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Dynamic Random Intersection Graph: Dynamic Local Convergence and Giant Structure

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ABSTRACT

Random intersection graphs containing an underlying community structure are a popular choice for modeling real-world networks. Given the group memberships, the classical random intersection graph is obtained by connecting individuals when they share at least one group. We extend this approach and make the communities *dynamic* by letting them alternate between an active and inactive phase. We analyse the new model, delivering results on degree distribution, local convergence, largest connected component, and maximum group size, paying particular attention to the dynamic description of these properties. We also describe the connection between our model and the bipartite configuration model, which is of independent interest.

1 | Introduction and Main Results

1.1 | Introduction

Networks are present in many areas of everyday life. We distinguish for instance social networks such as acquaintance networks, technological networks such as the worldwide web, or biological networks such as neural networks (see [1] for an extensive overview). Networks can be investigated in terms of many aspects, one of them being the existence and shape of local communities.

Local communities are subnetworks that have, on average, more connections than the network as a whole. They are crucial components of many real-life networks, such as social networks or the Internet, and they naturally give rise to high clustering (see [2, Chapters 7 and 11])—one of the most fiercely investigated network properties, that many systems seem to share.

There are many reasons why communities appear in networks. It might be because of a set of common features shared by a certain number of individuals (for example the same nationality) or some underlying geometry (for example living in the same city). In our model, we intuitively associate communities with social groups that people can belong to, such as families, groups of friends, commuters on the same bus and so forth. However, the model can also be relevant to other types of networks with similar structures.

Due to their complexity, real-world networks are often modeled with the help of *random graphs*. There are many ways of implementing a community structure such as described above. A classic choice is the random intersection graph (RIG), first introduced in [3]. The distinctive feature of intersection graph models is a double layer. We first establish a bipartite graph, with vertices on one side and communities on the other. After drawing edges between the two groups, we obtain a resulting graph by connecting two vertices if and only if they both belong to the same community. This procedure is called the one-mode projection. Throughout the years, multiple suggestions on how to generate such bipartite graphs with group memberships appeared [4]. This includes pre-assigning the number of group memberships to every vertex and then connecting them to groups in a uniform manner (uniform RIG [5, 6] or generalized RIG [7–11]), generating group membership via the bipartite configuration model, that is, assigning half-edges to individuals and groups and then connecting them uniformly at random [12–15] or via the bipartite Norros–Reitu model [16], or performing independent percolation on the complete bipartite graph (binomial RIG [3, 17, 18] or inhomogeneous RIG [19, 20]).

Our approach shares some similarities with the mentioned inhomogeneous RIG since we also assign weight w_i to every vertex i and the probability that vertex i belongs to a certain community depends on this weight. However, we add a novel *dynamic* factor by letting all communities go through active and inactive phases. We argue that such a modification is relevant for real-world networks since many of our regular social contacts are temporary. For instance, we usually do not spend all our days with our colleagues, and we only meet our closest friends a few times a week for a couple of hours. Other examples of temporary social interactions are concerts or rides with public transport. Our model is also a natural generalization of static graphs with communities, as the scenario in which all of the groups are present all the time is a special case.

The dynamics we implement in our model significantly differ from the well-investigated dynamic of graphs whose size evolves in time, such as preferential attachment models [21–24]. The size of our graph is *static*, but the connections between individuals and communities (and hence, in the resulting graph the connections between individuals themselves) keep evolving. This makes the model more similar to graphs with dynamic bond percolation [25, 26], or evolving configuration models [27, 28].

Contribution of the article: In this article, we investigate degree distribution, local convergence, and behavior of the largest connected component. A main innovation of our work is the methodology required to give a dynamic description of a random graph. We investigate its dynamic local convergence and, in particular, we introduce the concept of a dynamic largest connected component from the perspective of a uniformly chosen vertex, looking at it as a process in time. We then show in which way it is related to the dynamic local limit. To the best of our knowledge, the concept of dynamic local limit has only been discussed in the recent paper [29] where the authors adopt a different approach and treat a different model. We also develop an auxiliary result on the relationship between our model and the bipartite configuration model described in [14], that can be of independent interest (see Appendix A). This result can be thought of as a bipartite/community version of the relationship between the configuration model and generalized random graph (GRG) (see [22, Theorem 7.18] and also Section 2 in this article).

Outline of the article: We introduce our model and all necessary assumptions in Section 1.2. We state our main results in Section 1.3 and provide a discussion in Section 1.4. We describe the overview of the proofs and state some secondary results in Section 2. In Section 3, we prove the main results presented in Section 1.3. For the conceptually straightforward proofs of the remaining results, we refer the reader to relevant Appendices.

1.2 | Model and Notation

In this section, we intuitively and formally introduce the model and list some necessary assumptions that will hold throughout the article.

Vertices and groups: Let $[n] = \{1, \dots, n\}$ denote the set of vertices and equip each of these vertices with a unique weight w_i, where $\boldsymbol{w} = (w_i)_{i\in[n]}$ are deterministic weights. Further, let $[n]_k$ denote the set of subsets of size k of $[n]$ and let $\cup_{k\geq 2}[n]_k$—a union of all k-element subsets of $[n]$ with $k \geq 2, k \in \mathbf{N}$—denote the set of groups.

Group dynamics: Every group $a \in \cup_{k\geq 2}[n]_k$ will alternate between an ON and OFF state independently of all other groups, following a continuous-time Markov process. The holding times, that is, the time that a group spends in each of the states, are exponentially distributed with rates

$$\lambda^a_{\text{ON}} = 1 \text{ and } \lambda^a_{\text{OFF}} = \frac{f(|a|)\prod_{i\in a} w_i}{\ell_n^{|a|-1}} \tag{1}$$

respectively, where $\boldsymbol{w} = (w_i)_{i\in[n]}$ are certain vertex weights, $\ell_n = \sum_{i\in[n]} w_i$ is the total weight and $|a|$ denotes the size of a group $a \in \cup_{k\geq 2}[n]_k$. Naturally, $f(|a|)$ is a function of a group's size and can be chosen in a flexible way. The stationary distribution $\boldsymbol{\pi} = [\pi_{\text{ON}}, \pi_{\text{OFF}}]$ of these Markov chains is given by

$$\begin{aligned} \pi^a_{\text{ON}} &= \frac{\lambda^a_{\text{OFF}}}{\lambda^a_{\text{ON}} + \lambda^a_{\text{OFF}}} = \frac{f(|a|)\prod_{i\in a} w_i}{\ell_n^{|a|-1} + f(|a|)\prod_{i\in a} w_i} \text{ and} \\ \pi^a_{\text{OFF}} &= \frac{\lambda^a_{\text{ON}}}{\lambda^a_{\text{ON}} + \lambda^a_{\text{OFF}}} = \frac{\ell_n^{|a|-1}}{\ell_n^{|a|-1} + f(|a|)\prod_{i\in a} w_i} \end{aligned} \tag{2}$$

We will write "a is ON at time t" to denote that a group a is in an ON state at time t, and abbreviate this to "a is ON" when the time is clear from the context. We will also write "a switches ON" to denote the event of changing status from being in an OFF state to being in an ON state, and analogously for reversed roles of ON and OFF. Moreover, we initialize the group statuses with probabilities corresponding to the stationary distribution, that is, at time $s = 0$, each group is ON with probability π^a_{ON} and OFF with probability π^a_{OFF}, independently of all other groups. To obtain our dynamic random intersection graph, we draw an edge between all the vertices in groups that are ON, so that, in the dynamic random graph, the groups represent dynamic cliques.

Having described the dynamic random intersection graph for given weights, we next formulate the conditions on the weights that we will assume throughout the article.

Assumptions on weights: We first define what the empirical distribution of the weights is:

Definition 1.1. (Empirical vertex weights distribution). We define the empirical distribution function of the vertex weights as

$$F_n(x) = \frac{1}{n}\sum_{i\in[n]} \mathbb{1}_{\{w_i \le x\}}, \text{ for } x \ge 0 \tag{3}$$

F_n can be interpreted as the distribution function of the weight of a vertex chosen from $[n]$ uniformly at random. We denote such a uniformly chosen vertex by o_n and its weight by $W_n = w_{o_n}$. We impose the following conditions on the vertex weights:

Condition 1.1. (Regularity condition for vertex weight). There exists a distribution function F such that, as $n \to \infty$, the following conditions hold:

a. Weak convergence of vertex weight:

$$W_n \xrightarrow{d} W \tag{4}$$

where W_n and W have distribution functions F_n and F, respectively. Equivalently, for any x for which $x \mapsto F(x)$ is continuous,

$$\lim_{n\to\infty} F_n(x) = F(x) \tag{5}$$

b. Convergence of average vertex weight:

$$\lim_{n\to\infty} \mathbf{E}[W_n] = \mathbf{E}[W] \tag{6}$$

where W_n and W have distribution functions F_n and F, respectively. Further, we assume that $\mathbf{E}[W] > 0$.

c. Convergence of second moment of vertex weights:

$$\lim_{n\to\infty} \mathbf{E}[W_n^2] = \mathbf{E}[W^2] < \infty \tag{7}$$

Remark 1.1. For the time being, we assume Condition 1.1c. However, the results treated in this article are also true without it, which is very convenient for applications. We explain how Condition 1.1c can be lifted in Remark B.3.

Assumptions on the dependence on the group sizes: We take

$$f(|a|) = |a|!p_{|a|} \tag{8}$$

where $(p_k)_{k\ge 2}$ is the probability mass function of the group sizes. A particularly important case is a power-law group-size distribution where p_k is approximately proportional to $k^{-(\alpha+1)}$ for k large. We denote

$$\mu = \sum_{k=2}^{\infty} k p_k \tag{9}$$

and assume $\mu < \infty$. We also assume that the second moment of the group-size distribution is finite, so that $\alpha > 2$, that is,

$$\mu_{(2)} = \sum_{k=2}^{\infty} k^2 p_k < \infty \tag{10}$$

These assumptions are necessary for the graph to be sparse, that is, the average degree remains uniformly bounded, which becomes clear after reading Appendix B, where we derive convergence of the average degree and group-size.

The above description fully defines the dynamic random intersection graph that we will investigate. Below, we will first discuss the relation between the stationary and dynamic settings, after which we will give a reformulation of the static and dynamic models in terms of static and dynamic bipartite graphs. This reformulation will be essential in the remainder of our proof.

Stationary vs dynamic models: Due to the Markovian nature of our groups, our models can be examined in two scenarios: under the stationary distribution, and dynamically, for every time $s \in [0, T]$ with T fixed, incorporating continuous switching of the groups between the ON and OFF states. To make it clear which situation we are referring to, we will call the graph created via the stationary distribution the *stationary* or *static* graph, and we will refer to the graphs incorporating the group dynamic as *dynamic* graphs. Below we define the two scenarios separately and we also introduce different notation for each of the cases, which we will use consistently throughout the article.

The static bipartite graph: As mentioned before, there are two layers in our model: the underlying bipartite structure consisting of the set of vertices on the one hand, together with the set of groups on the other, and the resulting intersection graph connecting vertices that meet in the same group. Below, we will think of the two layers as corresponding to left- and right-vertices, thus effectively turning our random intersection graph model into a bipartite model. This reformulation will prove to be very convenient. We start with defining the first layer corresponding to the vertices in our random intersection graph. We again split between the static and dynamic bipartite graphs.

The stationary (static) bipartite GRG, denoted $\mathrm{BGRG}_n(\boldsymbol{w})$, consists of two bipartite sets of vertices: $[n]$ and $\cup_{k\ge2}[n]_k$, with $k \ge 2$. We intuitively think of them as left- and right-vertices respectively. $\mathrm{BGRG}_n(\boldsymbol{w})$ is formed by drawing edges between the left-vertices and right-vertices (groups) that are ON: with probability π^a_{ON} we draw edges between the group $a \in \cup_{k\ge2}[n]_k$ and each of its vertices from $[n]$, independently of all other groups. Analogously, with probability π^a_{OFF}, there are no edges between the group $a \in \cup_{k\ge2}[n]_k$ and each of its vertices. In the stationary (static) scenario, groups do not alternate between ON and OFF states, that is, they are always either in an ON or in an OFF state. To differentiate degrees in the underlying and the resulting graph, we write $d_i^{(l)}$ for the degree of a left-vertex $i \in [n]$ in $\mathrm{BGRG}_n(\boldsymbol{w})$ and $d_a^{(r)}$ for the degree of a right-vertex $a \in \cup_{k\ge2}[n]_k$ in $\mathrm{BGRG}_n(\boldsymbol{w})$. We will often refer to them as the left- and right-degree, respectively. The left-degrees denote the number of groups that vertices belong to, while the right-degrees are the number of left-vertices connected to a particular right-vertex, that is, the group sizes. Hence,

$$d_i^{(l)} = \sum_{k=2}^{\infty} \sum_{a\in[n]_k : a\ni i} \mathbb{1}_{\{a \text{ is ON}\}} \tag{11}$$

where $\mathbb{1}_A$ denotes the indicator of an event A, and

$$d_a^{(r)} = |a| \cdot \mathbb{1}_{\{a \text{ is ON}\}} \tag{12}$$

We also quantify the number of all active groups as

$$\sum_{k=2}^{\infty} \sum_{a \in [n]_k} \mathbb{1}_{\{a \text{ is ON}\}} \tag{13}$$

As one can see, we allow every collection of k vertices, with $k \in [2, n]$, to form a group.

The static intersection graph: The second layer—the resulting static random intersection graph denoted $\mathrm{DRIG}_n(\boldsymbol{w})$—has vertex set $[n] = \{1, \dots, n\}$. It is formed from $\mathrm{BGRG}_n(\boldsymbol{w})$ by drawing an edge between two vertices $i, j \in [n]$ if they are in at least one ON group together in $\mathrm{BGRG}_n(\boldsymbol{w})$. Hence, $\mathrm{DRIG}_n(\boldsymbol{w})$ is a projection of $\mathrm{BGRG}_n(\boldsymbol{w})$, the random multi-graph given by the edge multiplicities $X(i,j))_{i,j \in [n]}$, such that

$$X(i,j) = \sum_{k=2}^{\infty} \sum_{a \in [n]_k} \mathbb{1}_{\{i \text{ in } a\} \cap \{j \text{ in } a\}} \mathbb{1}_{\{a \text{ is ON}\}} \tag{14}$$

We often refer to the above procedure as "the community projection" or "the one-mode projection." Note that in our set-up, the community projection assigns multiple edges to a pair of vertices when they share more than one ON group in $\mathrm{BGRG}_n(\boldsymbol{w})$. However, we remark that in our sparse setting such a situation is quite unlikely. Let d_i denote the degree of a vertex $i \in [n]$ in $\mathrm{DRIG}_n(\boldsymbol{w})$. Then

$$d_i = \sum_{k=2}^{\infty} \sum_{a \in [n]_k : a \ni i} (|a| - 1) \mathbb{1}_{\{a \text{ is ON}\}} \tag{15}$$

The dynamic bipartite graph process: We denote the dynamic bipartite GRG by $(\mathrm{BGRG}_n^s(\boldsymbol{w}))_{s \in [0,T]}$. It is a dynamic graph process in which at time $s = 0$ every group is ON with probability π_{ON}^a and OFF with probability π_{OFF}^a, independently of all other groups. For $s > 0$, groups keep switching ON and OFF, according to the evolution of the continuous-time Markov Chains explained before. Analogously to the static $\mathrm{BGRG}_n(\boldsymbol{w})$, the edges are drawn between a group a and all of its vertices whenever the group is ON, and are removed when the group switches OFF (hence, for all $s \in [0, T]$, $\mathrm{BGRG}_n^s(\boldsymbol{w})$ is equal in distribution to the static $\mathrm{BGRG}_n(\boldsymbol{w})$).

The dynamic intersection graph process: We denote the dynamic random intersection graph process by $(\mathrm{DRIG}_n^s(\boldsymbol{w}))_{s \in [0,T]}$. $(\mathrm{DRIG}_n^s(\boldsymbol{w}))_{s \in [0,T]}$ is obtained from $(\mathrm{BGRG}_n^s(\boldsymbol{w}))_{s \in [0,T]}$ similarly to the way in which $\mathrm{DRIG}_n(\boldsymbol{w})$ is obtained from $\mathrm{BGRG}_n(\boldsymbol{w})$: for every $s \in [0, T]$, an edge is drawn between $i, j \in [n]$ if they are in at least one ON group together at time s in $(\mathrm{BGRG}_n^s(\boldsymbol{w}))_{s \in [0,T]}$. Hence, $(\mathrm{DRIG}_n^s(\boldsymbol{w}))_{s \in [0,T]}$ is again a projection of $(\mathrm{BGRG}_n^s(\boldsymbol{w}))_{s \in [0,T]}$, the random dynamic multi-graph given by the edge multiplicities $(X^s(i,j))_{i,j \in [n], s \in [0,T]}$ such that for each $s \in [0, T]$,

$$X^s(i,j) = \sum_{k=2}^{\infty} \sum_{a \in [n]_k} \mathbb{1}_{\{i \text{ in } a\} \cap \{j \text{ in } a\}} \mathbb{1}_{\{a \text{ is ON at time } s\}} \tag{16}$$

Let $d_i(s)$ denote the degree of a vertex $i \in [n]$ at time s in $(\mathrm{DRIG}_n^s(\boldsymbol{w}))_{s \in [0,T]}$. Then

$$d_i(s) = \sum_{k=2}^{\infty} \sum_{a \in [n]_k : a \ni i} (|a| - 1) \mathbb{1}_{\{a \text{ is ON at time } s\}} \tag{17}$$

Uniformly chosen vertices: Throughout the article we often discuss results with respect to a uniformly chosen vertex. We denote a uniformly chosen vertex in the bipartite graph by V_n^b and in the intersection graph by o_n. Note that this notation does not specify whether we refer to the stationary, dynamic, or any other type of graph that may appear throughout the article. If such an indication is needed, then we will make it clear by stating an appropriate name of the graph, according to the notation introduced in the previous paragraphs. We also denote the degree of a uniformly chosen vertex in $\mathrm{BGRG}_n(\boldsymbol{w})$ by $D_n^{(l)}$ and the degree of a uniformly chosen ON group by $D_n^{(r)}$. We denote the degree of a uniformly chosen vertex in $\mathrm{DRIG}_n(\boldsymbol{w})$ by D_n.

1.3 | Main Results

In this section, we state our main results. We first investigate the behavior of our model in stationarity. We start with the description of **local convergence**, that is, the convergence of the neighborhood counts. We explain this notion in more detail in Section 2.2.1. We continue with the description of the largest connected component. Next, we proceed to analyse the dynamic situation. We state our results on dynamic local convergence and dynamic giant membership process. We close by discussing the dynamics of the largest group that is ON.

1.3.1 | Stationarity

It turns out that under the stationary distribution, and given appropriate conditions, our underlying graph—$\mathrm{BGRG}_n(\boldsymbol{w})$—is related to the bipartite configuration model BCM, denoted $\mathrm{BCM}_n(\boldsymbol{d})$ in this article to accentuate the degree parameter, investigated for instance in [13, 14, 30]. Hence, it is possible to transfer the results on the local convergence and the largest connected component from [13, 14] to our model. The link between the two models and the transfer of results are explained in detail in further sections. We now state its most important consequences: the results on the static local limit and largest connected component.

Static local limit: Similarly, as in $\mathrm{BCM}_n(\boldsymbol{d})$, the neighborhood of a uniformly chosen vertex in $\mathrm{BGRG}_n(\boldsymbol{w})$ resembles a mixture of two branching processes, each of them corresponding to offspring distributions of left- and right-vertices. Then, the neighborhood of a uniformly chosen vertex in $\mathrm{DRIG}_n(\boldsymbol{w})$ resembles a community projection (see (14)) of the left-partition of this mixture. We summarize these statements in the following theorem, the limiting objects themselves are explained in detail in Section 2.2.2:

Theorem 1.1. (Local convergence of BGRGn(w) and DRIGn(w)). *Consider* $\mathrm{BGRG}_n(\boldsymbol{w})$ *under Condition 1.1. As* $n \to \infty$, $(\mathrm{BGRG}_n(\boldsymbol{w}), V_n^b)$ *converges locally in probability to* $(\mathrm{BP}_\gamma, 0)$, *where* $(\mathrm{BP}_\gamma, 0)$ *is a mixture of two branching processes. Consequently for* $\mathrm{DRIG}_n(\boldsymbol{w})$ *under Condition 1.1, as* $n \to \infty$, $(\mathrm{DRIG}_n(\boldsymbol{w}), o_n)$ *converges locally in probability to* (CP, o), *where* (CP, o) *is a random rooted graph.*

We prove Theorem 1.1 in Appendix B.4.

Static largest connected component: Denote the largest connected cluster (from now on also referred to as "the giant component" or shortly "the giant") in $\mathrm{DRIG}_n(\boldsymbol{w})$ by $\mathscr{C}_1$, and the second largest connected cluster by $\mathscr{C}_2$ (breaking ties arbitrarily). The random variables $\tilde{D}^{(l)}$ and $\tilde{D}^{(r)}$ are strongly connected to the offspring distributions present in the local limit of the underlying $\mathrm{BGRG}_n(\boldsymbol{w})$ and are explained in detail in Section 2.2.2. Denote the probability generating function of a random variable Y taking values in $\mathbf{N}$ by $G_Y : [0,1] \to [0,1]$, that is,

$$G_Y(z) = \mathbf{E}[z^Y], z \in [0,1] \tag{18}$$

Furthermore, we write $\xrightarrow{\mathbf{P}}$ for convergence in probability. The following theorem gives a precise condition for the existence of the giant component in $\mathrm{DRIG}_n(\boldsymbol{w})$:

Theorem 1.2. (Giant component in $\mathrm{DRIG}_n(\boldsymbol{w})$). *Consider $\mathrm{DRIG}_n(\boldsymbol{w})$. There exists $\eta_l \in [0,1]$, the smallest solution of the fixed point equation*

$$\eta_l = G_{\tilde{D}^{(r)}}(G_{\tilde{D}^{(l)}}(\eta_l)) \tag{19}$$

and $\xi_l = 1 - G_{D^{(l)}}(\eta_l) \in [0,1]$ such that

$$\frac{|\mathscr{C}_1|}{n} \xrightarrow{\mathbf{P}} \xi_l \tag{20}$$

Furthermore, $\xi_l > 0$ exactly when

$$\mathbf{E}[\tilde{D}^{(l)}]\mathbf{E}[\tilde{D}^{(r)}] = \frac{\mathbf{E}[W^2](\mu_{(2)} - \mu)}{\mathbf{E}[W]} > 1 \tag{21}$$

We call the above the supercritical case. In this case, $\mathscr{C}_1$ is unique, in the sense that $|\mathscr{C}_2|/n \xrightarrow{\mathbf{P}} 0$, where $\mathscr{C}_2$ is the second largest component.

We give a more detailed definition of $\mathscr{C}_1$ and related objects in Section 2. We prove Theorem 1.2 in Appendix B.5.

1.3.2 | Dynamic Local Convergence

We start with the description of the local limit of our graph in the dynamic scenario. This concept is rather new in the study of the local convergence of graphs and to the best of our knowledge it has not attracted much attention yet (see the discussion later on).

The concept behind dynamic local convergence explained: To examine the dynamic behavior, we rely on a well-known characterization of weak convergence of probability measures on $(D([0,T],S), \mathcal{D})$ equipped with the Skorokhod J_1 topology, where S is a separable metric space, $D([0,T],S)$ is the space of càdlàg functions $f : [0,T] \to S$ and $\mathcal{D}$ denotes the Borel σ-field generated by $D([0,T],S)$. Weak convergence of probability measures in such a space is guaranteed by convergence of finite-dimensional distributions together with tightness (see [31, Theorem 13.1]), which naturally implies convergence of random processes with sample paths in $D([0,T],S)$. Our dynamic graph processes can be perceived as a stochastic process, from $[0,T]$ onto the space of rooted graphs. Hence, in the remaining of the article, whenever we refer to weak convergence of dynamic graph processes, we mean weak convergence in $D([0,T],S)$ in the Skorokhod J_1 topology, with $S = (\mathcal{G}^\star, \mathrm{d}_{\mathcal{G}^\star})$, where $\mathcal{G}^\star$ is the space of rooted graphs and $\mathrm{d}_{\mathcal{G}^\star}$ is a metric on it. We describe this metric space of rooted graphs in detail in Section 2. When looking at a rooted dynamic graph $\left((\mathrm{DRIG}_n^s(\boldsymbol{w}), o_n)\right)_{s\in[0,T]}$, we choose the root o_n once, uniformly at random, and then investigate how the graph around it evolves in time. This yields one of our main results—the local convergence of the dynamic intersection graph seen as a stochastic process in time:

Theorem 1.3. (Dynamic local limit of DRIGns(w)). *Under Condition 1.1, as $n \to \infty$, the dynamic intersection rooted graph process $\left((\mathrm{DRIG}_n^s(\boldsymbol{w}), o_n)\right)_{s\in[0,T]}$ converges locally weakly in $D([0,T],S)$, with $S = (\mathcal{G}^\star, \mathrm{d}_{\mathcal{G}^\star})$, in the Skorokhod J_1 topology to $((\mathrm{CP}^s, o))_{s\in[0,T]}$, where, for every $s \in [0,T]$, $((\mathrm{CP}^s, o))_{s\in[0,T]}$ is a stochastic process of random rooted marked graphs.*

We prove Theorem 1.3 in Section 3.7.

The limiting dynamic object: Due to the one-mode projection needed to obtain $\mathrm{DRIG}_n^s(\boldsymbol{w})$, the limiting object is also rather involved and we fully explain what it is later (see Section 2.4). Now we only provide an informal description. The limit $((\mathrm{CP}^s, o))_{s\in[0,T]}$ depends heavily on the limit of $((\mathrm{BGRG}_n^s(\boldsymbol{w}), o_n))_{s\in[0,T]}$. In turn, the latter is a dynamic version of $(\mathrm{BP}_\gamma, 0)$—the limit of the static $\mathrm{BGRG}_n(\boldsymbol{w})$. We denote this dynamic limiting process by $((\mathrm{BP}_\gamma^s, 0))_{s\in[0,T]}$. As we have mentioned in the text before Theorem 1.1, $(\mathrm{BP}_\gamma, 0)$ is a rooted mixture, with "0" referring to the root, of two marked branching processes: one corresponding to the offspring distribution of the left-partition and the other one to the offspring distribution of the right-partition. Then, at time $s = 0$, $((\mathrm{BP}_\gamma^s, 0))_{s\in[0,T]}$ is equal in distribution to $(\mathrm{BP}_\gamma, 0)$. For $s > 0$, $((\mathrm{BP}_\gamma^s, 0))_{s\in[0,T]}$ continuously evolves according to the following dynamic: every right-vertex is removed at rate 1. At the same time, new groups arrive at the root and the other present left-vertices at rates proportional to their weights. Each of these groups has size k with probability kp_k/μ. These groups get attached with all their current descendants. $((\mathrm{CP}^s, o))_{s\in[0,T]}$ is then a community projection of the left-partition of $((\mathrm{BP}_\gamma^s, 0))_{s\in[0,T]}$ just like $(\mathrm{DRIG}_n^s(\boldsymbol{w}))_{s\in[0,T]}$ is a community projection of $(\mathrm{BGRG}_n^s(\boldsymbol{w}))_{s\in[0,T]}$ (see (16)).

Remark 1.2. (Dynamic generalized random graphs). Note that if we only allow groups of size 2, that is, we set

$$f(|a|) = \begin{cases} 2, & |a| = 2 \\ 0, & |a| \neq 2 \end{cases} \tag{22}$$

(recall (8)), then the stationary edge probability for any two vertices $i, j \in [n]$ equals $\frac{2w_i w_j}{\ell_n + 2w_i w_j}$, which corresponds to the edge probability in the GRG with vertex weights $2w_i$ for $i \in [n]$ (see Section 2.1 for more information on the static GRG and [29] for a related dynamic random graph model). Furthermore, if we also fix $w_i = \frac{n\lambda}{2(n-\lambda)}$ for all $i \in [n]$, then the edge probability between any two vertices equals $\frac{\lambda}{n}$. Hence, in this setting, our dynamic random graph model transforms into the classical dynamic Erdős-Rényi random graph, investigated for example in [32, 33]. Therefore Theorem 1.3 implies dynamic local convergence of the respective dynamic versions of these random graphs as well.

Dynamic degrees: Knowing the dynamic local limit, we can extract the information on the degree of a uniformly chosen vertex in $\mathrm{DRIG}_n^s(\boldsymbol{w})$ for every $s \in [0, T]$.

Degree in $\mathrm{DRIG}_n^s(\boldsymbol{w})$: Denote the number of active groups of size k containing a vertex i at time $s \in [0, T]$ by $C_k^s(i)$. Further, denote the number of groups of size k containing a vertex i at time $s \in [0, T]$ in the limiting structure $((\mathrm{CP}^s, o))_{s\in[0,T]}$ by $\overline{C}_k^s(i)$. Recall that we denote the degree of a uniformly chosen vertex o_n by D_n. Thus, naturally, we denote the dynamic degree of a uniformly chosen vertex at time s by D_n^s such that

$$(D_n^s)_{s\in[0,T]} = \left(\sum_{k=2}^{\infty} (k-1) C_k^s(o_n) \right)_{s\in[0,T]} \tag{23}$$

Since the degree distribution is a functional of the local limit we deduce the following convergence of the degree process:

Corollary 1.1. *As $n \to \infty$,*

$$(D_n^s)_{s\in[0,T]} \xrightarrow{\mathrm{d}} \left(\sum_{k=2}^{\infty} (k-1) \overline{C}_k^s(o) \right)_{s\in[0,T]} \tag{24}$$

in the Skorokhod J_1 topology on $D([0, T], \mathbf{R}_+)$. At time $s = 0$, $(\overline{C}_k^0(o))_{k\geq 2}$ are independent, Poisson distributed variables with parameters $(k p_k W)_{k\geq 2}$.

1.3.3 | Dynamic Giant Component

Denote the giant component in $(\mathrm{DRIG}_n^s(\boldsymbol{w}))_{s\in[0,T]}$ at time s by $\mathscr{C}_1^s$ and the connected component of the root o at time s in the limiting structure $((\mathrm{CP}^s, o))_{s\in[0,T]}$ (see Theorem 1.3) by $\mathscr{C}^s(o)$. We examine the behavior of the process $J_n(s) = \mathbb{1}_{\{o_n \in \mathscr{C}_1^s\}}$. From the behavior of the static giant (see Theorem 1.2), pointwise for every s, as $n \to \infty$,

$$J_n(s) \xrightarrow{\mathrm{d}} \mathbb{1}_{\{|\mathscr{C}^s(o)| = \infty\}} \tag{25}$$

However, we want to investigate the behavior of $J_n(s)$ as a stochastic process in time. It turns out that this question can be linked to local neighborhoods in our graph and answered thanks to local convergence and the "almost local" description of the giant in Theorem 1.2:

Theorem 1.4. (Dynamic giant component). *As $n \to \infty$,*

$$\left(\mathbb{1}_{\{o_n \in \mathscr{C}_1^s\}} \right)_{s\in[0,T]} \xrightarrow{\mathrm{d}} \left(\mathbb{1}_{\{|\mathscr{C}^s(o)| = \infty\}} \right)_{s\in[0,T]} \tag{26}$$

in the Skorokhod J_1 topology on $D([0, T], \{0, 1\})$ with $T \in \mathbf{R}_+$.

We provide the proof of Theorem 1.4 in Section 3.8.

1.3.4 | Maximal Group Size

Define

$$K_{\max}^{\{0\}} = \max_{a \in \cup_{k\geq 2}[n]_k : a \text{ is ON at time } 0} |a| \tag{27}$$

and $K_{\max}^{(0,T]}$ is the maximum group size in the set of groups that switch ON in the time interval $(0, T]$, respectively. We define for every $T \geq 0$

$$K_{\max}^{[0,T]} = \max \left\{ K_{\max}^{\{0\}}, K_{\max}^{(0,T]} \right\} \tag{28}$$

with $K_{\max}^{\{0\}}$ and $K_{\max}^{(0,T]}$ independent, as a consequence of the fact that different groups arrive independently. We further define $K_{\max}^{(s,T]}$ for every $s \in [0, T)$ as the maximum group size in the set of groups that switched ON in time interval $(s, T]$. Hence, for every $s \in [0, T)$,

$$K_{\max}^{[0,T]} = \max \left\{ K_{\max}^{(0,s]}, K_{\max}^{(s,T]} \right\} \tag{29}$$

and $K_{\max}^{(0,s]}, K_{\max}^{(s,T]}$ are independent. Now define $\kappa_{\max}^{[0,T]}$ such that, for every $T \geq 0$,

$$\kappa_{\max}^{[0,T]} = \max \left\{ \kappa_{\max}^{\{0\}}, \kappa_{\max}^{(0,T]} \right\} \tag{30}$$

with $\kappa_{\max}^{\{0\}}$ and $\kappa_{\max}^{(0,T]}$ independent and

$$\mathbf{P}\left(\kappa_{\max}^{\{0\}} \leq k \right) = e^{-c_p k^{-\alpha} \mathbf{E}[W]} \tag{31}$$

and

$$\mathbf{P}\left(\kappa_{\max}^{(0,T]} \leq k \right) = e^{-c_p T k^{-\alpha} \mathbf{E}[W]} \tag{32}$$

Hence, for every $T \geq 0$,

$$\mathbf{P}\left(\kappa_{\max}^{[0,T]} \leq k \right) = e^{-c_p (T+1) k^{-\alpha} \mathbf{E}[W]} \tag{33}$$

and the evolution of the whole process is such that for every $s_1, s_2, \dots, s_u$ such that $0 < s_1 < \cdots < s_u \leq T$, we have

$$\kappa_{\max}^{(0,T]} = \max \left\{ \kappa_{\max}^{(0,s_1]}, \kappa_{\max}^{(s_1,s_2]}, \dots, \kappa_{\max}^{(s_u,T]} \right\} \tag{34}$$

where, for non-overlapping time intervals $(s_1, s_2]$ and $(s_3, s_4]$, $\kappa_{\max}^{(s_1,s_2]}$, $\kappa_{\max}^{(s_3,s_4]}$ are independent, and for all $i, j \in \mathbf{N}$, $\kappa_{\max}^{(s_i,s_j]}$ has the same distribution as $\kappa_{\max}^{(0,s_j-s_i]}$. We show that the largest group size converges in distribution as a stochastic process to $(\kappa_{\max}^{[0,T]})_{T\geq 0}$:

Theorem 1.5. (Maximum group size). *If the group-size distribution is a power law, that is, if*

$$\sum_{l\geq k} p_l = c_p k^{-\alpha}(1 + o(1)), \qquad \text{as } k \to \infty \tag{35}$$

for some $\alpha > 3$, then, as $n \to \infty$,

$$\left(\frac{K_{\max}^{[0,T]}}{n^{1/\alpha}} \right)_{T\geq 0} \xrightarrow{\mathrm{d}} (\kappa_{\max}^{[0,T]})_{T\geq 0} \tag{36}$$

in the Skorokhod J_1 topology on $(D[0, T], \mathbf{R}_+)$.

We prove Theorem 1.5 in Section 3.9.

Remark 1.3. Note that one could also investigate a slightly different dynamic process, namely $(n^{-1/\alpha} K_{\max}^s)_{s\in[0,T]}$, where $K_{\max}^s$ is the maximal group that is ON at time s. We conjecture that a proof of convergence of such a process should be closely related to the proof of Theorem 1.5 that we present in the next section.

However, the dynamics of $(n^{-1/\alpha}K^s_{\max})_{s\in[0,T]}$ are significantly more involved: when the so far largest group switches OFF, then the largest ON group becomes the previously second largest one. Hence, to analyse such a process it would be necessary to keep track of the large ON groups as a process of infinite length.

1.4 | Discussion

In this section, we comment on the advantages and limitations of our model and results, and present open problems.

The dynamic nature of our model: An undeniably interesting feature of our model is the introduction of temporary connections between vertices. Intuitively, it makes sense that a dynamic model should describe real-world networks more accurately. The dynamic we introduce is also interesting purely from the perspective of random graph theory. The framework for the dynamic local limit we provide is an alternative to the one in [29]. However, the limit itself is a rather complex construction and its interpretation is not very straightforward.

Other work on bipartite graphs with group structure: To derive results on the static $\mathrm{BGRG}_n(\boldsymbol{w})$, which later on lead to results on $\mathrm{DRIG}_n(\boldsymbol{w})$, we heavily rely on [13, 14]. The authors of these papers derive statements on local convergence and giant components in the bipartite configuration model with communities, which can be transferred to our model by showing an appropriate relation between the bipartite configuration model conditioned on simplicity and the bipartite GRG conditioned on its degree sequence. We provide more details in Section 2.

Other work on dynamic local convergence: To the best of our knowledge, the only paper that treats dynamic local convergence is a recent pre-print by Dort and Jacob [29]. The approach of the authors is different from ours: we are taking a classic approach by treating the dynamic graph as a stochastic process onto the space of rooted graphs with a traditional local metric, whereas they define a metric that incorporates time. Contrary to [29], we also consider the dynamic giant component.

Group sizes: One of the beneficial features of our model is the existence of big groups (see Theorem 1.5). Such groups are an important factor in real-world social networks, which are highly clustered. We set the model in a way that allows for flexibility in the choice of the group-size distribution, as one can consider various $(p_k)_{k\geq 2}$, not necessarily heavy-tailed ones. We are aware that many of our proof techniques (see Appendix B) require p_k to have a finite second moment, which might not seem ideal. However, the finite second moment is needed to obtain a sparse graph, that is, a graph with a bounded average degree.

Choice of parameters and alternative interpretation: The model is quite flexible in the choice of parameters, as we do not determine the weight variables or the group-size distribution and only keep some general assumptions about them. However, the model does not allow for the one-to-one transfer of degree distribution from real-world data, as opposed to the bipartite configuration model with communities (studied in [14]). Moreover, at first glance, choosing the stationary distribution as in (2) might not seem intuitive. However, there is a nice intuitive description of this model. Our model is very closely connected to a Poisson process dynamic: take a situation, where we form a new group according to a Poisson process with intensity ℓ_n. When a group is formed, we choose its size $k \geq 2$ according to some size distribution $(p_k)_{k\geq 2}$. (This actually requires the choice of $f(k) = k!p_k$ in (8), which, in turn, gives rise to the factor 2 for the weights in (22).) Lastly, from all the groups of the chosen size, we pick the one to appear proportionally to products of weights, that is, with probability

$$\frac{\prod_{i\in a} w_i}{\sum_{b\in[n]_k}\prod_{j\in b} w_j}$$

for all $a \in [n]_k$. It can be shown that our model and the model we just described yield the same degree sequences and hence, produce very similar graphs. However, our model conditioned on its degree sequence has the advantage of being uniform over all bipartite graphs with such degree sequence, which proves to be a very useful feature.

Relationship to the bipartite configuration model: An interesting byproduct of our article is a relationship between the underlying bipartite structure in our model and the bipartite configuration model. More mathematical details regarding this relationship are stated in the next section and in the Appendix A. This raises the question of whether the dynamic versions of these models are also similarly related.

2 | Overview of the Proofs

In this section, we provide the ideas behind the proofs of our main results. We include shorter and straightforward proofs. We also state auxiliary results that we think might be of independent interest.

2.1 | Bipartite Generalized Random Graphs and Configuration Models are Equivalent

Two of the most popular approaches to modeling real-world networks are the GRG and the configuration model. The GRG, was introduced in 2006 by Britton, Deijfen and Martin-Löf (see [34]). In this model, each vertex $i \in [n]$ is given a weight w_i and the probability that there is an edge between vertex i and vertex j is equal to

$$p_{ij} = \frac{w_i w_j}{\ell_n + w_i w_j}$$

with $\ell_n = \sum_{i\in[n]} w_i$, just like in our model. Naturally, assigning edge probabilities according to weights can also be done in a different way. For a more general version see [35], for related models see the Chung–Lu model (for instance [36]) or the Norros–Reitu model [37]. For an overview of results on the classic GRG see [22, Chapter 6].

In contrast, in the configuration model (CM), the degrees of the vertices are fixed upfront. The concept of the configuration model originates in the early works of Bollobás (see [38]). Since then, various configuration models have been proposed but in its most standard formulation, the configuration model refers to a uniform pairing of half-edges, which can be represented in a form of a graph by assigning an appropriately determined number of half-edges to every vertex and then connecting them uniformly to form edges. A graph obtained in this way conditioned on being simple is uniform over the space of all simple graphs with a given degree sequence [2]. Such a model was popularized and intensively studied by Molloy and Reed (see [39, 40]). For an overview of results for the classic configuration model see [22, Chapter 7]. Again, there are many modifications of the classic configuration model such as the configuration model with households [41].

Despite their differences, it turns out that the GRG conditioned on its degree sequence and the configuration model conditioned on being simple are equal in distribution (see [22, Theorem 7.18]), which we will (and have in the title of this section) often shortly refer to as "equivalent." Also note that the static $\mathrm{BGRG}_n(\boldsymbol{w})$ introduced by us can be perceived as a bipartite version of the GRG (hence the name $\mathrm{BGRG}_n(\boldsymbol{w})$), where also certain communities are present. It turns out that this model is accordingly equivalent to a bipartite configuration model with communities $\mathrm{BCM}_n(\boldsymbol{d})$, introduced and studied in [13, 14], under the same conditions that guarantee equivalence of the classic GRG and the configuration model. More precisely, it turns out that, under such conditions, both $\mathrm{BGRG}_n(\boldsymbol{w})$ and $\mathrm{BCM}_n(\boldsymbol{d})$ from [14] are simple uniform random graphs (i.e., their distribution is uniform over all simple graphs with the prescribed degree sequence), and hence have the same distribution.

This relationship is one of the most important building blocks of the proofs of our results. Thanks to it, we can transfer results proved in [13, 14] to our graph. As these auxiliary statements can be of independent interest, we present them below, starting with $\mathrm{BGRG}_n(\boldsymbol{w})$. However, we first introduce some necessary notation. Note that we can encode the distribution of $\mathrm{BGRG}_n(\boldsymbol{w})$ via a sequence of indicator random variables: take $x = (x_a)_{a\in\cup_{k\geq 2}[n]_k}$—a sequence of 0s and 1s—and $X = (X_a)_{a\in\cup_{k\geq 2}[n]_k}$—a sequence of independent random variables describing the existence of particular groups that is,

$$\mathbf{P}(X_a = 1) = 1 - \mathbf{P}(X_a = 0) = \pi^a_{\mathrm{ON}} \tag{37}$$

Further, denote

$$d^{(l)}_i(X) = \sum_{a\in\cup_{k\geq 2}[n]_k : a\ni i} X_a, \text{and } d^{(r)}_a(X) = |a| \cdot X_a \tag{38}$$

Analogously, we define $d^{(l)}_i(x) = \sum_{a\in\cup_{k\geq 2}[n]_k : a\ni i} x_a, d^{(r)}_a(x) = |a| \cdot x_a$. Now let us explain what we mean by conditioning on a degree sequence in the case of $\mathrm{BGRG}_n(\boldsymbol{w})$. For the left-degree sequence, it simply means assigning some $d^{(l)}_i$ for all $i \in [n]$. However, the construction is slightly different for the right-degree sequence. Note that we want to leave some randomness and match the setting of the $\mathrm{BGRG}_n(\boldsymbol{w})$ to the one of the $\mathrm{BCM}_n(\boldsymbol{d})$, in which the uniform matching determines which vertices will be in a group. Hence, when prescribing the right-degrees, we only specify *how many* groups of a particular size will be ON, and not what their elements are: for every $k \geq 2$, fix $a_k \in \mathbf{N}$ and denote $A_k = \#\{\text{ON groups of size } k\}$. Thus, prescribing the right-degree sequence means that we prescribe that $A_k = a_k$ for all $k \geq 2$, that is, exactly a_k groups out of all $[n]_k$ possible groups of size k are ON. We now explain in detail how to extract such a general degree sequence describing only how many right-vertices of a particular degree exist, as recorded in $(d^{(r)}_a)_a$—the degree sequence specifying precisely which groups are ON, while not recording the vertices that are in groups that are ON.

Fix a bipartite graph encoded by $x = (x_a)_{a\in\cup_{k\geq 2}[n]_k}$, and let a_k denote the number of right-vertices of degree k. To facilitate future notation and clarify the link between $\mathrm{BGRG}_n(\boldsymbol{w})$ and $\mathrm{BCM}_n(\boldsymbol{d})$, we denote the total number of ON groups as $M = \sum_{k\geq 2} a_k$ and we denote the right-vertices by $[M] = \{1, 2, \dots, M\}$. By convention, fix $a_1 = 1$ and denote $s_k = \sum_{l=1}^{k} a_l$. Denote the right-degree sequence of $x = (x_a)_{a\in\cup_{k\geq 2}[n]_k}$ by

$$d^{(r)}_j(x) = k \qquad \text{for all } j \in [s_{k-1}, s_k) \tag{39}$$

Then, we have the following link between the $\mathrm{BGRG}_n(\boldsymbol{w})$ and simple uniform bipartite graphs:

Theorem 2.1. (BGRGn(w) conditioned on degree sequence is uniform). $\mathrm{BGRG}_n(\boldsymbol{w})$ *conditioned on* $\{d^{(l)}_i(X) = d^{(l)}_i \forall i \in [n], d^{(r)}_j(X) = k \forall j \in [s_{k-1}, s_k)\}$, *is uniform over all simple bipartite graphs with degree sequence* $(\boldsymbol{d}^{(l)}, \boldsymbol{d}^{(r)}) = \left((d^{(l)}_i)_{i\in[n]}, (d^{(r)}_j)_{j\in[M]}\right)$.

We prove Theorem 2.1 in Appendix A. We now explain how the $\mathrm{BCM}_n(\boldsymbol{d})$ is created:

Definition 2.1. (The bipartite configuration model). Denote the set of left-vertices by $[n] = \{1, \dots, n\}$ and the set of right-vertices by $[M] = \{1, \dots, M\}$. Equip each vertex with half-edges such that the total number of left half-edges equals the total number of right half-edges. Considering the half-edges as tokens to form edges, perform a uniform matching of half-edges in a manner analogous to the configuration model. Specifically, in each step, uniformly at random, choose one left half-edge and one right half-edge, and connect them to form an edge. The bipartite configuration model $\mathrm{BCM}_n(\boldsymbol{d})$ is then determined by the set of edges that arise through this procedure.

Note that in the case of the $\mathrm{BCM}_n(\boldsymbol{d})$, the groups are thought of as some right-vertices that left-vertices connect to, rather than all possible *combinations* of left-vertices, like in our dynamic random intersection graph. Therefore the number of these groups is fixed upfront, but their group members are *not*. It will later become clear under which conditions the $\mathrm{BCM}_n(\boldsymbol{d})$ and the $\mathrm{BGRG}_n(\boldsymbol{w})$ are equivalent, despite such differences. Having that in mind, an equivalent result to Theorem 2.1 follows for the $\mathrm{BCM}_n(\boldsymbol{d})$ conditioned on simplicity:

Theorem 2.2. (BCMn(d) conditioned on being simple is uniform). *For any double degree sequence* $\boldsymbol{d} = (\boldsymbol{d}^{(l)}, \boldsymbol{d}^{(r)}) = \left((d^{(l)}_i)_{i\in[n]}, (d^{(r)}_j)_{j\in[M]}\right)$, *and conditionally on the event* $\{\mathrm{BCM}_n(\boldsymbol{d})$ *is a simple graph*$\}$, $\mathrm{BCM}_n(\boldsymbol{d})$ *is a uniform simple bipartite graph with degree sequence* $\boldsymbol{d}$.

As a natural consequence, $\mathrm{BGRG}_n(\boldsymbol{w})$ conditioned on its degree sequences, and $\mathrm{BCM}_n(\boldsymbol{d})$ conditioned on simplicity, have the same distribution. We properly state this result in the following theorem, however, we first note that under some extra assumptions, an even stronger connection between the two graphs can be shown. As this connection plays a crucial role in many of our proofs, we include it in the theorem. The mentioned assumptions are as follows:

Condition 2.1. (Regularity conditions). Let G_n denote a bipartite graph with the set of left-vertices $[n]$ on one side and the set of right-vertices $[M]$ on the other side. The random variables $D_n^{(l)}$ and $D_n^{(r)}$ have distribution function $F_n^{(l)}$ and $F_n^{(r)}$ respectively, given by

$$F_n^{(l)}(x) = \frac{1}{n}\sum_{i\in[n]} \mathbb{1}_{\{d_i^{(l)}\le x\}} \text{ and}$$
$$F_n^{(r)}(x) = \frac{1}{n}\sum_{j\in[M]} \mathbb{1}_{\{d_j^{(r)}\le x\}} \tag{40}$$

We will call the following assumptions on these distribution functions the *regularity conditions* on the degree distributions:

a. There exist random variables $D^{(l)}, D^{(r)}$ such that, as $n\to\infty$ and for every $l\ge 0, k\ge 2$,

$$\mathbf{P}(D_n^{(l)} = l|G_n) \xrightarrow{\mathbf{P}} \mathbf{P}(D^{(l)} = l) \text{ and}$$
$$\mathbf{P}(D_n^{(r)} = k|G_n) \xrightarrow{\mathbf{P}} \mathbf{P}(D^{(r)} = k) \tag{41}$$

where $(\cdot|G_n)$ denotes conditioning with respect to a realization of a random graph.

b. Moreover, $\mathbf{E}[D^{(l)}] < \infty$, $\mathbf{E}[D^{(r)}] < \infty$ and, as $n\to\infty$,

$$\mathbf{E}[D_n^{(l)}|G_n] \xrightarrow{\mathbf{P}} \mathbf{E}[D^{(l)}] \text{ and } \mathbf{E}[D_n^{(r)}|G_n] \xrightarrow{\mathbf{P}} \mathbf{E}[D^{(r)}] \tag{42}$$

Additionally, we put a constraint on the second moment of the degree random variables:

c. $\mathbf{E}\left[(D^{(l)})^2\right], \mathbf{E}\left[(D^{(r)})^2\right] < \infty$ and, as $n\to\infty$,

$$\mathbf{E}\left[(D_n^{(l)})^2|G_n\right] \xrightarrow{\mathbf{P}} \mathbf{E}\left[(D^{(l)})^2\right] \text{ and}$$
$$\mathbf{E}\left[(D_n^{(r)})^2|G_n\right] \xrightarrow{\mathbf{P}} \mathbf{E}\left[(D^{(r)})^2\right] \tag{43}$$

With all the above in mind, the following result relates $\mathrm{BGRG}_n(\boldsymbol{w})$ and $\mathrm{BCM}_n(\boldsymbol{d})$:

Theorem 2.3. (Relation between BGRGn(w) and BCMn(d)). *Let $d_i^{(l)}$ be the degree of left-vertex i in $\mathrm{BGRG}_n(\boldsymbol{w})$, $d_j^{(r)}$ the degree of an ON group j in $\mathrm{BGRG}_n(\boldsymbol{w})$, and let $\boldsymbol{D} = (\boldsymbol{d}^{(l)}, \boldsymbol{d}^{(r)}) = ((d_i^{(l)})_{i\in[n]}, ((d_j^{(r)})_{j\in[M]})$. Then,*

$$\mathbf{P}(\mathrm{BGRG}_n(\boldsymbol{w}) = G|\boldsymbol{D} = \boldsymbol{d})$$
$$= \mathbf{P}(\mathrm{BCM}_n(\boldsymbol{d}) = G|\mathrm{BCM}_n(\boldsymbol{d}) \text{ simple}) \tag{44}$$

Let $\mathcal{E}_n$ be a subset of multi-graphs such that $\mathbf{P}(\mathrm{BCM}_n(\boldsymbol{d}) \in \mathcal{E}_n) \xrightarrow{\mathbf{P}} 1$ when $\boldsymbol{d}$ satisfies Condition 2.1. Assume that the degree sequence $\boldsymbol{D}$ of $\mathrm{BGRG}_n(\boldsymbol{w})$ satisfies Condition 2.1. Then also $\mathbf{P}(\mathrm{BGRG}_n(\boldsymbol{w}) \in \mathcal{E}_n) \to 1$.

Note that [13, 14] impose Condition 2.1 on the $\mathrm{BCM}_n(\boldsymbol{d})$ and all the results therein hold under them. When it comes to $\mathrm{BGRG}_n(\boldsymbol{w})$, it can be shown that Condition 1.1, which we have assumed in Section 1.2, implies Condition 2.1. Therefore all the results in Section 1.3 are stated under Condition 1.1. The proof of this implication is rather elementary and hence deferred to the Appendix A. See also Section 2.2.2.

The above theorem is a bipartite equivalent of the relationship between the classic generalized random graph $\mathrm{GRG}_n(\boldsymbol{w})$ and the configuration model $\mathrm{CM}_n(\boldsymbol{d})$ (see for instance [22, Theorem 7.18]). The idea behind the proof is also similar. We first show that $\mathrm{BGRG}_n(\boldsymbol{w})$ conditioned on the degree sequence is uniform. After that, we establish that $\mathrm{BCM}_n(\boldsymbol{d})$ conditioned on simplicity is also uniform. Finally, we use both statements to prove the desired result. The proofs of the auxiliary steps, as well as of the final Theorem 2.3, can be found in Appendix A.

2.2 | Static Local Limit and Giant Component

In this section, we investigate the local convergence of our graph under the stationary distribution. We introduce and describe in more detail the limiting local objects of the underlying $\mathrm{BGRG}_n(\boldsymbol{w})$ and of the resulting $\mathrm{DRIG}_n(\boldsymbol{w})$. Further on, we examine the proportion of vertices that are in the giant connected component. We state a phase transition in the size of the largest component in terms of the model parameters and give the explicit criterion under which a unique giant component exists.

2.2.1 | Brief Overview of Local Convergence

Before stating our results we briefly define local convergence in probability and local marked convergence in probability. Local convergence was introduced in [42] and a few years later, independently, in [43]. It describes the resemblance of the neighborhood of a vertex chosen uniformly at random to a certain limiting (possibly random) graph. To formalize this resemblance we introduce the notion of neighborhood and isomorphism on graphs:

Definition 2.2. (Rooted graph, rooted isomorphism and r-neighborhood).

i. We call a pair (G, o) a rooted graph if G is a locally finite, connected graph and o is a distinguished vertex of G. We denote the space of rooted graphs by $\mathcal{G}_\star$.

ii. We say that the rooted graphs (G_1, o_1), (G_2, o_2) are rooted isomorphic if there exists a graph-isomorphism between G_1 and G_2 that maps o_1 to o_2. We denote this by $(G_1, o_1) \simeq (G_2, o_2)$.

iii. For $r \in \mathbf{N}$, we define $B_r(G, o)$, the (closed) r-ball around o in G, or r-neighborhood of o in G, as the subgraph of G spanned by all vertices of graph distance at most r from o. We think of $B_r(G, o)$ as a rooted graph with root o.

The notion of graph isomorphism enables defining a metric on the space of rooted graphs:

Definition 2.3. (Metric on rooted graphs). Let (G_1, o_1) and (G_2, o_2) be two rooted connected graphs, and write $B_r(G_i, o_i)$ for the neighborhood of vertex $o_i \in V(G_i)$. Let

$$R^\star = \sup\{r : B_r(G_1, o_1) \simeq B_r(G_2, o_2)\} \tag{45}$$

and define

$$\mathrm{d}_{\mathcal{G}_\star}((G_1,o_1),(G_2,o_2)) = \frac{1}{R^\star+1} \tag{46}$$

The space $\mathcal{G}_\star$ of rooted graphs under the metric $\mathrm{d}_{\mathcal{G}_\star}$ modulo equivalences of isomorphic graphs is separable and complete and thus Polish (for a proof see [44, Appendix A]), which will later prove useful in the case of dynamic local convergence. We now define local convergence:

Definition 2.4. (Local convergence in probability). Let $G_n = ([n], \mathcal{E}(G_n))$ be a sequence of random graphs, and let $o_n \sim \mathrm{Unif}[[n]]$, that is, o_n is uniformly chosen from $[n]$ independently of G_n. Let (G,o) denote a random element (with arbitrary distribution) in the set of rooted graphs, which we call a random rooted graph. We say that $(G_n)_{n\geq 1}$ converges locally in probability to (G,o), if for any fixed rooted graph (H,o') and $r \in \mathbf{N}$, as $n\to\infty$,

$$\begin{aligned}\mathbf{P}(B_r(G_n,o_n)\simeq (H,o')|G_n) &:= \frac{1}{n}\sum_{i\in[n]}\mathbb{1}_{\{B_r(G_n,i)\simeq(H,o')\}}\\ &\xrightarrow{\mathbf{P}} \mathbf{P}(B_r(G,o)\simeq(H,o'))\end{aligned} \tag{47}$$

where $\mathbf{P}$ on the right-hand side refers to the law of (G,o). We say that (G,o) is the local limit in probability of $(G_n)_{n\geq 1}$.

Thus, intuitively, local convergence is defined as the convergence of the proportion of vertices whose neighborhoods have some specified structure. For further reading about local convergence, see for instance [44, Chapters 2–5] and the references therein for examples of local limits of various random graph models. Since we will actually need a more general setting of marked graphs and their convergence, we now briefly present some of the theory behind that.

Marked graphs and marked local convergence: Marks allow us to include additional information about vertices and/or edges such as directions, colors, and so on. In particular, we use marks to indicate the belonging of a vertex to a certain partition (of left- or right-vertices) in the underlying $\mathrm{BGRG}_n(\boldsymbol{w})$ and to denote the ON and OFF times of edges in the dynamic graphs, based on the group activity.

Definition 2.5. (Marked graphs). Let $\mathcal{G}$ denote the set of all locally finite multi-graphs on a countable (finite or countably infinite) vertex set. A marked multi-graph is a multi-graph $G = (\mathcal{V}(G), \mathcal{E}(G))$, $G\in\mathcal{G}$, together with a set $\mathcal{M}(G)$ of marks taking values in a complete separable metric space Ξ, called the mark space, and containing the special symbol $\emptyset$ which is to be interpreted as "no mark." $\mathcal{M}$ maps from $\mathcal{V}(G)$ and $\mathcal{E}(G)$ to Ξ. Images in Ξ are called marks. Each edge is given two marks, one associated with ("at") each of its endpoints, in particular, $\mathcal{M}(v)\in\Xi$ for $v\in\mathcal{V}(G)$, and for $\{u,v\}\in\mathcal{E}(G)$, $\mathcal{M}(\{u,v\},u)\in\Xi$ and $\mathcal{M}(\{u,v\},v)\in\Xi$. We denote the set of graphs with marks from the mark space Ξ by $\mathcal{G}(\Xi)$.

We generalize Definitions 2.2 and 2.3 to the setting of marked rooted graphs:

Definition 2.6. (Rooted marked graph and r-neighborhood).

i. We choose a vertex o in a marked graph $(G,\mathcal{M}(G))$ to be distinguished as the root. We denote the rooted marked graph by $(G,o,\mathcal{M}(G))$.
   We also denote the set of rooted marked graphs by $\mathcal{G}_\star(\Xi)$. We call a random element of $\mathcal{G}_\star(\Xi)$ (with an arbitrary joint distribution) a random rooted marked graph.
ii. The (closed) ball $B_r(G,o,\mathcal{M}(G))$ can be defined analogously to the unmarked graph ball (Definition 2.2iii), by restricting the mark function to the subgraph as well.

Definition 2.7. (Metric on marked rooted graphs). Let d_Ξ be a metric on the space of marks Ξ. Let

$$\begin{aligned}R^\star = \sup\{r : & B_r(G_1,o_1)\simeq B_r(G_2,o_2)\\ &\text{and there exists } \phi \text{ such that}\\ &\mathrm{d}_\Xi((\mathcal{M}_1(i),\mathcal{M}_2(\phi(i)))\leq 1/r\forall i\in V(B_r(G_1,o_1))\\ &\mathrm{d}_\Xi(\mathcal{M}_1((i,j)),\mathcal{M}_2(\phi(i,j)))\leq 1/r\forall\{i,j\}\in E(B_r(G_1,o_1))\}\end{aligned} \tag{48}$$

with $\phi : V(B_r(G_1,o_1))\to V(B_r(G_2,o_2))$ running over all rooted isomorphisms between $B_r(G_1,o_1)$ and $B_r(G_2,o_2)$ that map o_1 to o_2. Then define

$$\mathrm{d}_{\mathcal{G}_\star}((G_1,o_1,\mathcal{M}(G_1)),(G_2,o_2,\mathcal{M}(G_2))) = \frac{1}{R^\star+1} \tag{49}$$

This turns $\mathcal{G}_\star(\Xi)$ into a Polish space, that is, a complete, separable metric space.

Definition 2.7 puts a metric structure on marked rooted graphs. With this metric topology in hand, we can simply adapt all convergence statements to this setting. Hence, we generalize Equation (47) as follows:

Definition 2.8. (Local convergence in probability of marked graphs with continuous marks). Let $(G_n,\mathcal{M}(G_n))_{n\in\mathbf{N}}$, with $G_n = ([n],\mathcal{E}(G_n))$ and $(G_n,\mathcal{M}(G_n))\in\mathcal{G}_\star(\Xi)$, be a sequence of (finite) random marked graphs such that $n\to\infty$ and let $o_n\sim\mathrm{Unif}[[n]]$ independently of $(G_n,\mathcal{M}(G_n))$. Let $\mathbf{P}(\cdot|(G_n,\mathcal{M}(G_n)))$ denote conditional probability with respect to the marked graph (o_n is the free variable). We say that $(G_n,o_n,\mathcal{M}(G_n))_{n\in\mathbf{N}}$ converges locally in probability to a (possibly) random element $(G,o,\mathcal{M}(G))\in\mathcal{G}_\star(\Xi)$ if for any fixed rooted graph $(H,o',\mathcal{M}(H))$ and $r\in\mathbf{N}$, as $n\to\infty$,

$$\begin{aligned}&\mathbf{P}\Big(\mathrm{d}_{\mathcal{G}_\star}\big((G_n,o_n,\mathcal{M}(G_n)),(H,o',\mathcal{M}(H))\big)\\ &\quad\leq\frac{1}{r+1}|(G_n,\mathcal{M}(G_n))\Big)\\ &\quad:=\frac{1}{n}\sum_{i\in[n]}\mathbb{1}_{\left\{\mathrm{d}_{\mathcal{G}_\star}((G_n,i,\mathcal{M}(G_n)),(H,o',\mathcal{M}(H)))\leq\frac{1}{r+1}\right\}}\\ &\quad\xrightarrow{\mathbf{P}}\mathbf{P}\Big(\mathrm{d}_{\mathcal{G}_\star}\big((G,o,\mathcal{M}(G)),(H,o',\mathcal{M}(H))\big)\leq\frac{1}{r+1}\Big)\end{aligned} \tag{50}$$

for all continuity points $(H,o',\mathcal{M}(H))$ of the limiting distribution $\mathbf{P}$. We say that $(G,o,\mathcal{M}(G))$ is the local limit in probability of $(G_n,\mathcal{M}(G_n))_{n\geq 1}$.

2.2.2 | Static Local Convergence of the Dynamic Random Intersection Graph

To prove static local convergence of $\mathrm{DRIG}_n(\boldsymbol{w})$, we first look at static local convergence of $\mathrm{BGRG}_n(\boldsymbol{w})$. The authors of [13, 14] derive results on the local convergence and the giant component of $\mathrm{BCM}_n(\boldsymbol{d})$ under the assumption that $\mathrm{BCM}_n(\boldsymbol{d})$ fulfils Condition 2.1a. From the previous section we know that under Condition 2.1a,b, results that apply to $\mathrm{BCM}_n(\boldsymbol{d})$ also apply to $\mathrm{BGRG}_n(\boldsymbol{w})$. Hence, it suffices to show that our model under stationarity and Condition 1.1 fulfils Condition 2.1a,b and argue that Condition 2.1c is in fact not necessary to obtain local convergence and the giant component in $\mathrm{BGRG}_n(\boldsymbol{w})$.

Due to the one-mode projection present in [14] and in our model, the statements about the resulting intersection graphs, that is, of RIGC from [14] and $\mathrm{DRIG}_n(\boldsymbol{w})$, automatically follow from results shown for $\mathrm{BCM}_n(\boldsymbol{d})$ and $\mathrm{BGRG}_n(\boldsymbol{w})$ respectively. Since verification of the regularity conditions is quite elementary and the remaining results follow directly from [13, 14], the proofs of all static results can be found in Appendix B. Here we only describe the limiting object and state the theorems.

The static limiting object $(\mathrm{BP}_\gamma, 0)$: We start by introducing $(\mathrm{BP}_\gamma, 0)$, the local limit in probability of $\mathrm{BGRG}_n(\boldsymbol{w})$. Naturally, as we are dealing with two types of vertices—the left and the right ones—a typical neighborhood in this graph will be different depending on the type of the root. However, it is not possible to determine whether a uniformly chosen root was a left- or a right-vertex just on the basis of its neighborhood. Hence, we introduce marks to keep track of different types of vertices. Let $\Xi^b = \{l, r, \emptyset\}$ be the set of marks. We mark left-vertices as l and right-vertices as r. Formally,

$$\mathcal{M}_n^b(x) = \begin{cases} l & \text{if} \quad x \in [n] \\ r & \text{if} \quad x \in [n]_{k\geq 2} \\ \emptyset & \text{if} \quad x \text{ is an edge in } \mathrm{BGRG}_n(\boldsymbol{w}) \end{cases} \tag{51}$$

Now we introduce the limiting object $(\mathrm{BP}_\gamma, \mathcal{M}^\gamma, 0)$, the local limit of the $\mathrm{BGRG}_n(\boldsymbol{w})$ equipped with the mark function $\mathcal{M}_n^b$, while $(\mathrm{BP}_\gamma, 0)$ is then obtained by ignoring the mark function. Define a mixing variable γ as

$$\mathbf{P}(\gamma = l) = \frac{1}{1+\overline{M}} \text{ and } \mathbf{P}(\gamma = r) = \frac{\overline{M}}{1+\overline{M}} \tag{52}$$

where $\overline{M}$ is the limit in probability of M_n/n. See Theorem B.5 for the calculation of $\overline{M}$. Then, $(\mathrm{BP}_\gamma, \mathcal{M}^\gamma, 0)$ is a mixture of two marked ordered BP-trees, $(\mathrm{BP}_l, \mathcal{M}^l, 0)$ and $(\mathrm{BP}_r, \mathcal{M}^r, 0)$:

$$(\mathrm{BP}_\gamma, \mathcal{M}^\gamma, 0) \overset{d}{=} \mathbb{1}_{\{\gamma=l\}}(\mathrm{BP}_l, \mathcal{M}^l, 0) + \mathbb{1}_{\{\gamma=r\}}(\mathrm{BP}_r, \mathcal{M}^r, 0) \tag{53}$$

where $(\mathrm{BP}_l, \mathcal{M}^l, 0)$ describes the neighborhood of a left-vertex and $(\mathrm{BP}_r, \mathcal{M}^r, 0)$ of a right-vertex. Hence, $(\mathrm{BGRG}_n(\boldsymbol{w}), \mathcal{M}_n^b, V_n^{(l)})$ converges locally in probability to $(\mathrm{BP}_l, \mathcal{M}^l, 0)$, and $(\mathrm{BGRG}_n(\boldsymbol{w}), \mathcal{M}_n^b, V_n^{(r)})$ converges locally in probability to $(\mathrm{BP}_r, \mathcal{M}^r, 0)$, where $V_n^{(l)}$ and $V_n^{(r)}$ denote vertices chosen uniformly from the set of all left- and right-vertices respectively. The mixing variable γ can thus be re-interpreted as the random mark of the root.

Before we proceed, we need to introduce the size-biased and shift version of a random variable:

Definition 2.9. For an $\mathbb{N}$-valued random variable X with $\mathbf{E}[X] < \infty$, we define its size-biased distribution $X^\star$ and the shift variable $\tilde{X}$ by their probability mass functions, for all $k \in \mathbb{N}$,

$$\mathbf{P}(X^\star = k) = \frac{k\mathbf{P}(X = k)}{\mathbf{E}[X]} \text{ and} \\ \mathbf{P}(\tilde{X} = k) = \mathbf{P}(X^\star - 1 = k) \tag{54}$$

Now we can continue with the description of the random ordered marked tree $(\mathrm{BP}_l, \mathcal{M}^l, 0)$ itself. We consider a discrete-time branching process where the offspring of any two individuals are independent. We then give the individuals in even and odd generations marks l and r, respectively. Generation 0 contains the root alone and the root's offspring distribution is $D^{(l)}$ (the limit of the degree of a uniformly chosen left-vertex, see Condition 2.1). In consecutive generations, the offspring distribution of individuals marked with l will be $\tilde{D}^{(l)}$ and of individuals marked with r will be $\tilde{D}^{(r)}$. $(\mathrm{BP}_r, \mathcal{M}^r, 0)$ is defined analogously with reversed roles of l and r.

Static local limit of $\mathrm{DRIG}_n(\boldsymbol{w})$: Having specified the local limit of the underlying $\mathrm{BGRG}_n(\boldsymbol{w})$, we proceed to the limit of the resulting graph $\mathrm{DRIG}_n(\boldsymbol{w})$.

The static limiting object (CP, o): The limit that we denote by (CP, o) is a random rooted graph and the "community projection" (see (14)) of $(\mathrm{BP}_l, \mathcal{M}^l, 0)$ in the same way that $\mathrm{DRIG}_n(\boldsymbol{w})$ is the "community projection" of the underlying $\mathrm{BGRG}_n(\boldsymbol{w})$: it extracts only vertices marked as l and builds links between pairs of vertices that are connected to the same vertex with mark r. Let us accentuate that even though this limit is not a tree, it relies on the tree-like structure of the underlying $\mathrm{BGRG}_n(\boldsymbol{w})$. This constructs the local limit (CP, o) of $\mathrm{DRIG}_n(\boldsymbol{w})$.

2.2.3 | Degree Distribution

The average degree in $\mathrm{DRIG}_n(\boldsymbol{w})$ is asymptotically a sum of rescaled Poisson variables whose rates depend on the limiting weight variable W (see Condition 1.1) and the group-size distribution $(p_k)_k$:

Corollary 2.1. (Convergence of degree of a random vertex in $\mathrm{DRIG}_n(\boldsymbol{w})$). *Let* $\boldsymbol{w}$ *satisfy Condition 1.1. Then,*

$$D_n \xrightarrow{d} \sum_{l\geq 2} (l-1)X_l \tag{55}$$

where X_l *is a mixed-Poisson variable with mixing distribution* $lp_l W$*, that is, such that*

$$\mathbf{P}(X_l = k) = \mathbf{E}\left[e^{-lp_l W}\frac{(lp_l W)^k}{k!}\right] \tag{56}$$

Corollary 2.1 is a direct consequence of the static local convergence and we can actually prove a stronger result about the degree distribution in the static graph. Define

$$Q_k^{(n)} = \frac{1}{n} \sum_{i \in [n]} \mathbb{1}_{\{d_i = k\}} \quad (57)$$

Then, the following theorem shows that $(Q_k^{(n)})_{k \geq 0}$ converges in total variation distance:

Theorem 2.4. (Degree sequence in DRIGn(w)). *For every* $\varepsilon > 0$, *as* $n \to \infty$,

$$\mathbf{P}\left(\sum_{k=0}^{\infty} |Q_k^{(n)} - q_k| > \varepsilon\right) \to 0 \quad (58)$$

where $q_k = \mathbf{P}\left(\sum_{l \geq 2}(l-1)X_l = k\right)$ *with* $(X_l)_{l \geq 2}$ *independent mixed-Poisson variables with mixing distribution* $l p_l W$ *as in Corollary 2.1.*

Theorem 2.4 is proven in Appendix B.3.

We next investigate the sparsity of our model by investigating the average degree:

Theorem 2.5. (Convergence of average degree in DRIGn(w)). *As* $n \to \infty$,

$$\mathbf{E}[D_n | G_n] \xrightarrow{\mathbf{P}} (\mu_{(2)} - \mu)\mathbf{E}[W] \quad (59)$$

The proof of this result (see Appendix B.3) shows why Conditions 9 and 10 are necessary for the sparsity of our model.

2.2.4 | Static Giant Component

Since real-world networks tend to be highly connected and a large fraction of individuals often lies in a single connected component, it is useful to study the behavior of this component. We denote the cluster or connected component of a vertex $i \in [n]$ in the graph $G = ([n], \mathcal{E}(G))$ by $\mathscr{C}(i)$. We denote the graph distance in G, that is, the minimal number of edges in a path linking i and j, by $\text{dist}_G(i, j)$. We define connected components and the giant in a graph as follows:

Definition 2.10. (Giant connected component). Denote

$$\mathscr{C}(i) = \{j \in [n] : \text{dist}_G(i, j) < \infty\} \quad (60)$$

Let $\mathscr{C}_1$ denote the largest connected component (also referred to as "the giant component" or shortly "the giant"), that is, let $\mathscr{C}_1$ satisfy

$$|\mathscr{C}_1| = \max_{i \in [n]} |\mathscr{C}(i)| \quad (61)$$

where $|\mathscr{C}(i)|$ denotes the number of vertices in $\mathscr{C}(i)$ and we break ties arbitrarily.

Another popular question is the existence of a component containing a linear proportion of vertices—the so-called giant component problem. It was first studied by Erdős and Rényi [45] and has since been investigated on multiple other models (for instance the Chung–Lu model [46, 47], or configuration model [39, 40, 48, 49]).

Due to the structure of intersection graphs, the giant component exists when it exists in the underlying bipartite graph. Hence, the results on the giant component in $\text{DRIG}_n(\boldsymbol{w})$ follow from the results on the giant component in $\text{BGRG}_n(\boldsymbol{w})$. Similarly, as in the case of local convergence, thanks to the link between our model under stationarity and the $\text{BCM}_n(\boldsymbol{d})$ and the fact that the regularity conditions we impose on the weights variables imply regularity conditions on degrees in $\text{BGRG}_n(\boldsymbol{w})$, we are allowed to transfer the statements on the giant component for $\text{BCM}_n(\boldsymbol{d})$ and RIGC, proven in [13]. We again state the results and prove them in more detail in Appendix B.5.

Static giant component in the $\text{BGRG}_n(\boldsymbol{w})$: We start with the bipartite graph. Denote the giant component in $\text{BGRG}_n(\boldsymbol{w})$ by $\mathscr{C}_{1,b}$. This object is studied in the next theorem:

Theorem 2.6. (Giant component in $\text{BGRG}_n(\boldsymbol{w})$). *Under the supercriticality condition* $\mathbf{E}[\tilde{D}^{(l)}]\mathbf{E}[\tilde{D}^{(r)}] > 1$, *as* $n \to \infty$,

$$\frac{|\mathscr{C}_{1,b} \cap [n]|}{n} \xrightarrow{\mathbf{P}} \xi_l \quad (62)$$

where $\xi_l = 1 - G_{D^{(l)}}(\eta_l) \in [0,1]$ *and* $\eta_l \in [0,1]$ *is the smallest solution of the fixed point equation*

$$\eta_l = G_{\tilde{D}^{(r)}}(G_{\tilde{D}^{(l)}}(\eta_l)) \quad (63)$$

In this case, $\mathscr{C}_{1,b}$ *is unique in the sense that* $|\mathscr{C}_{2,b}|/n \xrightarrow{\mathbf{P}} 0$, *where* $\mathscr{C}_{2,b}$ *is the second largest component.*

We prove Theorem 2.6 in Appendix B.5.

Static giant component in the $\text{DRIG}_n(\boldsymbol{w})$: The statement on the giant in $\text{DRIG}_n(\boldsymbol{w})$ follows immediately. See Theorem 1.2.

Remark 2.1. The fact that the fixed point equation (19) shows up here can be intuitively explained as follows: For a vertex to be in the giant component, its local neighborhood has to survive. In the results on the local convergence we have shown that local neighborhoods are locally tree-like and are well approximated by branching processes with offspring distributions $\tilde{D}^{(l)}$ and $\tilde{D}^{(r)}$. Hence, the fixed point equation follows from the general theory of branching processes and their extinction probability.

2.3 | The Union Graph

Having shown multiple results for the static situation, we proceed to the dynamic setting. In order to describe the graph dynamically for every time point s, it is helpful to first look collectively at everything that happens during a time interval $[0, T]$, for T fixed. For that reason, we create a union graph $\text{BGRG}_n^{[0,T]}(\boldsymbol{w})$ (and accordingly, a resulting $\text{DRIG}_n^{[0,T]}(\boldsymbol{w})$) which includes any group that was ON at time $T = 0$, but also all the groups that ever switched ON within the time interval $(0, T]$.

Group probabilities in $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$: Note that

$$\begin{aligned}
&\mathbf{P}(a \text{ ON within } [0,T]) \\
&= \pi_{\mathrm{ON}}^a + \pi_{\mathrm{OFF}}^a \mathbf{P}(a \text{ switches ON within } (0,T]) \\
&= \pi_{\mathrm{ON}}^a + \pi_{\mathrm{OFF}}^a (1 - \mathbf{P}(a \text{ never ON within } (0,T])) \\
&= \pi_{\mathrm{ON}}^a + \pi_{\mathrm{OFF}}^a \left(1 - e^{-T\lambda_{\mathrm{OFF}}^a}\right) \wedge 1 \\
&= \frac{f(|a|)\prod_{i\in a} w_i}{\ell_n^{|a|-1} + f(|a|)\prod_{i\in a} w_i} + \frac{\ell_n^{|a|-1}}{\ell_n^{|a|-1} + f(|a|)\prod_{i\in a} w_i} \\
&\times \left(1 - e^{-T\frac{f(|a|)\prod_{i\in a} w_i}{\ell_n^{|a|-1}}}\right) \wedge 1 \le \pi_{\mathrm{ON}}^a (1+T) \wedge 1
\end{aligned} \tag{64}$$

using the fact that $1 - e^{-x} \le x$. Hence, we see that even though the group probability in the union graph is somewhat complicated, it can be bounded from above by $\pi_{\mathrm{ON}}^a(1+T)$, which is similar to the group probability in the static graph. Thus, instead of deriving convergence results directly for $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$, it is more convenient to couple it with a graph that is closer to the static graph.

2.3.1 | The Rescaled Bipartite Graph

Remember that since $\mathrm{BGRG}_n(\boldsymbol{w})$ is uniform and, assuming Condition 1.1, it fulfils the required degree regularity conditions, we can relate it to the $\mathrm{BCM}_n(\boldsymbol{d})$ model from [13, 14] and hence deduce its local convergence. It is not difficult to see that a graph with group ON probability $\mathbf{P}(a \text{ is ON}) = \pi_{\mathrm{ON}}^a(1+T)$ would also satisfy equivalent regularity conditions. However, a graph with $\mathbf{P}(a \text{ is ON}) = \pi_{\mathrm{ON}}^a(1+T)$, $\mathbf{P}(a \text{ is OFF}) = 1 - \pi_{\mathrm{ON}}^a(1+T)$ will not *exactly* be uniform (which is easy to see after inspecting the proofs of Proposition A.1 and Theorem 2.1 in Appendix A). Therefore, we define a graph with group probability also closely related to $\pi_{\mathrm{ON}}^a(1+T)$, but with a more convenient structure:

Definition of $\mathrm{BGRG}_n^{(T)}(\boldsymbol{w})$: We introduce $\mathrm{BGRG}_n^{(T)}(\boldsymbol{w})$, a graph similar to the static $\mathrm{BGRG}_n(\boldsymbol{w})$ but with slightly modified group probabilities: we fix the holding times to be exponentially distributed with rates

$$\lambda_{\mathrm{ON}}^a = 1 \text{ and } \lambda_{\mathrm{OFF}}^a = \frac{(1+T)f(|a|)\prod_{i\in a} w_i}{\ell_n^{|a|-1}} \tag{65}$$

Hence, the new stationary distribution $\pi^{(T)} = [\pi_{\mathrm{ON}}^{(T)}, \pi_{\mathrm{OFF}}^{(T)}]$ is given by

$$\begin{aligned}
\pi_{\mathrm{ON}}^{a,(T)} &= \frac{(1+T)f(|a|)\prod_{i\in a} w_i}{\ell^{|a|-1} + (1+T)f(|a|)\prod_{i\in a} w_i} \quad \text{and} \\
\pi_{\mathrm{OFF}}^{a,(T)} &= \frac{\ell_n^{|a|-1}}{\ell_n^{|a|-1} + (1+T)f(|a|)\prod_{i\in a} w_i}
\end{aligned} \tag{66}$$

for every $a \in \cup_{k\ge 2}[n]_k$. We again impose Condition 1.1 on the weights $\boldsymbol{w} = (w_i)_{i\in[n]}$ and assume finite first and second moment of the group-size distribution $(p_k)_{k\ge 2}$, taking $f(k) = k!p_k$ (see (9) and (10)). It turns out (see Remark B.4) that $\mathrm{BGRG}_n^{(T)}(\boldsymbol{w})$ conditioned on its degree sequence is uniform and that its degree sequences fulfil the same regularity conditions as the degree sequences of $\mathrm{BGRG}_n(\boldsymbol{w})$. The limiting variables of left- and right-degrees are also analogous to the limiting degree variables in $\mathrm{BGRG}_n(\boldsymbol{w})$, with Poisson parameters rescaled by the factor $T+1$ (for an explicit statement of the regularity conditions, see Remark B.4). Hence, $\mathrm{BGRG}_n^{(T)}(\boldsymbol{w})$ is just like $\mathrm{BGRG}_n(\boldsymbol{w})$ with slightly bigger group probabilities and thus, it asymptotically behaves in the same way. To draw the same conclusion about the union graph $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$, it then suffices to show that it is asymptotically equivalent to $\mathrm{BGRG}_n^{(T)}(\boldsymbol{w})$, which we explain further in the next section.

2.3.2 | Asymptotic Equivalence of Multi-Graphs

In this section, we briefly introduce the theory of asymptotic equivalence of graph sequences. In particular, we extend the condition determining when two inhomogeneous random graphs are asymptotically equivalent to the case of random multi-graphs. We start by introducing the notion of asymptotic equivalence for general random variables:

Definition 2.11. (Asymptotic equivalence of random variables). Let $(\mathcal{X}_n, \mathcal{F}_n)_{n\ge 1}$ be a sequence of arbitrary measurable spaces. Let (X_n) and (Y_n) be two sequences of random variables with values in $\mathcal{X}_n$. The sequence $(X_n)_n$ is asymptotically equivalent to $(Y_n)_n$ when for every sequence of measurable sets F_n (i.e., $F_n \in \mathcal{F}_n$), we have $\mathbf{P}(X_n \in F_n) - \mathbf{P}(Y_n \in F_n) \to 0$ as $n \to \infty$.

The definition of asymptotic equivalence is naturally extended to graphs by calling two graphs asymptotically equivalent, when the sequences of Bernoulli random variables that uniquely determine their edge statuses are asymptotically equivalent. We broaden this notion to bipartite multi-graphs whose distribution can also be uniquely encoded by a sequence of Bernoulli random variables from $\{0,1\}^{\sum_{k\ge 2}\binom{n}{k}}$, this time corresponding to group statuses (multi-edge statuses). Analogously, we call such multi-graphs asymptotically equivalent if these Bernoulli sequences are asymptotically equivalent as in Definition 2.11. In all examples below, the Bernoulli variables involved are independent. In the following theorem, we give a criterion guaranteeing that two bipartite multigraph sequences are asymptotically equivalent. The section applies results by [50]. We denote $(\pi_a)_{a\in\cup_{k\ge 2}[n]_k}$ for the group probabilities in some bipartite graph $\mathrm{BRG}_n(\boldsymbol{\pi})$ for which the probability that a group a is present equals π_a and all groups exist independently of each other.

Theorem 2.7. (Asymptotic equivalence of bipartite multi-graphs). *Let* $\mathrm{BRG}_n(\boldsymbol{\pi})$ *and* $\mathrm{BRG}_n(\hat{\boldsymbol{\pi}})$ *be two random bipartite graphs with group probabilities* $\boldsymbol{\pi} = (\pi_a)_{a\in\cup_{k\ge 2}[n]_k}$ *and* $\hat{\boldsymbol{\pi}} = (\hat{\pi}_a)_{a\in\cup_{k\ge 2}[n]_k}$ *respectively. If there exists* $\hat{\varepsilon} > 0$ *such that* $\max_{a\in\cup_{k\ge 2}[n]_k} \hat{\pi}_a \le 1 - \hat{\varepsilon}$, *then* $\mathrm{BRG}_n(\boldsymbol{\pi})$ *and* $\mathrm{BRG}_n(\hat{\boldsymbol{\pi}})$ *are asymptotically equivalent if*

$$\lim_{n\to\infty} \sum_{a\in\cup_{k\ge 2}[n]_k} \frac{(\pi_a - \hat{\pi}_a)^2}{\hat{\pi}_a} = 0 \tag{67}$$

In particular, $\mathrm{BRG}_n(\boldsymbol{\pi})$ and $\mathrm{BRG}_n(\hat{\boldsymbol{\pi}})$ are asymptotically equivalent when they can be coupled in such a way that $\mathbf{P}(\mathrm{BRG}_n(\boldsymbol{\pi}) \ne \mathrm{BRG}_n(\hat{\boldsymbol{\pi}})) = o(1)$. Indeed, there is a strong relationship between the asymptotic equivalence and coupling, which becomes obvious after the proof. We prove Theorem 2.7 in Section 3.1.

Having explained what is meant by asymptotic equivalence, we state our main equivalence result:

Theorem 2.8. (Asymptotic equivalence of BGRGn[0,T] (w) and BGRG n(T)(w)). *Under Condition 1.1, the random graphs* $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$ *and* $\mathrm{BGRG}_n^{(T)}(\boldsymbol{w})$ *are asymptotically equivalent.*

We prove Theorem 2.8 in Section 3.2. Thanks to the equivalence, all results that we derive for $\mathrm{BGRG}_n^{(T)}(\boldsymbol{w})$ automatically hold for $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$.

2.3.3 | Local Limit of Union Graphs

We now state the results on the local convergence of the union graph, starting with the bipartite one:

Local limit of $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$: Following the appropriate statement for the rescaled graph $\mathrm{BGRG}_n^{(T)}(\boldsymbol{w})$, we conclude that the bipartite union graph fulfils the equivalent of Condition 2.1a. Moreover, the limiting variables of left- and right-degrees in $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$, denoted further on as $\tilde{D}^{(l),[0,T]}$ and $\tilde{D}^{(r),[0,T]}$ respectively, are just the limiting variables of left- and right-degrees in $\mathrm{BGRG}_n(\boldsymbol{w})$ with Poisson parameters and proportion of groups of size k rescaled by the factor $T+1$. Hence, the union graph asymptotically behaves in the same way as the stationary graph with accordingly larger edge probabilities.

The limiting object $(\mathrm{BP}_\gamma^{[0,T]}, o)$: The local limit of $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$ is again a mixture of two branching processes corresponding to two types of vertices and everything is analogous to the limit of $\mathrm{BGRG}_n(\boldsymbol{w})$, with $D^{(l),[0,T]}$ and $D^{(r),[0,T]}$ taking the place of $D^{(l)}$ and $D^{(r)}$ as the offspring distribution of the root, $\tilde{D}^{(l),[0,T]}$ replacing $\tilde{D}^{(l)}$ as the offspring distribution of the rest of the l-vertices and $\tilde{D}^{(r),[0,T]}$ taking the place of $\tilde{D}^{(r)}$ as the offspring distribution of the rest of the r-vertices.

Local limit of $\mathrm{DRIG}_n^{[0,T]}(\boldsymbol{w})$: The local limit $(\mathrm{CP}^{[0,T]}, o)$ of $\mathrm{DRIG}_n^{[0,T]}(\boldsymbol{w})$ can be constructed from the left-partition of the local limit of $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$ via the appropriate community projection in exactly the same way as the local limit of $\mathrm{DRIG}_n(\boldsymbol{w})$ was constructed via the community projection from the local limit of $\mathrm{BGRG}_n(\boldsymbol{w})$ (see the previous section):

Theorem 2.9. (Local limit of BGRGn[0,T](w) and DRIG n[0,T](w)). *Assume that Condition 1.1 holds. Then* $\left(\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w}), V_n^b\right)$ *converges locally in probability to* $(\mathrm{BP}_\gamma^{[0,T]}, o)$*, where* $(\mathrm{BP}_\gamma^{[0,T]}, o)$ *is described above. Also* $\left(\mathrm{DRIG}_n^{[0,T]}(\boldsymbol{w}), o_n\right)$ *converges locally in probability to* $(\mathrm{CP}^{[0,T]}, o)$*, where* $(\mathrm{CP}^{[0,T]}, o)$ *is described above.*

We prove Theorem 2.9 in Section 3.3. From this analysis we also obtain results on the degrees $D_n^{[0,T]}$ and its expectation as in Corollary 2.1 and Theorem 2.4. We refrain from stating those.

2.3.4 | Marked Union Graph

Marks in $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$: We remark that the union graph is already dynamic, as it takes dynamically appearing groups into account. However, it does not equal the actual dynamic graph, as it does not take into account whether certain groups were active **at the same time**. Therefore it might show interactions that were never made at any time in $[0,T]$. Yet, it enables tracking the actual interactions between vertices. For this purpose, we add marks along the edges of $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$ indicating the (first) switch ON and switch OFF times within $[0,T]$. These times are determined by the activity of the groups responsible for the creation of these edges. We first mark the groups (right-vertices): write σ_{ON}^a to denote the first time that a group a switches ON within $[0,T]$, σ_{OFF}^a to denote the first time it switches OFF in $(0,T]$ and $\sigma_{\mathrm{ON}}^a = 0$ if a is ON at time 0. If $\sigma_{\mathrm{ON}}^a > T$ no time marks are given, and if $\sigma_{\mathrm{ON}}^a \le T, \sigma_{\mathrm{OFF}}^a > T$, only σ_{ON}^a is assigned. Hence, the new mark-set is $\Xi^d = \{l, r, r \times [0,\infty), r \times [0,\infty) \times (0,\infty)\}$. We then transfer the marks to the edges, that is, every edge in $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$ copies the marks of the right-vertex it is adjacent to. The left-vertices are still only marked with "l."

We define the distance metric d_Ξ on the above set of marks in $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$ in the following way: For any $s, s_1, s_2, t, t, t_1, t_2 \in [0,\infty)$,

$$\begin{aligned}
&\mathrm{d}_\Xi(l, r\times\cdots) = \mathrm{d}_\Xi(r,(r,s)) = \mathrm{d}_\Xi(r,(r,s,u)) \\
&\quad = \mathrm{d}_\Xi((r,s),(r,s,t)) = T \\
&\mathrm{d}_\Xi((r,s),(r,t)) = |s-t| \\
&\mathrm{d}_\Xi((r,s_1,t_1),(r,s_2,t_2)) = |s_1-s_2| \vee |t_1-t_2|
\end{aligned} \tag{68}$$

The above marks are also well-behaved: the marks of group a that is ON in the union graph are independent for different a and they converge in distribution with respect to the probability measure of the union graph to some limiting marks $(t_{\mathrm{ON}}, t_{\mathrm{OFF}})$, whose distribution is described in the following lemma:

Lemma 2.1. (Convergence of the law of the edge marks.). *Let* $F_{n|T}^{\mathrm{ON,OFF}}$ *denote the joint law of* $\left(\sigma_{\mathrm{ON}}^a, \sigma_{\mathrm{OFF}}^a\right)$ *for* $a \in \cup_{k\ge2}[n]_k$*, conditioned on the fact that such* $a \in \cup_{k\ge2}[n]_k$ *is* ON *during* $[0,T]$*, that is,*

$$F_{n|T}^{\mathrm{ON,OFF}}(s_1,s_2) = \mathbf{P}(\sigma_{\mathrm{ON}}^a \le s_1, \sigma_{\mathrm{OFF}}^a \le s_2 | \mathrm{ON\ at\ some\ point\ in\ } [0,T]) \tag{69}$$

where $s_1 \in [0,T]$ *and* $s_2 \ge s_1$*. Then, as* $n\to\infty$*,*

$$F_{n|T}^{\mathrm{ON,OFF}}(s_1,s_2) \to F_T^{\mathrm{ON,OFF}}(s_1,s_2) \tag{70}$$

with

$$F_T^{\mathrm{ON,OFF}}(s_1,s_2) = \frac{1-e^{-s_2+s_1}+s_1}{1+T} \tag{71}$$

Consequently,

$$\left(\sigma_{\mathrm{ON}}^a, \sigma_{\mathrm{OFF}}^a\right) \xrightarrow{\mathrm{d}} (t_{\mathrm{ON}}, t_{\mathrm{OFF}}) \tag{72}$$

where $(t_{\mathrm{ON}}, t_{\mathrm{OFF}})$ *has joint cumulative distribution function* $F_T^{\mathrm{ON,OFF}}$*.*

We prove Lemma 2.1 in Section 3.4.

Marks in $\mathrm{DRIG}_n^{[0,T]}(\boldsymbol{w})$: We have just described the assignment of marks to the underlying bipartite union graph $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$. We now explain how to do it for $\mathrm{DRIG}_n^{[0,T]}(\boldsymbol{w})$. Recall that the left-vertices in $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$ were only labeled as "l," that is, they did not contain information about the time activity. However, $\mathrm{DRIG}_n^{[0,T]}(\boldsymbol{w})$—the community projection (see (14)) of $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$—consists of left-vertices only. Hence, we need to transfer the marks accordingly when performing the community projection: if k left-vertices $i_1, i_2, \ldots, i_k \in [n]$, $k \in \mathbf{N}$ are connected to some right-vertex $\hat{a}$ with marks $(\sigma_{\mathrm{ON}}^{\hat{a}}, \sigma_{\mathrm{OFF}}^{\hat{a}})$, (which are also inherited by edges connecting each of $i_1, i_2, \ldots, i_k \in [n]$ with $\hat{a}$, according to the description at the beginning of this section), then every two-element combination of $i_1, i_2, \ldots, i_k \in [n]$ is connected by an edge with marks $(\sigma_{\mathrm{ON}}^{\hat{a}}, \sigma_{\mathrm{OFF}}^{\hat{a}})$ in $\mathrm{DRIG}_n^{[0,T]}(\boldsymbol{w})$. Whether we choose to later transfer these marks to $i_1, i_2, \ldots, i_k$ as well is irrelevant. Recall our argumentation that in the setting that we have chosen, it is unlikely for two vertices to meet in more than one group (see paragraph "The static intersection graph" in Section 1.3). However, in the rare case that the same vertices are connected to two or more distinct groups in $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$, the mark of the edge connecting these vertices in $\mathrm{DRIG}_n^{[0,T]}(\boldsymbol{w})$ appends all marks from the bipartite edges. This means the edge in the intersection graph will carry the combined $\bigcup_{\hat{a}}(\sigma_{\mathrm{ON}}^{\hat{a}}, \sigma_{\mathrm{OFF}}^{\hat{a}})$ marks, where the union is taken over each group $\hat{a}$ connecting the given vertices in the bipartite graph.

The limit of the marked union graphs: Lemma 2.1 is crucial in showing the dynamic local convergence. Thanks to the facts that edge marks are independent, as all the groups switch ON and OFF independently of each other, and that they converge in distribution to limiting marks $(t_{\mathrm{ON}}, t_{\mathrm{OFF}})$ (as shown in Lemma 2.1) we know that they converge jointly for all groups and all $s \in [0, T]$. Hence, the marked bipartite union graph will converge to the marked limit of $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$, which will imply that also the marked intersection union graph converges:

Theorem 2.10. (Local limit of marked BGRGn[0,T](w) and DRIG n[0,T](w)). *Under Condition 1.1, as* $n \to \infty$, $\left(\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w}), V_n^b, ((\sigma_{\mathrm{ON}}^a, \sigma_{\mathrm{OFF}}^a))_{a:a\ \mathrm{ON\ in}\ [0,T]}\right)$ *converges locally in probability to* $(\mathrm{BP}_\gamma^{[0,T]}, o, (t_{\mathrm{ON}}, t_{\mathrm{OFF}}))$, *where* $\left(\mathrm{BP}_\gamma^{[0,T]}, o, (t_{\mathrm{ON}}, t_{\mathrm{OFF}})\right)$ *is a marked version of* $(\mathrm{BP}_\gamma^{[0,T]}, o)$. *It follows that the marked random intersection graph* $\left(\mathrm{DRIG}_n^{[0,T]}(\boldsymbol{w}), o_n, ((\sigma_{\mathrm{ON}}^a, \sigma_{\mathrm{OFF}}^a))_{a:a\ \mathrm{ON\ in}\ [0,T]}\right)$ *converges locally in probability to* $(\mathrm{CP}^{[0,T]}, o, (t_{\mathrm{ON}}, t_{\mathrm{OFF}}))$, *where* $(\mathrm{CP}^{[0,T]}, o, (t_{\mathrm{ON}}, t_{\mathrm{OFF}}))$ *is a marked version of* $(\mathrm{CP}^{[0,T]}, o)$ *where the marks are i.i.d. with distribution given in (72).*

We prove Theorem 2.10 in Section 3.5.

Remark 2.2. (Marks in $(\mathrm{CP}^{[0,T]}, o, (t_{\mathrm{ON}}, t_{\mathrm{OFF}}))$). Similarly as in the static or unmarked union case, the limit $(\mathrm{CP}^{[0,T]}, o, (t_{\mathrm{ON}}, t_{\mathrm{OFF}}))$ is constructed from the $\left(\mathrm{BP}_l^{[0,T]}, o, (t_{\mathrm{ON}}, t_{\mathrm{OFF}})\right)$ via a community projection. Hence, the time marks have to be appropriately transferred to $(\mathrm{CP}^{[0,T]}, o, (t_{\mathrm{ON}}, t_{\mathrm{OFF}}))$ during this community projection. This happens in the same way as it happened for the marked $\mathrm{DRIG}_n^{[0,T]}(\boldsymbol{w})$, which we have explained in the paragraph "Marks in $\mathrm{DRIG}_n^{[0,T]}(\boldsymbol{w})$" before Theorem 2.10.

2.4 | Dynamic Local Convergence

Switching pace: Note that we have defined σ_{ON}^a and σ_{OFF}^a as the first switch-ON and switch-OFF times within the time interval $[0, T]$ and we did not comment on the possibility that some groups might switch ON again during this period of time. We address this issue by arguing that it is unlikely to encounter such a group in a neighborhood of a uniformly chosen left-vertex, which is the subject of the following lemma:

Lemma 2.2. (Existence of groups that switch ON more than once in the union graph). *Denote the neighborhood of a uniformly chosen left-vertex in* $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$ *by* $B_r^{[0,T]}(V_n^{(l)})$. *As* $n \to \infty$,

$$\mathbf{P}(\exists a \in B_r^{[0,T]}(V_n^{(l)}) : a \text{ ON twice in } [0,T]) \to 0 \tag{73}$$

We prove Lemma 2.2 in Section 3.6.

Lemma 2.2 implies that groups that switch ON more than once in the union graph do not contribute significantly to the structure of a neighborhood of a uniformly chosen vertex. Hence, the neighborhood in the union graph neglecting these groups is a good approximation of the actual neighborhood and it can be used to construct the dynamic graph. We summarize this statement in the following corollary:

Corollary 2.2. *Denote the neighborhood of a uniformly chosen left-vertex in* $\mathrm{BGRG}_n^s(\boldsymbol{w})$ *by* $B_r^s(V_n^{(l)})$ *and the neighborhood of a uniformly chosen left-vertex in* $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$ *restricted to groups that switch* ON *only once in* $[0, T]$ *and are present at time* s *by* $\tilde{B}_r^{s,[0,T]}(V_n^{(l)})$. *As* $n \to \infty$, *for every finite* $r > 0$ *and every* $s \in [0, T]$, *with high probability (whp)*

$$\left\{ B_r^s(V_n^{(l)}) = \tilde{B}_r^{s,[0,T]}(V_n^{(l)}) \right\} \tag{74}$$

We refrain from formally proving Corollary 2.2 and instead provide a short justification here: the fact that $\tilde{B}_r^{s,[0,T]}(V_n^{(l)})$ is contained in $B_r^s(V_n^{(l)})$ follows immediately. The other direction follows directly from Lemma 2.2.

Thus, for every $s \in [0, T]$, $\mathrm{BGRG}_n^s(\boldsymbol{w})$ is a subgraph of $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$ containing only these groups that are active at time s, that is, the groups a with $(\sigma_{\mathrm{ON}}^a, \sigma_{\mathrm{OFF}}^a)$ such that $s \in [\sigma_{\mathrm{ON}}^a, \sigma_{\mathrm{OFF}}^a]$. $\mathrm{DRIG}_n^s(\boldsymbol{w})$ is then a community projection (see (14)) of such a $\mathrm{BGRG}_n^s(\boldsymbol{w})$. Hence, the convergence of the union graph and joint convergence of edge marks, guaranteed by the independence of the marks and their convergence in distribution, yield convergence of finite-dimensional distributions of the dynamic graph $\mathrm{BGRG}_n^s(\boldsymbol{w})$ for every $s \in [0, T]$. Thanks to that, we can describe local limits of $(\mathrm{BGRG}_n^s(\boldsymbol{w}))_{s\in[0,T]}$ and consecutively of $(\mathrm{DRIG}_n^s(\boldsymbol{w}))_{s\in[0,T]}$.

When looking at $(\mathrm{BGRG}_n^s(\boldsymbol{w}))_{s\in[0,T]}$ as a process in time, we choose a random root o_n *only* once and then we investigate the evolution of its neighborhood in time. Since such a process encounters discontinuities with respect to the local metric, the convergence of finite-dimensional distributions is not sufficient to deduce the convergence of the entire process. However, as we have explained in more detail before Theorem 1.3, if we treat $(\text{BGRG}_n^s(\boldsymbol{w}))_{s\in[0,T]}$ for every $s \in [0,T]$ as a function from $[0,T]$ into the Polish space of rooted graphs $\mathcal{G}_\star$ with the $\mathrm{d}_{\mathcal{G}_\star}$ metric, it suffices to add a suitable tightness criterion to deduce the dynamic convergence of the process in time (see [31, Chapter 13] or/and [51, Chapter 16]), that is, weak convergence in $D([0,T],S)$ in the Skorokhod J_1 topology, with $S = (\mathcal{G}^\star, \mathrm{d}_{\mathcal{G}^\star}))$. Details are given in the proof of Theorem 1.3 in Section 3.7.

The limiting tree with a fixed weight: To facilitate the description of the dynamic local limit, we start by introducing (BP_l, v, w_v). This object is closely related to $(\text{BP}_l, 0)$, with the difference that here we fix the root and its weight to be v and w_v respectively. The offspring distribution of this root is then $\text{Poi}(\mu w_v)$. We also keep track of the weights of the remaining left-vertices in the tree, which are i.i.d. copies of $W^\star$, and hence the offspring distribution of these vertices is $\text{Poi}(\mu W_i^\star)$ for each vertex i.

Distinguished vertices: We also introduce the notion of *distinguished* vertices. By the "left-root" we mean simply a root that is a left-vertex.

Definition 2.12. (Distinguished vertices).

1. The left-root is the distinguished vertex of every right-vertex that appears as it offspring.
2. For all other right-vertices, at time 0, the unique left-vertex that is closest to the left-root is the distinguished vertex.
3. The distinguished vertex of the right-vertex that was created or switched ON at v is v.
4. The distinguished vertices of every other right-vertex in the subtree of v is that unique left-vertex that is closest to v.

In the context of limiting rooted trees, the distinguished vertex of a right-vertex a can be intuitively understood as the parent of that right-vertex.

Dynamic limiting objects: Having explained $(\text{BP}_\gamma, 0)$, $(\text{BP}_\gamma^{[0,T]}, 0)$ and its marked version in detail, we can now describe the dynamic $((\text{BP}_\gamma^s, 0))_{s\in[0,T]}$ more precisely. We do that considering the two partitions separately. As we have already mentioned in Section 1.3, at time $s = 0$, $\big((\text{BP}_l^s, 0)\big)$ is equal in distribution to $(\text{BP}_l, 0)$. The weight w_0 of the root has law W and the weights $(w_v)_v$ of all other left-vertices are i.i.d. copies of $W^\star$. These weights are recorded accordingly. For $s > 0$, the dynamic local limit evolves through two simultaneous processes: the addition and removal of right-vertices. We now describe these processes in more detail:

The right-vertex addition: A right-vertex of degree k is added at the root at rate $k p_k w_v$ at every present left-vertex v. The other $(k-1)$ left-vertices added as children of a newly attached right-vertex of size k receive weights that are i.i.d. copies of $W^\star$, and these weights are recorded. Every such vertex u upon arrival is attached to the tree along with (BP_l, u, w_u).

The right-vertex removal: Each right-vertex a having a distinguished vertex v_a is removed at rate 1. Along with a, all left-vertices attached to a except for v_a are also removed, as well as all their children (i.e., their whole tree). The limiting object continuously evolves according to the dynamic defined by the above processes.

The dynamic right-partition $((\text{BP}_r^s, 0))_{s\in[0,T]}$ is far less interesting. At time $s = 0$, it is equal in distribution to $(\text{BP}_r, 0)$. However, since we know from Lemma 2.2 that in the limit we can neglect right-vertices that switch ON more than once, the root of this copy of $(\text{BP}_r, 0)$ will be removed at rate 1 and its local limit will be an empty graph for the remaining time.

The dynamic local limit of the intersection graph $((\text{CP}^s, o))_{s\in[0,T]}$ is naturally a (dynamic) community projection of $((\text{BP}_l^s, 0))_{s\in[0,T]}$. Thus, at time $s = 0$, it is a community projection of (CP^0, o) with the weights assigned as described previously, that is, the root having weight w_0 with law W and all the other vertices having weights $(w_v)_v$ distributed as i.i.d. copies of $W^\star$. For $s > 0$, cliques of $k-1$ vertices are added at rate $k p_k w_v$ at every vertex v. Each of the $k-1$ vertices in these cliques is attached along with (CP_l, v, w_v)—a community projection of (BP_l, v, w_v). Simultaneously, each k-clique is removed at rate 1 along with its subtree and except for the root, or the distinguished vertex the closest to the root.

Remark 2.3. (The subgraph of the limit of the marked union graph versus the dynamic limit). Note that we have constructed the dynamic local limit by deriving the local limit of the marked union graph and then extracting corresponding subgraphs for each $s \in [0,T]$ (see the text below Corollary 2.2 for heuristics, and Section 3.7 for more details). However, we do not mention this construction as the background of the dynamic limiting object explained above and directly describe its dynamic instead. We now give a heuristic explanation of why both approaches will generate the same dynamic limit.

Without loss of generality, we give our argument for the root and its offspring distribution only, as the behavior of the vertices in consecutive generations is analogous. Recall that the degree of the root (given that the root is a left-vertex) in the local limit of $\text{BGRG}_n^{[0,T]}(\boldsymbol{w})$ is Poisson distributed with parameter $(1+T)W\mu$. Now, in the above construction, the degree of the root at time 0 is Poisson distributed with parameter $W\mu$ and consecutive children arrive at rate $W\mu$, which implies there are $\text{Poi}(TW\mu)$ new arrivals up to time T. Thus, the total degree of the root in the dynamic object up to time T is $\text{Poi}((1+T)W\mu)$-distributed and we conclude that the offspring distributions in the two constructions coincide.

It remains to compare the distribution of the arrival and departure times of the children. It is not difficult to verify that with $(t_{\text{ON}}, t_{\text{OFF}})$ with a joint distribution as in (71) we have $t_{\text{OFF}} - t_{\text{ON}} \sim Exp(1)$, which corresponds to the departure times of the dynamic objects. Now, note that the marginal distribution $\mathbf{P}(t_{\text{ON}} \leq s_1)$ is approximately equal to $\lim_{s_2\to\infty} \mathbf{P}(t_{\text{ON}} \leq s_1, t_{\text{OFF}} \leq s_2 | t_{\text{ON}} < T) = (s_1+1)/(T+1)$ which implies that t_{ON} is uniformly distributed. Note that the joint cumulative distribution function is conditioned on the presence in the union graph. If we condition the arrival times of the root's offspring in the dynamic limiting object on the number of these arrivals, due the properties of interarrival times during a Poisson process, we obtain that they are also uniformly distributed. Hence, conditionally on the roots in both constructions having the same number of offspring, the laws that govern the exact arrival and departure times of this offspring are the same. Thus, the objects generated via both of these constructions coincide.

Remark 2.4. (Dynamic generalized random graphs). Recall Remark 1.2, where we have explained how to obtain the dynamic generalized and Erdős-Rényi random graphs from our model, and we concluded that Theorem 1.3 can be extended to them. Hence, it is also possible to specify dynamic limiting objects of these graphs: in the limit of the dynamic GRG, the root is equipped with the weight w_0 with law W and the remaining vertices are equipped with $(w_v)_v$ distributed as i.i.d. copies of $W^\star$. At time $s = 0$, the root has a $\mathrm{Poi}(w_0)$ number of children and every other vertex v has a $\mathrm{Poi}(w_v)$ number of children. Then, every edge is broken at rate 1 and, on the other hand, i.i.d. copies of the limiting tree from time $s = 0$ keep arriving at the root at rate w_0. Analogously, copies of the $\mathrm{Poi}(w_v)$ branching tree keep arriving at rate w_v for each other vertex v, and the vertices that arrive within them again have weights distributed as i.i.d. copies of $W^\star$. The limit of the dynamic Erdős-Rényi random graph behaves analogously: at time $s = 0$ it is equal in distribution to a $\mathrm{Poi}(\lambda)$ branching process tree. Then, every edge is removed at rate 1 and a new copy of the $\mathrm{Poi}(\lambda)$ tree arrives at every vertex at rate λ.

2.5 | Dynamic Giant Component

We want to show that the process $(J_n(s))_{s\in[0,T]}$ with $J_n(s) = \mathbb{1}_{\{o_n\in\mathscr{C}_1^s\}}$ converges to another appropriate indicator process. Since both of these processes encounter discontinuities, we need to use the Skorokhod J_1 topology once again in order to obtain the desired convergence. The Skorokhod J_1 topology on $D[0,T]$—the space of càdlàg functions on $[0,T]$—is given by a metric d^0 (see [31, eq. 12.16]), which takes care of the time deformation present in processes with discontinuities. For more explanation, see a well-known characterization of weak convergence in $D[0,T]$ (see [31]).

The most important step in our proofs is **localization of the giant**, that is, noticing that the sequence of processes $(J_n(s))_{s\in[0,T]}$ is close in distribution to the sequence of processes $(J_n^{(r)}(s))_{s\in[0,T]}$, with

$$J_n^{(r)}(s) = \mathbb{1}_{\left\{\partial B_r^{G_n^s}(o_n)\neq\emptyset\right\}} \tag{75}$$

where $\partial B_r^{G_n^s}(o_n)$ denotes the set of vertices at distance r from the root o_n in $(\mathrm{BGRG}_n^s(\boldsymbol{w}))_{s\in[0,T]}$ at time s. Indeed, for two distinct time points $s_1, s_2 \in [0,T]$,

$$\begin{aligned}\mathbf{P}(J_n(s_1) = J_n(s_2) = 1) &= \mathbf{P}(J_n^{(r)}(s_1) = J_n^{(r)}(s_2) = 1)\\ &\quad + \mathbf{P}(J_n(s_1) = J_n(s_2) = 1) - \mathbf{P}(J_n^{(r)}(s_1) = J_n^{(r)}(s_2) = 1)\end{aligned} \tag{76}$$

Denote the complement of an event A by A^c. Using the fact that for any two events A and B, $|\mathbf{P}(A) - \mathbf{P}(B)| \le \mathbf{P}(A\setminus B) = \mathbf{P}(A\cap B^c)$ and subsequently applying De Morgan's law we obtain

$$\begin{aligned}&\left|\mathbf{P}(J_n(s_1) = J_n(s_2) = 1) - \mathbf{P}(J_n^{(r)}(s_1) = J_n^{(r)}(s_2) = 1)\right|\\ &\le \mathbf{P}(J_n(s_1) = J_n(s_2) = 1, \{J_n^{(r)}(s_1) = J_n^{(r)}(s_2) = 1\}^c)\\ &= \mathbf{P}(J_n(s_1) = J_n(s_2) = 1, J_n^{(r)}(s_1) \neq 1 \mathrm{OR} J_n(s_1)\\ &= J_n(s_2) = 1, J_n^{(r)}(s_2) \neq 1)\\ &\le \mathbf{P}(J_n(s_1) = J_n(s_2) = 1, J_n^{(r)}(s_1) \neq 1)\\ &\quad + \mathbf{P}(J_n(s_1) = J_n(s_2) = 1, J_n^{(r)}(s_2) \neq 1)\end{aligned}$$

where the last inequality follows from the union bound. Then,

$$\begin{aligned}&\mathbf{P}(J_n(s_1) = J_n(s_2) = 1, J_n^{(r)}(s_1) \neq 1) + \mathbf{P}(J_n(s_1)\\ &= J_n(s_2) = 1, J_n^{(r)}(s_2) \neq 1)\\ &\le \mathbf{P}(J_n(s_1) = 1, J_n^{(r)}(s_1) \neq 1) + \mathbf{P}(J_n(s_2)\\ &= 1, J_n^{(r)}(s_2) \neq 1) \le 2\mathbf{P}\left(J_n^{(r)}(s_1) \neq J_n(s_1)\right)\end{aligned}$$

where the last step follows from stationarity. Taking $n \to \infty$, thanks to the static local limit and our result on the static giant component (see Theorem 2.6), that is, the fact that pointwise for any s, we have

$$\frac{|\mathscr{C}_1^s|}{n} \xrightarrow{\mathbf{P}} \mathbf{P}(|\mathscr{C}^s(o)| = \infty) \tag{77}$$

where $\mathbf{P}$ is the law of the local limit. We obtain

$$\lim_{n\to\infty} \mathbf{P}\left(J_n^{(r)}(s_1) \neq J_n(s_1)\right) = \mathbf{P}\left(\mathbb{1}_{\{\partial B_r^{G^s}(o)\neq\emptyset\}} \neq \mathbb{1}_{\{|\mathscr{C}^s(o)|=\infty\}}\right) \tag{78}$$

which, after additionally taking $r \to \infty$, yields

$$\begin{aligned}\lim_{r\to\infty}\limsup_{n\to\infty} &\left|\mathbf{P}(J_n(s_1) = J_n(s_2) = 1)\right.\\ &\left. - \mathbf{P}(J_n^{(r)}(s_1) = J_n^{(r)}(s_2) = 1)\right| = 0\end{aligned}$$

Thus,

$$\lim_{n\to\infty} \mathbf{P}(J_n(s_1) = J_n(s_2) = 1) = \lim_{r\to\infty}\lim_{n\to\infty} \mathbf{P}(J_n^{(r)}(s_1) = J_n^{(r)}(s_2) = 1)$$

Thanks to this link, we can deduce the convergence of the dynamic giant process from the dynamic local weak convergence (see Theorem 1.3), which states that, as $n \to \infty$,

$$(J_n^{(r)}(s))_{s\in[0,T]} \xrightarrow{\mathrm{d}} (\mathcal{J}^{(r)}(s))_{s\in[0,T]} \tag{79}$$

In the proof of Theorem 1.4 we show how to extend the above argument to all finite-dimensional distributions. As a result, the convergence of all finite-dimensional distributions derived via localization paired with the tightness of the process will guarantee convergence of $(J_n(s))_{s\in[0,T]}$. We remark that this technique is not restricted to our model and it can be applied to any other dynamic graph for which (77) and (79) hold.

2.6 | Dynamic Largest Group

We investigate the behavior of the process $(n^{-1/\alpha}K_{\max}^{[0,T]})_{T\ge 0}$, where $K_{\max}^{[0,T]}$ is the maximum group size in the time interval $[0,T]$. It is an increasing process describing the largest group observed by a certain time point T. As such, the process in question also encounters discontinuities, just like the previous dynamic processes we have described, and hence, to deduce its convergence, we once again use the theory of convergence in Skorokhod topology.

Remark 2.5. Note that in the case of the maximal group present in the union graph, we do not run into the same problem of creating connections that do not exist as was the case with degrees of vertices. Hence, the largest group ever active in the union graph is at the same time the largest group ever active in the dynamic graph.

Remark 2.6. (Maximum group size in the static graph). The maximum group size in the static graph at any fixed time has the same distribution as $K^{\{0\}}_{\max}$, thus, it also scales as $n^{1/\alpha}$.

3 | Proofs of the Main Results

Here we provide proofs of all mentioned results, unless we previously stated we would prove them in the Appendix.

3.1 | Proof of the Condition for Asymptotic Equivalence for Bipartite Multi-Graphs

Proof of Theorem 2.7. Note that $\mathrm{BRG}_n(\boldsymbol{\pi})$ and $\mathrm{BRG}_n(\hat{\boldsymbol{\pi}})$ can be entirely encoded by the group presences, just like simple random graphs are encoded by the edge presences. Thus, the asymptotic equivalence of two bipartite multi-graphs $\mathrm{BRG}_n(\boldsymbol{\pi})$ and $\mathrm{BRG}_n(\hat{\boldsymbol{\pi}})$ is equivalent to the asymptotic equivalence of two sequences of independent Bernoulli random variables with success probabilities $\boldsymbol{\pi} = (\pi_a)_{a\in\cup_{k\geq 2}[n]_k}$ and $\hat{\boldsymbol{\pi}} = (\hat{\pi}_a)_{a\in\cup_{k\geq 2}[n]_k}$. By [50, Theorem 2.2] and under the assumption that there exists $\hat{\varepsilon} > 0$ such that $\max_{a\in\cup_{k\geq 2}[n]_k} \hat{\pi}_a \leq 1-\hat{\varepsilon}$, such sequences are asymptotically equivalent if

$$\lim_{n\to\infty} \sum_{a\in\cup_{k\geq 2}[n]_k} \frac{(\pi_a - \hat{\pi}_a)^2}{\hat{\pi}_a} = 0 \tag{80}$$

3.2 | Proof of Asymptotic Equivalence of the Union Graph and the Rescaled Graph

Proof of Theorem 2.8. Recall (64) and (66). To verify condition (67) we first compute

$$\begin{aligned}
0 &\leq \pi^{a,[0,T]}_{\mathrm{ON}} - \pi^{a,(T)}_{\mathrm{ON}} \\
&\leq \frac{(1+T)f(|a|)\prod_{i\in a} w_i}{\ell^{|a|-1} + f(|a|)\prod_{i\in a} w_i} - \frac{(1+T)f(|a|)\prod_{i\in a} w_i}{\ell^{|a|-1} + (1+T)f(|a|)\prod_{i\in a} w_i} \\
&\leq \frac{(1+T)^2 f^2(|a|)(\prod_{i\in a} w_i)^2}{\ell^{|a|-1}(\ell^{|a|-1} + (1+T)f(|a|)\prod_{i\in a} w_i)}
\end{aligned} \tag{81}$$

where we have used the fact that $1-e^{-x} \leq x$. Hence, by the fact that $k! < k^k$ and that for any $k \geq 2$ and any $c \geq 1$,

$$\begin{aligned}
\sum_{j_1<\cdots<j_k\in[n]} \frac{(w_{j_1}\cdots w_{j_k})^c}{(\ell_n^{k-1})^c} &= \frac{1}{k!}\sum_{j_1,\ldots,j_k\in[n]} \frac{(w_{j_1}\cdots w_{j_k})^c}{(\ell_n^{k-1})^c} \\
&\leq \frac{(\sum_{j\in[n]} w_j^c)^k}{k!(\ell_n^{k-1})^c}
\end{aligned} \tag{82}$$

we have that

$$\begin{aligned}
&\sum_{a\in\cup_{k\geq 2}[n]_k} \frac{(\pi^{a,[0,T]}_{\mathrm{ON}} - \pi^{a,(T)}_{\mathrm{ON}})^2}{\pi^{a,(T)}_{\mathrm{ON}}} \\
&\leq (1+T)^3 \sum_{k=2}^{\infty} \sum_{j_1<\cdots<j_k\in[n]} \frac{(k!)^3 (p_k)^3 (w_{j_1}\cdots w_{j_k})^3}{(\ell_n^{k-1})^3} \\
&\leq (1+T)^3 \sum_{k=2}^{\infty} k^4 p_k^3 \frac{1}{\ell_n^{k-1}} \left(\frac{k^2}{\ell_n}\right)^{k-2} \left(\frac{\mathbf{E}[W_n^3]}{\mathbf{E}[W_n]}\right)^k = o(1)
\end{aligned} \tag{83}$$

which can be shown using suitable truncation arguments: one with respect to the group size and one with respect to the weights. For the first truncation, we can fix a sequence $b_n \to \infty$ and show that the contribution from groups a with $|a| > b_n$ vanishes. Then take $b_n = o(\sqrt{n})$ to bound (83) for a with $|a| \leq b_n$. For the second truncation, we eliminate vertices with large weights in a similar manner. For more technical details see Appendix B, where analogous truncation arguments occur frequently. Hence, for some sequence $\varepsilon_n \to 0$ as $n \to \infty$,

$$\mathbf{P}\left(\sum_{a\in\cup_{k\geq 2}[n]_k} \frac{(\pi^{a,[0,T]}_{\mathrm{ON}} - \pi^{a,(T)}_{\mathrm{ON}})^2}{\pi^{a,(T)}_{\mathrm{ON}}} \geq \varepsilon_n \right) \to 0 \tag{84}$$

The desired equivalence of $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$ and $\mathrm{BGRG}_n^{(T)}(\boldsymbol{w})$ follows.

3.3 | Proof of Local Convergence of the Union Graph

Proof of Theorem 2.9. Hence, it also turns out that the limiting degree sequences in the union graph satisfy similar properties as the ones in the static graph, with the Poisson parameter and proportion of groups of size k rescaled by the factor $T+1$. Hence, the bipartite union graph asymptotically behaves like the static bipartite graph with slightly larger edge probabilities and converges locally in probability to a related limiting object with accordingly larger offspring distributions.

3.4 | Proof of the Law of the Marks

Proof of Lemma 2.1. We compute the law of the marks of a fixed group a taking into account two possible starting states:

$$\begin{aligned}
&\mathbf{P}(\sigma^a_{\mathrm{ON}} \leq s_1, \sigma^a_{\mathrm{OFF}} \leq s_2 | a \text{ ON in } [0,T]) \\
&= \frac{\mathbf{P}(\sigma^a_{\mathrm{ON}} \leq s_1, \sigma^a_{\mathrm{OFF}} \leq s_2, a \text{ ON in } [0,T])}{\mathbf{P}(a \text{ ON in } [0,T])} \\
&= \frac{\mathbf{P}(\sigma^a_{\mathrm{ON}} \leq s_1, \sigma^a_{\mathrm{OFF}} \leq s_2)}{\mathbf{P}(a \text{ ON in } [0,T])} \\
&= \frac{\mathbf{P}(\sigma^a_{\mathrm{ON}} = 0, \sigma^a_{\mathrm{OFF}} \leq s_2) + \mathbf{P}(a \text{ OFF at } 0, \sigma^a_{\mathrm{ON}} \leq s_1, \sigma^a_{\mathrm{OFF}} \leq s_2)}{\mathbf{P}(a \text{ ON in } [0,T])}
\end{aligned} \tag{85}$$

We compute all three ingredients separately:

Step 1.

$$\begin{aligned}
\mathbf{P}(\sigma^a_{\mathrm{ON}} = 0, \sigma^a_{\mathrm{OFF}} \leq s_2) &= \pi^a_{\mathrm{ON}} \mathbf{P}(a \text{ switches OFF in } (0, s_2]) \\
&= \pi^a_{\mathrm{ON}}(1 - e^{-s_2})
\end{aligned} \tag{86}$$

Step 2.

$$\mathbf{P}(a \text{ OFF at } 0, \sigma^a_{\mathrm{ON}} \le s_1, \sigma^a_{\mathrm{OFF}} \le s_2) = \pi^a_{\mathrm{OFF}}\mathbf{P} \\ (a \text{ switches ON in } (0, s_1], a \text{ switches OFF in } (\sigma^a_{\mathrm{ON}}, s_2]) \tag{87}$$

The second term can be computed as

$$\begin{aligned}\mathbf{P}&(a \text{ switches ON in } (0,s_1] \text{ and switches OFF in } (\sigma^a_{\mathrm{ON}}, s_2]) \\ &= \int_0^{s_1} \mathbf{P}(Exp(\lambda^a_{\mathrm{ON}}) \le s_2 \\ &\quad - Exp(\lambda^a_{\mathrm{OFF}}) | Exp(\lambda^a_{\mathrm{OFF}}) = x) f_{Exp(\lambda^a_{\mathrm{OFF}})}(x)\mathrm{d}x \\ &= \int_0^{s_1} (1 - e^{-\lambda^a_{\mathrm{ON}}(s_2 - x)})\lambda^a_{\mathrm{OFF}} e^{-\lambda^a_{\mathrm{OFF}} x}\mathrm{d}x\end{aligned} \tag{88}$$

Splitting the terms and using the fact that $\int_0^{s_1} \lambda^a_{\mathrm{OFF}} e^{-\lambda^a_{\mathrm{OFF}} x}\mathrm{d}x = \mathbf{P}(Exp(\lambda^a_{\mathrm{OFF}}) \le s_1)$ we obtain

$$\begin{aligned}\mathbf{P}&(a \text{ switches ON in } (0,s_1] \text{ and switches OFF in } (\sigma^a_{\mathrm{ON}}, s_2]) \\ &= \mathbf{P}(\mathrm{Exp}(\lambda^a_{\mathrm{OFF}}) \le s_1) - \lambda^a_{\mathrm{OFF}} \int_0^{s_1} e^{-(s_2 - x)} e^{-\lambda^a_{\mathrm{OFF}} x}\mathrm{d}x \\ &= 1 - e^{-\lambda^a_{\mathrm{OFF}} s_1} - \frac{\lambda^a_{\mathrm{OFF}} e^{-s_2}}{\lambda^a_{\mathrm{OFF}} - 1}(1 - e^{-s_1(\lambda^a_{\mathrm{OFF}} - 1)})\end{aligned} \tag{89}$$

Hence,

$$\begin{aligned}\mathbf{P}&(a \text{ OFF at } 0, \sigma^a_{\mathrm{ON}} \le s_1, \sigma^a_{\mathrm{OFF}} \le s_2) \\ &= \pi^a_{\mathrm{OFF}}\left(1 - e^{-\lambda^a_{\mathrm{OFF}} s_1} - \frac{\lambda^a_{\mathrm{OFF}}}{\lambda^a_{\mathrm{OFF}} - 1}(e^{-s_2} - e^{-s_1 \lambda^a_{\mathrm{OFF}}} e^{s_1 - s_2})\right)\end{aligned} \tag{90}$$

Step 3. For the probability in the denominator of (85), we recall (64) once more.

Gathering all three steps together, the expression in (85) becomes

$$\begin{aligned}\mathbf{P}&(\sigma^a_{\mathrm{ON}} \le s_1, \sigma^a_{\mathrm{OFF}} \le s_2 | a \text{ ON in } [0,T]) \\ &= \frac{\pi^a_{\mathrm{ON}}(1 - e^{-s_2})}{\pi^a_{\mathrm{ON}} + \pi^a_{\mathrm{OFF}}(1 - e^{-\lambda^a_{\mathrm{OFF}} T})} \\ &\quad + \frac{\pi^a_{\mathrm{OFF}}(1 - e^{-\lambda^a_{\mathrm{OFF}} s_1} - \frac{\lambda^a_{\mathrm{OFF}}}{\lambda^a_{\mathrm{OFF}} - 1}(e^{-s_2} - e^{-s_1 \lambda^a_{\mathrm{OFF}}} e^{s_1 - s_2}))}{\pi^a_{\mathrm{ON}} + \pi^a_{\mathrm{OFF}}(1 - e^{-\lambda^a_{\mathrm{OFF}} T})}\end{aligned} \tag{91}$$

We will now take the limit of the two fractions separately as $n \to \infty$. We simplify

$$\frac{\pi^a_{\mathrm{ON}}(1 - e^{-s_2})}{\pi^a_{\mathrm{ON}} + \pi^a_{\mathrm{OFF}}(1 - e^{-\lambda^a_{\mathrm{OFF}} T})} = \frac{1 - e^{-s_2}}{1 + 1/(\lambda^a_{\mathrm{OFF}})(1 - e^{-\lambda^a_{\mathrm{OFF}} T})} \tag{92}$$

As we know, $\lim_{n\to\infty} \lambda^a_{\mathrm{OFF}} = \lim_{n\to\infty} \frac{f(|a|)\prod_{i\in a} w_i}{\ell_n^{|a|-1}} = 0$. Substituting $x = \frac{f(|a|)\prod_{i\in a} w_i}{\ell_n^{|a|-1}}$, we obtain, by L'Hospital's rule,

$$\lim_{n\to\infty} \lambda^a_{\mathrm{OFF}}{}^{-1}(1 - e^{-\lambda^a_{\mathrm{OFF}} T}) = \lim_{x\to 0} \frac{1 - e^{-xT}}{x} = \lim_{x\to 0} \frac{Te^{-xT}}{1} = T \tag{93}$$

so that

$$\lim_{n\to\infty} \frac{\pi^a_{\mathrm{ON}}(1 - e^{-s_2})}{\pi^a_{\mathrm{ON}} + \pi^a_{\mathrm{OFF}}(1 - e^{-\lambda^a_{\mathrm{OFF}} T})} = \frac{1 - e^{-s_2}}{1 + T} \tag{94}$$

Now we compute the limit as $n \to \infty$ of the second term in (91). We again simplify, dividing by π^a_{OFF}, to obtain

$$\begin{aligned}&\frac{1 - e^{-\lambda^a_{\mathrm{OFF}} s_1} + \frac{\lambda^a_{\mathrm{OFF}}}{1 - \lambda^a_{\mathrm{OFF}}}(e^{-s_2} - e^{-s_1 \lambda^a_{\mathrm{OFF}}} e^{s_1 - s_2})}{\lambda^a_{\mathrm{OFF}} + 1 - e^{-\lambda^a_{\mathrm{OFF}} T}} \\ &= \frac{1 - e^{-\lambda^a_{\mathrm{OFF}} s_1}}{\lambda^a_{\mathrm{OFF}} + 1 - e^{-\lambda^a_{\mathrm{OFF}} T}} \\ &\quad + \frac{\lambda^a_{\mathrm{OFF}}(e^{-s_2} - e^{-s_1 \lambda^a_{\mathrm{OFF}}} e^{s_1 - s_2})}{(1 - \lambda^a_{\mathrm{OFF}})(\lambda^a_{\mathrm{OFF}} + 1 - e^{-\lambda^a_{\mathrm{OFF}} T})} =: A_n + B_n\end{aligned} \tag{95}$$

Then, using the same substitution as previously,

$$\lim_{n\to\infty} A_n = \lim_{x\to 0} \frac{1 - e^{-xs_1}}{x + 1 - e^{-xT}} = \lim_{x\to 0} \frac{s_1 e^{-xs_1}}{1 + Te^{-xT}} = \frac{s_1}{1 + T} \tag{96}$$

and

$$\lim_{n\to\infty} B_n = \lim_{x\to 0} \frac{xe^{-s_2} - xe^{-s_1 x} e^{s_1 - s_2}}{1 - e^{-xT} - x^2 + xe^{-xT}} = \frac{e^{-s_2} - e^{s_1 - s_2}}{1 + T} \tag{97}$$

Combining these limits yields

$$\begin{aligned}&\lim_{n\to\infty} \frac{\pi^a_{\mathrm{OFF}}(1 - e^{-\lambda^a_{\mathrm{OFF}} s_1} - \frac{\lambda^a_{\mathrm{OFF}}}{\lambda^a_{\mathrm{OFF}} - 1}(e^{-s_2} - e^{-s_1 \lambda^a_{\mathrm{OFF}}} e^{s_1 - s_2}))}{\pi^a_{\mathrm{ON}} + \pi^a_{\mathrm{OFF}}(1 - e^{-\lambda^a_{\mathrm{OFF}} T})} \\ &= \frac{s_1 + e^{-s_2} - e^{s_1 - s_2}}{1 + T}\end{aligned} \tag{98}$$

Thus, by (91),

$$\lim_{n\to\infty} \mathbf{P}(\sigma^a_{\mathrm{ON}} \le s_1, \sigma^a_{\mathrm{OFF}} \le s_2 | a \text{ ON in } [0,T]) = \frac{1 - e^{s_1 - s_2} + s_1}{1 + T} \tag{99}$$

as required.

3.5 | Proof of Local Limit of Marked Union Graphs

Proof of Theorem 2.10. Theorem 2.9 shows local convergence of the unmarked $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$, which means that for any fixed rooted graph (H, o') and $r \in \mathbf{N}$,

$$\begin{aligned}\mathbf{P}(B_r(G_n^{[0,T]}, V_n^{(l)}) \simeq (H, o') | G_n) &:= \frac{1}{|G_n|} \sum_{i\in[n]} \mathbb{1}_{\{B_r(G_n^{[0,T]}, i) \simeq (H, o')\}} \\ &\xrightarrow{\mathbf{P}} \mathbf{P}(B_r(\mathrm{BP}_l^{[0,T]}, o) \simeq (H, o'))\end{aligned}$$

where we have written $(G_n^{[0,T]}, o)$ instead of $\mathrm{BGRG}_n^{[0,T]}$ for the sake of simplicity of notation in this proof. If the marked version of the union graph converges locally in probability, then for any fixed marked rooted graph $(H, o', (\overline{m}_1, \overline{m}_2))$, where $(\overline{m}_1, \overline{m}_2)$ are marks, and $r \in \mathbf{N}$,

$$\begin{aligned}&\mathbf{P}\Big(\mathrm{d}_{\mathcal{G}_\star}((G_n^{[0,T]}, V_n^{(l)}, ((\sigma^a_{\mathrm{ON}}, \sigma^a_{\mathrm{OFF}}))_{a:a \text{ ON in } [0,T]}), (H, o', (\overline{m}_1, \overline{m}_2))) \\ &\quad \le \frac{1}{r+1} | (G_n^{[0,T]}, ((\sigma^a_{\mathrm{ON}}, \sigma^a_{\mathrm{OFF}}))_{a:a \text{ ON in } [0,T]})\Big) \\ &\quad := \frac{1}{n} \sum_{i\in[n]} \mathbb{1}_{\{\mathrm{d}_{\mathcal{G}_\star}((G_n^{[0,T]}, i, ((\sigma^a_{\mathrm{ON}}, \sigma^a_{\mathrm{OFF}}))_{a:a \text{ ON in } [0,T]}), (H, o', (\overline{m}_1, \overline{m}_2))) \le \frac{1}{r+1}\}} \\ &\quad \xrightarrow{\mathbf{P}} \mathbf{P}\Big(\mathrm{d}_{\mathcal{G}_\star}((\mathrm{BP}_l^{[0,T]}, o, (t_{\mathrm{ON}}, t_{\mathrm{OFF}})), (H, o', (\overline{m}_1, \overline{m}_2))) \le \frac{1}{r+1}\Big)\end{aligned}$$

Note that the edge marks are independent since all the groups switch ON and OFF independently of each other. Hence, if the marks converge in distribution to some limiting marks they will converge jointly for all groups and all $s \in [0,T]$. In Lemma 2.1 we have shown that they indeed converge and also that the marks of all groups present within $[0,T]$ are identically distributed. This implies that we can couple each pair of marks with their limiting marks so that they are appropriately close to each other. Hence, the proportion of vertices whose neighborhoods look like $(H, o', (\overline{m}_1, \overline{m}_2))$ must converge to the probability that neighborhoods in $B_r(\mathrm{BP}_l^{[0,T]}, o, (t_{\mathrm{ON}}, t_{\mathrm{OFF}}))$ look like $(H, o', (\overline{m}_1, \overline{m}_2))$, which precisely means that the marked version of $(\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w}), V_n^{(l)})$ converges. Convergence of the marked version of $(\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w}), V_n^{(r)})$ to $(\mathrm{BP}_r^{[0,T]}, o, (t_{\mathrm{ON}}, t_{\mathrm{OFF}}))$ follows automatically, which implies that $(\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w}), V_n^b, ((\sigma_{\mathrm{ON}}^a, \sigma_{\mathrm{OFF}}^a))_{a:a \text{ ON in } [0,T]})$ converges to $(\mathrm{BP}_\gamma^{[0,T]}, o, (t_{\mathrm{ON}}, t_{\mathrm{OFF}}))$. Next, the convergence of the marked intersection graph follows from the convergence of the underlying bipartite structure, as community projection preserves distances in the marked graph.

3.6 | Proof of the Switching Pace of the Groups in the Union Graph

Proof of Lemma 2.2. For any finite $r > 0$, we investigate

$$\begin{aligned}&\mathbf{P}(\exists a \in B_r^{[0,T]}(V_n^{(l)}) : a \text{ ON twice in } [0,T]) \\ &\quad = \mathbf{E}\big[\mathbf{P}(\exists a \in B_r^{[0,T]}(V_n^{(l)}) : a \text{ ON twice in } [0,T] | G_n^{[0,T]})\big]\end{aligned} \tag{100}$$

where we denote the r-neighborhood of a uniformly chosen vertex in $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$ by $B_r^{[0,T]}(V_n^{(l)})$ and for simplification we write $G_n^{[0,T]}$ instead of $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$ to denote conditioning on the union graph. Applying the union bound yields

$$\begin{aligned}&\mathbf{P}(\exists a \in B_r^{[0,T]}(V_n^{(l)}) : a \text{ ON twice in } [0,T] | G_n^{[0,T]}) \\ &\quad \le 1 \wedge \sum_{a \in B_r^{[0,T]}(V_n^{(l)})} \mathbf{P}(a \text{ ON twice in } [0,T] | G_n^{[0,T]})\end{aligned} \tag{101}$$

We first apply a suitable truncation (for more details see similar cases, for instance, the proofs of Theorem B.3 or Remark B.3): we truncate the maximum vertex weight by a_n and the maximum group size by b_n, for example with $a_n = \log(n), b_n = \log(n)$, to obtain

$$\begin{aligned}&\sum_{a \in B_r^{[0,T]}(V_n^{(l)})} \mathbf{P}(a \text{ ON twice in } [0,T] | G_n^{[0,T]}) \\ &\quad = \sum_{a \in B_r^{[0,T]}(V_n^{(l)})} \mathbf{P}(a \text{ ON twice in } [0,T] | G_n^{[0,T]}) \mathbb{1}_{\{\max_i w_i \le a_n, \max_a |a| \le b_n\}} \\ &\quad\quad + o_{\mathbf{P}}(1)\end{aligned} \tag{102}$$

where the maxima are taken w.r.t. a vertex $i \in B_r^{[0,T]}(V_n^{(l)})$ and group $a \in B_r^{[0,T]}(V_n^{(l)})$. The remainder is small since Condition 1.1 implies that our union graph is sparse, and hence observing large groups and vertices with large weights in a uniformly chosen r-neighborhood is unlikely. We now compute

$$\begin{aligned}&\mathbf{P}(a \text{ switches ON twice in } [0,T] | G_n^{[0,T]}) \\ &\quad = \frac{\mathbf{P}(a \text{ switches ON twice in } [0,T] | a \text{ ON in } [0,T])}{\mathbf{P}(a \text{ ON in } [0,T])} \\ &\quad = \frac{\mathbf{P}(a \text{ switches ON twice in } [0,T])}{\mathbf{P}(a \text{ ON in } [0,T])}\end{aligned} \tag{103}$$

where the first equality follows from the independence of groups. We have

$$\begin{aligned}&\mathbf{P}(a \text{ switches ON twice in } [0,T]) \\ &\quad = \pi_{\mathrm{ON}}^a \mathbf{P}(a \text{ switches OFF and ON again in } [0,T]) \\ &\quad\quad + \pi_{\mathrm{OFF}}^a \mathbf{P}(a \text{ switches ON, then OFF and ON again in } [0,T])\end{aligned} \tag{104}$$

Recall that the times that groups spend in the ON and OFF states are exponentially distributed with rates $Exp(\lambda_{\mathrm{ON}}^a)$ and $Exp(\lambda_{\mathrm{OFF}}^a)$ respectively. Hence, using the fact that for all $x, 1 - e^{-x} \le x$,

$$\begin{aligned}&\mathbf{P}(a \text{ switches OFF and ON again in } [0,T]) \\ &\quad = \mathbf{P}(Exp(\lambda_{\mathrm{ON}}^a) + Exp(\lambda_{\mathrm{OFF}}^a) \le T) \\ &\quad \le \mathbf{P}(Exp(\lambda_{\mathrm{OFF}}^a) \le T) = (1 - e^{-\lambda_{\mathrm{OFF}}^a T}) \le T\lambda_{\mathrm{OFF}}^a \wedge 1\end{aligned} \tag{105}$$

and, applying the same inequality again,

$$\begin{aligned}&\mathbf{P}(a \text{ switches ON, then OFF and ON again in } [0,T]) \\ &\quad = \mathbf{P}(Exp(\lambda_{\mathrm{OFF}}^a) + Exp(\lambda_{\mathrm{ON}}^a) + Exp'(\lambda_{\mathrm{OFF}}^a) \le T) \\ &\quad \le \mathbf{P}(Exp(\lambda_{\mathrm{OFF}}^a) + Exp'(\lambda_{\mathrm{OFF}}^a) \le T) = 1 - e^{-T\lambda_{\mathrm{OFF}}^a} \\ &\quad\quad - T\lambda_{\mathrm{OFF}}^a e^{-T\lambda_{\mathrm{OFF}}^a} \le T\lambda_{\mathrm{OFF}}^a (1 - e^{-T\lambda_{\mathrm{OFF}}^a}) \le (T\lambda_{\mathrm{OFF}}^a)^2 \wedge 1\end{aligned} \tag{106}$$

Substituting (105) and (106) into (104) and using that $\pi_{\mathrm{ON}}^a \le \lambda_{\mathrm{OFF}}^a, \pi_{\mathrm{OFF}}^a \le 1$ yields

$$\mathbf{P}(a \text{ switches ON twice in } [0,T]) \le (T + T^2)(\lambda_{\mathrm{OFF}}^a)^2 \wedge 1 \tag{107}$$

To obtain the probability in the denominator of (103) recall (64). Combining all of the above, we arrive at the bound

$$\mathbf{P}(a \text{ switches ON twice in } [0,T] | G_n^{[0,T]}) \le \frac{(T + T^2)(\lambda_{\mathrm{OFF}}^a)^2}{1 - e^{-T\lambda_{\mathrm{OFF}}^a}} \wedge 1 \tag{108}$$

Thus, substituting (108) into (101) and invoking (102) yields

$$\begin{aligned}&\mathbf{E}\big[\mathbf{P}(\exists a \in B_r^{[0,T]}(V_n^{(l)}) : a \text{ ON twice in } [0,T] | G_n^{[0,T]})\big] \\ &\quad \le \mathbf{E}\left[\left(\sum_{a \in B_r^{[0,T]}(V_n^{(l)})} \left(\frac{(T + T^2)(\lambda_{\mathrm{OFF}}^a)^2}{1 - e^{-T\lambda_{\mathrm{OFF}}^a}} \wedge 1\right) \right.\right. \\ &\quad\quad \left.\left. \cdot \mathbb{1}_{\{\max_i w_i \le a_n, \max_a |a| \le b_n\}}\right) \wedge 1\right]\end{aligned} \tag{109}$$

Note that by the local limit of $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$ shown in Theorem 2.9 we know that $|B_r^{[0,T]}(V_n^{(l)})|$ is tight. We also know that λ_{OFF}^a is small for n large for a group a with $\max_{i \in a} w_i \le a_n$ and

$|a| \le b_n$ and for x small it holds that $1 - e^{-x} \ge x/2$. Therefore, as $n \to \infty$,

$$\sum_{a \in B_r^{[0,T]}(V_n^{(l)})} \left(\frac{(T+T^2)(\lambda_{\mathrm{OFF}}^a)^2}{1 - e^{-T\lambda_{\mathrm{OFF}}^a}} \wedge 1 \right) \cdot \mathbb{1}_{\{\max_{i\in a} w_i \le a_n, \max|a| \le b_n\}} \le \sum_{a \in B_r^{[0,T]}(V_n^{(l)})} 2(\lambda_{\mathrm{OFF}}^a \wedge 1)\mathbb{1}_{\{\max_{i\in a} w_i \le a_n, \max|a|\le b_n\}} \xrightarrow{\mathbf{P}} 0 \tag{110}$$

Hence, by applying the dominated convergence theorem to (109) we conclude

$$\mathbf{P}(\exists a \in B_r^{[0,T]}(V_n^{(l)}) : a \text{ ON twice in } [0,T])) = o(1) \tag{111}$$

3.7 | Proof of Dynamic Local Limit of the Intersection Graph

Proof of Theorem 1.3. We first show the dynamic local weak convergence of $(\mathrm{BGRG}_n^s(w))_{s\in[0,T]}$ with a left-root, that is,

$$\left((\mathrm{BGRG}_n^s, V_n^{(l)}) \right)_{s\in[0,T]} \xrightarrow{\mathrm{d}} \left((\mathrm{BP}_l^s, o) \right)_{s\in[0,T]} \tag{112}$$

The proof follows in two steps: we first show convergence of finite-dimensional distributions and then tightness—the two conditions required for the weak convergence of processes with càdlàg sample paths from $[0,T]$ to a Polish space [31, Theorem 13.1]. As in separable and complete metric spaces tightness is equivalent to relative compactness in distribution, we verify the latter (in the form of two suitable conditions) as it is more convenient to work with. For the convenience of the reader, we reproduce the results we apply in Appendix C (see Theorem C.3).

Condition i: Convergence of finite-dimensional distributions. Once more, for the sake of simplicity of notation, throughout the proof we abbreviate $\mathrm{BGRG}_n^s(\boldsymbol{w})$ to G_n^s. We need to show that, for all $s_1 \le s_2 \le \cdots \le s_k \in [0,T]$

$$\begin{aligned}\mathbf{P}(\forall j \in [k] : B_r(G_n^{s_j}, V_n^{(l)}) &\simeq (H_j, o)) = \frac{1}{n}\mathbf{E}\left[\sum_{i\in[n]} \mathbb{1}_{\left\{B_r(G_n^{s_j}, i) \simeq (H_j, o)\right\}} \right] \\ &\to \mathbf{P}(\forall j \in [k] : B_r(\mathrm{BP}_l^{s_j}, o) \simeq (H_j, o))\end{aligned} \tag{113}$$

The convergence follows immediately from the convergence of the marked union graph. Indeed, if the marked union graph converges, appropriate marked-graph isomorphisms must hold. In particular, since marks converge,

$$\mathbf{P}(s \in [\sigma_{\mathrm{ON}}^a, \sigma_{\mathrm{OFF}}^a]) \to \mathbf{P}(s \in [t_{\mathrm{ON}}, t_{\mathrm{OFF}}]) \tag{114}$$

Thus, the local convergence of the marked union graph implies that a neighborhood of a uniformly chosen vertex in the marked union graph $(\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w}), V_n^{(l)}, ((\sigma_{\mathrm{ON}}^a, \sigma_{\mathrm{OFF}}^a))_{a:a \text{ ON in } [0,T]})$ resembles a neighborhood in the marked $(\mathrm{BP}_l^{[0,T]}, o, (t_{\mathrm{ON}}, t_{\mathrm{OFF}}))$. Take $s \in [0,T]$ and consider a neighborhood of a uniformly chosen vertex in the subgraph of the marked rooted graph $(\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w}), V_n^{(l)}, ((\sigma_{\mathrm{ON}}^a, \sigma_{\mathrm{OFF}}^a))_{a:a \text{ ON in } [0,T]})$ that is obtained by restricting to those groups a such that $s \in [\sigma_{\mathrm{ON}}^a, \sigma_{\mathrm{OFF}}^a]$ and a is in the connected component of V_n^l at time s. Then, given the local convergence of the market union graph, for every $s \in [0,T]$ such a neighborhood must resemble a neighborhood of o in the subgraph of $(\mathrm{BP}_l^{[0,T]}, o, ((t_{\mathrm{ON}}, t_{\mathrm{OFF}})))$ incorporating accordingly only right-vertices a such that $s \in [t_{\mathrm{ON}}, t_{\mathrm{OFF}}]$ and a is in the connected component of o at time s.

Condition ii: Tightness condition on the limiting process. Since the convergence of finite-dimensional distributions combined with tightness yields process convergence for random processes with càdlàg sample paths from $[0,T]$ to a separable space, it remains to show that $(\mathrm{BGRG}_n^s(\boldsymbol{w}), V_n^{(l)})_{s\in[0,T]}$ is tight with respect to the Skorokhod J_1 topology. In separable and complete spaces, tightness is equivalent to relative compactness in distribution and hence it can be verified by checking two convenient conditions (see Appendix C for a brief summary of the results we use, taken from [31, Chapter 13] and [51, Chapter 16]). We now treat the first of them. We want to show that, for all $\varepsilon > 0$ and as $\delta \searrow 0$,

$$\mathbf{P}(\mathrm{d}_{\mathcal{G}_\star}((G^T, o), (G^{T-\delta}, o)) > \varepsilon) \to 0 \tag{115}$$

which is equivalent to showing that, for all $\varepsilon > 0$ and as $\delta \searrow 0$,

$$\mathbf{P}(B_{1/\varepsilon}(G^T, o) \not\simeq B_{1/\varepsilon}(G^{T-\delta}, o)) \to 0 \tag{116}$$

From the proof of Condition i,

$$\lim_{\delta\searrow 0} \mathbf{P}(B_{1/\varepsilon}(G^T, o) \not\simeq B_{1/\varepsilon}(G^{T-\delta}, o)) = \lim_{\delta\searrow 0}\lim_{n\to\infty} \mathbf{P}(B_{1/\varepsilon}(G_n^T, V_n^{(l)}) \not\simeq B_{1/\varepsilon}(G_n^{T-\delta}, V_n^{(l)})) \tag{117}$$

Recall that the only aspect that can cause changes in neighborhoods is group activation and deactivation. Note that the groups that changed their status within $[T-\delta, T]$ must in particular be ON during $[T-\delta, T]$, which means that they are in the union graph of this time frame. Therefore, as it turns out to be useful for the upper bound, we now condition on the union graph over $[T-\delta, T]$, that is, the graph consisting of $G_n^{T-\delta}$ and all the groups that switched ON within $(T-\delta, T]$. For simplicity, we denote this union graph by $G_n^{[T-\delta,T]}$. We also denote the r-neighborhood of $V_n^{(l)}$ in this union graph by $B_r^{[T-\delta,T]}(V_n^{(l)})$. We compute

$$\begin{aligned}\mathbf{P}(B_{1/\varepsilon}(G_n^T, V_n^{(l)}) \not\simeq B_{1/\varepsilon}(G_n^{T-\delta}, V_n^{(l)}) | G_n^{[T-\delta,T]}) &= \mathbf{P}(\exists a \in B_{1/\varepsilon}^{[T-\delta,T]}(V_n^{(l)}) : a \text{ switches OFF in } [T-\delta, \delta] | G_n^{[T-\delta,T]}) \\ &\quad + \mathbf{P}(\exists a \in B_{1/\varepsilon}^{[T-\delta,T]}(V_n^{(l)}) : a \text{ switches ON in } [T-\delta, \delta] | G_n^{[T-\delta,T]})\end{aligned} \tag{118}$$

Hence,

$$\begin{aligned}\mathbf{P}(B_{1/\varepsilon}(G_n^T, V_n^{(l)}) \not\simeq B_{1/\varepsilon}(G_n^{T-\delta}, V_n^{(l)}) | G_n^{[T-\delta,T]}) &= (1 - e^{-\delta \#\{a \in B_{1/\varepsilon}^{[T-\delta,T]}(V_n^{(l)})\}}) \wedge 1 \\ &\quad + \left(1 - e^{-\delta \sum_{a\in B_{1/\varepsilon}^{[T-\delta,T]}(V_n^{(l)})} \lambda_{\mathrm{OFF}}^a}\right) \wedge 1\end{aligned} \tag{119}$$

Note that we can bound each of the expressions in brackets above by applying the inequality $1 - e^{-x} \leq x$, which holds for any x. Thus,

$$\begin{aligned}&\mathbf{P}(B_{1/\varepsilon}(G_n^T, V_n^{(l)}) \not\simeq B_{1/\varepsilon}(G_n^{T-\delta}, V_n^{(l)}) | G_n^{[T-\delta,T]}) \\ &\quad \leq \delta \#\{a \in B_{1/\varepsilon}^{[T-\delta,T]}(V_n^{(l)})\} \wedge 1 + \delta \sum_{a \in B_{1/\varepsilon}^{[T-\delta,T]}(V_n^{(l)})} \lambda_{\mathrm{OFF}}^a \wedge 1\end{aligned} \tag{120}$$

Further, by using a similar reasoning as in the proof of Lemma 2.2, we will show that each of the multipliers of δ in the above sum is bounded whp. Fix a large constant b_δ, and consider

$$\mathscr{D} = \left\{ |B_{1/\varepsilon}^{[T-\delta,T]}(V_n^{(l)})| \leq b_\delta \right\} \tag{121}$$

Recall that in Theorem 2.9 we derived local convergence of the union graph, which guarantees that the size of the neighborhood of a uniformly chosen vertex in the union graph is a tight random variable, that is, for every $\varepsilon > 0$ and every $\delta > 0$, we can find a b_δ sufficiently large, such that $\mathbf{P}(\mathscr{D}) \geq 1 - \varepsilon$. For the second term in (120), recall (102) from the proof of Lemma 2.2 which guarantees that whp we can truncate the maximum weight and the maximum group size in $B_{1/\varepsilon}^{[T-\delta,T]}(V_n^{(l)})$ by a_n and b_n respectively, with $a_n = \log(n), b_n = \log(n)$. Hence, for every $\varepsilon > 0$ and every $\delta > 0$, we can find $c_\delta(n)$ with $c_\delta(n) = o(n)$, such that $\mathbf{P}(\mathscr{E}) \geq 1 - \varepsilon$, where

$$\mathscr{E} = \left\{ \max_{a \in B_{1/\varepsilon}^{[T-\delta,T]}(V_n^{(l)})} \lambda_{\mathrm{OFF}}^a \leq c_\delta(n) \right\} \tag{122}$$

Then, given $G_n^{[T-\delta,T]}$ for which $\mathscr{D}$ and $\mathscr{E}$ hold, the second term in (120) can be bounded by a function of b_δ and $c_\delta(n)$:

$$\mathbf{P}(B_{1/\varepsilon}(G_n^T, V_n^{(l)}) \not\simeq B_{1/\varepsilon}(G_n^{T-\delta}, V_n^{(l)}) | G_n^{[T-\delta,T]}) \leq \delta b_\delta (1 + c_\delta(n)) \tag{123}$$

After taking the expectation of (123) with respect to the union graph we obtain

$$\begin{aligned}&\mathbf{P}\big(B_{1/\varepsilon}(G_n^T, V_n^{(l)}) \not\simeq B_{1/\varepsilon}(G_n^{T-\delta}, V_n^{(l)}); \mathscr{D} \cap \mathscr{E}\big) \\ &\quad \leq \mathbf{E}\left[\delta b_\delta (1 + c_\delta(n)) \mathbb{1}_{\mathscr{D} \cap \mathscr{E}}\right] = \delta b_\delta (1 + c_\delta(n)) \mathbf{P}(\mathscr{D} \cap \mathscr{E})\end{aligned} \tag{124}$$

Note that with our choice of $c_\delta(n)$, for any $\delta > 0$, $c_\delta(n)$ vanishes as $n \to \infty$. Additionally, for every b_δ, δb_δ can be made arbitrarily small by taking δ small. Thus, for every b_δ and $c_\delta(n)$, the entire expression under the first expectation in (124) can be made arbitrarily small by taking $n \to \infty$ and subsequently $\delta \searrow 0$, as in (117). Hence, by the dominated convergence theorem, also (124) can be made arbitrarily small for every b_δ and $c_\delta(n)$, by taking $n \to \infty$ and subsequently $\delta \searrow 0$. As we argued that for such choices of b_δ and $c_\delta(n)$, the events $\mathscr{D}$ and $\mathscr{E}$ each hold with probability at least $1 - \varepsilon$, we conclude that (116) holds by taking $n \to \infty$ and subsequently $\delta \searrow 0$, as well as ε small.

Condition iii: Tightness condition on the original process. We now check the second condition guaranteeing relative compactness in distribution, and hence tightness. For our dynamic graph process, this translates to verifying if, for all $T > 0$, $\varepsilon, \eta > 0$, there exists $n_0 \geq 1$ and $\delta > 0$ such that for all $n \geq n_0$

$$\begin{aligned}\mathbf{P}\Bigg(&\sup_{(s,s_1,s_2) \in \mathscr{S}_\delta} \min[\mathrm{d}_{\mathcal{G}_\star}((G_n^{s_1}, V_n^{(l)}), (G_n^s, V_n^{(l)})) \\ &\mathrm{d}_{\mathcal{G}_\star}((G_n^s, V_n^{(l)}), (G_n^{s_2}, V_n^{(l)}))] > \varepsilon \Bigg) \leq \eta\end{aligned} \tag{125}$$

with $\mathscr{S}_\delta = \{(s, s_1, s_2) : s \in [s_1, s_2], |s_2 - s_1| \leq \delta\}$. Note that the above is equivalent to

$$\begin{aligned}\mathbf{P}\big(&\exists s \in [s_1, s_2], s_2 - s_1 < \delta : B_{1/\varepsilon}(G_n^{s_1}, V_n^{(l)}) \not\simeq B_{1/\varepsilon}(G_n^s, V_n^{(l)}) \\ &B_{1/\varepsilon}(G_n^s, V_n^{(l)}) \not\simeq B_{1/\varepsilon}(G_n^{s_2}, V_n^{(l)})\big) \leq \eta\end{aligned} \tag{126}$$

We partition $[0, T]$ into intervals of length δ and introduce

$$\begin{aligned}S_l = \{&\text{two jumps changes in the neighborhood of } V_n^{(l)} \\ &\text{in the } l\text{th interval of length } \delta\}\end{aligned} \tag{127}$$

where by a "change" we mean that a relevant isomorphism does not hold anymore, as in (126), that is,

$$\begin{aligned}&\{\text{there is a change in the neighborhood of } V_n^{(l)} \text{ in } [a, b]\} \\ &\quad = \left\{\exists s \in [a, b] : B_{1/\varepsilon}(G_n^a, V_n^{(l)}) \not\simeq B_{1/\varepsilon}(G_n^s, V_n^{(l)})\right\}\end{aligned}$$

Note that thanks to the stationarity, the probability of a change in $[0, s]$ and then another change in $[s, \delta]$ can be bounded by the probability of two changes in $[0, \delta]$. Hence, by the union bound,

$$(3.47) \leq \mathbf{P}\left(\bigcup_{l=1}^{T/\delta} S_l\right) \leq \frac{T}{\delta} \mathbf{P}(S_1) \tag{128}$$

where S_1 is then accordingly the event of two changes in the time interval $[0, \delta]$. We now proceed analogously as in the proof of the previous condition. Recall that the only aspect that can cause changes in neighborhoods is group activation and deactivation. Note that the groups that changed their status within $[0, T]$ must in particular be ON during $[0, T]$, which means they are in the union graph $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$. Therefore, as it turns out to be useful for the upper bound, we now condition on the union graph. For simplicity, we denote the union graph by $G_n^{[0,T]}$. We also denote the r-neighborhood of $V_n^{(l)}$ in the union graph by $B_r^{[0,T]}(V_n^{(l)})$. We compute

$$\begin{aligned}\mathbf{P}(S_1 | G_n^{[0,T]}) = &\mathbf{P}(\exists a_1 \neq a_2 \in B_{1/\varepsilon}^{[0,T]}(V_n^{(l)}) : a_1 \\ &\quad a_2 \text{ switch OFF in } [0, \delta] | G_n^{[0,T]}) \\ &+ \mathbf{P}(\exists a_1 \neq a_2 \in B_{1/\varepsilon}^{[0,T]}(V_n^{(l)}) : a_1 \text{ switches OFF} \\ &\quad a_2 \text{ switches ON in } [0, \delta] | G_n^{[0,T]}) \\ &+ \mathbf{P}(\exists a_1 \neq a_2 \in B_{1/\varepsilon}^{[0,T]}(V_n^{(l)}) : a_1 \\ &\quad a_2 \text{ switch ON in } [0, \delta] | G_n^{[0,T]})\end{aligned} \tag{129}$$

Hence,

$$\begin{aligned}\mathbf{P}(S_1|G_n^{[0,T]}) &= (1-e^{-\delta\#\{a\in B_{1/\varepsilon}^{[0,T]}(V_n^{(l)})\}})^2\\ &+(1-e^{-\delta\#\{a\in B_{1/\varepsilon}^{[0,T]}(V_n^{(l)})\}})\left(1-e^{-\delta\sum_{a\in B_{1/\varepsilon}^{[0,T]}(V_n^{(l)})}\lambda_{\mathrm{OFF}}^a}\right)\wedge 1\\ &+\left(1-e^{-\delta\sum_{a\in B_{1/\varepsilon}^{[0,T]}(V_n^{(l)})}\lambda_{\mathrm{OFF}}^a}\right)^2\wedge 1\end{aligned}\tag{130}$$

The above can be bounded analogously as for (119) in the proof of Condition ii, that is, we first apply $1-e^{-x}\le x$ to each of the expressions; then, we fix a large constant $\hat{b}_\delta$ and a constant $\hat{c}_\delta(n)$ with $\hat{c}_\delta(n)=o(n)$ and consider

$$\hat{\mathscr{D}}=\left\{|B_{1/\varepsilon}^{[0,T]}(V_n^{(l)})|\le\hat{b}_\delta\right\}\text{ and}\qquad \hat{\mathscr{E}}=\left\{\max_{a\in B_{1/\varepsilon}^{[0,T]}(V_n^{(l)})}\lambda_{\mathrm{OFF}}^a\le\hat{c}_\delta(n)\right\}\tag{131}$$

As previously, using local convergence of the union graph (Theorem 2.9) and (102) from the proof of Lemma 2.2, we remark that for every $\varepsilon>0$ and every $\delta>0$, we can find a $\hat{b}_\delta$ sufficiently large and $\hat{c}_\delta(n)$ with $\hat{c}_\delta(n)=o(n)$, such that $\mathbf{P}(\hat{\mathscr{D}})\ge 1-\varepsilon$ and $\mathbf{P}(\hat{\mathscr{E}})\ge 1-\varepsilon$. Then, given $\hat{\mathscr{D}}$ and $\hat{\mathscr{E}}$,

$$\mathbf{P}(S_1|G_n^{[0,T]})\le\delta^2\hat{b}_\delta^2(1+\hat{c}_\delta(n)+\hat{c}_\delta^2(n))\tag{132}$$

After taking the expectation of (132) with respect to the union graph, and substituting the result into (128), we obtain

$$\begin{aligned}&\mathbf{P}\big(\exists s\in[s_1,s_2],s_2-s_1<\delta:B_{1/\varepsilon}(G_n^{s_1},V_n^{(l)})\not\simeq B_{1/\varepsilon}(G_n^{s},V_n^{(l)}),\\ &\quad B_{1/\varepsilon}(G_n^{s},V_n^{(l)})\not\simeq B_{1/\varepsilon}(G_n^{s_2},V_n^{(l)});\hat{\mathscr{D}}\cap\hat{\mathscr{E}}\big)\\ &\le\frac{T}{\delta}(\delta^2\hat{b}_\delta^2(1+\hat{c}_\delta(n)+\hat{c}_\delta^2(n))\mathbf{P}(\hat{\mathscr{D}}\cap\hat{\mathscr{E}}))\\ &=T\delta\hat{b}_\delta^2(1+\hat{c}_\delta(n)+\hat{c}_\delta^2(n))\mathbf{P}(\hat{\mathscr{D}}\cap\hat{\mathscr{E}})\end{aligned}\tag{133}$$

By the same reasoning as in the proof of Condition ii, for n large, the above can be made arbitrarily small for every $\hat{b}_\delta$ and every $\hat{c}_\delta(n)$ with $\hat{c}_\delta(n)=o(n)$, by taking δ small. As we argued that for such choices of $\hat{b}_\delta$ and $\hat{c}_\delta(n)$, the events $\hat{\mathscr{D}}$ and $\hat{\mathscr{E}}$ each hold with probability at least $1-\varepsilon$, we conclude that (126) holds for n large by taking δ and ε small as a function of η.

Consequence: For every $s\in[0,T]$, $\mathrm{DRIG}_n^s(\boldsymbol{w})$ can be built from $\mathrm{BGRG}_n^s(\boldsymbol{w})$ via a community projection, which preserves graph isomorphism and tightness. Hence, its convergence follows from the just shown convergence of $\mathrm{BGRG}_n^s(\boldsymbol{w})$, and its local limit, (CP^s,o), is a community projection of the limit of $\mathrm{BGRG}_n^s(\boldsymbol{w})$.

Remark 3.1. In the proof of Theorem 1.3 we show dynamic local weak convergence. However, we argue that in the same manner, we could derive dynamic local convergence in probability, which means that

$$\frac{1}{n}\sum_{i\in[n]}\mathbb{1}_{\{(\mathrm{BGRG}_n^s(\boldsymbol{w}))_{s\in[0,T]}\in\mathcal{A}\}}\xrightarrow{\mathbf{P}}\mathbf{P}((G^s,o)_{s\in[0,T]}\in\mathcal{A})\tag{134}$$

for all events $\mathcal{A}$ measurable with respect to $D(\mathcal{G}_\star,[0,T])$ (i.e., the probability measures on $\mathcal{G}_\star$ converge in probability). Indeed, note that neighborhood processes of two independent uniformly chosen vertices are i.i.d. stochastic processes, and hence converge to i.i.d. copies of the limiting graph. Thus, (134) holds.

3.8 | Proof of Convergence of the Dynamic Giant Membership Process

Proof of Theorem 1.4. Note again that the giant component in the intersection graph is whp equal to the giant component in the underlying bipartite random graph. Hence, in this proof, we only focus on the underlying bipartite random graph. To show the desired convergence it suffices to show that $(J_n(s))_{s\in[0,T]}$ and $(\mathcal{J}(s))_{s\in[0,T]}$ satisfy Conditions i–iii from Theorem C.3.

Condition i: Convergence of finite-dimensional distributions. Note that thanks to our results on the static giant component (see Theorem 2.6), pointwise for any $s\in[0,T]$,

$$\mathbf{P}(J_n(s)=1)=\mathbf{P}(o_n\in\mathscr{C}_1^s)=\frac{\mathbf{E}[|\mathscr{C}_1^s|]}{n}\xrightarrow{n\to\infty}\mathbf{P}(|\mathscr{C}^s(o)|=\infty)\tag{135}$$

where $\mathscr{C}^s(o)$ denotes the connected component of the root o in (G^s,o) and (G^s,o) is the limiting rooted graph at time s. Furthermore, as a consequence of the dynamic local weak convergence (Theorem 1.3), as $n\to\infty$,

$$\begin{aligned}(J_n^{(r)}(s))_{s\in[0,T]}&=\left(\mathbb{1}_{\{\partial B_r^{G_n^s}(o_n)\neq\emptyset\}}\right)_{s\in[0,T]}\xrightarrow{\mathrm{d}}\left(\mathbb{1}_{\{\partial B_r^{G^s}(o)\neq\emptyset\}}\right)_{s\in[0,T]}\\ &=(\mathcal{J}^{(r)}(s))_{s\in[0,T]}\end{aligned}\tag{136}$$

with $\partial B_r^{G_n^s}(o_n)=\{v\in[n]:\mathrm{d}_{G_n^s}(o_n,v)=r\}$, that is, the set of vertices at graph distance r from the root. Since we know what is happening in local neighborhoods jointly for all $s\in[0,T]$ we try to link the distribution of $(J_n(s))_{s\in[0,T]}$ to the distribution of $(J_n^{(r)}(s))_{s\in[0,T]}$. For any $r>0$ and for all $\{s_1,\dots,s_k\}\in[0,T]$,

$$\begin{aligned}\mathbf{P}(J_n(s_1)=\cdots=J_n(s_k)=1)&=\mathbf{P}(J_n^{(r)}(s_1)=\cdots=J_n^{(r)}(s_k)=1)\\ &+\mathbf{P}(J_n(s_1)=\cdots=J_n(s_k)=1)-\mathbf{P}(J_n^{(r)}(s_1)\\ &=\cdots=J_n^{(r)}(s_k)=1)\end{aligned}\tag{137}$$

We look at the difference of probabilities in (137) and apply the same reasoning as for the two-dimensional case presented in Section 2.5:

$$\begin{aligned}&\left|\mathbf{P}(J_n(s_1)=\cdots=J_n(s_k)=1)-\mathbf{P}(J_n^{(r)}(s_1)=\cdots=J_n^{(r)}(s_k)=1)\right|\\ &\le\mathbf{P}(J_n(s_1)=\cdots=J_n(s_k)=1,\{J_n^{(r)}(s_1)=\cdots=J_n^{(r)}(s_k)=1\}^c)\\ &=\mathbf{P}\left(J_n(s_1)=\cdots=J_n(s_k)=1,\bigcup_{i=1}^k\{J_n^{(r)}(s_i)\neq 1\}\right)\\ &=\mathbf{P}\left(\bigcup_{i=1}^k\{J_n(s_1)=\cdots=J_n(s_k)=1,J_n^{(r)}(s_i)\neq 1\}\right)\\ &\le\mathbf{P}\left(\bigcup_{i=1}^k\{J_n^{(r)}(s_i)\neq J_n(s_i)\}\right)\end{aligned}\tag{138}$$

Note that by the static local limit and (135),

$$\lim_{r\to\infty}\lim_{n\to\infty}\mathbf{P}\left(\bigcup_{i=1}^{k}\left\{J_n^{(r)}(s_i)\neq J_n(s_i)\right\}\right) \leq k\cdot\lim_{r\to\infty}\lim_{n\to\infty}\mathbf{P}(J_n^{(r)}(s)\neq J_n(s))=0 \tag{139}$$

Hence,

$$\lim_{r\to\infty}\lim_{n\to\infty}\left|\mathbf{P}(J_n(s_1)=\cdots=J_n(s_k)=1)-\mathbf{P}(J_n^{(r)}(s_1) =\cdots=J_n^{(r)}(s_k)=1)\right|=0 \tag{140}$$

and thus,

$$\lim_{n\to\infty}\mathbf{P}(J_n(s_1)=\cdots=J_n(s_k)=1)=\lim_{r\to\infty}\lim_{n\to\infty}\mathbf{P}(J_n^{(r)}(s_1) =\cdots=J_n^{(r)}(s_k)=1) \tag{141}$$

However, by (136),

$$\mathbf{P}(J_n^{(r)}(s_1)=\cdots=J_n^{(r)}(s_k)=1)\xrightarrow{n\to\infty}\mathbf{P}(\mathcal{J}^{(r)}(s_1) =\cdots=\mathcal{J}^{(r)}(s_k)=1) \xrightarrow{r\to\infty}\mathbf{P}(|\mathscr{C}^{G^{s_1}}(o)|=\cdots=|\mathscr{C}^{G^{s_k}}(o)|=\infty) \tag{142}$$

Hence,

$$\mathbf{P}(J_n(s_1)=\cdots=J_n(s_k)=1)\xrightarrow{n\to\infty}\mathbf{P}(|\mathscr{C}^{G^{s_1}}(o)| =\cdots=|\mathscr{C}^{G^{s_k}}(o)|=\infty)$$

We will now show that any other combination of values of the finite-dimensional distribution can be expressed in terms of sequences of 1's. Note that $\{J_n(s)=0\}=\{J_n(s)=1\}^c$ for any $s\geq 0$. Furthermore, by exclusion-inclusion, for all events A and B,

$$\mathbf{P}(A\cap B^c)=\mathbf{P}(A)-\mathbf{P}(A\cap B) \tag{143}$$

Write $\{J_n(s_1)=\cdots=J_n(s_{k-1})=1, J_n(s_k)=0\}=A\cap B^c$ with $A=\{J_n(s_1)=\cdots=J_n(s_{k-1})=1\}$ and $B=\{J_n(s_k)=1\}$. Hence,

$$\begin{aligned}\mathbf{P}(J_n(s_1)=\cdots=J_n(s_{k-1})=1, J_n(s_k)=0)\\ =\mathbf{P}(J_n(s_1)=\cdots=J_n(s_{k-1})=1)\\ -\mathbf{P}(J_n(s_1)=\cdots=J_n(s_k)=1)\end{aligned} \tag{144}$$

We can extend this argument to sequences involving more 0's to conclude that, for any $l\in\{1,\ldots,k-1\}$,

$$\mathbf{P}(J_n(s_1)=\cdots=J_n(s_l)=1, J_n(s_{l+1})=\cdots=J_n(s_k)=0) =\sum_{S\subseteq\{l+1,\ldots,k\}}(-1)^{|S|}\mathbf{P}(J_n(s_i)=1\forall i\in\{1,\ldots,l\}\cup S) \tag{145}$$

where the sum runs over all possible subsets S of $\{l+1,\ldots,k\}$. Thus, having proved the claim for the case $\{J_n(s_1)=\cdots=J_n(s_k)=1\}$, we can conclude the claim for any other combination of values of the finite-dimensional distribution.

Condition ii: Tightness condition on the limiting process. We want to show that, for all $\varepsilon>0$, as $\delta\searrow 0$,

$$\mathbf{P}(|\mathcal{J}(T)-\mathcal{J}(T-\delta)|>\varepsilon)\to 0 \tag{146}$$

Since $(\mathcal{J}(s))_{s\in[0,T]}$ is an indicator process, the difference in absolute value between any two points of the process equals either 0 or 1. Hence $\mathbf{P}(|\mathcal{J}(T)-\mathcal{J}(T-\delta)|>\varepsilon)$ is equivalent to $\mathbf{P}(|\mathcal{J}(T)-\mathcal{J}(T-\delta)|=1)$, which is equivalent to $\mathbf{P}(\mathcal{J}(T-\delta)=0,\mathcal{J}(T)=1)+\mathbf{P}(\mathcal{J}(T-\delta)=1,\mathcal{J}(T)=0)$. We investigate these two factors separately. From the proof of Condition i,

$$\begin{aligned}\lim_{\delta\searrow 0}\mathbf{P}(\mathcal{J}(T-\delta)=0,\mathcal{J}(T)=1)\\ =\lim_{\delta\searrow 0}\lim_{n\to\infty}\mathbf{P}(J_n(T-\delta)=0,J_n(T)=1)\\ =\lim_{\delta\searrow 0}\lim_{r\to\infty}\lim_{n\to\infty}\mathbf{P}(J_n^{(r)}(T-\delta)=0,J_n^{(r)}(T)=1)\end{aligned} \tag{147}$$

We compute

$$\begin{aligned}\mathbf{P}(J_n^{(r)}(T-\delta)=0,J_n^{(r)}(T)=1)\\ =\mathbf{P}\left(\partial B_r^{G_n^{T-\delta}}(V_n^{(l)})=\emptyset,\partial B_r^{G_n^{T}}(V_n^{(l)})\neq\emptyset\right)\end{aligned} \tag{148}$$

which means that the boundary of the r-neighborhood of a uniformly chosen vertex is empty at time $T-\delta$ but non-empty at time point T. For that to happen there has to be a change in groups' statuses. Similarly as in the proof of Theorem 1.3, we use the link with the union graph to obtain

$$\begin{aligned}\mathbf{P}\left(\partial B_r^{G_n^{T-\delta}}(V_n^{(l)})=\emptyset,\partial B_r^{G_n^{T}}(V_n^{(l)})\neq\emptyset|G_n^{[0,T]}\right)\\ =\mathbf{P}(\exists a\in B_r^{[0,T]}(V_n^{(l)}):a\text{ switches ON in }[T-\delta,T]|G_n^{[0,T]})\\ =\left(1-e^{-\delta\sum_{a\in B_r^{[0,T]}(V_n^{(l)})}\lambda_{\text{OFF}}^a}\right)\wedge 1\end{aligned} \tag{149}$$

Hence,

$$\mathbf{P}(J_n^{(r)}(T-\delta)=0,J_n^{(r)}(T)=1)\leq\mathbf{E}_{G_n^{[0,T]}}\left[\delta|B_r^{[0,T]}(V_n^{(l)})|\right] \tag{150}$$

which can be bounded in the same way as the terms in (120) and hence converges to 0 as $n\to\infty$ and, subsequently, $r\to\infty$ and $\delta\searrow 0$. We can compute the complementary probability, $\mathbf{P}(\partial B_r^{G_n^{T-\delta}}(V_n^{(l)})\neq\emptyset,\partial B_r^{G_n^{T}}(V_n^{(l)})=\emptyset)$, using similar reasoning. Note that, conveniently, the probability of switching off is the same for all groups. Thanks to this and the independence of groups we obtain

$$\begin{aligned}\mathbf{P}(\partial B_r^{G_n^{T-\delta}}(V_n^{(l)})\neq\emptyset,\partial B_r^{G_n^{T}}(V_n^{(l)})=\emptyset|G_n^{T-\delta})\\ =\mathbf{P}(\text{all }a\in\partial B_r^{G_n^{T-\delta}}(V_n^{(l)})\text{ switch OFF}|G_n^{T-\delta})\\ =\prod_{a\in\partial B_r^{G_n^{T-\delta}}(V_n^{(l)})}(1-e^{-\delta})\end{aligned} \tag{151}$$

and thus,

$$\begin{aligned}\mathbf{P}(\partial B_r^{G_n^{T-\delta}}(V_n^{(l)})\neq\emptyset,\partial B_r^{G_n^{T}}(V_n^{(l)})=\emptyset)\\ =\mathbf{E}_{G_n^{T-\delta}}\left[(1-e^{-\delta})^{\#\{a:a\in\partial B_r^{G_n^{T-\delta}}(V_n^{(l)})\}}\right]\end{aligned} \tag{152}$$

With arguments similar to those used before to bound (120), we can show that the above vanishes as $n \to \infty$, $r \to \infty$ and $\delta \searrow 0$. Combining (149) and (152) we conclude that condition (146) holds.

Condition iii: Tightness condition on the original process. We want to show that for any $\varepsilon, \eta > 0$ there exists $n_0 \geq 1$ and $\delta > 0$ such that, for all $n \geq n_0$,

$$\mathbf{P}\left(\sup_{(s,s_1,s_2)\in\mathcal{S}_\delta} \min\left(\left|J_n(s) - J_n(s_1)\right|, \left|J_n(s_2) - J_n(s)\right| \right) > \varepsilon \right) \leq \eta \tag{153}$$

with $\mathcal{S}_\delta = \{(s,s_1,s_2) : s \in [s_1,s_2], |s_2 - s_1| \leq \delta\}$. Note that since $(J_n(s))_{s\in[0,T]}$ is an indicator process, $\min(\left|J_n(s) - J_n(s_1)\right|, \left|J_n(s_2) - J_n(s)\right|) > \varepsilon$ if and only if $\left|J_n(s) - J_n(s_1)\right| = \left|J_n(s_2) - J_n(s)\right| = 1$, which is equivalent to $J_n(s) \neq J_n(s_1), J_n(s_2) \neq J_n(s)$. This means that one of two mutually exclusive events occurs: either $J_n(s_1) = J_n(s_2) = 1$ and $J_n(s) = 0$, or $J_n(s_1) = J_n(s_2) = 0$ and $J_n(s) = 1$. Note that we can skip the supremum since for any $s \in [s_1, s_2]$ the value of $\left|J_n(s) - J_n(s_1)\right|$ and $\left|J_n(s_2) - J_n(s)\right|$ is at most 1. Taking this all into consideration, (153) becomes

$$\begin{aligned}&\mathbf{P}(\exists s, s_1, s_2 \in \mathcal{S}_\delta : J_n(s_1) = J_n(s_2) = 1, J_n(s) = 0 \\ &\quad \text{or } J_n(s_1) = J_n(s_2) = 0, J_n(s) = 1)\end{aligned} \tag{154}$$

We apply the same approach as in the proof of Condition iii for Theorem 1.3: we partition $[0,T]$ into intervals of length δ and denote

$$\begin{aligned}\mathbf{P}(R_l) = \mathbf{P}(&\text{two changes of } (J_n(s))_{s\in[0,T]} \\ &\text{in the } l\text{th interval of length } \delta)\end{aligned} \tag{155}$$

where by change's we mean that the indicator process $(J_n(s))_{s\in[0,T]}$ switches from 0 to 1 or the other way around. Then, by stationarity,

$$(3.75) = \mathbf{P}\left(\bigcup_{l=1}^{T/\delta} R_l \right) \leq \tfrac{T}{\delta} \mathbf{P}(R_1) \tag{156}$$

From the proof of the Condition i of Theorem 1.4 we know that for some n_0 big enough for all $n \geq n_0$ and some $s \in [0, \delta]$,

$$\begin{aligned}&\mathbf{P}(J_n(0) = J_n(\delta) = 1, J_n(s) = 0) \\ &\quad = \lim_{r\to\infty} \mathbf{P}(J_n^{(r)}(0) = J_n^{(r)}(\delta) = 1, J_n^{(r)}(s) = 0) \\ &\quad = \lim_{r\to\infty} \mathbf{P}\Bigg(\mathbb{1}_{\left\{\partial B_r^{G_n^0}(V_n^{(l)})\neq\emptyset\right\}} = \mathbb{1}_{\left\{\partial B_r^{G_n^\delta}(V_n^{(l)})\neq\emptyset\right\}} \\ &\quad = 1, \mathbb{1}_{\left\{\partial B_r^{G_n^s}(V_n^{(l)})\neq\emptyset\right\}} = 0 \Bigg)\end{aligned} \tag{157}$$

and naturally, the analogous will hold for the complementary probability $\mathbf{P}(J_n(0) = J_n(\delta) = 0, J_n(s) = 1)$. Hence, from the proof of Condition iii from Theorem 1.3 it follows that

$$\mathbf{P}(R_1) \leq \mathbf{P}(S_1) = o(\delta) \tag{158}$$

with

$$\begin{aligned}\mathbf{P}(S_l) = \mathbf{P}(&\text{two changes in the neighborhood of} \\ &V_n^{(l)} \text{ in the } l\text{th interval of length } \delta)\end{aligned} \tag{159}$$

as in (127) in the proof of Theorem 1.3. Thus, the required condition holds.

Conclusion: Since all three conditions of Theorem C.3 hold, the convergence follows.

3.9 | Proof of Convergence of the Size of the Largest Group in $[0, T]$

Proof of Theorem 1.5. The proof consists of two parts. We start by deriving convergence in distribution for $K_{\max}^{[0,T]}/n^{\frac{1}{\alpha}}$ and afterwards proceed to show that $(K_{\max}^{[0,T]}/n^{\frac{1}{\alpha}})_{T\geq0}$ and the limiting process $(\kappa_{\max}^{[0,T]})_{T\geq0}$ satisfy conditions of Theorem C.3, which will yield the desired convergence.

Part 1: Convergence in distribution. To shorten the computations in the next part of the proof, we first derive convergence in distribution of the random variable $K_{\max}^{[0,T]}/n^{\frac{1}{\alpha}}$. We fix $T \geq 0$ and compute

$$\begin{aligned}\mathbf{P}(K_{\max}^{[0,T]} \leq kn^{1/\alpha}) &= \mathbf{P}(\max\{K_{\max}^{\{0\}}, K_{\max}^{(0,T]}\} \leq kn^{1/\alpha}) \\ &= \prod_{l>kn^{1/\alpha}} \prod_{a\in[n]_l} \pi_{\text{OFF}}^a \cdot \mathbf{P}(a \text{ never ON in } (0,T])\end{aligned} \tag{160}$$

Hence,

$$\begin{aligned}\mathbf{P}(K_{\max}^{[0,T]} \leq kn^{1/\alpha}) &= \prod_{l>kn^{1/\alpha}} \prod_{a\in[n]_l} \frac{\ell_n^{l-1}}{\ell_n^{l-1} + l! p_l \prod_{i\in a} w_i} \\ &\quad\times \prod_{l>kn^{1/\alpha}} \prod_{a\in[n]_l} e^{-\frac{l! p_l \prod_{i\in a} w_i}{\ell_n^{l-1}} T}\end{aligned} \tag{161}$$

Note that

$$\begin{aligned}\prod_{l>kn^{1/\alpha}} \prod_{a\in[n]_l} e^{-\frac{l! p_l \prod_{i\in a} w_i}{\ell_n^{l-1}} T} &= \prod_{l>kn^{1/\alpha}} e^{-T \sum_{j_1<\cdots<j_l\in[n]} \frac{l! p_l \prod_{i\in a} w_i}{\ell_n^{l-1}}} \\ &= \prod_{l>kn^{1/\alpha}} e^{-T \frac{l! p_l \ell_n}{l!} \sum_{j_1,\ldots,j_l\in[n]} \frac{w_{j_1}\cdots w_{j_l}}{\ell_n^l}} = e^{-T\ell_n \sum_{l>kn^{1/\alpha}} p_l} + o(1) \\ &= e^{-c_p(T+1)\ell_n (kn^{1/\alpha})^{-\alpha}} + o(1)\end{aligned}$$

where c_p is such that $\sum_{l>kn^{1/\alpha}} p_l = c_p(kn^{1/\alpha})^{-\alpha}(1 + o(1))$ (recall (35)), which plugged into (161) yields

$$\begin{aligned}\mathbf{P}(K_{\max}^{[0,T]} \leq kn^{1/\alpha}) &= e^{-c_p(T+1)\ell_n (kn^{1/\alpha})^{-\alpha}} \\ &\quad\times \prod_{l>kn^{1/\alpha}} \prod_{a\in[n]_l} \frac{\ell_n^{l-1}}{\ell_n^{l-1} + l! p_l \prod_{i\in a} w_i} + o(1)\end{aligned} \tag{162}$$

By a computation analogous to (162),

$$\prod_{l>kn^{1/\alpha}}\prod_{a\in[n]_l}\frac{\ell_n^{l-1}}{\ell_n^{l-1}+l!p_l\prod_{i\in a}w_i}=\prod_{l>kn^{1/\alpha}}\prod_{a\in[n]_l}\frac{1}{1+\frac{l!p_l\prod_{i\in a}w_i}{\ell_n^{l-1}}}$$
$$=e^{-\sum_{l>kn^{1/\alpha}}\sum_{a\in[n]_l}\frac{l!p_l\prod_{i\in a}w_i}{\ell_n^{l-1}}}+o(1)=e^{-c_p\ell_n(kn^{1/\alpha})^{-\alpha}}+o(1) \tag{163}$$

Further, since $\mathbf{E}[W_n]\to\mathbf{E}[W]$ as $n\to\infty$,

$$\lim_{n\to\infty}\ell_n(kn^{1/\alpha})^{-\alpha}=\lim_{n\to\infty}\frac{\ell_n}{k^\alpha n}=\lim_{n\to\infty}\frac{\mathbf{E}[W_n]}{k^\alpha}=\frac{\mathbf{E}[W]}{k^\alpha} \tag{164}$$

we obtain that, for every $T\geq 0$,

$$\lim_{n\to\infty}\mathbf{P}\left(\frac{K_{\max}^{[0,T]}}{n^{1/\alpha}}\leq k\right)=e^{-c_p(T+1)k^{-\alpha}\mathbf{E}[W]} \tag{165}$$

Note that $g(k)=e^{-c_p(T+1)k^{-\alpha}\mathbf{E}[W]}$ is a cumulative distribution function of the Fréchet distribution.

Part 2: Verifying appropriate conditions. We again check if our processes fulfil Conditions i–iii of Theorem C.3.

Condition i: Convergence of the finite-dimensional distribution. We want to show that for all $\{s_1,\dots,s_u\}\in[0,t]$ and $k_1,k_2,\dots,k_u\in\mathbf{N}$, as $n\to\infty$,

$$\mathbf{P}\left(\frac{K_{\max}^{[0,s_1]}}{n^{1/\alpha}}\leq k_1,\frac{K_{\max}^{[0,s_2]}}{n^{1/\alpha}}\leq k_2,\dots,\frac{K_{\max}^{[0,s_u]}}{n^{1/\alpha}}\leq k_u\right)$$
$$\to\mathbf{P}\left(\kappa_{\max}^{[0,s_1]}\leq k_1,\kappa_{\max}^{[0,s_2]}\leq k_2,\dots,\kappa_{\max}^{[0,s_u]}\leq k_u\right) \tag{166}$$

Note that for every $s_i,s_j\in[0,t]$ with $s_i<s_j$ it holds that $\mathbf{P}(K_{\max}^{[0,s_i]}>K_{\max}^{[0,s_j]})=0$. We perform the following loop:

$$\begin{aligned}&i=1,C_i=\{k_1,\dots,k_u\}\\&\bar{k}_i=\min C_i,\bar{u}_i=\arg\min\bar{k}_i\\&C_{i+1}=C_i\setminus\{(k_j)_{j=1}^{\bar{u}_i}\}\\&\text{Iterate for }C_i\end{aligned} \tag{167}$$

Note that $(\bar{k}_i)_{i\geq 1}$ is a non-decreasing sequence. Hence, for any input of the above loop $k_1,\dots,k_u$ it follows that

$$\mathbf{P}\left(\frac{K_{\max}^{[0,s_1]}}{n^{1/\alpha}}\leq k_1,\frac{K_{\max}^{[0,s_2]}}{n^{1/\alpha}}\leq k_2,\dots,\frac{K_{\max}^{[0,s_u]}}{n^{1/\alpha}}\leq k_u\right)$$
$$=\mathbf{P}\left(\frac{K_{\max}^{[0,s_1]}}{n^{1/\alpha}}\leq\bar{k}_1,\frac{K_{\max}^{[0,s_2]}}{n^{1/\alpha}}\leq\bar{k}_2,\dots,\frac{K_{\max}^{[0,s_u]}}{n^{1/\alpha}}\leq\bar{k}_u\right) \tag{168}$$

Hence, we further assume that we are working with non-decreasing sequences. Note that, by definition (see (29)), for a partition $\{0,s_1,s_2,\dots,s_{u-1},s_u\}$ of the time interval $[0,s_u]$,

$$K_{\max}^{[0,s_u]}=\max\{K_{\max}^{[0,s_1]},K_{\max}^{(s_1,s_2]},\dots,K_{\max}^{(s_{u-1},s_u]}\} \tag{169}$$

Thus,

$$\begin{aligned}&\mathbf{P}(K_{\max}^{[0,s_1]}\leq k_1n^{1/\alpha},K_{\max}^{[0,s_2]}\leq k_2n^{1/\alpha},\dots,K_{\max}^{[0,s_u]}\leq k_un^{1/\alpha})\\&=\mathbf{P}(K_{\max}^{[0,s_1]}\leq k_1n^{1/\alpha},\max\{K_{\max}^{[0,s_1]},K_{\max}^{(s_1,s_2]}\}\\&\leq k_2n^{1/\alpha},\dots,\max\{K_{\max}^{[0,s_1]},K_{\max}^{(s_1,s_2]},\dots,K_{\max}^{(s_{u-1},s_u]}\}\leq k_un^{1/\alpha})\\&=\mathbf{P}(K_{\max}^{[0,s_1]}\leq k_1n^{1/\alpha},K_{\max}^{[0,s_1]}\leq k_2n^{1/\alpha},K_{\max}^{(s_1,s_2]}\\&\leq k_2n^{1/\alpha},\dots,K_{\max}^{[0,s_1]}\leq k_un^{1/\alpha},K_{\max}^{(s_1,s_2]}\leq k_un^{1/\alpha}\\&\quad,\dots,K_{\max}^{(s_{u-1},s_u]}\leq k_un^{1/\alpha})\\&=\mathbf{P}(K_{\max}^{[0,s_1]}\leq k_1n^{1/\alpha},K_{\max}^{(s_1,s_2]}\leq k_2n^{1/\alpha},\dots,K_{\max}^{(s_{u-1},s_u]}\leq k_un^{1/\alpha})\end{aligned} \tag{170}$$

Note that $K_{\max}^{[0,s_1]},(K_{\max}^{(s_i,s_{i+1}]})_i$ are all independent, since the intervals $[0,s_1],(s_1,s_2],\dots,(s_i,s_{i+1}]$ for $i\geq 1$ do not overlap. Hence, using the computation from Part 1 we obtain

$$\begin{aligned}&\mathbf{P}(K_{\max}^{[0,s_1]}\leq k_1n^{1/\alpha},K_{\max}^{(s_1,s_2]}\leq k_2n^{1/\alpha},\dots,K_{\max}^{(s_{u-1},s_u]}\leq k_un^{1/\alpha})\\&=\mathbf{P}(K_{\max}^{[0,s_1]}\leq k_1n^{1/\alpha})\prod_{i=2}^{u}\mathbf{P}(K_{\max}^{(s_{i-1},s_i]}\leq k_in^{1/\alpha})\\&=\mathbf{P}(K_{\max}^{[0,s_1]}\leq k_1n^{1/\alpha})\prod_{i=2}^{u}\\&\times\left(\prod_{l>k_in^{1/\alpha}}\prod_{a\in[n]_l}\mathbf{P}(a\text{ never ON in }(s_{i-1},s_i])\right)\\&=e^{-c_p(s_1+1)\mathbf{E}[W_n]k_1^{-\alpha}}\prod_{l>k_1n^{1/\alpha}}\prod_{a\in[n]_l}\frac{\ell_n^{l-1}}{\ell_n^{l-1}+l!p_l\prod_{i\in a}w_i}\\&\times\prod_{i=2}^{u}e^{-(s_i-s_{i-1})\mathbf{E}[W_n]k_i^{-\alpha}}\\&\xrightarrow{n\to\infty}e^{-c_p(s_1+1)\mathbf{E}[W]k_1^{-\alpha}}\prod_{i=2}^{u}e^{-(s_i-s_{i-1})\mathbf{E}[W]k_i^{-\alpha}}\end{aligned} \tag{171}$$

Note that, since $k_1\leq k_2\leq\dots\leq k_u$,

$$\begin{aligned}&e^{-c_p(s_1+1)\mathbf{E}[W]k_1^{-\alpha}}\prod_{i=2}^{u}e^{-(s_i-s_{i-1})\mathbf{E}[W]k_i^{-\alpha}}\\&=\mathbf{P}(\kappa_{\max}^{[0,s_1]}\leq k_1)\prod_{i=2}^{u}\mathbf{P}(\kappa_{\max}^{(s_{i-1},s_i]}\leq k_i)\\&=\mathbf{P}(\kappa_{\max}^{[0,s_1]}\leq k_1,\kappa_{\max}^{(s_1,s_2]}\leq k_2,\dots,\kappa_{\max}^{(s_{u-1},s_u]}\leq k_u)\\&=\mathbf{P}(\kappa_{\max}^{[0,s_1]}\leq k_1,\max\{\kappa_{\max}^{[0,s_1]},\kappa_{\max}^{(s_1,s_2]}\}\\&\leq k_2,\dots,\max\{\kappa_{\max}^{[0,s_1]},\kappa_{\max}^{(s_1,s_2]},\dots,\kappa_{\max}^{(s_{u-1},s_u]}\}\leq k_u)\\&=\mathbf{P}(\kappa_{\max}^{[0,s_1]}\leq k_1,\kappa_{\max}^{[0,s_2]}\leq k_2,\dots,\kappa_{\max}^{[0,s_u]}\leq k_u)\end{aligned} \tag{172}$$

This proves the desired convergence from (166).

Condition ii: Tightness of the limiting process. We want to show that

$$\lim_{\delta\searrow 0}\mathbf{P}(\left|\kappa_{\max}^{[0,T]}-\kappa_{\max}^{[0,T-\delta]}\right|>\varepsilon)=0 \tag{173}$$

which is equivalent to

$$\lim_{\delta\searrow 0}\mathbf{P}(\kappa_{\max}^{[0,T]}>\kappa_{\max}^{[0,T-\delta]}+\varepsilon)=0 \tag{174}$$

By the proof of Condition i

$$\lim_{\delta\searrow 0}\mathbf{P}(\kappa_{\max}^{[0,T]}>\kappa_{\max}^{[0,T-\delta]}+\varepsilon)=\lim_{\delta\searrow 0}\lim_{n\to\infty}\mathbf{P}(K_{\max}^{[0,T]}>K_{\max}^{[0,T-\delta]}+\varepsilon n^{1/\alpha}) \tag{175}$$

We compute

$$\begin{aligned}\mathbf{P}(K_{\max}^{[0,T]}>K_{\max}^{[0,T-\delta]}+\varepsilon n^{1/\alpha})&=\sum_{k=2}^{\infty}\\ &\times\mathbf{P}(\text{group bigger than } k+\varepsilon n^{1/\alpha} \text{ switches ON in}\\ &\times(T-\delta,T])\mathbf{P}(K_{\max}^{[0,T-\delta]}=k)\\ &=\sum_{k=2}^{\infty}(1-e^{-\delta\ell_n\sum_{l>k+\varepsilon n^{1/\alpha}}p_l})\mathbf{P}(K_{\max}^{[0,T-\delta]}=k)\\ &\le\delta\ell_n\sum_{k=2}^{\infty}(k+\varepsilon n^{1/\alpha})^{-\alpha}\mathbf{P}(K_{\max}^{[0,T-\delta]}=k)\\ &\le\delta\ell_n\sum_{k=2}^{\infty}(\varepsilon n^{1/\alpha})^{-\alpha}\mathbf{P}(K_{\max}^{[0,T-\delta]}=k)\\ &=\frac{\delta\ell_n}{\varepsilon^{\alpha}n}=\frac{\delta\mathbf{E}[W_n]}{\varepsilon^{\alpha}}\end{aligned} \tag{176}$$

Substituting this into (175) yields

$$\lim_{\delta\searrow 0}\mathbf{P}(\kappa_{\max}^{[0,T]}>\kappa_{\max}^{[0,T-\delta]}+\varepsilon)=\lim_{\delta\searrow 0}\lim_{n\to\infty}\tfrac{\delta\mathbf{E}[W_n]}{\varepsilon^{\alpha}}=\lim_{\delta\searrow 0}\tfrac{\delta\mathbf{E}[W]}{\varepsilon^{\alpha}}=0 \tag{177}$$

Condition iii: Tightness of the original process. We want to show that for any $\varepsilon,\eta>0$ there exists $n_0\ge 1$ and $\delta>0$ such that, for all $n\ge n_0$,

$$\mathbf{P}\left(\sup_{(s,s_1,s_2)\in\mathcal{S}_\delta}\min\left(\left|\tfrac{K_{\max}^{[0,s]}}{n^{1/\alpha}}-\tfrac{K_{\max}^{[0,s_1]}}{n^{1/\alpha}}\right|,\left|\tfrac{K_{\max}^{[0,s_2]}}{n^{1/\alpha}}-\tfrac{K_{\max}^{[0,s]}}{n^{1/\alpha}}\right|\right)>\varepsilon\right)\le\eta \tag{178}$$

Note that

$$\begin{aligned}&\mathbf{P}\left(\sup_{(s,s_1,s_2)\in\mathcal{S}_\delta}\min\left(\left|\frac{K_{\max}^{[0,s]}}{n^{1/\alpha}}-\frac{K_{\max}^{[0,s_1]}}{n^{1/\alpha}}\right|,\left|\frac{K_{\max}^{[0,s_2]}}{n^{1/\alpha}}-\frac{K_{\max}^{[0,s]}}{n^{1/\alpha}}\right|\right)>\varepsilon\right)\\ &\le\mathbf{P}\Big(\exists s\in[s_1,s_2], s_2-s_1<\delta:\\ &\quad\min\left(\left|K_{\max}^{[0,s]}-K_{\max}^{[0,s_1]}\right|,\left|K_{\max}^{[0,s_2]}-K_{\max}^{[0,s]}\right|\right)>\varepsilon n^{1/\alpha}\Big)\end{aligned} \tag{179}$$

For the minimum of two terms to be bigger than $\varepsilon n^{1/\alpha}$, both of them have to be bigger than $\varepsilon n^{1/\alpha}$. As we are dealing with a non-decreasing random variable, this can only happen if a group which is bigger by $\varepsilon n^{1/\alpha}$ than the so far largest group switches ON in $(s_1,s]$ and then the same happens in $(s,s_2]$. Once again we partition $[0,T]$ into intervals of length δ and apply stationarity to deduce

$$\begin{aligned}&\mathbf{P}\left(\sup_{(s,s_1,s_2)\in\mathcal{S}_\delta}\min\left(\left|\frac{K_{\max}^{[0,s]}}{n^{1/\alpha}}-\frac{K_{\max}^{[0,s_1]}}{n^{1/\alpha}}\right|,\left|\frac{K_{\max}^{[0,s_2]}}{n^{1/\alpha}}-\frac{K_{\max}^{[0,s]}}{n^{1/\alpha}}\right|\right)>\varepsilon\right)\\ &\le\frac{T}{\delta}\mathbf{P}(T_1)\end{aligned} \tag{180}$$

where T_l is the event that the dynamic largest group process encounters two changes in the lth time interval of length δ, where a change means that a bigger than the so far the biggest group switches ON. We compute

$$\begin{aligned}&\mathbf{P}(T_1)\le\mathbf{P}(K_{\max}^{[0,s]}>K_{\max}^{[0,s_1]}+\varepsilon n^{1/\alpha},K_{\max}^{[0,s_2]}>K_{\max}^{[0,s]}+\varepsilon n^{1/\alpha})\\ &\sum_{k=2}^{\infty}\sum_{l=k+1}^{\infty}\mathbf{P}(a:|a|=l+\varepsilon n^{1/\alpha}\text{ switches ON in }(s_1,s])\mathbf{P}(a:|a|\\ &\quad>l+\varepsilon n^{1/\alpha}\text{ switches ON in }(s,s_2])\mathbf{P}(K_{\max}^{[0,T-\delta]}=k)\end{aligned} \tag{181}$$

From previous points we know that for $l\ge 2$ and any $s\in[0,T]$, $\mathbf{P}(a:|a|>l+\varepsilon n^{1/\alpha}\text{ switches ON in }(s,s+\delta])\le\delta\ell_n(\varepsilon n^{1/\alpha})^{-\alpha}$. Substituting this bound in the above calculation yields

$$\begin{aligned}\mathbf{P}(T_1)&\le\sum_{k=2}^{\infty}\sum_{l=k+1}^{\infty}\delta\ell_n(\varepsilon n^{1/\alpha})^{-\alpha}\mathbf{P}(a:|a|\\ &=l+\varepsilon n^{1/\alpha}\text{ switches ON in }(s_1,s])\mathbf{P}(K_{\max}^{[0,s_1]}=k)\\ &=\delta\ell_n(\varepsilon n^{1/\alpha})^{-\alpha}\le\sum_{k=2}^{\infty}\mathbf{P}(a:|a|\\ &=k+\varepsilon n^{1/\alpha}\text{ switches ON in }(s,s_2])\mathbf{P}(K_{\max}^{[0,s_1]}=k)\\ &\le\left(\delta\ell_n(\varepsilon n^{1/\alpha})^{-\alpha}\right)^2=\frac{\delta^2(\mathbf{E}[W_n])^2}{\varepsilon^{2\alpha}}\end{aligned} \tag{182}$$

Note that for n_0 big enough we have that $\mathbf{E}[W_n]$ is close to $\mathbf{E}[W]$ for any $n\ge n_0$. Hence, if we take $\delta<\frac{\eta\varepsilon^{2\alpha}}{(\mathbf{E}[W])^2}$ the condition will hold for any pair $\eta,\varepsilon>0$.

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Data Availability Statement

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

References

1. M. Girvan and M. E. J. Newman, "Community Structure in Social and Biological Networks," *Proceedings of the National Academy of Sciences* 99, no. 12 (2002): 7821–7826, https://doi.org/10.1073/pnas.122653799.

2. M. E. J. Newman, *Networks* (Oxford, UK: Oxford University Press, 2010).

3. K. B. Singer, "Random Intersection Graphs" (PhD thesis, Johns Hopkins University ProQuest, 1996).

4. M. Bloznelis, E. Godehardt, J. Jaworski, V. Kurauskas, and K. Rybarczyk, "Recent Progress in Complex Network Analysis: Properties of Random Intersection Graphs," in *Data Science, Learning by Latent Structures, and Knowledge Discovery* (Heidelberg: Springer, 2015), 79–88.

5. S. R. Blackburn and S. Gerke, "Connectivity of the Uniform Random Intersection Graph," *Discrete Mathematics* 309, no. 16 (2009): 5130–5140, https://doi.org/10.1016/j.disc.2009.03.042.

6. K. Rybarczyk, "Diameter, Connectivity, and Phase Transition of the Uniform Random Intersection Graph," *Discrete Mathematics* 311, no. 17 (2011): 1998–2019, https://doi.org/10.1016/j.disc.2011.05.029.

7. M. Bloznelis, "Component Evolution in General Random Intersection Graphs," *SIAM Journal on Discrete Mathematics* 24, no. 2 (2010): 639–654, https://doi.org/10.1137/080713756.

8. M. Bloznelis, "Degree and Clustering Coefficient in Sparse Random Intersection Graphs," *Annals of Applied Probability* 23, no. 3 (2013): 1254–1289, https://doi.org/10.1214/12-AAP874.

9. M. Bloznelis, "Degree-Degree Distribution in a Power Law Random Intersection Graph With Clustering," *Internet Mathematics* 1, no. 1 (2017): 25, https://doi.org/10.24166/im.03.2017.

10. M. Bloznelis, J. Jaworski, and V. Kurauskas, "Assortativity and Clustering of Sparse Random Intersection Graphs," *Electronic Journal of Probability* 18, no. 38 (2013): 24, https://doi.org/10.1214/EJP.v18-2277.

11. E. Godehardt and J. Jaworski, "Two Models of Random Intersection Graphs for Classification," in *Exploratory Data Analysis in Empirical Research* (Berlin: Springer, 2003), 67–81.

12. E. Coupechoux and M. Lelarge, "Contagions in Random Networks With Overlapping Communities," *Advances in Applied Probability* 47, no. 4 (2015): 973–988, https://doi.org/10.1239/aap/1449859796.

13. R. van der Hofstad, J. Komjáthy, and V. Vadon, "Phase Transition in Random Intersection Graphs With Communities," *Random Structures & Algorithms* 60, no. 3 (2022): 406–461, https://doi.org/10.1002/rsa.21063.

14. R. van der Hofstad, J. Komjáthy, and V. Vadon, "Random Intersection Graphs With Communities," *Advances in Applied Probability* 53, no. 4 (2021): 1061–1089, https://doi.org/10.1017/apr.2021.12.

15. M. E. J. Newman, "Properties of Highly Clustered Networks," *Physical Review E* 68, no. 2 (2003): 131–159.

16. F. Ball, D. Sirl, and P. Trapman, "Epidemics on Random Intersection Graphs," *Annals of Applied Probability* 24, no. 3 (2014): 1081–1128, https://doi.org/10.1214/13-AAP942.

17. J. A. Fill, E. R. Scheinerman, and K. B. Singer-Cohen, "Random Intersection Graphs When m = w(n): An Equivalence Theorem Relating the Evolution of the G(n, m, p) and G(n, p) Models," *Random Structures & Algorithms* 16 (2000): 156–176, https://doi.org/10.1002/(SICI)1098-2418(200003)16:2<156::AID-RSA3>3.0.CO;2-H.

18. M. Karonski, E. R. Scheinerman, and K. B. Singer-Cohen, "On Random Intersection Graphs: The Subgraph Problem," *Combinatorics, Probability and Computing* 8, no. 1-2 (1999): 131–159, https://doi.org/10.1017/S0963548398003459.

19. M. Bloznelis and J. Damarackas, "Degree Distribution of an Inhomogeneous Random Intersection Graph," *Electronic Journal of Combinatorics* 20, no. 3 (2013): P3, https://doi.org/10.37236/2786.

20. M. Deijfen and W. Kets, "Random Intersection Graphs With Tunable Degree Distribution and Clustering," *Probability in the Engineering and Informational Sciences* 23.4 (2009): 661–674, https://doi.org/10.1017/S0269964809990064.

21. A.-L. Barabási and R. Albert, "Emergence of Scaling in Random Networks," *Science* 286, no. 5439 (1999): 509–512, https://doi.org/10.1126/science.286.5439.509.

22. R. van der Hofstad, *Random Graphs and Complex Networks*, vol. 1 (Cambridge, UK: Cambridge University Press, 2016).

23. H. A. Simon, "On a Class of Skew Distribution Functions," *Biometrika* 42 (1955): 425–440, https://doi.org/10.1093/biomet/42.3-4.425.

24. G. U. Yule, "A Mathematical Theory of Evolution, Based on the Conclusions of Dr. J. C. Willis, F. R. S," *Philosophical Transactions of the Royal Society of London, Series B* 213 (1925): 21–87.

25. O. Häggström, Y. Peres, and J. Steif, "Dynamical Percolation," *Annales de l'Institut Henri Poincare (B) Probability and Statistics* 33, no. 4 (1997): 497–528, https://doi.org/10.1016/S0246-0203(97)80103-3.

26. J. Steif, "A Survey of Dynamical Percolation," in *Fractal Geometry and Stochastics IV*, vol. 61 (Basel, Switzerland: Birkhäuser Verlag, 2009), 145–174.

27. L. Avena, H. Güldaş, R. van der Hofstad, and F. den Hollander, "Random Walks on Dynamic Configuration Models: A Trichotomy," *Stochastic Processes and their Applications* 129.9 (2019): 3360–3375, https://doi.org/10.1016/j.spa.2018.09.010.

28. L. Avena, H. Güldaş, R. van der Hofstad, F. den Hollander, and O. Nagy, "Linking the Mixing Times of Random Walks on Static and Dynamic Random Graphs," *Stochastic Processes and their Applications* 153 (2022): 145–182, https://doi.org/10.1016/j.spa.2022.07.009.

29. L. Dort and E. Jacob, "Local Weak Limit of Dynamical Inhomogeneous Random Graphs," *arXiv preprint arXiv*:2303.17437 (2023).

30. J. L. Guillaume and M. Latapy, "Bipartite Graphs as Models of Complex Networks," *Physica A: Statistical Mechanics and its Applications* 371 (2006): 795–813.

31. P. Billingsley, *Convergence of Probability Measures* (New York: John Wiley & Sons, 1999).

32. M. Altmann, "Susceptible-Infected-Removed Epidemic Models With Dynamic Partnerships," *Journal of Mathematical Biology* 6, no. 33 (1995): 661–675, https://doi.org/10.1007/BF00298647.

33. P. Sousi and S. Thomas, "Cutoff for Random Walk on Dynamical Erdös-Rényi Graph," *Annales de l'Institute Henri Poincare, Probabilites et Statistiques* 56 (2020): 2745–2773.

34. T. Britton, M. Deijfen, and A. Martin-Löf, "Generating Simple Random Graphs With Prescribed Degree Distribution," *Journal of Statistical Physics* 124, no. 6 (2006): 1377–1397, https://doi.org/10.1007/s10955-006-9168-x.

35. B. Bollobás, S. Janson, and O. Riordan, "The Phase Transition in Inhomogeneous Random Graphs," *Random Structures & Algorithms* 31, no. 1 (2007): 3–122, https://doi.org/10.1002/rsa.20168.

36. F. Chung and L. Lu, *Complex Graphs and Networks* (Providence, RI: American Mathematical Society, 2006).

37. I. Norros and H. Reittu, "On a Conditionally Poissonian Graph Process," *Advances in Applied Probability* 38, no. 1 (2006): 59–75, https://doi.org/10.1239/aap/1143936140.

38. B. Bollobás, "A Probabilistic Proof of an Asymptotic Formula for the Number of Labelled Regular Graphs," *European Journal of Combinatorics* 1, no. 4 (1980): 311–316, https://doi.org/10.1016/S0195-6698(80)80030-8.

39. M. Molloy and B. Reed, "A Critical Point for Random Graphs With a Given Degree Sequence," *Random Structures & Algorithms* 6, no. 2-3 (1995): 161–179, https://doi.org/10.1002/rsa.3240060204.

40. M. Molloy and B. Reed, "The Size of the Giant Component of a Random Graph With a Given Degree Sequence," *Combinatorics, Probability and Computing* 7, no. 3 (1998): 295–305, https://doi.org/10.1017/S0963548398003526.

41. P. Trapman, "On Analytical Approaches to Epidemics on Networks," *Theoretical Population Biology* 71, no. 2 (2007): 160–173.

42. I. Benjamini and O. Schramm, "Recurrence of Distributional Limits of Finite Planar Graphs," *Electronic Journal of Probability* 6, no. 23 (2001): 1–13, https://doi.org/10.1214/EJP.v6-96.

43. D. Aldous and J. M. Steele, "The Objective Method: Probabilistic Combinatorial Optimization and Local Weak Convergence," in *Probability on Discrete Structures*, vol. 110 (Berlin: Springer, 2004), 1–72.

44. R. van der Hofstad, "Random Graphs and Complex Networks" (2024), http://www.win.tue.nl/~rhofstad/NotesRGCNII.pdf.

45. P. Erdös and A. Rényi, "On the Evolution of Random Graphs," *Mathematical Research Letters of the Hungarian Academy of Sciences* 5 (1960): 17–61.

46. F. Chung and L. Lu, "Connected Components in Random Graphs With Given Expected Degree Sequences," *Annals of Combinatorics* 6, no. 2 (2002): 125–145, https://doi.org/10.1007/PL00012580.

47. F. Chung and L. Lu, "The Volume of the Giant Component of a Random Graph With Given Expected Degrees," *SIAM Journal on Discrete Mathematics* 20, no. 2 (2006): 395–411, https://doi.org/10.1137/050630106.

48. B. Bollobás and O. Riordan, "An Old Approach to the Giant Component Problem," *Journal of Combinatorial Theory Series B* 113 (2015): 236–260, https://doi.org/10.1016/j.jctb.2015.03.002.

49. S. Janson and M. J. Luczak, "A New Approach to the Giant Component Problem," *Random Structures & Algorithms* 34, no. 2 (2009): 197–216, https://doi.org/10.1002/rsa.20231.

50. S. Janson, "Asymptotic Equivalence and Contiguity of Some Random Graphs," *Random Structures & Algorithms* 36 (2010): 26–45, https://doi.org/10.1002/rsa.20297.

51. O. Kallenberg, *Foundations of Modern Probability* (New York: Springer-Verlag, 2002).

52. O. Angel, R. van der Hofstad, and C. Holmgren, "Limit Laws for Self-Loops and Multiple Edges in the Configuration Model," *Annales de l'Institute Henri Poincare, Probabilites et Statistiques* 55.3 (2016): 1509–1530, https://doi.org/10.1214/18-AIHP926.

Appendix A

Proof of the Link Between $\mathrm{BGRG}_n(\boldsymbol{w})$ and $\mathrm{BCM}_n(\boldsymbol{d})$ From Van Der Hofstad, Komjáthy, and Vadon

Here we prove the theorem that lies at the heart of our results—the fact that, under certain conditions, $\mathrm{BGRG}_n(\boldsymbol{w})$ and $\mathrm{BCM}_n(\boldsymbol{d})$ are equivalent. The proof follows in four steps: first, we show that the $\mathrm{BGRG}_n(\boldsymbol{w})$ conditioned on its degree sequence is uniform. Second, we show that $\mathrm{BCM}_n(\boldsymbol{d})$ conditioned on simplicity is uniform. We also state regularity conditions that allow us to draw an even stronger link between the two models. Finally, we conclude that under such circumstances $\mathrm{BGRG}_n(\boldsymbol{w})$ and $\mathrm{BCM}_n(\boldsymbol{d})$ are equivalent.

A.1 | $\mathrm{BGRG}_n(\boldsymbol{w})$ Conditioned on Degree Sequence is Uniform

We adapt the derivation of a similar result for $\mathrm{GRG}_n(\boldsymbol{w})$ (see [22, Section 6.6]). Note that $\mathrm{BGRG}_n(\boldsymbol{w})$ is entirely determined by the group activity, that is, if we know which group is active we automatically know which vertices are in it. Hence, we can encode the probability of $\mathrm{BGRG}_n(\boldsymbol{w})$ taking a particular form via a sequence of indicator random variables: recall from Section 2.1 that we let $x = (x_a)_{a\in\cup_{k\geq 2}[n]_k}$ be a sequence of 0s and 1s, and $X = (X_a)_{a\in\cup_{k\geq 2}[n]_k}$ be a sequence of independent random variables describing the existence of particular groups, that is,

$$\mathbf{P}(X_a = 1) = 1 - \mathbf{P}(X_a = 0) = \pi^a_{\mathrm{ON}} \tag{A1}$$

Further, recall that we denote

$$d^{(l)}_i(X) = \sum_{a\in\cup_{k\geq 2}[n]_k : a\ni i} X_a, \text{ and } d^{(r)}_a(X) = |a|\cdot X_a \tag{A2}$$

Analogously, $d^{(l)}_i(x) = \sum_{a\in\cup_{k\geq 2}[n]_k : a\ni i} x_a$, $d^{(r)}_a(x) = |a|\cdot x_a$. Then, we have the following identification of the law of $\mathrm{BGRG}_n(\boldsymbol{w})$:

Proposition A.1. ($\mathrm{BGRG}_n(\boldsymbol{w})$ as a function of left- and right-degrees.). *The probability that the sequence $X = (X_a)_{a\in\cup_{k\geq 2}[n]_k}$ takes a form $x = (x_a)_{a\in\cup_{k\geq 2}[n]_k}$ can be expressed as a function of left- and right-degree sequences $(\boldsymbol{d}^{(l)}, \boldsymbol{d}^{(r)}) = ((d^{(l)}_i)_{i\in[n]}, (d^{(r)}_a)_{a\in\cup_{k\geq 2}[n]_k})$:*

$$\mathbf{P}(X = x) = H((d^{(l)}(x), d^{(r)}(x)))\cdot\left(\prod_{a\in\cup_{k\geq 2}[n]_k}(1+\lambda^a_{\mathrm{OFF}})\right)^{-1} \tag{A3}$$

where H satisfies

$$H((d^{(l)}(x), d^{(r)}(x))) = \prod_{i\in[n]} w_i^{d^{(l)}_i(x)} \prod_{a\in\cup_{k\geq 2}[n]_k} \frac{f(d^{(r)}_a(x))}{\ell_n^{d^{(r)}_a(x)(1-\frac{1}{|a|})}} \tag{A4}$$

Proof. Taking $X = (X_a)_{a\in\cup_{k\geq 2}[n]_k}$ and $x = (x_a)_{a\in\cup_{k\geq 2}[n]_k}$ as above, we obtain

$$\begin{aligned}\mathbf{P}(X = x) &= \prod_{a\in\cup_{k\geq 2}[n]_k}(\pi^a_{\mathrm{ON}})^{x_a}(1-\pi^a_{\mathrm{ON}})^{1-x_a}\\ &= \prod_{a\in\cup_{k\geq 2}[n]_k}\left(\prod_{i\in a}\frac{f(|a|)^{\frac{1}{|a|}}w_i}{\ell_n^{\frac{|a|-1}{|a|}}}\right)^{x_a}\\ &\times\prod_{a\in\cup_{k\geq 2}[n]_k}\frac{1}{1+\frac{f(|a|)\prod_{i\in a}w_i}{\ell_n^{|a|-1}}}\end{aligned} \tag{A5}$$

Note that $\frac{\prod_{i\in a}f(|a|)w_i}{\ell_n^{|a|-1}} = \lambda^a_{\mathrm{OFF}}$ and hence we can abbreviate

$$\begin{aligned}\mathbf{P}(X = x) &= \left(\prod_{a\in\cup_{k\geq 2}[n]_k}(1+\lambda^a_{\mathrm{OFF}})\right)^{-1}\\ &\times\prod_{a\in\cup_{k\geq 2}[n]_k}\left(\prod_{i\in a}w_i\right)^{x_a}\left(\frac{f(|a|)}{\ell_n^{|a|-1}}\right)^{x_a}\end{aligned} \tag{A6}$$

We observe that

$$\begin{aligned}\prod_{a\in\cup_{k\geq 2}[n]_k}\left(\prod_{i\in a}w_i\right)^{x_a} &= \prod_{a\in\cup_{k\geq 2}[n]_k}\prod_{i\in a}w_i^{x_a} = \prod_{i\in[n]}\prod_{a:a\ni i}w_i^{x_a}\\ &= \prod_{i\in[n]}w_i^{\sum_{a\in\cup_{k\geq 2}[n]_k : a\ni i}x_a} = \prod_{i\in[n]}w_i^{d^{(l)}_i(x)}\end{aligned} \tag{A7}$$

is a function of the left-degrees. Similarly, extending the definition of the function of group-size distribution by fixing $f(0) = 1$,

$$\begin{aligned}\prod_{a\in\cup_{k\geq 2}[n]_k}\left(\frac{f(|a|)}{\ell_n^{|a|-1}}\right)^{x_a} &= \prod_{a\in\cup_{k\geq 2}[n]_k}\frac{(f(|a|))^{x_a}}{\ell_n^{(|a|-1)x_a}}\\ &= \prod_{a\in\cup_{k\geq 2}[n]_k}\frac{f(d^{(r)}_a(x))}{\ell_n^{d^{(r)}_a(x)(1-\frac{1}{|a|})}}\end{aligned} \tag{A8}$$

is a function of the right-degrees, as $d^{(r)}_a(x) = |a|\cdot x_a$. After substituting (A7) and (A8) into (A6) the claim follows. □

Recall the way of prescribing the double degree sequence described in Section 2. Given Proposition A.1 it is not difficult to show that the static bipartite graph conditioned on its degree sequence in such a way is uniform over all simple bipartite graphs with given degree sequence, as stated in Theorem 2.1:

Proof of Theorem 2.1. With $x = (x_a)_{a \in \cup_{k\geq 2}[n]_k}$ satisfying $d_i^{(l)}(x) = d_i^{(l)}$ for all $i \in [n]$ and $d_j^{(r)}(x) = k$ for $j \in [s_{k-1}, s_k)$ and 0 otherwise, we can write

$$\begin{aligned}
&\mathbf{P}\Big(X = x | d_i^{(l)}(X) = d_i^{(l)} \forall i \in [n], d_j^{(r)}(X) = k \forall j \in [s_{k-1}, s_k)\Big) \\
&= \frac{\mathbf{P}(X = x)}{\mathbf{P}(d_i^{(l)}(X) = d_i^{(l)} \forall i \in [n], d_j^{(r)}(X) = k \forall j \in [s_{k-1}, s_k))} \\
&= \frac{\mathbf{P}(X = x)}{\sum_{\{y: d_i^{(l)}(y) = d_i^{(l)} \forall i \in [n], d_j^{(r)}(y) = k \forall j \in [s_{k-1}, s_k)\}} \mathbf{P}(X = y)} \\
&= \frac{G(a)^{-1} H((d^{(l)}(x), d^{(r)}(x)))}{\sum_{\{y: d_i^{(l)}(y) = d_i^{(l)} \forall i \in [n], d_j^{(r)}(y) = k \forall j \in [s_{k-1}, s_k)\}} G(a)^{-1} H((d^{(l)}(y), d^{(r)}(y)))} \\
&= \frac{H((d^{(l)}, d^{(r)}))}{\sum_{\{y: d_i^{(l)}(y) = d_i^{(l)} \forall i \in [n], d_j^{(r)}(y) = k \forall j \in [s_{k-1}, s_k)\}} H((d^{(l)}, d^{(r)}))} \\
&= \frac{1}{\#\{y : d_i^{(l)}(y) = d_i^{(l)} \forall i \in [n], d_j^{(r)}(y) = k \forall j \in [s_{k-1}, s_k)\}}
\end{aligned} \tag{A9}$$

which means that the distribution is uniform over all bipartite graphs with the prescribed left- and right-degree sequences.

A.2 | Bipartite Graph With Communities Conditioned on Simplicity is Uniform

Note that $\mathrm{BCM}_n(\boldsymbol{d})$ from [14] does not have a community structure. Hence, we prove the equivalence between $\mathrm{BGRG}_n(\boldsymbol{w})$ and $\mathrm{BCM}_n(\boldsymbol{d})$ with complete communities. Before proving the main result, that is, the fact that the bipartite configuration model is uniform given simplicity, we need an auxiliary proposition (which is analogous to a similar result for $\mathrm{CM}_n(\boldsymbol{d})$—see [22, Proposition 7.7]):

Proposition A.2. (The law of $\mathrm{BCM}_n(\boldsymbol{d})$). *Denote by $G = (x_{ij})_{i \in [n], j \in [M]}$ a bipartite multigraph on left-vertices $i \in [n]$ and right-vertices $j \in [M]$, such that $d_i^{(l)} = \sum_{j \in [M]} x_{ij}$ and $d_j^{(r)} = \sum_{i \in [n]} x_{ij}$, where x_{ij} is the number of edges between $i \in [n]$ and $j \in [M]$. Then,*

$$\mathbf{P}(\mathrm{BCM}_n(\boldsymbol{d}) = G) = \frac{1}{h_n!} \frac{\prod_{i \in [n]} d_i^{(l)}! \prod_{j \in [M]} d_j^{(r)}!}{\prod_{i \in [n], j \in [M]} x_{ij}!} \tag{A10}$$

with $h_n = \sum_{i \in [n]} d_i^{(l)} = \sum_{j \in [M]} d_j^{(r)}$. By $\boldsymbol{d}$ in $\mathrm{BCM}_n(\boldsymbol{d})$ we mean a double degree sequence $(\boldsymbol{d}^{(l)}, \boldsymbol{d}^{(r)}) = ((d_i^{(l)})_{i \in [n]}, (d_j^{(r)})_{j \in [M]})$.

Remark A.1. Note that [13, (2.38)] yields the same formula as (A10). However, the authors of [13] deliver this result in a form of a remark, giving justification rather than formal proof. We provide a formal proof.

Proof. We start by computing the number of all possible matchings between the left and right sides. Imagine we want to assign a right-half-edge to every left-half-edge uniformly at random. For the first fixed left-half-edge we have h_n choices of available right-half-edges. For the second left-half-edge, $h_n - 1$ choices, and so on. It is not hard to see that the number of all such matchings is $h_n!$. Hence,

$$\mathbf{P}(\mathrm{BCM}_n(\boldsymbol{d}) = G) = \frac{1}{h_n!} \# N(G) \tag{A11}$$

where $N(G)$ is the number of configurations that, after identifying the vertices, result in the multigraph G. Note that permuting half-edges incident to vertices will give rise to the same pairs of left- and right-vertices, hence the same multigraph G, but yet, when it comes to half-edges, it is a different configuration. The number of such permutations is $\prod_{i \in [n]} d_i^{(l)}! \cdot \prod_{j \in [M]} d_j^{(r)}!$. However, some of these permutations yield the same half-edge pairings. If two half-edges of the left-vertex $i \in [n]$ are paired to two half-edges of the right-vertex $j \in [M]$ and we permute all of them, we will have the same half-edges being matched again. Thus, we divide by $\prod_{i \in [n], j \in [M]} x_{ij}!$ to compensate for the "double-counting" caused by multiple connections. □

Using this result we can prove the main theorem of the section (which is an adaptation of a similar result for $\mathrm{CM}_n(\boldsymbol{d})$—see [22, Proposition 7.15]):

Proof of Theorem 2.2. Since, by (A10), $\mathbf{P}(\mathrm{BCM}_n(\boldsymbol{d}) = G)$ is the same for every bipartite simple graph G, also conditional probability $\mathbf{P}(\mathrm{BCM}_n(\boldsymbol{d}) = G | \mathrm{BCM}_n(\boldsymbol{d})$is simple) is the same for every bipartite simple graph G. Hence, for any bipartite degree sequence $(\boldsymbol{d}^{(l)}, \boldsymbol{d}^{(r)}) = ((d_i^{(l)})_{i \in [n]}, (d_j^{(r)})_{j \in [M]})$ and conditionally on the event $\{\mathrm{BCM}_n(\boldsymbol{d})$ is a bipartite simple graph$\}$, $\mathrm{BCM}_n(\boldsymbol{d})$ is a uniform bipartite simple graph with bipartite degree sequence $(\boldsymbol{d}^{(l)}, \boldsymbol{d}^{(r)}) = ((d_i^{(l)})_{i \in [n]}, (d_j^{(r)})_{j \in [M]})$.

A.3 | Relation Between $\mathrm{BGRG}_n(\boldsymbol{w})$ and $\mathrm{BCM}_n(\boldsymbol{d})$

Thanks to Theorems 2.1 and 2.2 we can show that the static bipartite GRG and the bipartite configuration model under certain conditions yield a certain graph G with the same probability. However, as we mentioned in the proof overview section, if we assume a few regularity conditions on the degree sequences, we can deduce a stronger link determining when certain events happen with high probability for $\mathrm{BCM}_n(\boldsymbol{d})$ and $\mathrm{BGRG}_n(\boldsymbol{w})$. These necessary regularity conditions are stated in Section 2, see Condition 2.1. If we assume Condition 2.1 for $\mathrm{BCM}_n(\boldsymbol{d})$, then the following is a natural consequence of Theorem 2.2 (analogously to a similar result for $\mathrm{CM}_n(\boldsymbol{d})$—see [22, Corollary 7.17]):

Corollary A.1. (Uniform graphs with given degree sequence and $\mathrm{BCM}_n(\boldsymbol{d})$). *Assume that $(\boldsymbol{d}^{(l)}, \boldsymbol{d}^{(r)}) = ((d_i^{(l)})_{i \in [n]}, (d_j^{(r)})_{j \in [M]})$ satisfies Condition 2.1. Then, an event $\mathcal{E}_n$ occurs with high probability for a uniform simple bipartite random graph with degrees $\boldsymbol{d} = (\boldsymbol{d}^{(l)}, \boldsymbol{d}^{(r)}) = ((d_i^{(l)})_{i \in [n]}, (d_j^{(r)})_{j \in [M]})$ when it occurs with high probability for $\mathrm{BCM}_n(\boldsymbol{d})$.*

Proof. Let $\mathrm{UG}_n(\boldsymbol{d})$ denote a uniform simple bipartite random graph with degrees $\boldsymbol{d} = (\boldsymbol{d}^{(l)}, \boldsymbol{d}^{(r)})$. Let $\mathcal{E}_n$ be a subset of multi-graphs such that $\lim_{n\to\infty} \mathbf{P}(\mathrm{BCM}_n(\boldsymbol{d}) \in \mathcal{E}_n^c) = 0$, where, as previously (see Section 2.5), the superscript "c" denotes a complement. We need to prove that then also $\lim_{n\to\infty} \mathbf{P}(\mathrm{UG}_n(\boldsymbol{d}) \in \mathcal{E}_n^c) = 0$. By Theorem 2.2,

$$\begin{aligned}
\mathbf{P}((\mathrm{UG}_n(\boldsymbol{d}) \in \mathcal{E}_n^c) &= \mathbf{P}(\mathrm{BCM}_n(\boldsymbol{d}) \in \mathcal{E}_n^c | \mathrm{BCM}_n(\boldsymbol{d}) \text{ is simple}) \\
&= \frac{\mathbf{P}(\mathrm{BCM}_n(\boldsymbol{d}) \in \mathcal{E}_n^c, \mathrm{BCM}_n(\boldsymbol{d}) \text{ is simple})}{\mathbf{P}(\mathrm{BCM}_n(\boldsymbol{d}) \text{ is simple})} \\
&\leq \frac{\mathbf{P}(\mathrm{BCM}_n(\boldsymbol{d}) \in \mathcal{E}_n^c)}{\mathbf{P}(\mathrm{BCM}_n(\boldsymbol{d}) \text{ is simple})}
\end{aligned} \tag{A12}$$

By assumption we have that $\lim_{n\to\infty} \mathbf{P}(\mathrm{BCM}_n(\boldsymbol{d}) \in \mathcal{E}_n^c) = 0$. Moreover, by [52, Theorem 1.10 (1.45)], for which the conditions are satisfied by Condition 2.1, it follows

$$\liminf_{n\to\infty} \mathbf{P}(\mathrm{BCM}_n(\boldsymbol{d}) \text{ is simple}) > 0 \tag{A13}$$

so that $\mathbf{P}((\mathrm{UG}_n(\boldsymbol{d}) \in \mathcal{E}_n^c) \to 0$. □

Now we can prove the final result on the link between the two models (which is analogous to a similar result for $\mathrm{GRG}_n(\boldsymbol{w})$ and $\mathrm{CM}_n(\boldsymbol{d})$—see [22, Theorem 7.18]):

Proof of Theorem 2.3. Equality in (44) follows from Theorems 2.1 and 2.2 for every simple bipartite graph G with degree sequence $\boldsymbol{d}$. Indeed, these results imply that $\mathrm{BGRG}_n(\boldsymbol{w})$ conditionally on $\boldsymbol{D} = \boldsymbol{d}$ as well as $\mathrm{BCM}_n(\boldsymbol{d})$ conditionally on being simple are uniform simple random graphs with degree sequence $\boldsymbol{d}$. Further, by (44) we have that

$$\begin{aligned}
&\mathbf{P}(\mathrm{BGRG}_n(\boldsymbol{w}) \in \mathcal{E}_n | \boldsymbol{D} = \boldsymbol{d}) \\
&\quad = \mathbf{P}(\mathrm{BCM}_n(\boldsymbol{d}) \in \mathcal{E}_n | \mathrm{BCM}_n(\boldsymbol{d}) \text{ simple})
\end{aligned}$$

We rewrite

$$\begin{aligned}
\mathbf{P}(\mathrm{BGRG}_n(\boldsymbol{w}) \in \mathcal{E}_n^c) &= \mathbf{E}[\mathbf{P}(\mathrm{BGRG}_n(\boldsymbol{w}) \in \mathcal{E}_n^c | \boldsymbol{D})] \\
&= \mathbf{E}[\mathbf{P}(\mathrm{BCM}_n(\boldsymbol{D}) \in \mathcal{E}_n^c | \mathrm{BCM}_n(\boldsymbol{D}) \text{ simple})] \\
&\leq \mathbf{E}\left[\frac{\mathbf{P}_n(\mathrm{BCM}_n(\boldsymbol{D}) \in \mathcal{E}_n^c)}{\mathbf{P}_n(\mathrm{BCM}_n(\boldsymbol{D}) \text{ simple})} \wedge 1\right]
\end{aligned} \tag{A14}$$

where $\mathbf{E}$ is the expectation w.r.t. the degree sequence $\boldsymbol{D}$ and $\mathbf{P}_n$ stands for the conditional law given the degrees $\boldsymbol{D}$. We assumed that $\mathbf{P}(\mathrm{BCM}_n(\boldsymbol{D}) \in \mathcal{E}_n^c) \xrightarrow{\mathbf{P}} 0$. Since $\boldsymbol{D}$ satisfies Condition 2.1, by [52, Theorem 1.10 (1.45)] it follows

$$\liminf_{n\to\infty} \mathbf{P}(\mathrm{BCM}_n(\boldsymbol{D}) \text{ is simple}) > 0 \tag{A15}$$

Hence, by the dominated convergence theorem, we conclude that $\mathbf{P}(\mathrm{BGRG}_n(\boldsymbol{w}) \in \mathcal{E}_n^c) \to 0$.

Appendix B

Regularity Conditions of the Static Bipartite Graph and Consequences

Theorem 2.3 shows that if the degree sequences of $\mathrm{BCM}_n(\boldsymbol{d})$ and $\mathrm{BGRG}_n(\boldsymbol{w})$ satisfy Condition 2.1a,b, then if some event $\mathcal{E}_n$ happens with high probability for $\mathrm{BCM}_n(\boldsymbol{d})$, it also happens with high probability for $\mathrm{BGRG}_n(\boldsymbol{w})$. The results derived for $\mathrm{BCM}_n(\boldsymbol{d})$ in [13, 14] hold precisely under Condition 2.1a,b. Hence, if $\mathrm{BGRG}_n(\boldsymbol{w})$ also satisfies these regularity conditions, we can transfer all the results on the local convergence and the giant component from [13, 14].

In this section, we show that Condition 1.1 we have assumed implies that the degree sequence of $\mathrm{BGRG}_n(\boldsymbol{w})$ satisfies the required conditions. We will also argue why it is possible to drop Condition 2.1ii. Before proceeding to the proofs, we state one consequence of Condition 1.1 that we frequently make use of in the following sections:

Corollary B.1. *Condition 1.1a,b imply that* $\max_{i\in[n]} w_i = o(n)$ *and Condition 1.1a–c imply that* $\max_{i\in[n]} w_i = o(\sqrt{n})$.

Since the above corollary is the same as for the $\mathrm{GRG}_n(\boldsymbol{w})$, we omit its proof.

B.1 | Convergence of Left-Degrees in $\mathrm{BGRG}_n(\boldsymbol{w})$

Throughout this section we assume Condition 1.1a–c; however, we later argue that (c) can be lifted. To show the convergence of the left-degrees in $\mathrm{BGRG}_n(\boldsymbol{w})$, we first need the following auxiliary result:

Theorem B.1. (Poisson approximation of the number of k-cliques containing a vertex). *Let $C_k(i)$ denote the number of groups of size k containing vertex $i \in [n]$. There exists a coupling $(\hat{C}_k(i), \hat{Z}_{i,k})$ of $\hat{C}_k(i)$—a random variable with the same distribution as $C_k(i)$—and a Poisson random variable $Z_{i,k}$ with parameter $kp_k w_i$, such that*

$$\mathbf{P}(\hat{C}_k(i) \neq \hat{Z}_{i,k}) \leq \frac{2(k(k-1)p_k)^2 w_i^2}{\ell_n}\left(\frac{k-1}{\ell_n}\right)^{k-2}\left(\frac{\mathbf{E}[W_n^2]}{\mathbf{E}[W_n]}\right)^{k-1} \tag{B1}$$

Proof. We adapt the proof of a similar result for $\mathrm{GRG}_n(\boldsymbol{w})$ (see [22, Theorem 6.7]). Note that

$$C_k(i) = \sum_{a\in[n]_k : a\ni i} \mathbb{1}_{\{a \text{ is ON}\}} \tag{B2}$$

Hence, $C_k(i)$ is a sum of independent Bernoulli random variables and by [14, Theorem 2.10] we know that there exists a Poisson random variable $\hat{Y}_{i,k}$ with parameter

$$\lambda_{i,k} = \sum_{a\in[n]_k : a\ni i} \pi_{\mathrm{ON}}^a \tag{B3}$$

and a random variable $\hat{C}_k(i)$ with the same distribution as $C_k(i)$, such that

$$\mathbf{P}(\hat{C}_k(i) \neq \hat{Y}_{i,k}) \leq \sum_{a\in[n]_k : a\ni i} (\pi_{\mathrm{ON}}^a)^2 \tag{B4}$$

We have

$$\begin{aligned}\sum_{a\in[n]_k : a\ni i} (\pi_{\mathrm{ON}}^a)^2 &= \sum_{a\in[n]_k : a\ni i}\left(\frac{k!p_k w_i \prod_{j\in a, j\neq i} w_j}{\ell_n^{k-1} + k!p_k w_i \prod_{j\in a, j\neq i} w_j}\right)^2\\ &\leq (k!p_k)^2 w_i^2 \sum_{a\in[n]_k : a\ni i}\left(\frac{\prod_{j\in a, j\neq i} w_j}{\ell_n^{k-1}}\right)^2\\ &\leq (k!p_k)^2 w_i^2 \frac{(\sum_{j\in[n]} w_j^2)^{k-1}}{(k-1)!(\ell_n^2)^{k-1}}\\ &= \frac{(k-1)!(kp_k)^2 w_i^2}{\ell_n^{k-1}}\frac{(\mathbf{E}[W_n^2])^{k-1}}{(\mathbf{E}[W_n])^{k-1}}\end{aligned} \tag{B5}$$

where we have used (82) similarly as in Section 3.2. Let $\varepsilon_{i,k} = kp_k w_i - \lambda_{i,k} \geq 0$. Take $\hat{V}_{i,k} \sim \mathrm{Poi}(\varepsilon_{i,k})$ and write $\hat{Z}_{i,k} = \hat{Y}_{i,k} + \hat{V}_{i,k}$. By the Markov inequality,

$$\mathbf{P}(\hat{Y}_{i,k} \neq \hat{Z}_{i,k}) = \mathbf{P}(\hat{V}_{i,k} \neq 0) = \mathbf{P}(\hat{V}_{i,k} \geq 1) \leq \mathbf{E}[\hat{V}_{i,k}] = \varepsilon_{i,k} \tag{B6}$$

We note that

$$\begin{aligned}\varepsilon_{i,k} = kp_k w_i - \lambda_{i,k} &= \frac{k!p_k w_i}{(k-1)!}\cdot 1 - \sum_{a\in[n]_{k\geq 2} : a\ni i} \pi_{\mathrm{ON}}^a\\ &= \frac{k!p_k w_i}{(k-1)!}\sum_{j_1,\dots,j_{k-1}\in[n]} \frac{w_{j_1}\cdots w_{j_{k-1}}}{\ell_n^{k-1}} + o(1)\\ &\quad - \sum_{j_1<\cdots<j_{k-1}\in[n]} \frac{k!p_k w_i w_{j_1}\cdots w_{j_{k-1}}}{\ell_n^{k-1} + k!p_k w_i w_{j_1}\cdots w_{j_{k-1}}}\\ &= \sum_{j_1<\cdots<j_{k-1}\in[n]} k!p_k w_i w_{j_1}\cdots w_{j_{k-1}}\\ &\quad \times\left(\frac{1}{\ell_n^{k-1}} - \frac{1}{\ell_n^{k-1} + k!p_k w_i w_{j_1}\cdots w_{j_{k-1}}}\right) + o(1)\\ &= \sum_{j_1<\cdots<j_{k-1}\in[n]} \frac{(k!p_k)^2 w_i^2 w_{j_1}^2\cdots w_{j_{k-1}}^2}{\ell_n^{k-1}(\ell_n^{k-1} + k!p_k w_i w_{j_1}\cdots w_{j_{k-1}})} + o(1)\\ &\leq \sum_{j_1<\cdots<j_{k-1}\in[n]} \frac{(k!p_k)^2 w_i^2 w_{j_1}^2\cdots w_{j_{k-1}}^2}{\ell_n^{2(k-1)}} + o(1)\\ &\leq \frac{(k-1)!(kp_k)^2 w_i^2}{\ell_n^{k-1}}\frac{(\sum_{j\in[n]} w_j^2)^{k-1}}{\ell_n^{k-1}} + o(1)\end{aligned} \tag{B7}$$

where we have again used (82). Using the fact that $k! \leq k^k$ this yields

$$\begin{aligned}\mathbf{P}(\hat{C}_k(i) \neq \hat{Z}_{i,k}) &\leq \mathbf{P}(\hat{C}_k(i) \neq \hat{Y}_{i,k}) + \mathbf{P}(\hat{Y}_{i,k} \neq \hat{Z}_{i,k})\\ &\leq \frac{2(k-1)!(kp_k)^2 w_i^2}{\ell_n^{k-1}}\frac{(\sum_{j\in[n]} w_j^2)^{k-1}}{\ell_n^{k-1}} + o(1)\\ &\leq 2(k(k-1)p_k)^2\left(\frac{k-1}{\ell_n}\right)^{k-2}\frac{w_i^2}{\ell_n}\left(\frac{\mathbf{E}[W_n^2]}{\mathbf{E}[W_n]}\right)^{k-1} + o(1)\end{aligned} \tag{B8}$$

which proves the desired result. □

Thanks to the coupling derived above we conclude that the degree of a uniformly chosen left-vertex converges:

Theorem B.2. (Left-degree in bipartite graph $\mathrm{BGRG}_n(\boldsymbol{w})$). *Let $D_n^{(l)}$ denote the degree of a uniformly chosen left-vertex in $\mathrm{BGRG}_n(\boldsymbol{w})$. Then, as $n \to \infty$,*

$$D_n^{(l)} \xrightarrow{\mathrm{d}} D^{(l)} \tag{B9}$$

where $D^{(l)}$ is a mixed-Poisson variable with mixing parameter $W\mu$, with $\mu = \sum_{k\geq 2} kp_k$.

Proof. We adapt the proof of a similar result for $\mathrm{GRG}_n(\boldsymbol{w})$ (see [22, Corollary 6.9]). We have that $d_i^{(l)} = \sum_{k=2}^{\infty} C_k(i)$ for any vertex $i \in [n]$. Fix b and consider a truncated degree $d_i^{(l),b} = \sum_{k=2}^{b} C_k(i)$. We know from Theorem B.1 and the fact that $\max_{i\in[n]} w_i = o(\sqrt{n})$ (Corollary B.1), that for all $k \in [2, b]$, $C_k(i)$ is close in distribution to a Poisson variable with parameter $kp_k w_i$. Hence, for a uniformly chosen vertex o_n from $[n]$, the number of groups of size k containing this vertex, for all $k \in [2, b]$, is close to a Poisson variable with parameter $kp_k w_{o_n}$, where w_{o_n} is the weight of a uniformly chosen vertex. Such a variable follows a mixed-Poisson distribution with mixing distribution w_{o_n}, and w_{o_n} is distributed like W_n. We know that a mixed-Poisson random variable converges to a limiting mixed-Poisson random variable when the mixing distribution converges. Since we have assumed convergence of W_n to a limiting variable W in Condition 1.1, it follows that for all $k \in [2, b]$, $C_k(o_n)$ converges to a Poisson random variable with parameter $kp_k W$. Thus, the truncated $D_n^{(l),b}$ has the same distribution as the sum over $k \in [2, b]$ of independent Poisson variables with parameters $kp_k W$ and for b large enough converges to a Poisson variable with parameter $W\mu$ as $n \to \infty$. We will now show that the probability that any $C_k(o_n)$ with $k > b$ is positive vanishes for b large enough. By the Markov inequality,

$$\limsup_n \mathbf{P}\left(\sum_{k\geq b} C_k(o_n) \geq 1\right) \leq \limsup_n \mathbf{E}\left[\sum_{k\geq b} C_k(o_n)\right] \\ \leq \limsup_n \mathbf{E}[W_n] \sum_{k\geq b} kp_k = o(1) \tag{B10}$$

for $b \to \infty$, since we have assumed $\sum_{k=2}^{\infty} kp_k < \infty$ and $W_n \to W$ as $n \to \infty$. Since therefore, with high probability, the only contribution to $D_n^{(l)}$ comes from the truncated $D_n^{(l)b}$, we conclude that $D_n^{(l)}$ also converges to a Poisson variable with parameter $W\mu$. □

Below we use the second-moment method to show that also the expected degree of a uniformly chosen left-vertex converges:

Theorem B.3. *Let $D_n^{(l)}$ denote the degree of a uniformly chosen left-vertex in $\mathrm{BGRG}_n(\boldsymbol{w})$. Then, as $n \to \infty$,*

$$\mathbf{E}[D_n^{(l)} | G_n] \xrightarrow{\mathbf{P}} \mu \mathbf{E}[W] \tag{B11}$$

Proof. Note that $\mathbf{E}[D_n^{(l)} | G_n] = \frac{1}{n}\sum_{i\in[n]} d_i^{(l)}$ is the same as $\frac{1}{n}\sum_{a\in\cup_{k\geq 2}[n]_k} d_a^{(r)} = \frac{1}{n}\sum_{a\in\cup_{k\geq 2}[n]_k} |a| \cdot \mathbb{1}_{\{a \text{ is ON}\}}$, which is a sum of independent variables. To avoid more assumptions on the moments of the group-size distribution, we fix a sequence $b_n \to \infty$ as $n \to \infty$ and use a truncation with respect to the group size, that is,

$$\sum_{k=2}^{\infty} \sum_{a\in[n]_k} |a| \mathbb{1}_{\{a \text{ is ON}\}} = \sum_{k=2}^{b_n} \sum_{a\in[n]_k} |a| \mathbb{1}_{\{a \text{ is ON}\}} \\ + \sum_{k=b_n+1}^{\infty} \sum_{a\in[n]_k} |a| \mathbb{1}_{\{a \text{ is ON}\}} \tag{B12}$$

For further notational convenience, denote $w_a = \prod_{i\in a} w_i$. We compute

$$\mathbf{E}\left[\frac{1}{n}\sum_{k=b_n+1}^{\infty} \sum_{a\in[n]_k} |a| \mathbb{1}_{\{a \text{ is ON}\}}\right] = \frac{1}{n}\sum_{k>b_n} k \sum_{a\in[n]_k} \frac{k! p_k w_a}{\ell_n^{k-1} + k! p_k w_a} \\ \leq \frac{\ell_n}{n} \sum_{k>b_n} kp_k = o(1) \tag{B13}$$

for every $b_n \to \infty$, since we have assumed $\sum_k kp_k < \infty$. Hence, the contribution from groups larger than b_n will vanish in probability, that is,

$$\sum_{k=2}^{\infty} \sum_{a\in[n]_k} |a| \mathbb{1}_{\{a \text{ is ON}\}} = \sum_{k=2}^{b_n} \sum_{a\in[n]_k} |a| \mathbb{1}_{\{a \text{ is ON}\}} + o(1) \tag{B14}$$

Note that

$$\frac{k! p_k w_a}{\ell_n^{k-1} + k! p_k w_a} = \frac{k! p_k w_a}{\ell_n^{k-1}} - \frac{(k! p_k)^2 (w_a)^2}{\ell_n^{k-1}(\ell_n^{k-1} + k! p_k w_a)} \tag{B15}$$

Hence,

$$\mathbf{E}\left[\frac{1}{n}\sum_{i\in[n]} d_i^{(l)}\right] \leq \frac{1}{n}\sum_{k=2}^{b_n} k \sum_{a\in[n]_k} \frac{k! p_k w_a}{\ell_n^{k-1}} \leq \frac{\ell_n}{n}\sum_{k=2}^{\infty} kp_k \\ = \mathbf{E}[W_n]\mu \xrightarrow{n\to\infty} \mathbf{E}[W]\mu \tag{B16}$$

Further, using the fact that $k! \leq k^k$ and (82) from Section 3.2,

$$\sum_{a\in[n]_k} \frac{(k! p_k)^2 (w_a)^2}{\ell_n^{k-1}(\ell_n^{k-1} + k! p_k w_a)} \leq \sum_{a\in[n]_k} \frac{(k! p_k)^2 (w_a)^2}{(\ell_n^{k-1})^2} \\ \leq k!(p_k)^2 \frac{(\sum_{i\in[n]} w_i^2)^k}{\ell_n^{k-2}\ell_n^k} = (kp_k)^2 \left(\frac{k}{\ell_n}\right)^{k-2} \left(\frac{\mathbf{E}[W_n^2]}{\mathbf{E}[W_n]}\right)^k \tag{B17}$$

and thus

$$\mathbf{E}\left[\frac{1}{n}\sum_{i\in[n]} d_i^{(l)}\right] \geq \frac{\ell_n}{n}\sum_{k=2}^{b_n} kp_k \\ - \frac{1}{n}\sum_{k=2}^{b_n} k(kp_k)^2 \left(\frac{k}{\ell_n}\right)^{k-2} \left(\frac{\mathbf{E}[W_n^2]}{\mathbf{E}[W_n]}\right)^k \\ \geq \frac{\ell_n}{n}\sum_{k=2}^{b_n} kp_k - \frac{b_n}{n}\sum_{k=2}^{b_n} (kp_k)^2 \left(\frac{b_n}{\ell_n}\right)^{k-2} \left(\frac{\mathbf{E}[W_n^2]}{\mathbf{E}[W_n]}\right)^k \\ = \frac{\ell_n}{n}\sum_{k=2}^{b_n} kp_k - o(1) \xrightarrow{n\to\infty} \mathbf{E}[W]\mu \tag{B18}$$

if we choose $b_n = o(n)$ and since we have assumed $\mu_{(2)} = \sum_k k^2 p_k < \infty$. Using the independence and the fact that the variance of an indicator random variable is smaller than or equal to its expectation, we compute

$$\mathrm{Var}\left[\frac{1}{n}\sum_{i\in[n]} d_i^{(l)}\right] \leq \frac{\ell_n}{n^2}\sum_{k=2}^{\infty} k^2 p_k = \frac{\mathbf{E}[W_n]}{n}\sum_{k=2}^{\infty} k^2 p_k = o(1) \tag{B19}$$

since we have assumed that $\sum_{k=2}^{\infty} k^2 p_k < \infty$. Taking n large enough so that $\left|\mathbf{E}[\frac{1}{n}\sum_{i\in[n]} d_i^{(l)}] - \mu\mathbf{E}[W]\right| \leq \frac{\varepsilon}{2}$, by Chebyshev's inequality

$$\mathbf{P}\left(\left|\frac{1}{n}\sum_{i\in[n]} d_i^{(l)} - \mu\mathbf{E}[W]\right| > \varepsilon\right) \\ \leq \mathbf{P}\left(\left|\frac{1}{n}\sum_{i\in[n]} d_i^{(l)} - \mathbf{E}\left[\frac{1}{n}\sum_{i\in[n]} d_i^{(l)}\right]\right| > \frac{\varepsilon}{2}\right) \\ \leq \frac{4}{\varepsilon^2}\mathrm{Var}\left[\frac{1}{n}\sum_{i\in[n]} d_i^{(l)}\right] = o(1) \tag{B20}$$

□

Remark B.1. The proofs above show why our assumptions on the first and second moment of the group-size distribution in (9) and (10) are necessary.

B.2 | Convergence of Right Degrees in $\mathrm{BGRG}_n(\boldsymbol{w})$

We want to show that the degree of a uniformly chosen right-vertex converges. We first need two auxiliary results:

Theorem B.4. (Convergence of the number of groups of fixed size). *Denote $A_k = \#\{a \text{ON} : |a| = k\}$. For $k \geq 2$ and as $n \to \infty$,*

$$\frac{A_k}{n} \xrightarrow{\mathbf{P}} p_k \mathbf{E}[W] \tag{B21}$$

Proof. To prove the desired statement we again use the second-moment method. We have

$$A_k = \sum_{a \in [n]_k} \mathbb{1}_{\{\text{group } a \text{ is ON}\}} \tag{B22}$$

and thus,

$$\mathbf{E}[A_k] = \sum_{a \in [n]_k} \pi^a_{\text{ON}} = \sum_{j_1 < \cdots < j_k \in [n]} \frac{k! p_k w_{j_1} \cdots w_{j_k}}{\ell_n^{k-1} + k! p_k w_{j_1} \cdots w_{j_k}} \tag{B23}$$

Using (B15) we arrive at

$$\mathbf{E}\left[\frac{A_k}{n}\right] \leq \frac{1}{n \cdot k!} \sum_{j_1, \ldots, j_k \in [n]} \frac{k! p_k w_{j_1} \cdots w_{j_k}}{\ell_n^{k-1}} = \frac{p_k \ell_n}{n}$$
$$\times \sum_{j_1, \ldots, j_k \in [n]} \frac{w_{j_1} \cdots w_{j_k}}{\ell_n^k} = p_k \mathbf{E}[W_n] + o(1) \to p_k \mathbf{E}[W]$$

as $n \to \infty$. Since k is fixed, by (B15) and (B17),

$$\begin{aligned}\mathbf{E}\left[\frac{A_k}{n}\right] &\geq p_k \mathbf{E}[W_n] - \frac{(k p_k)^2}{n} \left(\frac{k}{\ell_n}\right)^{k-2} \left(\frac{\mathbf{E}[W_n^2]}{\mathbf{E}[W_n]}\right)^k \\ &= p_k \mathbf{E}[W_n] - o(1)\end{aligned} \tag{B24}$$

since we have assumed $\mu_{(2)} = \sum_k k^2 p_k < \infty$. Therefore,

$$p_k \mathbf{E}[W_n] - o(1) \leq \mathbf{E}\left[\tfrac{A_k}{n}\right] \leq p_k \mathbf{E}[W_n] \tag{B25}$$

Moreover, since A_k is a sum of indicator random variables it holds that $\text{Var}[A_k] \leq \mathbf{E}[A_k]$, which yields, for all $k \geq 2$,

$$\text{Var}\left(\tfrac{A_k}{n}\right) = \tfrac{1}{n^2} \text{Var}[A_k] \leq \tfrac{p_k \mathbf{E}[W_n]}{n} = o(1) \tag{B26}$$

Take n big enough so that $\left|\mathbf{E}[\frac{A_k}{n}] - p_k \mathbf{E}[W]\right| \leq \frac{\varepsilon}{2}$. Then

$$\mathbf{P}\left(\left|\frac{A_k}{n} - p_k \mathbf{E}[W]\right| > \varepsilon\right) \leq \mathbf{P}\left(\left|\frac{A_k}{n} - \mathbf{E}\left[\frac{A_k}{n}\right]\right| > \frac{\varepsilon}{2}\right)$$
$$\leq \frac{4}{\varepsilon^2} \text{Var}\left[\frac{A_k}{n}\right] = o(1)$$

□

Theorem B.5. (Convergence of the groups to vertices ratio). *Denote $M_n = \#\{a \in [n]_k : a \text{ is ON}\}$. As $n \to \infty$,*

$$\frac{M_n}{n} \xrightarrow{\mathbf{P}} \mathbf{E}[W] \tag{B27}$$

Proof. We have

$$M_n = \sum_{k=2}^{\infty} \sum_{a \in [n]_k} \mathbb{1}_{\{a \text{ is ON}\}} = \sum_{k=2}^{\infty} A_k = \sum_{k=2}^{b_n} A_k + \sum_{k > b_n} A_k \tag{B28}$$

where b_n is a sequence diverging to infinity as $n \to \infty$. Note that, using (B15),

$$\mathbf{E}\left[\frac{\sum_{k > b_n} A_k}{n}\right] \leq \mathbf{E}[W_n] \sum_{k > b_n} p_k = o(1) \tag{B29}$$

for each $b_n \to \infty$. Thus,

$$M_n = \sum_{k=2}^{b_n} A_k + o_{\mathbf{P}}(1) \tag{B30}$$

Again by (B15),

$$\mathbf{E}\left[\frac{M_n}{n}\right] = \sum_{k=2}^{b_n} \mathbf{E}\left[\frac{A_k}{n}\right] \leq \frac{\ell_n}{n} \sum_{k=2}^{\infty} p_k = \mathbf{E}[W_n] \to \mathbf{E}[W] \tag{B31}$$

and on the other hand, by (B15) and (B17),

$$\begin{aligned}\mathbf{E}\left[\frac{M_n}{n}\right] &\geq \mathbf{E}[W_n] \sum_{k=2}^{b_n} p_k - \frac{1}{n} \sum_{k=2}^{b_n} (k p_k)^2 \left(\frac{k}{\ell_n}\right)^{k-2} \left(\frac{\mathbf{E}[W_n^2]}{\mathbf{E}[W_n]}\right)^k \\ &= \mathbf{E}[W_n] \sum_{k=2}^{b_n} p_k - o(1) \xrightarrow{n \to \infty} \mathbf{E}[W]\end{aligned} \tag{B32}$$

if we choose $b_n = o(n)$. Further, since each A_k is a sum of independent indicators,

$$\frac{1}{n^2} \text{Var}(M_n) = \frac{1}{n^2} \sum_{k=2}^{b_n} \text{Var}(A_k) \leq \frac{1}{n^2} \sum_{k=2}^{\infty} \mathbf{E}[A_k] \leq \frac{\ell_n}{n^2} = o(1) \tag{B33}$$

Taking again n big enough so that $\left|\mathbf{E}[\frac{M_n}{n}] - \mathbf{E}[W]\right| \leq \frac{\varepsilon}{2}$. Then

$$\mathbf{P}\left(\left|\frac{M_n}{n} - \mathbf{E}[W]\right| > \varepsilon\right) \leq \mathbf{P}\left(\left|\frac{M_n}{n} - \mathbf{E}\left[\frac{M_n}{n}\right]\right| > \frac{\varepsilon}{2}\right)$$
$$\leq \frac{4}{\varepsilon^2} \text{Var}\left(\frac{M_n}{n}\right) = o(1)$$

□

With the above, we can conclude convergence of the degree of a uniformly chosen right-vertex:

Theorem B.6. (Convergence of the degree of a uniformly chosen group). *Recall that we denote the degree of a uniformly chosen group $a \in [n]_{k \geq 2}$ by $D_n^{(r)}$. As $n \to \infty$,*

$$\mathbf{P}(D_n^{(r)} = k | G_n) \xrightarrow{\mathbf{P}} p_k \tag{B34}$$

Proof. Note that $\mathbf{P}(D_n^{(r)} = k | G_n) = A_k / M_n$. From Theorems B.4 and B.5 we know that $n^{-1} A_k$ and $n^{-1} M_n$ converge in probability, which implies convergence of the joint vector $(n^{-1} A_k, n^{-1} M_n)$. Hence, the convergence of the ratio is guaranteed by the continuous mapping theorem:

$$\frac{A_k}{M_n} = \frac{A_k / n}{M_n / n} \xrightarrow{\mathbf{P}} \frac{p_k \mathbf{E}[W]}{\mathbf{E}[W]} = p_k \tag{B35}$$

□

It also follows easily that the expected degree of a uniformly chosen right-vertex converges.

Corollary B.2. (Convergence of the first moment of the degree of a uniformly chosen group). *It follows from the previous that*

$$\mathbf{E}[D_n^{(r)} | G_n] \xrightarrow{\mathbf{P}} \mu \tag{B36}$$

Proof. Note that

$$\mathbf{E}[D_n^{(r)} | G_n] = \frac{\sum_{k=2}^{\infty} \sum_{a \in [n]_k} d_a^{(r)}}{M_n} = \frac{\sum_{k=2}^{\infty} k A_k}{M_n} \tag{B37}$$

It is easy to show by the second-moment method and a suitable truncation, analogously to the previous results, that $\sum_{k=2}^{\infty} k A_k / n \xrightarrow{\mathbf{P}} \mu \mathbf{E}[W]$ and we already showed that $\frac{M_n}{n} \xrightarrow{\mathbf{P}} \mathbf{E}[W]$. The claim follows again thanks to the joint convergence and continuous mapping theorem. □

B.3 | Convergence of the Degree Distribution in $\mathrm{DRIG}_n(\boldsymbol{w})$

Remark B.2. Note that the construction of $\mathrm{DRIG}_n(\boldsymbol{w})$ allows for multiple edges, as two vertices $i, j \in [n]$ might meet in more than one group. However, as we have argued earlier in Section 1.3, it is easy to show that it will not happen with high probability in a neighborhood of a uniformly chosen vertex and hence is negligible as long as local convergence is concerned.

B.3.1 | Expected Average Degree

Proof of Theorem 2.5. Write $\mathbf{E}[D_n|G_n] = \frac{1}{n}\sum_{i\in[n]} d_i$, where d_i is the degree of vertex $i \in [n]$. Since we want to use the second-moment method and d_i, d_j for some $i, j \in [n]$ are not independent, it is more convenient to express $\mathbf{E}[D_n|G_n]$ in terms of groups, which are independent. Note that $\sum_{i\in[n]} d_i$ is nothing else than twice the number of all edges in $\mathrm{DRIG}_n(\boldsymbol{w})$. As $\mathrm{DRIG}_n(\boldsymbol{w})$ is constructed from $\mathrm{BGRG}_n(\boldsymbol{w})$, it is a collection of k-cliques, whose ON or OFF status is determined by the ON and OFF processes of groups $a \in \cup_{n\geq 2}[n]_k$ present in $\mathrm{BGRG}_n(\boldsymbol{w})$. Hence,

$$\begin{aligned}\sum_{i\in[n]} d_i &= \sum_{k=2}^{\infty}\sum_{a\in[n]_k} 2\cdot\frac{|a|(|a|-1)}{2}\mathbb{1}_{\{a \text{ is ON}\}}\\ &= \sum_{k=2}^{\infty}\sum_{a\in[n]_k}|a|(|a|-1)\mathbb{1}_{\{a \text{ is ON}\}}\end{aligned} \tag{B38}$$

since the number of edges in a k-clique equals $k(k-1)/2$. To avoid more assumptions on the moments of the group-size distribution, we once more fix a sequence $b_n \to \infty$ as $n \to \infty$ and use the truncation with respect to the group size, that is,

$$\begin{aligned}\sum_{k=2}^{\infty}\sum_{a\in[n]_k}|a|(|a|-1)\mathbb{1}_{\{a \text{ is ON}\}} &= \sum_{k=2}^{b_n}\sum_{a\in[n]_k}|a|(|a|-1)\mathbb{1}_{\{a \text{ is ON}\}}\\ &+ \sum_{k>b_n}\sum_{a\in[n]_k}|a|(|a|-1)\mathbb{1}_{\{a \text{ is ON}\}}\end{aligned} \tag{B39}$$

Once again denote $w_a = \prod_{i\in a} w_i$. Using upper bounds derived in previous proofs we compute

$$\begin{aligned}&\mathbf{E}\left[\frac{1}{n}\sum_{k>b_n}\sum_{a\in[n]_k}|a|(|a|-1)\mathbb{1}_{\{a \text{ is ON}\}}\right]\\ &= \frac{1}{n}\sum_{k>b_n}k(k-1)\sum_{a\in[n]_k}\frac{k!p_k w_a}{\ell_n^{k-1}+k!p_k w_a}\\ &\leq \frac{\ell_n}{n}\sum_{k>b_n}k(k-1)p_k = o(1)\end{aligned} \tag{B40}$$

for every $b_n \to \infty$, since we have assumed that $\sum_k k^2 p_k < \infty$. Hence, the contribution from groups larger than b_n will vanish in probability, that is,

$$\begin{aligned}&\sum_{k=2}^{\infty}\sum_{a\in[n]_k}|a|(|a|-1)\mathbb{1}_{\{a \text{ is ON}\}}\\ &= \sum_{k=2}^{b_n}\sum_{a\in[n]_k}|a|(|a|-1)\mathbb{1}_{\{a \text{ is ON}\}} + o_{\mathbf{P}}(1)\end{aligned} \tag{B41}$$

Thus, again applying previously derived upper bounds,

$$\begin{aligned}\mathbf{E}\left[\frac{1}{n}\sum_{i\in[n]}d_i\right] &= \frac{1}{n}\sum_{k=2}^{b_n}\sum_{a\in[n]_k}|a|(|a|-1)\mathbf{E}[\mathbb{1}_{\{a \text{ is ON}\}}]\\ &+ o(1) = \frac{1}{n}\sum_{k=2}^{b_n}k(k-1)\sum_{a\in[n]_k}\pi^a_{\mathrm{ON}} + o(1)\\ &\leq \frac{\ell_n}{n}\sum_{k=2}^{b_n}k(k-1)p_k + o(1) \leq \mathbf{E}[W_n](\mu_{(2)}-\mu)\\ &+ o(1) \xrightarrow{n\to\infty} (\mu_{(2)}-\mu)\mathbf{E}[W]\end{aligned} \tag{B42}$$

On the other hand, using (B15),

$$\begin{aligned}\mathbf{E}\left[\frac{1}{n}\sum_{i\in[n]}d_i\right] &\geq \mathbf{E}[W_n]\sum_{k=2}^{b_n}k(k-1)p_k\\ &-\frac{1}{n}\sum_{k=2}^{b_n}k(k-1)(kp_k)^2\left(\frac{k}{\ell_n}\right)^{k-2}\left(\frac{\mathbf{E}[W_n^2]}{\mathbf{E}[W_n]}\right)^k\\ &\geq \mathbf{E}[W_n]\sum_{k=2}^{b_n}k(k-1)p_k - \frac{b_n^2}{n}\sum_{k=2}^{b_n}(kp_k)^2\left(\frac{b_n}{\ell_n}\right)^{k-2}\left(\frac{\mathbf{E}[W_n^2]}{\mathbf{E}[W_n]}\right)^k\\ &= \mathbf{E}[W_n]\sum_{k=2}^{b_n}k(k-1)p_k - o(1) \xrightarrow{n\to\infty} (\mu_{(2)}-\mu)\mathbf{E}[W]\end{aligned} \tag{B43}$$

if we choose $b_n = o(n)$. We now compute the variance:

$$\begin{aligned}\mathrm{Var}\left(\frac{1}{n}\sum_{i\in[n]}d_i\right) &= \frac{1}{n^2}\sum_{k=2}^{b_n}\sum_{a\in[n]_k}|a|^2(|a|-1)^2\mathrm{Var}[\mathbb{1}_{\{a \text{ is ON}\}}]\\ &\leq \frac{1}{n^2}\sum_{k=2}^{b_n}\sum_{a\in[n]_k}|a|^2(|a|-1)^2\mathbf{E}[\mathbb{1}_{\{a \text{ is ON}\}}]\\ &\leq \frac{\ell_n}{n^2}\sum_{k=2}^{b_n}k^2(k-1)^2p_k\\ &\leq \frac{\ell_n b_n^2}{n^2}\sum_{k=2}^{b_n}k(k-1)p_k = o(1)\end{aligned} \tag{B44}$$

if we choose $b_n = o(\sqrt{n})$. Thus, taking n large enough so that $\left|\mathbf{E}[\frac{1}{n}\sum_{i\in[n]}d_i] - (\mu_{(2)}-\mu)\mathbf{E}[W]\right| \leq \frac{\varepsilon}{2}$, we obtain

$$\begin{aligned}&\mathbf{P}\left(\left|\frac{1}{n}\sum_{i\in[n]}d_i - (\mu_{(2)}-\mu)\mathbf{E}[W]\right| > \varepsilon\right)\\ &\leq \mathbf{P}\left(\left|\frac{1}{n}\sum_{i\in[n]}d_i - \mathbf{E}\left[\frac{1}{n}\sum_{i\in[n]}d_i\right]\right| > \frac{\varepsilon}{2}\right)\\ &\leq \frac{4}{\varepsilon^2}\mathrm{Var}\left(\frac{1}{n}\sum_{i\in[n]}d_i\right) = o(1)\end{aligned}$$

B.3.2 | Degree Sequence

Recall from Section 2.2.3 that we defined the degree sequence as

$$Q_k^{(n)} = \frac{1}{n}\sum_{i\in[n]}\mathbb{1}_{\{d_i=k\}} \tag{B45}$$

where d_i is the degree of vertex $i \in [n]$ in $\mathrm{DRIG}_n(\boldsymbol{w})$. We will now prove Theorem 2.4.

Proof of Theorem 2.4. We adapt the proof of a similar result for $\mathrm{GRG}_n(\boldsymbol{w})$ (see [22, Theorem 6.10]). Since $(q_k)_{k\geq 0}$ is a probability mass function,

$$\sum_{k\geq 0}|Q_k^{(n)} - q_k| = 2d_{\mathrm{TV}}(Q^{(n)}, q) \xrightarrow{\mathbf{P}} 0 \tag{B46}$$

if and only if $\max_{k\geq 0}|Q_k^{(n)} - q_k| \xrightarrow{\mathbf{P}} 0$, where d_{TV} denotes the total variation distance. Therefore we have to show that $\mathbf{P}(\max_{k\geq 0}|Q_k^{(n)} - q_k| \geq \varepsilon)$ vanishes for every $\varepsilon > 0$. Note that,

$$\mathbf{P}(\max_{k\geq 0}|Q_k^{(n)} - q_k| \geq \varepsilon) \leq \sum_{k\geq 0}\mathbf{P}(|Q_k^{(n)} - q_k| \geq \varepsilon) \tag{B47}$$

We also have that $\mathbf{E}[Q_k^{(n)}] = \mathbf{P}(D_n = k) = \mathbf{P}(\sum_{l=2}^{\infty}(l-1)C_l(o_n) = k)$ and thus, by the previous results (see Theorem B.1) we know that $\lim_{n\to\infty} \mathbf{P}(D_n = k) = q_k$. Hence, for n sufficiently large,

$$\max_k |\mathbf{E}[Q_k^{(n)}] - q_k| \le \frac{\varepsilon}{2} \tag{B48}$$

Thus, again for n sufficiently large,

$$\begin{aligned}\mathbf{P}\left(\max_{k\ge 0}|Q_k^{(n)} - q_k| \ge \varepsilon\right) &\le \sum_{k\ge 0}\mathbf{P}\left(|Q_k^{(n)} - \mathbf{E}[Q_k^{(n)}]| \ge \frac{\varepsilon}{2}\right) \\ &\le \frac{4}{\varepsilon^2}\sum_{k\ge 0}\mathrm{Var}(Q_k^{(n)})\end{aligned} \tag{B49}$$

where the last follows from the Chebyshev inequality. We have

$$\begin{aligned}\mathrm{Var}(Q_k^{(n)}) \le \frac{1}{n^2}\sum_{i\in[n]}[\mathbf{P}(d_i = k) - \mathbf{P}(d_i = k)^2] + \frac{1}{n^2} \\ \times \sum_{i,j\in[n],i\ne j}[\mathbf{P}(d_i = d_j = k) - \mathbf{P}(d_i = k)\mathbf{P}(d_j = k)]\end{aligned} \tag{B50}$$

Note that

$$\begin{aligned}\sum_{k\ge 0}\frac{1}{n^2}\sum_{i\in[n]}[\mathbf{P}(d_i = k) - \mathbf{P}(d_i = k)^2] \\ \le \sum_{k\ge 0}\frac{1}{n^2}\sum_{i\in[n]}\mathbf{P}(d_i = k) = \frac{1}{n} = o(1)\end{aligned} \tag{B51}$$

We want to show that the second term in (B50) vanishes too, when summed overall k. Note that $d_i = \sum_{a: a\ni i}(|a|-1)\mathbb{1}_{\{a \text{ is ON}\}}$. Hence, the correlation between d_i and d_j is due to groups containing both i and j. We write

$$d_{i\setminus j} = \sum_{k=2}^{\infty}\sum_{a\in[n]_k : a\ni i, a \not\ni j}(|a|-1)\mathbb{1}_{\{a \text{ is ON}\}} \tag{B52}$$

and we define $d_{j\setminus i}$ analogously. We also define

$$d_{i,j} = \sum_{k=2}^{\infty}\sum_{a\in[n]_k : a\ni i,j}(|a|-1)\mathbb{1}_{\{a \text{ is ON}\}} \tag{B53}$$

Then (d_i, d_j) has the same law as $(d_{i\setminus j} + d_{i,j}, d_{j\setminus i} + d_{i,j})$. Now let us introduce random variable $\hat{\mathbb{1}}_{\{a \text{ is ON}\}}$ such that $\hat{\mathbb{1}}_{\{a \text{ is ON}\}} \overset{d}{=} \mathbb{1}_{\{a \text{ is ON}\}}$ and $\hat{\mathbb{1}}_{\{a \text{ is ON}\}}$ is independent of $(\mathbb{1}_{\{a \text{ is ON}\}})_{a\in\cup_{k\ge 2}[n]_k}$. Thus,

$$\hat{d}_{i,j} = \sum_{k=2}^{\infty}\sum_{a\in[n]_k : a\ni i,j}(|a|-1)\hat{\mathbb{1}}_{\{a \text{ is ON}\}} \overset{d}{=} d_{i,j} \tag{B54}$$

and then $(d_{i\setminus j} + d_{i,j}, d_{j\setminus i} + \hat{d}_{i,j})$ are *independent* random variables with the same marginals as d_i, d_j. Hence

$$\begin{aligned}\mathbf{P}(d_i = d_j = k) &= \mathbf{P}((d_{i\setminus j} + d_{i,j}, d_{j\setminus i} + d_{i,j}) = (k,k)) \\ \mathbf{P}(d_i = k)\mathbf{P}(d_j = k) &= \mathbf{P}((d_{i\setminus j} + d_{i,j}, d_{j\setminus i} + \hat{d}_{i,j}) = (k,k))\end{aligned} \tag{B55}$$

Therefore,

$$\begin{aligned}&\mathbf{P}(d_i = d_j = k) - \mathbf{P}(d_i = k)\mathbf{P}(d_j = k) \\ &= \mathbf{P}((d_{i\setminus j} + d_{i,j}, d_{j\setminus i} + d_{i,j}) = (k,k)) \\ &\quad - \mathbf{P}((d_{i\setminus j} + d_{i,j}, d_{j\setminus i} + \hat{d}_{i,j}) = (k,k)) \\ &\le \mathbf{P}(\{(d_{i\setminus j} + d_{i,j}, d_{j\setminus i} + d_{i,j}) \\ &\quad = (k,k)\} \setminus \{(d_{i\setminus j} + d_{i,j}, d_{j\setminus i} + \hat{d}_{i,j}) = (k,k)\}) \\ &= \mathbf{P}((d_{i\setminus j} + d_{i,j}, d_{j\setminus i} + d_{i,j}) = (k,k) \\ &\quad (d_{i\setminus j} + d_{i,j}, d_{j\setminus i} + \hat{d}_{i,j}) \ne (k,k))\end{aligned} \tag{B56}$$

When the above happens, it must be that $d_{i,j} \ne \hat{d}_{i,j}$, so there exists such $a \ni i, j$ that $\mathbb{1}_{\{a \text{ is ON}\}} \ne \hat{\mathbb{1}}_{\{a \text{ is ON}\}}$. If then, for some a, $\hat{\mathbb{1}}_{\{a \text{ is ON}\}} = 1$, then $\mathbb{1}_{\{a \text{ is ON}\}} = 0$ and $d_{i\setminus j} + d_{i,j} = k$; If $\mathbb{1}_{\{a \text{ is ON}\}} = 1$, then $\hat{\mathbb{1}}_{\{a \text{ is ON}\}} = 0$ and $d_{j\setminus i} + \hat{d}_{i,j} = k - (|a|-1)$. Hence,

$$\begin{aligned}&\mathbf{P}(d_i = d_j = k) - \mathbf{P}(d_i = k)\mathbf{P}(d_j = k) \\ &\quad \le \sum_{a\ni i,j}\mathbf{P}(a \text{ is ON})[\mathbf{P}(d_i = k) + \mathbf{P}(d_j = k - |a| + 1)]\end{aligned} \tag{B57}$$

This yields

$$\begin{aligned}\sum_{k\ge 0}\mathrm{Var}(Q_k^{(n)}) &\le o(1) + \sum_{k\ge 0}\frac{1}{n^2}\sum_{i,j\in[n],i\ne j}\sum_{a\ni i,j}\mathbf{P}(a \text{ is ON})[\mathbf{P}(d_i = k) \\ &\qquad + \mathbf{P}(d_j = k - |a| + 1)] \\ &\le o(1) + \frac{2}{n^2}\sum_{i,j\in[n]}\sum_{a\ni i,j}\frac{|a|!p_{|a|}w_a}{\ell_n^{a-1}}\end{aligned} \tag{B58}$$

where we once again denote $w_a = \prod_{i\in a} w_i$. Since

$$\begin{aligned}\sum_{a\ni i,j}\frac{|a|!p_{|a|}w_a}{\ell_n^{a-1}} &= \sum_{l=2}^{\infty}\sum_{v_1<\cdots<v_{l-2}}\frac{l!p_l w_i w_j w_a}{\ell_n^{l-1}} \\ &= \frac{w_i w_j}{\ell_n}\sum_{l=2}^{\infty}l(l-1)p_l\sum_{v_1,\ldots,v_{l-2}}\frac{w_{v_1}\cdots w_{v_l}}{\ell_n^{l-2}} \\ &= \frac{w_i w_j}{\ell_n}(\mu_{(2)} - \mu)\end{aligned} \tag{B59}$$

we obtain

$$\begin{aligned}\sum_{k\ge 0}\mathrm{Var}(Q_k^{(n)}) &\le o(1) + \frac{2}{n^2}\sum_{i,j\in[n]}\frac{w_i w_j}{\ell_n}(\mu_{(2)} - \mu) \\ &= o(1) + (\mu_{(2)} - \mu)\frac{1}{n^2}\sum_{i\in[n]}\frac{w_i^2}{\ell_n} = o(1) \\ &\quad + \frac{(\mu_{(2)} - \mu)\mathbf{E}[W_n^2]}{n^2} = o(1)\end{aligned} \tag{B60}$$

Remark B.3. (Eliminating conditions on higher moments by weight truncation). Note that Theorem 2.3 is only valid for the $\mathrm{BGRG}_n(\boldsymbol{w})$ with weights $(w_i)_{i\in[n]}$ satisfying Condition 1.1a–c, as it requires a finite second moment of the degree of a uniformly chosen vertex. However, we argue that the local convergence statement can easily be extended to the $\mathrm{BGRG}_n(\boldsymbol{w})$ not satisfying the latter via a truncation argument. To do so, we adapt a similar argument from $\mathrm{GRG}_n(\boldsymbol{w})$ (see the proof of [44, Theorem 4.23]). Namely, we can truncate the weights of all vertices by some $K > 0$, that is, introduce $\mathrm{BGRG}_n^{(K)}(\boldsymbol{w})$ with weights $(w_i^{(K)})_{i\in[n]}$ such that

$$w_i^{(K)} = w_i \wedge K \tag{B61}$$

Note that if $(w_i)_{i\in[n]}$ in $\mathrm{BGRG}_n(\boldsymbol{w})$ satisfies Condition 1.1a,b, then $(w_i^{(K)})_{i\in[n]}$ in $\mathrm{BGRG}_n^{(K)}(\boldsymbol{w})$ satisfies Condition 1.1a–c. Hence, the second moments of $D_n^{(l),(K)}$ and $D_n^{(r),(K)}$ are finite and all results on $\mathrm{BCM}_n(\boldsymbol{d})$ can be transferred to $\mathrm{BGRG}_n^{(K)}(\boldsymbol{w})$ thanks to Theorem 2.3. This means that $(\mathrm{BGRG}_n^{(K)}(\boldsymbol{w}), V_n^{(l)})$ converges locally in probability to some limiting (G, o). We now show that this implies that also $(\mathrm{BGRG}_n(\boldsymbol{w}), V_n^{(l)})$ converges locally in probability to (G, o). By local convergence, for any fixed rooted graph (H, o') and $r \in \mathbf{N}$,

$$\frac{1}{n}\sum_{i\in[n]}\mathbb{1}_{\{B_r(G_n^{(K)}, i)\simeq(H,o')\}} \xrightarrow{\mathbf{P}} \mathbf{P}(B_r(G, o) \simeq (H, o')) \tag{B62}$$

We write

$$\frac{1}{n}\sum_{i\in[n]}\mathbb{1}_{\{B_r(G_n,i)\simeq(H,o')\}}$$
$$=\frac{1}{n}\sum_{i\in[n]}\mathbb{1}_{\{B_r(G_n,i)\simeq(H,o'),B_r(G_n,i)\text{ contains only } j \text{ such that } w_j\le K\}} \tag{B63}$$
$$+\frac{1}{n}\sum_{i\in[n]}\mathbb{1}_{\{B_r(G_n,i)\simeq(H,o'),B_r(G_n,i)\text{ contains } j \text{ with } w_j>K\}}$$

and note that

$$\lim_{K\to\infty}\limsup_{n\to\infty}\mathbf{E}\left[\frac{1}{n}\sum_{i\in[n]}\mathbb{1}_{\{B_r(G_n,i)\simeq(H,o'),B_r(G_n,i)\text{ contains } j \text{ with } w_j>K\}}\right]$$
$$\le\lim_{K\to\infty}\limsup_{n\to\infty}\mathbf{P}(B_r(G_n,V_n^{(l)})\text{ contains } j : w_j>K)=o(1) \tag{B64}$$

by Condition 1.1a,b. Thus, for K large enough,

$$\frac{1}{n}\sum_{i\in[n]}\mathbb{1}_{\{B_r(G_n,i)\simeq(H,o')\}}$$
$$\overset{\mathrm{d}}{=}\frac{1}{n}\sum_{i\in[n]}\mathbb{1}_{\{B_r(G_n,i)\simeq(H,o'),B_r(G_n,i)\text{ contains only } j:w_j\le K\}}+o_{\mathbf{P}}(1) \tag{B65}$$
$$=\frac{1}{n}\sum_{i\in[n]}\mathbb{1}_{\{B_r(G_n^{(K)},i)\simeq(H,o')\}}+o_{\mathbf{P}}(1)\xrightarrow{\mathbf{P}}\mathbf{P}(B_r(G,o)\simeq(H,o'))$$

Having shown local convergence of $\mathrm{BGRG}_n(\boldsymbol{w})$ without Condition 1.1c, we transfer the result on the giant component in a similar manner. Denote the size of the giant component in $\mathrm{BGRG}_n^{(K)}(\boldsymbol{w})$ by $|\mathscr{C}_{\max}^{(K)}|$ and the size of the giant component in $\mathrm{BGRG}_n(\boldsymbol{w})$ by $|\mathscr{C}_{\max}|$. Transferring results from [13] we obtain, as $n\to\infty$,

$$\frac{|\mathscr{C}_{\max}^{(K)}|}{n}\xrightarrow{\mathbf{P}}\xi \tag{B66}$$

Naturally,

$$\frac{|\mathscr{C}_{\max}|}{n}\ge\frac{|\mathscr{C}_{\max}^{(K)}|}{n}\xrightarrow{\mathbf{P}}\xi=\mathbf{P}(|\mathscr{C}(o)|=\infty) \tag{B67}$$

On the other hand, denote $Z_{\ge k}=\frac{1}{n}\sum_{i\in[n]}\mathbb{1}_{\{|\mathscr{C}(i)|\ge k\}}$ and note that on the event $\{Z_{\ge k}\ge 1\}$,

$$\frac{|\mathscr{C}_{\max}|}{n}\le\frac{Z_{\ge k}}{n} \tag{B68}$$

By local convergence in probability,

$$\frac{1}{n}Z_{\ge k}=\mathbf{E}\left[\mathbb{1}_{\{|\mathscr{C}(V_n^{(l)})|\ge k\}}|G_n\right]\xrightarrow{\mathbf{P}}\xi_{\ge k}=\mathbf{P}(|\mathscr{C}(o)|\ge k) \tag{B69}$$

Note that $\lim_{k\to\infty}\xi_{\ge k}=\xi$. Thus, by (B69), for every $\varepsilon>0$,

$$\limsup_{n\to\infty}\mathbf{P}\left(\frac{|\mathscr{C}_{\max}|}{n}\ge\xi+\varepsilon\right)\le\lim_{n\to\infty}\mathbf{P}\left(\frac{1}{n}Z_{\ge k}\ge\xi+\varepsilon\right)$$
$$\le\lim_{k\to\infty}\lim_{n\to\infty}\mathbf{P}\left(\frac{1}{n}Z_{\ge k}\ge\xi_{\ge k}+\varepsilon\right)=0 \tag{B70}$$

and the desired statement follows. Hence, the most important results that we treat in this article, that is, local convergence and the existence of giant component, are also true for $\mathrm{BGRG}_n(\boldsymbol{w})$ only satisfying Condition 1.1a,b.

B.4 | Static Local Convergence of $\mathrm{BGRG}_n(\boldsymbol{w})$ and $\mathrm{DRIG}_n(\boldsymbol{w})$

Local convergence of $\mathrm{BCM}_n(\boldsymbol{d})$: Having verified that $\mathrm{BGRG}_n(\boldsymbol{w})$ fulfils the necessary regularity conditions we can now conveniently transfer results on the local convergence from [14]. For the comfort of the reader, we quote the statement of the original result on the local convergence of $\mathrm{BCM}_n(\boldsymbol{d})$ ([14, Theorem 2.14]), which states that under Condition 2.1, as $n\to\infty$, (BCM_n, V_n^b) converges locally in probability to $(\mathrm{BP}_\gamma, 0)$. The limiting object—$(\mathrm{BP}_\gamma, 0)$—is a mixture of two branching processes with the root 0. The two processes are needed because of the bipartite structure of $\mathrm{BCM}_n(\boldsymbol{d})$—each of them corresponds to contributions made to the limit by left- and right-vertices respectively. As the structures of local limits of $\mathrm{BCM}_n(\boldsymbol{d})$ and $\mathrm{BGRG}_n(\boldsymbol{w})$ are very similar, we do not describe the first one in more detail and direct the reader to Section 2.2 where the latter is explained.

Local convergence of RIGC: In [14] the local convergence of the resulting intersection graph is a consequence of the convergence of the underlying bipartite graph. Thus, the same will take place for our model.

Again for the reader's convenience, we first quote the statement of the original result from [14] (see [14, Theorem 2.8]: Under Condition 2.1, as $n\to\infty$, (RIGC, V_n^l) converges locally in probability to (CP, o). As we already mentioned, the convergence of RIGC follows from the convergence of $\mathrm{BCM}_n(\boldsymbol{d})$. Therefore, the limiting object (CP, o) is a community projection of the limiting object $(\mathrm{BP}_l, 0)$ just like RIGC is a community projection of $\mathrm{BCM}_n(\boldsymbol{d})$ (see (2.2) in [14]).

Static local limit of $\mathrm{BGRG}_n(\boldsymbol{w})$ **and** $\mathrm{DRIG}_n(\boldsymbol{w})$: We proceed by proving the local convergence of $\mathrm{BGRG}_n(\boldsymbol{w})$ and $\mathrm{DRIG}_n(\boldsymbol{w})$ in Theorem 1.1:

Proof of Theorem 1.1. Thanks to Theorem 2.3 and the fact that under Condition 1.1 $\mathrm{BGRG}_n(\boldsymbol{w})$ fulfils Condition 2.1a (see Appendix B.1 and B.2), we can transfer [14, Theorem 2.8] to $\mathrm{BGRG}_n(\boldsymbol{w})$, obtaining the above. The limiting left- and right-degrees $D^{(l)}$ and $D^{(r)}$ are the ones derived in Appendix B.1 and B.2.

The result for $\mathrm{DRIG}_n(\boldsymbol{w})$ is equivalent to [14, Theorem 2.8] and it is a consequence of the relationship between the resulting $\mathrm{DRIG}_n(\boldsymbol{w})$ and the underlying $\mathrm{BGRG}_n(\boldsymbol{w})$. The convergence of intersection graphs is preserved by the community projection that transforms the underlying bipartite structures into them. Hence, naturally, the local limit of $\mathrm{DRIG}_n(\boldsymbol{w})$ is a community projection (see (14)) of the local limit of $\mathrm{BGRG}_n(\boldsymbol{w})$.

B.5 | Static Giant Component

B.5.1 | Giant Component in $\mathrm{BGRG}_n(\boldsymbol{w})$

In [13], results on the giant component of $\mathrm{BCM}_n(\boldsymbol{d})$ are again shown under Condition 2.1. Hence, assuming Condition 1.1, thanks to Theorem 2.3 we can transfer them to our situation. We start with the giant component of the underlying bipartite graph. For the reader's convenience, we state the original result from [13] adapted to the notation we use in this article. Recall that for the distinction we denote the largest connected component in the underlying bipartite graphs by $\mathscr{C}_{1,b}$. Also recall that $\xi_l=1-G_{D^{(l)}}(\eta_l)\in[0,1]$, where $\eta_l\in[0,1]$ is the smallest solution of the fixed point equation

$$\eta_l=G_{\tilde{D}^{(r)}}(G_{\tilde{D}^{(l)}}(\eta_l)) \tag{B71}$$

Then, we have the following results on $\mathscr{C}_{1,b}$ in $\mathrm{BCM}_n(\boldsymbol{d})$:

Theorem B.7. (The largest component of the BCMn(d) [13, Theorem 2.11]). *Consider* $\mathrm{BCM}_n(\boldsymbol{d})=\mathrm{BCM}_n(\boldsymbol{d})(\boldsymbol{d})$ *under Condition 2.1 and further assume that* $\overline{V}_2+\bar{A}_2<2$, *where* $\overline{V}_k=\frac{1}{n}\#\{i\in[n]:d_i^{(l)}=k\}$ *and* $\bar{A}_k=\frac{1}{M_n}\#\{a:|a|=k\}$. *Under the supercriticality condition* $\mathbf{E}[\tilde{D}^{(l)}]\mathbf{E}[\tilde{D}^{(r)}]>1$, *we have that* $\xi_l>0$, $\eta_l<1$ *and* $\eta_r=G_{\tilde{D}^{(l)}}(\eta_l)<1$. *Then, as* $n\to\infty$,

$$\frac{|\mathscr{C}_{1,b}\cap[n]|}{n}\xrightarrow{\mathbf{P}}\xi_l \tag{B72}$$

$$\frac{|\mathscr{C}_{1,b}\cap\overline{V}_k|}{n}\xrightarrow{\mathbf{P}}p_k(1-\eta_l^k) \tag{B73}$$

In this case, $\mathscr{C}_{1,b}$ *is unique in the sense that* $|\mathscr{C}_{2,b}|/(n+M_n)\xrightarrow{\mathbf{P}}0$, *where* $\mathscr{C}_{2,b}$ *is the second largest component. If the supercriticality condition does not hold, then* $|\mathscr{C}_{1,b}|/(n+M_n)\xrightarrow{\mathbf{P}}0$.

Given this, we obtain Theorem 2.6:

Proof of Theorem 2.6. The result is automatically transferred from Theorem B.7. We are allowed to do that because in Theorem 2.3 we have linked $\mathrm{BCM}_n(\boldsymbol{d})$ satisfying Condition 2.1 to $\mathrm{BGRG}_n(\boldsymbol{w})$ satisfying Condition 2.1 and we also showed that $\mathrm{BGRG}_n(\boldsymbol{w})$ satisfies these conditions earlier in this section. Note that the condition $\overline{V}_2 + \bar{A}_2 < 2$ made in [13] is always satisfied in our model (as $V_0 > 0$) and does not have to be assumed for Theorem 2.6 to hold.

B.5.2 | Giant Component in $\mathrm{DRIG}_n(\boldsymbol{w})$

Once again, we want to transfer results from [13], namely the result on the giant component in the resulting intersection graph. Again for the comfort of the reader, we quote the original result with notation adapted to ours:

Theorem B.8. (Size of the largest component [13, Theorem 2.6]). *Consider* RIGC *under Condition 2.1, and further assume that* $\overline{V}_2 + \bar{A}_2 < 2$, *where* $\overline{V}_k = \frac{1}{n}\#\{i \in [n] : d_i^{(l)} = k\}$ *and* $\bar{A}_k = \frac{1}{M_n}\#\{a : |a| = k\}$. *Then, there exists* $\eta_l \in [0,1]$, *the smallest solution of the fixed point equation*

$$\eta_l = G_{\tilde{D}^{(r)}}(G_{\tilde{D}^{(l)}}(\eta_l)) \tag{B74}$$

and $\xi_l = 1 - G_{D^{(l)}}(\eta_l) \in [0,1]$ *such that*

$$\frac{|\mathscr{C}_1|}{n} \xrightarrow{\mathbf{P}} \xi_l \tag{B75}$$

Furthermore, $\xi_l > 0$ *exactly when*

$$\mathbf{E}[\tilde{D}^{(l)}]\mathbf{E}[\tilde{D}^{(r)}] > 1 \tag{B76}$$

In this case, $\mathscr{C}_1$ *is unique, in the sense that* $|\mathscr{C}_2|/n \xrightarrow{\mathbf{P}} 0$, *where* $\mathscr{C}_2$ *is the second largest component*.

Hence, we obtain that our model has a giant component as well when $\mathbf{E}[W^2](\mu_{(2)} - \mu)/\mathbf{E}[W] > 1$, as in Theorem 1.2:

Proof of Theorem 1.2. The convergence in (20) can be transferred from (B75) because of the graph equivalence from Theorem 2.3 and since imposing Condition 1.1 on the weights implies that $\mathrm{DRIG}_n(\boldsymbol{w})$ fulfils Condition 2.1 (as we have shown in Appendix B.1 and B.2). However, since, as in the case of RIGC, the giant component in the resulting graph will exist only if the giant component in the underlying graph exists this result can be also deduced from Theorem 2.6. Note again that the condition $\overline{V}_2 + \bar{A}_2 < 2$ made in [13] is always satisfied in our model (as $V_0 > 0$) and does not have to be assumed for Theorem 1.2 to hold. A nice feature of our model is the fact that we have a more explicit form of the limiting left- and right-degrees than the authors of [13, 14]. Hence, we can express the supercriticality condition as in (21): by definition of the shift random variable $\tilde{D}^{(r)}$ (see (2.9)),

$$\begin{aligned}\mathbf{E}[\tilde{D}^{(r)}] &= \sum_{k=1} k\mathbf{P}(\tilde{D}^{(r)} = k) = \sum_{k=1} k(k+1)\frac{\mathbf{P}(D^{(r)} = k+1)}{\mathbf{E}[D^{(r)}]}\\ &= \frac{1}{\mu}\sum_{k=1} k(k+1)p_{k+1}\\ &= \frac{1}{\mu}\sum_{l=2}(l-1)lp_l = \frac{\mu_{(2)} - \mu}{\mu}\end{aligned} \tag{B77}$$

where we have used (B34) and (B36) when substituting explicit expressions for the probability mass function and expected value of the random variable $D^{(r)}$. Similarly,

$$\mathbf{E}[\tilde{D}^{(l)}] = \sum_{k=1} k(k+1)\frac{\mathbf{P}(D^{(l)} = k+1)}{\mathbf{E}[D^{(l)}]} \tag{B78}$$

We know by Corollary B.2 that $D^{(l)}$ is a mixed-Poisson variable with rate $W\mu$. Hence, we need to condition on the weight variable W to compute its expectation:

$$\mathbf{E}[D^{(l)}] = \mathbf{E}[\mathbf{E}(\mathrm{Poi}(w\mu)|W] = \mathbf{E}[\mu W] = \mu\mathbf{E}[W] \tag{B79}$$

Plugging it into (B78) yields

$$\begin{aligned}\mathbf{E}[\tilde{D}^{(l)}] &= \frac{1}{\mu\mathbf{E}[W]}\sum_{k=1} k(k+1)\mathbf{P}(D^{(l)} = k+1)\\ &= \frac{1}{\mu\mathbf{E}[W]}\sum_{l=2}(l-1)l\mathbf{E}[\mathbf{E}[\mathbf{P}(\mathrm{Poi}(w\mu) = l)|W]]\end{aligned} \tag{B80}$$

After substituting $\mathbf{P}(\mathrm{Poi}(w\mu) = l) = \frac{(w\mu)^l e^{-w\mu}}{l!}$ we obtain

$$\begin{aligned}\mathbf{E}[\tilde{D}^{(l)}] &= \frac{1}{\mu\mathbf{E}[W]}\sum_{l=2}(l-1)l\mathbf{E}[\mathbf{E}[\frac{(w\mu)^l e^{-w\mu}}{l!}|W]]\\ &= \frac{1}{\mu\mathbf{E}[W]}\mathbf{E}\left[\mathbf{E}\left[\sum_{l=2}\frac{(W\mu)^l}{(l-2)!}e^{-W\mu}|W\right]\right]\\ &= \frac{1}{\mu\mathbf{E}[W]}\mathbf{E}[\mathbf{E}[e^{-W\mu}\cdot e^{W\mu}(W\mu)^2|W]]\\ &= \frac{\mu^2\mathbf{E}[W^2]}{\mu\mathbf{E}[W]} = \frac{\mu\mathbf{E}[W^2]}{\mathbf{E}[W]}\end{aligned} \tag{B81}$$

Therefore, the condition (21) becomes

$$\frac{\mathbf{E}[W^2](\mu_{(2)} - \mu)}{\mathbf{E}[W]} > 1 \tag{B82}$$

Remark B.4. (Results for $\mathrm{BGRG}_n^{(T)}(\boldsymbol{w})$). We claim that $\mathrm{BGRG}_n^{(T)}(\boldsymbol{w})$ fulfils the same conditions that guaranteed convergence of the $\mathrm{BGRG}_n(\boldsymbol{w})$, that is, uniformity and regularity conditions of the degree sequences. Indeed, note that by replacing $f(|a|) = |a|!p_{|a|}$ by $f(|a|) = (1+T)|a|!p_{|a|}$, all the proofs from Appendix B follow analogously. Hence, we do not explicitly repeat them here. However, for the comfort of the reader we now state the corresponding regularity conditions as the limiting variables are necessary to understand some of our statements on the union graph:

1. Denote the degree of a uniformly chosen left-vertex in $\mathrm{BGRG}_n^{(T)}(\boldsymbol{w})$ by $D_n^{(l),(T)}$. Then, as $n \to \infty$,

$$D_n^{(l),(T)} \to D^{(l),(T)} \text{ and } \mathbf{E}[D_n^{(l),(T)}|G_n] \xrightarrow{\mathbf{P}} \mu(1+T)\mathbf{E}[W] \tag{B83}$$

where $D^{(l),(T)}$ is a Poisson variable with parameter $W\mu(1+T)$, with $\mu = \sum_k kp_k$.

2. Denote the degree of a uniformly chosen right-vertex in $\mathrm{BGRG}_n^{(T)}(\boldsymbol{w})$ by $D_n^{(r),(T)}$. Then, as $n \to \infty$,

$$\mathbf{P}(D_n^{(r),(T)} = k|G_n) \xrightarrow{\mathbf{P}} p_k \text{ and } \mathbf{E}[D_n^{(r),(T)}|G_n] \xrightarrow{\mathbf{P}} \mu \tag{B84}$$

Naturally, the above automatically transfers to $\mathrm{BGRG}_n^{[0,T]}(\boldsymbol{w})$, as it is asymptotically equivalent to $\mathrm{BGRG}_n^{(T)}(\boldsymbol{w})$.

Appendix C

Convergence of Processes With Discontinuities of the First Kind

To prove the dynamic results for processes of giant membership and size of the largest group we apply a well-known criterion for convergence of processes in Skorokhod J_1 topology, that we now quote with a minor change of replacing $[0,1]$—the domain used by [31]—with $[0,t]$. Since the mentioned processes have different codomains ($[0,1]$ in case of the giant membership process and $\mathbf{R}_+$ in case of the largest group process) we refrain from specifying the codomain in what follows, stressing that the cited results stay the same in both cases.

Theorem C.1. ([31, Theorem 13.1]). *For any sequence of probability measures* $\mathbf{P}_n$ *in* $D[0,T]$, *it holds that if the finite-dimensional distributions* *of* $\mathbf{P}_n$ *converge to finite-dimensional distributions of* $\mathbf{P}$ *and* $\mathbf{P}_n$ *is tight, then* $\mathbf{P}_n$ *converges weakly to* $\mathbf{P}$ *as* $n \to \infty$.

Naturally, the convergence of probability measures in $D[0, T]$ will imply the convergence of random elements with sample paths in $D[0, T]$. Since tightness might sometimes be difficult to verify, we make use of the following well-known result that allows us to check relative compactness in distribution instead:

Theorem C.2. ([51, Theorem 16.3]). *For any sequence of random elements* $\xi_1, \xi_2, \ldots$ *in a metric space* S, *tightness implies relative compactness in distribution, and the two conditions are equivalent when* S *is separable and complete*.

An extension of the Arzellà-Ascoli result to $D[0, T]$ provides a convenient criterion for relative compactness in distribution. From there, it is possible to derive a couple of alternative criteria. We refrain from providing all of them and state the following result on weak convergence of processes with discontinuities with the criterion that was the most suitable for us. For a more detailed overview of other possibilities, we refer the reader to [31, 51].

Theorem C.3. ([31, Theorem 13.3]). *Assume a sequence of processes* $(X_n(s))_{s\in[0,T]}$ *and a process* $(\mathcal{X}(s))_{s\in[0,T]}$ *in* $D[0, T]$, *satisfy the following conditions*:

i. *For all* $\{s_1, \ldots, s_k\} \in [0, T] : (X_n(s_1), \ldots, X_n(s_k)) \xrightarrow{d} (\mathcal{X}(s_1), \ldots, \mathcal{X}(s_k))$ *as* $n \to \infty$.
ii. $\mathcal{X}(T) - \mathcal{X}(T - \delta) \xrightarrow{\mathbf{P}} 0$ *as* $\delta \to 0$.
iii. *For every* $\varepsilon, \eta > 0$ *there exists* $n_0 \geq 1$ *and* $\delta > 0$ *such that for all* $n \geq n_0$

$$\mathbf{P}\left(\sup_{s,s_1,s_2 : s\in[s_1,s_2], s_2-s_1<\delta} \min\left(\left|X_n(s) - X_n(s_1)\right|, \left|X_n(s_2) - X_n(s)\right|\right) > \varepsilon\right) \leq \eta \tag{C1}$$

Then, as $n \to \infty$, $(X_n(s))_{s\in[0,T]}$ *converges weakly to* $(\mathcal{X}(s))_{s\in[0,T]}$ *in Skorokhod* J_1 *topology on* $D[0, T]$.

Thanks to the fact that the space of rooted graphs equipped with the local topology is Polish, we can use the same criteria to extend the notion of local convergence to the dynamic setting [31, Chapter 12]. mentions that all the results relevant for processes in $D([0, 1], [0, 1])$ easily extend to more general processes with sample paths in $D([0, T], S)$—the space of càdlàg functions $f : [0, T] \to S$, where S is separable and complete. Naturally, in this case the Euclidian distances in Theorem C.3 need to be replaced with the local metric. Alternatively, the required results can be found in [51, Chapter 16].