) = 1 \text{ for all } n \ge 1.$

We find in [16]: From Miklos Kristof, Mar 19 2007:

Let L(n) = A000032(n) = Lucas numbers. Then for a >= b and odd b, F(a+b) - F(a-b) = F(a) * L(b).

So $F_{2n+2}L_{2n+1} = F_{4n+3} - F_1 = F_{4n+3} - 1$. But $\text{Tot}^{\text{FIB}}(F_n - 1) = 1$ is a well-known formula.

Remark 4.5 An alternative proof of Theorem 4.3 can be given with Theorem 4.4.

From Proposition 5.1 we know that a number N with $\beta(N) = d_L \dots d_R$ in Λ_{2n} has -R(N) = 2n. According to Lemma 4.2: $F_{2n+2} \in \Lambda_{2n}$. So Theorem 4.4 leads to

$$\operatorname{Tot}^{\nu}(F_{2n+2}) = \operatorname{Tot}^{\operatorname{FIB}}(F_{2n+2} F_{2n+2}) = \operatorname{Tot}^{\operatorname{FIB}}(F_{2n+2}^2).$$

To finish the alternative proof, one needs to know that $\operatorname{Tot^{FIB}}(F_{2n}^2) = F_{2n-1}$ for all $n \geq 1$. This can be proved similarly to the proof of the main result of Stockmeyers paper [14]. The extra trick is to jointly prove the formula $\operatorname{Tot^{FIB}}(F_{2n}^2) = F_{2n-1}$ together with the formula $\operatorname{Tot^{FIB}}(F_{2n+1}^2 - 2) = F_{2n}$.

the formula $\text{Tot}^{\text{FIB}}(F_{2n}^2) = F_{2n-1}$ together with the formula $\text{Tot}^{\text{FIB}}(F_{2n+1}^2 - 2) = F_{2n}$. The main tool of the proof is Klarner's result from [9]: $\text{Tot}^{\text{FIB}}(n) = \text{Tot}^{\text{FIB}}(n-F_m) + \text{Tot}^{\text{FIB}}(F_{m+1}-n-2)$, which holds for $m \geq 4$ and $F_m \leq n < F_{m+1} - 1$.

The proof by induction applies Klarner's identity with $n = F_m^2$, respectively with $n = F_m^2 - 2$. Here the identities (5): $F_m^2 = F_{2m-2} + F_{m-2}^2$ and (6): $F_m^2 = F_{2m-1} - F_{m-1}^2$ from [14] make the induction step work.

5 Comparing Knott expansions and natural expansions

It is not hard to see that the double golden mean shift—in general combined with more golden mean shifts—can be applied if and only if the expansion ends in $10^{s}1$, where s is odd. So the difference between the Knott expansions and the natural expansions is made more explicit by part a) of the following result.

Proposition 5.1 a) A number N is in Λ_{2n} if and only if $\beta(N) = ...10^s1$, where s is odd, and N is in Λ_{2n+1} if and only if $\beta(N) = ...10^s1$, where s is even.

b) Let $\beta(N) = L(N)...R(N)$. A number N in Λ_{2n} has -R(N) = 2n, a number N in Λ_{2n+1} has -R(N) = 2n + 2.

Proposition 5.1 will be proved by induction. Thus we need recursions to let the proof work. These are given in the paper [7], from which we repeat the following.

To obtain recursive relations, the interval $\Lambda_{2n+1} = [L_{2n+1} + 1, L_{2n+2} - 1]$ has to be divided into three subintervals. These three intervals are

$$\begin{split} I_n := & [L_{2n+1}+1,\,L_{2n+1}+L_{2n-2}-1],\\ J_n := & [L_{2n+1}+L_{2n-2},\,L_{2n+1}+L_{2n-1}],\\ K_n := & [L_{2n+1}+L_{2n-1}+1,\,L_{2n+2}-1]. \end{split}$$

It will be convenient to extend the monoid of words of 0's and 1's to the corresponding free group. So, for example, $(01)^{-1}0001 = 1^{-1}001$.

Theorem 5.2 [Recursive structure theorem, [7]]

I For all $n \ge 1$ and $k = 0, ..., L_{2n-1}$ one has $\beta(L_{2n} + k) = \beta(L_{2n}) + \beta(k) = 10 ... 0 \beta(k) 0 ... 01$.

II For all $n \ge 2$ and $k = 1, ..., L_{2n-2} - 1$

$$I_n: \quad \beta(L_{2n+1}+k) = 1000(10)^{-1}\beta(L_{2n-1}+k)(01)^{-1}1001,$$

$$K_n: \quad \beta(L_{2n+1}+L_{2n-1}+k) = 1010(10)^{-1}\beta(L_{2n-1}+k)(01)^{-1}0001.$$

Moreover, for all $n \geq 2$ and $k = 0, \ldots, L_{2n-3}$

$$J_n: \beta(L_{2n+1} + L_{2n-2} + k) = 10010(10)^{-1}\beta(L_{2n-2} + k)(01)^{-1}001001.$$

Proof of Proposition 5.1: To start the induction, we note that

$$\begin{split} &\Lambda_2 = [3,4]; \quad \beta(3) = 100 \cdot 01, \; \beta(4) = 101 \cdot 01, \\ &\Lambda_3 = [5,6]; \quad \beta(5) = 1000 \cdot 1001, \; \beta(6) = 1010 \cdot 0001. \end{split}$$

For the even intervals we have that $\beta(L_{2n}) = 10^{2n} \cdot 0^{2n-1}1$, so the expansion of the first element ends indeed in $10^{s}1$, where s is odd. Note also that $R(L_{2n}) = 2n$, and this property will hold for all $L_{2n} + k$, $k = 0, \ldots, L_{2n-1}$ since the sum $\beta(L_{2n}) + \beta(k)$ in **I** does not change the length of the negative part. Moreover, since the length of the negative part of each $\beta(k)$ in the sum $\beta(L_{2n}) + \beta(k)$ is even (by the induction hypothesis for part **b**), the expansion must end in $10^{s}1$ with s odd, simply because the difference of two even numbers is even.

For the odd intervals we have to consider the three cases from II.

For I_n : we know that $\beta(L_{2n-1}+k)$ ends in 01, so $\beta(L_{2n+1}+k)$ ends in 1001. For part **b**): the length of the negative part is increased by 2.

For K_n : $L_{2n-1} + k$ is from an odd interval, so the expansion ends in $10^{2t}1$ from some t > 0. But then the expansion of $L_{2n+1} + L_{2n-1} + k$ ends in $10^{2t}1 (01)^{-1}0001 = 10^{2t-1}0001 = 10^{2t+2}1$. For part **b**): the length of the negative part is increased by 2.

For J_n : obviously $\beta(L_{2n+1} + L_{2n-2} + k)$ ends in 1001. For part **b**): the length of the negative part is 2n - 2 + 4 = 2n + 2.

References

- [1] G. Bergman, A number system with an irrational base, Math. Mag. 31 (1957), 98–110.
- [2] Bicknell and Fielder, The number of Representations of N Using Distinct Fibonacci Numbers, Counted by Recursive Formulas, Fibonacci Quart. 37.1, Fibonacci Quart. (1999),47–60.
- [3] S. Chow and T. Slattery, On Fibonacci partitions, Journal of Number Theory, Volume 225, August 2021, 310–326.
- [4] L. Carlitz, Fibonacci representations, Fib. Quart. 6 (1968), 193–220.
- [5] L. Carlitz, R. Scoville, V. E. Hoggatt, Jr., Fibonacci representations, Fibonacci Quart. 10 (1972), 1–28.
 [Also see L. Carlitz, R. Scoville, V. E. Hoggatt, Jr., Addendum to the paper: Fibonacci representations, Fibonacci Quart. 10 (1972), 527–530.]
- [6] F.M. Dekking, Base phi representations and golden mean beta-expansions, Fib. Quart. 58 (2020), 38–48.
- [7] F.M. Dekking, How to add two numbers in base phi, Fib.Quart. <u>59</u> (2021), 19–22.
- [8] D. A. Klarner, Partitions of n into distinct Fibonacci numbers, Fibonacci Quart. 6 (1968), 235-243.
- [9] D. A. Klarner, Representations of N as a Sum of Distinct Elements from Special Sequences, The Fibonacci Quarterly, 4.4 (1966), 289–306, 322.
- [10] Ron Knott, The Lucas Numbers: A formula for the Lucas Numbers involving Phi and phi. http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/lucasNbs.html
- [11] Ron Knott, Using Powers of Phi to represent Integers (Base Phi): Powers of Phi, Base Phi representations, http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/phigits.html
- [12] C.G. Lekkerkerker, Voorstelling van natuurlijke getallen door een som van Fibonacci, Simon Stevin 29 (1952), 190–195.
- [13] M. Lothaire, Algebraic combinatorics on words, Encyclopedia of Mathematics and its Applications 90, Cambridge University Press, 2002.
- [14] P. K. Stockmeyer, A smooth tight upper bound for the Fibonacci representation function R(n), Fibonacci Quart. 46/47 (2008/2009), 103–106.
- [15] E. Zeckendorf, Réprésentation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas, Bull. Soc. Roy. Sci. Liège 41 (1972), 179–182.
- [16] On-Line Encyclopedia of Integer Sequences, founded by N. J. A. Sloane, electronically available at http://oeis.org.
- [17] N. Robbins, Fibonacci partitions, Fibonacci Quart. 34 (1996), 306–313.