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Random restrictions of high-rank tensors and polynomial maps

(Extended abstract)

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Abstract

Motivated by a problem in computational complexity, we consider the behavior of rank functions for tensors and polynomial maps under random coordinate restrictions. We show that, for a broad class of rank functions called *natural rank functions*, random coordinate restriction to a dense set will typically reduce the rank by at most a constant factor.

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1 Introduction

Different but equivalent definitions of matrix rank have been generalized to truly different rank functions for tensors. Although they have proved useful in a variety of applications, the basic theory of these rank functions, describing for instance their interrelations and elementary properties, is still far from complete. Without going into the definitions, we mention below a number of these rank functions to indicate some of the contexts in which they have appeared.

The *slice rank* of a tensor was introduced by Tao [15, 16] to reformulate the breakthrough proof of the cap set conjecture due to Croot, Lev and Pach [3] and Ellenberg and Gijswijt [4]. Slice rank is generalized by the *partition rank*, which was introduced by

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Naslund to prove bounds on the size of subsets of $\mathbb{F}_q^n$ without k-right corners [12], as well as provide exponential improvements on the Erdős–Ginzburg–Ziv constant [11]. The *analytic rank* is based on a measure of equidistribution for multilinear forms associated to tensors over finite fields, and was introduced by Gowers and Wolf to study solutions to linear systems of equations in large subsets of finite vectors spaces [6]. *Geometric rank*, defined and studied by Kopparty, Moshkovitz and Zuiddam in the context of algebraic complexity theory [10], gives a natural analogue of analytic rank for tensors over infinite fields.

Closely related to these rank functions for tensors are notions of rank for multivariate polynomials. A notion of polynomial rank akin to the partition rank of tensors was used already in the '80s by Schmidt in work on algebraic geometry [13], and has since been re-discovered and proven useful on several occasions. Work on the Inverse Theorem for the Gowers uniformity norms led Green and Tao to define the notion of *degree rank* [7], which quantifies how hard it is to express the considered polynomial as a function of lower-degree polynomials; this notion was shown to be closely linked to equidistribution properties of multivariate polynomials over prime fields $\mathbb{F}_p$. Tao and Ziegler [17] later studied the relationship between the degree rank of a polynomial and its *analytic rank*, defined as the (tensor) analytic rank of its associated homogeneous multilinear form, and exploited their close connection in order to prove the general case of the Gowers Inverse Theorem over $\mathbb{F}_p^n$.

Recent work on constant-depth Boolean circuits by Buhrman, Neumann and the present authors gave rise to a problem on equidistribution properties of higher-dimensional polynomial maps under biased input distributions [1]. This motivated a new notion of analytic rank for (high-dimensional) polynomial maps and prompted the study of rank under random coordinate restrictions, which is the topic of this work.

Common to the tensors, polynomials and polynomial maps considered here is that they can be viewed as maps on $\mathbb{F}^X$, where $\mathbb{F}$ is a given field and X is a finite set indexing the variables. The main question we address is whether, if a map ϕ on $\mathbb{F}^X$ has high rank, then most of its coordinate restrictions $\phi_{|I}$ on $\mathbb{F}^I$ also have high rank for dense subsets $I \subseteq X$ (where we also respect the product structure of X in the case of tensors). Our main results show that this is the case for all "natural" rank functions, which include all those mentioned above.

2 The matrix case

It is instructive to first consider the case of matrices, which is simpler and illustrates the spirit of our main results. For a matrix $A \in \mathbb{F}^{n\times n}$ and subsets $I, J \subseteq [n]$, denote by $A_{|I\times J}$ the sub-matrix of A induced by the rows in I and columns in J. Given $\sigma \in (0,1)$, consider a random set $I \subseteq [n]$ containing each element independently with probability σ; we write $I \sim [n]_\sigma$ when I is distributed as such. Note that, if $I \sim [n]_\rho$ and $J \sim [n]_\sigma$ are independent, then $I \cup J \sim [n]_\eta$ with $\eta = 1 - (1-\rho)(1-\sigma)$.

Proposition 2.1. *For every $\sigma \in (0,1]$ there exists $\kappa \in (0,1]$ such that for every matrix*

$A \in \mathbb{F}^{n \times n}$ *we have*

$$\Pr_{I \sim [n]_\sigma}\big[\operatorname{rk}(A_{|I\times I}) \geq \kappa \cdot \operatorname{rk}(A)\big] \geq 1 - 2e^{-\kappa \operatorname{rk}(A)}.$$

Proof: Write $\rho = 1 - \sqrt{1-\sigma}$ and let $J, J' \sim [n]_\rho$ be independent random sets; note that $J \cup J' \sim [n]_\sigma$. Let $r = \operatorname{rk}(A)$, and fix a set $S \subseteq [n]$ of r linearly independent rows of A. By the Chernoff bound [8], the probability that the set J satisfies $|J \cap S| < \rho r/2$ is at most $e^{-\rho r/8}$.

Now let $B := A_{|(J\cap S)\times[n]}$ be the (random) sub-matrix of A formed by the rows in $J \cap S$. Since its rows are linearly independent, the rank of B is precisely $|J \cap S|$; let $T \subseteq [n]$ be a set of $|J \cap S|$ linearly independent columns of B. Then the probability that $|J' \cap T| < \rho|T|/2$ is at most $e^{-\rho|T|/8}$, and the rank of $B_{|(J\cap S)\times(J'\cap T)} = A_{|(J\cap S)\times(J'\cap T)}$ is equal to $|J' \cap T|$. It follows from the union bound and monotonicity of rank under restrictions that, with probability at least $1 - 2e^{-\rho^2 r/16}$, the principal sub-matrix of A induced by $J \cup J'$ has rank at least $\rho^2 r/4$. The result now follows since $J \cup J' \sim [n]_\sigma$. □

3 Main results

Here we generalize Proposition 2.1 to tensors and polynomial maps for rank functions that satisfy a few natural properties, namely "sub-additivity", "monotonicity", a "Lipschitz condition" and, in the case of polynomial maps, "symmetry" (see below for the precise definitions). Those functions which satisfy these properties are called *natural rank functions*; we note that all notions of rank mentioned in the Introduction are natural rank functions.

Since our results are independent of the field considered (which can be finite or infinite), we will always denote it by $\mathbb{F}$ and suppress statements of the form "let $\mathbb{F}$ be a field" or "for every field $\mathbb{F}$".

3.1 Tensors

We begin by considering the case of tensors.

Definition 3.1. For finite sets $X_1, \ldots, X_d \subset \mathbb{N}$, a d-tensor is a map $T : X_1 \times \cdots \times X_d \to \mathbb{F}$. We will associate with any d-tensor T a multilinear map $\mathbb{F}^{X_1} \times \cdots \times \mathbb{F}^{X_d} \to \mathbb{F}$ and an element of $\mathbb{F}^{X_1} \otimes \cdots \otimes \mathbb{F}^{X_d}$ in the obvious way, and also denote these objects by T.

For a tensor T as in Definition 3.1 and subsets $I_1 \subseteq X_1, \ldots, I_d \subseteq X_d$, denote $I_{[d]} = I_1 \times \cdots \times I_d$ and write $T_{|I_{[d]}}$ for the restriction of T to $I_{[d]}$. If T is viewed as an element of $\mathbb{F}^{X_1} \otimes \cdots \otimes \mathbb{F}^{X_d}$, then $T_{|I_{[d]}}$ is simply a sub-tensor.

We denote the set of d-tensors over $\mathbb{F}$ with finite support by $(\mathbb{F}^\infty)^{\otimes d}$; note that the tensors defined on finite sets naturally embed into this set, and that the rank functions for tensors discussed above are invariant under this embedding. The notions of tensor rank we will consider here are those called *natural rank functions* as defined below:

Definition 3.2. We say that $\mathrm{rk} : (\mathbb{F}^\infty)^{\otimes d} \to \mathbb{R}_+$ is a natural rank function if it satisfies the following properties:

1. Sub-additivity:
   $\mathrm{rk}(T + S) \le \mathrm{rk}(T) + \mathrm{rk}(S)$ for all $T, S \in (\mathbb{F}^\infty)^{\otimes d}$.

2. Monotonicity under restrictions:
   $\mathrm{rk}\left(T_{|I_{[d]}}\right) \le \mathrm{rk}(T)$ for all $T \in (\mathbb{F}^\infty)^{\otimes d}$ and all sets $I_1, \dots, I_d \subset \mathbb{N}$.

3. Restriction Lipschitz property:
   $\mathrm{rk}\left(T_{|J_{[d]}}\right) \le \mathrm{rk}\left(T_{|I_{[d]}}\right) + \sum_{i=1}^d |J_i \setminus I_i|$ for all $T \in (\mathbb{F}^\infty)^{\otimes d}$ and all sets $I_1 \subseteq J_1, \dots, I_d \subseteq J_d$.

Our main result in this setting concerns how natural rank functions behave under random coordinate restrictions. Intuitively, it shows that random restrictions of high-rank tensors will also have high rank with high probability. It can be formally stated as follows:

Theorem 3.3. *For every $d \in \mathbb{N}$ and $\sigma \in (0, 1]$, there exist constants $C, \kappa > 0$ such that the following holds. For every natural rank function $\mathrm{rk} : (\mathbb{F}^\infty)^{\otimes d} \to \mathbb{R}_+$ and every d-tensor $T \in \bigotimes_{i=1}^d \mathbb{F}^{n_i}$ we have*

$$\mathrm{Pr}_{I_1 \sim [n_1]_\sigma, \dots, I_d \sim [n_d]_\sigma}\left[\mathrm{rk}\left(T_{|I_{[d]}}\right) \ge \kappa \cdot \mathrm{rk}(T)\right] \ge 1 - Ce^{-\kappa\, \mathrm{rk}(T)}.$$

From this theorem one can easily deduce a more symmetric version, which is valid in the standard case of "cubic" tensors where every row is indexed by the same set:

Corollary 3.4. *For every $d \in \mathbb{N}$ and $\sigma \in (0, 1]$, there exist constants $C, \kappa > 0$ such that the following holds. For every natural rank function $\mathrm{rk} : (\mathbb{F}^\infty)^{\otimes d} \to \mathbb{R}_+$ and every d-tensor $T \in (\mathbb{F}^n)^{\otimes d}$ we have*

$$\mathrm{Pr}_{I \sim [n]_\sigma}\left[\mathrm{rk}\left(T_{|I^d}\right) \ge \kappa \cdot \mathrm{rk}(T)\right] \ge 1 - Ce^{-\kappa\, \mathrm{rk}(T)}.$$

3.2 Polynomial maps

Next we consider the setting of polynomials and higher-dimensional polynomial maps.

Definition 3.5. A polynomial map is an ordered tuple $\phi(x) = \big(f_1(x), \dots, f_k(x)\big)$ of polynomials $f_1, \dots, f_k \in \mathbb{F}[x_1, \dots, x_n]$. We identify with ϕ a map $\mathbb{F}^n \to \mathbb{F}^k$ in the natural way. The degree of ϕ is the maximum degree of the f_i.

For a polynomial map $\phi : \mathbb{F}^n \to \mathbb{F}^k$ and a set $I \subseteq [n]$, define the restriction $\phi_{|I} : \mathbb{F}^I \to \mathbb{F}^k$ to be the map given by $\phi_{|I}(y) = \phi(\bar{y})$, where $\bar{y} \in \mathbb{F}^n$ agrees with y on the coordinates in I and is zero elsewhere.

We denote the space of all polynomial maps $\phi : \mathbb{F}^n \to \mathbb{F}^k$ of degree at most d by $\mathrm{Pol}_{\le d}(\mathbb{F}^n, \mathbb{F}^k)$, and write

$$\mathrm{Pol}_{\le d}(\mathbb{F}^\infty, \mathbb{F}^k) = \bigcup_{n \in \mathbb{N}} \mathrm{Pol}_{\le d}(\mathbb{F}^n, \mathbb{F}^k).$$

The notions of rank we consider are defined below:

Definition 3.6. We say that $\mathrm{rk} : \mathrm{Pol}_{\leq d}(\mathbb{F}^\infty, \mathbb{F}^k) \to \mathbb{R}_+$ is a natural rank function if it satisfies the following properties:

1. Symmetry:
   $\mathrm{rk}(\phi) = \mathrm{rk}(-\phi)$ for all $\phi \in \mathrm{Pol}_{\leq d}(\mathbb{F}^\infty, \mathbb{F}^k)$.

2. Sub-additivity:
   $\mathrm{rk}(\phi + \gamma) \leq \mathrm{rk}(\phi) + \mathrm{rk}(\gamma)$ for all $\phi, \gamma \in \mathrm{Pol}_{\leq d}(\mathbb{F}^\infty, \mathbb{F}^k)$.

3. Monotonicity under restrictions:
   $\mathrm{rk}(\phi_{|I}) \leq \mathrm{rk}(\phi)$ for all $\phi \in \mathrm{Pol}_{\leq d}(\mathbb{F}^\infty, \mathbb{F}^k)$ and all sets $I \subset \mathbb{N}$.

4. Restriction Lipschitz property:
   $\mathrm{rk}(\phi_{|I \cup J}) \leq \mathrm{rk}(\phi_{|I}) + |J|$ for all $\phi \in \mathrm{Pol}_{\leq d}(\mathbb{F}^\infty, \mathbb{F}^k)$ and all sets $I, J \subset \mathbb{N}$.

Our second main result shows that random restrictions of a high-rank polynomial map will also have high rank with high probability. Its formal statement is given as follows:

Theorem 3.7. *For every $d \in \mathbb{N}$ and $\sigma, \varepsilon \in (0, 1]$, there exist constants $\kappa = \kappa(d, \sigma) > 0$ and $R = R(d, \sigma, \varepsilon) \in \mathbb{N}$ such that the following holds. For every natural rank function* rk : $\mathrm{Pol}_{\leq d}(\mathbb{F}^\infty, \mathbb{F}^k) \to \mathbb{R}_+$ *and every map $\phi \in \mathrm{Pol}_{\leq d}(\mathbb{F}^n, \mathbb{F}^k)$ with* $\mathrm{rk}(\phi) \geq R$*, we have*

$$\mathrm{Pr}_{I \sim [n]_\sigma} \big[\mathrm{rk}(\phi_{|I}) \geq \kappa \cdot \mathrm{rk}(\phi) \big] \geq 1 - \varepsilon.$$

3.3 The proofs

Whereas the proof of the matrix case (Proposition 2.1) uses in an essential way the fact that a rank-r matrix contains a full-rank $r \times r$ submatrix, an analogous property is not known to be true in general for tensors and polynomial maps. In fact, it was shown by Gowers that such a property is *false* in the case of slice rank for 3-tensors (see [9, Proposition 3.1]). Karam [9] recently studied the extent for which similar but quantitatively weaker properties hold for tensor rank functions, but the quantitative bounds obtained are still insufficient for an argument akin to that of Proposition 2.1 to work.

The proofs of our main theorems must then proceed differently from the simpler case of matrices. Our proof of Theorem 3.3 (the tensor case) uses instead ideas from probability theory, in particular concerning concentration inequalities on product spaces; it relies mainly on an inequality of Talagrand [14, Theorem 3.1.1].

The proof of Theorem 3.7 (for polynomial maps) is again very different from the tensor case, which implicitly makes use of multilinearity; it relies instead on results from the analysis of Boolean functions, in particular Friedgut's Junta Theorem [5], taken together with elementary (but somewhat involved) combinatorial arguments. The full proofs can be found in the full version of our paper [2].

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