Iterated Elimination of Weakly Dominated Strategies in Well-Founded Games

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Recently, in [3], we studied well-founded games, a natural extension of finite extensive games with perfect information in which all plays are finite. We extend here, to this class of games, two results concerned with iterated elimination of weakly dominated strategies, originally established for finite extensive games.

The first one states that every finite extensive game with perfect information and injective payoff functions can be reduced by a specific iterated elimination of weakly dominated strategies to a trivial game containing the unique subgame perfect equilibrium. Our extension of this result to well-founded games admits transfinite iterated elimination of strategies. It applies to an infinite version of the centipede game. It also generalizes the original result to a class of finite games that may have several subgame perfect equilibria.

The second one states that finite zero-sum games with n outcomes can be solved by the maximal iterated elimination of weakly dominated strategies in $n-1$ steps. We generalize this result to a natural class of well-founded strictly competitive games.

1 Introduction

This paper is concerned with the iterated elimination of weakly dominated strategies (IEWDS) in the context of natural class of infinite extensive games with perfect information. While simple examples show that the deletion of weakly dominated strategies may result in removal of a unique Nash equilibrium, IEWDS has some merit if it results in solving a game. It is for instance used to show that the so-called "beauty contest" game has exactly one Nash equilibrium (see, e.g., [7, Chapter 5]). Other games can be solved this way, see, e.g., [11, pages 63, 110-114].

This procedure was also studied in the realm of finite extensive games with perfect information. In [8] the correspondence between the outcomes given by the iterated elimination of weakly dominated strategies and backward induction was investigated in the context of binary voting agendas with sequential voting. More recently, this procedure was studied in [16] in the context of supermodular games.

For arbitrary games two important results were established. The first one states, see [11], that in such games with injective payoff functions (such games are sometimes called *generic*) a specific iterated elimination of weakly dominated strategies (that mimics the backward induction) yields a trivial game which contains the unique subgame perfect equilibrium. It was noticed in [4] that this result holds for a slightly more general class of games *without relevant ties*.[1]

[1] All mentioned concepts are explained in Sections 2, 4, and 5. We did not find any precise proofs in the literature. The proof is briefly sketched in [11, pages 108-109] and summarized in [4, pages 48-49] as follows: "if backward induction deletes action a at node x, delete all the strategies reaching x and choosing a". We provided in [2] a detailed proof of the stronger result of [4] in which we clarified how the backward induction algorithm needs to be modified to achieve the desired outcome.

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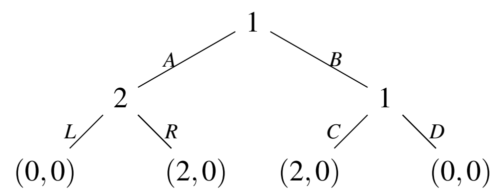
The second result, due to [6], is concerned with finite extensive zero-sum games. It states that such games can be reduced to a trivial game by the 'maximal' iterated elimination of weakly dominated strategies in $n-1$ steps, where n is the number of outcomes.[2]

In [3] we studied a natural extension of finite extensive games with perfect information in which one assumes that all plays are finite. We called these games well-founded games.[3] The subject of this paper is to extend the above two results to well-founded games. In both cases some non-trivial difficulties arise.



	L	R
AC	$(0,0)$	$(2,0)$
AD	$(0,0)$	$(2,0)$
BC	$(2,0)$	$(2,0)$
BD	$(0,0)$	$(0,0)$

Figure 1: An extensive game G and the corresponding strategic game $\Gamma(G)$

Example 1 Consider the extensive game G and the corresponding strategic game $\Gamma(G)$ given in Figures 1. G has three subgame perfect equilibria which are all payoff equivalent: $\{(AC,R),(BC,L),(BC,R)\}$. We can observe that in $\Gamma(G)$ no sequence of iterated elimination of weakly dominated strategies results in a trivial game that contains all the subgame perfect equilibria in G. To see this, first note that the strategies L and R of player 2 are never weakly dominated irrespective of the elimination done with respect to the strategies of player 1. Also, note that the strategy BD of player 1 is strictly dominated by BC in $\Gamma(G)$. Thus the only possibility of reducing $\Gamma(G)$ to a trivial game is to eliminate all strategies of player 1 except BC. But this results in the elimination of (AC,R) which is a subgame perfect equilibrium in G. □

This might suggest that one should limit oneself to extensive games with a unique subgame perfect equilibrium. Unfortunately, this restriction does not work either as shown in Example 2. Additional complication arises when the game has no subgame perfect equilibrium as shown in 3.

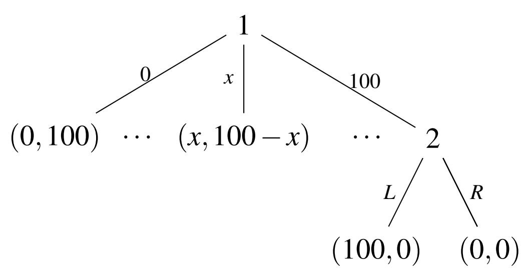


Figure 2: A game G with a unique SPE

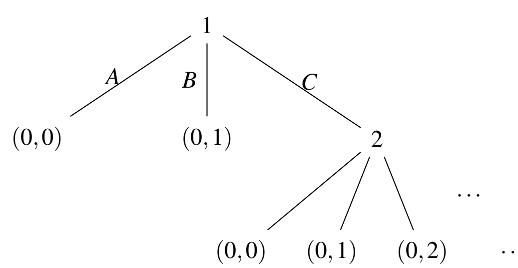


Figure 3: A game G with no SPE

Example 2 Consider a 'trimmed version' of the ultimatum game from [3] given in Figure 2, in which for each $x \in [0,100]$ the root has a direct descendant x. This game has a unique subgame perfect equilibrium, namely $(100,L)$. Consider an iterated elimination of weakly dominated strategies. For each strategy of player 1 the strategies L and R of player 2 yield the same payoff. So these two strategies are never

[2]An alternative proof given in [17] shows that the result holds for the larger class of strictly competitive games. In [2] we clarified that the original proof also holds for this class of games.

[3]In the economic literature such games are sometimes called 'games with finite horizon'.

eliminated. Further, strategy 100 of player 1 is never eliminated either, since for any strategy $x < 100$ we have $p_1(x,L) = x < 100 = p_1(100,L)$ and $p_1(x,R) = x > 0 = p_1(100,R)$. So the joint strategies $(100,L)$ and $(100,R)$ are never eliminated and they are not payoff equivalent. (In fact, each iterated elimination of weakly dominated strategies yields the game with the sets of strategies $\{100\}$ and $\{L,R\}$.) □

Example 3 Consider the well-founded game G given in Figure 3. Clearly G has no subgame perfect equilibrium. Further, strategies A and B of player 1 yield the same outcome for him, so cannot be eliminated by any iterated elimination of weakly dominated strategies. Thus any result of such an elimination contains at least two outcomes, $(0,0)$ and $(0,1)$. So G cannot be reduced to a trivial game. □

To address these issues, we introduce the concept of an *SPE-invariant* well-founded game. These are games in which subgame perfect equilibria exist and moreover in each subgame such equilibria are payoff equivalent. Then we show that the first result can be extended to such games. In view of the above examples it looks like the strongest possible generalization of the original result. In particular, it applies to an infinite version of the well-known centipede game of [15].

This result calls for a careful extension of the iterated elimination of weakly dominated strategies to infinite games: its stages have to be indexed by ordinals and one has to take into account that the outcome can be the empty game.

When limited to finite games, our theorem extends the original result. In particular it applies to the class of extensive games that satisfy the ***transference of decisionmaker indifference (TDI)*** condition due to [10], a class that includes strictly competitive games. We also show that the well-founded games with finitely many outcomes that satisfy the TDI condition are SPE-invariant. Also when extending the second result, about strictly competitive games, to well-founded games one has to be careful. The original proof crucially relies on the fact that finite extensive zero-sum games have a value. Fortunately, as we showed in [3], well-founded games with finitely many outcomes have a subgame perfect equilibrium, so a fortiori a Nash equilibrium, which suffices to justify the relevant argument (Lemma 21 in Section 5).

By carefully checking of the crucial steps of the original proof we extend the original result to a class of well-founded strictly competitive games that includes *almost constant* games, in which for all but finitely many leaves the outcome is the same. It remains an open problem whether this result holds for all strictly competitive games with finitely many outcomes.

IEWDS is one of the early approaches applied to analyze strategies and extensive games. It does not take into account epistemic reasoning of players in the presence of assumptions such as common knowledge of rationality. The vast literature on this subject, starting with [5] and [12], led to identification of several more informative ways of analyzing finite extensive games with imperfect information. We just mention here two representative references. In [4] Pearce's notion of *extensive form rationalizability* (EFR) was studied and it was shown that for extensive games without relevant ties it coincides with the IEWDS. A more general notion of common belief in future rationality was studied in [13] that led to identification of a new iterative elimination procedure called *backward dominance*.

In our paper IEWDS is defined as a transfinite elimination procedure. A number of papers, starting with [9], analyzed when such a transfinite elimination of strategies cannot be reduced to an iteration over ω steps. In our framework it is a simple consequence of the fact that the ranks of the admitted game trees can be arbitrary ordinals. In particular, an infinite version of the centipede game considered in Example 12 requires more than ω elimination rounds.

2 Preliminaries

2.1 Strategic games

A ***strategic game*** $H = (H_1, \ldots, H_n, p_1, \ldots, p_n)$ consists of a set of players $\{1, \ldots, n\}$, where $n \geq 1$, and for each player i, a set H_i of ***strategies*** along with a ***payoff function*** $p_i : H_1 \times \cdots \times H_n \to \mathbb{R}$.

We call each element of $H_1 \times \cdots \times H_n$ a ***joint strategy*** of players $1, \ldots, n$, denote the ith element of $s \in H_1 \times \cdots \times H_n$ by s_i, and abbreviate the sequence $(s_j)_{j \neq i}$ to s_{-i}. We write (s'_i, s_{-i}) to denote the joint strategy in which player's i strategy is s'_i and each other player's j strategy is s_j. Occasionally we write (s_i, s_{-i}) instead of s. Finally, we abbreviate the Cartesian product $\times_{j \neq i} H_j$ to H_{-i}.

Given a joint strategy s, we denote the sequence $(p_1(s), \ldots, p_n(s))$ by $p(s)$ and call it an ***outcome*** of the game. We say that H ***has*** k ***outcomes*** if $|\{p(s) \mid s \in H_1 \times \cdots \times H_n\}| = k$ and call a game ***trivial*** if it has one outcome. If one of the sets H_i is empty, we call the game ***empty*** and ***non-empty*** otherwise. Unless explicitly stated, all used strategic games are assumed to be non-empty. We say that two joint strategies s and t are ***payoff equivalent*** if $p(s) = p(t)$.

We call a joint strategy s a ***Nash equilibrium*** if $\forall i \in \{1, \ldots, n\} \forall s'_i \in H_i : p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i})$. When the number of players and their payoff functions are known we can identify the game H with the set of strategies in it.

By a ***subgame*** of a strategic game H we mean a game obtained from H by removing some strategies. Given a set $\mathscr{J}$ of subgames of a strategic game H we define $\bigcap \mathscr{J}$ as the subgame of H in which for each player i his set of strategies is $\bigcap_{J \in \mathscr{J}} J_i$. Also, given two subgames H' and H'' of a strategic game H we write $H' \subseteq H''$ if for each player i, $H'_i \subseteq H''_i$.

Consider two strategies s_i and s'_i of player i in a strategic game H. We say that s_i ***weakly dominates*** s'_i (or equivalently, that s'_i is ***weakly dominated by*** s_i) in H if $\forall s_{-i} \in H_{-i} : p_i(s_i, s_{-i}) \geq p_i(s'_i, s_{-i})$ and $\exists s_{-i} \in H_{-i} : p_i(s_i, s_{-i}) > p_i(s'_i, s_{-i})$.

In what follows, given a strategic game we consider, possibly transfinite, sequences of sets of strategies. They are written as $(\rho_\alpha, \alpha < \gamma)$, where α ranges over all ordinals smaller than some ordinal γ. Given two such sequences $\rho := (\rho_\alpha, \alpha < \gamma)$ and $\rho' := (\rho'_{\alpha'}, \alpha' < \gamma')$, we denote by (ρ, ρ') their concatenation (which is indexed by $\gamma + \gamma'$), by ρ^β the subsequence $(\rho_\alpha, \alpha < \beta)$ of ρ, and for $\alpha < \beta$ by $\rho^{\beta - \alpha}$ the subsequence such that $(\rho^\alpha, \rho^{\beta-\alpha}) = \rho^\beta$. Further, we write $H \to^\rho H'$ to denote the fact that the game H' is the outcome of the iterated elimination from the non-empty game H of the sets of strategies that form ρ. In each step all eliminated strategies are weakly dominated in the current game. As a result H' may be empty. The relation $\to^\rho$ is defined as follows.

If $\rho = (\rho_0)$, that is, if $\gamma = 1$, then $H \to^\rho H'$ holds if each strategy in the set ρ_0 is weakly dominated in H and H' is the outcome of removing from H all strategies from ρ_0. If γ is a successor ordinal > 1, say $\gamma = \delta + 1$, and $H \to^{\rho'} H'$, $H' \to^{(\rho_\delta)} H''$, where H' is non-empty, and $\rho' := (\rho_\alpha, \alpha < \delta)$, then $H \to^\rho H''$. Finally, if γ is a limit ordinal and for all $\beta < \gamma$, $H \to^{\rho^\beta} H^\beta$, then $H \to^\rho \bigcap_{\beta < \gamma} H^\beta$. In general, the strategic game H from which we eliminate strategies will be a subgame of a game $\Gamma(G)$, where G is an extensive game (to be defined shortly). It will be then convenient to allow in ρ strategies from $\Gamma(G)$. In the definition of $H \to^\rho H'$ we then disregard the strategies from ρ that are not from H. In the proofs below we rely on the following observations about the $\to^\rho$ relation, the proofs of which we omit.

Note 4

(i) Suppose $H \to^\rho H'$ and $H' \to^{\rho'} H''$, where H' is non-empty. Then $H \to^{(\rho, \rho')} H''$.

(ii) Suppose $H \to^\rho H'$, where $\rho = (\rho_\alpha, \alpha < \gamma)$ and γ is a limit ordinal. Suppose further that for a sequence of ordinals $(\alpha_\delta)_{\delta < \varepsilon}$ converging to γ we have $H \to^{\rho^{\alpha_\delta}} H^{\alpha_\delta}$ for all $\delta < \varepsilon$. Then $H' = \bigcap_{\delta < \varepsilon} H^{\alpha_\delta}$.

2.2 Well-founded games

We recall from [3] the definition of a well-founded game. A ***tree*** is an acyclic directed connected graph, written as (V,E), where V is a non-empty set of nodes and E is a possibly empty set of edges. An ***extensive game with perfect information*** $(T, turn, p_1, \ldots, p_n)$ consists of a set of players $\{1,\ldots,n\}$, where $n \geq 1$ along with the following. A ***game tree***, which is a tree $T := (V,E)$ with a ***turn function*** $turn : V \setminus Z \to \{1,\ldots,n\}$, where Z is the set of leaves of T. For each player i a ***payoff function*** $p_i : Z \to \mathbb{R}$, for each player i. The function *turn* determines at each non-leaf node which player should move. The edges of T represent possible ***moves*** in the considered game, while for a node $v \in V \setminus Z$ the set of its children $C(v) := \{w \mid (v,w) \in E\}$ represents possible ***actions*** of player $turn(v)$ at v.

We say that an extensive game with perfect information is ***finite***, ***infinite***, or ***well-founded*** if, respectively, its game tree is finite, infinite, or well-founded. Recall that a tree is called ***well-founded*** if it has no infinite paths. *From now on by an* **extensive game** *we mean a well-founded extensive game with perfect information.*

For a node u in T we denote the subtree of T rooted at u by T^u. In the proofs we shall often rely on the concept of a *rank* of a well-founded tree T, defined inductively as follows, where v is the root of T:

$$rank(T) := \begin{cases} 0 & \text{if } T \text{ has one node} \\ sup\{rank(T^u)+1 \mid u \in C(v)\} & \text{otherwise,} \end{cases}$$

where $sup(X)$ denotes the least ordinal larger than all ordinals in the set X.

For an extensive game $G := (T, turn, p_1, \ldots, p_n)$ let $V_i := \{v \in V \setminus Z \mid turn(v) = i\}$. So V_i is the set of nodes at which player i moves. A ***strategy*** for player i is a function $s_i : V_i \to V$, such that $(v, s_i(v)) \in E$ for all $v \in V_i$. We denote the set of strategies of player i by S_i. Let $S = S_1 \times \cdots \times S_n$. As in the case of the strategic games we use the '$-i$' notation, when referring to sequences of strategies or sets of strategies.

Each joint strategy $s = (s_1, \ldots, s_n)$ determines a rooted path $play(s) := (v_1, \ldots, v_m)$ in T defined inductively as follows. v_1 is the root of T and if $v_k \notin Z$, then $v_{k+1} := s_i(v_k)$, where $turn(v_k) = i$. So when the game tree consists of just one node, v, we have $play(s) = v$. Informally, given a joint strategy s, we can view $play(s)$ as the resulting *play* of the game. For each joint strategy s the rooted path $play(s)$ is finite since the game tree is assumed to be well-founded. Denote by $leaf(s)$ the last element of $play(s)$. To simplify the notation we just write everywhere $p_i(s)$ instead of $p_i(leaf(s))$.

With each extensive game $G := (T, turn, p_1, \ldots, p_n)$ we associate a strategic game $\Gamma(G)$ defined as follows. $\Gamma(G) := (S_1, \ldots, S_n, p_1, \ldots, p_n)$, where each S_i is the set of strategies of player i in G. In the degenerate situation when the game tree consists of just one node, each strategy is the empty function, denoted by $\emptyset$, and there is only one joint strategy, namely the n-tuple $(\emptyset, \ldots, \emptyset)$ of these functions. In that case we just stipulate that $p_i(\emptyset, \ldots, \emptyset) = 0$ for all players i. All notions introduced in the context of strategic games can now be reused in the context of an extensive game G simply by referring to the corresponding strategic form $\Gamma(G)$. In particular, the notion of a Nash equilibrium is well-defined.

The ***subgame*** of an extensive game $G := (T, turn, p_1, \ldots, p_n)$, rooted at the node w and denoted by G^w, is defined as follows. The set of players is $\{1, \ldots, n\}$, the game tree is T^w. The *turn* and payoff functions are the restrictions of the corresponding functions of G to the nodes of T^w. We call G^w a ***direct subgame*** of G if w is a child of the root v.

Note that some players may 'drop out' in G^w, in the sense that at no node of T^w it is their turn to move. Still, to keep the notation simple, it is convenient to admit in G^w all original players in G.

Each strategy s_i of player i in G uniquely determines his strategy s_i^w in G^w. Given a joint strategy $s = (s_1, \ldots, s_n)$ of G we denote by s^w the joint strategy $(s_1^w, \ldots, s_n^w)$ in G^w. Further, we denote by S_i^w the set of strategies of player i in the subgame G^w and by S^w the set of joint strategies in this subgame.

Finally, a joint strategy s of G is called a ***subgame perfect equilibrium*** in G if for each node w of T, the joint strategy s^w of G^w is a Nash equilibrium in the subgame G^w. We denote by $SPE(G)$ the set of subgame perfect equilibria in G. Finally, we say that a game is ***SPE-invariant*** if it has a subgame perfect equilibrium and in each subgame of it all subgame perfect equilibria are payoff equivalent.

We shall often use the following result.

Theorem 5 ([3]) *Every extensive game with finitely many outcomes has a subgame perfect equilibrium.*

3 Preliminary lemmas

In this section we present a sequence of lemmas needed to prove our first main result. In the proofs we often switch between a game and its direct subgames.

Consider an extensive game $G := (T, turn, p_1, \ldots, p_n)$ with the root v and a child w of v. For each player j to each of his strategy t_j in a direct subgame G^w there corresponds a natural set $[t_j]$ of his strategies in the game G defined by $[t_j] := \{s_j \mid t_j = s_j^w \text{ and } s_j(v) = w \text{ if } j = turn(v)\}$. So for a player j, $[t_j]$ is the set of his strategies in G the restriction of which to G^w is t_j, with the additional proviso that if $j = turn(v)$, then each strategy in $[t_j]$ selects w at the root v. We call $[t_j]$ the *lifting* of t_j to the game G. The following lemma clarifies the relevance of lifting.

Lemma 6 *Consider a direct subgame G^w of G. Suppose that the strategy t_j is weakly dominated in G^w. Then each strategy in $[t_j]$ is weakly dominated in G.*

Proof. Suppose that t_j is weakly dominated in G^w by some strategy u_j. Take a strategy v_j in $[t_j]$. We show that v_j is weakly dominated in G by the strategy w_j in $[u_j]$ that coincides with v_j on all the nodes that do not belong to G^w. So w_j is obtained from v_j by replacing in it v_j^w, i.e., t_j, by u_j. Below s_{-j} denotes a sequence of strategies in G of the opponents of player j.

Case 1. $j = turn(v)$.

By the choice of u_j for all s_{-j} $p_j(t_j, s_{-j}^w) \leq p_j(u_j, s_{-j}^w)$ and for some s_{-j} $p_j(t_j, s_{-j}^w) < p_j(u_j, s_{-j}^w)$. Further, by the definition of $[\cdot]$ we have $v_j(v) = w$, so for all s_{-j} we have $p_j(v_j, s_{-j}) = p_j(t_j, s_{-j}^w)$ and $p_j(u_j, s_{-j}^w) = p_j(w_j, s_{-j})$, so the claim follows.

Case 2. $j \neq turn(v)$.

Let $i = turn(v)$. Take some s_{-j}. If $s_i(v) = w$, then $p_j(v_j, s_{-j}) = p_j(t_j, s_{-j}^w)$ and $p_j(w_j, s_{-j}) = p_j(u_j, s_{-j}^w)$. Thus $p_j(v_j, s_{-j}) \leq p_j(w_j, s_{-j})$ by the choice of u_j and w_j. Further, if $s_i(v) \neq w$, then $p_j(v_j, s_{-j}) = p_j(w_j, s_{-j})$ by the choice of w_j.

Choose an arbitrary s_{-j} such that $s_i(v) = w$ and $p_j(t_j, s_{-j}^w) < p_j(u_j, s_{-j}^w)$. By the choice of s_i we have $p_j(v_j, s_{-j}) = p_j(t_j, s_{-j}^w)$ and $p_j(w_j, s_{-j}) = p_j(u_j, s_{-j}^w)$, so $p_j(v_j, s_{-j}) < p_j(w_j, s_{-j})$. Thus the claim follows. $\square$

We now extend the notation $[\cdot]$ to sets of strategies and sequences of sets strategies. First, given a set of strategies A in a direct subgame G^w of G we define $[A] := \bigcup_{s_j \in A} [s_j]$. Next, given a sequence ρ of sets of strategies of players, each set taken from a direct subgame of G, we denote by $[\rho]$ the corresponding sequence of sets of strategies of players in G obtained by replacing each element A in ρ by $[A]$.

Given a set A of strategies of players in a direct subgame G^w we define the corresponding set of strategies in the game G by putting $\langle A \rangle = \{s_j \mid s_j^w \in A\}$. Thus for a set A of strategies in a direct subgame G^w, the set $\langle A \rangle$ differs from $[A]$ in that we do include in the former set strategies s_j for which $s_j(v) \neq w$. Given a set A of strategies of player j in the subgame G^w, we call $\langle A \rangle$ an *extension* of A to the game G.

Further, given a subgame H of $\Gamma(G^w)$, we define $\langle H\rangle$ as the subgame of $\Gamma(G)$ in which for each player j we have $\langle H\rangle_j = \langle H_j\rangle$.

In what follows we need a substantially strengthened version of Lemma 6 that relies on the following concept. Given an extensive game G with a root v, we say that a non-empty subgame J of $\Gamma(G)$ ***does not depend on*** a direct subgame G^w if for any strategy s_j from J any modification of it on the non-leaf nodes of G^w or on v if $turn(v) = j$ is also in J. Note that in particular $\Gamma(G)$ does not depend on any of its direct subgame and that for any non-empty subgame H of a direct subgame G^w of G the subgame $\langle H\rangle$ does not depend on any other direct subgame of G.

Lemma 7 *Consider a direct subgame G^w of G, subgames H and H' of $\Gamma(G^w)$ and a set A of strategies in H. Suppose that $H \to^A H'$ and that the subgame J of $\Gamma(G)$ does not depend on G^w. Then $J \cap \langle H\rangle \to^{[A]} J \cap \langle H'\rangle$.*

Proof. Take a strategy v_j in $[A]$. For some strategy t_j from A that is weakly dominated in H by some strategy u_j we have $v_j \in [t_j] \cap J_j$. Select a strategy w_j in $[u_j]$ that coincides with v_j on the nodes that do not belong to G^w. So w_j is a modification of v_j on the non-leaf nodes of G^w and consequently, by the assumption about J, it is in J_j. Further, w_j is in $\langle H\rangle$, since u_j is from H.

We claim that v_j is weakly dominated in $J \cap \langle H\rangle$ by w_j. Below s_{-j} denotes a sequence of strategies of the opponents of player j in the original game G.

Case 1. $j = turn(v)$.

By the choice of u_j for all s_{-j} such that $s^w_{-j} \in H_{-j}$ $p_j(t_j, s^w_{-j}) \leq p_j(u_j, s^w_{-j})$ and for some s_{-j} such that $s^w_{-j} \in H_{-j}$ $p_j(t_j, s^w_{-j}) < p_j(u_j, s^w_{-j})$. By the definition of 'does not depend on' and the fact that $j = turn(v)$ we can also assume that the latter s_{-j} is from J_{-j} by stipulating that $s_{-j} = t_{-j}$ for an arbitrary joint strategy t from J.

Further, by the definition of $[\cdot]$ we have $v_j(v) = w$, so for all s_{-j} such that $s^w_{-j} \in H_{-j}$ we have $p_j(v_j, s_{-j}) = p_j(t_j, s^w_{-j})$ and $p_j(u_j, s^w_{-j}) = p_j(w_j, s_{-j})$. Hence for all s_{-j} $p_j(v_j, s_{-j}) \leq p_j(w_j, s_{-j})$ and for some s_{-j} such that $s_{-j} \in J_{-j}$ and $s^w_{-j} \in H_{-j}$ (i.e., for some $s_{-j} \in (J \cap \langle H\rangle)_{-j}$) $p_j(v_j, s_{-j}) < p_j(w_j, s_{-j})$. This establishes the claim.

Case 2. $j \neq turn(v)$.

Let $i = turn(v)$. Take some s_{-j}. If $s_i(v) = w$, then $p_j(v_j, s_{-j}) = p_j(t_j, s^w_{-j})$ and $p_j(w_j, s_{-j}) = p_j(u_j, s^w_{-j})$. Thus $p_j(v_j, s_{-j}) \leq p_j(w_j, s_{-j})$ by the choice of u_j and w_j. Further, if $s_i(v) \neq w$, then $p_j(v_j, s_{-j}) = p_j(w_j, s_{-j})$ by the choice of w_j. So for all s_{-j} we have $p_j(v_j, s_{-j}) \leq p_j(w_j, s_{-j})$.

Choose an arbitrary s_{-j} such that $s_i(v) = w$, $s^w_{-j} \in H_{-j}$, and $p_j(t_j, s^w_{-j}) < p_j(u_j, s^w_{-j})$. Additionally, we can claim that $s_{-j} \in J_{-j}$ by stipulating that $s_{-j} = t_{-j}$ for an arbitrary joint strategy t from J. Then $s_{-j} \in (J \cap \langle H\rangle)_{-j}$.

By the choice of s_i we have $p_j(v_j, s_{-j}) = p_j(t_j, s^w_{-j})$ and $p_j(w_j, s_{-j}) = p_j(u_j, s^w_{-j})$, so $p_j(v_j, s_{-j}) < p_j(w_j, s_{-j})$. This establishes the claim for this case. □

We continue with some lemmas concerned with the relation $\to^\rho$.

Lemma 8 *Consider a direct subgame G^w of G. Suppose that for some sequence ρ of sets of strategies of players in G^w and a subgame H of $\Gamma(G^w)$, $\Gamma(G^w) \to^\rho H$. Suppose further that the subgame J of $\Gamma(G)$ does not depend on G^w. Then $J \to^{[\rho]} J \cap \langle H\rangle$.*

Proof. We proceed by transfinite induction on the length γ of $\rho = (\rho_\alpha, \alpha < \gamma)$.

Case 1. $\gamma = 1$.

By Lemma 7 $J \cap \langle\Gamma(G^w)\rangle \to^{[\rho_0]} J \cap \langle H\rangle$, so the claim holds since $\langle\Gamma(G^w)\rangle = \Gamma(G)$ and $J \cap \Gamma(G) = J$.

Case 2. γ is a successor ordinal > 1.

Suppose $\gamma = \delta + 1$. Then $\rho = (\rho', \rho_\delta)$, where $\rho' := (\rho_\alpha, \alpha < \delta)$. By definition for some H' we have $\Gamma(G^w) \to^{\rho'} H'$ and $H' \to^{\rho_\delta} H$. By the induction hypothesis $J \to^{[\rho']} J \cap \langle H' \rangle$ and by Lemma 7 $J \cap \langle H' \rangle \to^{[\rho_\delta]} J \cap \langle H \rangle$, so the claim follows by Note 4(i), since $[\rho] = ([\rho'], [\rho_\delta])$.

Case 3. γ is a limit ordinal.

By definition for some games H^β, where $\beta < \gamma$, we have $\Gamma(G^w) \to^{\rho^\beta} H^\beta$ and $H = \bigcap_{\beta<\gamma} H^\beta$, where—recall—$\rho^\beta = (\rho_\alpha, \alpha < \beta)$. By the induction hypothesis for all $\beta < \gamma$, we have $J \to^{[\rho^\beta]} J \cap \langle H^\beta \rangle$. So by definition $J \to^{[\rho]} J \cap \langle H \rangle$, since $J \cap \langle H \rangle = \bigcap_{\beta<\gamma} \langle J \cap H^\beta \rangle$ as $\langle H \rangle = \bigcap_{\beta<\gamma} \langle H^\beta \rangle$. □

Lemma 9 *Consider an extensive game G with the root v. Suppose that $(w_\alpha, \alpha < \gamma)$ is a sequence of children of v and that for all $\alpha < \gamma$, ρ_α is a sequence of sets of strategies in the direct subgame G^{w_α}. Suppose further that for each $\alpha < \gamma$ $\Gamma(G^{w_\alpha}) \to^{\rho_\alpha} H^{w_\alpha}$, where each game H^{w_α} is non-empty. Let ρ be the concatenation of the sequences $(\rho_\alpha, \alpha < \gamma)$. Then $\Gamma(G) \to^{[\rho]} \bigcap_{\alpha<\gamma} \langle H^{w_\alpha} \rangle$.*

By assumption each H^{w_α} is a non-empty subgame of $\Gamma(G^{w_\alpha})$, so each $\langle H^{w_\alpha} \rangle$ is a non-empty subgame of $\Gamma(G)$, and consequently $\bigcap_{\alpha<\gamma} \langle H^{w_\alpha} \rangle$ is also a non-empty subgame of $\Gamma(G)$.

Informally, suppose that for each direct subgame G^{w_α} of G we can reduce the corresponding strategic game $\Gamma(G^{w_\alpha})$ to a non-empty game H^{w_α}. Then the strategic game $\Gamma(G)$ can be reduced to a strategic game the strategies of which are obtained by intersecting for each player the extensions of his strategy sets in all games H^{w_α}. To establish this lemma we do not assume that $(w_\alpha, \alpha < \gamma)$ contains all children of v, which makes it possible to proceed by induction.

Proof. We proceed by transfinite induction on the length γ of ρ.

Case 1. $\gamma = 1$. Follows from Lemma 8 with $J = \Gamma(G)$.

Case 2. γ is a successor ordinal > 1.

Suppose $\gamma = \delta + 1$. By the induction hypothesis $\Gamma(G) \to^{[\rho^\delta]} \bigcap_{\alpha<\delta} \langle H^{w_\alpha} \rangle$, where ρ^δ is the concatenation of the sequences $(\rho_\alpha, \alpha < \delta)$. We also have by assumption $\Gamma(G^{w_\delta}) \to^{\rho_\delta} H^{w_\delta}$.

Note that the subgame $\bigcap_{\alpha<\delta} \langle H^{w_\alpha} \rangle$ of $\Gamma(G)$ does not depend on G^{w_δ}, so by Lemma 8 we have that $\bigcap_{\alpha<\delta} \langle H^{w_\alpha} \rangle \to^{[\rho_\delta]} \bigcap_{\alpha<\delta} \langle H^{w_\alpha} \rangle \cap \langle H^{w_\delta} \rangle$. By Note 4($i$) the claim follows.

Case 3. γ is a limit ordinal.

By the induction hypothesis for all $\beta < \gamma$ $\Gamma(G) \to^{[\rho^\beta]} \bigcap_{\alpha<\beta} \langle H^{w_\alpha} \rangle$, where ρ^β is the concatenation of the sequences $(\rho_\alpha, \alpha < \beta)$. Then by Note 4($ii$) and by definition $\Gamma(G) \to^{[\rho]} \bigcap_{\beta<\gamma} \bigcap_{\alpha<\beta} \langle H^{w_\alpha} \rangle$. But $\bigcap_{\beta<\gamma} \bigcap_{\alpha<\beta} \langle H^{w_\alpha} \rangle = \bigcap_{\alpha<\gamma} \langle H^{w_\alpha} \rangle$, so the claim follows. □

The next lemma shows that when each subgame H^{w_α} of $\Gamma(G^{w_\alpha})$ is trivial, under some natural assumptions the subgame $\bigcap_{\alpha<\gamma} \langle H^{w_\alpha} \rangle$ of $\Gamma(G)$ can then be reduced in one step to a trivial game.

Lemma 10 *Consider an extensive game G with the root v. Suppose that*

(a) G has a subgame perfect equilibrium and all subgame perfect equilibria of G are payoff equivalent,

(b) for all $w \in C(v)$, $SPE(G^w) \subseteq H^w$, where H^w is a trivial subgame of $\Gamma(G^w)$.

Then for some set of strategies A we have $\bigcap_{w \in C(v)} \langle H^w \rangle \to^A H'$, where H' a trivial game and $SPE(G) \subseteq H'$.

Proof. Let $H := \bigcap_{w \in C(v)} \langle H^w \rangle$. Note that H is a non-empty subgame of $\Gamma(G)$.

Denote the unique outcome in the game H^w by val^w, i.e., for all joint strategies s in H^w we have $p(s) = val^w$. Then the possible outcomes in H are val^w, where $w \in C(v)$. More precisely, suppose that $i = turn(v)$. Then if s is a joint strategy in H, then $p(s) = val^w$, where $s_i(v) = w$.

Take two strategies t'_i and t''_i of player i in H with $t'_i(v) = w_1$ and $t''_i(v) = w_2$ such that $val_i^{w_1} < val_i^{w_2}$. This means that for any joint strategies s_{-i} from H_{-i} we have $p_i(t'_i, s_{-i} < p_i(t''_i, s_{-i}$, so t'_i is weakly dominated in H by t''_i (actually, even strictly dominated).

By assumption *(a)* G has a subgame perfect equilibrium, so by Corollary 7 of [3] $\max\{val_i^w \mid w \in C(v)\}$ exists. Denote it by val_i and let $W := \{w \in C(v) \mid val_i^w = val_i\}$. So W is the set of children w of v for which the corresponding value val_i^w is maximal. Finally, let A be the set of strategies t_i of player i in H such that $t_i(v) \notin W$.

By the above observation about t'_i and t''_i all strategies in A are weakly dominated in H. By removing them from H we get a game H' with the unique payoff val_i for player i. To prove that H' is trivial consider two joint strategies s and t in H'. Suppose that $s_i(v) = w_1$ and $t_i(v) = w_2$. Then $w_1, w_2 \in W$, $s^{w_1} \in H^{w_1}$, $t^{w_2} \in H^{w_2}$, $p(s) = p(s^{w_1})$, and $p(t) = p(t^{w_2})$.

By Theorem 8 of [3] subgame perfect equilibria u' and u'' in G exist such that $u'_i(v) = w_1$, $(u')^{w_1}$ is a subgame perfect equilibrium in G^{w_1}, $u''_i(v) = w_2$, and $(u'')^{w_2}$ is a subgame perfect equilibrium in G^{w_2}. Then $p(u') = p((u')^{w_1})$ and $p(u'') = p((u'')^{w_2})$, so $p((u')^{w_1}) = p((u'')^{w_2})$ by assumption *(a)*. Further, by assumption *(b)* both $(u')^{w_1} \in H^{w_1}$ and $(u'')^{w_2} \in H^{w_2}$, so since both subgames are trivial, $p(s^{w_1}) = p((u')^{w_1})$ and $p(t^{w_2}) = p((u')^{w_2})$. Consequently $p(s) = p(t)$, which proves that H' is trivial.

To prove that $SPE(G) \subseteq H'$ consider a subgame perfect equilibrium s in G. Take some $u \in C(v)$. By assumption *(b)*, $s^u \in H^u$, so $p_i(s^u) = val_i^u$ and, by the definition of $\langle \cdot \rangle$, $s \in H$. Suppose that $s_i(v) = w$. By Corollary 7 of [3] $val_i^w = val_i$, i.e., $s_i(v) \in W$. This means that $s_i \notin A$ and thus $s \in H'$. □

4 SPE-invariant games

We can now prove the desired result.

Theorem 11 *Consider an SPE-invariant extensive game G. There exists a sequence ρ of strategies of players in G and a subgame H of $\Gamma(G)$ such that $\Gamma(G) \to^{\rho} H$, H is trivial and $SPE(G) \subseteq H$.*

Proof. We proceed by induction on the rank of the game tree of G. For game trees of rank 0 all strategies are empty functions, so $\Gamma(G)$ is a trivial game with the unique joint strategy $(\emptyset, \ldots, \emptyset)$ and $SPE(G) = \{(\emptyset, \ldots, \emptyset)\}$, so the claim holds. Suppose that the rank of the game tree of G is $\alpha > 0$ and assume that claim holds for all extensive games with the game trees of rank smaller than α.

Let v be the root of G. Each direct subgame of G is SPE-invariant, so by the induction hypothesis for all $w \in C(v)$ there exists a sequence ρ^w of strategies of players in G^w and a subgame H^w of $\Gamma(G^w)$ such that $\Gamma(G^w) \to^{\rho^w} H^w$, H^w is trivial and $SPE(G^w) \subseteq H^w$. The claim now follows by Lemmas 9 and 10. □

The following example illustrates the use of this theorem. An extensive game is called ***generic*** if each payoff function is an injective.

Example 12 Recall that the centipede game, introduced in [15] (see also [11, pages 106-108]), is a two-players extensive game played for an even number of periods. We define it inductively as follows. The game with 2 periods is depicted in Figure 4. Here and below the argument of each non-leaf is the player whose turn is to move, and the leaves are followed by players' payoffs. The moves are denoted by

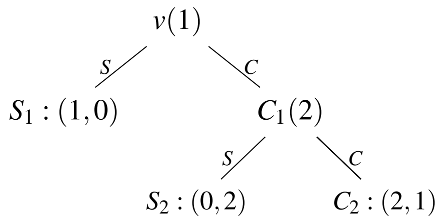


Figure 4: Centipede game with 2 periods

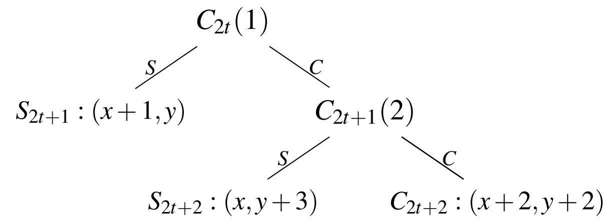


Figure 5: From t to $t+2$ periods

the letters C and S. The game with $2t+2$ periods is obtained from the game with $2t$ periods by replacing the leaf C_{2t} by the tree depicted in Figure 5.

By the the result of [11, pages 108-109]) each centipede game can be reduced by an iterated elimination of weakly dominated strategies to a trivial game which contains the unique subgame perfect equilibrium, with the outcome $(1,0)$. We now show that the same holds for an infinite version of the centipede game G in which player 2 begins the game by selecting an even number $2t > 0$. Subsequently, the centipede version with $2t$ periods is played.

Note that G is SPE-invariant. Indeed, G has infinitely many subgame perfect equilibria (one for each first move of player 2), but each of them yields the outcome $(1,0)$. Moreover, each subgame of G is either a centipede game with $2t$ periods for some $t > 0$, or a subgame of such a game. So each subgame of G is a finite generic game and thus has a unique subgame perfect equilibrium.

By Theorem 11 we can reduce G by an infinite iterated elimination of weakly dominated strategies to a trivial game which contains all its subgame perfect equilibria. Note that the strategy elimination sequence constructed in the proof of this theorem consists of for more than ω steps. $\square$

For finite extensive games, Theorem 11 extends the original result reported in [11, pages 108-109]. Namely, the authors prove the corresponding result for finite extensive games that are generic. In such games a unique subgame perfect equilibrium exists, while we only claim that the game is SPE-invariant.

To clarify the relevance of this relaxation let us mention two classes of well-founded extensive games that are SPE-invariant and that were studied for finite extensive games. Following [4] we say that an extensive game $(T, turn, p_1, \ldots, p_n)$ is ***without relevant ties*** if for all non-leaf nodes u in T the payoff function p_i, where $turn(u) = i$, is injective on the leaves of T^u. This is a more general property than being generic. The relevant property for finite extensive games is that a game without relevant ties has a unique subgame perfect equilibrium, see [2] for a straightforward proof. In the case of well-founded games a direct modification of this proof, that we omit, shows that every extensive game without relevant ties has at most one subgame perfect equilibrium. Further, if a game is without relevant ties, then so is every subgame of it, so we conclude that well-founded games without relevant ties are SPE-invariant.

Next, following [10] we say that an extensive game $(T, turn, p_1, \ldots, p_n)$ satisfies the ***transference of decisionmaker indifference (TDI)*** condition if:

$$\begin{aligned}&\forall i \in \{1, \ldots, n\}\, \forall r_i, t_i \in S_i\, \forall s_{-i} \in S_{-i}\\ &[p_i(leaf(r_i, s_{-i})) = p_i(leaf(t_i, s_{-i})) \rightarrow p(leaf(r_i, s_{-i})) = p(leaf(t_i, s_{-i}))].\end{aligned}$$

where S_i is the set of strategies of player i. Informally, this condition states that whenever for some player i, two of his strategies r_i and t_i are indifferent w.r.t. some joint strategy s_{-i} of the other players then this indifference extends to all players.

Strategic games that satisfy the TDI condition are of interest because of the main result of [10] which states that in finite games that satisfy this condition iterated elimination of weakly dominated strategies is order independent.[4] The authors also give examples of natural games that satisfy this condition. Also strictly competitive games studied in the next section satisfy this condition.

The following result extends an implicit result of [10] to well-founded games.

Theorem 13 *Consider an extensive game G. Suppose that G has finitely many outcomes and G satisfies the TDI condition. Then G is SPE-invariant.*

Proof. We reduce the game G to a finite game H as follows. First, consider the set of all leaves of the game tree T of G that are the ends of the plays corresponding with a subgame perfect equilibrium. Next, for each outcome associated with a subgame perfect equilibrium retain in this set just one leaf with this outcome. By assumption the resulting set L is finite.

Next, order the leaves arbitrarily. Following this ordering remove all leaves with an outcome already associated with an earlier leaf, but ensuring that the leaves from L are retained. Let M be the resulting set of leaves. Finally, remove all nodes of T from which no leaf in M can be reached.

The resulting tree corresponds to a finite extensive game H in which all the outcomes possible in G are present. Further, all the leaves of H are also leaves of G, so H satisfies the TDI condition since G does. So by Theorem 12 of [2] (that is implicit in [10]) all subgame perfect equilibria of H are payoff equivalent.

Further, by Theorem 5 G has a subgame perfect equilibrium. Consider two subgame perfect equilibria s and t in G with the outcomes $p(s)$ and $p(t)$. By construction two subgame perfect equilibria s' and t' in H exist such that $p(s) = p(s')$ and $p(t) = p(t')$. We conclude that all subgame perfect equilibria of G are payoff equivalent.

To complete the proof it suffice to note that if an extensive game G satisfies the TDI condition, then so does every subgame of it. Indeed, consider a subgame G^w of G. Let $i = turn(w)$ and take $r_i^w, t_i^w \in S_i^w$ and $s_{-i}^w \in S_{-i}^w$. Extend these strategies to the strategies r_i, t_i and s_{-i} in the game G in such a way that w lies both on $play(r_i, s_{-i})$ and on $play(t_i, s_{-i})$. Then $p(r_i^w, s_{-i}^w) = p(r_i, s_{-i})$ and $p(t_i^w, s_{-i}^w) = p(t_i, s_{-i})$, so the claim follows. □

Corollary 14 *The claim of Theorem 11 holds for extensive games with finitely many outcomes that satisfy the TDI condition.*

Conjecture Every extensive game that satisfies the TDI-condition is SPE-invariant.

If the conjecture is true, Theorem 11 holds for all extensive games that satisfy the TDI condition. An example of a game with infinitely many outcomes that satisfies the TDI condition is the infinite version of the centipede game from Example 12.

5 Strictly competitive extensive games

In some games, for instance, the infinite version of the centipede game from Example 12, infinite rounds of elimination of weakly dominated strategies are needed to solve the game. In this section, we focus on maximal elimination of weakly dominated strategies and identify a subclass of extensive games for which we can provide a finite bound on the number of elimination steps required to solve the game. The outcome is our second main result which is a generalization of the following result due to [6] to a class of well-founded games.

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[4] Alternative proofs of this result were given in [1] and [17].

Theorem Every finite extensive zero-sum game with n outcomes can be reduced to a trivial game by the maximal iterated elimination of weakly dominated strategies in $n-1$ steps.

We first present some auxiliary results. Their proofs follow our detailed exposition in [2] of the proofs in [6] generalized to strictly competitive games, now appropriately modified to infinite games.

5.1 Preliminary results

We denote by H^1 the subgame of H obtained by the elimination of all strategies that are weakly dominated in H, and put $H^0 := H$ and $H^{k+1} := (H^k)^1$, where $k \geq 1$. Abbreviate the phrase 'iterated elimination of weakly dominated strategies' to IEWDS. If for some k, H^k is a trivial game we say that H ***can be solved by the IEWDS***.

In infinite strategic games with finitely many outcomes it is possible that all strategies of a player are weakly dominated as shown in the Example 15. Then by definition, H^1 is an empty game. We define a class of games, called WD-admissible games in which this does not happen.

Example 15 Consider the following infinite zero-sum strategic game with two outcomes:

	A	B	C	D	$\ldots$
A	$0,0$	$0,0$	$0,0$	$0,0$	$\ldots$
B	$0,0$	$1,-1$	$0,0$	$0,0$	$\ldots$
C	$0,0$	$1,-1$	$1,-1$	$0,0$	$\ldots$
D	$0,0$	$1,-1$	$1,-1$	$1,-1$	$\ldots$
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$

This game has a Nash equilibrium, namely (A,A), but each strategy of the row player is weakly dominated. So after one round of elimination the empty game is reached. □

Consider a strategic game H. We say that a strategy is ***undominated*** if no strategy weakly dominates it. Next, we say that H is ***WD-admissible*** if for all subgames H' of it the following holds: *each strategy is undominated or is weakly dominated by an undominated strategy*. Intuitively, a strategic game H is WD-admissible if in every subgame H' of it, for every strategy s_i in H' the relation 'is weakly dominated' in H' has a maximal element above s_i. The crucial property of WD-admissible games is formalised in the following lemma whose proof follows directly by induction.

Lemma 16 *Let* $H := (H_1,\ldots,H_n,p_1,\ldots,p_n)$ *be a WD-admissible strategic game and for* $k \geq 1$*, let* $H^k := (H_1^k,\ldots,H_n^k,p_1,\ldots,p_n)$*. Then* $\forall i \in \{1,\ldots,n\}\ \forall s_i \in H_i\ \exists t_i \in H_i^k\ \forall s_{-i} \in H_{-i}^k : p_i(t_i,s_{-i}) \geq p_i(s_i,s_{-i})$.

A two player strategic game $H = (H_1,H_2,p_1,p_2)$ is called ***strictly competitive*** if $\forall i \in \{1,2\}\ \forall s,s' \in S$: $p_i(s) \geq p_i(s')$ iff $p_{-i}(s) \leq p_{-i}(s')$. For $i \in \{1,2\}$ we define $maxmin_i(H) := \max_{s_i \in H_i} \min_{s_{-i} \in H_{-i}} p_i(s_i,s_{-i})$. We allow $-\infty$ and ∞ as minima and maxima, so $maxmin_i(H)$ always exists. When $maxmin_i(H)$ is finite we call any strategy s_i^* such that $\min_{s_{-i} \in H_{-i}} p_i(s_i^*,s_{-i}) = maxmin_i(H)$ a ***security strategy*** for player i in H.

We shall reuse the following auxiliary results from [2].

Note 17 *Let* $H = (H_1,H_2,p_1,p_2)$ *be a strictly competitive strategic game. Then*

$$\forall i \in \{1,2\}\ \forall s,s' \in S : p_i(s) = p_i(s') \textit{ iff } p_{-i}(s) = p_{-i}(s').$$

This simply means that every strictly competitive strategic game satisfies the TDI condition.

Lemma 18 *Consider a strictly competitive strategic game H with a Nash equilibrium s. Suppose that for some $i \in \{1,2\}$, t_i weakly dominates s_i. Then (t_i,s_{-i}) is also a Nash equilibrium.*

Lemma 19 *Consider a strictly competitive strategic game H with two outcomes that has a Nash equilibrium. Then H^1 is a trivial game.*

The following result is standard (for the used formulation see, e.g., [14, Theorem 5.11, page 235]).

Theorem 20 *Consider a strictly competitive strategic game H.*

(i) All Nash equilibria of H yield the same payoff for player i, namely $maxmin_i(H)$.

(ii) All Nash equilibria of H are of the form (s_1^*, s_2^*) where each s_i^* is a security strategy for player i.

By modifying the proof of Corollary 5 from [2] appropriately, we have the following.

Lemma 21 *Consider a WD-admissible strictly competitive strategic game H that has a Nash equilibrium. Then H^1 has a Nash equilibrium, as well, and for all $i \in \{1,2\}$, $maxmin_i(H) = maxmin_i(H^1)$.*

5.2 A bound on IEWDS

We now move on to a discussion of extensive games. We say that an extensive game G is ***WD-admissible*** (respectively, ***strictly competitive***) if $\Gamma(G)$ is WD-admissible (respectively, strictly competitive). We write $\Gamma^k(G)$ instead of $(\Gamma(G))^k$, $\Gamma_i(G)$ instead of $(\Gamma(G))_i$, and $\Gamma_i^k(G)$ instead of $(\Gamma^k(G))_i$. So $\Gamma^0(G) = \Gamma(G)$. Further, for a strictly competitive game $H = (H_1, H_2, p_1, p_2)$ with finitely many outcomes for each player i we define the following three sets: $p_i^{max}(H) := \max_{s \in S} p_i(s)$, $win_i(H) := \{s_i \in H_i \mid \forall s_{-i} \in H_{-i}\, p_i(s_i, s_{-i}) = p_i^{max}(H)\}$ and $lose_{-i}(H) = \{s_{-i} \in H_{-i} \mid \exists s_i \in H_i\, p_i(s_i, s_{-i}) = p_i^{max}(H)\}$. By the assumption about H, $p_i^{max}(H)$ is finite.

We can then prove the following generalization of the crucial Lemma 1 and Theorem 1 from [6], where the proofs are analogous to that of Lemma 18 and Theorem 19 in [2].

Lemma 22 *Let G be a WD-admissible strictly competitive extensive game with finitely many outcomes. For all $i \in \{1,2\}$ and for all $k \geq 0$, if $win_i(\Gamma^k(G)) = \emptyset$ then $lose_{-i}(\Gamma^k(G)) \cap \Gamma_{-i}^{k+2}(G) = \emptyset$.*

Lemma 22 implies that if for all $i \in \{1,2\}$, $win_i(\Gamma^k(G)) = \emptyset$ then two further rounds of eliminations of weakly dominated strategies remove from $\Gamma^k(G)$ at least two outcomes.

This allows us to establish the following result. The proof is almost the same as the one given in [2, Theorem 19] for the finite extensive games. We reproduce it here for the convenience of the reader.

Theorem 23 *Let G be a WD-admissible strictly competitive extensive game with at most m outcomes. Then $\Gamma^{m-1}(G)$ is a trivial game.*

Proof. We prove a stronger claim, namely that for all $m \geq 1$ and $k \geq 0$ if $\Gamma^k(G)$ has at most m outcomes, then $\Gamma^{k+m-1}(G)$ is a trivial game.

We proceed by induction on m. For $m = 1$ the claim is trivial. For $m = 2$ we first note that by Theorem 5 and Lemma 21 each game $\Gamma^k(G)$ has a Nash equilibrium. So the claim follows by Lemma 19. For $m > 2$ two cases arise.

Case 1. For some $i \in \{1,2\}$, $win_i(\Gamma^k(G)) \neq \emptyset$.

For player i every strategy $s_i \in win_i(\Gamma^k(G))$ weakly dominates all strategies $s_i' \notin win_i(\Gamma^k(G))$ and no strategy in $win_i(\Gamma^k(G))$ is weakly dominated. So the set of strategies of player i in $\Gamma^{k+1}(G)$ equals $win_i(\Gamma^k(G))$ and consequently $p_i^{max}(\Gamma^k(G))$ is his unique payoff in this game. By Note 17 $\Gamma^{k+1}(G)$, and hence also $\Gamma^{k+m-1}(G)$, is a trivial game.

Case 2. For all $i \in \{1,2\}$, $win_i(\Gamma^k(G)) = \emptyset$.

Take joint strategies s and t such that $p_1(s) = p_1^{\max}(\Gamma^k(G))$ and $p_2(t) = p_2^{\max}(\Gamma^k(G))$. By Note 17 the outcomes $(p_1(s), p_2(s))$ and $(p_1(t), p_2(t))$ are different since $m > 1$.

We have $s_2 \in lose_2(\Gamma^k(G))$ and $t_1 \in lose_1(\Gamma^k(G))$. Hence by Lemma 22 for no joint strategy s' in $\Gamma^{k+2}(G)$ we have $p_1(s') = p_1^{\max}(\Gamma^k(G))$ or $p_2(s') = p_2^{\max}(\Gamma^k(G))$.

So $\Gamma(G^{k+2})$ has at most $m-2$ outcomes. By the induction hypothesis $\Gamma(G^{k+m-1})$ is a trivial game. □

We now show that Theorem 23 holds for a large class of natural games. Call an extensive game ***almost constant*** if for all but finitely many leaves the outcome is the same. Note that every almost constant game has finitely many outcomes, but the converse does not hold. Indeed, it suffices to take a game with two outcomes, each associated with infinitely many leaves. The following general result holds.

Theorem 24 *Every almost constant extensive game is WD-admissible.*

Proof. We begin with two unrelated observations. Call a function $p : A \to B$ ***almost constant*** if for some b we have $p(a) = b$ for all but finitely many $a \in A$.

Observation 1. Consider two sequences of some elements $(v_0, v_1, \ldots)$ and $(w_0, w_1, \ldots)$ such that $v_j \neq v_k$, $v_j \neq w_k$, and $w_j \neq w_k$ for all $j \geq 0$ and $k > j$, and a function $p : \{v_0, v_1, \ldots\} \cup \{w_0, w_1, \ldots\} \to B$ such that $p(v_j) \neq p(w_j)$ for all $j \geq 0$. Then p is not almost constant.
Indeed, otherwise for some $k \geq 0$ the function $p : \{v_k, v_{k+1}, \ldots\} \cup \{w_k, w_{k+1}, \ldots\} \to B$ would be constant.

Observation 2. Take an extensive game. For some player i, consider two joint strategies (s_i, s_{-i}) and (s'_i, s'_{-i}). If $leaf(s_i, s_{-i}) = leaf(s'_i, s'_{-i})$ then $leaf(s_i, s_{-i}) = leaf(s'_i, s_{-i})$.

Indeed, consider any node w in $play(s_i, s_{-i})$ such that $turn(w) = i$. Then by assumption $s_i(w) = s'_i(w)$. This implies that $play(s_i, s_{-i}) = play(s'_i, s_{-i})$, which yields the claim.

Now consider an almost constant extensive game G. Take an arbitrary subgame H of $\Gamma(G)$. Suppose by contradiction that for some player i there exists an infinite sequence of strategies $s_i^0, s_i^1, s_i^2, \ldots$ such that for all $j \geq 0$, s_i^{j+1} weakly dominates s_i^j in H. By definition of weak dominance, for all $j \geq 0$ there exists $s_{-i}^j \in H_{-i}$ such that $p_i(s_i^j, s_{-i}^j) < p_i(s_i^{j+1}, s_{-i}^j)$. Let for $j \geq 0$, $v_j = leaf(s_i^j, s_{-i}^j)$ and $w_j = leaf(s_i^{j+1}, s_{-i}^j)$. By the above inequalities $p_i(v_j) \neq p_i(w_j)$ for all $j \geq 0$.

We now argue that $v_j \neq v_k$, $v_j \neq w_k$, and $w_j \neq w_k$ for all $j \geq 0$ and $k > j$. First, note that by the transitivity of the 'weakly dominates' relation we have the following.

- $p_i(s_i^j, s_{-i}^j) < p_i(s_i^{j+1}, s_{-i}^j) \leq p_i(s_i^k, s_{-i}^j)$,
- $p_i(s_i^j, s_{-i}^j) < p_i(s_i^{j+1}, s_{-i}^j) \leq p_i(s_i^{k+1}, s_{-i}^j)$,
- $p_i(s_i^{j+1}, s_{-i}^k) \leq p_i(s_i^k, s_{-i}^k) < p_i(s_i^{k+1}, s_{-i}^k)$.

This implies in turn, $leaf(s_i^j, s_{-i}^j) \neq leaf(s_i^k, s_{-i}^j)$, $leaf(s_i^j, s_{-i}^j) \neq leaf(s_i^{k+1}, s_{-i}^j)$, and $leaf(s_i^{j+1}, s_{-i}^k) \neq leaf(s_i^{k+1}, s_{-i}^k)$. So by Observation 2 we have the following.

- $v_j = leaf(s_i^j, s_{-i}^j) \neq leaf(s_i^k, s_{-i}^k) = v_k$,
- $v_j = leaf(s_i^j, s_{-i}^j) \neq leaf(s_i^{k+1}, s_{-i}^k) = w_k$,
- $w_j = leaf(s_i^{j+1}, s_{-i}^j) \neq leaf(s_i^{k+1}, s_{-i}^k) = w_k$.

By Observation 1, p_i is not almost constant, which contradicts the assumption that G is almost constant. By the transitivity of the 'weakly dominates' relation we conclude that G is WD-admissible. □

Corollary 25 *Let G be an almost constant strictly competitive extensive game with at most m outcomes. Then $\Gamma^{m-1}(G)$ is a trivial game.*

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