QSETH strikes again: finer quantum lower bounds for lattice problem, strong simulation, hitting set problem, and more

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Abstract

While seemingly undesirable, it is not a surprising fact that there are certain problems for which quantum computers offer no computational advantage over their respective classical counterparts. Moreover, there are problems for which there is no 'useful' computational advantage possible with the current quantum hardware [BMN+21]. This situation however can be beneficial if we *don't want* quantum computers to solve certain problems fast - say problems relevant to post-quantum cryptography. In such a situation, we would like to have evidence that it is difficult to solve those problems on quantum computers; *but what is their exact complexity?*

To do so one has to prove lower bounds, but proving unconditional time lower bounds has never been easy. As a result, resorting to conditional lower bounds has been quite popular in the classical community and is gaining momentum in the quantum community. In this paper, by the use of the QSETH framework [Buhrman-Patro-Speelman 2021], we are able to understand the quantum complexity of a few natural variants of CNFSAT, such as parity-CNFSAT or counting-CNFSAT, and also are able to comment on the non-trivial complexity of approximate-#CNFSAT; both of these have interesting implications about the complexity of (variations of) lattice problems, strong simulation and hitting set problem, and more.

In the process, we explore the QSETH framework in greater detail than was (required and) discussed in the original paper, thus also serving as a useful guide on how to effectively use the QSETH framework.

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1 Introduction

A popular classical hardness conjecture known as the Strong Exponential Time-Hypothesis (SETH) says determining whether an input CNF formula is satisfiable or not cannot be done in time $O(2^{n(1-\varepsilon)})$ for any constant $\varepsilon > 0$ [IP01, IPZ01]. Several fine-grained lower bounds based on SETH were shown ever since; see [Vas15, Wil18] for a summary of many such results. Some of the SETH-hard problems are building blocks for fine-grained cryptography [BRSV17, LLW19]. Following the introduction to SETH, hardness assumptions for variations of CNFSAT, for e.g., \oplus SETH (conjecturing complexity of parity CNFSAT, output of the parity of satisfying assignments to input CNF formula), #SETH (conjectures complexity of counting CNFSAT, output the count of the satisfying assignments), were also proposed and several other fine-grained lower bounds based on these variants of SETH were given [CDL⁺16].

When considering quantum computation the SETH conjecture is no longer true, as one can, using Grover's algorithm for unstructured search [Gro96], solve the CNFSAT problem in $O(2^{\frac{n}{2}} \cdot \text{poly}(n))$ time. However, conjectures such as #SETH or \oplus SETH are likely to still hold in the quantum setting because a Grover-like quantum speedup is not witnessed when the task is to compute the total number of satisfying assignments or to compute the parity of the total number of satisfying assignments. This situation can in some cases be exploited to give better quantum lower bounds than one would get from the conjectured quantum lower bound for the vanilla CNFSAT problem. This makes it at least as relevant (if not more) to study variants of CNFSAT and their implications in the quantum setting, as has been done classically. In fact, motivated by this exact observation, Buhrman, Patro, and Speelman [BPS21] introduced a framework of Quantum Strong Exponential-Time Hypotheses (QSETH) as quantum analogues to SETH, with a striking feature that allows one to 'technically' unify conjectures such as quantum analogues of \oplus SETH, #SETH, maj-SETH, etc. under one *umbrella* conjecture.

In this paper, we use the QSETH framework to 'generate' natural variations of QSETH such as \oplus QSETH, #QSETH, maj-QSETH, etc., which otherwise (in our opinion) would already be acceptable standalone conjectures in the quantum setting, and study some of their interesting implications. Additionally, we also use the QSETH framework in proving *non-trivial* quantum lower bounds for approximately counting the number of satisfying assignments to CNF formulas, a problem whose complexity has been of interest in the classical setting [DL21]; we study its quantum implications. See Section 2 for details. Proof of this result follows from a more detailed exploration of the QSETH framework than what was required in the original paper. Thus, as another contribution of this paper, we present a useful guide on how to effectively use the QSETH framework.

The QSETH framework In their framework, Buhrman et al. consider the problem in which one is given formula or circuit representation of a Boolean function $f : \{0,1\}^n \to \{0,1\}$ and is asked whether a property $P : \mathcal{D} \subseteq \{0,1\}^{2^n} \to \{0,1\}$ on the truth table¹ of this formula evaluates to 1. They conjectured that when the circuit representation is obfuscated enough then for *most* properties P (that are compression-oblivious properties as we will see in Definition 1.1), the time taken to compute P on the truth table of poly(n)-sized circuits is lower bounded by Q(P), i.e. the 1/3-bounded error quantum query complexity of P, on all bit strings of length 2^n .

¹Truth table of a formula ϕ on n variables, denoted by $tt(\phi)$, is a 2^n bit string derived in the following way $tt(\phi) = \bigcap_{a \in \{0,1\}^n} \phi(a)$; the symbol \circ denotes concatenation.

It is not hard to see that such a conjecture cannot be true for *all* properties. In principle, one can construct properties for which the above statement would not hold. For instance, consider a property P that is trivial on truth tables of small formulas (i.e., poly(n) size) but complicated on formulas of longer length. These kinds of properties are likely to have very high quantum query complexity, but in reality, it will be trivial to compute such a P on formulas of poly(n) size. In order to prevent such scenarios the authors in [BPS21] introduce the notion of compressionoblivious properties which they believe encompasses most of the naturally occurring properties. See Sections 2.2 and 2.3 of their paper for a detailed discussion on this topic. To give a bit of intuition, first consider the set of truth tables corresponding to the set of poly(n) size formulas on n variables and consider the set of all the 2^n bit strings. Compression-oblivious properties are those properties for which one cannot save in computational time to compute them on a string from the former set in comparison to computing the same property on strings from the latter set. More formally,

Definition 1.1 (Compression Oblivious Properties [BPS21]). Let Γ denote a class of representation such as poly-sized depth-2 circuits or poly-sized circuits of a more complex class. Then compression oblivious properties corresponding to the class Γ , denoted by $\mathcal{CO}(\Gamma)$, are the set of (partial or total) properties such that time taken to compute P on the truth table of the input circuit from Γ is at least the quantum query complexity of P on all strings, i.e. $\Omega(Q(P)^{1-\alpha})$ for all constants $\alpha > 0$.

With that, they conjecture the following.

Conjecture 1.2 (QSETH framework [BPS21]). For each class of representations such as polysized depth-2 circuits or poly-sized circuits of a more complex class, let us denote by Γ , for all compression oblivious properties P of class Γ , given a circuit $C \in \Gamma$ the time taken to compute P on the truth table of C is at least $\Omega(Q(P)^{1-\alpha})$ for all constants $\alpha > 0$.

It is good to note that the QSETH framework allows us to consider formulas of complicated classes, however, taking Γ as the class of all poly-sized CNF or DNF formulas suffices for the results presented in this paper. Therefore, using AC₂⁰ to denote the class of all poly-sized CNF and DNF formulas, we define AC₂⁰-QSETH as the following conjecture and use this as the main conjecture of our paper.

Conjecture 1.3 (AC₂⁰-QSETH). For all compression oblivious properties $P \in \mathcal{CO}(AC_2^0)$, given a circuit $C \in AC_2^0$ the time taken to compute P on the truth table of C is at least $\Omega(Q(P)^{1-\alpha})$ for all constants $\alpha > 0$.

One can now *almost* directly use AC_2^0 -QSETH instead of the general QSETH framework to understand the hardness of computing certain properties on truth tables of CNF formulas. The emphasis on almost is because of the following two reasons: firstly, there are certain properties that might be easy on CNF formulas and hard on DNF formulas which one has to formally verify, and secondly, one needs to ensure that the property we are interested in is compression oblivious. Moreover, for properties defined on N bit strings that can be computed in polylog(N) space and have quantum query complexity $\tilde{\omega}(\sqrt{N})$ it is not easy to show that they are compression oblivious, as doing so would separate P from PSPACE; see Theorem 9 in [BPS21] for more details. This means even for natural properties such as COUNT or PARITY one cannot easily check if they are compression oblivious. In such situations, we resort to conjecturing that the property is compression oblivious. See Figure 1 for a step-by-step guide on how we use the QSETH framework in a plug-and-play fashion.

In comparison to the original QSETH paper where the framework was introduced and applied to a more complex class of formulas,² this paper instead serves as a guide to using QSETH for the lowest level of formulas, i.e., poly-sized CNF and DNF formulas, in a more elaborate fashion.

Problem	Variants	Quantum lower bound	Reference
	Vanilla	$2^{\frac{n}{2}-o(n)}$	Corollary 2.1
	Parity	$2^{n-o(n)}$	Corollary 2.7
	Majority	$2^{n-o(n)}$	Corollary 2.8
	Strict Majority	$2^{n-o(n)}$	Corollary 2.8
	Count	$2^{n-o(n)}$	Theorem 2.6
CNFSAT	Count_q	$2^{n-o(n)}$	Corollary 2.10
	Δ -Additive error	$\left(\sqrt{\frac{2^n}{\Delta}} + \frac{\sqrt{\hat{h}(2^n - \hat{h})}}{\Delta}\right)^{1 - o(1)}$	Theorem 2.15
	$\gamma\text{-}\text{Multiplicative factor}$	$\left(\frac{1}{\gamma}\sqrt{\frac{2^n-\hat{h}}{\hat{h}}}\right)^{1-o(1)}$	Theorem 2.21
	Vanilla	$2^{\frac{n}{2}-o(n)}$	Section 4, $[ACL+20]$
	Parity	$2^{n-o(n)}$	Corollary 2.24
$k ext{-SAT}$	Count	$2^{n-o(n)}$	Corollary 2.24
$k = \Theta(\log(n))$	Count_q	$2^{n-o(n)}$	Corollary 2.24
	γ -Multiplicative factor	$\left(\frac{1}{\gamma}\sqrt{\frac{2^n-\hat{h}}{\hat{h}}}\right)^{1-o(1)}$	Corollary 2.25

Table 1: Overview of conditional lower bounds for variants of CNFSAT and k-SAT. The variable \hat{h} in the above table is an arbitrary natural number satisfying $\gamma \hat{h} \geq 1$. The multiplicative approximation factor γ is between 2^{-n} to 1, and the additive approximate factor Δ is between 2^{-1} to 2^n .

Summary and technical overview

- We zoom into Buhrman et al.'s QSETH framework at the lowest-level formula class, i.e., the class of polynomial-size CNFs and DNFs, and use it to study the quantum complexity of variations of CNFSAT problems. The QSETH framework is quite general which also makes it not entirely trivial to use it thus, we present a useful guide on how to effectively use the AC₂⁰-QSETH conjecture, for e.g., what lemmas need to be proved and what assumptions are needed to be made in order to understand the quantum complexity of CNFSAT and its variants; see Figure 1.
- We can categorise the several variants of CNFSAT in two ways. First classification can be done by the width of the CNF formulas, i.e., *k*-CNFs versus CNFs of unbounded clause length; second, by the property of the truth table one needs to compute. See the summary of all the variants and their respective quantum time lower bounds in Table 1 and see below for the overview of the techniques used.

 $^{^{2}}$ The authors in [BPS21] extensively used QSETH framework for branching programs or equivalently NC circuits to show non-trivial lower bounds for edit distance and longest common subsequence problems.

Problem	Variants	Quantum lower bound	Reference
	Exact (with n bits precision)	$2^{n-o(n)}$	Theorem 3.2
	Exact (with 2 bits precision)	$2^{n-o(n)}$	Corollary 3.3
STRONG SIMULATION	Δ -Additive error	$\left(\sqrt{\frac{1}{\Delta}} + \frac{\sqrt{\hat{h}(2^n - \hat{h})}}{2^n \Delta}\right)^{1 - o(1)}$	Corollary 3.5
	$\gamma\text{-}\text{Multiplicative factor}$	$\left(\frac{1}{\gamma}\sqrt{\frac{2^n-\hat{h}}{\hat{h}}}\right)^{1-o(1)}$	Theorem 3.7
CVP_p for $p \notin 2\mathbb{Z}$		$2^{\frac{n}{2}-o(n)}$	Section 4
	Vanilla	$2^{n-o(n)}$	Corollary 4.7
LATTICE COUNTING (for non-even norm)	γ -Multiplicative factor	$\left(\frac{1}{\gamma}\sqrt{\frac{2^n-\hat{h}}{\hat{h}}}\right)^{1-o(1)}$	Corollary 4.8
	q-count	$2^{n-o(n)}$	Corollary 4.10
	Vanilla	$n^{1-o(1)}$	$[ACL^+20, BPS21]$
	Parity	$n^{2-o(1)}$	Corollary 5.7
Orthogonal Vectors	Count	$n^{2-o(1)}$	Corollary 5.7
	γ -Multiplicative factor	$\left(\frac{1}{\gamma}\sqrt{\frac{n^2-\hat{h}}{\hat{h}}}\right)^{1-o(1)}$	Corollary 5.7
	Vanilla	$2^{\frac{n}{2}-o(n)}$	Corollary 5.4
	Parity	$2^{n-o(n)}$	Corollary 5.4
HITTING SET	Count	$2^{n-o(n)}$	Corollary 5.4
	γ -Multiplicative factor	$\left(\frac{1}{\gamma}\sqrt{\frac{2^n-\hat{h}}{\hat{h}}}\right)^{1-o(1)}$	Corollary 5.4
\oplus Set Cover		$2^{n-o(\overline{n})}$	Corollary 5.9

Table 2: Overview of lower bounds based on AC₂⁰-QSETH. The variable \hat{h} in the above table is an arbitrary natural number satisfying $\gamma \hat{h} \geq 1$. The multiplicative approximation factor γ is between 2^{-n} to 1, while the additive approximate factor Δ is between 2^{-n-1} to 1.

- To prove the quantum time lower bounds for the property variants of CNFSAT problem we invoke AC₂⁰-QSETH (Conjecture 1.3). But, AC₂⁰-QSETH conjectures the hardness of properties on a set of CNF and DNF formulas. For properties like COUNT, PARITY, MAJORITY, etc., it easily follows from De Morgan's laws that these properties are equally hard on both CNF and DNF formulas. However, such arguments no longer hold when the properties are approximate variants of count for which we give nontrivial proofs; see Sections 2.3 and 2.4.
- Additionally, we also use AC_2^0 -QSETH to understand quantum complexity of k-SAT and its property variants. As a first step we study the classical reduction from CNFSAT to k-SAT given by [CIP06] and observe that the $2^{\frac{n}{2}}$ quantum lower bound for k-SAT, for $k = \Theta(\log n)$, follows from the quantum lower bound of CNFSAT. Moreover, we make an important observation that this reduction by [CIP06] is count-preserving³ and can be used to conclude lower bounds for other counting variants of k-SAT. See Table 1 for the summary of these bounds.

³Should not be mistaken to be parsimonious, see Section 2.5 for details.

- Having (somewhat) understood the complexities of the above-mentioned variants of CN-FSAT, we then prove conditional quantum time lower bounds for lattice problem, strong simulation, orthogonal vectors, set cover, hitting set problem, and their respective variants; see Table 2.
 - The quantum $2^{\frac{n}{2}}$ time lower bound we present for CVP_p (for $p \notin 2\mathbb{Z}$) follows from a reduction from k-SAT to CVP_p by [BGS17, ABGS21] and from the hardness result of k-SAT we present. Though such a result would also trivially follow by using Aaronson et al.'s version of QSETH, we stress that our hardness result of k-SAT is based on basic-QSETH which is a more believable conjecture.⁴
 - Additionally, we also discuss the quantum complexity of the lattice counting problem (for non-even norm). We present a reduction, using a similar idea of [BGS17], from #k-SAT to the lattice counting problem and we show a 2^n time quantum lower bound for the latter when $k = \Theta(\log n)$. As mentioned earlier, we get a 2^n time quantum lower bound for #k-SAT, when $k = \Theta(\log n)$, using AC₂⁰-QSETH.
 - As another application to the bounds we get from the property variants of CNFSAT we look at the strong simulation problem. It was already established by [CHM21, VDN10] that strong simulation of quantum circuit is a #P-hard problem but in this work we give exact lower bounds for the same. Additionally, using the lower bounds of approximate counts of CNFSAT we are able to shed light on how hard it is to quantumly solve the strong simulation problem with additive and multiplicative error approximation.
 - Last but not least, we are also able to use the lower bounds for the property variants of CNFSAT to give interesting lower bounds for orthogonal vectors, hitting set problem and their respective variants. See Section 5 for more details.

Related work Our paper is a follow-up work to the original QSETH paper by [BPS21]; also the list of problems for which we show lower bounds does not overlap with the problems studied in [BPS21]. A basic version of QSETH was also introduced by Aaronson et al. [ACL⁺20] where they primarily used it to study the quantum complexity of closest pair and bichromatic pair problems; they also discuss the complexity of the (vanilla version of) orthogonal vector problem. Prior to this work, a quantum hitting set conjecture was proposed and its implications were discussed in Schoneveld's bachelor thesis [Sch22] but their definition of hitting set is different from ours. In our work, we observe that the parsimonious reduction from CNFSAT to hitting set (Definition 5.1) given by [CDL⁺16] is easily quantizable, using which we get a QSETH-based lower bound.

Structure of our paper In Section 2, we discuss the quantum complexity of variants of CN-FSAT problems, for e.g. #CNFSAT, \oplus CNFSAT, γ -#CNFSAT, etc., conditional on AC₂⁰-QSETH. The broad idea to show the hardness results of these variants is quite similar; the *several* lemmas only account for the properties being different. Using results from Section 2, we show quantum time lower bounds for strong simulation problem in Section 3, lattice problems in Section 4, and orthogonal vectors, hitting set and set cover in Section 5. We conclude with open questions and future directions in Section 6. The appendix section of this paper contains proofs of theorems that are long to state or lemmas of observational nature.

⁴If basic-QSETH from Buhrman et al.'s framework is false then Aaronson et al's QSETH is also false, but the implication in the other direction is not obvious.



Figure 1: Step-by-step guide on how to use the QSETH framework in a plug-n-play manner to show hardness results for P-CNFSAT. Here P can be any (partial or total) Boolean property.

2 Lower bounds for variants of CNFSAT using AC_2^0 -QSETH

We will now define several variants of CNFSAT problem and using AC_2^0 -QSETH understand the quantum complexity of all of them. The consequences of these hardness results, some of which follow immediately and the rest with some work, will be discussed in Sections 3 to 5. We begin with some common variants of CNFSAT problem which are also very well studied classically [CDL+16]; we do this in Section 2.2. And, proceed with some less popular variants (Sections 2.3 to 2.5) but with interesting consequences (presented in Sections 3 to 5).

2.1 Quantum complexity of CNFSAT

We first restate the quantum hardness of CNFSAT before delving into showing hardness results for its other variants. Interestingly, for the property $OR : \{0,1\}^{2^n} \to \{0,1\}$, where for $x \in \{0,1\}^{2^n}$ we define OR(x) = 1 if $|x| \ge 1$ and OR(x) = 0 whenever |x| = 0, we can explicitly prove that $OR \in CO(AC_2^0)$ in the following way: The adversarial set $\{x : x \in \{0,1\}^{2^n} \text{ and } |x| \in \{0,1\}\}$ gives the tight query bound for the property OR are truth tables of functions that can be represented by O(n) sized CNF formulas. Namely, these are given by the formulas that accept a single possible input, which can be constructed by using *n* clauses that each contain a single variable or its negation. Also, note that computing OR on truth tables of DNF formulas of poly(*n*) length can be computed in poly(*n*) time. Hence, using AC_2^0 -QSETH we can recover the following Basic-QSETH conjecture.

Corollary 2.1 (BASIC-QSETH [BPS21]). If AC_2^0 -QSETH (Conjecture 1.3) is true then there is no bounded error quantum algorithm that solves CNFSAT on n variables and m clauses in $O\left(2^{\frac{n(1-\delta)}{2}}m^{O(1)}\right)$ time, for any $\delta > 0$.

2.2 Quantum complexity of #CNFSAT, \oplus CNFSAT, $\#_q$ CNFSAT and maj-CNFSAT

Definition 2.2. Some popular variants of CNFSAT.

- 1. #CNFSAT: Given a CNF formula ϕ on n input variables, output t where $t = |\{x : x \in \{0,1\}^n \text{ and } \phi(x) = 1\}|$ is the number of satisfying assignments to ϕ .
- 2. \oplus CNFSAT: Given a CNF formula ϕ on n input variables, output t where $t = |\{x : x \in \{0,1\}^n \text{ and } \phi(x) = 1\}| \mod 2$ is the parity of the number of satisfying assignments to ϕ .
- 3. $\#_q$ CNFSAT: Given a CNF formula ϕ on n input variables and an integer $q \in [2^n] \setminus \{1\}$, output t where $t = |\{x : x \in \{0,1\}^n \text{ and } \phi(x) = 1\}| \mod q$ is the q-count of the number of satisfying assignments to ϕ .
- 4. MAJ-CNFSAT: Given a CNF formula ϕ on n input variables, output if majority of the assignments satisfy ϕ , i.e., if $t = |\{x : x \in \{0,1\}^n \text{ and } \phi(x) = 1\}|$ output 1 if $t \ge 2^{n-1}$ else output 0.
- 5. st-MAJ-CNFSAT: Given a CNF formula ϕ on n input variables, output if a strict majority of the assignments satisfy ϕ , i.e., if $t = |\{x : x \in \{0,1\}^n \text{ and } \phi(x) = 1\}|$ output 1 if $t > 2^{n-1}$ else output 0.

We now define the Boolean properties corresponding to items in Definition 2.2 for completeness.⁵

Definition 2.3. Some popular properties defined on Boolean bit strings.

- 1. COUNT: Let COUNT: $\{0,1\}^N \to [N] \cup \{0\}$ be a non-Boolean function such that $\text{COUNT}(x) = |\{i : x \in \{0,1\}^N \text{ and } x_i = 1\}|.$
- 2. PARITY: Let PARITY : $\{0,1\}^N \to \{0,1\}$ be a Boolean function with $\text{PARITY}(x) = |\{i : x \in \{0,1\}^N \text{ and } x_i = 1\}| \mod 2.$
- 3. COUNT_q: Let q be an integer and let COUNT_q : $\{0,1\}^N \to [q-1] \cup \{0\}$ be a non-Boolean function with COUNT_q(x) = $|\{i : x \in \{0,1\}^N \text{ and } x_i = 1\}| \mod q$.
- 4. MAJORITY: Let MAJORITY: $\{0,1\}^N \to \{0,1\}$ be a Boolean function with

$$MAJORITY(x) = \begin{cases} 1 & if \ t \ge \frac{N}{2}, \\ 0 & otherwise; \end{cases}$$

here $t = |\{i : x \in \{0,1\}^N \text{ and } x_i = 1\}|.$

And, there is also the following function almost similar to MAJORITY.

5. st-MAJORITY: Let st-MAJORITY: $\{0,1\}^N \to \{0,1\}$ be a Boolean function with

$$st\text{-MAJORITY}(x) = \begin{cases} 1 & if \ t > \frac{N}{2}, \\ 0 & otherwise; \end{cases}$$

here $t = |\{i : x \in \{0, 1\}^N \text{ and } x_i = 1\}|.$

Theorem 2.4 (Implicit in [BBC⁺01]). The bounded error quantum query complexity for COUNT, PARITY, MAJORITY and st-MAJORITY on inputs of N bit Boolean strings is $\Omega(N)$.

Proof. There is a seminal result by $[BBC^+01]$ where they show that the bounded-error quantum query complexity of a (total) Boolean function $f : \{0,1\}^N \to \{0,1\}$, denoted by Q(f) is lower bounded by the degree of a minimum-degree polynomial p that approximates f on all $X \in \{0,1\}^N$, i.e., $|p(X) - f(X)| \leq 1/3$; let us denote this degree by deg(f). Another important result by Paturi [Pat92] showed that if f is a non-constant, symmetric⁶ and a total Boolean function on $\{0,1\}^N$ then $deg(f) = \Theta(\sqrt{N(N - \Gamma(f))})$ where $\Gamma(f) = \min\{|2k - N + 1| : f_k \neq f_{k+1} \text{ and } 0 \leq k \leq N - 1\}$ and $f_k = f(X)$ for |X| = k.

Using the above two results we can show the following:

1. $\Gamma(\text{PARITY}) = 0$ for odd N and $\Gamma(\text{PARITY}) = 1$ whenever N is even. Either ways $Q(\text{PARITY}) = \Omega(\Gamma(f)) = \Omega(N)$.

⁶A symmetric Boolean function $f: \{0,1\}^N \to \{0,1\}$ implies f(X) = f(Y) for all X, Y whenever |X| = |Y|.

⁵Note that the QSETH framework though is defined on properties with Boolean outcomes it can be used for non-Boolean properties like COUNT or γ -COUNT as we can instead consider the threshold versions of these properties which are Boolean. In fact, we will soon see the inner working of all our hardness results actually rely on Boolean properties.

- 2. Similar to the above item $\Gamma(MAJORITY) = \Gamma(st-MAJORITY) = 0$ for odd N and $\Gamma(MAJORITY) = \Gamma(st-MAJORITY) = 1$ otherwise. Hence, $Q(MAJORITY) = \Omega(N)$ and $Q(st-MAJORITY) = \Omega(N)$.
- 3. Any of these above three properties can be computed from COUNT. Hence, the quantum query lower bound for COUNT is also $Q(\text{COUNT}) = \Omega(N)$.

As we don't yet know how to prove compression-obliviousness of complicated properties (Theorem 9 in [BPS21]) we instead conjecture that COUNT, PARITY, MAJORITY and st-MAJORITY are compression oblivious for poly-sized CNF and DNF formulas.

Conjecture 2.5. Let AC_2^0 denote the class of poly(n) sized CNF and DNF formulas on n input variables then the following properties

- 1. COUNT: $\{0,1\}^{2^n} \to [2^n] \cup \{0\},\$
- 2. PARITY : $\{0,1\}^{2^n} \to \{0,1\},\$
- 3. $\operatorname{COUNT}_q : \{0,1\}^{2^n} \to [q-1] \cup \{0\},\$
- 4. MAJORITY : $\{0,1\}^{2^n} \to \{0,1\}, and$
- 5. st-majority: $\{0,1\}^{2^n} \to \{0,1\}$

stated in Definition 2.3 are in $\mathcal{CO}(AC_2^0)$.

We can now invoke AC_2^0 -QSETH (as mentioned in Conjecture 1.3) to prove the quantum hardness for #CNFSAT, \oplus CNFSAT, MAJ-CNFSAT and st-MAJ-CNFSAT.

Theorem 2.6 (#QSETH). There is no bounded error quantum algorithm that solves #CNFSAT on n variables and m clauses in $O(2^{n(1-\delta)}m^{O(1)})$ time for any $\delta > 0$ unless AC_2^0 -QSETH (Conjecture 1.3) is false or COUNT $\notin CO(AC_2^0)$ (i.e., Item 1 of Conjecture 2.5 is false).

Proof. By way of contradiction let us assume that there exists a bounded-error quantum algorithm \mathcal{A} that solves #CNFSAT on n variables and m clauses in $O(2^{n(1-\delta)}m^{O(1)})$ time for some $\delta > 0$. Then given a circuit $C \in AC_2^0$ we do one of the following:

- if C is a poly-sized CNF formula then we use the algorithm \mathcal{A} to compute the number of satisfying assignments to C in $O(2^{n(1-\delta)}m^{O(1)})$ time. Or,
- if C is a poly-sized DNF formula then we first construct the negation of C, let us denote by $\neg C$, in poly(n) time; using De Morgan's law we can see that the resulting formula $\neg C$ will be a poly(n) CNF formula. Using \mathcal{A} we can now compute the number of satisfying assignments t to $\neg C$ in $O(2^{n(1-\delta)}m^{O(1)})$ time. The number of satisfying assignments to C would be then $2^n t$.

The existence of an algorithm such as \mathcal{A} would imply that AC_2^0 -QSETH is false. Hence, proved. \Box

Using similar arguments as in the proof of Theorem 2.6 we can conclude the following statements.

Corollary 2.7 (\oplus QSETH). There is no bounded error quantum algorithm that solves \oplus CNFSAT on n variables and m clauses in $O(2^{n(1-\delta)}m^{O(1)})$ time for any constant $\delta > 0$ unless AC_2^0 -QSETH (Conjecture 1.3) is false or PARITY $\notin CO(AC_2^0)$ (i.e., Item 2 of Conjecture 2.5 is false).

Corollary 2.8 (Majority QSETH). There are no bounded-error quantum algorithms that solve

1. MAJ-CNFSAT on n variables and m clauses in $O(2^{n(1-\delta)}m^{O(1)})$ time, or

2. st-MAJ-CNFSAT on n variables and m clauses in $O(2^{n(1-\delta)}m^{O(1)})$ time,

any constant $\delta > 0$ unless AC₂⁰-QSETH is false or MAJORITY $\notin CO(AC_2^0)$ (i.e., Item 4 of Conjecture 2.5 is false).

We can extend the above-mentioned arguments to compute the total number of assignments in \mathbb{F}_q for any $q \geq 2$ as well. To understand the complexity of $\#_q \text{CNFSAT}$ (Definition 2.2) we first need to understand the quantum query complexity of COUNT_q .

Lemma 2.9. Let $2 < q \leq \frac{N}{2}$ be an integer and let $\operatorname{COUNT}_q : \{0,1\}^N \to [q-1]$ be a function such that for any $x \in \{0,1\}^N$ the $\operatorname{COUNT}_q(x) = \operatorname{COUNT}(x) \mod q$. Then $Q(\operatorname{COUNT}_q) = \Omega(\sqrt{N(N-2q+1)})$.

Proof. Let DEC-COUNT_q be a decision version of the COUNT_q defined for all $x \in \{0,1\}^N$ as

$$DEC-COUNT_q(x) = \begin{cases} 1, \text{ if } COUNT_q(x) = q - 1, \\ 0, \text{ otherwise.} \end{cases}$$
(1)

Beals et al. showed that the bounded error quantum query complexity of any Boolean function is lower bounded by the approximate degree of that function [BBC⁺01]. When the function is nonconstant and symmetric then one can use Paturi's theorem to characterise the approximate degree of that function [Pat92]. It is easy to see that DEC-COUNT_q is a non-constant symmetric function. Combining both these results we get that $Q(\text{DEC-COUNT}_q) = \Omega(\sqrt{N(N - \eta(\text{DEC-COUNT}_q))})$; we compute the value of $\eta(\text{DEC-COUNT}_q)$ in the next paragraph.

For any symmetric Boolean function $f : \{0,1\}^N \to \{0,1\}$ the quantity $\eta(f)$ is defined as $\eta(f) = \min_k\{|2k - N + 1|\}$ such that $f_k \neq f_{k+1}$ and $f_k = f(x)$ for |x| = k with $1 \le k \le N - 1$. It is easy to see that DEC-COUNT_q(x) = 1 only for x with hamming weight |x| = rq - 1 where r is an integer and DEC-COUNT_q(x) = 0 elsewhere. Let r' be the integer such that $r'q - 1 \le \frac{N}{2} \le (r' + 1)q - 1$ then the k minimising $\eta(\text{DEC-COUNT}_q)$ is either r'q - 1 or (r' + 1)q - 1. With a back-of-the-envelope-calculation we can see that $\eta(\text{DEC-COUNT}_q) \le 2q - 1$ which in turn implies that $N - \eta(\text{DEC-COUNT}_q) \ge N - 2q + 1$. Therefore, $Q(\text{DEC-COUNT}_q) = \Omega(\sqrt{N(N - 2q + 1)})$.

As one can compute DEC-COUNT_q using an algorithm that computes $COUNT_q$, we therefore have $Q(COUNT_q) = \Omega(\sqrt{N(N-2q+1)})$.

Using Conjectures 1.3 and 2.5 we can now conclude the following.

Corollary 2.10 (#_qQSETH). Let $2 < q \le 2^{n-1}$. There is no bounded error quantum algorithm that solves #_qCNFSAT on n variables and m clauses in $O\left(\left(\sqrt{2^n(2^n-2q+1)}\right)^{1-\delta}m^{O(1)}\right)$ time for any constant $\delta > 0$ unless AC₂⁰-QSETH (Conjecture 1.3) is false or COUNT_q $\notin CO(AC_2^0)$ (i.e., Item 3 of Conjecture 2.5 is false).

Quantum complexity of Δ -add-#CNFSAT 2.3

Instead of the exact number of satisfying assignments to a formula, one might be interested in its additive error approximation. Towards that, we define the Δ -ADD-#CNFSAT as follows.

Definition 2.11 (Δ -ADD-#CNFSAT). Given an input a CNFSAT formula on n Boolean variables and m clauses. The problem is to output an additive error approximation of the total number of satisfying assignments, i.e., an integer d such that $|d - d'| < \Delta$ where d' is the number of assignments to these n variables that satisfy the given formula and $\Delta \in [1, 2^n)$.

This problem (Definition 2.11) can be viewed as computing the following property on the truth table of the given formula.

Definition 2.12 (Δ -ADD-COUNT). Given a Boolean string $x \in \{0,1\}^N$, the Δ -ADD-COUNT(x) is an integer such that $|\Delta$ -ADD-COUNT $(x) - |x|| < \Delta$ where $|x| = |\{i : x \in \{0,1\}^N \text{ and } x_i = 1\}|$ and $\Delta \in [1, N).$

The bounded-error quantum query complexity for computing Δ -ADD-COUNT was studied in [NW99]. They showed the following result.

Theorem 2.13 (Theorem 1.11 in [NW99]). Any quantum algorithm computing Δ -ADD-COUNT(x) for an $x \in \{0,1\}^N$ and $\Delta \in [1,N)$ (Definition 2.12) requires to make $\Omega\left(\sqrt{\frac{N}{\Delta}} + \frac{\sqrt{t(N-t)}}{\Delta}\right)$ quantum queries to x where t = |x|.

For values of $\Delta = o(\sqrt{t})$ we are unable to prove the compression-obliviousness of this property. Hence, we make the following conjecture.

Conjecture 2.14. Let AC_2^0 denote the class of poly(n) size CNF and DNF formulas on n input variables the property Δ -ADD-COUNT is compression oblivious corresponding to AC₂⁰, i.e. Δ -ADD-COUNT $\in \mathcal{CO}(\mathrm{AC}_2^0)$.

One can now establish the time lower bound for computing the Δ -ADD-COUNT on poly(n) sized CNF and DNF formulas. However, this doesn't automatically imply the same lower bound for the case when there are only CNF formulas to consider. Fortunately, Δ -ADD-COUNT is defined in such a way that computing this property is equally hard for both CNF and DNF formulas. More precisely, the following statement holds.

Theorem 2.15 (Δ -ADD-#QSETH). Let n be an integer and let $\Delta \in [\frac{1}{2}, 2^n)$. Conditional on AC_2^0 -QSETH (Conjecture 1.3) and Conjecture 2.14, there is no bounded error quantum algorithm that solves Δ -ADD-#CNFSAT on CNF formulas with n variables and m = poly(n) clauses in

$$O\left(\left(\sqrt{\frac{N}{\Delta}} + \frac{\sqrt{t(N-t)}}{\Delta}\right)^{1-\alpha} m^{O(1)}\right) \text{ time for any constant } \alpha > 0.$$

Proof. By way of contradiction let's assume that there is an algorithm \mathcal{A} such that given a CNF formula it can compute the Δ -ADD-COUNT on its truth table in $O\left(\left(\sqrt{\frac{N}{\Delta}} + \frac{\sqrt{t(N-t)}}{\Delta}\right)^{1-\beta} m^{O(1)}\right)$

time for some constant $\beta > 0$.

Then, given a poly(n) sized DNF formula on n variables and m clauses, let us denote that by ϕ , we can run Algorithm \mathcal{A} on $\neg \phi$ and use its output which is a Δ additive error approximation of the number of satisfying assignments to $\neg \phi$ to compute a Δ additive error approximation of the number of satisfying assignments to ϕ .

Let us denote the number of satisfying assignments of $\neg \phi$ by d' and the output of Algorithm \mathcal{A} by d. This means we have $|d - d'| < \Delta$. We claim that $2^n - d$ will be a Δ additive error approximation of $2^n - d'$, which is the number of satisfying assignments of ϕ ; $|(2^n - d) - (2^n - d')| = |d' - d| < \Delta$.

Therefore, a $O\left(\left(\sqrt{\frac{N}{\Delta}} + \frac{\sqrt{t(N-t)}}{\Delta}\right)^{1-\beta} m^{O(1)}\right)$ time (for some constant $\beta > 0$) algorithm for computing Δ -ADD-COUNT on truth table of CNF formulas also implies the same run time for computing Δ -ADD-COUNT on truth table of DNF formulas; this violates the combination of AC₂⁰-QSETH and Conjecture 2.14. Hence, proved.

2.4 Quantum complexity of γ -#CNFSAT and other related problems

One other approximation of the count of satisfying assignments is the multiplicative factor approximation, defined as follows.

Definition 2.16 (γ -#CNFSAT). Let $\gamma \in (0, 1)$. The γ -#CNFSAT is defined as follows. The input is a CNFSAT formula on n Boolean variables and m clauses. The problem is to output an integer d such that $(1 - \gamma)d' < d < (1 + \gamma)d'$ where d' is the number of assignments to these n variables that satisfy the given formula.⁷

The expression $(1 - \gamma)d' < d < (1 + \gamma)d'$, which is same as $(d' - \gamma d') < d < (d' + \gamma d')$, can be categorised into the following two cases.

- Case 1 is when $\gamma d' \leq 1$: in this regime, the algorithm solving γ -#CNFSAT is expected to return the value d' which is the *exact* count of the number of solutions to the CNFSAT problem and from Theorem 2.6 we know that there is no $O^*(2^{n(1-\varepsilon)})$ time algorithm, for any constant $\varepsilon > 0$, that can compute the exact number of solutions to input CNF formula; this is a tight lower bound.
- Case 2 is when $\gamma d' > 1$: in this regime, the algorithm solving γ -#CNFSAT is expected to return value d which is the γ -approximate relative count of the number of solutions to the CNFSAT problem. Using AC₂⁰-QSETH we are able to prove a lower bound for γ -#CNFSAT that accounts for the approximation factor γ also when $\gamma d' > 1$. We use the rest of this section towards proving this result.

In order to understand the hardness of γ -#CNFSAT we will first try to understand how hard it is to compute the following property on poly-sized CNF and DNF formulas. Let $f_{\ell,\ell'} : \mathcal{D} \to \{0,1\}$ with $\mathcal{D} \subset \{0,1\}^N$ be a partial function defined as follows

$$f_{\ell,\ell'} = \begin{cases} 1, & \text{if } |x| = \ell, \\ 0, & \text{if } |x| = \ell'. \end{cases}$$

⁷The same results hold if the approximation is defined with the equalities, i.e., $(1 - \gamma)d' \leq d \leq (1 + \gamma)d'$. An additional observation under this changed definition of γ -#CNFSAT is as follows. Given a CNF formula as input, the algorithm for γ -#CNFSAT outputs 0 only when there is no satisfying assignment to that formula. Hence, one can decide satisfiability of a given CNF formula using the algorithm for γ -#CNFSAT. Therefore, the same lower bound from BASIC-QSETH carries for this changed definition of γ -#CNFSAT.

Nayak and Wu in [NW99] analysed the minimum degree of any polynomial approximating $f_{\ell,\ell'}$ and by invoking the Beals et al. result [BBC⁺01] gave a lower bound on the quantum query complexity of $f_{\ell,\ell'}$ as mentioned in the following statement.

Corollary 2.17 (Rephrasing Corollary 1.2 in [NW99]). Let $f_{\ell,\ell'} : \mathcal{D} \to \{0,1\}$ where $\mathcal{D} \subset \{0,1\}^N$ and

$$f_{\ell,\ell'} = \begin{cases} 1, & \text{if } |x| = \ell, \\ 0, & \text{if } |x| = \ell'. \end{cases}$$

Let $\Delta_{\ell} = |\ell - \ell'|$ and let $p \in \{\ell, \ell'\}$ such that $|\frac{N}{2} - p|$ is maximized. Then the bounded error quantum query complexity of $f_{\ell,\ell'}$ is $\Omega\left(\sqrt{\frac{N}{\Delta_{\ell}}} + \frac{\sqrt{p(N-p)}}{\Delta_{\ell}}\right)$.

Using AC₂⁰-QSETH we will now show that for a choice of ℓ, ℓ' computing $f_{\ell,\ell'}$ on truth tables of CNF formulas of poly(n) size requires $\Omega\left(\sqrt{\frac{2^n}{\Delta_\ell}} + \frac{\sqrt{p(2^n-p)}}{\Delta_\ell}\right)$ time where $\Delta_\ell = |\ell - \ell'|$ and $p \in \{\ell, \ell'\}$ that maximises $|2^{n-1} - p|$. The only caveat (as also witnessed several times earlier) is that we cannot prove the compression obliviousness of $f_{\ell,\ell'}$ hence we state and use the following conjecture.

Conjecture 2.18. Let AC_2^0 denote the class of poly(n) size CNF and DNF formulas on n input variables then for any pair of integers $\ell \in [2^n] \cup \{0\}$ and $\ell' \in [2^n] \cup \{0\}$ the property $f_{\ell,\ell'}$ is compression oblivious corresponding to AC_2^0 , i.e. $f_{\ell,\ell'} \in \mathcal{CO}(AC_2^0)$.⁸

We can now prove the following.

Lemma 2.19. Let n be an integer and let $\ell \in [2^n] \cup \{0\}$ and $\ell' \in [2^n] \cup \{0\}$. Then conditional on Conjecture 2.18 and AC₂⁰-QSETH (Conjecture 1.3) then at least one of the following is true.

- There is no bounded error quantum algorithm that can compute $f_{\ell,\ell'}$ on the truth table of $\operatorname{poly}(n)$ size CNF formulas defined on n variables in $O^*\left(\left(\sqrt{\frac{2^n}{\Delta_\ell}} + \frac{\sqrt{p(2^n p)}}{\Delta_\ell}\right)^{1-\delta}\right)$ time for any constant $\delta > 0$.
- There is no bounded error quantum algorithm that can compute $f_{N-\ell,N-\ell'}$ on the truth table of poly(n) size CNF formulas defined on n variables in $O^*\left(\left(\sqrt{\frac{2^n}{\Delta_\ell}} + \frac{\sqrt{p(2^n-p)}}{\Delta_\ell}\right)^{1-\delta}\right)$ time for any constant $\delta > 0$.

Here $\Delta_{\ell} = |\ell - \ell'|$ and $p \in \{\ell, \ell'\}$ is such that $|2^{n-1} - p|$ is maximized.

Proof. Let N be an integer that we will fix later and let $f'_{\ell,\ell'}: \{0,1\}^N \to \{0,1\}$ be defined as follows

$$f'_{\ell,\ell'} = \begin{cases} 1, \text{ if } |x| = N - \ell, \\ 0, \text{ if } |x| = N - \ell'. \end{cases}$$

⁸Note that, there are some values of ℓ, ℓ' for which $f_{\ell,\ell'}$ will be provably compression oblivious, for e.g., $\ell = 1$ and $\ell' = 0$ would capture the OR property which is compression oblivious; see Section 2.1.

It is not hard to see $f'_{\ell,\ell'}$ is the same as function $f_{N-\ell,N-\ell'}$. Fortunately, both the functions $f_{N-\ell,N-\ell'}$ and $f_{\ell,\ell'}$ have the same value of Δ_{ℓ} and h where h = p(N-p). Therefore the bounded error quantum query complexity of $f'_{\ell,\ell'}$ is $\Omega\left(\sqrt{\frac{N}{\Delta_{\ell}}} + \frac{\sqrt{p(N-p)}}{\Delta_{\ell}}\right)$ where $\Delta_{\ell} = |\ell - \ell'|$ and $p \in \{\ell,\ell'\}$ such that $|\frac{N}{2} - p|$ is maximised; same as the bounded error quantum query complexity of $f_{\ell,\ell'}$ as mentioned in Corollary 2.17.

Moreover, as $f'_{\ell,\ell'}$ is the same function $f_{N-\ell,N-\ell'}$ it is therefore clear from Conjecture 2.18 that $f'_{\ell,\ell'} \in \mathcal{CO}(\mathrm{AC}_2^0)$ which means there is no bounded error quantum algorithm that can compute $f'_{\ell,\ell'}$ or $f_{\ell,\ell'}$ on truth tables of poly(n) size CNF or DNF formulas defined on n input variables in $O^*\left(\left(\sqrt{\frac{2^n}{\Delta_\ell}} + \frac{\sqrt{p(2^n-p)}}{\Delta_\ell}\right)^{1-\delta}\right)$ time for any constant $\delta > 0$ unless AC_2^0 -QSETH is false. We will now that can different an AC_2^0 QSETH this need to be a substitute to the conditional on AC_2^0 of AC_2^0 .

now show that conditional on AC_2^0 -QSETH this result holds even when we restrict ourselves to only poly(n) sized CNF formulas.

Having introduced $f'_{\ell,\ell'}$ we will now prove Lemma 2.19 using the following propositions.

- **Proposition A** There is no bounded error quantum algorithm that can compute $f_{\ell,\ell'}$ on truth table of CNF formulas defined on n variables in $O^*\left(\left(\sqrt{\frac{2^n}{\Delta_\ell}} + \frac{\sqrt{p(2^n p)}}{\Delta_\ell}\right)^{1-\delta}\right)$ time for any $\delta > 0$.
- **Proposition B** There is no bounded error quantum algorithm that can compute $f_{\ell,\ell'}$ on truth table of DNF formulas defined on n variables in $O^*\left(\left(\sqrt{\frac{2^n}{\Delta_\ell}} + \frac{\sqrt{p(2^n p)}}{\Delta_\ell}\right)^{1-\delta}\right)$ time for any $\delta > 0$.
- **Proposition C** There is no bounded error quantum algorithm that can compute $f'_{\ell,\ell'}$ on truth table of CNF formulas defined on n variables in $O^*\left(\left(\sqrt{\frac{2^n}{\Delta_\ell}} + \frac{\sqrt{p(2^n p)}}{\Delta_\ell}\right)^{1-\delta}\right)$ time for any $\delta > 0$.
- **Proposition D** There is no bounded error quantum algorithm that can compute $f'_{\ell,\ell'}$ on truth table of DNF formulas defined on n variables in $O^*\left(\left(\sqrt{\frac{2^n}{\Delta_\ell}} + \frac{\sqrt{p(2^n - p)}}{\Delta_\ell}\right)^{1-\delta}\right)$ time for any $\delta > 0$.

Conditional on Conjecture 2.18 and AC_2^0 -QSETH the following statements hold.

- Claim 1 At least one of the propositions A or B is true.
- Claim 2 At least one of the propositions C or D is true.
- Claim 3 At least one of the propositions A or C is true; by way of contradiction let us assume that both propositions A and C are false, this means there exist algorithms $\mathcal{A}, \mathcal{A}'$ that for an $\alpha > 0$ and $\alpha' > 0$ compute $f_{\ell,\ell'}$ and $f'_{\ell,\ell'}$ on the truth table of poly(n) size CNF formulas defined on n input variables in $O^*\left(\left(\sqrt{\frac{2^n}{\Delta_\ell}} + \frac{\sqrt{p(2^n-p)}}{\Delta_\ell}\right)^{1-\alpha}\right)$ time and in

 $O^*\left(\left(\sqrt{\frac{2^n}{\Delta_\ell}} + \frac{\sqrt{p(2^n - p)}}{\Delta_\ell}\right)^{1-\alpha'}\right) \text{ time, respectively. Moreover, if propositions A and C are false then from Claims 1 and 2 we can deduce that both B and D must be true which means there is no quantum algorithm that can compute <math>f_{\ell,\ell'}$ or $f'_{\ell,\ell'}$ on the truth table of $\operatorname{poly}(n)$ size DNF formulas on n input variables in $O^*\left(\left(\sqrt{\frac{2^n}{\Delta_\ell}} + \frac{\sqrt{p(2^n - p)}}{\Delta_\ell}\right)^{1-\delta}\right)$ time for any $\delta > 0$. However, given a DNF formula ϕ as an input to compute $f_{\ell,\ell'}$ on its truth table one can instead compute $f'_{\ell,\ell'}$ on the negation of ϕ , let us denote by $\neg \phi$, using algorithm \mathcal{A}' on $\neg \phi$ in $O^*\left(\left(\sqrt{\frac{2^n}{\Delta_\ell}} + \frac{\sqrt{p(2^n - p)}}{\Delta_\ell}\right)^{1-\alpha'}\right)$ time which is a contradiction. This means at least one of the two propositions A or C must be true which is exactly the statement of Lemma 2.19.

It is then not hard to see that for the values when $\ell + \ell' = N$ the following result holds.

Corollary 2.20. Let n be an integer and let $\ell \in [2^n] \cup \{0\}$ and $\ell' \in [2^n] \cup \{0\}$ such that $\ell + \ell' = 2^n$. Then there is no bounded error quantum algorithm that can compute $f_{\ell,\ell'}$ on the truth table of poly(n) size CNF formulas defined on n variables in $O^*\left(\left(\sqrt{\frac{2^n}{\Delta_\ell}} + \frac{\sqrt{\ell\ell'}}{\Delta_\ell}\right)^{1-\delta}\right)$ time for any constant $\delta > 0$ unless AC₂⁰-QSETH or Conjecture 2.18 is false; here $\Delta_\ell = |\ell - \ell'|$.

Inspired by arguments used in the proof of Theorem 1.13 in [NW99], we will now use Corollary 2.20 to prove the following result. Our result holds for $\gamma \in \left[\frac{1}{2^n}, \frac{1}{2}\right)$; this range of γ suffices for our reductions presented in the later sections.

Theorem 2.21 (γ -#QSETH). Let *n* be an integer and let $\gamma \in \left[\frac{1}{2^n}, \frac{1}{2}\right)$. Conditional on AC₂⁰-QSETH and Conjecture 2.18, there is no bounded error quantum algorithm that solves γ -#CNFSAT on *n* variables and m = poly(n) clauses

1. in
$$O\left(\left(\frac{1}{\gamma}\sqrt{\frac{2^n-\hat{h}}{\hat{h}}}\right)^{1-\delta}m^{O(1)}\right)$$
 time, for any constant $\delta > 0$, whenever $\gamma \hat{h} > 1$, and

2. in $O(2^{n(1-\delta)}m^{O(1)})$ time, for any constant $\delta > 0$, whenever $\gamma \hat{h} \leq 1$.

Here h denotes the number of satisfying assignments to the input CNF formula.

We prove the first part of Theorem 2.21 in the following way and use the result from Theorem 2.6 for the second part. Given a value of $\gamma \in \left[\frac{1}{2^n}, \frac{1}{2}\right)$ we will fix values of $\ell \in [2^n] \cup \{0\}$ and $\ell' \in [2^n] \cup \{0\}$ such that we are able to compute $f_{\ell,\ell'}$ on truth table of an input CNF formulas on n variables and m clauses using the algorithm that solves γ -#CNFSAT. Hence, showing a lower bound on γ -#CNFSAT using the lower bound result from Corollary 2.20.

Proof. Let $N = 2^n$ and let $\gamma \in \left[\frac{1}{N}, \frac{1}{2}\right)$. Let $\ell = \frac{N}{2} + \lceil \gamma t \rceil = \left\lceil \frac{N}{2} + \gamma t \rceil$ and $\ell' = \frac{N}{2} - \lceil \gamma t \rceil = \left\lfloor \frac{N}{2} - \gamma t \right\rfloor$; here $t \in [N]$ is a value that we will fix later but in any case, we have $1 \leq \lceil \gamma t \rceil < \frac{N}{2}$. With that, we

are ensured that $\gamma \ell > \frac{1}{2}$. We also make sure to choose values ℓ, ℓ' in such a way that $\gamma \ell' = \Omega(1)$. Clearly, $\ell + \ell' = N$ and $\Delta_{\ell} = |\ell - \ell'| = 2 \lceil \gamma t \rceil$ therefore by invoking the result from Corollary 2.20 we can say that for these values of ℓ, ℓ' there is no bounded error quantum algorithm that can solve $f_{\ell,\ell'}$ on truth table of CNF formulas in $O\left(\left(\sqrt{\frac{N}{\lceil \gamma t \rceil}} + \frac{\sqrt{\ell(N-\ell)}}{\lceil \gamma t \rceil}\right)^{1-\delta} m^{O(1)}\right)^{-1}$ time, for any $\delta > 0$; let us denote this claim by (*).

Let \mathcal{A} be an algorithm that computes γ -#CNFSAT to the relative error approximation γ , i.e., Algorithm \mathcal{A} returns a value h such that $(1 - \gamma)\hat{h} < h < (1 + \gamma)\hat{h}$ where \hat{h} is the actual number of satisfying assignments of the input CNF formula. Given $\ell = \frac{N}{2} + \lceil \gamma t \rceil$ and $\ell' = \frac{N}{2} - \lceil \gamma t \rceil$ there are values of $t \in [N]$ such that we will be able to distinguish whether the number of satisfying assignments to a formula is ℓ or ℓ' using Algorithm \mathcal{A} . As $\ell > \ell'$ in our setup, we want t such that $\ell'(1+\gamma) < \ell(1-\gamma)$; it is then necessary that $\gamma N < 2[\gamma t]$; let us denote this as Condition 1.

Now we set the values of ℓ and ℓ' . Given a value of $\gamma \in \left[\frac{1}{N}, \frac{1}{2}\right)$ set $\ell = \left[\frac{N}{2(1-\gamma)}\right]$ and set $\ell' = N - \ell$; this means $\frac{N}{2(1-\gamma)} \leq \ell < \frac{N}{2(1-\gamma)} + 1$ and $\frac{N(1-2\gamma)}{2(1-\gamma)} - 1 < \ell' \leq \frac{N(1-2\gamma)}{2(1-\gamma)}$ and $\frac{\gamma N}{(1-\gamma)} \leq \ell < \frac{N}{2(1-\gamma)}$ $|\ell - \ell'| < \frac{\gamma N}{(1-\gamma)} + 2$ therefore $2\gamma \ell - 2\gamma \leq |\ell - \ell'| < 2\gamma \ell + 2.9$ We know from claim (*) that all algorithms that (for these values of ℓ, ℓ') compute $f_{\ell,\ell'}$ on CNF formulas require $\Omega(L^{1-\delta})$ time for all $\delta > 0$ where $L = \frac{1}{\gamma} \sqrt{\frac{N-\ell}{\ell+1}} = \Omega\left(\frac{1}{\gamma} \sqrt{\frac{N-\ell}{\ell}}\right)$. Moreover, ℓ' when expressed in terms of ℓ and γ is as follows $(\ell - 1)(1 - 2\gamma) - 1 < \ell' \le \ell(1 - 2\gamma)$. Therefore, we can see that $L = \Omega\left(\frac{1}{\gamma}\sqrt{\frac{N-\ell}{\ell}}\right) =$ $\Omega\left(\frac{1-2\gamma}{\gamma}\sqrt{\frac{N-\ell'}{\ell'}}\right) = \Omega\left(\frac{1}{\gamma}\sqrt{\frac{N-\ell'}{\ell'}}\right); \text{ the last equality holds because } \frac{1-2\gamma}{\gamma}\sqrt{\frac{N-\ell'}{\ell'}} > c_0 \cdot \frac{1}{\gamma}\sqrt{\frac{N-\ell'}{\ell'}}$ where c_0 is some constant strictly less than 1.¹⁰

It is also easy to see that if $\ell = \lceil \frac{N}{2(1-\gamma)} \rceil$ were to be expressed as $\frac{N}{2} + \lceil \gamma t \rceil$ then for that value of $t \text{ we have } \lceil \gamma t \rceil \ge \frac{N\gamma}{2(1-\gamma)} > \frac{N\gamma}{2}$ which satisfies Condition 1 implying that this is in the case where we can use Algorithm \mathcal{A} to distinguish whether the number of satisfying assignments to a formula is ℓ or ℓ' . Hence given a CNF formula as input, we will be able to now use Algorithm \mathcal{A} to distinguish whether the number of satisfying assignments is ℓ or ℓ' . Let $T = \frac{1}{\gamma} \sqrt{\frac{N-\ell}{\ell}} + \frac{1}{\gamma} \sqrt{\frac{N-\ell'}{\ell'}} = O(\frac{1}{\gamma} \sqrt{\frac{N-\ell'}{\ell'}})$ because $\ell' < \ell$. If Algorithm \mathcal{A} can solve the γ -#CNFSAT on an input CNF that has \hat{h} number of satisfying assignments in $O((\frac{1}{\gamma}\sqrt{\frac{N-\hat{h}}{\hat{h}}})^{1-\delta}m^{O(1)})$ time, for some constant $\delta > 0$, then we are essentially computing $f_{\ell,\ell'}$ in $O(T^{1-\delta}m^{O(1)})$ time, which is a contradiction to claim (*); hence the first part of the statement of Theorem 2.21 proved.

Proof of the second part of this theorem follows from Theorem 2.6 as the regime $\gamma h \leq 1$ translates to exactly counting the number of satisfying assignments.

⁹To view the calculations in a less cumbersome way one can use the fact that asymptotically $\ell = \frac{N}{2(1-\gamma)}, \ell' =$ $\frac{N(1-2\gamma)}{2(1-\gamma)} \text{ and } |\ell - \ell'| = \frac{\gamma N}{(1-\gamma)} = 2\gamma \ell.$ ¹⁰Note that we need $c_0 < 1$ which means this would require that $\gamma \in [\frac{1}{N}, \frac{1-c_0}{2})$ instead of the range $\gamma \in [\frac{1}{N}, \frac{1}{2})$

that we considered earlier.

2.5 Quantum complexity of #k-SAT and other related problems

While AC_2^0 -QSETH can be used to comment on the quantum hardness of several properties on truth tables of CNF formulas, it however doesn't immediately allow for making similar hardness assumptions for bounded width k-CNF formulas. In fact, one has to be extremely careful while making such general hardness assumptions in the case of bounded width CNF formulas, such a thing can be witnessed in [AW21]. However, sometimes reductions from CNF-SAT to lattice problems [BGS17], require that the width of the input formulas gets encoded exponentially as the dimension of the vector of the lattice problems, which is undesirable when k approaches n, which however is tolerable for smaller k. Hence, we would like to say something interesting about hardness #k-SAT or $\oplus k$ -SAT say when $k = \Theta(\log n)$. Turns out that we can do so by basing the hardness on #QSETH.

To do so, we use the randomised classical algorithm by Rainer Schuler [Sch05] that solves CNF-SAT on *n* variables, *m* clauses in $O(\text{poly}(n)2^{n(1-1/(1+\log m))})$ expected time. This algorithm can be viewed as a Turing reduction from SAT with bounded clause density to SAT with bounded clause width, which was analyzed in [CIP06], a result that we directly use.

Algorithm 1 ReduceWidth_k(ψ)

Input: CNF formula ψ 1: if ψ has no clause of width > k then output ψ 2: 3: else let $C' = \{l_1, \ldots, l_{k'}\}$ be a clause of ψ of width k' > k4: $C = \{l_1, \ldots, l_k\}$ 5: $\psi_0 \leftarrow \psi - \{C'\} \cup \{C\}$ 6: $\psi_1 \leftarrow \psi \land \neg l_1 \land \neg l_2 \land \cdots \land \neg l_k$ 7: $\psi_1 \leftarrow \text{Remove variables corresponding to literals } l_1, \ldots, l_k \text{ from } \psi_1 \text{ by setting } l_1 = 0, \ldots, l_k =$ 8: 0 9: ReduceWidth_k(ψ_0) 10: ReduceWidth_k(ψ_1)

Algorithm 1, the ReduceWidth_k subroutine takes an input a CNF formula, say ψ , of width greater than k and then outputs a list of k-CNF formulas ψ_i where the solutions of the input formula is the union of the solutions of the output formulas, i.e., $\operatorname{sol}(\psi) = \bigcup_i \operatorname{sol}(\psi_i)$ where $\operatorname{sol}(\phi)$ denotes the set of satisfying assignments to a formula ϕ . In fact, it is not hard to see that the count of the number of satisfying assignments also is preserved, i.e., $|\operatorname{sol}(\psi)| = \sum_i |\operatorname{sol}(\psi_i)|$.

Lemma 2.22 (Implicit from Section 3.2 in [CIP06]). Algorithm 1 takes as input a CNF formula ψ on n input variables, m clauses, and is of width strictly greater than k and outputs a number of k-CNF formulas ψ_i each defined on at most n input variables and at most m clauses such that $|sol(\psi)| = \sum_i |sol(\psi_i)|$.

Proof. Let $\psi = C'_1 \wedge C'_2 \wedge \cdots \wedge C'_m$ be the input CNF formula to Algorithm 1. The algorithm finds the first clause C'_i that has its width k' > k. Let $C'_i = (l_1 \vee l_2 \vee \cdots \vee l_{k'})$ and $C_i = (l_1 \vee l_2 \vee \cdots \vee l_k)$. The algorithm then constructs two formulas $\psi_0 = \psi - \{C'_i\} \cup \{C_i\}$ and $\psi_1 = \psi \wedge \neg l_1 \wedge \neg l_2 \wedge \cdots \wedge \neg l_k$. Then recursively calls the subroutine on ψ_0 and ψ_1 . We now claim the following. Claim 2.23. $sol(\psi_0) \cap sol(\psi_1) = \emptyset$ and $sol(\psi) = sol(\psi_0) \cup sol(\psi_1)$, *i.e.*, $sol(\psi) = sol(\psi_0) \sqcup sol(\psi_1)$.

Proof. Let $x \in \operatorname{sol}(\psi_0)$ and then the clause $C_i(x) = (l_1(x) \vee \cdots \vee l_k(x))$ should evaluate to 1. Which means $\neg(l_1(x) \vee \cdots \vee l_k(x)) = 0$, which using De Morgan's laws implies $(\neg l_1(x) \wedge \cdots \wedge \neg l_k(x)) = 0$, which means that $\psi_1(x) = \psi(x) \wedge \neg l_1(x) \wedge \cdots \wedge \neg l_k(x) = 0$. Similar argument can be used to show that if $x \in \operatorname{sol}(\psi_1)$ then $x \notin \operatorname{sol}(\psi_0)$. Therefore, $\operatorname{sol}(\psi_0) \cap \operatorname{sol}(\psi_1) = \emptyset$.

What remains to show is $sol(\psi) = sol(\psi_0) \cup sol(\psi_1)$.

- If $x \in \operatorname{sol}(\psi_0)$ then $C_i(x) = 1$ which implies $C'_i(x) = 1$, therefore $x \in \operatorname{sol}(\psi)$. If $x \in \operatorname{sol}(\psi_1)$ then $\psi_1(x) = 1$, but $\psi_1(x) = \psi(x) \wedge \neg l_1(x) \wedge \neg l_2(x) \wedge \cdots \wedge \neg l_k(x)$ which means $\psi(x) = 1$ as well. Therefore, if $x \in \operatorname{sol}(\psi_0) \cup \operatorname{sol}(\psi_1)$ then $x \in \operatorname{sol}(\psi)$.
- If $x \in \operatorname{sol}(\psi)$ then $\psi(x) = 1$ which means $C'_i(x) = 1$. However, $C'_i(x) = (l_1(x) \lor \cdots \lor l_k(x)) \lor (l_{k+1}(x) \lor \cdots \lor l'_k(x))$. Which means either $(l_1(x) \lor \cdots \lor l_k(x)) = C_i(x) = 1$ or $(l_{k+1}(x) \lor \cdots \lor l'_k(x)) = 1$ or both evaluate to 1. If $C_i(x) = 1$ then $\psi_0(x) = 1$ which would mean $x \in \operatorname{sol}(\psi_0)$. If $C_i(x) = 0$ then $\psi_1(x) = \psi(x) \land (\neg C_i(x)) = 1$ which means $x \in \operatorname{sol}(\psi_1)$.

Therefore, $\operatorname{sol}(\psi) = \operatorname{sol}(\psi_0) \sqcup \operatorname{sol}(\psi_1)$.

Using the statement of Claim 2.23 we conclude that $sol(\psi) = \bigsqcup_i sol(\psi_i)$, consequently implying that $|sol(\psi)| = \sum_i |sol(\psi_i)|$.

Using Lemma 2.22 from this paper and Lemma 5 in [CIP06] we will now show the hardness of k-SAT and its popular counting variants when $k = \Theta(\log n)$.

Corollary 2.24. Conditional on AC₂⁰-QSETH (Conjecture 1.3) and Conjecture 2.5, there are no bounded error quantum algorithms that for $k = \Theta(\log n)$ solve

- 1. k-SAT in $O(2^{sn/2})$ time,
- 2. #k-SAT in $O(2^{sn})$ time,
- 3. $\oplus k$ -SAT in $O(2^{sn})$ time,
- 4. $\oplus_q k$ -SAT in $O(2^{sn})$ time,

for any constant s < 1, respectively.

Proof. Consider the ReduceWidth_k algorithm. Let p be some path of length t in the tree T of recursive calls to Algorithm 1. Let ψ_g be the output formula of width k on the path p. Let l, r be the number of left, right turns respectively on path p. Every left branch in the path reduces the width of exactly 1 clause to k, therefore $l \leq m$. One the other hand, with additional poly(n, m) time every right branch of path p reduces the number of variables by k, therefore $r \leq n/k$. Therefore number of paths in tree T with r right branches is at most $\binom{m+r}{r}$.

Now suppose there is an algorithm S that solves #k-SAT in 2^{ns} for some s < 1 for all $k = \Theta(\log n)$. We can use the same arguments from Lemma 5 in [CIP06] to show that this algorithm

S together with the ReduceWidth_k subroutine we can solve #CNF-SAT (ignoring poly(n) factors) in time at most

$$\sum_{r=0}^{n/k} \binom{m+r}{r} 2^{s(n-rk)}$$

$$\leq \sum_{r=0}^{n/k} \binom{m+\frac{n}{k}}{r} 2^{s(n-rk)}$$

$$= 2^{sn} \sum_{r=0}^{n/k} \binom{m+\frac{n}{k}}{r} \frac{1}{2^{srk}}$$

$$\leq 2^{sn} (1+\frac{1}{2^{sk}})^{m+\frac{n}{k}}$$

$$\leq 2^{sn} e^{\frac{1}{2^{sk}}(m+\frac{n}{k})} \qquad \text{since } (1+x) \leq e^{x}$$

$$\leq 2^{sn+\frac{4m}{2^{sk}}} \qquad \text{since we can assume that } m \geq \frac{n}{k}$$

For every s < 1 and m = poly(n) there is a setting of $k = c \log n$ for a constant c(s,m) such that $s + \frac{4m}{n2^{sk}} < 1$. Therefore, a 2^{ns} (for some s < 1) algorithm for #k-SAT for $k = \Theta(\log n)$ would refute #-QSETH (Theorem 2.6). The same arguments hold for k-SAT or \oplus k-SAT as well.

Note that, we *cannot* extend the same arguments for the MAJORITY or st-MAJORITY or additive error approximation of count. However, these arguments extend to the multiplicative factor approximation of the count.

Corollary 2.25. Let n be an integer and let $\gamma \in \left[\frac{1}{2^n}, \frac{1}{2}\right)$. Conditional on γ -#QSETH, there is no bounded error quantum algorithm that for $k = \Theta(\log n)$ solves γ -#k-SAT on n variables and $m = \operatorname{poly}(n)$ clauses

1. in
$$O\left(\left(\frac{1}{\gamma}\sqrt{\frac{2^n-\hat{h}}{\hat{h}}}\right)^{1-\delta}m^{O(1)}\right)$$
 time, for any constant $\delta > 0$, whenever $\gamma \hat{h} > 1$, and

2. in $O(2^{n(1-\delta)}m^{O(1)})$ time, for any constant $\delta > 0$, whenever $\gamma \hat{h} \leq 1$.

Here \hat{h} denotes the number of satisfying assignments to the input k-CNF formula.

3 Quantum strong simulation of quantum circuits

We use the phrase strong simulation problem to mean strong simulation of quantum circuits which is defined as follows.¹¹

Definition 3.1 (The strong simulation problem). Given a quantum circuit C on n qubits, a strong simulation of C computes the value of $|\langle x|C|0^n\rangle|$, up to a desired bits of precision, for some specified $x \in \{0,1\}^n$.¹²

¹¹Note that, this is different from *weak simulation* problem; a weak simulation samples from probability distribution $p(x) = |\langle 0^n | C | x \rangle|^2$.

¹²Though in some papers the strong simulation problem requires that we output $\langle x | C | 0^n \rangle$ instead of $| \langle x | C | 0^n \rangle |$, we use this definition because it is more comparable to the definition of the weak simulation problem. Also, the lower bound we present holds for either of these definitions.



Figure 2: The circuit C_{ϕ} .

For a quantum circuit C, computing $|\langle x| C | 0^n \rangle|$ exactly, to a precision of n bits, is #P-hard [CHM21, VDN10]. This means even a quantum computer will likely require exponential time to strongly simulate another quantum circuit. In this section, we prove a more precise quantum time bound for strongly simulating quantum circuits, both exactly and approximately; in the approximate case, we present complexity results for both multiplicative factor and additive error approximation. Our results extend the results by [HNS20] in two directions: firstly, we give explicit (conditional) bounds proving that it is hard to strongly simulate quantum circuits using quantum computers as well. Secondly, we also address the open question posed by [HNS20] on the (conditional) hardness of strong simulation up to accuracy $O(2^{-n/2})$, however, our results are based on a hardness assumption different from SETH or Basic QSETH.

The results presented in this section are based on two main components. Firstly, on the observation that the reduction from CNFSAT to the strong simulation problem given (in Theorem 3) by [HNS20] encodes the count of the number of satisfying assignments; this fact allows us to use the same reduction to reduce other variants of CNFSAT, such as #CNFSAT or \oplus CNFSAT, to the strong simulation problem, moreover, the same reduction also allows us to reduce γ -#CNFSAT and Δ -ADD-#CNFSAT to analogous variants of the strong simulation problem, respectively. As the second main component, we use the quantum hardness of these variants of CNFSAT problem discussed in Section 2.

We will first state the result of the exact quantum time complexity of the strong simulation problem and then use that result to later show how hard it is for a quantum computer to strongly simulate a quantum circuit with an additive error or a multiplicative factor approximation.

Theorem 3.2. There is no constant $\alpha > 0$ such that one can solve the strong simulation problem (*Definition 3.1*) up to a precision of n bits for an s-size circuit on n qubits in $2^{n(1-\alpha)}s^{O(1)}$ time, unless #QSETH (Theorem 2.6) is false.

The proof is identical to the proof of Theorem 3 by [HNS20] which we restate here for ease of reading.

Proof. Let ϕ be a CNF formula on n input variables and $m = \operatorname{poly}(n)$ clauses. Let C'_{ϕ} denote the reversible classical circuit, using TOFFOLI, CNOT and NOT gates, that *tidily* computes $\phi(x)$ for all $x \in \{0,1\}^{n.13}$ One can construct circuit C'_{ϕ} of width k with $n \leq k \leq n + 2(\lceil \log n \rceil + \lceil \log m \rceil)$ and size $s \leq 8 \times 3^{\lceil \log n \rceil + \lceil \log m \rceil}$ with sublinear space and polynomial time overhead; see Section 4.1 of [HNS20]. Let C''_{ϕ} denote the quantum analogue of the classical reversible circuit C'_{ϕ} , i.e., for the gates in C'_{ϕ} , TOFFOLI and CNOT remain unchanged and NOT gate is replaced by X gate; therefore, the width and size of C''_{ϕ} remains k and s, respectively.

Clearly, $C''_{\phi} : \mathcal{H} \to \mathcal{H}$ where \mathcal{H} denotes a 2^k dimensional Hilbert space. Then, it is not hard to see that $C''_{\phi} |x\rangle |0^{k-n}\rangle = |x\rangle |\phi(x)\rangle |0^{k-1-n}\rangle$. Let C_{ϕ} denote the quantum circuit in Figure 2; width of this circuit is still k and size is O(s). With a bit of calculation, it can be shown that $\langle 0^k | C_{\phi} | 0^k \rangle$ encodes the fraction of satisfying assignments to formula ϕ .

This means if there exists a constant $\alpha > 0$ such that strong quantum simulation of circuit C_{ϕ} on basis state $|x\rangle$ with $x = 0^k$ can be computed in $T = 2^{k(1-\alpha)}s^{O(1)}$ time up to k bits of precision, then essentially this would reveal the count of the number of satisfying assignments formula ϕ in time T. Plugging in the values of $k \le n + 2(\lceil \log n \rceil + \lceil \log m \rceil)$ and $s \le 8 \times 3^{\lceil \log n \rceil + \lceil \log m \rceil} = \text{poly}(n)$ we get $T \le 2^{n(1-\alpha)}n^{O(1)}$ time; this would refute #QSETH (Theorem 2.6). Hence, proving the statement of Theorem 3.2.

The following corollary also gives the same lower bound for 2-bits of precision but by using a different hardness assumption (which is Majority QSETH) addressing an open question by [HNS20].^{14,15}

Corollary 3.3. There is no $2^{n(1-\alpha)}s^{O(1)}$ time quantum algorithm, for any constant $\alpha > 0$, that can solve the strong simulation problem (Definition 3.1) up to 2 bits of precision for an s-size circuit on n qubits, unless Majority QSETH (Part 1 of Corollary 2.8) is false.

One can also try to solve the strong simulation problem with additive error approximation with the following definition.

Definition 3.4. (Δ' additive error strong simulation) Given a quantum circuit C on n qubits, output with $\Delta' \in [\frac{1}{2^{n+1}}, 1)$ additive error the value $d' = |\langle 0^n | C | x \rangle|$ for some specified $x \in \{0, 1\}^n$, *i.e.*, output a value d such that $|d - d'| < \Delta'$.

¹³A classical circuit $C: \{0,1\}^{n+w(n)+1} \to \{0,1\}^{n+w(n)+1}$ reversibly and tidily computes a function $f: \{0,1\}^n \to \{0,1\}$ if the following statements are true.

1. Circuit C reversibly computes f if C consists of reversible gates, such as {TOFFOLI, CNOT, NOT}, and

$$\forall x \in \{0,1\}^n, \exists W(x) \in \{0,1\}^{w(n)}, C(x,0^{w(n)},b) = (x,W(x),b \oplus f(x)).$$
(2)

2. Circuit C tidily computes f if

$$\forall x \in \{0,1\}^n, \forall b \in \{0,1\}, C(x,0^{w(n)},b) = (x,0^{w(n)},b \oplus f(x)).$$
(3)

 14 We must admit, this answer might not be an interesting enough answer to the open question by [HNS20] because same observation can also be made in the classical setting by basing the hardness assumption on *Majority SETH*. However, we still think it is an observation worthwhile to mention.

¹⁵Additionally, one needs to be careful about the definition of Majority SETH: mostly about what definition of Majority is being used because if the formulas under consideration are k-SAT formulas then Majority can be computed in polynomial time but Strict Majority still seems to require exponential time on k-SAT [AW21]. Hence, one way to circumvent that would be to define SETH also on CNF formulas of unbounded width, which is exactly how QSETH is defined.

Using the reduction mentioned in Theorem 3.2 and the conjectured hardness of Δ -ADD-#CNFSAT problem we can prove the following.¹⁶

Corollary 3.5. Conditional on Δ -ADD-#QSETH (Theorem 2.15), there is no bounded error quantum algorithm that can solve the strong simulation problem of an s-sized circuit C on n qubits to $\Delta' = \frac{\Delta}{2^n} \in \left[\frac{1}{2^{n+1}}, 1\right)$ additive error approximation in $\widetilde{O}\left(\left(\sqrt{\frac{2^n}{\Delta}} + \frac{\sqrt{\hat{h}(2^n - \hat{h})}}{\Delta}\right)^{1-\alpha} s^{O(1)}\right)$ time for any constant $\alpha > 0$, for a specified $x \in \{0,1\}^n$, upto n bits of precision; here $\hat{h} = \langle 0^n | C | x \rangle \cdot 2^n$.

It is beneficial to note that, we only get (at best) a poly(n) time quantum lower bound for the strong simulation problem when $\Delta' = \frac{\Delta}{2^n} = O(1)$. Fortunately, this lower bound matches the poly(n) time quantum upper bound for the strong simulation problem when $\Delta' = O(1)$, see Appendix A in the appendix for details. In fact, for some values of \hat{h} our lower bounds are also tight in terms of Δ' ; this reassures us of our proposed conjectures.

Additionally, we are also able to show strong simulation results with multiplicative error whose definition is as follows.

Definition 3.6. (Multiplicative γ -approximate strong simulation) Given a quantum circuit C on n qubits, output with γ multiplicative error the value $d' = |\langle 0^n | C | x \rangle|$ for some specified $x \in \{0,1\}^n$, *i.e.*, output a value d such that $(1 - \gamma)d' < d < (1 + \gamma)d'$.

The exact arguments in the proof of Theorem 3.2 can be used to prove the following statement.

Theorem 3.7. Conditional on $\gamma - \#QSETH$ (Theorem 2.21), there is no bounded-error quantum algorithm that can solve the strong simulation problem to multiplicative error approximation (Definition 3.6) of an input circuit C on n qubits with error parameter $\gamma \in [\frac{1}{2^n}, \frac{1}{2})$ in

1. in
$$O\left(\left(\frac{1}{\gamma}\sqrt{\frac{2^n-\hat{h}}{\hat{h}}}\right)^{1-\delta}s^{O(1)}\right)$$
 time, for any constant $\delta > 0$, whenever $\gamma \hat{h} \ge 1$, and

2. in $O(2^{n(1-\delta)}s^{O(1)})$ time, for any constant $\delta > 0$, whenever $\gamma \hat{h} < 1$.

Here $\hat{h} = \langle 0^n | C | x \rangle$ for a specific $x \in \{0, 1\}^n$ and s denotes the size of circuit C.

Proof. Let \mathcal{A} denote an algorithm that for a given s-sized quantum circuit C on k qubits and a given $x \in \{0,1\}^k$ computes $\langle 0^k | C | x \rangle$ with γ multiplicative error approximation for a $\gamma \in [\frac{1}{2^k}, \frac{1}{2})$ in time

1. in
$$O\left(\left(\frac{1}{\gamma}\sqrt{\frac{2^k-\hat{h}}{\hat{h}}}\right)^{1-\delta}s^{O(1)}\right)$$
 time, for a constant $\delta > 0$, whenever $\gamma \hat{h} \ge 1$, and

2. in $O(2^{k(1-\delta')}s^{O(1)})$ time, for a constant $\delta' > 0$, whenever $\gamma \hat{h} < 1$;

here $\hat{h} = \langle 0^k | C | x \rangle \cdot 2^n$.

Then, given a CNF formula ϕ on n input variables and m clauses, one can do the following to approximately count the number of satisfying assignments to ϕ : first use the poly(n) reduction as

¹⁶Note that, the value of k in relation to n is such that $2^k = \widetilde{O}(2^n)$; here k refers to the k used in the proof of Theorem 3.2.

in the proof of Theorem 3.2 to construct the quantum circuit C_{ϕ} of size s = poly(n) and width $k \leq n + 2(\lceil \log n \rceil + \lceil \log m \rceil)$. Then run algorithm \mathcal{A} on quantum circuit C_{ϕ} and $x = 0^k$ as inputs. The output would then be the γ multiplicative error approximation of the count of the number of satisfying assignments to ϕ . Depending on the value of $\gamma \cdot \hat{h}$, the run-time of this entire process is either $O\left(\operatorname{poly}(n) + \left(\frac{1}{\gamma}\sqrt{\frac{2^k - \hat{h}}{\hat{h}}}\right)^{1-\delta}s^{O(1)}\right) = O\left(\left(\frac{1}{\gamma}\sqrt{\frac{2^n - \hat{h}}{\hat{h}}}\right)^{1-\delta}n^{O(1)}\right)$ for some constant $\delta > 0$ or $O(2^{k(1-\delta')}s^{O(1)}) = O(2^{n(1-\delta')}n^{O(1)})$ time, for an constant $\delta' > 0$. Either way, violating the γ -#QSETH (Theorem 2.21).

4 Quantum lower bound for lattice counting and q-count problems

In this section, we would like to connect k-SAT to variants of lattice problems and then use the QSETH lower bound we have in Section 2.5 to give quantum fine-grained complexity for those lattice problems. For any set of n linearly independent vectors $\mathbf{B} = \{b_1, \ldots, b_n\}$ from \mathbb{R}^d , the lattice \mathcal{L} generated by basis \mathbf{B} is

$$\mathcal{L}(\mathbf{B}) = \left\{ \sum_{i=1}^n z_i \boldsymbol{b_i} : z_i \in \mathbb{Z} \right\}.$$

We call *n* the rank of the lattice \mathcal{L} and *d* the dimension. The vectors $\mathbf{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ forms a basis of the lattice. Given a basis \mathbf{B} , we use $\mathcal{L}(\mathbf{B})$ to denote the lattice generated by \mathbf{B} .

For $1 \leq p < \infty$ the ℓ_p -norm $\|\boldsymbol{x}\|_p$ of any vector $\boldsymbol{x} \in \mathbb{R}^d$ is defined by

$$\|\boldsymbol{x}\|_p := \left(\sum_{i=1}^d x_i^p\right)^{1/p}$$

Additionally, the ℓ_{∞} norm of such a vector \boldsymbol{x} is defined as $\|\boldsymbol{x}\|_{\infty} := \max_{i \in [d]} |x_i|$. The ℓ_p unit ball $B_p^d \subset \mathbb{R}^d$ is defined by $\{\boldsymbol{x} \in \mathbb{R}^d \mid \|\boldsymbol{x}\|_p \leq 1\}$.

Fine-grained complexity of lattice problems is quite widely studied in the classical case [BGS17, ABGS21, AC21, AS18, BP20, BPT22]. Lots of variants of lattice problems have been considered before, and the most well-studied one is the closest vector problem (with respect to ℓ_p norm).

Definition 4.1. For any $1 \leq p \leq \infty$, the Closest Vector Problem CVP_p is the search problem defined as: The input is a basis $\mathbf{B} \in \mathbb{R}^{d \times n}$ of the lattice \mathcal{L} and a target vector \mathbf{t} . The goal is to output a vector $\mathbf{v} \in \mathcal{L}$ such that $\|\mathbf{v} - \mathbf{t}\|_p = \min_{x \in \mathcal{L}} \|\mathbf{x} - \mathbf{t}\|_p$.

 CVP_p is known to have a 2^n SETH lower bound for any $p \notin 2\mathbb{Z}$ [BGS17, ABGS21], and for even p, there seems a barrier for showing a fine-grained reduction from k-SAT to CVP [AK22]. Kannan gave a $n^{\mathcal{O}(n)}$ -time algorithm for solving CVP_p for arbitrary $p \geq 1$ [Kan83], while the best-known algorithm for solving CVP_p with noneven p is still n^{cn} for some constant c. To get a conditional quantum lower bound for CVP_p for noneven p, given there is already a classical reduction from k-SAT to CVP_p using $2^k \cdot \operatorname{poly}(n)$ time (for noneven p) [BGS17, ABGS21], either one can directly use the QSETH framework by Aaronson et al. [ACL+20] to get a $2^{(0.5-\varepsilon)n}$ lower bound, or we can use Corollary 2.24 to get the same lower bound in our QSETH framework.¹⁷

¹⁷Basic QSETH assumption is weaker than the QSETH assumption in Aaronson et al [ACL⁺20], so our lower bound under basic QSETH assumption (Conjecture 1.3 and Corollary 2.1) will also imply a lower bound under their quantum SETH framework.

A natural question is invoked here: Can we have a $2^{(0.5+\varepsilon)n}$ quantum SETH lower bound for any (variants of) lattice problems? The answer is yes by using the framework and the problems introduced in Section 2.5 and by considering the counting variant of lattice problems. We begin by introducing the (approximate) lattice counting problem and some other related problems as follows:

Definition 4.2 (Lattice counting problem). For all $\gamma \geq 0$ and $1 \leq p \leq \infty$, the γ -approximate Vector Counting Problem γ -VCP_p is the counting problem defined as follows: The input is a lattice a basis $\mathbf{B} \in \mathbb{R}^{d \times n}$ of lattice $\mathcal{L}(\mathbf{B})$, target vector $\mathbf{t} \in \mathbb{R}^d$, and radius $r \in \mathbb{R}_+$. The problem is to output a value C satisfying $|(\mathcal{L} - \mathbf{t}) \cap r \cdot B_p^d| \leq C \leq (1 + \gamma) \cdot |(\mathcal{L} - \mathbf{t}) \cap r \cdot B_p^d|$. If $\gamma = 0$, we simply denote the problem as VCP_p.

The (approximate) lattice counting problem was first introduced by Stephens-Davidowitz as a promise problem [Ste16], and here we slightly modify the definition to make it a counting problem. We also generalize the lattice vector counting problem to the q-count problem, as follows.

Definition 4.3 (Lattice q-count problem). For all $\gamma \geq 1$, $1 \leq p \leq \infty$, and $q \in [2^n] \setminus \{1\}$, the lattice q-count Problem $\#_q$ -VCP_p is the lattice q-count problem defined as follows: The input is a lattice a basis $\mathbf{B} \in \mathbb{R}^{d \times n}$ of lattice $\mathcal{L}(\mathbf{B})$, target vector $\mathbf{t} \in \mathbb{R}^d$, and radius $r \in \mathbb{R}_+$. The problem is to output a value $C = |(\mathcal{L} - \mathbf{t}) \cap r \cdot B_p^d| \mod q$. If q = 2, we simply denote the problem as $\oplus VCP_p$.

One can consider the above two problems are the counting and the q-counting version of CVP_p , respectively. To connect these problems to the counting k-SAT, we should first introduce the following geometric tool introduced by Bennett, Golovnev, and Stephens-Davidowitz [BGS17].

Definition 4.4 (Isolating parallelepiped). Let k be an integer between 3 and n. For all $1 \le p \le \infty$, we say that $V \in \mathbb{R}^{2^k \times k}$ and $\mathbf{u} \in \mathbb{R}^{2^n}$ define a (p,k)-isolating parallelepiped if $\|\mathbf{u}\|_p > 1$ and $\|V\mathbf{x} - \mathbf{u}\|_p = 1$ for all $\mathbf{x} \in \{0,1\}^k \setminus \{0_k\}$.

For the sake of completeness, we will explain how to connect a k-CNF formula to an instance of VCP using the above object. The proof is very similar to the proof of Theorem 3.2 of [BGS17].

Theorem 4.5. Let k be an integer between 3 and n. Suppose we have a (p, k)-isolating parallelepiped (V, \mathbf{u}) for some $p = p(n) \in [1, \infty)$ and can make quantum queries to oracles $O_V : |i\rangle |s\rangle |0\rangle \rightarrow |i\rangle |s\rangle |0\rangle \rightarrow |i\rangle |s\rangle |0\rangle \rightarrow |i\rangle |u_i\rangle$ for $i \in [2^k]$ and $s \in [k]$. Then for every given input oracle $O_{\Phi} : |j\rangle |w\rangle |0\rangle \rightarrow |j\rangle |k\rangle |C_w(j)\rangle$ of k-CNF formula $\psi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ for $w \in [m]$ and $j \in [n]$, one can output oracles $O_{\mathbf{B}} : |h\rangle |j\rangle |0\rangle \rightarrow |h\rangle |j\rangle |\mathbf{B}_j(h)\rangle$ of basis $\mathbf{B} \in \mathbb{R}^{(m \cdot 2^k + n) \times n}$ for each $h \in [m \cdot 2^k + n]$ and $j \in [n]$, $O_t : |h\rangle |0\rangle \rightarrow |h\rangle |h\rangle |h\rangle$ of target vector $\mathbf{t} \in \mathbb{R}^{m \cdot 2^k + n}$ for each $h \in [m \cdot 2^k + n]$, and radius r such that $|sol(\psi)| = VCP_p(\mathbf{B}, \mathbf{t}, r)$, using poly(n, m) queries to O_V , O_V^{\dagger} , $O_{\mathbf{u}}$, $O_{\mathbf{v}}^{\dagger}$, $O_{\mathbf{v}}$, $O_{\mathbf{v}}^{\dagger}$, $O_{\mathbf{v}}$, $O_{\mathbf{v}}^{\dagger}$, and elementary gates.

Proof. Let $d = m \cdot 2^k + n$ and $V = \{v_1, \ldots, v_k\}$ with $v_s \in \mathbb{R}^{2^k}$ for every $s \in [k]$. The basis $\mathbf{B} \in \mathbb{R}^{d \times n}$ and target vector $\mathbf{t} \in \mathbb{R}^d$ in the output instance have the form:

$$\mathbf{B} = egin{pmatrix} \mathbf{B}_1 \ dots \ \mathbf{B}_m \ 2 \cdot m^{1/p} \cdot I_n \end{pmatrix}, \quad oldsymbol{t} = egin{pmatrix} t_1 \ dots \ \mathbf{h}_m \ \mathbf{h}_n \end{pmatrix},$$

with blocks $\mathbf{B}_w \in \mathbb{R}^{2^k \times n}$ that correspond to the clause $C_w = \bigvee_{s=1}^k \ell_{w,s}$ and $\mathbf{t}_w \in \mathbb{R}^{2^n}$ for each $w \in [m]$ and $j \in [n]$, the *j*th column $(\mathbf{B}_w)_j$ of block \mathbf{B}_w is

$$(\mathbf{B}_w)_j = \begin{cases} \boldsymbol{v}_{\boldsymbol{w}}, \text{ if } x_j \text{ is the } w \text{th literal of clause } w, \\ -\boldsymbol{v}_{\boldsymbol{w}}, \text{ if } \neg x_j \text{ is the } w \text{th literal of clause } w \\ 0_{2^d}, \text{ otherwise}, \end{cases}$$

and $\mathbf{t}_w = \mathbf{u} - \sum_{s \in N_w} \mathbf{v}_s$, where $N_w = \{s \in [k] : \ell_{w,s} \text{ is negative}\}$ is the set of the indices of negative literals in C_w . Also set $r = (mn + m)^{1/p}$.

Define the equation unitary $U_e : |a\rangle |b\rangle |0\rangle \rightarrow |a\rangle |b\rangle |\delta_{|a|,|b|}\rangle$ for each $a, b \in [d]$, and the location indicator unitary as for each $x_1, \ldots, x_n \in \{0, 1\}$,

$$U_{loc}: |x_1\rangle |x_2\rangle \dots |x_n\rangle |0\rangle \rightarrow \begin{cases} |x_1\rangle |x_2\rangle \dots |x_n\rangle |s\rangle, & \text{if only } x_s = 1 \text{ and all others are } 0, \\ |x_1\rangle |x_2\rangle \dots |x_n\rangle |0\rangle, & \text{otherwise,} \end{cases}$$

which can both be implemented via poly(n) elementary gates up to negligible error. Also, we define the sign literal indicator unitary as follows: for each $w \in [m], j \in [n]$,

$$U_{s\ell}: |w\rangle |j\rangle |0\rangle \to \begin{cases} |w\rangle |j\rangle |s\rangle, & \text{if } x_j \text{ is the sth literal of clause } C_w, \\ |w\rangle |j\rangle |-s\rangle, & \text{if } \neg x_j \text{ is the sth literal of clause } C_w, \\ |w\rangle |j\rangle |0\rangle, & \text{otherwise,} \end{cases}$$

which can also be implemented by using poly(m, n) applications of O_{Φ} and poly(m, n) many elementary gates.

We are now ready to show how to implement $O_{\mathbf{B}}$ and O_t . For input $|h\rangle |j\rangle |0\rangle |0\rangle |0\rangle |0\rangle |0\rangle$, let $|h\rangle = |h_1\rangle |h_2\rangle$ where h_2 is the last *n* bits of *h* and h_1 is the remaining prefix. Then we first apply U_e to the first, third, and fifth registers to have $|h_1\rangle |h_2\rangle |j\rangle |0\rangle |\delta_{h_1,0}\rangle |0\rangle |0\rangle$. After that, apply $U_{s\ell}$ to the first, third, and sixth registers, and apply O_V to the second, third, and seventh registers we get

$$|h_1\rangle |h_2\rangle |j\rangle |0\rangle |\delta_{h_1,0}\rangle |s\rangle |V_{js}\rangle;$$

note that $V_{js} = (\boldsymbol{v}_s)_j$. Finally, adding another ancilla register, we can store the value $V_{js} \cdot \delta_{h_{1,0}} + 2m^{1/p}(1-\delta_{h_{1,0}})\delta_{h-m\cdot 2^{k},j}$ in the last register. Uncomputing fourth to seventh registers, we have

$$|h_{1}\rangle|h_{2}\rangle|j\rangle|V_{js}\cdot\delta_{h_{1},0}+2m^{1/p}(1-\delta_{h_{1},0})\delta_{h-m\cdot2^{k},j}\rangle=|h\rangle|j\rangle|(\boldsymbol{v}_{s})_{j}\cdot\delta_{h_{1},0}+2m^{1/p}(1-\delta_{h_{1},0})\delta_{h-m\cdot2^{k},j}\rangle$$

and $(\boldsymbol{v}_s)_j \cdot \delta_{h_1,0} + 2m^{1/p}(1-\delta_{h_1,0})\delta_{h-m\cdot 2^k,j}$ is exactly the coefficient of $\mathbf{B}_j(h)$. One can see we only use poly(n,m) queries to O_V , O_V^{\dagger} , O_{Φ} , O_{Φ}^{\dagger} and elementary gates. We can also construct O_t using a similar strategy, which can also be done using at most poly(n,m) queries to O_V , O_V^{\dagger} , O_{Φ} , O_{Φ}^{\dagger} , O_u , O_u^{\dagger} , and elementary gates.

To see the correctness, consider $\boldsymbol{y} \in \mathbb{Z}^n$. If $\boldsymbol{y} \notin \{0,1\}^n$, then

$$\|\mathbf{B}\boldsymbol{y} - \boldsymbol{t}\|_p^p \ge \|2m^{1/p}I_n\boldsymbol{y} - m^{1/p}\mathbf{1}_n\|_p^p \ge m(n+2).$$

On the other hand, if $\boldsymbol{y} \in \{0,1\}^n$, then for each \mathbf{B}_w

$$\begin{split} \|\mathbf{B}_{w}\boldsymbol{y} - \boldsymbol{t}\|_{p}^{p} &= \|\sum_{s \in P_{w}} \boldsymbol{y}_{ind(\ell_{w,s})} \cdot \boldsymbol{v}_{s} - \sum_{s \in N_{w}} \boldsymbol{y}_{ind(\ell_{w,s})} \cdot \boldsymbol{v}_{s} - (u - \sum_{s \in N_{w}} \boldsymbol{y}_{ind(\ell_{w,s})} \cdot \boldsymbol{v}_{s})\|_{p}^{p} \\ &= \|\sum_{s \in P_{w}} \boldsymbol{y}_{ind(\ell_{w,s})} \cdot \boldsymbol{v}_{s} + \sum_{s \in N_{w}} (1 - \boldsymbol{y}_{ind(\ell_{w,s})}) \cdot \boldsymbol{v}_{s} - u\|_{p}^{p} \\ &= \|\sum_{s \in S_{w}(\boldsymbol{y})} \boldsymbol{v}_{s} - u\|_{p}^{p}, \end{split}$$

where $ind(\ell_{w,s})$ is the index of the variable underlying $\ell_{w,s}$, $P_w = \{s \in [k] : \ell_{w,s} \text{ is positive}\}$ is the set of the indices of positive literals in C_w , and $S_w(\boldsymbol{y}) = \{s \in P_w : \boldsymbol{y}_{ind(\ell_{w,s})} = 1\} \cup \{s \in N_w : \boldsymbol{y}_{ind(\ell_{w,s})} = 0\}$ is the indices of literals in C_w satisfied by \boldsymbol{y} . Because (V, \boldsymbol{u}) is a (p, k)-isolating parallelepiped, if $|S_w(\boldsymbol{y})| \neq 0$, then $\|\sum_{s \in S_w(\boldsymbol{y})} \boldsymbol{v}_s - \boldsymbol{u}\|_p^p = 1$, and it will be greater than 1 otherwise.

Also, $|S_w(y)| \neq 0$ if and only if C_w is satisfied. Therefore, we have that for every satisfying assignment y,

$$\|\mathbf{B}\boldsymbol{y}-\boldsymbol{t}\|_p^p = \sum_{k=1}^m \|\mathbf{B}_k\boldsymbol{y}-\boldsymbol{t}\|_p^p + mn = m + mn,$$

and for all other unsatisfying assignments \mathbf{y}' , $\|\mathbf{B}\mathbf{y}' - \mathbf{t}\|_p^p > m + mn$. As a result, every satisfying assignment of $\psi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ is the lattice points of lattice $\mathcal{L}(\mathbf{B})$ with distance $r = (mn+m)^{1/p}$ to the target vector t, which implies the answer of counting-SAT with input ψ is equal to $\mathrm{VCP}_p(\mathbf{B}, \mathbf{t}, r)$.

The theorem above shows how we can connect the counting k-SAT problem to the vector counting problem given access to a (p, k)-isolating parallelepiped. However, it is not always the case that we can compute such an isolating parallelepiped efficiently. Nevertheless, Aggarwal, Bennett, Golovnev, and Stephens-Davidowitz [BGS17, ABGS21] also showed the existence of isolating parallelepiped for some p, k and provided an efficient algorithm for computing them.

Theorem 4.6 ([ABGS21]). For $k \in \mathbb{Z}_+$ and computable $p = p(n) \in [1, \infty)$ if p satisfies either (1) $p \notin 2\mathbb{Z}$ or (2) $p \geq k$, there exists a (p,k)-isolating parallelepiped $V \in \mathbb{R}^{2^k \times k}$, $u \in \mathbb{R}^{2^k}$ and it is computable in time $poly(2^k)$.

Therefore, by choosing $k = \Theta(\log n)$ and combining Corollary 2.24 and Theorems 4.5 and 4.6, we can directly show a 2^n QSETH lower bound for VCP_p for all non-even p. Also, a similar idea works for the q-count and approximate counting of CNF-SAT: for any CNF-SAT formula ψ , using Algorithm 1 and Lemma 2.22, we can output a number of k-CNF formula $\psi_1 \dots, \psi_N$ such that $|\operatorname{sol}(\psi)| = \sum_{i \in [N]} |\operatorname{sol}(\psi_i)|$. Once we have an algorithm that solves $\gamma \#$ -k-SAT ($\#_q$ -k-SAT), we can use it to compute γ -#k-SAT(ψ_i) ($\#_q$ -k-SAT(ψ_i)) for all $i \in [N]$, and then by adding the outputs together, we can therefore get a valid solution to γ -#CNFSAT ($\#_q$ CNFSAT) with input ψ . By a similar argument of the proof of Corollary 2.24, we know N is bounded by $2^{o(n)}$. Combining the above arguments with Theorem 4.5, we have following corollaries.

Corollary 4.7. Let $p \in [1,\infty) \setminus 2\mathbb{Z}$. There is no bounded error quantum time algorithm that solves VCP_p in $O(2^{sn})$ time for a constant s < 1 unless #QSETH conjecture is false.

Corollary 4.8. Let $\gamma \in [\frac{1}{2^n}, \frac{1}{2})$, $\hat{h} \in [2^n]$, and $p \in [1, \infty) \setminus 2\mathbb{Z}$. There is no bounded error quantum time algorithm that solves γ -VCP_p,

1. in
$$O\left(\left(\frac{1}{\gamma}\sqrt{\frac{2^n-\hat{h}}{\hat{h}}}\right)^s\right)$$
 time, for any constant $s < 1$, whenever $\gamma \hat{h} > 1$, and

2. in $O(2^{sn})$ time, for any constant s < 1, whenever $\gamma \hat{h} \leq 1$

unless γ -#QSETH conjecture is false.

Corollary 4.9. Let $p \in [1, \infty) \setminus 2\mathbb{Z}$. There is no bounded error quantum time algorithm that solves $\oplus VCP_p$ in $O(2^{sn})$ time for a constant s < 1 unless $\oplus QSETH$ conjecture is false.

Corollary 4.10. Let $p \in [1, \infty) \setminus 2\mathbb{Z}$ and $q \in [2^n] \setminus \{1, 2\}$. There is no bounded error quantum time algorithm that solves $\#_q VCP_p$ in $O(2^{sn})$ time for a constant s < 1 unless $\#_q QSETH$ conjecture is false.

The following theorem also shows how to connect the approximate vector counting problem to the closest vector problem. The connection was already built up in Theorem 3.5 of [Ste16], while we can easily give a quadratic saving for the number of calls to CVP_p oracle by using some basic quantum tricks if quantum queries are allowed. We include the proof in Appendix B for completeness.

Theorem 4.11. Let $f(n) \ge 20$ be an efficiently computable function and $p \in [1, \infty)$. One can solve $f(n)^{-1}$ -VCP_p using $\mathcal{O}(f^2)$ quantum queries to CVP_p .

Note that the QSETH lower bound for f^{-1} -VCP_p depends on f. Therefore if one can show a reduction from f^{-1} -VCP_p to CVP_p using f^c for some constant c < 1, then it will end up with a better QSETH lower bound for CVP_p.

5 Hardness of Counting/Parity of OV, Hitting Set, and Set-Cover

In this section, we will discuss the consequences of Corollary 2.24 and Theorem 2.21 on some wellmotivated problems: Orthogonal Vectors, Hitting Set and Set Cover. Following are the definitions of Hitting Set and its variants.

Definition 5.1 (Hitting Set). For any integers n, m > 0, the Hitting Set problem is defined as follows: The input is a collection $\Sigma = (S_1, \dots, S_m)$, where $S_i \subset V$ and integer t > 0, the goal is to output a subset $S' \subset V$ such that $|S'| \leq t$ and $\forall i \in [m], |S' \cap S_i| > 0$.

Definition 5.2 (Variants of Hitting Set). For any integers n, m > 0 and $\gamma \in \left\lfloor \frac{1}{2^n}, \frac{1}{2} \right)$, the input is a collection $\Sigma = (S_1, \dots, S_m)$, where $S_i \subset V$ and integer t > 0.

- 1. In Count Hitting Set problem, the goal is to output $|S' \subset V : |S'| \le t, \forall i \in [m], |S' \cap S_i| > 0|$.
- 2. In Parity Hitting Set problem, the goal is to output $|S' \subset V : |S'| \le t, \forall i \in [m], |S' \cap S_i| > 0| \mod 2$.
- 3. In Majority Hitting Set, the goal is to output 1 if $|S' \subset V : |S'| \le t, \forall i \in [m], |S' \cap S_i| > 0| > 2^{n-1}$, otherwise output 0.
- 4. In γ approximation of count Hitting Set, the goal is to output an integer d such that $(1-\gamma)d' < d < (1+\gamma)d'$ where $d' = |S' \subset V : |S'| \le t, \forall i \in [m], |S' \cap S_i| > 0|$.

In [CDL⁺16], Cygan et al. showed a Parsimonious reduction between CNF-SAT and Hitting Set. By Parsimonious reduction, we mean a transformation from a problem to another problem that preserves the number of solutions.

Theorem 5.3 (Theorem 3.4 in [CDL⁺16]). For any sufficiently large integer n > 0 and constant $\varepsilon > 0$, there exists a polynomial time Parsimonious reduction from CNF-SAT on n variables to Hitting Set on $n(1 + \varepsilon)$ size universal set.

We get the following corollary.

Corollary 5.4. For any sufficiently large integer n > 0, constant $\delta > 0$ and $\gamma \in \left[\frac{1}{2^n}, \frac{1}{2}\right)$, there is no bounded error quantum algorithm

- 1. in $2^{\frac{n}{2}(1-\delta)}$ time for Hitting Set unless BASIC-QSETH (Corollary 2.1) is false.
- 2. in $2^{n(1-\delta)}$ time for Count Hitting Set unless #QSETH(Theorem 2.6) is false.
- 3. in $2^{n(1-\delta)}$ time for Parity Hitting Set unless $\oplus QSETH(Corollary 2.7)$ is false.
- 4. in $2^{n(1-\delta)}$ time for Majority Hitting Set unless Majority-QSETH(Corollary 2.8) is false.
- 5. $in\left(\frac{1}{\gamma}\sqrt{\frac{2^n-h}{h}}\right)^{1-\delta}$ time for γ approximation of count Hitting Set unless γ -#QSETH (as in **Theorem 2.21**) is false. Here h denote the number of hitting sets to the input instance.

We use our results form Section 2 to show fine-grained hardness of problems in complexity class P. More specifically, we study Orthogonal Vectors problem and its variants defined as follows.

Definition 5.5 (Orthogonal Vectors (OV)). For any integers d, n > 0, the Orthogonal Vectors (OV) problem is defined as follows: The input is two lists A and B, each consists of n vectors from $\{0,1\}^d$. The goal is to find vectors $a \in A, b \in B$ for which $\langle a, b \rangle = 0$.

Definition 5.6 (Variants of OV). For any integers d, n > 0 and $\gamma \in \left[\frac{1}{n^2}, \frac{1}{2}\right)$, the input is two lists A and B, each consists of n vectors from $\{0, 1\}^d$.

- 1. In Count OV problem, the goal is to output $|(a,b) : a \in A, b \in B, \langle a,b \rangle = 0|$ i.e. number of pairs (a,b) for which $a \in A, b \in B$ and $\langle a,b \rangle = 0$.
- 2. In Parity OV problem, the goal is to output $|(a,b) : a \in A, b \in B, \langle a,b \rangle = 0 \mod 2$.
- 3. In Majority OV, the goal is to output 1 if $|(a,b) : a \in A, b \in B, \langle a,b \rangle = 0| > n^2/2$, otherwise output 0.
- 4. In γ approximation of count OV, the goal is to output an integer d such that $(1 \gamma)d' < d < (1 + \gamma)d'$ where $d' = |(a, b) : a \in A, b \in B, \langle a, b \rangle = 0|$.

Orthogonal Vectors (OV) is an important computational problem that lies in complexity class P. It turns out to be one of the central problem to show fine-grained hardness of problems in P [Vas15, ABDN18]. Williams showed a reduction from a reduction from CNF-SAT to OV [Wil05]. We observe that Williams's reductions is Parsimonious. Therefore we also get fine-grained hardness of counting versions of OV. In [BPS21], authors showed O(n)-hardness for OV under basic QSETH assumption.

Corollary 5.7. For any sufficiently large integer n > 0, $\gamma \in \left[\frac{1}{n^2}, \frac{1}{2}\right)$ and any constant $\delta > 0$, there is no bounded error quantum algorithm

- 1. in $n^{2-\delta}$ time for Count OV unless #QSETH (Theorem 2.6) is false.
- 2. in $n^{2-\delta}$ time for Parity OV unless $\oplus QSETH$ (Corollary 2.7) is false.
- 3. in $n^{2-\delta}$ time for Majority OV unless Majority-QSETH (Corollary 2.8) is false.
- 4. $in\left(\frac{1}{\gamma}\sqrt{\frac{n^2-h}{h}}\right)^{1-\delta}$ time for γ approximation of count OV unless γ -#QSETH (Theorem 2.21) conjecture is false. Here h denote the number of hitting sets to the input instance.

We also show fine-grained hardness of parity Set-Cover defined as follows.

Definition 5.8 (parity Set-Cover). For any integers n, m > 0, the parity Set-Cover problem is defined as follows: The input is a collection $\Sigma = (S_1, \dots, S_m)$, where $S_i \subset V$ and integer t > 0, the goal is to output $|\{\mathcal{F} \subset \Sigma : \bigcup_{S \in \mathcal{F}} S = V, |\mathcal{F}| \leq t\}| \mod 2$.

In $[CDL^{+}16]$, Cygan et al. showed an efficient reduction from parity Hitting Set to parity Set Cover. We get the following corollary:

Corollary 5.9. For any sufficiently large integer n > 0 and t > 0, parity Set-Cover on a universal set of size n is 2^n hard unless $\oplus QSETH$ (Corollary 2.7) is false.

It's a big open to get an efficient fine-grained reduction from Hitting Set or CNF-SAT to Set-Cover. Note that Hitting Set and Set Cover are dual problems of each other but this reduction does not say anything interesting about the fine-grained hardness of Set-Cover.

6 Discussions and open questions

We believe that this paper opens up the possibility of concluding quantum time lower bounds for many other problems, both for other variants of CNFSAT and also for problems that are not immediately related to CNFSAT. While this is a natural broad future direction to explore, we also mention the following few directions for future work which are more contextual to this paper.

- One of the motivations to use AC_2^0 -QSETH in this paper is so that we can 'tie' certain conjectures, that would have otherwise been standalone conjectures, to one main conjecture. But in the process, we conjecture compression obliviousness of several properties. It would be nice if we could also have an 'umbrella' conjecture that allows one to establish compression obliviousness of several properties. For e.g., it would be nice if we could show that compression obliviousness of a natural property like COUNT or PARITY implies compression obliviousness of say Δ -ADD-COUNT.
- It will be interesting to see if it is possible to use the QSETH framework (or the AC_2^0 -QSETH conjecture) to give a single exponential lower bound for CVP in Euclidean norm (CVP₂).

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A Quantum upper bounds for strong simulation

In order to solve the Δ' additive error strong simulation problem we will be using the amplitude estimation algorithm given by [BHMT02] as a subroutine.

Theorem A.1 (Implicit in Theorem 12 by [BHMT02]). Given a natural number M and access to an (n + 1)-qubit unitary U satisfying

$$U \left| 0^{n} \right\rangle \left| 0 \right\rangle = \sqrt{a} \cdot e^{i\theta_{1}} \left| \phi_{1} \right\rangle \left| 1 \right\rangle + \sqrt{1 - a} \cdot e^{i\theta_{2}} \left| \phi_{0} \right\rangle \left| 0 \right\rangle,$$

where $|\phi_0\rangle$ and $|\phi_1\rangle$ are arbitrary n-qubit states, $\theta_1, \theta_2 \in [0, 2\pi]$ and 0 < a < 1, there exists a quantum algorithm that uses O(M) applications of U and U^{\dagger} and $\widetilde{O}(M)$ elementary gates, and outputs a state $|\Lambda\rangle$ such that after measuring that state, with probability $\geq 9/10$, the first register λ of the outcome satisfies

$$|\sqrt{a} - \lambda| \le \frac{100\pi}{M}.$$

Theorem A.2. Let $n \in \mathbb{N}$. There is a $O(\operatorname{poly}(n)\frac{1}{\Delta'})$ time quantum algorithm for solving the Δ' additive error strong simulation of Definition 3.4.

Proof. Let C and x denote the input, a quantum circuit on n input variables of size poly(n) and $x \in \{0,1\}^n$ be basis state, respectively. The task is to estimate the value of $|\langle x|C|0^n\rangle|$ with $\Delta' \in [\frac{1}{2^{n+1}}, 1)$ additive error approximation.

Let $|\psi\rangle = C |0^n\rangle = \sum_{i \in \{0,1\}^n} \alpha_i |i\rangle$ and $\forall i, \alpha_i \in \mathbb{C}$. Let U' denote the unitary $U' : |i\rangle |x\rangle |b\rangle \rightarrow |i\rangle |x\rangle |b \oplus (i = x)\rangle$ for some $i, x \in \{0, 1\}^n$ and $b \in \{0, 1\}$. It is easy to verify that combining C and U' we can construct a unitary U on 2n + 1 qubits such that

$$U |0^{2n}\rangle |0\rangle = \alpha_x |x\rangle |x\rangle |1\rangle + \sum_{i \neq x} \alpha_i |i\rangle |x\rangle |0\rangle,$$

and $|\alpha_x| = |\langle x | C | 0^n \rangle|$. Using the *amplitude estimation* algorithm from Theorem A.1 we can now estimate $|\langle x | C | 0^n \rangle|$ to an additive error of Δ' in $O(\text{poly}(n)\frac{1}{\Delta'})$ time.

B Proof of Theorem 4.11

To prove Theorem 4.11, we first introduce the following theorem.

Theorem B.1 ([Ste16], modified Theorem 3.2). Let **B** be a basis of a lattice $\mathcal{L}(\mathbf{B})$ and Q be a prime number. Consider the sparsification process: input any two vectors $\mathbf{z}, \mathbf{c} \in \mathbb{Z}_Q^n$, the sparsification process $Spar(\mathbf{B}, Q, \mathbf{z}, \mathbf{c})$ outputs a basis $\mathbf{B}_{Q,\mathbf{z}}$ of the sublattice $\mathcal{L}_{\mathbf{z}} \subset \mathcal{L} = \{\mathbf{u} \in \mathcal{L} : \langle \mathbf{z}, \mathbf{B}^{-1}\mathbf{u} \rangle = 0$ mod $Q\}$ and $\mathbf{w}_{\mathbf{z},\mathbf{c}} = \mathbf{B}\mathbf{c}$. Then for every $\mathbf{t} \in \mathbb{R}^n$, $\mathbf{x} \in \mathcal{L}$ with $N = |(\mathcal{L} - \mathbf{t}) \cap ||\mathbf{x} - \mathbf{t}|| \cdot B_p^n| \leq Q$, and CVP_p oracle, we have

$$\frac{1}{Q} - \frac{N}{Q^2} - \frac{N}{Q^{n-1}} \leq \Pr_{\boldsymbol{z}, \boldsymbol{c} \in \mathbb{Z}_Q^n}[CVP_p(\boldsymbol{t} + \boldsymbol{w}_{\boldsymbol{z}, \boldsymbol{c}}, \mathcal{L}_{\boldsymbol{z}}) = \boldsymbol{x} + \boldsymbol{w}_{\boldsymbol{z}, \boldsymbol{c}}] \leq \frac{1}{Q} + \frac{1}{Q^n},$$

and in particular,

$$\frac{N}{Q} - \frac{N^2}{Q^2} - \frac{N^2}{Q^{n-1}} \leq \Pr_{\boldsymbol{z}, \boldsymbol{c} \in \mathbb{Z}_Q^n} [\min_{\boldsymbol{u} \in \mathcal{L}_{\boldsymbol{z}}} \| (\boldsymbol{t} + \boldsymbol{w}_{\boldsymbol{z}, \boldsymbol{c}} - \boldsymbol{u} \|_p) \leq \| \boldsymbol{x} - \boldsymbol{t} \|_p] \leq \frac{N}{Q} + \frac{N}{Q^n}$$

Define U_{Spar} : $|\mathbf{B}, Q, \mathbf{z}, \mathbf{c}\rangle |0\rangle |0\rangle \rightarrow |\mathbf{B}, Q, \mathbf{z}, \mathbf{c}\rangle |\mathbf{B}_{Q, \mathbf{z}}\rangle |\mathbf{w}_{\mathbf{z}, \mathbf{c}}\rangle$, where $\mathbf{B}_{Q, \mathbf{z}}, \mathbf{w}_{\mathbf{z}, \mathbf{c}}$ are the output of $Spar(\mathbf{B}, Q, \mathbf{z}, \mathbf{c})$. Since given CVP_p oracle and a basis \mathbf{B} , the sparsification process can be efficiently done according to the construction, we can also implement the unitary U_{Spar} efficiently. Then we are ready to show the proof of Theorem 4.11.

Before we enter the proof of Theorem 4.11, we would like to introduce the promised version of lattice vector counting problem as follows:

Definition B.2 (Gap-VCP). For all $\gamma \geq 0$ and $1 \leq p \leq \infty$, the problem γ -approximate gap Vector Counting Problem γ -Gap-VCP_p is a promise problem defined as follows: The input is a lattice a basis $\mathbf{B} \in \mathbb{R}^{d \times n}$ of lattice $\mathcal{L}(\mathbf{B})$, target vector $\mathbf{t} \in \mathbb{R}^d$, radius $r \in \mathbb{R}_+$, and $N \geq 1$. It is a No instance if $|(\mathcal{L} - \mathbf{t}) \cap r \cdot B_p^d| \leq N$ and a Yes instance if $N > (1 + \gamma) \cdot |(\mathcal{L} - \mathbf{t}) \cap r \cdot B_p^d|$.

One can easily see if we can solve γ -Gap-VCP_p, then by using poly(n) calls of it we can solve γ -VCP_p. As a result, it suffices to show how to reduce γ -Gap-VCP_p to CVP_p in the following proof.

Proof of Theorem 4.11. Choose a prime $Q = \Theta(fN)$ and let O_{CVP} be the quantum CVP_p oracle. First we prepare the superposition state $\frac{1}{Q^{n/2}} \sum_{\boldsymbol{z}, \boldsymbol{c} \in \mathbb{Z}_Q^n} |\mathbf{B}, Q, \boldsymbol{z}, \boldsymbol{c}\rangle |0\rangle |\boldsymbol{t}\rangle |0\rangle$, apply U_{Spar} on the first six registers, and apply O_{CVP_p} on the fifth, sixth, seventh registers, and then apply subtraction unitary $U_{sub} : |\boldsymbol{a}\rangle |\boldsymbol{b}\rangle |0\rangle \rightarrow |\boldsymbol{a}\rangle |\boldsymbol{b}\rangle ||\boldsymbol{a} - \boldsymbol{b}||_p\rangle$ on the last three registers, we get

$$\frac{1}{Q^{n/2}}\sum_{\boldsymbol{z},\boldsymbol{c}\in\mathbb{Z}_Q^n} |\mathbf{B}, Q, \boldsymbol{z}, \boldsymbol{c}\rangle |\mathbf{B}_{Q,\boldsymbol{z}}\rangle |\boldsymbol{t} + \boldsymbol{w}_{\boldsymbol{z},\boldsymbol{c}}\rangle |CVP_p(\mathbf{B}_{Q,\boldsymbol{z}}, \boldsymbol{t} + \boldsymbol{w}_{\boldsymbol{z},\boldsymbol{c}})\rangle |\|\boldsymbol{t} + \boldsymbol{w}_{\boldsymbol{z},\boldsymbol{c}} - CVP_p(\mathbf{B}_{Q,\boldsymbol{z}}, \boldsymbol{t} + \boldsymbol{w}_{\boldsymbol{z},\boldsymbol{c}}))\|_p\rangle.$$

Let $r_{z,c} = \|\boldsymbol{t} + \boldsymbol{w}_{z,c} - CVP_p(\mathbf{B}_{Q,z}, \boldsymbol{t} + \boldsymbol{w}_{z,c}))\|_p$ and uncompute the first seven registers, we get

$$\frac{1}{Q^{n/2}}\sum_{\boldsymbol{z},\boldsymbol{c}\in\mathbb{Z}_Q^n}\ket{r_{\boldsymbol{z},\boldsymbol{c}}},$$

add another ancilla $|0\rangle$ at the end of the above state, and then apply r-threshold gate

$$U_r: |R\rangle |0\rangle \to \begin{cases} |R\rangle |1\rangle & \text{if } R \leq r \\ |R\rangle |0\rangle & \text{otherwise,} \end{cases}$$

on it, we get

$$\frac{1}{Q^{n/2}} \Big(\sum_{r_{\boldsymbol{z},\boldsymbol{c}} \leq r} |r_{\boldsymbol{z},\boldsymbol{c}}\rangle |1\rangle + \sum_{r_{\boldsymbol{z},\boldsymbol{c}} > r} |r_{\boldsymbol{z},\boldsymbol{c}}\rangle |0\rangle \Big) = \sqrt{a} |\phi_1\rangle |1\rangle + \sqrt{1-a} |\phi_0\rangle |0\rangle,$$

where $a = |\{(\boldsymbol{z}, \boldsymbol{c}) : r_{\boldsymbol{z}, \boldsymbol{c}} \leq r\}|/Q^n = \Pr_{\boldsymbol{z}, \boldsymbol{c}, \in \mathbb{Z}_Q^n}[r_{\boldsymbol{z}, \boldsymbol{c}} \leq r]$. Note that by Theorem B.1, if $|\mathcal{L} \cap (rB_p^n + \boldsymbol{t})| \leq N$, then

$$\Pr_{\boldsymbol{z},\boldsymbol{c},\in\mathbb{Z}_Q^n}[r_{\boldsymbol{z},\boldsymbol{c}}\leq r]\leq \frac{N}{Q}+\frac{N}{Q^n},$$

and if $|\mathcal{L} \cap (rB_p^n + t)| \ge \gamma N$, then

$$\Pr_{z,c,\in\mathbb{Z}_Q^n}[r_{\boldsymbol{z},\boldsymbol{c}}\leq r]\geq \frac{\gamma N}{Q}-\frac{\gamma^2 N^2}{Q^2}-\frac{\gamma^2 N^2}{Q^{n-1}}.$$

Observing that $\frac{\gamma N}{Q} - \frac{\gamma^2 N^2}{Q^2} - \frac{\gamma^2 N^2}{Q^{n-1}} - (\frac{N}{Q} + \frac{N}{Q^n}) = \Theta(f^{-1}(n)N/Q)$, we know that to distinguish the above two cases, it suffices to learn $a = \Pr_{\mathbf{z}, \mathbf{c}, \in \mathbb{Z}_Q^n} [r_{\mathbf{z}, \mathbf{c}} \leq r]$ with additive error $\Theta(f^{-1}(n)N/Q)$. Therefore, by using Theorem A.1, we can solve γ -Gap-VCP_p with $\gamma = f^{-1}(n)$ using $\mathcal{O}(fQ/N)$

Therefore, by using Theorem A.I, we can solve γ -Gap-VCP_p with $\gamma = f^{-1}(n)$ using O(fG) queries to O_{CVP_p} . Because $Q = \Theta(fN)$, we finish the proof.