Topological Bounds on the Price of Anarchy of Clustering Games on Networks

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We consider clustering games in which the players are embedded into a network and want to coordinate (or anti-coordinate) their strategy with their neighbors. The goal of a player is to choose a strategy that maximizes her utility given the strategies of her neighbors. Recent studies show that even very basic variants of these games exhibit a large Price of Anarchy: A large inefficiency between the total utility generated in centralized outcomes and equilibrium outcomes in which players selfishly maximize their utility.

Our main goal is to understand how structural properties of the network topology impact the inefficiency of these games. We derive *topological bounds* on the Price of Anarchy for different classes of clustering games. These topological bounds provide a more informative assessment of the inefficiency of these games than the corresponding worst-case Price of Anarchy bounds. More specifically, depending on the type of clustering game, our bounds reveal that the Price of Anarchy depends on the maximum subgraph density or the maximum degree of the graph. Among others, these bounds enable us to derive bounds on the Price of Anarchy for clustering games on Erdős-Rényi random graphs. Depending on the graph density, these bounds stand in stark contrast to the known worst-case Price of Anarchy bounds.

Additionally, we characterize the set of distribution rules that guarantee the existence of a pure Nash equilibrium or the convergence of best-response dynamics. These results are of a similar spirit as the work of Gopalakrishnan et al. [19] and complement work of Anshelevich and Sekar [4].

CCS Concepts: • **Theory of computation** → **Quality of equilibria**; **Network games;**

Additional Key Words and Phrases: Price of Anarchy, coordination games, clustering games, random graphs

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1 INTRODUCTION

Clustering games on networks constitute a class of strategic games in which the players are embedded into a network and want to coordinate, or anti-coordinate, their choices with their neighbors. These games capture several key characteristics encountered in applications such as opinion formation, technology adoption, information diffusion, or virus spreading on various types of networks, e.g., the Internet, social networks, and biological networks.

Different variants of clustering games have recently been studied intensively in the algorithmic game theory literature, both with respect to the existence and the inefficiency of equilibria, see, e.g., References [4, 5, 16, 20, 21, 24, 30, 33]. Unfortunately, several of these studies reveal that the strategic choices of the players may lead to equilibrium outcomes that are highly inefficient. Arguably the most prominent notion to assess the inefficiency of equilibria is the *Price of Anarchy (PoA)* [29], which refers to the worst-case ratio of the optimal social welfare and the social welfare of a (pure) Nash equilibrium. It is known that even the most basic clustering games exhibit a large, sometimes even unbounded, Price of Anarchy (see below for details). These negative results naturally trigger the following questions: Is this high inefficiency inevitable in clustering games on networks? Or, can we trace more precisely what causes a large inefficiency? These questions constitute the starting point of our investigations:

Our main goal in this article is to understand how structural properties of the network topology impact the Price of Anarchy in clustering games.

In general, our idea is that a more fine-grained analysis may reveal topological parameters of the network that can be used to derive more accurate bounds on the Price of Anarchy. Given the many applications of clustering games on different types of networks, our hope is that such topological bounds will be more informative than the corresponding worst-case bounds.

Clearly, this hope is elusive for a number of fundamental games on networks whose inefficiency is known to be *independent* of the network topology. Arguably, the most prominent example are selfish routing games whose Price of Anarchy has been analyzed in the seminal works by Roughgarden and Tardos [36] and Roughgarden [35]. But, in contrast to these games, clustering games exhibit a strong *locality property* induced by the network structure, i.e., the utility of each player is affected only by the choices of her direct neighbors in the network. This observation also motivates our choice of quantifying the inefficiency by means of topological parameters rather than other parameters of the game.

In this article, we derive topological bounds on the Price of Anarchy for different classes of clustering games. Our bounds reveal that the Price of Anarchy depends on different topological parameters in the case of symmetric and asymmetric strategy sets of the players: While they exhibit a dependency on the maximum subgraph density for symmetric clustering games, they reveal a dependency on the maximum degree for asymmetric clustering games. Using these topological bounds, we are able to derive improved bounds for certain special graph classes as simple corollaries.

We also use our topological bounds to obtain a precise understanding of the Price of Anarchy of clustering games on Erdős-Rényi random graphs [18]. Our results reveal that, depending on the density of the graph, the Price of Anarchy improves significantly over the known worst-case bounds. To the best of our knowledge, this is also the first work that addresses the inefficiency of equilibria on random graphs.[1]

The applicability of our topological Price of Anarchy bounds is not limited to the class of Erdős-Rényi random graphs. The main reason for using these graphs is that their structural properties

[1] Valiant and Roughgarden [37] study the Braess paradox in large random graphs; but their work has a different focus than ours (see Section 1.3 for more details).

are well-understood. In particular, our topological bounds can be applied directly to any graph class of interest for which the above topological parameters are well-understood.

Apart from our topological Price of Anarchy bounds, we also give a complete characterization of what type of *distribution rules*, which determine how utility generated by two adjacent players in the network is split when they (anti-)coordinate, guarantee the convergence of best-response dynamics in symmetric clustering games. These results extend and complement the results by Gopalakrishnan et al. [19] and Anshelevich and Sekar [4].

Altogether, our results give a more fine-grained view on clustering and coordination games.

1.1 Our Clustering Games

We study a generalization of the unifying model of *clustering games* introduced by Feldman and Friedler [16]: We are given an undirected graph $G = (V, E)$ on $n = |V|$ nodes whose edge set $E = E_c \cup E_a$ is partitioned into a set of *coordination* edges E_c and a set of *anti-coordination* edges E_a. The game is called a *coordination game* if all edges are coordination edges and an *anti-coordination game* (or *cut game*) if all edges are anti-coordination edges. Further, we are given a set $[c] = \{1, \dots, c\}$ of $c > 1$ colors and non-negative edge weights $\boldsymbol{w} = (w_e)_{e \in E}$.[2] Each node i corresponds to a player who chooses a color $s_i \in S_i$ from a set of colors $S_i \subseteq [c]$ that are available to her. We say that the game is *symmetric* if $S_i = [c]$ for all $i \in V$ and *asymmetric* otherwise. An edge $e = \{i, j\} \in E$ is *satisfied* if either (i) it is a coordination edge and both i and j choose the same color or (ii) it is an anti-coordination edge and i and j choose different colors. The goal of player i is to choose a color $s_i \in S_i$ such that the weight of all satisfied edges incident to i is maximized.

We consider a generalization of these games by incorporating additionally: (i) individual player preferences, as in Reference [33], and (ii) different distribution rules, as in Reference [4]: We assume that each player i has a *preference function* $q_i : S_i \rightarrow \mathbb{R}_{\geq 0}$ that encodes her preferences over the colors in S_i. Further, player i has a *split parameter* $\xi_{ij} \geq 0$ for every incident edge $e = \{i, j\}$ that determines the share she obtains from e: If e is satisfied, then i obtains a proportion of $\xi_{ij}/(\xi_{ij} + \xi_{ji})$ of the weight w_e of e. The utility $u_i(\boldsymbol{s})$ of player i with respect to strategy profile $\boldsymbol{s} = (s_1, \dots, s_n)$ is then the sum of the individual preference $q_i(s_i)$ and the total share of all satisfied edges incident to i. We consider the standard utilitarian *social welfare* objective $u(\boldsymbol{s}) = \sum_i u_i(\boldsymbol{s})$. We use $\bar{\xi}_e$ to denote the *disparity* of an edge $e = \{i, j\}$, defined as $\bar{\xi}_e = \max\{\xi_{ij}/\xi_{ji}, \xi_{ji}/\xi_{ij}\}$, and let $\bar{\xi} = \max_{e \in E} \bar{\xi}_e$ refer to the maximum disparity of all edges. We say that the game has the *equal-split distribution rule* if $\bar{\xi} = 1$; equivalently, $\xi_{ij} = \xi_{ji}$ for all $\{i, j\} \in E$.

Our clustering games generalize several other strategic games, which were studied extensively in the literature before. For example, they generalize *max cut games* and *not-all-equal satisfiability games* [20], *max k-cut games* [21], *coordination games* [5], *clustering games* [16], and *anti-coordination games* [30]. In turn, the class of clustering games that we study here can be generalized naturally. We consider several such extensions here. However, as we show, our results do not carry over to any of these (slightly) more general settings. In that sense, our results are best possible with respect to the generality of the class of clustering games considered here. We present these results in Appendix A.

1.2 Our Contributions

We derive results for symmetric and asymmetric clustering games. We only elaborate on our main findings for symmetric clustering games below; our results for the asymmetric case are detailed in Section 5. An overview of the bounds derived in this article is given in Table 1.

[2]We use $[k]$ to denote the set $\{1, \dots, k\}$ for a given integer $k \geq 1$.

Table 1. Overview of Our Topological Price of Anarchy Bounds for Symmetric and Asymmetric Clustering Games

SYMMETRIC CLUSTERING GAMES

Graph topology	**Coord. only**	**Indiv. pref.**	**Distr.** ξ	**Topological PoA** (our bounds)		**PoA** (prev. work)
arbitrary	✗	✓	+	$1+(1+\bar{\xi})\rho(G)$	(Thm. 1)	c [4, 16]
planar	✗	✓	+	$\leq 4+3\bar{\xi}$	(Cor. 2)	
arbitrary	✗	✓	**1**	$1+2\rho(G)$	(Cor. 4)	
arbitrary	✗	✓	**1**	$\leq 2+2\rho(G_c)$	(Thm. 5)	
sparse random	✓	✓	**1**	$\Theta(1)$	(Cor. 6)	
dense random	✓	✗	**1**	$\Omega(c)$	(Thm. 7)	

ASYMMETRIC COORDINATION GAMES

Graph topology	**Coord. only**	**Indiv. pref.**	**Distr.** ξ	(α,k)-**Topological PoA** (our bounds)		(α,k)-**PoA** (prev. work)
arbitrary	✓	✗	**1**	$\leq 2\alpha\Delta(G)$	(Thm. 14)	$\leq 2\alpha\frac{n-1}{k-1}$
arbitrary	✓	✗	**1**	$\geq \alpha(\frac{\Delta(G)}{k-1}-1)$	(Thm. 14)	
dense random	✓	✗	**1**	$\Omega(\alpha n)$		$\geq 2\alpha\frac{n-k}{k-1}+1$
sparse random	✓	✗	**1**	$\Theta\left(\frac{\alpha\ln(n)}{\ln\ln(n)}\right)$	(Thm. 15)	[33]
+ common color	✓	✗	**1**	$O(1)$	(Thm. 17)	

A "+" or "**1**" in the column "Distr. ξ" indicates whether the distribution rule ξ is positive or equal-split, respectively. $\bar{\xi}$ is the maximum disparity, and c is the number of colors. The parameters $\rho(G)$ and $\Delta(G)$ refer to the maximum subgraph density and maximum degree of G, respectively. The stated bounds for random graphs hold with high probability.

1. Topological bounds on the Price of Anarchy. We show that the Price of Anarchy for symmetric clustering games is bounded as a function of the *maximum subgraph density* of G, which is defined as $\rho(G) = \max_{S\subseteq V}\{|E[S]|/|S|\}$, where $|E[S]|$ is the number of edges in the subgraph induced by S. More specifically, we prove that PoA $\leq 1+(1+\bar{\xi})\rho(G)$ and that this bound is tight already for coordination games. Using this topological bound, we are able to show that the Price of Anarchy is at most $4+3\bar{\xi}$ for clustering games on planar graphs and $1+2\rho(G)$ for coordination games with equal-split distribution rule. We also derive a (qualitatively) refined bound of PoA $\leq 2+2\rho(G[E_c])$ for clustering games with equal-split distribution rule. In particular, this bound reveals that the maximum subgraph density with respect to the graph $G[E_c]$ (or simply G_c) induced by the *coordination edges* E_c *only*, is the crucial topological parameter determining the Price of Anarchy.

These bounds provide more refined insights than the known (tight) bound of PoA $\leq c$ (number of colors) on the Price of Anarchy for (i) symmetric coordination games with individual preferences and arbitrary distribution rule [4] and (ii) clustering games without individual preferences and equal-split distribution rule [16], both being special cases of our model. An important point to notice here is that this bound indicates that the Price of Anarchy is unbounded if the number of colors $c = c(n)$ grows as a function of n. In contrast, our topological bounds are *independent* of c. In particular, they provide improved bounds when c is large, while the maximum subgraph density is small. Moreover, our refined bound of $2+2\rho(G[E_c])$ mentioned above provides a nice qualitative bridge between the facts that for max-cut (or anti-coordination) games the Price of Anarchy is known to be constant, whereas for coordination games the Price of Anarchy might grow large.

2. Price of Anarchy for Random Coordination Games. By using our topological bounds, we are able to derive bounds on the Price of Anarchy for coordination games on random graphs. We focus on the *Erdős-Rényi random graph model* [18], also known as the $G(n,p)$-model, where each graph consists of n nodes and every edge is present independently with probability $p \in [0,1]$. More specifically, we show that with high probability the Price of Anarchy is constant for coordination games on sparse random graphs, i.e., $p = d/n$ for some constant $d > 0$, with equal-split distribution rule. In contrast, we show that with high probability the Price of Anarchy remains $\Omega(c)$ for dense random graphs, i.e., $p = d$ for some constant $0 < d \leq 1$. We leave it as an interesting open question to understand what happens for p in the intermediate regime between the sparse and dense extremes.

Note that our constant bound on the Price of Anarchy for sparse random graphs stands in stark contrast to the deterministic bound of PoA $= c$ [4, 16], which could increase with the size of the network. However, our bound for dense random graphs reveals that we cannot significantly improve upon this bound through randomization of the graph topology.

It is worth mentioning that all our results for random graphs hold against an *adaptive adversary* who can fix the input of the clustering game *knowing* the realization of the random graph. To obtain these results, we need to exploit some deep probabilistic results on the maximum subgraph density and the existence of perfect matchings in random graphs.

3. Convergence of Best-response Dynamics. In general, pure Nash equilibria are not guaranteed to exist for clustering games with *arbitrary* distribution rules ξ, even if the game is symmetric [4]. While some sufficient conditions for the existence of pure Nash equilibria, or the convergence of best-response dynamics, are known (see also Reference [4]), a complete characterization is elusive so far.

In this work, we instead obtain a complete characterization of the class of distribution rules that guarantee the convergence of best-response dynamics in clustering games on a fixed network topology. We prove that best-response dynamics converges if and only if ξ is a *generalized weighted Shapley distribution rule*. Our proof relies on the fact that there needs to be some form of *cyclic consistency* similar to the one used in Reference [19]. In fact, our characterization results regarding the existence of pure Nash equilibria and convergence of best-response dynamics are conceptually similar to the work of Chen et al. [11] and Gopalakrishnan et al. [19]. We refer to Section 4 for more details.

Prior to our work, the existence of pure Nash equilibria was known for certain special cases of coordination games only, namely, for symmetric coordination games with individual preferences and $c = 2$ [4], and for symmetric coordination games without individual preferences [16]. To the best of our knowledge, this is the first characterization of distribution rules in terms of best-response dynamics, which, in particular, applies to the settings in which pure Nash equilibria are guaranteed to exist for every distribution rule [4, 16].

1.3 Related Work

The literature on clustering and coordination games is vast; we only include references relevant to our model here. The proposed model above is a mixture of special cases of existing models considered in References [4, 5, 16, 33].

Anshelevich and Sekar [4] consider symmetric coordination games with individual preferences and (general) distribution rules. They show existence of *α-approximate k-strong equilibria*, *(α, k)-equilibria* for short, for various combinations; in particular, $(2, k)$-equilibria always exist for any k. Moreover, they show that the number of colors c is an upper bound on the PoA. Apt et al. [5] study asymmetric coordination games with unit weights, zero individual preferences, and equal-split distribution rules. They derive an almost complete picture of the existence of $(1, k)$-equilibria for different values of c. Feldman and Friedler [16] introduce a unified framework (as introduced above) for studying the (strong) Price of Anarchy in clustering games with individual preferences set to zero and equal-split distribution rules. In particular, they show that the number of colors is an upper bound on the PoA and that $2(n-1)/(k-1)$ is an upper bound on the $(1, k)$-PoA. Rahn and Schäfer [33] consider the more general setting of polymatrix coordination games with equal-split distribution rule, of which our asymmetric coordination games with individual preferences are a special case. They show a bound of $2\alpha(n-1)/(k-1)$ on the (α,k)-PoA and that an (α,k)-equilibrium is guaranteed to exist for any $\alpha \geq 2$ and any k.

There is also a vast literature on different variants of anti-coordination (or cut) games; see, e.g., References [21, 24] and the references therein, which are also captured by our clustering games. In a recent paper, Carosi and Monaco [10] consider so-called *k-coloring games.* Moreover, clustering and coordination games were also studied on directed graphs [5, 9]. Finally, certain coordination and clustering games can be seen as special cases of hedonic games [14]; we refer the reader to Reference [7] for, in particular, a survey of recent literature on (fractional) hedonic games. Identifying topological inefficiency bounds for these type of games, as well as for clustering games on directed graphs, could be an interesting direction for future work.

Regarding the study of the inefficiency of equilibria on random graphs, closest to our work seems to be the work by Valiant and Roughgarden [37]. They study the Braess paradox on large Erdős-Rényi random graphs and show that for certain settings the Braess paradox occurs with high probability as the size of the network grows large. The study of randomness in games has also received some attention in other settings; see, e.g., References [2, 6]. These are mostly settings with small strategy sets and random utility functions and are not comparable with ours. We only focus on randomness in the graph topology of the game.

In the case of equal-split distribution rules, our clustering games can also be modelled as congestion games [34]. The inefficiency of pure Nash equilibria in congestion games has received a lot of attention; see, e.g., References [1, 8, 12, 13, 26, 28] and references therein. However, none of these results are directly applicable to the clustering games considered in this work. Finally, our games are also a special case of so-called *distributed welfare games* as studied, e.g., by Marden and Wierman [32].

2 PRELIMINARIES

An instance of a *clustering game* is given by $\Gamma = (G, c, \boldsymbol{S}, \boldsymbol{\xi}, \boldsymbol{w}, \boldsymbol{q})$, where:

- $G = (V, E)$ is an undirected graph whose set of edges $E = E_c \cup E_a$ is partitioned into coordination edges E_c and anti-coordination edges E_a;
- $c \geq 2$ is the total number of colors that are available;
- $\boldsymbol{S} = (S_i)_{i \in V} = \times_{i \in V} S_i$ is the Cartesian product of the strategy sets, where $S_i \subseteq [c]$ with $|S_i| \geq 2$ is the subset of colors available to player $i \in V$;
- $\boldsymbol{\xi} = (\xi_{ij})$ is the *distribution rule*, where $\xi_{ij} \geq 0$ specifies a split parameter for every player $i \in V$ and every incident edge $\{i, j\} \in E$;
- $\boldsymbol{w} = (w_e)_{e \in E}$ specifies the edge weights, where $w_e \geq 0$ is the weight of edge $e \in E$;
- $\boldsymbol{q} = (q_i)_{i \in V}$ defines the players' individual preferences, where $q_i : S_i \to \mathbb{R}_{\geq 0}$ is the individual preference function of player $i \in V$.

Whenever we refer to a *clustering game* below, we assume that all of the above input parameters are non-trivial; we specify the respective restrictions otherwise.

Each node $i \in V$ corresponds to a player whose goal is to choose a color $s_i \in S_i$ from the set of colors available to her to maximize her utility function u_i. Given a strategy profile $\boldsymbol{s} = (s_1, \ldots, s_n) \in \boldsymbol{S}$, the utility of player i is defined as

$$u_i(\boldsymbol{s}) = q_i(s_i) + \sum_{\{i,j\} \in E_c : s_i = s_j} \frac{\xi_{ij}}{\xi_{ij} + \xi_{ji}} \cdot w_{ij} + \sum_{\{i,j\} \in E_a : s_i \neq s_j} \frac{\xi_{ij}}{\xi_{ij} + \xi_{ji}} \cdot w_{ij}.$$

We say that an edge $\{i,j\} \in E$ is *satisfied* in a strategy profile $\boldsymbol{s} \in \boldsymbol{S}$ if either (i) $s_i = s_j$ and $\{i,j\}$ is a coordination edge, or (ii) $s_i \neq s_j$ and $\{i,j\}$ is an anti-coordination edge.

We assume that the distribution rule $\boldsymbol{\xi}$ satisfies $\xi_{ij} + \xi_{ji} > 0$ for every edge $e = \{i,j\} \in E$; in particular, not both i and j have a zero split for edge e. We say that $\boldsymbol{\xi}$ is *positive* if $\xi_{ij} > 0$ and $\xi_{ji} > 0$ for all $e = \{i,j\} \in E$; we also write $\boldsymbol{\xi} > \boldsymbol{0}$. Further, $\boldsymbol{\xi}$ is called the *equal-split* distribution rule if $\xi_{ij} = \xi_{ji}$ for all $e = \{i,j\} \in E$; we also indicate this by $\boldsymbol{\xi} = \boldsymbol{1}$. The *disparity* of an edge $e = \{i,j\}$ is defined as $\bar{\xi}_e = \max\{\xi_{ij}/\xi_{ji}, \xi_{ji}/\xi_{ij}\}$ and we use $\bar{\xi} = \max_{e \in E} \bar{\xi}_e$ to denote the maximum disparity.

We say that the clustering game is *symmetric* if $S_i = [c]$ for every player $i \in V$ and *asymmetric* otherwise. If we focus on symmetric clustering games, then we omit the explicit reference of the strategy sets $\boldsymbol{S} = \times_{i \in V} S_i$ with $S_i = [c]$. A clustering game is called a *coordination game* if $E_a = \emptyset$ and an *anti-coordination game*, or *cut game*, if $E_c = \emptyset$. We use $n = |V|$ to refer to the number of players.

A strategy profile $\boldsymbol{s} = (s_1, \ldots, s_n) \in \boldsymbol{S}$ is an *α-approximate k-strong equilibrium* with $\alpha \geq 1$ and $k \in [n]$, or *(α,k)-equilibrium* for short, if for every set of players $K \subseteq V$ with $|K| \leq k$ and every deviation $\boldsymbol{s}'_K = (s'_i)_{i \in K}$, there is at least one player $j \in K$ such that $\alpha \cdot u_j(\boldsymbol{s}) \geq u_j(\boldsymbol{s}_{-K}, s'_K)$. That is, for any joint deviation of the players in K from strategy profile $\boldsymbol{s}$, there is at least one player that cannot improve her utility by more than a factor α.

Let (α,k)-NE(Γ) be the set of all (α,k)-equilibria of a game Γ. The *(α,k)-Price of Anarchy* of Γ is then defined as

$$(\alpha,k)\text{-PoA}(\Gamma) = \max_{s \in (\alpha,\, k)\text{-NE}(\Gamma)} \frac{u(\boldsymbol{s}^*)}{u(\boldsymbol{s})},$$

where $\boldsymbol{s}^*$ a strategy profile maximizing the *social welfare* objective $u(\boldsymbol{s}) = \sum_{i \in V} u_i(\boldsymbol{s})$. For a class of clustering games $\mathcal{G}$ the *(α,k)-Price of Anarchy* is given by (α,k)-PoA$(\mathcal{G}) = \sup_{\Gamma \in \mathcal{G}} (\alpha,k)$-PoA$(\Gamma)$. We only consider pairs (α,k) for which (α,k)-NE$(\Gamma) \neq \emptyset$ for all $\Gamma \in \mathcal{G}$. When $\alpha = 1$ and $k = 1$, we simply write PoA$(\cdot)$ instead of $(1,1)$-PoA$(\cdot)$.

2.1 Graph Theory

We consider undirected simple graphs $G = (V,E)$ where $E = E_c \cup E_a \subseteq \{\{i,j\} : i,j \in V\}$ is a partition of the edges in coordination and anti-coordination edges. We usually write $n = |V|$. For a subset $F \subseteq E$, we write $G[F]$ for the subgraph of G formed by the edges of F. For a subset $S \subseteq V$, we write $G[S] = (S, E[S])$ for the induced subgraph on S, where $E[S] = \{\{i,j\} \in E : i,j \in S\}$. We say that a node $i \in V$ is adjacent to an edge $e \in E$ if $i \in e$. A graph is *complete* if $E = \{\{i,j\} : i,j \in V\}$. Furthermore, a graph is *triangle-free* if it contains no complete induced subgraph on three nodes. Finally, a graph is *planar* if, informally speaking, it can be drawn in $\mathbb{R}^2$ without crossings; see, e.g., Reference [38] for a formal definition.

A *matching* $M \subseteq E$ is a collection of edges so every node in V is adjacent to at most one edge in M. A *perfect matching* M is a collection of edges such that every node in V is adjacent to precisely one edge; in particular, this means that $|M| = n/2$. A *maximum matching* is a matching so no other matching in G has larger cardinality.

The degree of a node $i \in V$ is defined as $d(i) = |\{j : \{i,j\} \in E\}|$ and the *maximum degree* of a graph G is defined as

$$\Delta(G) = \max_{i \in V} d(i).$$

The *maximum subgraph density* of a graph G is defined as

$$\rho(G) = \max_{S \subseteq V} \left\{ \frac{|E[S]|}{|S|} \right\}.$$

2.2 Random Clustering Games

In our probabilistic framework to study the Price of Anarchy of random clustering games, we use the well-known *Erdős-Rényi random graph model* [18], denoted by $G(n, p)$: There are n nodes and every undirected edge is present independently with probability $p = p(n) \in [0, 1]$. Although this model was first introduced by Gilbert, it is often referred to as the *Erdős-Rényi random graph model.* We say that a random graph is *sparse* if $p = d/n$ for some constant $d > 0$, and it is *dense* if $p = d$ for some constant $0 < d < 1$. In this article, we focus on random graph instances with equal-split distributions rules. Some of our results naturally extend to more general distribution rules, but we omit the details here, because they do not provide additional insights.

We continue with defining the Price of Anarchy for games with a random graph topology. Fix some probability $p = p(n) \in [0, 1]$ and let $\beta = \beta(n, c(n))$ be a given function. Define $\mathcal{G}_{G_n} = \{\Gamma : \Gamma = (G_n, c(n), \boldsymbol{\xi}, \boldsymbol{w}, \boldsymbol{q})\}$ as the set of all clustering games on random graph $G_n \sim G(n, p)$ with at most $c(n)$ available colors. We say that the *Price of Anarchy for random clustering games is at most β with high probability*, or *PoA*$(\mathcal{G}_{G_n}) \leq \beta$ for short, if

$$\mathbb{P}_{G_n \sim G(n,p)} \left(\text{PoA}(\mathcal{G}_{G_n}) \leq \beta\right) \geq 1 - o(1),$$

where the asymptotics in $o(1)$ is with respect to $n \to \infty$. We use a similar definition if we want to lower bound the Price of Anarchy.

Finally, for a constant β independent of n and c, we say that the *Price of Anarchy for random clustering games is β with high probability*, or *PoA*$(\mathcal{G}_{G_n}) \to \beta$ *for short*, if for all $\delta > 0$

$$\mathbb{P}_{G_n \sim G(n,p)} \left(|\text{PoA}(\mathcal{G}_{G_n}) - \beta| \leq \delta\right) \geq 1 - o(1),$$

where again the asymptotics in $o(1)$ is with respect to $n \to \infty$.

All our results for clustering games on random graphs hold with high probability.

2.3 Shapley Distribution Rules

We adapt the definition of Shapley distribution rules for resource allocation games [19] to our setting. A distribution rule $\boldsymbol{\xi}$ corresponds to a *generalized weighted Shapley distribution rule* if and only if there exists a permutation σ of the players in V and weight vector $\boldsymbol{\gamma} \in \mathbb{R}^V_{\geq 0}$ such that the following two conditions are satisfied for every edge $e = \{i, j\}$:

(i) If $\xi_{ij} = 0$, then $\sigma(i) < \sigma(j)$.
(ii) If $\xi_{ij}, \xi_{ji} > 0$, then $\frac{\xi_{ij}}{\xi_{ij}+\xi_{ji}} = \frac{\gamma_i}{\gamma_i+\gamma_j}$.

If all weights are strictly positive, then the resulting distribution rule is a *weighted Shapley distribution rule.* If $\gamma_i = \gamma_j$, then for all $i, j \in V$ the resulting distribution rule is an *unweighted Shapley distribution rule.* Note that this case corresponds to an equal-split distribution rule.

3 REFINED BOUNDS ON THE PRICE OF ANARCHY

In this section, we first establish our topological bound on the Price of Anarchy for symmetric clustering games and then use it to derive new bounds for some special cases as well as random clustering games.

3.1 Topological Price of Anarchy Bound

Our topological bound depends on the *maximum subgraph density* of G, which is defined as $\rho(G) = \max_{S \subseteq V} \{|E[S]|/|S|\}$, where $|E[S]|$ is the number of edges in the subgraph induced by S. Recall that $\bar{\xi}$ refers to the maximum disparity.

Theorem 1 (Density Bound). *Let $\Gamma = (G, c, \boldsymbol{\xi}, \boldsymbol{w}, \boldsymbol{q})$ be a symmetric clustering game with $\boldsymbol{\xi} > 0$. If a pure Nash equilibrium exists for the game Γ, then $PoA(\Gamma) \leq 1 + (1 + \bar{\xi})\rho(G)$ and this bound is tight in general.*

Proof. Let $\boldsymbol{s}$ and $\boldsymbol{s}^*$ be a Nash equilibrium and a social optimum, respectively. Consider an edge $\{i, j\} \in E$ and assume without loss of generality that $u_i(\boldsymbol{s}) \leq u_j(\boldsymbol{s})$, so $u_i(\boldsymbol{s}) = \min\{u_i(\boldsymbol{s}), u_j(\boldsymbol{s})\}$. If $\{i, j\}$ is a coordination edge, then $u_i(\boldsymbol{s}) \geq u_i(\boldsymbol{s}_{-i}, s_j) \geq \frac{\xi_{ij}}{\xi_{ij}+\xi_{ji}} \cdot w_{ij}$, where $(\boldsymbol{s}_{-i}, s_j)$ is the strategy profile in which player i deviates to the color of player j and all other players play according to $\boldsymbol{s}$. Suppose $\{i, j\}$ is an anti-coordination edge. If $s_i \neq s_j$, then we trivially have $u_i(\boldsymbol{s}) \geq \frac{\xi_{ij}}{\xi_{ij}+\xi_{ji}} \cdot w_{ij}$ by non-negativity of the weights and individual preferences. If $s_i = s_j$, then the same inequality holds by using the Nash condition for some color that is not s_j. Recall that such a color exists, because we assume that $|S_i| \geq 2$ for all i. In either case, we conclude that

$$w_{ij} \leq \left(1 + \frac{\xi_{ji}}{\xi_{ij}}\right) u_i(\boldsymbol{s}) \leq \left(1 + \max_{e \in E} \bar{\xi}_e\right) u_i(\boldsymbol{s}) = (1 + \bar{\xi}) u_i(\boldsymbol{s}). \tag{1}$$

Moreover, by exploiting that $\boldsymbol{s}$ is a Nash equilibrium and the non-negativity of the edge weights, we obtain for every $i \in V$, $u_i(\boldsymbol{s}) \geq u_i(\boldsymbol{s}_{-i}, s_i^*) \geq q_i(s_i^*)$.

Using that the sum of the weights of all satisfied edges in $\boldsymbol{s}^*$ is at most the sum of all edge weights, we obtain

$$u(\boldsymbol{s}^*) \leq \sum_{i \in V} q_i(s_i^*) + \sum_{e=\{i,j\} \in E} w_{ij} \leq \sum_{i \in V} u_i(\boldsymbol{s}) + (1 + \bar{\xi}) \sum_{\{i,j\} \in E} \min\{u_i(\boldsymbol{s}), u_j(\boldsymbol{s})\}.$$

If we can find a value M such that

$$\sum_{\{i,j\} \in E} \min\{u_i(\boldsymbol{s}), u_j(\boldsymbol{s})\} \leq M \cdot \sum_{i \in V} u_i(\boldsymbol{s}), \tag{2}$$

then it follows that $u(\boldsymbol{s}^*) \leq (1 + (1 + \bar{\xi}) \cdot M) u(\boldsymbol{s})$. We show that $M = \max_{S \subseteq V}\{|E[S]|/|S|\}$ satisfies Equation (2).

Let $N(i) = \{j \in V \,:\, \{i, j\} \in E\}$ be the set of neighbors of i. Define

$$m_i = \left|\{j \in N(i) \,:\, (u_i(\boldsymbol{s}) < u_j(\boldsymbol{s})) \text{ or } (u_i(\boldsymbol{s}) = u_j(\boldsymbol{s}) \text{ and } i < j)\}\right|$$

and note that $\sum_{i \in V} m_i = |E|$. We can assume without loss of generality that $\sum_{i \in V} u_i(\boldsymbol{s}) = 1$, since the expression in Equation (2) is invariant under multiplication with a constant positive scalar. Moreover, the players may be renamed such that $u_1(\boldsymbol{s}) \leq u_2(\boldsymbol{s}) \leq \cdots \leq u_n(\boldsymbol{s})$.

We continue by showing that M is an upper bound for the linear program (P) given below, in which the variables are the $u_i = u_i(\boldsymbol{s})$, and the m_i are considered constants. The dual program (D) is given on the right.

$$\begin{array}{lll}
\text{(P)} & \max & \sum_{i \in V} u_i m_i \\
 & \text{s.t.} & u_1 + u_2 + \cdots + u_n = 1 \\
 & & 0 \leq u_1 \leq u_2 \leq \cdots \leq u_n
\end{array}
\qquad
\begin{array}{lll}
\text{(D)} & \min & z \\
 & \text{s.t.} & -\pi_i + \pi_{i+1} + z = m_i \quad \text{for } i = 1, \ldots, n-1 \\
 & & -\pi_n + z = m_n \\
 & & \pi_i \geq 0 \quad \text{for } i = 1, \ldots, n \\
 & & z \in \mathbb{R}
\end{array}.$$

We now construct a feasible dual solution for (D). Set

$$z^* = \max_{l \in V} \left\{ \frac{\sum_{i=l}^{n-1} m_i}{n - l} \right\}.$$

We will often use that $(n - l)z^* \geq \sum_{i=l}^{n-1} m_i$ for any fixed l. In particular, with $l = n - 1$, we find $z^* \geq m_n$, so $\pi_n^* := z^* - m_n \geq 0$. Then, we define $\pi_{n-1}^* := \pi_n^* + z^* - m_{n-1} = 2z^* - (m_{n-1} + m_n) \geq 0$.

Using induction, it then easily follows that $\pi_i^* := \pi_{i+1}^* + z^* - m_i \geq 0$ for all $i = 1, \dots, n-2$ as well. We have constructed a feasible dual solution with objective function value z^*. Using weak duality, it follows that for any feasible primal solution $\boldsymbol{u} = (u_1, \dots, u_n)$, we have

$$\sum_{\{i,j\} \in E} u_i m_i \leq \max_{l \in V} \left\{ \frac{\sum_{i=l}^{n-1} m_i}{n-l} \right\} \leq \max_{S \subseteq V} \left\{ \frac{|E[S]|}{|S|} \right\},$$

since the term in middle is precisely the density of the induced subgraph on the nodes $l, \dots, n$. This completes the proof of the upper bound.

We continue with showing tightness, even for coordination games. Let $G = (L \cup R, E)$ be a complete bipartite graph between node-sets L and R, with $|L| = \ell$ and $|R| = r$, and assume that all edges in E are coordination edges. We show tightness using a weighted Shapley distribution rule. That is, for any value $\gamma = \max_{\{i,j\} \in E} \bar{\xi}_{ij}$, there is also some weighted Shapley distribution rule that attains this value. The nodes in L get a fixed weight γ, with $\gamma \geq 1$, and the nodes in R get a fixed weight 1.

We define $C = A \cup B \cup \{c_0\}$ where A contains colors $\{a_1, \dots, a_\ell\}$ and $B = \{b_1, \dots, b_r\}$. Assume that every $i \in L$ has an individual preference of $q_i(a_i) = q_i(c_0) = \gamma/(1+\gamma)$ for colors a_i and c_0, and every player $j \in R$ an individual preference of $q_j(b_j) = q_j(c_0) = 1/(1+\gamma)$ for colors b_j and c_0, and all other individual preferences for a player k and color $\bar{c}$ are $q_k(\bar{c}) = 0$. Furthermore, all edge weights are set to $w_{ij} = 1$. Consider the strategy profile $\boldsymbol{s}$ in which player $i \in L$ plays a_i, and $j \in R$ plays b_j. This profile is a Nash equilibrium with $u(\boldsymbol{s}) = \ell \cdot \gamma/(1+\gamma) + r \cdot 1/(1+\gamma)$. A social optimum evolves when every player plays color c_0. The resulting profile $\boldsymbol{s}^*$ has social welfare

$$u(\boldsymbol{s}^*) = \ell \cdot \frac{\gamma}{1+\gamma} + r \cdot \frac{1}{1+\gamma} + r \cdot \ell.$$

Here, the first two terms arise from the individual preferences of the nodes in L and R, respectively. The last term arises because coordination takes place between all pairs of nodes $(\ell, r) \in L \times R$; remember that we consider a complete bipartite graph. It then follows that

$$\frac{u(\boldsymbol{s}^*)}{u(\boldsymbol{s})} = 1 + \frac{r \cdot \ell}{\ell \cdot \gamma/(1+\gamma) + r \cdot 1/(1+\gamma)}.$$

By letting $r \to \infty$, we find a lower bound of $1 + \ell \cdot (1+\gamma)$. Note that for ℓ and r fixed, the densest subgraph is the whole graph and has density $r\ell/(\ell + r)$ that converges to ℓ as $r \to \infty$. □

We use our topological bound to derive deterministic bounds on the Price of Anarchy for two special cases of clustering games. Note that these bounds cannot be deduced from References [4, 16].

Corollary 2 (Planar Clustering Games). *Let $\Gamma = (G, c, \boldsymbol{\xi}, \boldsymbol{w}, \boldsymbol{q})$ be a symmetric clustering game on a planar graph G with $\boldsymbol{\xi} > 0$. If a pure Nash equilibrium exists for the game Γ, then PoA$(\Gamma) \leq 4 + 3\bar{\xi}$.*

Proof. By Euler's formula, $|E(H)| \leq 3|V(H)| - 6 \leq 3|V(H)|$ for any planar graph H, see, e.g., Reference [38, Corollary 13.4(i)]. Further, any induced subgraph H of a planar graph G is again planar, and, hence, the maximum subgraph density is also at most 3. Using this in Theorem 1 proves the claim for general planar graphs. □

For planar graphs that are also triangle-free, we can give a slightly better bound. The result in Corollary 3 also shows that the linear dependence on $\bar{\xi}$ in Corollary 2 is necessary.

Corollary 3 (Planar Triangle-free Clustering Games). *Let $\Gamma = (G, c, \boldsymbol{\xi}, \boldsymbol{w}, \boldsymbol{q})$ be a symmetric clustering game on a triangle-free planar graph G with $\boldsymbol{\xi} > 0$. If a pure Nash equilibrium exists for the game Γ, then PoA$(\Gamma) \leq 3 + 2\bar{\xi}$ and this bound is tight in general.*

Proof. It is known that $|E(H)| \leq 2|V(H)| - 4 \leq 2|V(H)|$ for any triangle-free planar graph H; see, e.g., Reference [38, Corollary 13.4(ii)]. Also here, any induced subgraph H of a triangle-free planar graph G is again triangle-free and planar. This then yields an upper bound of $1 + 2(1 + \bar{\xi}) = 3 + 2\bar{\xi}$. To show tightness, we can use a similar construction as in the proof of Theorem 1. We rely on the fact that for any ℓ and r there exists a bipartite (and therefore triangle-free) planar graph $G = (L \cup R, E)$ with $L = \{x_1, \ldots, x_\ell\}$ and $R = \{y_1, \ldots, y_r\}$, for which $|E| = 2(r + \ell) - 4$; see, e.g., Reference [15]. The graph can be constructed by connecting both x_1 and x_2 to all nodes in R and in addition also $x_3, \ldots, x_\ell$ to both r_1 and r_2. This gives in total $2r + 2(\ell - 2) = 2(r + \ell) - 4$ edges. One can then use exactly the same construction in Theorem 1, which now gives for the ratio between the utilities of the Nash equilibrium $\boldsymbol{s}$ and the social optimum $\boldsymbol{s}^*$

$$\frac{u(\boldsymbol{s}^*)}{u(\boldsymbol{s})} = 1 + \frac{2(r + \ell) - 4}{\ell \cdot \gamma/(1 + \gamma) + r \cdot 1/(1 + \gamma)},$$

i.e., we have replaced the factor $r\ell$, the total number of edges in a complete bipartite graph, with $2(r + \ell) - 4$. By letting $r \to \infty$, we find a lower bound of $1 + 2(1 + \gamma)$ that gives the desired result. □

Corollary 4 (Equal-split Coordination Games). *Let G be a given undirected graph, and let $\mathcal{G}_G$ be the set of all symmetric coordination games $\Gamma = (G, c, \mathbf{1}, \boldsymbol{w}, \boldsymbol{q})$ with equal-split distribution rule on G. Then $PoA(\mathcal{G}_G) = 1 + 2\rho(G)$.*

We emphasize that the bound in Corollary 4 is tight on *every* fixed graph topology G, rather than only in the *value* of $\rho(G)$.

Proof of Corollary 4. The upper bound follows directly from Theorem 1. We prove the lower bound by constructing an instance of a coordination game as follows: Let $S \subseteq V$ be arbitrary and consider the induced subgraph on S. Assume without loss of generality that $S = \{1, \ldots, \sigma\}$ with $\sigma = |S|$. Define the set of colors as $C = \{c_1, \ldots, c_\sigma\} \cup \{c_0\}$. We give every player $i \in S$ an individual preference of one for colors c_i and c_0 and zero for all other colors. Further, the individual preferences of all nodes in $V \setminus S$ are set to zero. The weight of all edges in $E[S]$ is set to 2 and the weight of all edges in $E \setminus E[S]$ is set to zero.

Consider a strategy profile $\boldsymbol{s}$ in which every player $i \in S$ chooses color c_i and every player $i \notin S$ chooses an arbitrary color. Then $\boldsymbol{s}$ is a Nash equilibrium with social welfare $u(\boldsymbol{s}) = |S|$. However, the strategy profile $\boldsymbol{s}^*$ in which every player chooses color c_0 is a social optimum with social welfare $u(\boldsymbol{s}^*) = |S| + 2|E[S]|$. This implies that $u(\boldsymbol{s}^*)/u(\boldsymbol{s}) = 1 + 2|E[S]|/|S|$. The result now follows by choosing S as a subset of maximum subgraph density. □

It is known that the Price of Anarchy of anti-coordination games is 2 (see, e.g., Reference [24]), which is not reflected by our bound in Theorem 1. Intuitively, this suggests that a large Price of Anarchy is caused by the coordination edges of the graph. Theorem 5 reveals that this intuition is correct: It shows that the maximum subgraph density with respect to the *coordination edges only* is the determining topological parameter. Note that it captures the bound of 2 for anti-coordination games.

Theorem 5. *Let $\Gamma = (G, c, \mathbf{1}, \boldsymbol{w}, \boldsymbol{q})$ be a symmetric clustering game with equal-split distribution rule. Then*

$$1 + 2\rho(G[E_c]) \leq PoA(\Gamma) \leq 2 + 2\rho(G[E_c]),$$

where $G[E_c]$ is the subgraph induced by the coordination edges E_c.

Proof. The proof is a modification of the proof of Theorem 1. Let s be a Nash equilibrium and s^* a socially optimal strategy profile. For notational convenience, we write $u_i = u_i(s)$ for $i \in V$. The proof relies on the following two claims:

Claim 1. *For any coordination edge $\{a, b\} \in E_c$, it holds that*

$$w_{ab} \leq 2\min\{u_a(s), u_b(s)\}. \tag{3}$$

Proof. Assume without loss of generality that $u_a(s) \leq u_b(s)$. Then

$$u_a(s) \geq u_i(s_{-a}, s_b) \geq \frac{1}{2} w_{ab},$$

where (s_{-a}, s_b) is the strategy profile in which player a deviates to the color of player b and all others play their strategy in s. Rewriting gives $w_{ab} \leq 2u_a(s)$. □

Claim 2. [3] *It holds that*

$$\sum_{i \in V} q_i(s_i^*) + \sum_{\{i,j\} \in E_a} w_{ij} \leq 2 \sum_{i \in V} u_i(s). \tag{4}$$

Proof. First note that for the Nash equilibrium s is holds that

$$u_i(s) \geq u_i(s_{-i}, s_i^*) \geq q_i(s_i^*) + \frac{1}{2} \sum_{j:\{i,j\} \in E_a, s_j \neq s_i^*} w_{ij}. \tag{5}$$

Also, for every player i and some fixed color ℓ_i with $\ell_i \neq s_i^*$, it holds that

$$u_i(s) \geq u_i(s_{-i}, \ell_i) \geq \frac{1}{2} \sum_{j:\{i,j\} \in E_a, s_j \neq \ell_i} w_{ij} \geq \frac{1}{2} \sum_{j:\{i,j\} \in E_a, s_j = s_i^*} w_{ij}. \tag{6}$$

The last inequality is true, as $\{j : \{i,j\} \in E_a, s_j = s_i^*\} \subseteq \{j : \{i,j\} \in E_a, s_j \neq \ell_i\}$, because $s_i^* \neq \ell_i$. Adding up Equations (5) and (6) yields

$$2u_i(s) \geq q_i(s_i^*) + \frac{1}{2} \sum_{j:\{i,j\} \in E_a} w_{ij}. \tag{7}$$

Adding up Equation (7) for every player i then yields Equation (4). To see this, one should observe that every edge $\{i, j\} \in E_a$ appears in precisely two summations, that of player i and j. □

Combining Equations (3) and (4), we find

$$\begin{aligned} u(s^*) &\leq \sum_{i \in V} q_i(s_i^*) + \sum_{\{i,j\} \in E_c} w_{ij} + \sum_{\{i,j\} \in E_a} w_{ij} \leq 2 \cdot \sum_{i \in V} u_i(s) + \sum_{\{i,j\} \in E_c} 2 \cdot \min\{u_i(s), u_j(s)\} \\ &\leq 2 \cdot u(s) + 2 \cdot \max_{S \subseteq V} \left\{ \frac{|E_c[S]|}{|S|} \right\} u(s), \end{aligned} \tag{8}$$

where the final step follows from similar arguments as in the proof of Theorem 1.

The lower bound can be achieved using a similar construction as in the proof of Corollary 4. □

[3] We are grateful to Dr. Wennan Zhu for suggesting this claim and for its proof as given here.

3.2 Price of Anarchy for Random Coordination Games

We now turn to our bounds for random coordination games. Recall that, for random graphs, we consider equal-split distribution rules only. We first show that for sparse random graphs the Price of Anarchy is constant with high probability.

Corollary 6 (Sparse Random Coordination Games). *Let $d > 0$ be a constant. Let $\mathcal{G}_{G_n}$ be the set of all symmetric coordination games $\Gamma = (G_n, c, \mathbf{1}, \boldsymbol{w}, \boldsymbol{q})$ on graph $G_n \sim G(n, d/n)$ with equal-split distribution rule. Then there is a constant $\beta = \beta(d)$ such that PoA $(\mathcal{G}_{G_n}) \to \beta$.*

Proof. Anantharam and Salez [3] prove that the maximum subgraph density of a random graph G_n approaches a constant $\beta = \beta(d)$ with high probability; approximations of this constant can be found in Reference [22]. Combining this with the bound in Corollary 4 proves the claim. □

As we show in Theorem 7, the result of Corollary 6 does not hold for sufficiently dense random graphs if the number of available colors grows large.

Theorem 7 (Dense Random Coordination Games). *Let $(c_n)_{n\in\mathbb{N}} \to \infty$ be a sequence of available colors and let $0 < d \leq 1$ be a constant independent of n. Let $\mathcal{G}_{G_n}(c_n)$ be the set of all symmetric coordination games $\Gamma = (G_n, c_n, \mathbf{1}, \boldsymbol{w}, \mathbf{0})$ on graph $G_n \sim G(n, d)$ with c_n common colors, equal-split distribution rule, and no individual preferences. Then there is a constant $\beta = \beta(d)$ such that PoA $(\mathcal{G}_{G_n}(c_n)) \geq \beta c_n$.*

We note that this lower bound holds even for coordination games without individual preferences (as studied in Reference [16]). Basically, this bound implies that, for dense graph topologies, we cannot significantly improve upon the Price of Anarchy bound of c by References [4, 16], even if we randomize the graph topology.

Proof of Theorem 7. We first construct a deterministic instance Γ with Price of Anarchy $\Omega(c_n)$ and then show that we can embed this construction into a random graph with high probability.

Consider a graph $G = (V, E)$ and let c be the number of available colors. Let $M = \{e_1, \ldots, e_\ell\} \subseteq E$ be a matching of size at most c. Let V_M be the set of nodes that are matched in M. Define the weight of an edge $e \in E$ as

$$w_e = \begin{cases} 2 & \text{if } e \in M \\ 1 & \text{if } e \in E[V_M] \setminus M \\ 0 & \text{otherwise} \end{cases},$$

where $E[V_M]$ is the set of edges of the induced subgraph on $V_M = \{i : i \in e \text{ for some } e \in M\}$, i.e, the induced subgraph of the nodes that are matched in M.

Consider the strategy profile $\boldsymbol{s}$ in which the nodes in e_a play color a, for $a = 1, \ldots, \ell$, and all other nodes play an arbitrary color, say, color ℓ. Note that ℓ distinct colors are used in this profile, which is possible because $\ell \leq c$ by assumption. Furthermore, we have $u(\boldsymbol{s}) = 2\ell$, as every edge in M is satisfied, and therefore contributes 1 to the social welfare of $\boldsymbol{s}$. We claim that $\boldsymbol{s}$ is a pure Nash equilibrium. To see this, first note that all nodes $i \notin V_M$ are only adjacent to edges with weight zero, and so playing color ℓ is a best response for them, independently of the colors played by the nodes in V_M. We next consider a node $i \in e_a \in M$ for an arbitrary $a = 1, \ldots, \ell$. We write j for the other node in e_a, i.e., $e_a = \{i, j\}$. Because $w_{e_a} = 2$, all individual preferences are zero, we only have coordination edges, and consider an equal-split distribution rule, the utility of player i in profile $\boldsymbol{s}$ is $u_i(\boldsymbol{s}) = 1$. To see that color a is a best response for player i, note that if she deviates to another color $b \in [\ell] \setminus \{a\}$, then she derives a utility of $1/2$ from every node in e_b she is adjacent to, as edges $e \in E[V_M] \setminus M$ have weight $w_e = 1$. Since $|e_b| = 2$, this means the maximum utility she can obtain is 1, which happens in case she is adjacent to both nodes in e_b.

Next, consider the strategy profile s^* in which all players choose a common color, say, color 1. Since every edge is then satisfied, and there are no individual preferences, it follows that the profile s^* is a social optimum. Because all edges $e \in E[V_M]$ have weight $w_e \in \{1, 2\}$, it holds that $u(s^*) \geq |E[V_M]|$. This then implies that

$$\text{PoA}(\Gamma) \geq \frac{|E[V_M]|}{2\ell} \geq \frac{|E[V_M]|}{2c}. \tag{9}$$

We next show how to embed the above deterministic construction in a random graph, with high probability. Let $G_n = (V_n, E_n) \sim G(n, d)$ and write $V_n = \{1, \dots, n\}$. Let

$$N = \min\{2c_n, 2\lfloor n/2 \rfloor\}.$$

Note that $2\lfloor n/2 \rfloor = n - 1$ if n is odd, and $2\lfloor n/2 \rfloor = n$ if n is even. Because we will be working with a matching, it is convenient to work with an even number of nodes, hence, this definition of N. Note that $N \to \infty$ whenever $n \to \infty$.

We claim that with high probability the induced subgraph on nodes $W_N = \{1, \dots, N\}$ contains (1) $\Omega(c_n^2)$ edges, and (2) a perfect matching.

We start by proving property (1). Note that

$$\mu = \mathbb{E}[E_n[W_N]] = d \cdot \binom{N}{2} = \Omega(c_n^2),$$

where the last equality holds by definition of N. Using Chernoff's bound, it follows that $\mathbb{P}\left(E_n[W_n] < \mu/2\right) \leq \exp(-\mu/8) = \exp(-\Omega(c_n^2)/8) \to 0$ as $n \to \infty$, since then $c_n \to \infty$ by assumption.

We continue with the proof of property (2). It uses the following result (see, e.g., Reference [17]): For every fixed $0 < d \leq 1$ it holds that

$$\lim_{n\to\infty} \mathbb{P}_{G_n \sim G(n,d)} \left(G_n \text{ contains a perfect matching}\right) = 1.$$

By applying this result to the random (induced) subgraph on W_N and using that c_n approaches infinity as $n \to \infty$, property (2) follows.

Now, if two events A_n and B_n, our properties (1) and (2), respectively, happen with high probability as $n \to \infty$, then their intersection $A_n \cap B_n$ also happens with high probability as $n \to \infty$. This means that the subgraph on W_N contains $\Omega(c_n^2)$ edges and a perfect matching, with high probability. Using the deterministic construction of the first part of the proof on the subgraph W_N then gives

$$\text{PoA}(\Gamma) \geq \frac{\Omega(c_n^2)}{4c_n} = \Omega(c_n).$$

This completes the proof. □

4 CONVERGENCE OF BEST-RESPONSE DYNAMICS

In this section, we derive our characterization results for the convergence of best-response dynamics in symmetric clustering games and for the existence of pure Nash equilibria in symmetric coordination games. Recall that best-response dynamics is said to *converge* if any sequence of player deviations, where in each step the deviating player chooses a most profitable deviation, converges in a finite number of steps to a pure Nash equilibrium. Basically, for symmetric clustering games our characterization shows that best-response dynamics is guaranteed to converge to a pure Nash equilibrium if and only if ξ is a generalized weighted Shapley distribution rule. For the special case of symmetric coordination games with $c \geq 3$, we can further strengthen this characterization result and show that a pure Nash equilibrium is guaranteed to exist if and only if ξ is a generalized weighted Shapley distribution rule. This complements a result of Anshelevich and Sekar [4].

4.1 Symmetric Clustering Games

We provide a characterization of distribution rules that guarantee the convergence of best-response dynamics in symmetric clustering games.

Theorem 8 (Best-response Convergence). *Let $\mathcal{G}_{G,c,\xi}$ be the set of all symmetric clustering games $\Gamma = (G, c, \boldsymbol{\xi}, \boldsymbol{w}, \boldsymbol{q})$ on a fixed graph G with c common colors and distribution rule $\boldsymbol{\xi}$. Then best-response dynamics is guaranteed to converge to a pure Nash equilibrium for every clustering game in $\mathcal{G}_{G,c,\xi}$ if and only if $\boldsymbol{\xi}$ corresponds to a generalized weighted Shapley distribution rule.*

In general, this characterization does not hold if the condition of "guaranteed convergence of best-response dynamics" is replaced by "guaranteed existence of a pure Nash equilibrium" (as in Reference [19] or Reference [11]): There are settings where on a fixed graph G, a pure Nash equilibrium is guaranteed to exist even if $\boldsymbol{\xi}$ is not a generalized weighted Shapley distribution rule, e.g., in the case of $c = 2$ or in coordination games with no individual preferences.

The proof of Theorem 8 relies on the following lemma. In the proofs of Lemma 9 and Theorem 8, all player or edge indices are always modulo n.

Lemma 9. *Consider a symmetric clustering game $(H, 2, \boldsymbol{\xi}, \boldsymbol{w}, \boldsymbol{q})$ on a cycle $H = \langle 1, \ldots, n\rangle$ with n players and $c = 2$ colors. If for every strategy profile $\boldsymbol{s}$ it is a best-response for every player i to choose a color that satisfies at least edge $\{i, i+1\}$, then there exists a best-response sequence that does not converge to a Nash equilibrium.*

Proof. We first construct an initial state $\boldsymbol{s}^0$ using only colors k_1 and k_2. Set $s_1^0 = k_1$, and iteratively, for $i = 2, \ldots, n-1$, set s_i^0 such that edge $\{i-1, i\}$ is satisfied (using only colors k_1 and k_2). The color for s_n is chosen in such a way that at least one of the edges $\{n-1, n\}$ or $\{n, 1\}$ is not satisfied (this can always be done, since if color k_1 would satisfy both edges, that color k_2 would satisfy neither, and vice versa). Now, either (i) precisely $n-1$ edges of the cycle are satisfied in $\boldsymbol{s}^0$ or (ii) precisely $n-2$ edges are satisfied in $\boldsymbol{s}^0$ (and two consecutive edges are not).

Case (i): *There are precisely $n-1$ edges satisfied.* This case is illustrated in Figure 1. Note that currently edge $\{n, 1\}$ is not satisfied. Therefore, by assumption, it is a best-response for player n to switch to its other color. But the situation after this switch is isomorphic to the starting profile $\boldsymbol{s}^0$, that is, if we would have started the numbering at node n instead of node 1. Therefore, we can repeat the same argument, and in particular, after $2n$ of such best-response steps, we are back in $\boldsymbol{s}^0$. Roughly speaking, during this process the unsatisfied edge moves over the whole cycle twice. After n steps, we are essentially also in the same situation as $\boldsymbol{s}^0$, but now with the roles of k_1 and k_2 interchanged.

Case (ii): *There are precisely $n-2$ edges satisfied except for the two consecutive edges $\{n-1, n\}$ and $\{n, 1\}$.* This case is illustrated in Figure 2. If player n would switch to its other color (which is a best-response move), then we would find a Nash equilibrium, however, we do not choose player n. Instead, we let player $n-1$ switch to its other color (which is a best-response move by assumption), then afterwards, it is a best-response for player $n-2$ to switch as well (to satisfy edge $\{n-2, n-1\}$), and we continue this in decreasing player order up until (and including) player 2. In particular, we are then in the situation where again precisely $n-2$ edges are satisfied except two consecutive edges, which are now $\{n, 1\}$ and $\{1, 2\}$. This situation is equivalent to the starting state $\boldsymbol{s}^0$, and in particular by repeating this process n times, we are back in $\boldsymbol{s}^0$. This completes the proof. □

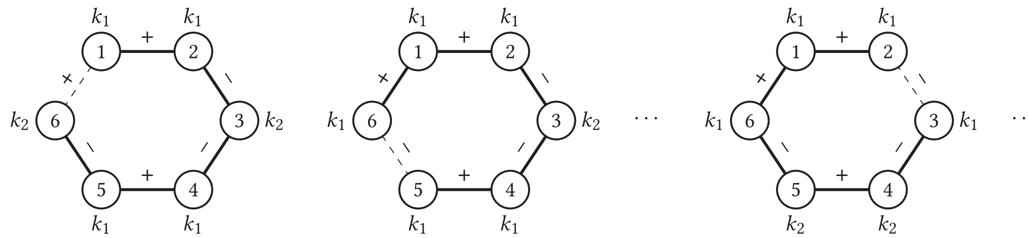

Fig. 1. This is an example with $n = 6$ players. The satisfied edges are bold, and the unsatisfied edges are dashed. The cycle on the left illustrates the initial state s^0, the middle one the situation after player n has deviated, and the right one illustrates the situation after players $n, n-1, \ldots, 3$ have deviated. The same steps are given for Case (ii) in Figure 2.

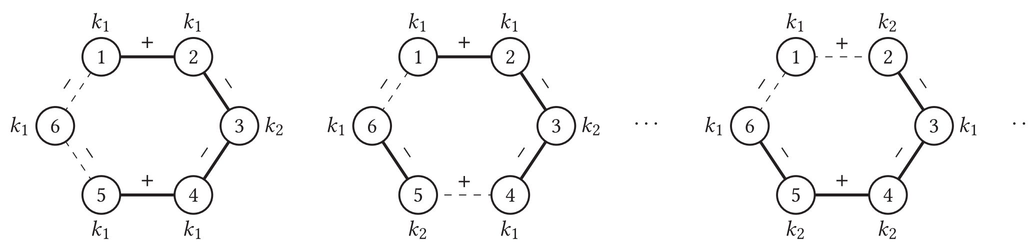

Fig. 2. This is an example with $n = 6$ players. The satisfied edges are bold, and the unsatisfied edges are dashed. The cycle on the left illustrates the initial state s^0, the middle one the situation after player $n - 1$ has deviated, and the right one illustrates the situation after players $n - 1, n - 2, \ldots, 2$ have deviated. The same steps are given for Case (i) in Figure 1.

Proof of Theorem 8. If ξ corresponds to a generalized weighted Shapley distribution rule, then the convergence of best-response dynamics follows immediately from the fact that the game can be modeled as a resource allocation game (see Appendix B). Such games, with generalized weighted Shapley distribution rules, are potential games; see, e.g., Reference [19] for details. We now continue with the other direction, i.e., assume that best-response dynamics always converge.

The idea of the proof is to show that the pair (G, ξ) has a certain nice structure if best-response dynamics is guaranteed to converge, from which it will be easy to conclude that ξ is a generalized weighted Shapley distribution rule. To define this nice structure, we need the help of an auxiliary digraph $D = (Q, A)$. The nodes in the set $Q = \{V_1, \ldots, V_r\}$ form a partition of the players in V, that is, $V = V_1 \cup V_2 \cup \cdots \cup V_r$. This partition is constructed by looking at the subgraph $G_{>0} = (V, E_{>0})$ of G given by edges $\{i, j\}$ from which both i and j get strictly positive utility if it is satisfied and the weight w_{ij} is non-zero, i.e.,

$$E_{>0}(G) = \{\{i, j\} : \xi_{ij},\ \xi_{ji} > 0 \text{ and } w_{ij} > 0\}.$$

We write $G^1_{>0}, \ldots, G^r_{>0}$ for the connected components of $G_{>0}$, with $V_t = V(G^t_{>0}) = \{v_{t1}, \ldots, v_{tz_t}\}$ and $E_t = E(G^t_{>0})$ for $t = 1, \ldots, r$. In other words, V_t consists of the z_t players that form the connected component $G^t_{>0}$. For given $a, b \in \{1, \ldots, r\}$, there is a directed arc $(V_a, V_b) \in A$ if and only if there exists an $i \in V_a$ and $j \in V_b$ such that $\xi_{ji} = 0$, and, thus, $\xi_{ij} > 0$, as $\xi_{ij} + \xi_{ji} > 0$. Note that, initially, it is not obvious whether D can have a self-loop or not. We next show that

(1) The digraph D does not have a directed cycle or self-loop, i.e., it is acyclic.
(2) Within every component $V_t = \{v_{t1}, \dots, v_{tz_t}\}$ there exist numbers $\gamma_1, \dots, \gamma_{z_t}$ such that

$$\frac{\xi_{ij}}{\xi_{ij} + \xi_{ji}} = \frac{\gamma_i}{\gamma_i + \gamma_j}$$

whenever $i, j \in V_t$.

The above two conditions (1) and (2) together are sufficient to guarantee that $\boldsymbol{\xi}$ is a generalized weighted Shapley distribution rule. To see this, first note that the weights $\gamma_1, \dots, \gamma_n$ yielded by condition (2) also satisfy condition (ii) in Section 2.3, because an edge $\{i, j\}$ with $\xi_{ij}, \xi_{ji} > 0$ is contained in exactly one of the components $G^1_{>0}, \dots, G^r_{>0}$, and so i and j are then always in the same V_t whenever $\xi_{ij}, \xi_{ji} > 0$. Furthermore, since D is acyclic, we can find a topological ordering of the partition sets $V_1, \dots, V_r$. By renaming the partition sets, we may assume without loss of generality that the ordering is $(V_1, V_2, \dots, V_r)$. The permutation

$$\sigma = (v_{11}, \dots, v_{1z_1}, v_{21}, \dots, v_{2z_2}, \dots, v_{r1} \dots v_{rz_r})$$

then satisfies condition (i) in Section 2.3, because $(V_1, \dots, V_r)$ is a topological ordering. That is, in σ, we start with the nodes in V_1, followed by those in V_2, and so on. The constructed vector $\boldsymbol{\gamma}$ and permutation σ then together show that $\boldsymbol{\xi}$ is a generalized weighted Shapley distribution rule. It therefore remains to show that the conditions (1) and (2) above are satisfied.

We start with showing that condition (1) is true. For sake of contradiction, first suppose that $D = (Q, A)$ has a self-loop, that is, $(V_t, V_t) \in A$ for some t. This means that there are nodes $v_h, v_1 \in V_t$ such that $\xi_{v_h v_1} > 0$ and $\xi_{v_1 v_h} = 0$. We write $e_h = \{v_h, v_1\}$ for the edge in the original graph G. Furthermore, as $G^t_{>0}$ is a connected component, there exists a simple path $(v_1, \dots, v_h)$ in $G^t_{>0}$ where both $\xi_{v_i v_{i+1}}, \xi_{v_{i+1} v_i} > 0$, for the edges $e_i = \{v_i, v_{i+1}\}$ for $i = 1, \dots, h-1$. We write $H = \langle v_1, v_2, \dots, v_{h-1}, v_h \rangle$ for the resulting simple cycle in the original graph G obtained by the path $(v_1, \dots, v_h)$ concatenated with the edge $\{e_h, e_1\}$. We will next create an instance of a symmetric clustering game, of which the relevant part takes place on the cycle H, for which best-response dynamics is not guaranteed to converge. Set $w_{v_1 v_2} = 1$ and iteratively define the weights $w_{v_i v_{i+1}}$ so

$$\frac{\xi_{v_i v_{i-1}}}{\xi_{v_i v_{i-1}} + \xi_{v_{i-1} v_i}} \cdot w_{v_{i-1} v_i} < \frac{\xi_{v_i v_{i+1}}}{\xi_{v_i v_{i+1}} + \xi_{v_{i+1} v_i}} \cdot w_{v_i v_{i+1}} \tag{10}$$

for $i = 2, \dots, h \bmod h$. All other edge weights, of edges not in H, are set to zero. Individual preferences $q_i(s_i)$ for every player are set to $K = \sum_{e \in E} w_e$ for two fixed colors k_1 and k_2, and zero otherwise. In particular this means that for all players $v_1, \dots, v_h$ on the cycle H, only colors k_1 and k_2 can potentially be a best response in any strategy profile. By construction, it will always be a best response for player v_i to satisfy edge $\{v_i, v_{i+1}\}$. For player $v_2, \dots, v_h$ this claim follows directly from Equation (10). For player v_1 it is true as she derives zero utility from edge $\{v_h, v_1\}$ if it is satisfied, because $\xi_{v1 v_h} = 0$, and strictly positive utility from $\{v_1, v_2\}$ if it is satisfied. Because of the fact that for every player v_i it is always a best response to satisfy $\{v_i, v_{i+1}\}$, we are in the situation of Lemma 9, and, thus, we may conclude that best-response dynamics is not guaranteed to converge.

Second, suppose that D contains a directed cycle. The argument showing that best-response dynamics is not guaranteed to converge is very similar to the case of a self-loop. We can now construct a cycle H in G that traverses multiple components $G^t_{>0}$, and, in particular, contains multiple edges from the set $E \setminus E_{>0}(G)$. Because of the fact that the edges in $E \setminus E_{>0}(G)$ form a directed cycle in D, the procedure in Equation (10) still can be used to define the edge weights for the edges in H to make sure that every player v_i always tries to satisfy the edge $\{v_i, v_{i+1}\}$. In particular, for any edge $\{v_i, v_{i+1}\}$ in which one of the two players receives the full share, this will always be v_i, because the edges of H that are contained in $E \setminus E_{>0}(G)$ form a directed cycle in D. If the cycle would not have been directed, then this would mean that for some edge $\{v_i, v_{i+1}\}$ player v_{i+1} would get the full share, i.e., $\xi_{v_i v_{i+1}} = 0$, and then the procedure in Equation (10) cannot be used as for that i, the right-hand side in Equation (10) would be zero. Individual preferences q_i and remaining edge weights, of edges not in H, can be chosen just as in the case of a self-loop in D.

We continue with proving condition (2). The proof is the same for every component so fix any $G^t_{>0}$. To make things easier notation-wise, we will write $V_t = \{1, \ldots, z\}$. The goal is to show that there exist numbers $\gamma_1, \ldots, \gamma_z$ such that

$$\frac{\xi_{ij}}{\xi_{ij} + \xi_{ji}} = \frac{\gamma_i}{\gamma_i + \gamma_j} \tag{11}$$

for every $\{i, j\} \in E_t$. It is clear that if we can find such a vector $(\gamma_1, \ldots, \gamma_z)$, and multiply every γ_i with a fixed constant $d > 0$, then the weights $d \cdot \gamma_i$ also satisfy Equation (11). In particular, this implies that we may fix γ_1 as we like without loss of generality. We next fix a spanning tree T in $G^t_{>0}$ to obtain the values of $\gamma_2, \ldots, \gamma_z$. We root the tree at the node 1, and set

$$\gamma_j = \frac{\xi_{r_2 r_1} \xi_{r_3 r_2} \cdots \xi_{r_{h_j} r_{h_j - 1}}}{\xi_{r_1 r_2} \xi_{r_2 r_3} \cdots \xi_{r_{h_j - 1} r_{h_j}}} \cdot \gamma_1,$$

where $(r_1, r_2, \ldots, r_{h_j})$ is the unique path from player $1 = r_1$ to player $j = r_{h_j}$ in T. Setting γ_j in this way guarantees that Equation (11) is satisfied for the edges of the spanning tree T. To see this, note that Equation (11) for an edge $\{i, j\} \in E(T)$ is equivalent to

$$\gamma_i = \frac{\xi_{ij}}{\xi_{ji}} \cdot \gamma_j. \tag{12}$$

Now, if there is some edge $e = \{v_1, v_2\} \in E(G^t_{>0}) \setminus E(T)$ with the property that Equation (11) is not satisfied, then we get

$$\gamma_{v_1} \neq \frac{\xi_{v_1 v_2}}{\xi_{v_2 v_1}} \cdot \gamma_{v_2}. \tag{13}$$

Let $H = \langle v_1, \ldots, v_h \rangle$ be the unique cycle in $T \cup e$ containing edge e. It then follows that

$$\xi(H) := \frac{\xi_{v_2 v_1} \xi_{v_3 v_2} \cdots \xi_{v_1 v_h}}{\xi_{v_1 v_2} \xi_{v_2 v_3} \cdots \xi_{v_h v_1}} \neq 1,$$

which can be seen by multiplying Equation (13) with the Equations in (12) for the edges on $E(H) \setminus e$. The expression $\xi(H) = 1$ is analogue to the cyclic consistency property of Gopalakrishnan et al. [19]. We will show that $\xi(H) \neq 1$ implies that best-response dynamics is not guaranteed to converge.

We let $e_i = \{v_i, v_{i+1}\}$ for $i = 1, \ldots, h$ (mod h). Assume that $\xi(H) < 1$. This can be done without loss of generality by changing the orientation of the cycle H if needed. Then there exists a constant $\tau > 0$ so

$$(1 + \tau)^n \cdot \xi(H) < 1.$$

Set $w_{v_1 v_2} = 1$ and iteratively define the weights w_{e_i} so

$$\frac{\xi_{v_i v_{i-1}}}{\xi_{v_i v_{i-1}} + \xi_{v_{i-1} v_i}} \cdot w_{e_{i-1}} = \frac{\xi_{v_i v_{i+1}}}{\xi_{v_i v_{i+1}} + \xi_{v_{i+1} v_i}} \cdot w_{e_i} \tag{14}$$

for $i = 2, \ldots, h$. We then define the weights $w'_{v_i v_{i+1}} = (1+\tau)^i \cdot w_{v_i v_{i+1}}$ for $i = 1, \ldots, h$. All other edge weights, of edges not in H, are set to zero. Individual preferences $q_i(s_i)$ for every player are set to $K = \sum_{e \in E} w_e$ for two fixed colors k_1 and k_2, and zero otherwise. This is similar to what we did in the proof of condition (1). For players $v_2, \ldots, v_h$ it is now always a best-response to choose, among k_1 and k_2, the color that satisfies edge $\{i, i+1\}$. This follows from the fact that the weights $w'_{v i v_{i+1}}$ satisfy

$$\frac{\xi_{v_i v_{i-1}}}{\xi_{v_i v_{i-1}} + \xi_{v_{i-1} v_i}} \cdot w'_{v_{i-1} v_i} < \frac{\xi_{v_i v_{i+1}}}{\xi_{v_i v_{i+1}} + \xi_{v_{i+1} v_i}} \cdot w'_{v_i v_{i+1}}.$$

This is the same idea used in the proof of condition (1), based on the inequality in (10). For player v_1 the argument is slightly more involved. Suppose it is not a best response for v_1 to satisfy the edge $\{v_1, v_2\}$. Then

$$\frac{\xi_{v_1 v_h}}{\xi_{v_1 v_h} + \xi_{v_h v_1}} \cdot w_{e_h} \cdot (1+\tau)^n \geq \frac{\xi_{v_1 v_2}}{\xi_{v_1 v_2} + \xi_{v_2 v_1}} \cdot w_{e_1}. \tag{15}$$

If we multiply all equalities in Equation (14) with each other for $i = 2, \ldots, h$, and the result also with Equation (15), then after simplification we find $(1+\tau)^n \xi(H) \geq 1$, which contradicts the choice of τ. As for all players $v_1, \ldots, v_h$ it is a best response to satisfy edge $\{v_i, v_{i+1}\}$, and we are again in the situation of Lemma 9. This leads to a contradiction and concludes the proof. □

Corollary 10. *The characterization in Theorem 8 also is true if $\mathcal{G}_{G,c,\xi}$ is replaced by either:*

(i) The set $\mathcal{H}_{G,c,\xi,\mathbf{0}} = \{\Gamma : \Gamma = (G, c, \boldsymbol{\xi}, \boldsymbol{w}, \mathbf{0})$ with $E_a = \emptyset\}$ of symmetric coordination games on G, with c common colors, and distribution rule $\boldsymbol{\xi}$, but without individual preferences q_i.

(ii) The set $\mathcal{G}_{G,2,\xi,\mathbf{0}} = \{\Gamma : \Gamma = (G, 2, \boldsymbol{\xi}, \boldsymbol{w}, \mathbf{0})\}$ of symmetric clustering games on graph G with 2 common colors, and distribution rule $\boldsymbol{\xi}$, but without individual preferences q_i.

The first setting corresponds to certain models in References [4, 16]. Also note that the second setting cannot hold true with $c \geq 3$ by considering a cycle of length three with only anti-coordination edges. That is, if there are no individual preferences q_i, then any best-response sequence will in at most three steps end up in a strategy profile in which all three players have a different color. Such a profile is always a pure Nash equilibrium in case there are no individual preferences, and so any best-response sequence will converge.

Proof of Corollary 10. To prove the first case, one should observe that if there are only coordination edges in the proof of Theorem 8, it is not necessary to set the individual preferences $q_i(s_i)$ to the sum of the w_e for two chosen colors k_1 and k_2, and zero otherwise. Instead, one can set all individual preferences equal to zero, i.e., let them play no role in the game. This is allowed, because when all players start with either color k_1 or k_2, then no player can have some other color $k \in [c] \setminus \{k_1, k_2\}$ as a best response, since there are only coordination edges and no individual preferences.

For the second case, note that when there are only two colors k_1 and k_2 available to the players, then every player will always play either of these colors. There is then no need to use the individual preferences q_i to force players to always use one of these colors as a best response. □

4.2 Symmetric Coordination Games

We next consider the special case of symmetric coordination games in which the common strategy set contains $c \geq 4$ colors. We can strengthen the characterization result of Theorem 8 in this case. We prove in Theorem 11 that a pure Nash equilibrium is guaranteed to exist if and only if $\boldsymbol{\xi}$ is a generalized weighted Shapley distribution rule. This complements a result of Anshelevich and Sekar [4].

Theorem 11. *Let $\mathcal{G}_{G,c,\xi} = \{\Gamma : \Gamma = (G, c, \xi, \boldsymbol{w}, \boldsymbol{q})\}$ be the set of all symmetric coordination games on G, with common strategy set $\{1, \ldots, c\}$ for $c \geq 4$ and distribution rule ξ. Then a pure Nash equilibrium is guaranteed to exist for every game in $\mathcal{G}_{G,c,\xi}$ if and only if ξ corresponds to a generalized weighted Shapley distribution rule.*

Our arguments are conceptually similar to those of Gopolakrishnan et al. [19], however, they are technically different. We elaborate on the connection between Theorem 11 and the work in Reference [19] in Appendix B. We essentially show a similar result as in Reference [19], but for a more restricted setting than the resource allocation games considered there. Nevertheless, the result in Theorem 11 allows us to fully characterize which distribution ξ guarantee equilibrium existence, thereby completing results of Anshelevich and Sekar [4], who only partially address this question.

In particular, Anshelevich and Sekar [4] provide an example showing that for general distribution rules, pure Nash equilibria are not guaranteed to exist. On the positive side, they show that if the distribution rule has the so-called *correlated coordination condition*, then pure Nash equilibria are guaranteed to exist. This condition is actually the same as saying that the local distribution rule corresponds to a weighted Shapley distribution rule, and the proof of Theorem 1 in Reference [4] is essentially a direct consequence of the work of Hart and Mas-Collel [23] who characterize the (weighted) Shapley value in terms of a (weighted) potential function. Theorem 11 allows us to precisely characterize which distribution rules ξ guarantee (pure) equilibrium existence in symmetric coordination games for arbitrary weight functions w and individual preferences q_i. In particular, we note that generalized weighted Shapley distribution rules (see preliminaries) still guarantee equilibrium existence. This follows from Reference [19] by observing that these coordination games are resource allocation games (see Appendix B). We then show that these distribution rules are also necessary in a certain sense, already in the case of four colors.

Proof of Theorem 11. The idea of the proof is similar to that of Theorem 8. We again consider the graph $G_{>0}$, with its connected components $G^t_{>0}$ for $t = 1, \ldots, r$, and the auxiliary graph $D = (Q, A)$, which is defined in the same way. Again, the goal will be to show that

(1) The digraph D does not have a directed cycle or self-loop, i.e., it is acyclic.
(2) Within every component $V_t = \{v_{t1}, \ldots, v_{tz_t}\}$ there exist numbers $\gamma_1, \ldots, \gamma_{z_t}$ such that

$$\frac{\xi_{ij}}{\xi_{ij} + \xi_{ji}} = \frac{\gamma_i}{\gamma_i + \gamma_j}$$

whenever $i, j \in V_t$.

For similar reasons as in the proof of Theorem 8, this is sufficient to conclude that ξ is a generalized weighted Shapley distribution rule. To prove conditions (1) and (2), the idea is again for both conditions to create an instance on a cycle H and derive a contradiction. For the proof here, we are trying to derive a stronger contradiction than that in the proof of Theorem 8, namely, that a pure Nash equilibrium does not exist as opposed to showing that best-response dynamics is not guaranteed to converge. However, we only have coordination edges and no anti-coordination edges, which makes the situation manageable.

We again start with the proof of condition (1) for the case that D contains a self-loop corresponding to a cycle $H = \langle v_1, \ldots, v_h \rangle$. We start by fixing $w_{v_1 v_2} = 1$ and then repeatedly choose $w_{v_i v_{i+1}}$ so

$$\frac{\xi_{v_i v_{i-1}}}{\xi_{v_i v_{i-1}} + \xi_{v_{i-1} v_i}} \cdot w_{v_{i-1} v_i} < \frac{\xi_{v_i v_{i+1}}}{\xi_{v_i v_{i+1}} + \xi_{v_{i+1} v_i}} \cdot w_{v_i v_{i+1}} \tag{16}$$

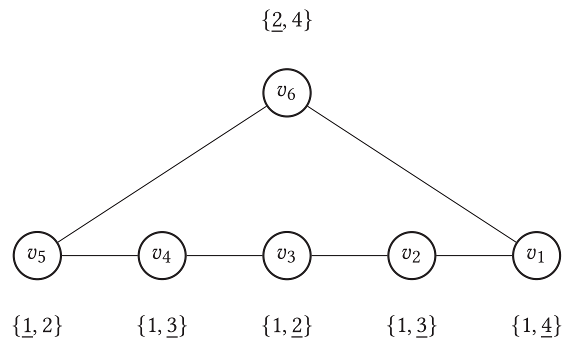

Fig. 3. Sketch of instance on cycle H in case D has self-loop. For every node, we have indicated its two colors for which it has a nonzero individual preference and have underlined its preferred color of the two.

for $i = 2, \dots, h$. Compared to the proof of Theorem 8, the individual preferences of the players will play an important role in this proof. Let $C = \{1, \dots, c\}$ and let $M = \sum_e w_e$. For some arbitrarily fixed $\delta > 0$, we define

$$q_{v_{h-1}}(j) = \begin{cases} M+\delta & \text{if } j=1 \\ M & \text{if } j=2 \\ 0 & \text{else,} \end{cases} \qquad q_{v_h}(j) = \begin{cases} M+\delta & \text{if } j=2 \\ M & \text{if } j=4 \\ 0 & \text{else,} \end{cases} \quad \text{and} \quad q_{v_1}(j) = \begin{cases} M+\delta & \text{if } j=1 \\ M & \text{if } j=4 \\ 0 & \text{else} \end{cases}.$$

The constant M is used to, roughly speaking, mimic asymmetric strategy sets. It will be important, later on, that δ can be chosen arbitrarily close to zero. The individual preferences for players $v_2, \dots, v_{h-2}$ are chosen differently. For $i = 2, \dots, h-2$, we choose δ_i such that

$$\frac{\xi_{v_i v_{i-1}}}{\xi_{v_i v_{i-1}} + \xi_{v_{i-1} v_i}} \cdot w_{v_{i-1} v_i} < \delta_i < \frac{\xi_{v_i v_{i+1}}}{\xi_{v_i v_{i+1}} + \xi_{v_{i+1} v_i}} \cdot w_{v_i v_{i+1}}, \tag{17}$$

which is possible because we have a strict inequality in (16). We then define

$$q_{v_i}(j) = \begin{cases} M+\delta_i & \text{if } j=3 \\ M & \text{if } j=1 \\ 0 & \text{else,} \end{cases} \quad \text{for } i = h-2, h-4, \dots \text{ and}$$

$$q_{v_i}(j) = \begin{cases} M+\delta_i & \text{if } j=2 \\ M & \text{if } j=1 \\ 0 & \text{else,} \end{cases} \quad \text{for } i = h-3, h-5, \dots$$

Note that we essentially only need four colors to define the individual preferences as above, meaning that this construction works for any $c \geq 4$. Player i's *preferred color* is the unique color for which its individual preference is the highest, for $i = 1, \dots, h$. Furthermore, it is obvious by the definition of M, that in any pure Nash equilibrium every player will play one of its two colors for which it has nonzero individual preference. Therefore, from now on, we will slightly abuse terminology and say that *every player only has two colors*, namely, those with nonzero individual preference. A sketch of the instance on H is given in Figure 3. All edges not in H, and all individual preferences of nodes not in H, are set to zero.

We will argue that the resulting instance has no pure Nash equilibrium $\boldsymbol{s}$.

Case 1: *Player v_{h-1} plays color 1 in $\boldsymbol{s}$.* Then player v_{h-2} will also play color 1 in $\boldsymbol{s}$. This claim is true, because if v_{h-2} would be playing color 3, then in particular the edge $\{v_{h-2}, v_{h-3}\}$ cannot be satisfied, as v_{h-3} does not have color 3. By construction of δ_{h-2} and the weight $w_{v_{h-2}v_{h-1}}$ it is then optimal for player v_{h-2} to player color 1 as well. Repeating this argument yields that $v_{h-2}, \dots, v_1$

will all play color 1 in $\boldsymbol{s}$. Note that in particular player v_1 obtains no utility from the edge $\{v_h, v_1\}$ as $\xi_{v_1 v_h} = 0$, because v_h and v_1 are the players that give rise to the self-loop in D. Since v_{h-1} and v_1 play color 1 in $\boldsymbol{s}$, player v_h cannot obtain any utility from the edges $\{v_h, v_{h-1}\}$ and $\{v_h, v_1\}$ and thus it will play its preferred color 2 in $\boldsymbol{s}$. However, with δ chosen small enough, it will then be optimal for player v_{h-1} to deviate to color 2. This contradicts the fact that $\boldsymbol{s}$ is a pure Nash equilibrium.

Case 2: *Player v_{h-1} plays color 2 in $\boldsymbol{s}$.* As player v_{h-2} cannot play color 2, it must be that player v_h also plays color 2, otherwise player v_{h-1} would not derive any utility from the edges $\{v_{h-1}, v_h\}$ and $\{v_{h-1}, v_{h-2}\}$ but then it would be optimal for v_{h-1} to play its preferred color 1. Because player v_{h-2} cannot play 2, and therefore not obtain any utility from the edge $\{v_{h-2}, v_{h-1}\}$, it will play its preferred color 3 in $\boldsymbol{s}$. Note that this is the optimal choice for v_{h-2} even if the edge $\{v_{h-2}, v_{h-3}\}$ would be satisfied by definition of δ_{h-2}. Repeating this argument, all players $v_{h-2}, \dots, v_1$ will play their preferred color. In particular, this means that v_1 will play color 4 in $\boldsymbol{s}$. However, then it will be optimal for player v_h to deviate to color 4 as well. This contradicts that $\boldsymbol{s}$ is a pure Nash equilibrium.

The completes the proof that D cannot have a self-loop. The case that D contains a cycle proceeds along similar lines, because of the same reasons as why these cases are similar in the proof of Theorem 8.

We continue with the proof of condition (2). Set $w_{v_1 v_2} = 1$ and iteratively define the weights w_{e_i} so

$$\frac{\xi_{v_i v_{i-1}}}{\xi_{v_i v_{i-1}} + \xi_{v_{i-1} v_i}} \cdot w_{e_{i-1}} = \frac{\xi_{v_i v_{i+1}}}{\xi_{v_i v_{i+1}} + \xi_{v_{i+1} v_i}} \cdot w_{e_i} \tag{18}$$

for $i = 2, \dots, h$. We then define the weights $w'_{v_i v_{i+1}} = (1+\tau)^i \cdot w_{v_i v_{i+1}}$ for $i = 1, \dots, h$, where τ is chosen similarly as in the proof of Theorem 8. All other edge weights, of edges not in H, are set to zero as well as individual preferences of all nodes not in H. The weights $w'_{v i v_{i+1}}$ for $i = 1, \dots, h-1$, satisfy

$$\frac{\xi_{v_i v_{i-1}}}{\xi_{v_i v_{i-1}} + \xi_{v_{i-1} v_i}} \cdot w'_{v_{i-1} v_i} < \frac{\xi_{v_i v_{i+1}}}{\xi_{v_i v_{i+1}} + \xi_{v_{i+1} v_i}} \cdot w'_{v_i v_{i+1}}.$$

The individual preferences are set similarly as in the the proof of condition (1) above, but this time, we choose δ_i, for $i = 2, \dots, h-2$, to satisfy

$$\frac{\xi_{v_i v_{i-1}}}{\xi_{v_i v_{i-1}} + \xi_{v_{i-1} v_i}} \cdot w'_{v_{i-1} v_i} < \delta_i < \frac{\xi_{v_i v_{i+1}}}{\xi_{v_i v_{i+1}} + \xi_{v_{i+1} v_i}} \cdot w'_{v_i v_{i+1}}.$$

By choosing δ, used in the definition of the individual preferences of v_{h-1}, v_h, and v_1, sufficiently small, we can carry out exactly the same argument as in the proof of condition (2) of Theorem 8 to make sure that v_1 prefers to satisfy $\{v_1, v_2\}$ over satisfying $\{v_h, v_1\}$. Using the same reasoning as in the case of condition (1) above, it can then be shown that the resulting instance has no pure Nash equilibrium. □

5 RESULTS FOR ASYMMETRIC COORDINATION GAMES

In this section, we present our results for asymmetric coordination games. We focus on coordination games with equal-split distribution rule and no individual preferences.

5.1 Approximate Nash Equilibria

Apt et al. [5] show that the $(1,1)$-PoA of coordination games is unbounded if $c \geq n+1$. Notably, this holds for arbitrary graph topologies with unit weights and without individual preferences. We slightly generalize this observation. We show that the Price of Anarchy is unbounded if and only if $c \geq \chi(G) + 1$, where $\chi(G)$ is the chromatic number of G.

Theorem 12. *Let $\mathcal{G}_G(c)$ be the set of all coordination games $\Gamma = \{G, c, (S_i)_{i\in V}, \mathbf{1}, \mathbf{1}, \mathbf{0}\}$ on graph G with c colors and equal-split distribution rule. Then, for any $\alpha \geq 1$, we have $(\alpha, 1)$-PoA$(\mathcal{G}_G(c)) = \infty$ if $c \geq \chi(G) + 1$ and finite if $c < \chi(G)$.*

Proof of Theorem 12. For a given coloring of G with colors $a_1, \ldots, a_{\chi(G)}$, we assign a strategy set of $\{a_i, a_0\}$ to all nodes that are colored with color $i \in \{1, \ldots, \chi(G)\}$. In particular, the strategy profile $\boldsymbol{s}$ in which every player chooses its color a_i from the coloring is a pure Nash equilibrium with utility $u(\boldsymbol{s}) = 0$, whereas the profile $\boldsymbol{s}^*$ in which every player chooses a_0 is a socially optimal profile with $u(\boldsymbol{s}^*) = |E(G)|$. This shows unboundedness if $c \geq \chi(G) + 1$.

If $c < \chi(G)$, then in every strategy profile $\boldsymbol{s}$ there is at least one edge $e \in E(G)$ such that its endpoints have the same color in $\boldsymbol{s}$. This shows boundedness. □

We can exploit the above insight to prove that if the number of colors c is a constant, then the Price of Anarchy is unbounded for sparse random graphs, while it is bounded by some constant for dense random graphs:

Theorem 13 (Constant Strategy Sets). *Let $\alpha \geq 1$ and let $c \geq 3$ be a given integer. Let $\mathcal{G}_{G_n,c}$ be the set of all coordination games $\Gamma = (G_n, c, (S_i)_{i\in V}, \mathbf{1}, \mathbf{1}, \mathbf{0})$ on graph $G_n \sim G(n,p)$ with strategy sets $S_i \subseteq [c]$ for every player i.*

- *Then there exists a constant $d = d(c)$ such that for $p \leq d/n$, we have*
$$\lim_{n\to\infty} \mathbb{P}_{G_n \sim G(n,p)} \left((\alpha, 1)\text{-}PoA\left(\mathcal{G}_{G_n,c}\right) = \infty\right) = 1.$$
- *In contrast, if $p \in (0,1)$ is constant, then there exists a constant $\beta_0 = \beta_0(p, c)$ such that*
$$\lim_{n\to\infty} \mathbb{P}_{G_n \sim G(n,p)} \left((\alpha, 1)\text{-}PoA\left(\mathcal{G}_{G_n,c}\right) \leq \beta_0\right) = 1.$$

Proof. For sparse Erdős-Rényi graphs $G \sim G(n, d/n)$ for constant $d > 0$, it is known that the chromatic number is a constant $\beta = \beta(d)$ with high probability; see Reference [31]. It is not hard to see that this constant is non-decreasing in d. However, for $G \sim G(n,p)$ with p constant independent of n, a standard argument shows that with high probability as $n \to \infty$, there is a constant $\beta = \beta(p)$ such that every subset of more than βn nodes contains $\Theta(n^2)$ edges. Since the maximum number of colors in a strategy set is bounded by c, there is at least one subset of n/c nodes such that all players in this set play the same color in a given Nash equilibrium $\boldsymbol{s}$. This means that the social cost of any Nash equilibrium is $\Theta(n^2)$. Moreover, since all edge-weights are one, it follows that the social cost of an optimal strategy profile is $O(n^2)$. This proves that there is a constant β_0 dependent on both c and p. □

5.2 Approximate k-strong Equilibria

In general, approximate Nash equilibria are not guaranteed to exist in asymmetric coordination games; see, e.g., Reference [5]. In this section, we consider the Price of Anarchy of (α, k)-equilibria with $k \geq 2$. It is known that the (α, k)-PoA of coordination games is between $2\alpha(n-1)/(k-1)+1-2\alpha$ and $2\alpha(n-1)/(k-1)$ for $k \geq 2$ [33]. In particular, the (α, k)-PoA grows like $\Theta(\alpha n)$ if k is a constant.

We derive a topological bound on the (α, k)-Price of Anarchy that depends on the maximum degree $\Delta(G)$ of the graph G.

Theorem 14 (Degree Bound). *Let $\alpha \geq 1, k \geq 2, c \geq 3$, and let G be an arbitrary graph. Let $\mathcal{G}_G(c)$ be the set of all coordination games $\Gamma = (G, c, (S_i)_{i\in V}, \mathbf{1}, \boldsymbol{w}, \mathbf{0})$ on graph G with c colors, equal-split distribution rule, and no individual preferences. Then*
$$\alpha \cdot \max\left\{1, \frac{\Delta(G)}{k-1} - 1\right\} \leq (\alpha, k)\text{-}PoA(\mathcal{G}_G(c)) \leq 2\alpha \cdot \Delta(G).$$

Proof. We first construct the lower bound. The following assumptions can be made without loss of generality:

— We assume that $\alpha = 1$. This is allowed, because an (α, k)-equilibrium is also an (α', k)-equilibrium for any $\alpha' \geq \alpha$.
— We assume that $\Delta(G) > k - 1$. Otherwise, we have a trivial lower bound of one.
— We assume that the game has $c = 3$ colors. This is allowed, because we consider asymmetric strategy sets. That is, in case $c > 3$ the colors in $\{4, 5, 6, .., c\}$ will not appear in any strategy set.

We consider a game with three colors $\{a, b, c\}$. Let $i \in V$ be a node of maximum degree, and let $j_1^*, \dots, j_{k-1}^* \in N(i)$ be $k-1$ fixed neighbors of i. We give players i and j_l^* for $l = 1, \dots, k-1$ a strategy set of $\{a, b\}$, and all other nodes a strategy set of $\{a, c\}$. Moreover, edges $\{i, j\}$ get a weight of $w_{ij} = 1$ for $j = j_1^*, \dots, j_{k-1}^*$, $w_{ij} = \alpha$ for all $j \in N(i) \setminus \{j_1^*, \dots, j_{k-1}^*\}$ and all other edges a weight of zero. It is not hard to see that the strategy profile $\boldsymbol{s}$ in which $s_v = b$ for $v = i, j_1^*, \dots, j_{k-1}^*$, and $s_v = c$ otherwise, is a k-equilibrium with utility $u(\boldsymbol{s}) = k - 1$. The strategy profile $\boldsymbol{s}^*$ in which all players choose color a is clearly a socially optimal state with utility $u(\boldsymbol{s}^*) = \alpha(\Delta(G) - k + 1) + k$. This proves the lower bound.

It remains to proof the upper bound. Consider an instance $\Gamma = (G, c, (S_i)_{i \in V}, \mathbf{1}, \boldsymbol{w}, \mathbf{0})$. Let $\boldsymbol{s}$ be an (α, k)-equilibrium and let $\boldsymbol{s}^*$ be an optimal strategy profile. Let $V = S \cup T$ be a partition of the node set, where $S = \{i \in V : u_i(\boldsymbol{s}) > 0\}$ and $T = \{i \in V : u_i(\boldsymbol{s}) = 0\}$.

Let $i, j \in T$ and suppose that $e = \{i, j\} \in E$. We claim that either $w_e = 0$, or e is unsatisfied in $\boldsymbol{s}^*$. Suppose that $w_e > 0$ and e is satisfied in $\boldsymbol{s}^*$. Then, in particular, it follows that i and j both have a color c' in their strategy set, i.e., $S_i \cap S_j \neq \emptyset$. Since $u_i(\boldsymbol{s}) = u_j(\boldsymbol{s}) = 0$, this means that they can (jointly) profitably deviate to c', contradicting the fact that $\boldsymbol{s}$ is a k-equilibrium. That is, either one of the players chose c' in $\boldsymbol{s}$, in which case the other player can deviate to c' to improve her utility, or i and j can jointly deviate to c', which is feasible, because $k \geq 2$.

The above implies that

$$u(\boldsymbol{s}^*) \leq \sum_{\{i,j\} \in E(\boldsymbol{s}^*) : \{i,j\} \cap S \neq \emptyset} w_{ij} = \sum_{\{i,j\} \in E(\boldsymbol{s}^*) : \{i,j\} \cap S \neq \emptyset \text{ and } w_{ij} > 0} w_{ij},$$

where $E(\boldsymbol{s}^*)$ is the set of satisfied edges in $\boldsymbol{s}^*$. We now show that the latter summation is at most $2\alpha\Delta(G) \cdot u(\boldsymbol{s})$, which completes the proof.

First, let $i \in S$ and $j \in T$, and suppose that $e = \{i, j\}$ is satisfied in $\boldsymbol{s}^*$ with $w_e > 0$. The fact that e is satisfied in $\boldsymbol{s}^*$ implies that i and j have a common color c' in their strategy sets. By definition, we have $u_j(\boldsymbol{s}) = 0$, so it must be that $\alpha \cdot u_i(\boldsymbol{s}) \geq w_{ij}/2$ otherwise i and j could (jointly) profitably deviate to c'. Second, let $i \in S$ and $j \in S$, and suppose that $e = \{i, j\}$ is satisfied in $\boldsymbol{s}^*$ with $w_e > 0$. Similar arguments imply that either $\alpha \cdot u_i(\boldsymbol{s}) \geq w_{ij}/2$ or $\alpha \cdot u_j(\boldsymbol{s}) \geq w_{ij}/2$ (or both).

In particular, this implies that the edges in $\{e \in E(\boldsymbol{s}^*) : w_e > 0 \text{ and } e \cap S \neq \emptyset\}$ can be partitioned into sets $E_1, \dots, E_{|S|}$ defined as $E_i = \{\{i, j\} : i \prec j \in N(i) \text{ and } \alpha \cdot u_i(\boldsymbol{s}) \geq w_{ij}/2\}$ for all $i \in S$, where $\prec$ is some total ordering on the nodes in S. That is, in case both $u_i(\boldsymbol{s}) \geq w_{ij}/2$ and $u_j(\boldsymbol{s}) \geq w_{ij}/2$, we assign edge $\{i, j\}$ to the node that is lower in the ordering $\prec$. Note that $|E_i| \leq \Delta(G)$. By definition of the set E_i, we now have that

$$\sum_{\{i,j\} \in E(\boldsymbol{s}^*) : \{i,j\} \cap S \neq \emptyset \text{ and } w_{ij} > 0} w_{ij} \leq 2\alpha \sum_{i \in S} \sum_{\{i,j\} \in E_i} u_i(\boldsymbol{s}) \leq 2\alpha\Delta(G) \sum_{i \in S} u_i(\boldsymbol{s}) = 2\alpha\Delta(G) \sum_{i \in V} u_i(\boldsymbol{s}),$$

where the last equality holds, because $u_i(\boldsymbol{s}) = 0$ for all $i \in T = V \setminus S$. □

We now use this result to bound the (α, k)-Price of Anarchy for random graphs. Note that, by exploiting the topological bound of Theorem 14, it suffices to bound the maximum degree of the corresponding random graph. The maximum degree of random graphs drawn according to the Erdős-Rényi random graph model is well understood; see, e.g., the work of Frieze and Karonski [17].

In particular, for dense random graphs with constant $p = d \in (0,1)$, the maximum degree of a random graph satisfies $\Delta(G) \sim \Theta(n)$; see, e.g., Reference [17, Chapter 3]. So, for these graphs the (α, k)-Price of Anarchy still grows like $\Omega(\alpha n)$, as in the worst case.

In contrast, we obtain an improved bound for sparse random graphs.

Theorem 15. *Let $\alpha \geq 1$, $k \geq 2$, and $d > 0$ be constants. Let $(c_n)_{n\in\mathbb{N}}$ be a sequence of integers with $c_n \geq 3$ for all n. Let $\mathcal{G}_{G_n}(c_n)$ be the set of all coordination games $\Gamma = (G_n, c_n, (S_i)_{i\in V}, \mathbf{1}, \boldsymbol{w}, \mathbf{0})$ on graph $G_n \sim G(n, d/n)$ with c_n colors, equal-split distribution rule and no individual preferences. Then*

$$(\alpha, k)\text{-}PoA(\mathcal{G}_{G_n}(c_n)) = \Theta\left(\frac{\alpha \ln(n)}{\ln\ln(n)}\right).$$

Proof. The bounds follow directly from Theorem 14 and the fact that for a random graph $G_n \sim G(n, d/n)$ with $p = d/n$, we have $\Delta(G_n) \approx \mathcal{O}(\ln(n)/\ln\ln(n))$ with high probability; see, e.g., Reference [17, Chapter 3]. □

If, in addition, the strategy sets are drawn according to a sequence of distributions that satisfy the so-called *common color property*, and all weights are equal to one, corresponding to the games studied in Reference [5], then we can even prove that the (α, k)-Price of Anarchy is bounded by a constant.

Definition 16 (Common Color Property). For a sequence of integers $(c_n)_{n\in\mathbb{N}}$, we say that a sequence of probability distributions $(\mathcal{F}_n)_{n\in\mathbb{N}}$ over $2^{[c_n]} \setminus \emptyset$ satisfies the *common color property* if there exists some constant $d_0 > 0$, independent of n, such that for $A_n^1, A_n^2 \sim \mathcal{F}_n$, $\inf_n \mathbb{P}\left(A_n^1 \cap A_n^2 \neq \emptyset\right) \geq d_0$.

Intuitively, the common color property requires that with positive probability any two players have a color in common in their strategy sets. In particular, this condition is satisfied if we draw the strategy sets uniformly at random from $2^{[c]} \setminus \emptyset$ with $d_0 = \frac{1}{2}$. We remark that in the deterministic setting the Price of Anarchy does not improve if all players have a color in common [33].

Theorem 17. *Let $\alpha \geq 1$, $k \geq 2$, and $d > 0$ be constants. Let $(c_n)_{n\in\mathbb{N}}$ be a sequence of integers with $c_n \geq 3$ for all n and let $(\mathcal{F}_n)_{n\in\mathbb{N}}$ be a sequence of strategy set distributions satisfying the common color property. Let $\mathcal{G}_{G_n,(S_i)_{i\in V}}(c_n)$ be the set of all coordination games $\Gamma = (G_n, c_n, (S_i)_{i\in V}, \mathbf{1}, \mathbf{1}, \mathbf{0})$ on graph $G_n \sim G(n, d/n)$ with c_n colors, strategy set $S_i \sim \mathcal{F}_n$ for every i, equal-split distribution rule, unit weights, and no individual preferences. Then there exists a constant $\beta = \beta(d, \alpha)$ such that $(\alpha, k)\text{-}PoA(\mathcal{G}_{G_n,(S_i)_{i\in V}}(c_n)) \leq \beta$.*

The proof of Theorem 17 relies on the following probabilistic result regarding the maximum size of a matching in Erdős-Rényi random graphs [25]:

Lemma 18 ([25]). *Let $\delta > 0$ be fixed. Then there is a constant $\mu^* = \mu^*(d)$ such that*

$$\lim_{n\to\infty} \mathbb{P}_{G_n\sim G(n,d/n)}\left(|\mu(G_n)/n - \mu^*| \geq \delta\right) = 0,$$

where $\mu(G_n)$ is the size of a maximum matching in G_n.

Proof of Theorem 17. Let $M(G_n)$ be a maximum matching in G_n of size $\mu(G_n)$. For a fixed edge $e = \{i, j\} \in M(G_n)$, the probability that the strategy sets S_i and S_j of players i and j satisfy $S_i \cap S_j \neq \emptyset$ is at least d_0 by the common color property. Here, we implicitly use that the strategy sets are drawn independently from the graph topology. Combining this with the lemma above, it follows that there exists a constant $\beta_0 = \beta_0(d_0)$ such that with high probability there exist $\beta_0 n$ pairwise node-disjoint edges in G_n for which the players corresponding to the endpoints have a common color in their strategy set. This follows from standard Chernoff bound arguments similarly as in the proof of Theorem 7. As a consequence, $u_i(s) + u_j(s) \geq 1/(2\alpha)$ for any (α, k)-equilibrium, because otherwise player i and j could jointly deviate, as $k \geq 2$, to their common color.

This implies that there exists a constant $\beta_1 = \beta_1(d_0, \alpha)$ such that with high probability

$$(\alpha, k)\text{-PoA}\left(\mathcal{G}_{G_n,(S_i)}(c_n)\right) \leq \beta_1 \frac{\mu(G_n)}{n}.$$

Finally, using again standard Chernoff bound arguments as in the proof of Theorem 7, it follows that $\mu(G_n)/n \leq \beta_2(d)$ with high probability. This completes the proof. □

The statement of Theorem 17 does not hold for $k = 1$. To see this, consider the uniform distribution over strategy sets $\{s_0, s_1\}, \dots, \{s_0, s_n\}$. In the strategy profile where every player picks her color different from s_0, at most a constant number of edges will be satisfied with high probability. Thus, $(\alpha, 1)$-PoA $\geq \beta n$ for some β with high probability.

APPENDICES

A ON POSSIBLE EXTENSIONS OF COORDINATION GAMES

We discuss some possible (natural) extensions of the coordination game model introduced in Section 2. However, we show that the results obtained in Section 3 and/or Section 4 no longer hold for these extensions.

A.1 Global Distribution Rules

A first natural generalization would be to look at more "global" distribution rules. However, already for slight generalizations of local distribution rules, it can be shown that there exist distribution rules that do not correspond to a generalized weighted Shapley distribution rule, but still guarantee the existence of a pure Nash equilibrium.

For example consider *edge-based distribution rules* defined by a function $g_e : N \times 2^N \rightarrow \mathbb{R}$ for every $e = \{a, b\} \in E$ determining shares $g_e(i, S) = \xi_{i,e,S}$ so if players a and b play the same color, and $\{a, b\} \subseteq S \subseteq N$ is the set of all players that also play that common color, then player $i \in S$ receives a share of

$$\frac{\xi_{i,e,S}}{\sum_{j \in S} \xi_{j,e,S}} w_{ab}$$

of the edge weight w_{ab}, that is, his utility in strategy profile s is

$$u_i(s) = q_i(s_i) + \sum_{e \in E: e \subseteq C_{s_i}} \frac{\xi_{i,e,C_{s_i}}}{\sum_{j \in C_{s_i}} \xi_{j,e,C_{s_i}}} w_e,$$

where C_{s_i} is the set of all players choosing color $k = s_i$. For example, this captures the case of egalitarian sharing in which every edge weight is shared equally between all players choosing the same color (if $\xi_{i,e,S} = 1$ for all $i \in S$).

We show that there exists an edge-based distribution rule, not corresponding to a generalized weighted Shapley value, that guarantees the existence of a pure Nash equilibrium. Consider the graph $G = (V, E)$ with $V = \{1, 2, 3\}$ and $E = \{1, 2\}$. We define $\xi_{3,\{1,2\},S} = 0$ for $S = \{1, 2, 3\}$, that is, player 3 never gets a share of edge $\{1, 2\}$. Moreover, we define $\xi_{1,\{1,2\},\{1,2\}} = \xi_{2,\{1,2\},\{1,2\}} = 1$, but $\xi_{1,\{1,2\},\{1,2,3\}} = 1$ and $\xi_{1,\{1,2\},\{1,2,3\}} = 3$. That is, if players 1 and 2 play a common color, and player 3 a different color, then the edge-weight is split evenly, whereas if player 3 also plays the same color, then the shares are $1/4 w_{12}$ for player 1, and $3/4 w_{12}$ for player 2. Roughly speaking, although player 3 never receives a share from edge w_{12}, he does in fact influence how the edge-weight is split between players 1 and 2.

For any fixed number of colors, and sets of individual preferences, and any weight w_{12}, it can be shown that any better-response sequence converges to a pure Nash equilibrium. This claim follows by observing that player 3 is not influenced by players 1 and 2, and so in any best response sequence she will appear at most once (in the corresponding step, she will deviate to the color yielding the highest individual preference). After this step, players 1 and 2 will always converge to a pure Nash equilibirum, as their distribution rule corresponds to a weighted Shapley distribution rule.

A.2 Hypergraph Coordination Games

Another natural extension would be to consider hypergraph coordination games where edges can have size larger than two as well. However, here the Price of Anarchy immediately becomes unbounded already on instances with one hyperedge of size three. This can easily be seen by constructing a symmetric instance with edge weight 1, without individual preferences, and three common colors. If all three players choose a different color, then that strategy profile s is a pure Nash equilibrium with $u(s) = 0$. If all players play the same color, then the resulting strategy profile s^* is a social optimum with $u(s^*) = 1$.

A.3 Color-dependent Edge Weights

Another possible extension would be to introduce color-dependent edge weights, so the edge-weights split between two players can differ, depending on the common color that they have. The characterizations in Theorems 8 and 11 still hold. The results in Theorems 8 and 11 are even stronger, since we can obtain the characterization already in the special case that the edge-weights are actually color-independent. However, the Price of Anarchy becomes unbounded already for symmetric coordination games on a graph with one edge. To see this claim, assume that the players forming the endpoints of the edge have no individual preferences and that $c = 2$. For the first color, we set the edge weight $w_{e,1} = 0$, and for the second color, we set $w_{e,2} = 1$. If both players play color 1, then we obtain a pure Nash equilibrium s with $u(s) = 0$, and if both players play color 2, then we obtain the social optimum s^* with $u(s^*) = 1$.

B RESOURCE ALLOCATION GAMES

A resource allocation game [19, 32] $\mathcal{G} = (N, R, (S_i)_{i \in N}, (W_r)_{r \in R}, (f_r)_{r \in R})$ is given by a set $N = \{1, \dots, n\}$ of players, a set $R = \{1, \dots, m\}$ of resources, and *strategy sets* $S_i \subseteq 2^R$ for players $i \in N$. Moreover, $W_r : 2^N \to \mathbb{R}$ denotes the welfare function for resource $r \in R$ and f_r the distribution rule of resource $r \in R$. A *distribution rule* $f^W : N \times 2^N$ for welfare function W is mapping with $f(i, S) = 0$ if $i \notin S \subseteq N$. We assume that the f_r^W are *efficient*, meaning that $\sum_{i \in S} f_r(i, S) = W_r(S)$ for all $S \subseteq N$.

For a given strategy profile $\boldsymbol{s} = (s_1, \dots, s_n) \in S$ the utility (pay-off) of player i is defined as

$$u_i(\boldsymbol{s}) = \sum_{r \in s_i} f_r(i, N_r(\boldsymbol{s}))$$

with $N_r(\boldsymbol{s}) = \{i \in N : r \in s_j\}$ the set of players using resource r in profile $\boldsymbol{s}$.

A well-known result from cooperative game theory states that for any fixed welfare function W, there exist real numbers $(\beta_T^W)_{T \subseteq N}$ such that

$$W(S) = \sum_{T \subseteq N} \beta_T^W g_T(S),$$

where, for $T \subseteq 2^N$, $g_T : 2^N \to \mathbb{R}$ is the welfare functions given by $g_T(S) = 1$ if $T \subseteq S$ and zero otherwise. A distribution rule f^W is said to have a *base decomposition* [19] if it can be written as

$$f^W(i, S) = \sum_{T \subseteq N} \beta_T^W f^T(i, S),$$

where $f^T(i, S)$ is given by $f^T(i, S) = 0$ if $T \nsubseteq S$, and, if $T \subseteq S$, $f^T(i, S) = \omega^T$ where $\omega(i) = 0$ if $i \notin S$, and $\omega(i) > 0$ for at least one $i \in S$. This is equivalent to saying that the distribution rules f^T for the welfare functions g_T is a generalized weighted Shapley distribution rule [19].

B.1 Clustering Games as Resource Allocation Games

For a fixed graph $G = (V, E_c \cup E_a)$, distribution rule $\boldsymbol{\xi}$, and $c \in \mathbb{N}$, any game in $\mathcal{G}(G, c, \boldsymbol{\xi})$ can be modeled as a resource allocation game. That is, for every $\Gamma \in \mathcal{G}$, there exists a resource allocation game $\Psi = (N, R, (S_i)_{i \in N}, W, f^W)$ with a one-to-one correspondence between the strategy profiles of Γ and Ψ that preserves improving moves. Here, every resource is equipped with welfare function

$$W(S) = \sum_{T \subseteq \{V, E\}} \beta_T^W g_T, \tag{19}$$

where $\beta_T^W = 1$ if $T \in V$ or $T \in E$ with $\tau(T) = 1$, and $\beta_T^W = -1$ if $T \in E$ with $\tau(T) = 0$. Note that the welfare function W is independent of $\boldsymbol{w}$ and $\boldsymbol{q}$. Moreover, the distribution rule f^W has a base decomposition given by $\boldsymbol{\xi}$. That is, the value β_T^W for $T = \{i\}$ with $i \in V$ is always given to player i, and for $T \in E$, the corresponding weight $\beta_W^T \in \{-1, 1\}$ is split among the players in T according to $\boldsymbol{\xi}$ (note that this yields an efficient distribution rule).

The modeling of a clustering game as a resource allocation game is done by including many copies of a single resource, a technique also used by Gopalakrishnan et al. [19]. The details of this procedure are not hard to derive and left to the reader at this point.

B.2 Interpretation of Theorem 11

Gopalakrishnan et al. [19] show the impressive result that, for any fixed welfare function W, if a distribution rule f^W guarantees the existence of a pure Nash equilibrium in *every* resource allocation game $(N, R, (S_i)_{i \in N}, W, f^W)$, for arbitrary N, R, and $(S_i)_{i \in N})$, then the distribution rule f^W must be a generalized weighted Shapley distribution rule. We refer the reader to Reference [19] for the formal definition of generalized weighted Shapley distribution rules for general resource allocation games. Moreover, any generalized weighted Shapley distribution rule guarantees pure Nash equilibrium existence [19].

Roughly speaking, they first show that if an equilibrium is guaranteed to exist in every game where resources are equipped with welfare function W, then the distribution rule f^W must have a base-decomposition (as introduced above). They then continue by showing that generalized weighted Shapley distribution rules (which are base-decomposable by definition) are the only ones guaranteeing existence among all base-decomposable distribution rules.

In Theorem 11, we essentially give an alternative, but also stronger, proof for this final step of the proof of Gopalakrishnan et al. [19], in the (very) special case where W is of the form (19) and $\beta_W^T > 0$ for all $T \in \{V, E\}$. That is, we show that if a pure Nash equilibrium is always guaranteed to exist in a coordination game with individual preferences, where there are three common strategies (colors), then the distribution rule must be a generalized weighted Shapley distribution rule. This then implies the result of Gopalakrishnan et al. [19], since coordination games with individual preferences essentially form a subclass of all resource allocation games where resources are equipped with W (using the modeling of clustering games as resource allocation games mentioned before).

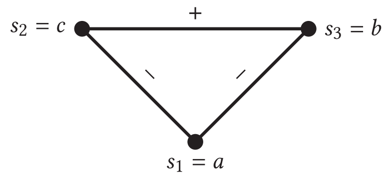

Fig. 4. Counter-example for Theorem 1 in the case of a clustering game with anti-coordination edges.

However, Example 19 below illustrates that, in general, this is not true if $\beta_W^T < 0$ for some $T \in E$. That is, if certain coefficients β_W^T are negative, then in general it does not suffice to focus on the subclass of corresponding clustering games to derive that ξ must be a generalized weighted Shapley distribution rule. In this case, one has to make use of more complex resource allocation games, i.e., more complex than clustering games with individual preferences, to guarantee that ξ is a generalized weighted Shapley distribution rule (the resource allocation games used by Gopalakrishnan et al. [19] for this final step are indeed more complex than clustering games in this case).

Example 19. Consider the instance in Figure 4, and let ξ be some arbitrary local distribution rule. Fix arbitrary weights w_{12}, w_{23}, and w_{31} and individual preferences q_l^i for $i = 1, 2, 3$ and $l \in C = \{1, \ldots, c'\}$. (We use c' here to denote the number of strategies in the common strategy set instead of c.) We claim that a pure Nash equilibrium always exists.

Consider the strategy profile in Figure 4 and assume without loss of generality that a is the color for which player 1 his individual preference is maximal, i.e., $a = \arg\max\{q_1(l) : l \in C\}$. If there is some profile $\mathbf{s}' = (a, c, b)$ with $a \neq b, c$ (but possibly $b = c$) where c and b are best-responses for, respectively, players 2 and 3, then we find a pure Nash equilibrium by definition of a.

It now suffices to show that for any profile of the form (a, c, b) where either player 2 or 3 has a best-response to a, we can always perform a sequence of best-response moves that end up in a pure Nash equilibrium. Assume without loss of generality that a is a best-response for player 2 in the profile (a, c, b). This in particular implies that

$$a = \arg\max\{q_2(l) : l \in C\}.$$

We now consider player 3 in the profile (a, a, b).

(1) **Player 3 only has a as best-response.** Then, we let player 3 switch to a to get the profile (a, a, a). Note that a is still a best-response for player 2 as well, since w_{23} is non-negative and edge $\{2, 3\}$ is now satisfied as well. To summarize, both players 2 and 3 are playing a best-response in the profile (a, a, a). Now, if player 1 has a best-response different from a, say, c, then in particular a remains a best-response for both players 2 and 3 in the profile (c, a, a), since edges $\{1, 2\}$ and $\{1, 3\}$ are anti-coordination edges and their weights are non-negative. That is, (c, a, a) is a pure Nash equilibrium.

(2) **Player 3 has only b as best-response.** Then both players 2 and 3 are playing a best-response in the profile (a, a, b). Now suppose player 1 has a different best-response than a.
   (i) **Player 1 has a best-response to some color $d \neq b$.** Then a is still a best-response for player 2 in the profile (d, a, b). If player 3 now has a best-response other than b, then it must be a, otherwise he would have had a response better than b in the profile (a, a, b) as well. Clearly, in the profile (a, a, d) both players 2 and 3 are playing a best-response. If player 1 still has a best-response, then it must be b (otherwise, he would have had a different best-response than d before). The profile (a, a, b) is a pure Nash equilibrium.
   (ii) **Player 1 only has a best-response to b.** Then a is still a best-response for player 2 in (b, a, b). Suppose that player 3 has a best-response other than b. If a is a best-response for

player 3, then we reach the pure Nash equilibrium (b, a, a), since player 2 clearly plays a best-response, and player 1 cannot have a better response, otherwise deviating to b in the profile (a, a, b) was not a best-response.

Therefore, suppose player 3 has a best-response different from a, say, e. Then b is still a best-response for player 1. If player 2 has a better response than a, then it must be e, otherwise a would not have been a best-response in the initial profile. Clearly, player 3 plays a best-response in the profile (b, e, e). If player 1 still has a better response than b, then it must be a, otherwise b would not have been a best-response in the profile (b, a, b). The resulting profile (a, e, e) is a pure Nash equilibrium, since player 3 cannot play a better response, otherwise e would not have been a best-response in the profile (a, b, b).

(3) **Player 3 has $c \neq a, b$ as best-response.** Player 2 now cannot have a best-response to some color $f \neq c$, otherwise a would not have been a best-response in the initial profile (a, b, b). Therefore, suppose that c is a best-response. Then the resulting profile (a, c, c) is a pure Nash equilibrium, since player 1 has maximum possible utility, and player 3 clearly has no better response than c, otherwise c would not have been a best-response in (a, a, b). We can now assume to be in the profile (a, a, c) in which players 2 and 3 play a best-response.

(i) **Player 1 has a best-response to $d \neq c$.** Then either the resulting profile (d, a, c) is a pure Nash equilibrium or player 3 still has a best-response to a, but then the resulting profile (d, a, a) is a pure Nash equilibrium.

(ii) **Player 1 has a best-response to c.** Then player 2 still has a as best-response. Suppose that player 3 now has a better response. If it a, then the resulting profile (a, a, c) is a pure Nash equilibrium. Therefore, suppose that player 3 has a better response to some color $g \neq a$. Then c is still a best-response for player 1. If a is also still best-response for player 2, then (c, a, g) is a pure Nash equilibrium. Therefore, suppose that player 2 has a better response. Then this must be g (similar reasoning as before). Clearly, in the profile (c, g, g) player 3 is still playing a best-response. Suppose that player 2 still has a better-response, then this must be a. The profile (a, g, g) is a pure Nash equilibrium.

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