


# Discreteness of Asymptotic Tensor Ranks

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## Abstract

Tensor parameters that are amortized or regularized over large tensor powers, often called “asymptotic” tensor parameters, play a central role in several areas including algebraic complexity theory (constructing fast matrix multiplication algorithms), quantum information (entanglement cost and distillable entanglement), and additive combinatorics (bounds on cap sets, sunflower-free sets, etc.). Examples are the asymptotic tensor rank, asymptotic slice rank and asymptotic subrank. Recent works (Costa–Dalai, Blatter–Draisma–Rupniewski, Christandl–Gismundo–Zuiddam) have investigated notions of discreteness (no accumulation points) or “gaps” in the values of such tensor parameters.

We prove a general discreteness theorem for asymptotic tensor parameters of order-three tensors and use this to prove that (1) over any finite field (and in fact any finite set of coefficients in any field), the asymptotic subrank and the asymptotic slice rank have no accumulation points, and (2) over the complex numbers, the asymptotic slice rank has no accumulation points.

Central to our approach are two new general lower bounds on the asymptotic subrank of tensors, which measures how much a tensor can be diagonalized. The first lower bound says that the asymptotic subrank of any concise three-tensor is at least the cube-root of the smallest dimension. The second lower bound says that any concise three-tensor that is “narrow enough” (has one dimension much smaller than the other two) has maximal asymptotic subrank.

Our proofs rely on new lower bounds on the maximum rank in matrix subspaces that are obtained by slicing a three-tensor in the three different directions. We prove that for any concise tensor, the product of any two such maximum ranks must be large, and as a consequence there are always two distinct directions with large max-rank.

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## 1 Introduction

Tensor parameters that are amortized or regularized over large tensor powers, often called “asymptotic” tensor parameters, play a central role in several areas of theoretical computer science including algebraic complexity theory (constructing fast matrix multiplication algorithms [11, 5], and barriers for such constructions [2, 14]), quantum information (entanglement cost and distillable entanglement [56, 54]), and additive combinatorics (bounds on cap sets [53], sunflower-free sets [46], etc.). Examples are the asymptotic tensor rank (famous for its connection to the matrix multiplication exponent), the asymptotic subrank, and the asymptotic slice rank. These asymptotic tensor parameters are of the form  $\underline{F}(T) = \lim_{n \rightarrow \infty} F(T^{\boxtimes n})^{1/n}$  for some integer-valued function  $F$  (e.g. tensor rank, subrank or slice rank), where  $\boxtimes$  denotes the Kronecker product of tensors. The computation of these parameters, which in some applications is the main goal and in others is done to bound other parameters of interest, has turned out to be very challenging, and many questions about them are open despite much interest.

The fundamental question whether the matrix-multiplication exponent  $\omega$  equals 2 is closely related to the question whether asymptotic tensor rank is integral-valued. Contrary to matrix rank, some asymptotic tensor parameters may indeed take non-integral values. For instance, the asymptotic slice rank of the  $W$ -tensor (which appears in the study of sunflower-free sets) equals  $2^{h(1/3)} \approx 1.88$  [52], where  $h$  is the binary entropy function, and the asymptotic slice rank of the cap set tensor (which appears in the study of arithmetic progression-free sets or cap sets) is known to be the non-integral value  $\approx 2.755$  over the finite field  $\mathbb{F}_3$  [31, 53, 39, 52].

This raises the fundamental question, for a given function  $F$ , what values  $\underline{F}(T)$  can take when varying  $T$  over all tensors of order three with arbitrary dimensions (over any fixed field). More generally, what is the structure (geometric, algebraic, topological, etc) of the set of values

$$\{\underline{F}(T) : T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}, n_1, n_2, n_3 \in \mathbb{N}\}?$$

Does  $\underline{F}(T)$  have accumulation points, that is, are there non-trivial sequences of tensors  $T_1, T_2, \dots$  such that  $\underline{F}(T_i)$  converges? Or is it discrete? What “gaps” are there between the possible values? Even when  $\mathbb{F}$  is a finite field, the answers to these questions are a priori not clear at all.

In this paper we prove *discreteness* of asymptotic tensor rank, asymptotic subrank and asymptotic slice rank in several regimes. This means that the values of each of these parameters have no accumulation points. In fact, the proof of discreteness of asymptotic tensor rank (over any finite field or finite coefficient set in any field) follows from a simple

argument. Using that same simple argument, together with several new results about tensors, we obtain discreteness for the other parameters. In particular, as a core ingredient, we prove a new result about diagonalizability of tensors. This comes in the form of a lower bound on the asymptotic subrank that relies only on the dimensions of the tensor (as opposed to the well-known laser method for fast matrix multiplication, which relies on much more information about the tensor). As another core ingredient we prove new results about matrix subspaces and their max-rank and min-rank. In particular, we prove that the max-ranks of the matrix spaces obtained by slicing a tensor in the three different directions are strongly related, in such a way that at least two of them must be large.

Our discreteness results show that there is a surprising rigidity in the asymptotic behaviour of tensors. The discreteness of the above parameters gives rise to the phenomenon of “rounding” or “boosting” (upper or lower) bounds on them to the next possible value (although making this effective requires more knowledge of the possible values than just knowing discreteness). The discreteness of asymptotic tensor rank<sup>1</sup> implies, for instance, that the asymptotic rank of the matrix multiplication tensor is bounded away from (or equal to) the asymptotic rank of any other tensor. In particular, the matrix multiplication exponent  $\omega$  is an isolated point among the exponents of all bilinear maps. If such a tensor is “close enough” to the matrix multiplication tensor, its exponent must “snap” to  $\omega$ . Similar statements hold for the other asymptotic parameters. In the context of combinatorial applications, this may moreover lead to limitations for the asymptotic slice rank to improve on existing results.

Before stating our results in detail, we will first discuss the various asymptotic ranks, their applications and context.

## Matrix multiplication and asymptotic tensor rank

Determining the computational complexity of matrix multiplication is a fundamental problem in algebraic complexity theory. This complexity is controlled by the famous matrix multiplication exponent  $\omega$ , which is defined as the infimum over nonnegative real numbers  $\tau$  such that any two  $n \times n$  matrices can be multiplied using  $\mathcal{O}(n^\tau)$  arithmetic operations [11, 5]. The naive matrix multiplication algorithm gives the upper bound  $\omega \leq 3$ . In 1969, Strassen proved that  $\omega \leq 2.81$  [49]. Since then, using many different techniques, the best upper bound has been brought down to  $\omega \leq 2.371552$  [42, 3, 30, 58]. There is a tantalizing possibility (and many have conjectured) that  $\omega = 2$ , and routes have been proposed that aim to prove this [19, 18, 20, 7, 8]. It is just as intriguing to consider the possibility that  $\omega > 2$  and  $\omega$  giving rise to a new fundamental constant, and there has been much work on this lower bound direction as well [12, 41].

Not only can we currently not determine the value of  $\omega$ , or decide whether  $\omega = 2$  or  $\omega > 2$ , there is a much more relaxed problem that we cannot solve. Indeed  $\omega$  is naturally described in terms of tensors as the logarithm of the asymptotic tensor rank of the matrix multiplication tensor  $\langle 2, 2, 2 \rangle \in \mathbb{F}^4 \otimes \mathbb{F}^4 \otimes \mathbb{F}^4$ , that is  $\omega = \log \underline{\mathbb{R}}(\langle 2, 2, 2 \rangle)$ , and thus  $\omega > 2$  is equivalent to  $\underline{\mathbb{R}}(\langle 2, 2, 2 \rangle) > 4$ . The following much more relaxed problem is open:

► **Problem 1** ([11, Open Problem 15.5]). Prove that there is a tensor  $T \in \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$  with  $\underline{\mathbb{R}}(T) > n$ .

<sup>1</sup> Here we work in the regime where we fix the field to be any finite field, or we use any finite set of coefficients in any field.

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It is possible that for every concise tensor  $T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$  we have  $\underline{\mathbb{R}}(T) = n$  (which would in particular imply  $\omega = 2$ ) and that the image of  $\underline{\mathbb{R}}$  over all tensors is simply  $\mathbb{N}$ . This naturally leads to the (very general) question: What is the structure (geometric, algebraic, topological, etc) of the set

$$S = \{\underline{\mathbb{R}}(T) : T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}, n_1, n_2, n_3 \in \mathbb{N}\}?$$

Is there anything we can prove about  $S$  without resolving Problem 1 or determining  $\omega$ ? Not much is known.

One known structural result is that  $S$  is closed under applying any univariate polynomial with non-negative integer coefficients [57, Theorem 4.8]. This statement applies in fact more generally, and in particular also to asymptotic subrank and asymptotic slice rank. This thus says that  $S$  has “many” elements.

Our discreteness result says that  $S$  does not have “too many” elements, and for asymptotic tensor rank over a finite field<sup>2</sup> the proof is surprisingly simple. Here is a sketch: Let  $T_1, T_2, \dots$  be any sequence of tensors such that  $T_i \in \mathbb{F}^{a_i} \otimes \mathbb{F}^{b_i} \otimes \mathbb{F}^{c_i}$  and such that  $\underline{\mathbb{R}}(T_i)$  takes infinitely many values. We may assume that every  $T_i$  is concise, meaning it does not fit into any smaller tensor space. Then  $\underline{\mathbb{R}}(T_i) \geq \max\{a_i, b_i, c_i\}$ . Since by assumption  $\mathbb{F}$  is finite, there are only finitely many tensors per format  $a_i \times b_i \times c_i$ , so the set of triples  $\{(a_i, b_i, c_i) : i \in \mathbb{N}\}$  is infinite. In particular,  $\max\{a_i, b_i, c_i\}$  is unbounded, and so  $\underline{\mathbb{R}}(T_i)$  is unbounded and cannot converge, which proves the claim.

While this argument is very simple, a much more subtle argument and new technical results will be needed to deal with the other tensor parameters that we consider.

### Asymptotic subrank and asymptotic slice rank

Besides tensor rank, there are many other notions of rank of a tensor that play a role in applications, for instance the subrank, slice rank, analytic rank [35, 43, 17], geometric rank [40], and G-stable rank [26]. We will focus here on the asymptotic subrank and asymptotic slice rank. The subrank was introduced by Strassen [50] in the study of matrix multiplication algorithms. The subrank  $\mathbb{Q}(T)$  is the size of the largest diagonal tensor that can be obtained from  $T$  by taking linear combinations of the slices in the three different directions (i.e. “Gaussian elimination” for tensors). The slice rank was introduced by Tao [53] to give a tensor proof of the cap set problem after the first proof of Gijswijt and Ellenberg [31]. The slice rank  $\text{SR}(T)$  is the smallest number of tensors with flattening rank one whose sum is  $T$ . Tao proved that  $\mathbb{Q}(T) \leq \text{SR}(T)$ . Recent works have shown that analytic rank, geometric rank [16, 17], and G-stable rank [26] are all equal to slice rank, up to a multiplicative constant. These results imply that the asymptotic versions of these parameters are all equal to the asymptotic slice rank, warranting our focus on it.

**Slice rank method in combinatorics.** The proof of the longstanding cap set problem [53, 48] (and other related results [46]) can be thought of as upper bounding the independence number of powers of a hypergraph, by constructing a tensor that “fits” on the hypergraph and then computing the slice rank of the powers of the tensor, that is, the asymptotic slice rank  $\underline{\text{SR}}(T)$ . Knowing (the structure of) the set

$$S = \{\underline{\text{SR}}(T) : T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}, n_1, n_2, n_3 \in \mathbb{N}\}$$

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<sup>2</sup> Or in fact over any finite set of coefficients coming from an arbitrary field.

thus gives information on what bounds one can prove on the size of combinatorial objects using the slice rank method. One of our main results is that asymptotic slice rank is discrete, not only over every finite field, but even over the complex numbers. The latter crucially requires a result from [15] that characterizes asymptotic slice rank in terms of representation-theoretic objects called moment polytopes. In fact it is known by now that the four smallest value in  $S$  are  $0, 1, \approx 1.88, 2, \approx 2.68$  (see next section) and our result says that also the larger values are discrete.

**Matrix multiplication barriers.** Besides the aforementioned combinatorial problems, the asymptotic subrank and asymptotic slice rank appear in several “barrier results” for matrix multiplication algorithms [2, 14, 7]. Matrix multiplication algorithms are usually constructed by a reduction of matrix multiplication to another bilinear map, and these barrier results say what properties that intermediate bilinear map must have to obtain certain upper bounds on  $\omega$ , or to reach  $\omega = 2$ . These properties can be phrased in terms of asymptotic subrank or asymptotic slice rank. In particular, the barrier of [14] states that to reach  $\omega = 2$ , an intermediate tensor  $T$  must have  $\underline{R}(T) = \underline{Q}(T)$ , which has led to further research to find tensors with large asymptotic subrank [6]. Our discreteness result says that asymptotic subrank is discrete over every finite field. We do not get this result over the complex numbers because the analogous representation-theoretic ingredient from above is missing here. Intriguingly, it is possible that

$$S = \{\underline{Q}(T) : T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}, n_1, n_2, n_3 \in \mathbb{N}\}$$

equals the analogous set for asymptotic slice rank, that is, the following is open:

► **Problem 2.** Prove that asymptotic slice rank equals asymptotic subrank.

We do as a side-result prove a new relation between asymptotic subrank and asymptotic slice rank (which we will discuss in more detail in the next section).

## Previous work on discreteness of asymptotic ranks

Several works, among which some very recent ones, have investigated notions of discreteness in the values of tensor parameters.

Strassen [51, Lemma 3.7] proved that for any  $k$ -tensor  $T$  over any field, the asymptotic subrank (and, as a consequence of his method, also the asymptotic slice rank) of  $T$  is equal to 0, equal to 1, or at least  $2^{2/k}$ . This result established the first “gaps” in asymptotic tensor parameters. Costa and Dalai [23] proved that, for any  $k$ -tensor  $T$  over any field, the asymptotic slice rank of  $T$  is equal to 0, equal to 1 or at least  $2^{h(1/k)}$  where  $h$  is the binary entropy function<sup>3</sup>. Christandl, Gesmundo and Zuiddam [13] extended the result of Costa and Dalai by proving that, for any  $k$ -tensor  $T$  over any field, the asymptotic subrank and asymptotic partition rank of  $T$  are equal to 0, equal to 1 or at least  $2^{h(1/k)}$  (which is a tight bound). Additionally, they prove that for any 3-tensor  $T$  over any field, the asymptotic subrank and asymptotic slice rank of  $T$  are equal to 0, equal to 1, equal to  $2^{h(1/3)} \approx 1.88$  or at least 2. Gesmundo and Zuiddam [34] extended this result by proving that the next possible value after 2 is  $\approx 2.68$ .

<sup>3</sup> The binary entropy function is defined for  $p \in (0, 1)$  by  $h(p) = -p \log_2 p - (1 - p) \log_2 (1 - p)$  and  $h(0) = h(1) = 0$ .

Blatter, Draisma and Rupniewski [10] proved that for any function on  $k$ -tensors, over any finite field, that is normalized and monotone the set of values that this function takes is well-ordered. The asymptotic subrank and asymptotic slice rank<sup>4</sup> are examples of such functions. This means that the values of any such function do not have accumulation points “from above”, but leaves open the possibility that there are accumulation points “from below”.

Christandl, Vrana and Zuiddam [15] proved that the asymptotic slice rank over the complex numbers takes only finitely many values on tensors of any fixed format, and thus only countably infinite many values in general. This is done by characterizing the asymptotic slice rank as an optimization over the moment polytope of the tensor and using the result that there are only finitely many such polytopes per tensor format.

Blatter, Draisma and Rupniewski [9] proved that for any “algebraic” tensor invariant over the complex numbers the related asymptotic parameter takes only countably many values. This implies in particular that the asymptotic subrank, asymptotic slice rank, asymptotic geometric rank, and asymptotic partition rank (all over the complex numbers) take only countably many values.

## New results

In this paper:

- We prove two general lower bounds on the asymptotic subrank of concise tensors that depend only on the dimensions of the tensor. The first says that the asymptotic subrank of any concise tensor is at least the cube-root of its smallest dimension. The second says that the asymptotic subrank of any “narrow enough” tensor (meaning that one dimension is much smaller than the others) is maximal.
- We use those lower bounds to prove that over any finite set of coefficients in any field the asymptotic subrank has no accumulation points (i.e., is discrete). We moreover prove that over any finite set of coefficients and over the complex numbers the asymptotic slice rank has no accumulation points. A much simpler argument gives that the asymptotic rank is discrete over any finite set of coefficients.
- As a core ingredient for the above, we prove optimal relations among the maximal rank of any matrix in the span of the slices of a tensor, when considering slicings in the three different directions.
- With similar techniques, we prove an upper bound on the difference between asymptotic subrank and asymptotic slice rank.

### 1.1 Discreteness of asymptotic tensor parameters

We will now discuss our results in more detail. We begin with some basic definitions (which we discuss in more detail in the Preliminaries subsection in the full version of the paper). Let  $\mathbb{F}$  be any field. Let  $T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$  be an order-three tensor over  $\mathbb{F}$  with dimensions  $(n_1, n_2, n_3)$ . The subrank of  $T$ , denoted by  $Q(T)$ , is the largest number  $r$  such that there are linear maps  $L_i : \mathbb{F}^{n_i} \rightarrow \mathbb{F}^r$  such that  $(L_1 \otimes L_2 \otimes L_3)T = \sum_{i=1}^r e_i \otimes e_i \otimes e_i$ . In other words, the subrank measures how much a tensor can be diagonalized. The flattenings of  $T$  are the elements in  $(\mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2}) \otimes \mathbb{F}^{n_3}$ ,  $\mathbb{F}^{n_1} \otimes (\mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3})$  and  $\mathbb{F}^{n_2} \otimes (\mathbb{F}^{n_1} \otimes \mathbb{F}^{n_3})$ , obtained by naturally grouping the tensor legs of  $T$ . We say  $T$  has slice rank one if at least one of

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<sup>4</sup> Over finite fields, since we do not know whether the limit  $\lim_{n \rightarrow \infty} \text{SR}(T^{\boxtimes n})^{1/n}$  exists in general, when we say asymptotic slice rank we will mean  $\liminf_{n \rightarrow \infty} \text{SR}(T^{\boxtimes n})^{1/n}$ .

the flattenings has rank one. The slice rank of  $T$ , denoted by  $\text{SR}(T)$  is the smallest number  $r$  such that  $T$  is a sum of  $r$  tensors with slice rank one. The asymptotic subrank of  $T$  is defined as  $\underline{\text{Q}}(T) = \lim_{n \rightarrow \infty} \text{Q}(T^{\boxtimes n})^{1/n}$  where  $\boxtimes$  is the Kronecker product on tensors. The limit exists and equals the supremum by Fekete's lemma, since  $\text{Q}$  is super-multiplicative. The asymptotic slice rank of  $T$  we define as  $\underline{\text{SR}}(T) = \liminf_{n \rightarrow \infty} \text{SR}(T^{\boxtimes n})^{1/n}$ . (Over the complex numbers, it is known that the liminf can be replaced by a limit. Over other fields, however this is generally not known.)

### 1.1.1 Discreteness of asymptotic slice rank and asymptotic subrank

We prove discreteness (no accumulation points) for the asymptotic slice rank and asymptotic subrank, in several regimes, as follows.

► **Theorem 3.** *Over any finite set of coefficients in any field, the asymptotic subrank and the asymptotic slice rank each have no accumulation points.*

Theorem 3 improves the result of Blatter, Draisma and Rupniewski [10] that the asymptotic subrank and asymptotic slice rank over any finite field have no accumulation points “from above”, that is, have well-ordered sets of values. Indeed our result rules out all accumulation points (so also those “from below”).

► **Theorem 4.** *Over the complex numbers, the asymptotic slice rank has no accumulation points.*

Theorem 4 improves the result of Blatter, Draisma and Rupniewski [9] and Christandl, Vrana and Zuiddam [15] that the asymptotic slice rank over the complex numbers takes only countably many values. Our result is indeed strictly stronger, as countable sets may a priori have accumulation points.

Our results shed light on the recent results of Costa and Dalai [23] and Christandl, Gesmundo and Zuiddam [13] that found gaps between the smallest values of the asymptotic slice rank and asymptotic subrank, and answers positively (in some regimes) the question stated in [13] asking whether the values will always be “gapped”.

Besides the above, we prove discreteness theorems over arbitrary fields for two sub-classes of all tensors. Namely we consider the class of oblique tensors, which are the tensors whose support in some basis is an antichain, and the tight tensors, whose support in some basis can be characterized by algebraic equation (details in the full version of the paper). The set of tight tensors is a strict subset of the set of oblique tensors, which is a strict subset of all tensors. Both classes originate in the work of Strassen [50, 51, 52]. Examples of tight tensors include the well-known matrix multiplication tensors. Tight tensors also play a central role in the laser method of Strassen to construct fast matrix multiplication algorithms. Our discreteness theorems in these regimes are as follows.

► **Theorem 5.** *Over any field, on tight tensors, the asymptotic subrank and asymptotic slice rank (which are equal) have no accumulation points.*

► **Theorem 6.** *Over any field, on oblique tensors, the asymptotic slice rank has no accumulation points.*

### 1.1.2 General discreteness theorem

We prove the above discreteness theorems as an application of a general discreteness theorem that we discuss now. This general theorem gives discreteness for real-valued tensor parameters that satisfy several conditions. To describe these conditions we need the notion of equivalence

of tensors. We say two tensors  $S \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$  and  $T \in \mathbb{F}^{m_1} \otimes \mathbb{F}^{m_2} \otimes \mathbb{F}^{m_3}$  are *equivalent* if there are linear maps  $A_i : \mathbb{F}^{n_i} \rightarrow \mathbb{F}^{m_i}$  such that  $(A_1 \otimes A_2 \otimes A_3)S = T$  and there are linear maps  $B_i : \mathbb{F}^{m_i} \rightarrow \mathbb{F}^{n_i}$  such that  $(B_1 \otimes B_2 \otimes B_3)S = T$ . A tensor  $T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$  is called *concise* if the three flattenings obtained by grouping two of the three tensor legs together, in the three possible ways, each have maximal rank. This means essentially that  $T$  cannot be embedded into a smaller tensor space.

► **Theorem 7.** *Let  $\mathbb{F}$  be any field. Let  $\mathcal{C}$  be any subset of tensors over  $\mathbb{F}$  such that for every  $T \in \mathcal{C}$  there is an  $S \in \mathcal{C}$  that is concise and equivalent to  $T$ . Let  $f : \mathcal{C} \rightarrow \mathbb{R}_{\geq 0}$  be any function such that*

- (i)  *$f$  does not change under equivalence of tensors*
- (ii)  *$f(T) \geq Q(T)$  for every  $T \in \mathcal{C}$*
- (iii) *For every  $n_1, n_2, n_3 \in \mathbb{N}$ ,  $f$  takes finitely many values on  $\mathcal{C} \cap (\mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3})$*
- (iv) *For every  $n_1, n_2, n_3 \in \mathbb{N}$ ,  $f(T) \leq \min_i n_i$  for every  $T \in \mathcal{C} \cap (\mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3})$ .*

*Then the set of values that  $f$  takes on  $\mathcal{C}$  has no accumulation points.*

We make some remarks on condition (iii) of Theorem 7. When applying Theorem 7 to a given real-valued function  $f$  on tensors, it will depend very much on the regime we are working in whether condition (iii) is non-trivial or not. In particular, over finite fields, condition (iii) is trivial, because then  $\mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$  contains only finitely many elements. Another trivial situation is when  $f$  takes only integral values, as there are only finitely many integers between 0 and  $\min_i n_i$  (in this case the conclusion of the theorem is also trivial). However, when  $f$  is not integral (say  $f$  is the asymptotic slice rank or asymptotic subrank) and the field is not finite (say  $\mathbb{F}$  is the complex numbers) condition (iii) can be very non-trivial. For instance, our application of Theorem 7 to asymptotic slice rank over the complex numbers (leading to Theorem 4) relies on the representation-theoretic characterization of this parameter in terms of moment polytopes [15] and a result from invariant theory that there are only finitely many such polytopes per choice of  $(n_1, n_2, n_3)$ .

Our proof of Theorem 7 relies mainly on new lower bounds on the asymptotic subrank  $Q(T)$  of concise tensors  $T$ , which we will discuss next. Intuitively, these lower bounds will ensure (using condition (ii)) that for any infinite sequence of tensors  $T_i$  the value of  $f(T_i)$  gets “pushed up” so much that it either cannot converge, or eventually becomes constant (when  $\min_i n_i$  is bounded).

## 1.2 Lower bounds on asymptotic subrank

Having discussed our discreteness theorems, we will now discuss two results that are central in the proof of the discreteness theorems, and of independent interest. These results are about lower bounds on the asymptotic subrank.

The general goal of these results, in the context of the proof of the general discreteness theorem, is to establish that if we have a sequence of tensors  $T_i \in \mathbb{F}^{a_i} \otimes \mathbb{F}^{b_i} \otimes \mathbb{F}^{c_i}$  for  $i \in \mathbb{N}$  such that the set of triples  $\{(a_i, b_i, c_i) : i \in \mathbb{N}\}$  is infinite, then the asymptotic subrank of these tensors must be either unbounded, or eventually constant and integral. We will discuss this in detail in the proof of the general discreteness theorem.

### 1.2.1 Previous work

Strassen [52] building on the work of Coppersmith and Winograd [22] introduced a method to prove (optimal) asymptotic subrank lower bounds for a subclass of structured tensors called “tight” tensors. This method formed an integral part of the laser method for constructing fast matrix multiplication algorithms, and similar ideas have also been applied in the context of additive combinatorics [39].



### 1.2.2 Concise tensors have large asymptotic subrank

For concise tensors  $T$  we prove a cube-root lower bound on the asymptotic subrank  $\underline{Q}(T)$  in terms of the smallest dimension of the tensor.

► **Theorem 8.** *Let  $T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$  be concise. Then  $\underline{Q}(T) \geq (\min_i n_i)^{1/3}$ .*

We emphasize that Theorem 8 does not rely on any special structure of the tensor  $T$ , unlike previous methods like the laser method that rely on  $T$  being “tight”.

We do not know whether the lower bound  $(\min_i n_i)^{1/3}$  in Theorem 8 is optimal. The best upper bound we know is from an example of a concise tensor  $T \in \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$  such that  $\underline{Q}(T) = 2\sqrt{n-1}$  (Strassen’s null-algebra [52, p. 168], see some details in the full version of this paper).

Alternatively, Theorem 8 can be phrased without the conciseness condition if we replace the dimension  $n_i$  by the flattening rank  $R(T_i)$ , as follows: Let  $T$  be any tensor. Then  $\underline{Q}(T) \geq (\min_i R_i(T))^{1/3}$ .

For symmetric tensors we can prove the following stronger bound. (In fact, we can prove this stronger bound for a larger class of tensors which we call “pivot-matched”, as we will explain in the full version of the paper). We recall that a tensor  $T = \sum_{i,j,k} T_{i,j,k} e_i \otimes e_j \otimes e_k \in \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$  is called symmetric if permuting the three tensor legs does not change the tensor, that is, for every  $(i_1, i_2, i_3) \in [n]^3$  and every permutation  $\sigma \in S_3$  it holds that  $T_{i_1, i_2, i_3} = T_{i_{\sigma(1)}, i_{\sigma(2)}, i_{\sigma(3)}}$ .

► **Theorem 9.** *Let  $T \in \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$  be concise and symmetric. Then  $\underline{Q}(T) \geq n^{1/2}$ .*

We do not know whether the lower bound  $n^{1/2}$  in Theorem 9 is optimal.

Again, alternatively, Theorem 9 can be phrased without conciseness if the dimension  $n$  is replaced by the flattening rank  $R_1(T)$  (for symmetric tensors the three flattening ranks are equal): Let  $T \in \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$  be symmetric. Then  $\underline{Q}(T) \geq R_1(T)^{1/2}$ .

### 1.2.3 Narrow tensors have maximal asymptotic subrank

Theorem 8 implies that if  $n_1, n_2$  and  $n_3$  all grow, and  $T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$  is any concise tensor, then  $\underline{Q}(T)$  must also grow. This leaves open what happens in the regime where one of the  $n_i$  is constant. We will consider the “narrow” regime where  $n_3 = c$  is constant, and one of the dimensions  $n_1, n_2$  is large enough. Here we prove that the asymptotic subrank is maximal, that is, matches the upper bound  $c$ .

► **Theorem 10.** *For every integer  $c \geq 2$  there is an  $N(c) \in \mathbb{N}$  such that for every  $n_1, n_2$  with  $\max\{n_1, n_2\} > N(c)$  and every concise tensor  $T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^c$  we have  $\underline{Q}(T) = c$ .*

Moreover, for the case  $c = 2$  we prove with a direct construction that  $N(2) = 2$  and that asymptotic subrank can be replaced by subrank.

► **Theorem 11.** *Let  $n_1, n_2 > 2$ . Let  $T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^2$  be concise. Then  $\underline{Q}(T) = 2$ .*

### 1.2.4 Lower bound on asymptotic subrank in terms of slice rank

Besides the aforementioned bounds on the asymptotic subrank, we use some of the same methods to prove a lower bound on the asymptotic subrank in terms of the asymptotic slice rank.

Slice rank was introduced by Tao [53]. He proved that for every tensor  $T$  we have  $\text{SR}(T) \geq \underline{Q}(T)$ . The gap between  $\text{SR}(T)$  and  $\underline{Q}(T)$  can be large, namely for generic tensors  $T \in \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$  (over algebraically closed fields  $\mathbb{F}$ ) it is known that  $\text{SR}(T) =$

$n$  while  $Q(T) = \Theta(\sqrt{n})$  [28]. It is, however, not known whether there can be a large gap between  $\text{SR}(T^{\boxtimes m})$  and  $Q(T^{\boxtimes m})$  when  $m$  is large. In particular, it is possible that  $\lim_{n \rightarrow \infty} Q(T^{\boxtimes n})^{1/n} = \lim_{n \rightarrow \infty} \text{SR}(T^{\boxtimes n})^{1/n}$ . Strassen's work implies this equality for the subset of tight tensors [52]. Over the complex numbers, the limit  $\lim_{n \rightarrow \infty} \text{SR}(T^{\boxtimes n})^{1/n}$  is also known to exist and has a characterization in terms of moment polytopes [15].

We prove the following:

► **Theorem 12.** *Suppose the limit  $\text{SR}(T) = \lim_{n \rightarrow \infty} \text{SR}(T^{\boxtimes m})^{1/m}$  exists. Then*

$$Q(T) \geq \text{SR}(T)^{2/3}.$$

Our proof of Theorem 12 consists of proving that the border subrank of the third power of  $T$  is bounded from below in terms of the slice rank of  $T$ , and applying a field-agnostic Flanders-type lower bound on the max-rank of Haramaty and Shpilka [36].

### 1.3 Max-rank and min-rank of slice spans

Our lower bounds on asymptotic subrank discussed in the previous subsection rely on results we prove about the ranks of elements in the span of the slices of a tensor. These may be of independent interest and we discuss some of them here.

#### 1.3.1 Max-ranks of slice spans

Our proof of Theorem 8 relies (among other ingredients) on the notion of the max-rank of matrix subspaces, and the relation between max-ranks of matrix subspaces obtained by slicing a tensor in the three possible directions.

For any matrix subspace  $\mathcal{A} \subseteq \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2}$ , we let  $\text{max-rank}(\mathcal{A})$  denote the largest matrix rank of any element of  $\mathcal{A}$ . To any tensor  $T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$  we may associate three matrix subspaces  $\mathcal{A}_1 \subseteq \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$ ,  $\mathcal{A}_2 \subseteq \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_3}$  and  $\mathcal{A}_3 \subseteq \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2}$ , obtained by taking the span of the slices of  $T$  in each of the three possible directions. We denote the max-ranks of these spaces by  $Q_i(T) = \text{max-rank}(\mathcal{A}_i)$ , for  $i \in [3]$ . We prove that for a concise tensor  $T$  the max-ranks  $Q_i(T)$  cannot all be small, in the following sense.

► **Theorem 13.** *Let  $T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$  be concise. Let  $i, j, k \in [3]$  be distinct. Suppose that  $|\mathbb{F}| > n_k$ . Then  $Q_i(T) Q_j(T) \geq n_k$ .*

We have explicit examples of families of tensors (provided later) that show how Theorem 13 is essentially optimal:

- For every  $n$  that is a square, there is a concise tensor  $T \in \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$  such that for all  $i$  we have  $\sqrt{n} \leq Q_i(T) \leq 2\sqrt{n}$  (see the tensor in the full version of the paper). In particular, for all  $i \neq j$  we have  $n \leq Q_i(T) Q_j(T) \leq 4n$ .
- For every  $n$ , there is a concise tensor  $T \in \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$  such that  $Q_1(T) = Q_3(T) = n$  and  $Q_2(T) = 2$ , so that  $Q_2(T) Q_3(T) = 2n$  (Strassen's null algebra [52, p. 168]).
- For every  $c$  and for every  $n$  that is a multiple of  $c$ , there is a concise tensor  $T \in \mathbb{F}^n \otimes \mathbb{F}^n \otimes \mathbb{F}^n$  such that  $Q_1(T) = n$ ,  $Q_2(T) \leq c+1$  and  $Q_3(T) \leq n/c+1$  (by a generalisation of Strassen's null algebra, see the full version of this paper). In particular,  $Q_2(T) Q_3(T) \leq \frac{(c+1)}{c} n + c + 1$ .

It follows from Theorem 13 (by a straightforward argument) that  $Q_i(T)$  must be large for at least two directions  $i$ , in the following sense:

► **Corollary 14.** *Let  $T \in \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2} \otimes \mathbb{F}^{n_3}$  be concise. There are distinct  $\ell_1, \ell_2 \in \{1, 2, 3\}$  such that  $Q_{\ell_1}(T) \geq (\max_i n_i)^{1/2}$  and  $Q_{\ell_2}(T) \geq (\min_i n_i)^{1/2}$ .*

From Corollary 14 we can prove a preliminary asymptotic subrank lower bound  $\underline{Q}(T) \geq (\min_i n_i)^{1/4}$ , using the (easy to prove) fact that  $\underline{Q}(T)^2 \geq Q_i(T)Q_j(T)$  for any distinct  $i, j \in [3]$ . Proving our stronger cube-root lower bound  $\underline{Q}(T) \geq (\min_i n_i)^{1/3}$  of Theorem 8 requires slightly more work.

Work on max-rank (and its relation to dimension) goes back to Dieudonné [29], Meshulam [44] and [32]. Another relevant line of work here is on commutative and non-commutative rank, which has established strong connections between max-rank and non-commutative rank [21, 33, 27].

### 1.3.2 Min-ranks of slice spans

Our proof of Theorem 10 relies on a careful analysis of the min-rank of matrix subspaces, the relation to subrank and the behaviour of min-rank under powering.

For any matrix subspace  $\mathcal{A} \subseteq \mathbb{F}^{n_1} \otimes \mathbb{F}^{n_2}$ , we let  $\text{min-rank}(\mathcal{A})$  denote the smallest matrix rank of any nonzero element of  $\mathcal{A}$ . We prove several properties of the min-rank, of which we give a rough outline here (for the precise description we see the full version of the paper):

- If the slices of a tensor have large min-rank, then the tensor has large subrank.
- Any concise tensor has a slice of large rank.
- If a matrix subspace has large max-rank, then we can transform the subspace in a natural fashion such that it has large min-rank and all elements are diagonal.
- Min-rank is super-multiplicative under tensor product, as long as at least one of the matrix subspaces is diagonal.

A careful combination of the above ingredients leads to a proof of Theorem 10.

The min-rank has been investigated before in several different contexts. Amitsur [4] used min-rank to characterize properties of rings of operators. Meshulam and Semrl [45] used the min-rank to study properties of operator spaces. In quantum information the rank is a measure of entanglement. Spaces of bipartite states which are all entangled are spaces of matrices over the complex numbers with min-rank strictly greater than 1. They were investigated in [55] and [47]. In [38] a slightly different angle was taken, analysing random subspaces. It was shown that most random subspaces have almost maximal min-rank, which was used for superdense coding in [1], and was summarised in [37]. Generalising both of these lines of work, in [25] the question of dimension versus min-rank was addressed as number of qubits versus guaranteed entanglement in a subspace of states, and their construction is used in a follow-up paper [24] to show there are counterexamples to the additivity of  $p$ -Rényi entropies, for all  $p \leq p_0$  for some small constant  $p_0 < 1$ , utilising the fact that the 0-Rényi entropy is the min-rank.

This is an extended abstract. The full version can be found at:

<https://arxiv.org/abs/2306.01718>

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