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A solution to the multidimensional additive homological equation

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Abstract. We prove that, for a finite-dimensional real normed space V, every bounded mean zero function $f \in L_\infty([0,1];V)$ can be written in the form $f = g \circ T - g$ for some $g \in L_\infty([0,1];V)$ and some ergodic invertible measure preserving transformation T of $[0,1]$. Our method moreover allows us to choose g, for any given $\varepsilon > 0$, to be such that $\|g\|_\infty \leqslant (S_V+\varepsilon)\|f\|_\infty$, where S_V is the Steinitz constant corresponding to V.

Keywords: additive homological equation, coboundary problem, Kwapień's theorem, Steinitz constant, measure preserving transformation.

§ 1. Introduction

Given a bounded mean zero function f on $[0,1]$, the question is whether there exist a measure preserving transformation T and a bounded function g such that

$$f = g \circ T - g, \tag{1.1}$$

with equality holding almost everywhere. We call (1.1) the homological equation, and while it has been extensively studied in the scalar-valued setting, little is known about this equation for vector-valued functions. Below, we will study the homological equation for vector-valued functions.

We always assume that the interval $[0,1]$ is equipped with the standard Lebesgue measure λ. Equation (1.1), also known as the coboundary equation, was studied by Anosov for a fixed operator T in [1], where it was demonstrated that such an equation, with f continuous or even analytic on the torus, may have a measurable, but not integrable, solution. This study dates back to a comment made by Kolmogorov in [2] that there is no such a "good" solution. We note by [1], Theorem 1, if f is integrable and if its homological equation has a measurable solution g for some T, then f is mean zero. For a closely related variant of this problem, Bourgain [3] showed that, for a compact abelian group G with finitely many components, any mean zero function $f \in L^p(G)$, where $p \in (1,\infty)$, can be decomposed as

$$f = \sum_{j=1}^{J} (f_j - \tau(a_j) f_j),$$

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for $f_j \in L^p(G)$, $a_j \in G$, and the standard translation operator τ. Moreover, Bourgain proved that this result is sharp, and estimated the range of the index J.

Browder [4], Theorem 2, also studied when the homological equation has a solution $g \in L_\infty[0,1]$, for a given function $f \in L_\infty[0,1]$ and a given transformation T. He showed that a necessary and sufficient for solvability of this equation is that the norms $\|\sum_{j=0}^{k} f \circ T^j\|_\infty$ should be uniformly bounded for all $k \geqslant 1$.

In [5], it was shown that, for every real-valued mean-zero $f \in L_\infty[0,1]$, there is an ergodic transformation T such that (1.1) admits a solution $g \in L_\infty[0,1]$. In [6], this result was strengthened to show that, for $1 \leqslant p \leqslant \infty$, for any real-valued mean zero $f \in L_p[0,1]$, there exists a solution $g \in L_{p-1}[0,1]$ for some ergodic T.

Theorem 0.1 in [7] shows that, for real-valued mean zero $f \in L_\infty[0,1]$, we can choose g such that $\|g\|_\infty \leqslant (1+\varepsilon)\|f\|_\infty$. This result (with a weaker estimate) was announced earlier in [8], however, the proof there goes through only for $f \in C[0,1]$. Theorem 0.1 in [7] provides an upper bound for $\|g\|_\infty$, which is important for certain applications in the theory of symmetric functionals (see, for example, [9]) and singular traces (see, for example, [10]). Unlike [5], Theorem 1.1 says nothing about the ergodicity of T.

Theorem 1.1 (see [7], Theorem 0.1). *Let $f \in L_\infty[0,1]$ be a real-valued mean zero function. Then, for any $\varepsilon > 0$, there exist a* $\operatorname{mod} 0$ *automorphism T of* $[0,1]$ *and a function $g \in L_\infty[0,1]$, $\|g\|_\infty \leqslant (1+\varepsilon)\|f\|_\infty$, such that $f = g \circ T - g$.*

Here and throughout, a $\operatorname{mod} 0$ automorphism is defined as follows.

Definition 1.1. Given two measure spaces $(\Omega, \mathcal{A}, \mu)$ and $(\Omega', \mathcal{A}', \mu')$. A mapping $T\colon \Omega \to \Omega'$ is called a $\operatorname{mod} 0$ *isomorphism* if $T\colon \Omega \setminus N \to \Omega' \setminus N'$ is a bijection for nullsets $N \in \mathcal{A}$, $N' \in \mathcal{A}'$ such that both T and T^{-1} are measurable, and $\mu'(T(A)) = \mu(A)$, for all $A \in \mathcal{A}$, with $A \subseteq \Omega \setminus N$. When the two measure spaces are equal, this T is called a $\operatorname{mod} 0$ *automorphism.*

A natural question here is whether Theorem 1.1 carries over to complex-valued mean zero functions? This question may be equivalently restated for mean zero functions taking values in $\mathbb{R}^2$, and, even further, for mean zero $\mathbb{R}^d$-valued functions for an arbitrary positive integer d. Another question is whether the transformation T may be chosen to be ergodic. In this paper, we answer these questions affirmatively, and prove the following result.

Theorem 1.2. *Let $f \in L_\infty([0,1]; V)$ be a V-valued mean zero function on a finite-dimensional real normed space V. Then, for any $\varepsilon > 0$, there exist an ergodic* $\operatorname{mod} 0$ *automorphism T of* $[0,1]$ *and a function $g \in L_\infty([0,1]; V)$ such that*

$$\|g\|_\infty \leqslant (S_V + \varepsilon)\|f\|_\infty$$

(here S_V is the Steinitz constant corresponding to V) and

$$f = g \circ T - g.$$

This theorem holds for all measure spaces which are $\operatorname{mod} 0$ isomorphic to the interval $[0,1]$ with respect to the Lebesgue measure. We note that if we would fix some basis for V and apply Theorem 1.1 to the component functions of f, this would only yield that $f_i = g_i \circ T_i - g_i$ for $i = 1, \dots, \dim(V)$. In this case, we could have $T_i \neq T_j$ for $i \neq j$, and so Theorem 1.2 does not follow from Theorem 1.1 and hence extends it. We also show that the resulting transformation is ergodic.

However, it was not at all clear how to prove Theorem 1.2, as the machinery of [7] and [5] cannot be extended to the case of complex-valued functions (or to more general $\mathbb{R}^d$-valued functions). The proof of [5] for real-valued functions is split into a proof for step-functions and a proof for those with infinitely many values. However, this trick does not apply to $\mathbb{R}^d$-valued functions. The method of [7] also involves splitting into step-functions and the functions for which the pre-image of any point (except one point) is a nullset. Even though this method cannot be carried over, to the full extent, it turns out that there exists some smaller class of functions for which the result may be extended, albeit with some difficulty. Using this argument, we were able to prove our Theorem 1.2 in full.

The constant S_V mentioned in the theorem is the Steinitz constant corresponding to the space V, which arises from Steinitz's rearrangement lemma (see, for example, [11], Lemma 2.1.3). This constant is defined as the smallest number such that, for all finite collections of vectors $v_1, \dots, v_n$ in V with sum $\sum_{i=1}^n v_i = 0$, there exists a permutation π such that $\| \sum_{j=1}^k v_{\pi(i)} \| \leqslant S_V \max_i \|v_i\|$ for all $k = 1, \dots, n$ (see [11]). To show that the Steinitz constant and the rearrangement lemma are closely related to the additive homological equation, we will give an equivalent definition. Let Ω_n be a finite set of n elements equipped with a counting measure. Then S_V can also be equivalently defined as the smallest number such that, for $n \geqslant 1$, and for all mean zero $f \in L_\infty(\Omega_n, V)$, there exist an (ergodic) automorphism T of Ω_n and a set of positive measure $X \subseteq \Omega_n$ such that $\| \sum_{j=0}^k f \circ T^k \|_{L_\infty(X;V)} \leqslant S_V \|f\|_\infty$ for all $k = 1, 2, \dots$. As a consequence of Theorem 1.2 we have the following result, which is a natural continuous analogue of Steinitz's rearrangement lemma.

Theorem 1.3. *Let V be a finite-dimensional real normed space, let $\varepsilon > 0$, and let $f \in L_\infty([0,1]; V)$ be a V-valued mean zero function. Then there exist an ergodic* $\operatorname{mod} 0$ *automorphism T of $[0,1]$ and a set $X \subset [0,1]$ of positive measure such that $\left\| \sum_{j=0}^k f \circ T^j \right\|_{L_\infty(X;V)} \leqslant (S_V + \varepsilon) \|f\|_\infty$ for all $k = 1, 2, \dots$.*

Note that this result extends Theorem 2 in [4] to the class of $f \in L_\infty([0,1]; V)$ as well. This means that, for any $f \in L_\infty([0,1]; V)$ and any measure preserving T, there is a solution $g \in L_\infty([0,1]; V)$ if and only if the norms $\left\| \sum_{j=0}^k f \circ T^j \right\|_\infty$ are uniformly bounded for $k \geqslant 1$.

An immediate corollary of our main result is the following extension of Kwapień's Theorem 1.1 to the case of complex-valued mean zero functions.

Corollary 1.1. *Let $f \in L_\infty[0,1]$ be a complex-valued mean zero function. Then, for any $\varepsilon > 0$, there exist an ergodic* $\operatorname{mod} 0$ *automorphism T and a function $g \in L_\infty[0,1]$ such that $\|g\|_\infty \leqslant (\sqrt{5}/2 + \varepsilon) \|f\|_\infty$ and $f = g \circ T - g$.*

Let us briefly discuss our method of the proof of Theorem 1.2, and outline the structure of the paper. The proof of the main theorem involves three key steps.

In § 2, we will present the basic facts, definitions, and notation used in the paper. We will recall the definition of the Steinitz constant S_V, and its fundamental properties, and introduce affinely homogeneous and affinely partially homogeneous functions.

After this, we will present the first key step of the proof of Theorem 1.2. The following lemma is fundamental to Kwapień's proof [8]. We will need an extension of this result.

Lemma 1.1. *If* $(a_{i,j})_{n\times m}$ *is a matrix with real entries satisfying* $|a_{i,j}|\leqslant C$, $i = 1,\dots,n$, $j = 1,\dots,m$, *and* $\sum_{j=1}^{m} a_{i,j} = 0$ *for* $i = 1,\dots,n$, *then there exist permutations* $\sigma_1,\dots,\sigma_n$ *of the integers* $\{1,\dots,m\}$ *such that*

$$\left|\sum_{i=1}^{k} a_{i,\sigma_i(j)}\right| \leqslant 2C, \qquad k = 1,\dots,n, \quad j = 1,\dots,m.$$

We generalize this result in Theorem 3.3 to the case where the real entries $a_{i,j}$ are replaced by vectors from V. Our extension of Kwapieńn's lemma is the main result in § 3. This result is then used in § 6 to solve the homological equation for continuous functions on Cantor sets (see Theorem 6.1).

In § 4, we show that the functions we consider can be decomposed into affinely partially homogeneous functions. In § 5, we prove several "shrinking lemmas", which refine Lusin's theorem (Theorem 2.1), and which are required in the proof of our main result for affinely homogeneous functions.

In § 6, we prove that the main result holds for continuous mean-zero functions over the Cantor set (Theorem 6.1). In § 7, using the result for continuous functions on the Cantor set, we solve the homological equation for the subclass of $L_\infty([0,1];V)$ consisting of affinely homogeneous functions. For these functions, using the machinery developed in § 5, we construct subsets of positive measure which are homeomorphic to the Cantor set, and such that the restriction of f to such subsets is mean zero and continuous. We can then apply the result for continuous functions to solve the equation for this class of functions. Note that the transformation constructed here is not ergodic.

Finally, in § 8, we complete the proof of Theorems 1.2 and 1.3. However, in order to prove these main results, we need different tools since the method for affinely homogeneous functions cannot be used in general, and also since we want T to be ergodic. Our proof for general functions does however use the results that we developed for affinely homogeneous functions. Indeed, in Lemma 8.3 we use results from § 4 and § 7 to construct a partition of the domain, and a transformation satisfying certain properties. In the final part of the proof of the theorem, we apply this lemma inductively to obtain transformations $T^{(1)}, T^{(2)}, \dots$. Using these transformations, we construct an ergodic transformation T and a function g that solve the equation.

1.1. Novelty and necessity of affinely homogeneous function techniques. It is worth pointing out that, although the constructions for affinely homogeneous functions and continuity on Cantor sets bear some analogy to [7], the proof for general functions is totally different. Indeed, the proof of [7], Theorem 0.1, is based on splitting the case of a general f into (roughly speaking) two cases, where f is simple and where the pre-image of each point (except one) is a nullset. A quick analysis shows that such a splitting is impossible for $\mathbb{R}^d$-valued functions. This observation called for a new approach, which is most visible in the proof of our Theorem 1.2 in § 8, and in Lemma 8.3.

Let us briefly discuss why these earlier techniques are not applicable to the general result. According to [7] and [5], a solution of the equation for real-valued functions involves difficulties associated with step-functions. In [5], this is bypassed by restricting to the case where f takes infinitely many values, and using a different method for step-functions. In [7], the domain is split into the parts on which f is mean zero and either behaves in a "non-constant" manner or is a step-function with two steps. The homological equation is solved separately on these domains. In our present setting of $\mathbb{R}^d$-valued functions, the issue with step-functions gets more complicated, as some difficulties arise with affine subspaces. We solve the equation by extending methods from [7] and using a new technique.

To show the difficulty of dealing with general functions, we fix $\alpha \in (0,1)$, and consider the mean zero function $f \in L_\infty([0,1];\mathbb{R}^2)$ given by

$$f = (f_1, f_2), \qquad f_1 = (1-\alpha)\chi_{[0,\alpha]} - \alpha\chi_{(\alpha,1]}, \quad f_2(t) = t - \frac{1}{2}.$$

A solution g, T solving the equation $f = g \circ T - g$ would directly provide us with a solution for the first coordinate function f_1. But since the function f takes infinitely many values, we cannot apply any extension of the method of [5], as it cannot be applied to step-function f_1. An extension of the method of [7] also fails, as the Cantor set construction cannot be carried out for f with irrational α. This motivates our new approach, which is implemented in § 8 and involves our construction for affinely homogeneous functions. There is a link between our approach in § 8 and the method in [5], though these methods are different.

1.2. Failure of Theorems 1.2 and 1.3 for infinite-dimensional vector spaces. Let us give an example showing that the results of Theorems 1.2 и 1.3 fail to hold in the case $\dim(V) = \infty$.

In the space $\mathbb{R}^d$, $d > 1$, equipped with Euclidean norm, consider the vertices of the regular $(d-1)$-dimensional simplex centred at zero:

$$x_k = (a_{k,i}), \qquad k = 1, \dots, d, \quad a_{k,i} = \delta_{ki} - \frac{1}{d}.$$

We have

$$\|x_k\|^2 = \frac{d-1}{d^2} + \frac{(d-1)^2}{d^2} = \frac{d-1}{d} < 1, \qquad \sum_{k=1}^{d} x_k = 0.$$

We assume that d is even and y is the sum of $d/2$ such vertices (with possible repetition!). Let us estimate the norm $\|y\|$ from below. At least $d/2$ of the components of y are equal to $d/2 \cdot (-1/d) = -1/2$. Hence, $\|y\|^2 \geqslant d/2 \cdot 1/4 = d/8$, that is, $\|y\| \geqslant \sqrt{d/8}$. For every $n \geqslant 1$, we define $d_n = 2^n$ and choose $r_n > 0$ satisfying

$$\sum_{n=1}^{\infty} r_n^2 \leqslant 1, \qquad 2^{(n-3)/2} r_n \to \infty.$$

Next, for every $n > 1$, let $x_1^n, \dots, x_{d_n}^n \in \mathbb{R}^{d_n}$ be defined as above but multiplied by a constant factor depending on n only, so that we have $\|x_k^n\| = r_n$. Finally, we define the space $V = \bigoplus_{n=1}^{\infty} \mathbb{R}^{d_n}$ as a Hilbertian sum. We now set

$$f_n \colon [0,1] \to \mathbb{R}^{d_n}\colon \quad f_n\biggl(\biggl[\frac{i-1}{d_n}, \frac{i}{d_n}\biggr)\biggr) = x_i^n, \qquad i = 1, \dots, d_n, \quad f_n(1) = x_{d_n}^n.$$

We have

$$f_n \in L_\infty([0,1], \mathbb{R}^{d_n}), \qquad \|f_n\|_\infty \leqslant r_n, \qquad \int f_n \, d\lambda = 0.$$

Setting $f = \bigoplus_{n=1}^\infty f_n$, we obtain

$$f \in L_\infty([0,1], V), \qquad \|f\|_\infty \leqslant 1, \qquad \int f \, d\lambda = 0.$$

Assume that T is a mod 0 automorphism of $[0,1]$ such that

$$\sup_k \biggl\| \sum_{i=0}^k f \circ T^i \biggr\|_\infty = C < \infty.$$

Then

$$\sup_k \biggl\| \sum_{i=0}^k f_n \circ T^i \biggr\|_\infty \leqslant C \quad \forall\, n.$$

We shall show that this is not the case, thereby obtaining a contradiction with the assumption on existence of such a T. For almost every $t \in [0,1]$, the element

$$\sum_{i=0}^{d_n/2-1} f_n \circ T^i(t)$$

coincides with the sum of $d_n/2$ elements from the set $\{x_1^n, \dots, x_{d_n}^n\}$, as $f_n \circ T^i(t) \in \{x_1^n, \dots, x_{d_n}^n\}$ for almost every $t \in [0,1]$. So its norm cannot be smaller than

$$\sqrt{\frac{d_n}{8}}\, r_n = 2^{(n-3)/2} r_n.$$

Therefore, $C \geqslant 2^{(n-3)/2} r_n \to \infty$, yielding the required contradiction. Hence, $\sup_k \bigl\| \sum_{i=0}^k f \circ T^i \bigr\|_\infty = \infty$. Therefore, there is no function $g \in L_\infty([0,1], V)$ satisfying $f = g \circ T - g$. Indeed, otherwise we would have

$$\sup_k \biggl\| \sum_{i=0}^k f \circ T^i \biggr\|_\infty = \sup_k \|g \circ T^{k+1} - g\|_\infty \leqslant 2\|g\|_\infty.$$

§ 2. Preliminaries

2.1. Three fundamental theorems. The following version of Lusin's theorem is given in [12], Theorem 2.2.10.

Theorem 2.1. *Let $D \subseteq [0,1]$ be Borel-measurable and let $f\colon D \to \mathbb{R}$ be Borel-measurable. If $\varepsilon > 0$, then there exists a compact subset $K \subseteq D$ such that $\lambda(D \setminus K) < \varepsilon$ and the restriction of f to K is continuous.*

The following fundamental fact can be obtained by combining Theorems 9.3.4 and 9.5.1 from [12].

Theorem 2.2. *Let $A, B \subseteq [0,1]$ be some subsets of equal positive measure. Then there exists a* mod 0 *isomorphism $T\colon A \to B$.*

We shall also require the following Lyapunov's theorem (see [13], Theorem 2.c.9).

Theorem 2.3. *Let* $\{\mu_i\}_{i=1}^d$ *be a set of finite (not necessarily positive) non-atomic measures on the measurable space* (Ω, Σ)*. Then the set*

$$\{(\mu_1(X), \dots, \mu_d(X)) \colon X \in \Sigma\}$$

is convex and compact in $\mathbb{R}^d$*.*

2.2. The space $L_\infty(D; V)$. Throughout, $(V, \|\cdot\|)$ will denote a finite-dimensional normed vector space over $\mathbb{R}$. Let D be a Lebesgue measurable subset of $[0,1]$ equipped with Lebesgue measure λ, and let $f \colon D \to V$ be a measurable mapping. A vector $r \in V$ is said to be an *essential value* of the function f if $\lambda(f^{-1}(U)) > 0$ for any neighbourhood U of the vector r. The symbol $\sigma(f)$ stands for the set of all essential values of f (the usage of this symbol is justified by the fact that, for a function $f \in L_\infty[0,1]$, the set of all its essential values coincides with the spectrum of the element f in the C^*-algebra $L_\infty[0,1]$).

By $L_\infty(D; V)$ we denote the linear space of all measurable mappings $f \colon D \to V$ with bounded $\sigma(f)$. As usual, we will identify any two mappings if they are equal almost everywhere (that is, the space $L_\infty(D; V)$ consists of classes of measurable mappings equal almost everywhere).

We say that a function $f \in L_\infty(D; V)$ is *simple* if $f = \sum_{i=1}^\infty r_i \chi_{X_i}$, where $r_i \in V$, $i = 1, 2, \dots$, and $\{X_i\}_{i=1}^\infty$ is a partition of D into measurable subsets.

We define a norm on $L_\infty(D; V)$ by setting, for $f \in L_\infty(D; V)$,

$$\|f\|_\infty = \sup\{\|r\| \colon r \in \sigma(f)\}.$$

For every $f \in L_\infty(D; V)$, the integral $\int f \, d\lambda \in V$ is defined in a standard way. If $\int f \, d\lambda = 0$, then the function f is said to be *mean zero*.

We shall frequently use the notation

$$\fint_X f \, d\lambda = \frac{\int_X f \, d\lambda}{\lambda(X)},$$

that is, $\fint_X f \, d\lambda$ is the mean value of f on the set X.

We will sometimes use the Euclidean norm; the Euclidean inner products will be denoted by we use $(\cdot, \cdot)$.

2.3. Affinely homogeneous functions. For an arbitrary set $X \subset V$, the symbol $\operatorname{Aff}(X)$ denotes the *affine hull* in V generated by X, that is

$$\operatorname{Aff}(X) = \biggl\{\sum_{i=1}^k a_i x_i \colon x_i \in X,\, a_i \in \mathbb{R},\, \sum_{i=1}^k a_i = 1\biggr\}.$$

Recall that any affine subspace in V can be viewed as a set $\{x + V_0\}$, where x is some point in V and V_0 is a linear subspace in V. The dimension of such an affine subspace is defined as that of the subspace V_0. In particular, every point in V is an affine subspace of dimension 0.

We will say that a function $f \in L_\infty(D; V)$ is *affinely homogeneous* if, for every proper affine subspace $W \subsetneq \operatorname{Aff}(\sigma(f))$, we have $\lambda(f^{-1}(W)) = 0$.

Note that a real-valued function is affinely homogeneous if and only if it either is constant, or satisfies $\lambda(f^{-1}(\{y\})) = 0$ for all $y \in \mathbb{R}$.

It is easy to see that any affinely homogeneous simple function is constant. Indeed, if a simple function has two distinct essential values, say a and b, then $\lambda(f^{-1}(a)) > 0$ and $\lambda(f^{-1}(b)) > 0$. Since $\{a\} \subsetneq \mathrm{Aff}(\sigma(f))$ and $\{b\} \subsetneq \mathrm{Aff}(\sigma(f))$ are proper affine subspaces of $\mathrm{Aff}(\sigma(f))$, we arrive at a contradiction.

More generally, for any affinely homogeneous function f, we have $\mathrm{Aff}(\sigma(f|_A)) = \mathrm{Aff}(\sigma(f))$ for every subset $A \subseteq D$ of positive measure.

A function $f \in L_\infty(D;V)$ is said to be *affinely partially homogeneous* if D can be split into at most $d+1$ measurable subsets, where $d = \dim(V)$ such that (the reduction of) f is affinely homogeneous on each of this subsets. For example, the function $f = (1-a)\chi_{[0,a)} - a\chi_{[a,1]} \in L_\infty([0,1];\mathbb{R})$ is affinely partially homogeneous for any $a \in (0,1)$.

2.4. The Steinitz constant. For any real finite-dimensional normed space V, there exists a smallest number S_V (called the Steinitz constant) such that, for every $r_1,\dots,r_n \in V$, $\sum_{i=1}^n r_i = 0$,

$$\Bigg\|\sum_{i=1}^{k} r_{\pi(i)}\Bigg\| \leqslant S_V \max\{\|r_i\| : i=1,\dots,n\}, \qquad k=1,\dots,n,$$

for some permutation π of the set $\{1,\dots,n\}$ (see [14]). This constant depends generally both on the dimension of V and on the norm of the space.

According to [15], $S_V \leqslant \dim(V)$ (for a detailed proof, see [11], Lemma 2.1.3). Trivially, we have $S_{\mathbb{R}} = 1$. In [16], Remark 3, it is stated that "Applying the same method as in the proof of Lemma 2, one can show that the Steinitz constant of an n-dimensional space is not greater than $n-1+1/n$". If we equip $\mathbb{R}^d$ with the Euclidean norm, then $S_{\mathbb{R}^d} \geqslant (\sqrt{d+3})/2$ (see [15]) and s $S_{\mathbb{R}^2} = S_{\mathbb{C}} = \sqrt{5}/2$ (see [16], Theorem 2, [17]). For other estimates of $S_{\mathbb{R}^d}$ for the Euclidean norm, see [18], Remark 8, Added in proof.

Let us explain the appearance of the Steinitz constant by proving that the main result holds for mean zero functions $f \in L_\infty(\Omega_n;V)$, where a finite measure space $\Omega_n = \{1,\dots,n\}$ is equipped with a counting measure. Indeed, as $\sum_{i=1}^n f(i) = 0$, by definition of the Steinitz constant there exists a permutation π of $\{1,\dots,n\}$ such that

$$\Bigg\|\sum_{i=1}^{m} f(\pi(i))\Bigg\| \leqslant S_V\|f\|_\infty, \qquad m=1,\dots,n.$$

We can then define a cyclic permutation σ of Ω_n by $\sigma(\pi(j)) = \pi(j+1)$ for $j = 1,\dots,n-1$ and $\sigma(\pi(n)) = \pi(1)$. We set $g(\pi(k)) = \sum_{i=1}^{k-1} f(\pi(i))$ for $k=2,\dots,n$ and $g(\pi(1)) = 0$. Now $g\circ\sigma - g = f$ and $\|g\|_\infty \leqslant S_V\|f\|_\infty$, proving the result. It can be seen that this argument can also be applied to simple functions $f \in L_\infty([0,1];V)$ of the form

$$f = \sum_{k=1}^{n} r_k\chi_{I_k}, \qquad I_k = \left[\frac{k-1}{n}, \frac{k}{n}\right), \quad r_k \in V, \quad k=1,\dots,d, \quad \sum_{k=1}^{n} r_k = 0,$$

as they can be identified with mean zero functions $\widetilde{f}$ from $L_\infty(\Omega_n;V)$ given by $\widetilde{f}(k) = r_k$. It remains to define $T = T(I_k) = I_{\sigma(k)}$, and consider the simple function g given by $g|_{I_{\pi(k)}} = \sum_{i=1}^{k-1} r_{\pi(i)}$ for $k = 2, \dots, n$ and $g|_{I_{\pi(1)}} = 0$. This gives us $g \circ \sigma - g = f$ and $\|g\|_\infty \leqslant S_V \|f\|_\infty$.

§ 3. A multidimensional version of Kwapień's lemma

The main result of this section is Theorem 3.3. Its proof is based on the following known results. Below, $\operatorname{Conv}(X)$ is the convex hull of a set $X \subset V$.

Theorem 3.1 (see [19], Theorem 3). *Let V be a d-dimensional real normed space with unit ball B^d, let $C_i \subset B^d$, and let $0 \in \operatorname{Conv}(C_i)$, $i = 1, 2, \dots$. Then there exist elements $c_i \in C_i$, $i = 1, 2, \dots$ such that*

$$\left\| \sum_{i=1}^{p} c_i \right\| \leqslant 2d, \qquad p = 1, 2, \dots .$$

Theorem 3.2 (see [15], Theorem 1, [11], Lemma 2.1.3). *Let V be a d-dimensional real normed space, $\|x_i\| \leqslant 1$, $i = 1, \dots, n$, and $x_1 + \dots + x_n = x$. Then there exists an permutation π such that, for all natural indices $k \leqslant n$,*

$$\left\| \sum_{i=1}^{k} x_{\pi(i)} - \frac{k-d}{n} x \right\| \leqslant d.$$

Now we are ready to prove the following lemma.

Lemma 3.1. *Let V be a d-dimensional real normed space, let $\{a_{i,j}\}_{i,j=1}^{n,m}$ be vectors from V with $\|a_{i,j}\| \leqslant 1$, $i = 1, \dots, n$, $j = 1, \dots, m$,*

$$\sum_{j=1}^{m} a_{i,j} = 0, \qquad i = 1, \dots, n,$$

and let $p \leqslant m$ be a natural number. Then the set $\{1, \dots, m\}$ contains subsets $I_1, \dots, I_n$ such that

$$|I_1| = \dots = |I_n| = p$$

and

$$\left\| \sum_{i=1}^{k} \sum_{j \in I_i} a_{i,j} \right\| \leqslant 4d^2, \qquad k = 1, \dots, n.$$

Proof. For every fixed $i = 1, \dots, n$, we have $\sum_{j=1}^{m} a_{ij} = 0$ by the assumption. By Theorem 3.2, replacing the collection of vectors $x_1, \dots, x_n$ by the collection $a_{i,1}, \dots, a_{i,m}$, we infer the existence of a permutation π of the set $\{1, \dots, m\}$ such that $\left\| \sum_{j=1}^{k} a_{i,\pi(j)} \right\| \leqslant d$, $k = 1, \dots, m$. Relabelling the vectors a_{ij}, $j = 1, \dots, m$, we may assume without loss of generality that, for every $i = 1, \dots, n$,

$$\left\| \sum_{j=1}^{l} a_{i,j} \right\| \leqslant d, \qquad l = 1, \dots, m.$$

Let m_1 be the least common multiple of the numbers m and p, and let $m_2 = m_1/p$. Consider the mapping α from $\{1, \dots, m_1\}$ onto $\{1, \dots, m\}$ which maps a number j to the remainder of the division on m provided that j is not a scalar multiple of m, and to m, otherwise.

We now replace the matrix $\{a_{ij}\}_{i,j=1}^{n,m}$ by $\{a'_{i,j}\}_{i,j=1}^{n,m_1}$, where $a'_{i,j} = a_{i,\alpha(j)}$. In other words, any column of the matrix $\{a_{i,j}\}_{j=1}^{m}$ is repeated m_1/m times.

Note that the matrix $\{a'_{i,j}\}_{i,j=1}^{n,m_1}$ also satisfies the same assumptions as the original matrix $\{a_{i,j}\}_{i,j=1}^{n,m}$.

We now set $b_{i,j} = \sum_{r=(j-1)p+1}^{jp} a'_{i,r}$, $j = 1, \dots, m_2$, $i = 1, \dots, n$. Let us show that, for all i, j,

$$\|b_{i,j}\| \leqslant 2d.$$

If $\alpha((j-1)p+1)$, $\alpha((j-1)p+2)$, $\dots$, $\alpha(jp)$ increases, we have

$$\|b_{i,j}\| = \left\| \sum_{r=1}^{\alpha(jp)} a_{ir} - \sum_{r=1}^{\alpha((j-1)p)} a_{ir} \right\| \leqslant 2d.$$

Otherwise,

$$m \in \{\alpha((j-1)p+1), \alpha((j-1)p+2), \dots, \alpha(jp)\},$$

that is, $\{\alpha((j-1)p+1), \alpha((j-1)p+2), \dots, \alpha(jp)\}$ consists of two disjoint sets, $\{m-k+1, m-k+2, \dots, m\}$ and $\{1, 2, \dots, p-k\}$.

Hence

$$\|b_{i,j}\| = \left\| \sum_{r=m-k+1}^{m} a_{i,r} + \sum_{r=1}^{p-k} a_{i,r} \right\| = \left\| - \sum_{r=1}^{m-k} a_{i,r} + \sum_{r=1}^{p-k} a_{i,r} \right\| \leqslant 2d.$$

We also have $\sum_{j=1}^{m_2} b_{i,j} = \sum_{r=1}^{m_1} a'_{i,r} = (m_1/m) \sum_{j=1}^{m} a_{i,j} = 0$, $i \in \{1, \dots, n\}$, and so $0 \in \operatorname{Conv}\{b_{i,j} \colon j \in \{1, \dots, m_2\}\}$ for all $i \in \{1, \dots, n\}$.

By Theorem 3.1, there exist indices j_i such that, for all $k = 1, \dots, n$,

$$\left\| \sum_{i=1}^{k} b_{i,j_i} \right\| \leqslant 4d^2.$$

Since $b_{i,j_i} = \sum_{r=(j_i-1)p+1}^{j_i p} a'_{i,r} = \sum_{r=(j_i-1)p+1}^{j_i p} a_{i,\alpha(r)} = \sum_{j \in I_i} a_{i,j}$, where $I_i = \alpha(\{(j_i-1)p+1, (j_i-1)p+2, \dots, j_i p\})$, the above estimate yields the assertion and completes the proof.

We will now use Lemma 3.1 to obtain a similar result for non-mean-zero vectors.

Lemma 3.2. *Let V be a d-dimensional real normed space, let $(v_{i,j})_{1\leqslant i\leqslant n,\, 1\leqslant j\leqslant m}$ be vectors in V, $\|v_{i,j}\| \leqslant 1$, let $p \in \{1, \dots, m\}$, and let $x_k = \sum_{i=1}^{k} \sum_{j=1}^{m} v_{i,j}/m$. Then, for each $i = 1, \dots, n$, there is an index set $I_i \subseteq \{1, \dots, m\}$ such that*

$$|I_k| = p \quad \forall\, k \in \{1, \dots, n\}, \qquad \left\| \sum_{i=1}^{k} \sum_{j \in I_i} v_{i,j} - p x_k \right\| \leqslant 8d^2 \quad \forall\, k \in \{1, \dots, n\}.$$

Proof. We define $(v'_{i,j})_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}$ by

$$v'_{i,j} = \frac{1}{2} v_{i,j} - \frac{1}{2m} \sum_{k=1}^{m} v_{i,k}.$$

Note that $\sum_{j=1}^m v'_{i,j} = 0$ for all $i \in \{1,\dots,n\}$, and so

$$\|v'_{i,j}\| \leqslant \frac{1}{2}\|v_{i,j}\| + \frac{1}{2m}\sum_{k=1}^m \|v_{i,k}\| \leqslant \frac{1}{2} + \frac{m}{2m} = 1.$$

Using Lemma 3.1, we can find sets I_i, $i = 1,\dots,n$, such that

$$|I_k| = p \quad \forall\, k \in \{1,\dots,n\}, \qquad \Big\|\sum_{i=1}^k \sum_{j\in I_i} v'_{i,j}\Big\| \leqslant 4d^2 \quad \forall\, k \in \{1,\dots,n\}.$$

Hence

$$\Big\|\sum_{i=1}^k \sum_{j\in I_i} v_{i,j} - px_k\Big\| = \Big\|\sum_{i=1}^k \sum_{j\in I_i} v_{i,j} - \sum_{i=1}^k \frac{|I_k|}{m}\sum_{j=1}^m v_{i,j}\Big\|$$
$$= \Big\|\sum_{i=1}^k \sum_{j\in I_i}\Big(v_{i,j} - \frac{1}{m}\sum_{t=1}^m v_{i,t}\Big)\Big\| = 2\Big\|\sum_{i=1}^k \sum_{j\in I_i} v'_{i,j}\Big\| \leqslant 2\cdot 4d^2 = 8d^2$$

for all $k \in \{1,\dots,n\}$. Lemma 3.2 is proved.

Now we can generalize Lemma 1.1 for vectors from V.

Theorem 3.3. *Let V be a d-dimensional real normed space, let $(v_{i,j})_{1\leqslant i\leqslant n,\,1\leqslant j\leqslant m}$ be vectors in V, $\|v_{i,j}\| \leqslant 1$, and let $x_k = (1/m)\sum_{i=1}^k\sum_{j=1}^m v_{i,j}$ for all $k \in \{1,\dots,n\}$. Then there exist permutations $(\pi_i)_{1\leqslant i\leqslant n}$ of the set $\{1,\dots,m\}$ such that $\big\|\sum_{i=1}^k v_{i,\pi_i(j)} - x_k\big\| \leqslant 8d^2/\log 1.5$ for all k and all j.*

Proof. Let us show that the required permutations can be constructed by partitioning the input vectors into two sets of almost equal size via Lemma 3.2, after which we recursively construct suitable permutations for both parts of the partition. We then combine these two permutations into one permutation, and show that this permutation satisfies the required properties.

For $m = 1$ the required result is trivial, because in this case $\sum_{i=1}^k v_{i,1} - x_k = 0$ for all k.

Let $m = 2$. Then

$$\Big(v_{i,1} - \frac{v_{i,1}+v_{i,2}}{2}\Big) + \Big(v_{i,2} - \frac{v_{i,1}+v_{i,2}}{2}\Big) = 0$$

for all $i = 1,\dots,n$. By Theorem 3.1, there exist $j_i \in \{1,2\}$ such that

$$\Big\|\sum_{i=1}^k \Big(v_{i,j_i} - \frac{v_{i,1}+v_{i,2}}{2}\Big)\Big\| \leqslant 4d.$$

Setting $\pi_i(1) = j_i$, $\pi_i(2) = 3 - j_i$, we have $\big\|\sum_{i=1}^k v_{i,\pi_i(1)} - x_k\big\| \leqslant 4d$ for all k and $v_{i,\pi_i(2)} - (v_{i,1}+v_{i,2})/2 = -(v_{i,\pi_i(1)} - (v_{i,1}+v_{i,2})/2)$. Hence $\big\|\sum_{i=1}^k v_{i,\pi_i(2)} - x_k\big\| \leqslant 4d$ for all k. We have

$$\int_0^{\log_{1.5}(2)-1} \Big(\frac{2}{3}\Big)^x\,dx = \frac{1}{4(\log 3 - \log 2)} > \frac{1}{2},$$

and hence

$$4d < 8d^2 \int_0^{\log_{1.5}(2)-1} \left(\frac{2}{3}\right)^x dx,$$

which gives that

$$\left\|\sum_{i=1}^{k} v_{i,\pi_i(j)} - x_k\right\| \leqslant 8d^2 \int_0^{\log_{1.5}(2)-1} \left(\frac{2}{3}\right)^x dx.$$

Next, we will prove by induction on m that, for a given set of input vectors $(v_{i,j})_{1\leqslant i\leqslant n,\, 1\leqslant j\leqslant m}$ with $\|v_{i,j}\| \leqslant 1$, there exist permutations $(\pi_i)_{1\leqslant i\leqslant n}$ of $\{1,\ldots,m\}$ such that

$$\left\|\sum_{i=1}^{k} v_{i,\pi_i(j)} - x_k\right\| \leqslant 8d^2 \int_0^{\log_{1.5}(m)-1} \left(\frac{2}{3}\right)^x dx.$$

The conclusion of Theorem 3.3 would then follow by replacing the above integral by the integral from 0 to ∞, which is equal to $1/\log 1.5$.

For $m = 2$, the required inequality was established above. For $m > 2$, assume that the required result holds up to $m-1$ inclusively. By Lemma 3.2, there exist subsets $(I_i)_{1\leqslant i\leqslant n}$ of $\{1,\ldots,m\}$ such that $|I_i| = p := \lceil m/2\rceil$ and, for $\delta_k := \sum_{i=1}^{k}\sum_{j\in I_i} v_{i,j}$,

$$\|\delta_k - p x_k\| \leqslant 8d^2 \quad \forall\, k \in \{1,\ldots,n\}. \tag{3.1}$$

Let $\delta'_k := \sum_{i=1}^{k}\sum_{j\in\{1,\ldots,m\}\setminus I_i} v_{i,j}$. We claim that $\delta_k + \delta'_k = m x_k$. Indeed, we have

$$\delta_k + \delta'_k = \sum_{i=1}^{k}\sum_{j\in I_i} v_{i,j} + \sum_{i=1}^{k}\sum_{j\in\{1,\ldots,m\}\setminus I_i} v_{i,j} = \sum_{i=1}^{k}\sum_{j=1}^{m} v_{i,j} = m x_k,$$

and so

$$\|\delta'_k - (m-p)x_k\| = \|m x_k - \delta_k - (m-p)x_k\| = \|\delta_k - p x_k\| \leqslant 8d^2. \tag{3.2}$$

For each $i \in \{1,\ldots,n\}$, let π'_i be a permutation of $\{1,\ldots,m\}$ that maps the set $\{1,\ldots,p\}$ to I_i. Let $(v^{(1)}_{i,j})_{1\leqslant i\leqslant n,\, 1\leqslant j\leqslant p}$ be defined by $v^{(1)}_{i,j} = v_{i,\pi'_i(j)}$. By the induction hypothesis, we can find permutations $\pi^{(1)}_i$ of $\{1,\ldots,p\}$ such that, for all $k \in \{1,\ldots,n\}$ and all $j \in \{1,\ldots,p\}$,

$$\left\|\sum_{i=1}^{k} v^{(1)}_{i,\pi^{(1)}_i(j)} - \frac{1}{p}\,\delta_k\right\| \leqslant 8d^2 \int_0^{\log_{1.5}(p)-1} \left(\frac{2}{3}\right)^x dx.$$

Similarly, we define $(v^{(2)}_{i,j})_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m-p}$, by $v^{(2)}_{i,j} = v_{i,\pi'_i(j+p)}$. Using the induction hypothesis, we can find permutations $\pi^{(2)}_i$ of $\{1,\ldots,m-p\}$ such that, for all $k \in \{1,\ldots,n\}$ and all $j \in \{1,\ldots,m-p\}$,

$$\left\|\sum_{i=1}^{k} v^{(2)}_{i,\pi^{(2)}_i(j)} - \frac{1}{m-p}\,\delta'_k\right\| \leqslant 8d^2 \int_0^{\log_{1.5}(m-p)-1} \left(\frac{2}{3}\right)^x dx.$$

We set

$$\pi_i(j) = \begin{cases} \pi'_i(\pi_i^{(1)}(j)), & j \leqslant p, \\ \pi'_i(\pi_i^{(2)}(j-p)+p), & j > p, \end{cases}$$

and define

$$p_j = \begin{cases} p, & j \leqslant p, \\ m-p, & j > p, \end{cases} \qquad \Delta_i(j) = \begin{cases} \dfrac{1}{p_j}\,\delta_i, & j \leqslant p, \\ \dfrac{1}{p_j}\,\delta'_i, & j > p. \end{cases}$$

In two cases $j \leqslant p$ and $j > p$, applying (3.1) and (3.2), respectively, we obtain $\|\Delta_i(j) - x_i\| \leqslant 8d^2/p_j$ for all $i \in \{1,\dots,n\}$ and all $j \in \{1,\dots,m\}$.

For $j \in \{1,\dots,p\}$, for all $k \in \{1,\dots,n\}$, we have

$$\begin{aligned} \biggl\|\sum_{i=1}^{k} v_{i,\pi_i(j)} - \Delta_k(j)\biggr\| &= \biggl\|\sum_{i=1}^{k} v^{(1)}_{i,\pi_i^{(1)}(j)} - \frac{1}{p}\,\delta_k\biggr\| \\ &\leqslant 8d^2 \int_0^{\log_{1.5}(p)-1} \left(\frac{2}{3}\right)^x dx = 8d^2 \int_0^{\log_{1.5}(p_j)-1} \left(\frac{2}{3}\right)^x dx. \end{aligned}$$

Similarly, for $j \in \{p+1,\dots,m\}$, we have

$$\begin{aligned} \biggl\|\sum_{i=1}^{k} v_{i,\pi_i(j)} - \Delta_k(j)\biggr\| &= \biggl\|\sum_{i=1}^{k} v^{(2)}_{i,\pi_i^{(2)}(j-p)} - \frac{1}{p}\delta'_k\biggr\| \\ &\leqslant 8d^2 \int_0^{\log_{1.5}(m-p)-1} \left(\frac{2}{3}\right)^x dx = 8d^2 \int_0^{\log_{1.5}(p_j)-1} \left(\frac{2}{3}\right)^x dx. \end{aligned}$$

Combining these two inequalities, we have, for all $k \in \{1,\dots,n\}$ and all $j \in \{1,\dots,m\}$,

$$\begin{aligned} \biggl\|\sum_{i=1}^{k} v_{i,\pi_i(j)} - x_k\biggr\| &\leqslant \biggl\|\sum_{i=1}^{k} v_{i,\pi_i(j)} - \Delta_k(j)\biggr\| + \|\Delta_k(j) - x_k\| \\ &\leqslant 8d^2 \int_0^{\log_{1.5}(p_j)-1} \left(\frac{2}{3}\right)^x dx + 8d^2\,\frac{1}{p_j} \\ &= 8d^2 \int_0^{\log_{1.5}(p_j)-1} \left(\frac{2}{3}\right)^x dx + 8d^2 \left(\frac{2}{3}\right)^{\log_{1.5} p_j} \\ &\leqslant 8d^2 \int_0^{\log_{1.5}(p_j)} \left(\frac{2}{3}\right)^x dx \leqslant 8d^2 \int_0^{\log_{1.5}(m)-1} \left(\frac{2}{3}\right)^x dx. \end{aligned}$$

Note that the last inequality follows from the fact that $m/p_j \geqslant 3/2$, and hence $\log_{1.5}(m) - \log_{1.5}(p_j) = \log_{1.5}(m/p_j) \geqslant 1$. Theorem 3.3 is proved.

§ 4. Decomposition for bounded functions into affinely partially homogeneous functions

We will need several well-known Carathéodory's results. The first lemma below can be found in [20], Theorem 8.11.

Lemma 4.1. *Let $B \subset \mathbb{R}^d$, $d < \infty$. Then any element $\xi \in \operatorname{Conv}(B)$ can be decomposed as the convex combination of at most $d+1$ elements from B.*

For the following two results we refer to [21], Corollary IV.1.13, and [21], Corollary IV.3.11, respectively.

Theorem 4.1. *The convex hull of the closure of a bounded subset in $\mathbb{R}^d$, $d < \infty$, coincides with the closure of the convex hull of this subset.*

Theorem 4.2. *The closed convex hull of a set $A \subseteq \mathbb{R}^d$ equals the intersection of all closed half-spaces containing it.*

We begin with the following general (and probably well-known) result.

Proposition 4.1. *Let $\{\xi_i\}_{i\in I} \subset \mathbb{R}^d$, $d < \infty$, $\{\alpha_i\}_{i\in I} \subset \mathbb{R}_+ \setminus \{0\}$, $\operatorname{card}(I) \leqslant \aleph_0$, $0 < \|\xi_i\| \leqslant 1$ (here $\|\cdot\|$ is the Euclidean norm), $\sum_i \alpha_i \leqslant 1$, $\sum_i \alpha_i \xi_i = 0$. Then there exist indices $i_1, \dots, i_m \in I$, $1 \leqslant m \leqslant d+1$, and scalars $0 < \beta_k \leqslant \alpha_{i_k}$, $k = 1, \dots, m$, such that $\sum_{k=1}^m \beta_k \xi_{i_k} = 0$.*

Proof. Without loss of generality, we may assume that

$$\dim(\operatorname{Span}\{\xi_i \colon i \in I\}) = d, \qquad \sum_i \alpha_i = 1.$$

Let $B = \{\xi_i\}_{i\in I}$, $C = \operatorname{Conv}(B)$. By Theorem 4.1, $\overline{C} = \operatorname{Conv}(\overline{B})$. Therefore, $0 \in \overline{C} = \operatorname{Conv}(\overline{B})$.

For any set $X \subset \mathbb{R}^d$, its support function h_X is defined by

$$h_X(\eta) = \sup\{(\eta, \xi) \colon \xi \in X\}.$$

Let $\mathbb{S}^{d-1} = \{\eta \in \mathbb{R}^d \colon \|\eta\| = 1\}$, that is $\mathbb{S}^{d-1}$ is the sphere in $\mathbb{R}^d$ centred at zero with radius 1.

Since for $\eta \in \mathbb{S}^{d-1}$ the closed half-space $H_\eta := \{\xi \colon (\eta, \xi) \leqslant h_X(\eta)\}$ contains X, and since every closed half-space H that contains X is contained in H_η for some $\eta \in \mathbb{S}^{d-1}$, it follows from Theorem 4.2 that

$$\overline{\operatorname{Conv}(X)} = \bigcap_{\eta\in\mathbb{S}^{d-1}} \{\xi \colon (\eta, \xi) \leqslant h_X(\eta)\}.$$

We show that $\overline{C}$ contains a ball of radius $r_0 > 0$ centred at the origin. Indeed, the function $h_{\overline{C}}$ is continuous on the unit sphere $\mathbb{S}^{d-1}$. Since $\mathbb{S}^{d-1}$ is compact, there exists a point $\eta_0 \in \mathbb{S}^{d-1}$, at which $h_{\overline{C}}$ reaches the minimum. Assume that $h_{\overline{C}}(\eta_0) \leqslant 0$. Hence $(\eta_0, \xi) \leqslant 0$ for any $\xi \in B$. The equality $\sum_i \alpha_i \xi_i = 0$ implies that $\sum_i \alpha_i (\eta, \xi_i) = 0$, and, thus, $(\eta_0, \xi) = 0$ for any $\xi \in B$. This contradicts the fact that $\dim(\operatorname{Span}(B)) = d$. Therefore, $r_0 := h_{\overline{C}}(\eta_0) > 0$. Since $\overline{C} = \bigcap_{\eta\in\mathbb{S}^{d-1}} \{\xi \colon (\eta, \xi) \leqslant h_{\overline{C}}(\eta)\}$, it follows that $\overline{C}$ contains a ball with a radius r_0 centred at 0.

Since $\overline{B}$ is compact, there exists $n \in \mathbb{N}$ such that $B_n := \{\xi_i\}_{i=1}^n$ is a $r_0/3$-net in $\overline{B}$.

Let $\eta \in \mathbb{S}^{d-1}$. There exists a vector $\xi \in \overline{B}$ such that

$$(\eta, \xi) = h_{\overline{B}}(\eta) = h_{\overline{C}}(\eta) \geqslant r_0.$$

Let now $\xi' \in B_n$ be such that $\|\xi - \xi'\| < r_0/3$. We have

$$|(\eta, \xi') - (\eta, \xi)| \leqslant \|\xi - \xi'\| < \frac{r_0}{3}, \qquad (\eta, \xi) \geqslant r_0.$$

Hence $|(\eta, \xi')| = (\eta, \xi')$, and, therefore,

$$(\eta, \xi') \geqslant (\eta, \xi) - |(\eta, \xi') - (\eta, \xi)| > r_0 - \frac{r_0}{3} > \frac{r_0}{2}.$$

Thus, $h_{B_n}(\eta) \geqslant r_0/2$. Therefore, $\mathrm{Conv}(B_n)$ contains the ball of radius $r_0/2$ centred at 0. In particular, the point 0 is a convex combination of the vectors $\{\xi_i\}_{i=1}^n$.

By Lemma 4.1, there exist $\xi_{i_1}, \dots, \xi_{i_m} \in B_n$, $m \leqslant d+1$ such that $0 = \sum_{k=1}^m \beta'_k \xi_{i_k}$, $\beta'_k \in \mathbb{R}_+$, $\sum_{k=1}^m \beta'_k = 1$. Finally, setting

$$\beta_k = \beta'_k \gamma, \qquad \gamma := \min\{\alpha_{i_k} : k = 1, \dots, m\},$$

we complete the proof of Proposition 4.1.

In the following lemma, we partition the domain of a function f so that, on each partition subset P, the function $f|_P$ is affinely homogeneous.

Lemma 4.2. *Let $f \in L_\infty(D; \mathbb{R}^d)$. Then there exists at most countable partition $\{P_i\}_{i \in I}$ of D consisting of measurable subsets of non-zero measure so that every $f|_{P_i}$ is affinely homogeneous.*

Proof. Consider the collection $\mathcal{A}$ of all families $\{D_i\}_{i \in I}$ of disjoint measurable subsets of D of positive measure for which $f|_{D_i}$ is affinely homogeneous. We order this collection by inclusion. Then, by Zorn's lemma we can find a maximal element $\{P_i\}_{i \in I} \in \mathcal{A}$. We claim that this is the required partition. Let $X = D \setminus \bigcup_{i \in I} P_i$. Suppose that $\lambda(X) > 0$. Since the set $\{0, 1, \dots, d\}$ is finite, there exists a minimal k for which there exists an affine linear subspace $W \subseteq \mathrm{Aff}(\sigma(f|_X))$, $\dim(W) = k$ and $\lambda(f^{-1}(W) \cap X) > 0$. Setting $P_0 = f^{-1}(W) \cap X$, we obtain that $f|_{P_0}$ is affinely homogeneous. However, this contradicts the maximality of $\{P_i\}_{i \in I}$. We conclude that $\lambda(X) = 0$, hence $\{P_i\}_{i \in I}$ is a partition of D. Lemma 4.2 is proved.

Theorem 4.3. *Let $f \in L_\infty([0,1]; \mathbb{R}^d)$, $\int f \, d\lambda = 0$. Then there exists at most countable partition of $[0,1]$ into measurable subsets $X_1, X_2, \dots$ such that*

(i) $\int_{X_n} f \, d\lambda = 0$, $n = 1, 2, \dots$;

(ii) *for any $n = 1, 2, \dots$, the function $f|_{X_n}$ is affinely partially homogeneous.*

Proof. Let $\{X_i\}_{i \in I}$ be a maximal collection of disjoint subsets of $[0,1]$ of positive measure satisfying (i) and (ii). Such collection exists by application of Zorn's lemma. Let $D = [0,1] \setminus \bigcup_{i \in I} X_i$. We show that $\lambda(D) = 0$. Namely, suppose that $\lambda(D) > 0$. Let $\{D_i\}_{i \in I}$ be the partition of D established in Lemma 4.2. We have

$$0 = \sum_{i \in I} \lambda(D_i) ⨍_{D_i} f \, d\lambda.$$

Now, by Proposition 4.1, for some $1 \leqslant m \leqslant d+1$, we can find $i_1, \dots, i_m \in I$ and $0 < \lambda_j \leqslant \lambda(D_{i_j})$ such that

$$0 = \sum_{j=1}^m \lambda_j ⨍_{D_{i_j}} f \, d\lambda.$$

We set $\lambda'_j = \lambda_j/\lambda(D_{i_j})$ so that $0 = \sum_{j=1}^m \lambda'_j \int_{D_{i_j}} f\,d\lambda$. Now, we can define non-atomic measures $\{\mu_i\}_{i=1}^d$ as $\mu_i(E) = \int_E f_i\,d\lambda$ for every Lebesgue measurable set $E \subset [0,1]$ and apply Theorem 2.3. As a result, we obtain measurable subsets $D'_{i_j} \subset D_{i_j}$ of non-zero measure with $\int_{D'_{i_j}} f\,d\lambda = \lambda'_j \int_{D_{i_j}} f\,d\lambda$. Now we set $X = \bigcup_{j=1}^m D'_{i_j}$, so that $\int_X f\,d\lambda = 0$. Next, by the properties of D_{i_j} and since $m \leqslant d+1$, $f|_X$ is affinely partially homogeneous. However, then the collection $\{X_i\}_{i\in I} \cup \{X\}$ would contradict the maximality of $\{X_i\}_{i\in I}$. We thus conclude that $\lambda(D) = 0$, and hence $\{X_i\}_{i\in I}$ is a partitions of $[0,1]$. Theorem 4.3 is proved.

§ 5. Shrinking lemmas

5.1. Obtaining positive constants. The following lemma will be required in the proof of Lemma 5.2. This lemma will be established for general mean zero integrable functions.

In the following lemma, $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^k$, and $(\,\cdot\,,\cdot\,)$ is the inner product. The norm $\|\cdot\|_1$ on $L_1(D;\mathbb{R}^k)$ is defined via the Euclidean norm $\|\cdot\|$ on $\mathbb{R}^k$. For $v \in \mathbb{R}^d$ and $f \in L_1(D;\mathbb{R}^d)$, by (v,f) we denote the function $t \mapsto (v, f(t))$, that is, the composition of f with the inner product. We also write $|f|$ for the function $t \mapsto \|f(t)\|$.

Lemma 5.1. *Let $D \subseteq [0,1]$ be a set of positive measure, and let $f \in L_1(D;\mathbb{R}^d)$ be such that $\int_D f\,d\lambda = 0$. Then there exist α, $\beta_{\min}$, $\beta_{\max}$, $\tau > 0$ such that $\lambda(\{(v,f)/(\|v\|\,|f|) > \alpha\} \cap \{\beta_{\min} < |f| < \beta_{\max}\}) > \tau$ for any non-zero $v \in \operatorname{Span}(\sigma(f))$.*

Proof. We argue by induction on the dimension d. The result for $d = 0$ is trivial, because in this case there are no non-zero vectors. For a fixed $d \geqslant 1$, assume that we have already proved the required result for $0 \leqslant j \leqslant d-1$. Let $f \in L_1([0,1];\mathbb{R}^d)$ be mean zero. Suppose first that $\operatorname{Span}(\sigma(f)) \neq \mathbb{R}^d$. By choosing an orthonormal basis for $\operatorname{Span}(\sigma(f))$ we can consider f as a mean zero function in $L_1(D;\mathbb{R}^k)$, where

$$k = \dim \operatorname{Span}(\sigma(f)).$$

By the induction hypothesis, we have $\alpha, \beta_{\min}, \beta_{\max}, \tau > 0$ such that, for every nonzero $v \in \operatorname{Span}(\sigma(f))$, the required result holds. Hence this result also holds with the same constants if we again consider f as a function in $L_1([0,1];\mathbb{R}^d)$, proving the required result in this case.

We can now assume that $\operatorname{Span}(\sigma(f)) = \mathbb{R}^d$. Under this assumption, $(v,f) \neq 0$ for any non-zero $v \in \mathbb{R}^d$.

Let us now find the numbers α, $\beta_{\min}$, $\beta_{\max}$, τ.

Setting $D_0 := D \setminus \{f = 0\}$, we have $\lambda(D_0) > 0$ and $\int_{D_0} f\,d\lambda = 0$. For non-zero $v \in \mathbb{R}^d$,

$$(v, f|_{D_0}) \neq 0.$$

Let $\mathbb{S}^{d-1}$ denote the $(d-1)$-dimensional unit sphere. For $v \in \mathbb{S}^{d-1}$, let the bounded function $h_v\colon D_0 \to \mathbb{R}$ be defined by $h_v = (v,f)/|f|$. By the Cauchy–Schwartz inequality, for $v, w \in \mathbb{S}^{d-1}$, we have,

$$|h_v^+ - h_w^+| = \frac{1}{|f|}|(v,f)^+ - (w,f)^+| \leqslant \frac{1}{|f|}|(v-w,f)| \leqslant \|v-w\|.$$

Hence the map $v \mapsto \|h_v^+\|_{L_\infty(D_0)}$ is continuous. We set

$$\alpha = \frac{1}{2} \min_{v \in \mathbb{S}^{d-1}} \|h_v^+\|_{L_\infty(D_0)}.$$

This number is possible by compactness of $\mathbb{S}^{d-1}$. We claim that

$$\|h_v^+\|_{L_\infty(D_0)} > 0$$

for every $v \in \mathbb{S}^{d-1}$. Indeed, suppose for a moment that this is not the case. Then $(v, f|_{D_0}) \leqslant 0$ almost everywhere. Now, since $\int_{D_0} f \, d\lambda = 0$, we have

$$\int_{D_0} (v, f) \, d\lambda = \left(v, \int_{D_0} f \, d\lambda \right) = 0.$$

This implies that $(v, f|_{D_0}) = 0$ almost everywhere, contradicting the above inequality $(v, f|_{D_0}) \neq 0$. Hence $\|h_v^+\|_{L_\infty(D_0)} > 0$, implying that $\alpha > 0$.

Now, for $v \in \mathbb{S}^{d-1}$, we define

$$\tau_v = \lambda(\{h_v > \alpha\}), \qquad \tau = \frac{1}{2} \inf_{v \in \mathbb{S}^{d-1}} \tau_v.$$

Note that $\tau_v > 0$ for all $v \in \mathbb{S}^{d-1}$, since $\|h_v^+\|_{L_\infty(D_0)} > \alpha$. We claim that $\tau > 0$.

Suppose that (v_n) is a sequence in $\mathbb{S}^{d-1}$ such that $\tau_{v_n} \to 0$. By compactness of $\mathbb{S}^{d-1}$, we can assume that v_n converges to some $v \in \mathbb{S}^{d-1}$. Choose $\varepsilon > 0$. Since $\{h_v > \alpha + 1/j\}$ increases to $\{h_v > \alpha\}$ as $j \to \infty$, and since D_0 has finite measure, we can choose a sufficiently small $\delta > 0$ such that

$$\lambda(\{h_v > \alpha\} \setminus \{h_v > \alpha + \delta\}) < \varepsilon.$$

Now, since $h_{v_n} \to h_v$ in $L_\infty(D_0)$, by the Cauchy–Schwarz inequality we can find N such that, for $n \geqslant N$,

$$\|h_v - h_{v_n}\|_\infty < \delta.$$

Now, for $n \geqslant N$, we have

$$\begin{aligned} \tau_v - \tau_{v_n} &\leqslant \lambda\big(\{h_v > \alpha\} \setminus (\{h_{v_n} > \alpha\})\big) \leqslant \lambda(\{h_v > \alpha\} \setminus \{h_v > \alpha + \delta\}) \\ &\quad + \lambda(\{h_v > \alpha + \delta\} \setminus \{h_{v_n} > \alpha\}) < \varepsilon + 0 = \varepsilon. \end{aligned}$$

As $\tau_{v_n} \to 0$, we have $\tau_v \leqslant \varepsilon$. Since ε is arbitrary, this means that $\tau_v = 0$, a contradiction. Hence, no such sequence (v_n) exist. Therefore, $\tau > 0$.

We now choose a small $\beta_{\min} > 0$ and a large $\beta_{\max} > 0$ such that

$$\lambda(D_0 \setminus \{\beta_{\min} < |f| < \beta_{\max}\}) < \frac{1}{2}\tau.$$

This is possible because $\lambda(D_0 \cap \{f = 0\}) = 0$ and D_0 has finite measure.

Now, for non-zero $v \in \mathbb{R}^d = \operatorname{Span}(\sigma(f))$, we have

$$\begin{aligned}\lambda\bigg(\bigg\{\frac{(v,f)}{\|v\|\,|f|} > \alpha\bigg\} \cap \{\beta_{\min} < |f| < \beta_{\max}\}\bigg)& \\ = \lambda(\{h_{v/\|v\|} > \alpha\} \cap \{\beta_{\min} < |f| < \beta_{\max}\})& \\ \geqslant \lambda(\{h_{v/\|v\|} > \alpha\}) - \lambda(D_0 \setminus \{\beta_{\min} < |f| < \beta_{\max}\})& \\ \geqslant \tau_{v/\|v\|} - \frac{1}{2}\tau \geqslant 2\tau - \frac{1}{2}\tau > \tau,&\end{aligned}$$

verifying the claim. Lemma 5.1 is proved

5.2. Changing the mean zero condition for subsets. We will now deal with a domain $D \subseteq [0,1]$ of positive measure, and a mean zero function $f \in L_1(D;V)$. The following lemma allows us to obtain a slightly smaller compact subset $E \subseteq D$ for which $f|_E$ is continuous and mean zero. This result will be needed in the proof of Lemma 5.4.

Lemma 5.2. *Let V be a finite-dimensional normed real space, let $D \subseteq [0,1]$ be of positive measure, and let $f \in L_1(D;V)$ be mean zero. Then, for $\varepsilon > 0$, there is $\delta > 0$ such that, for every measurable subset $D' \subseteq D$ with $\lambda(D \setminus D') < \delta$, and every vector $u \in \operatorname{Span}(\sigma(f))$ with $\|u\| \leqslant \delta$, there is a compact subset $E \subseteq D' \cap (\inf D, \sup D)$, $\lambda(D \setminus E) < \varepsilon$, such that $\int_E (f+u)\,d\lambda = 0$, and that $f|_E$ is continuous.*

Proof. As the norms on any given finite-dimensional vector space are equivalent, we can assume without loss of generality that V is $\mathbb{R}^d$ with the Euclidean norm.

Let D, f and ε be as stated. Applying Lemma 5.1 to D and f, we find positive constants α, $\beta_{\min}$, $\beta_{\max}$ and τ from this lemma. In particular, $\beta_{\min} \in (0, \beta_{\max})$ and $\alpha \in (0,1)$, hence we can define $\gamma := \sqrt{1 - \alpha^2\beta_{\min}/(4\beta_{\max})} \in (0,1)$ and $\rho := \alpha/(2\beta_{\max}(1-\gamma)) > 0$.

We introduce a continuous non-decreasing function $I_{\sup}\colon [0, \lambda(D)] \to [0, \|f\|_1]$ by

$$I_{\sup}(s) = \sup_{U \subseteq D,\, \lambda(U) = s} \int_U |f|\,d\lambda.$$

We also set

$$\delta' := \frac{1}{2}\min\bigg\{\frac{\tau}{4(1+\rho)}, \frac{\tau\beta_{\max}}{2}, \frac{\tau}{2\rho}, \frac{\varepsilon}{2(1+\rho)}, \beta_{\max}, \frac{\alpha\beta_{\min}}{8}\bigg\} > 0, \tag{5.1}$$

$$\delta := \frac{1}{2}\min\{\delta', I_{\sup}^{-1}(\delta')\} > 0. \tag{5.2}$$

Now, we choose a measurable subset $D' \subseteq D$ such that $\lambda(D \setminus D') < \delta$ and find $u \in \operatorname{Span}(\sigma(f))$ such that $\|u\| \leqslant \delta$. For $i = 1, \dots, d$, let f_i be the coordinate functions of f_i with respect to the standard basis. By Theorem 2.1, there exist compact subsets $K_i \subseteq D' \cap (\inf D, \sup D)$ such that $\lambda(D' \setminus K_i) < \delta/d$ and $f_i|_{K_i}$ is continuous, $i = 1, \dots, d$. Now if $K := \bigcap_{i=1}^d K_i$ then $f|_K$ is continuous and bounded, as K is compact. Hence

$$\lambda(D \setminus K) = \lambda(D \setminus D') + \lambda(D' \setminus K) \leqslant \delta + d\,\frac{\delta}{d} = 2\delta.$$

We now set $E_0 := K$ and define

$$v_0 := \int_{E_0} (f+u)\, d\lambda.$$

Since f is mean zero on D, we have

$$\|v_0\| \leqslant \left\| \int_{E_0} u\, d\lambda \right\| + \left\| \int_{D\setminus E_0} f\, d\lambda \right\| \leqslant \lambda(E_0)\|u\| + I_{\sup}(\lambda(D\setminus E_0)) \leqslant \delta + I_{\sup}(2\delta) \leqslant 2\delta'.$$

We will now inductively define compact sets $(E_j)_{j\geqslant 1}$ and mutually disjoint sets $(A_j)_{j\geqslant 1}$ in D, and define the vectors $v_j := \int_{E_j} (f+u)\, d\lambda$ such that, for $j \geqslant 1$, the following holds:

1) $E_j = E_{j-1} \setminus A_j$;
2) $A_j \subseteq E_{j-1} \cap \{\beta_{\min} < |f| < \beta_{\max}\} \cap \{(v_{j-1}, f)/(\|v_{j-1}\|\, |f|) > \alpha\}$;
3) $\lambda(A_j) = \alpha\|v_{j-1}\|/(2\beta_{\max})$;
4) $\|v_j\| \leqslant \gamma^j \|v_0\|$;
5) $\lambda(E_0 \setminus E_j) \leqslant \rho\|v_0\|$.

Assume that the required E_l and v_l are constructed for $l < j$ and A_l is constructed for $0 < l < j$. Let us construct E_j, A_j, and v_j. Assume first that $v_{j-1} = 0$. We define $A_j := \varnothing$ and $E_j := E_{j-1}$ so that $v_j = v_{j-1} = 0$ and $\lambda(E_0 \setminus E_j) = \lambda(E_0 \setminus E_{j-1}) \leqslant \rho\|v_0\|$. In this case, all the conditions are satisfied, and we are done. So, can assume that $v_{j-1} \neq 0$. Since $v_{j-1} \in \operatorname{Span}(\sigma(f))$ is non-zero, we have

$$\begin{aligned}
&\lambda\left(E_{j-1} \cap \{\beta_{\min} < |f| < \beta_{\max}\} \cap \left\{ \frac{(v_{j-1}, f)}{\|v_{j-1}\|\, |f|} > \alpha \right\}\right) \\
&\quad > \lambda\left(\{\beta_{\min} < |f| < \beta_{\max}\} \cap \left\{ \frac{(v_{j-1}, f)}{\|v_{j-1}\|\, |f|} > \alpha \right\}\right) - \lambda(D \setminus E_{j-1}) \\
&\quad \geqslant \tau - \lambda(D\setminus E_0) - \lambda(E_0 \setminus E_{j-1}) \geqslant \tau - 2\delta - \rho\|v_0\| \geqslant \tau - (2+2\rho)\delta' \geqslant \frac{1}{2}\tau.
\end{aligned}$$

Now for $r \in [0,1]$, we set

$$B_r = (0,r) \cap E_{j-1} \cap \{\beta_{\min} < |f| < \beta_{\max}\} \cap \left\{ \frac{(v_{j-1}, f)}{\|v_{j-1}\|\, |f|} > \alpha \right\}.$$

Since $f|_K$ is continuous, $E_{j-1} \cap \{\beta_{\min} < |f| < \beta_{\max}\}$ is open in E_{j-1}. On this set, f does not vanish. In particular, $(v_{j-1}, f)/(\|v_{j-1}\|\, |f|)$ is continuous on this set. Thus, the set

$$E_{j-1} \cap \{\beta_{\min} < |f| < \beta_{\max}\} \cap \left\{ \frac{(v_{j-1}, f)}{\|v_{j-1}\|\, |f|} > \alpha \right\}$$

is open in $E_{j-1} \cap \{\beta_{\min} < |f| < \beta_{\max}\}$, and therefore, it is also open in E_{j-1}. So, the sets B_r are open in E_{j-1} for every $r \in [0,1]$.

By the induction step, since $\alpha, \gamma \in (0,1)$, and using the bound on $\|v_0\|$ and the definition of δ', we have

$$\frac{\alpha\|v_{j-1}\|}{2\beta_{\max}} \leqslant \frac{\alpha\gamma^{j-1}\|v_0\|}{2\beta_{\max}} \leqslant \frac{\|v_0\|}{2\beta_{\max}} \leqslant \frac{2\delta'}{2\beta_{\max}} < \frac{1}{2}\tau.$$

Now, since $\lambda(B_0) = 0$ and $\lambda(B_1) \geqslant \tau/2$, we can find $r_0 \in [0, 1)$ such that

$$\lambda(B_{r_0}) = \frac{\alpha\|v_{j-1}\|}{2\beta_{\max}}.$$

We set $A_j := B_{r_0}$ and $E_j := E_{j-1} \setminus A_j$, so that E_j is compact and so that conditions (1), (2) and (3) are met.

Next, we define $v_j := \int_{E_j}(f + u)\, d\lambda = v_{j-1} - \int_{A_j}(f + u)\, d\lambda$, and write $w := \int_{A_j}(f + u)\, d\lambda$. We have

$$\|w\| \leqslant \int_{A_j} (|f| + \delta)\, d\lambda \leqslant \lambda(A_j)(\beta_{\max} + \delta) \leqslant 2\beta_{\max}\lambda(A_j) = \alpha\|v_{j-1}\|. \tag{5.3}$$

By definition of A_j (step 2) of the induction),

$$\int_{A_j} (v_{j-1}, f)\, d\lambda > \alpha\|v_{j-1}\| \int_{A_j} |f|\, d\lambda, \tag{5.4}$$

$$\int_{A_j} |f|\, d\lambda > \beta_{\min}\lambda(A_j). \tag{5.5}$$

We have

$$\begin{aligned}
\|v_j\|^2 &= \|v_{j-1} - w\|^2 = \|v_{j-1}\|^2 + \|w\|^2 - 2(v_{j-1}, w) \\
&= \|v_{j-1}\|^2 + \|w\|^2 - 2\int_{A_j} (v_{j-1}, f)\, d\lambda - 2\int_{A_j} (v_{j-1}, u)\, d\lambda \\
&\overset{(5.4)}{\leqslant} \|v_{j-1}\|^2 + \|w\|^2 - 2\alpha\|v_{j-1}\| \int_{A_j} |f|\, d\lambda + 2\delta\|v_{j-1}\|\lambda(A_j) \\
&= \|v_{j-1}\|^2 + \|w\|^2 - 2\alpha\|v_{j-1}\| \int_{A_j} (|f| + \delta)\, d\lambda + 4\delta\|v_{j-1}\|\lambda(A_j) \\
&\leqslant \|v_{j-1}\|^2 + (\|w\| - 2\alpha\|v_{j-1}\|) \int_{A_j} (|f| + \delta)\, d\lambda + 4\delta\|v_{j-1}\|\lambda(A_j) \\
&\overset{(5.3)}{\leqslant} \|v_{j-1}\|^2 - \alpha\|v_{j-1}\| \int_{A_j} (|f| + \delta)\, d\lambda + 4\delta\|v_{j-1}\|\lambda(A_j) \\
&\leqslant \|v_{j-1}\|^2 - \alpha\|v_{j-1}\| \int_{A_j} |f|\, d\lambda + 4\delta\|v_{j-1}\|\lambda(A_j) \\
&\overset{(5.5)}{\leqslant} \|v_{j-1}\|^2 - \alpha\|v_{j-1}\|\, \beta_{\min}\lambda(A_j) + 4\delta\|v_{j-1}\|\lambda(A_j) \\
&= \|v_{j-1}\|^2\Big(1 - (\alpha\beta_{\min} - 4\delta)\frac{\lambda(A_j)}{\|v_{j-1}\|}\Big) \overset{(5.1),(5.2)}{\leqslant} \|v_{j-1}\|^2\Big(1 - \frac{\alpha\beta_{\min}}{2}\,\frac{\lambda(A_j)}{\|v_{j-1}\|}\Big) \\
&= \|v_{j-1}\|^2\Big(1 - \frac{\alpha\beta_{\min}}{2}\,\frac{\alpha}{2\beta_{\max}}\Big) = \gamma^2\|v_{j-1}\|^2.
\end{aligned}$$

Hence, we obtain $\|v_j\| \leqslant \gamma\|v_{j-1}\| \leqslant \gamma^j\|v_0\|$, which verifies (4).

At last, we have

$$\lambda(E_0 \setminus E_j) = \sum_{n=1}^{j} \lambda(A_n) = \frac{\alpha}{2\beta_{\max}} \sum_{n=1}^{j} \|v_{n-1}\| \leqslant \frac{\alpha}{2\beta_{\max}} \sum_{n=1}^{j} \gamma^{n-1} \|v_0\|$$
$$\leqslant \frac{\alpha\|v_0\|}{2\beta_{\max}} \sum_{n=0}^{\infty} \gamma^n = \frac{\alpha\|v_0\|}{2\beta_{\max}(1-\gamma)} = \rho\|v_0\|$$

and so the inductive construction is complete. Setting $E = \bigcap_{j=0}^{\infty} E_j$, we obtain a compact subset of $D' \cap (\inf D, \sup D)$ such that $\int_E (f+u)\,d\lambda = \lim_{j\to\infty} v_j = 0$. Furthermore,

$$\lambda(D \setminus E) = \lambda(D \setminus E_0) + \sup_{j\geqslant 1} \lambda(E_0 \setminus E_j) \leqslant 2\delta + \rho\|v_0\| \leqslant (2+2\rho)\delta' < \varepsilon.$$

Moreover, $E \subseteq K$ implies that $f|_E$ is continuous. Lemma 5.2 is proved

5.3. Arbitrary shrinking and rational splitting. In the following lemma, for a set K and a mean zero function $f \in L_\infty(K;V)$, we find a compact subset $E \subseteq K$ of prescribed measure such that $\int_E f\,d\lambda = 0$.

Lemma 5.3 (arbitrary shrinking). *Let $K \subseteq [0,1]$ be a compact set of positive measure, and let $f \in L_\infty(K;V)$ be mean zero. Then, for $r \in (0, \lambda(K))$, there is a compact set $E \subseteq K \cap (\inf K, \sup K)$ such that $\lambda(K \setminus E) = r$ and $\int_E f\,d\lambda = 0$.*

Proof. Consider the collection $\mathcal{A}$ of all compact subsets E of $K \cap (\inf K, \sup K)$ such that $\int_E f\,d\lambda = 0$ and $\lambda(K \setminus E) \leqslant r$. Note that this collection is nonempty by Lemma 5.2. Let $\mathcal{A}$ be equipped with the order inverse to the inclusion order. The elements of $\mathcal{A}$ whose symmetric difference is a nullset will be identified. Now, for a chain $\{E_i\}_{i\in I}$ in $\mathcal{A}$ (note that I will either be finite or countable), the set $E' := \bigcap_{i\in I} E_i$ is a compact subset of $K \cap (\inf K, \sup K)$, $\lambda(K \setminus E') = \sup_{i\in I} \lambda(K \setminus E_i) \leqslant r$, and $\int_{E'} f\,d\lambda = 0$. Hence $E' \in \mathcal{A}$ is an upper bound for the chain. Therefore, by Zorn's lemma, there exists a maximal element E in $\mathcal{A}$. Suppose that $\lambda(K \setminus E) < r$. Then, setting $\varepsilon = r - \lambda(K \setminus E) > 0$ and applying Lemma 5.2, we obtain a compact subset $\widetilde{E} \subseteq E$, $\lambda(E \setminus \widetilde{E}) < \varepsilon$ and such that $\int_{\widetilde{E}} f\,d\lambda = 0$. We thus have $\lambda(K \setminus \widetilde{E}) < \lambda(K \setminus E) + \varepsilon = r$. However, this contradicts the maximality of E. Hence $\lambda(K \setminus E) = r$, proving Lemma 5.3.

In the following lemma we construct a subset which, in addition to satisfying the properties required in Lemma 5.3, also asserts that certain ratios are dyadic rationals.

Lemma 5.4 (rational splitting). *Let V be a finite-dimensional real vector space, let $N \in \mathbb{N}$, let $K = K_1 \cup \cdots \cup K_N \subseteq [0,1]$ be of positive measure, where $K_i \subseteq [0,1]$ are such that $\lambda(K_i \cap K_j) = 0$ whenever $i \neq j$, and let $f \in L_\infty(K;V)$ be mean zero and such that, for $i \geqslant 2$, there exists a subset $B_i \subseteq K_i$ of positive measure such that $\lambda(f|_{B_i}^{-1}(W)) = 0$ for every proper affine subspace $W \subsetneq \operatorname{Aff}(\sigma(f))$. Then, for any $R \in (0, \lambda(K))$, there exists a set $E = E_1 \cup \cdots \cup E_N$ such that $E_i \subseteq K_i \cap (\inf K_i, \sup K_i)$ is compact, $\lambda(E) = R$, $\int_E f\,d\lambda = 0$, and $\lambda(E_i)/\lambda(E) \in \mathbb{Q}_2$ for all $1 \leqslant i \leqslant N$. Here, $\mathbb{Q}_2$ is the set of all dyadic rationals.*

Proof. We argue by induction on N. For $N = 1$ we have $K = K_1$ and we can simply apply Lemma 5.3. Indeed, $\lambda(E_1)/\lambda(E) = 1 \in \mathbb{Q}_2$, which proves the assertion for $N = 1$. Thus, let $N \geqslant 2$ and assume that the assertion holds for $N - 1$. We show that it also holds for N. Let $K = K_1 \cup \cdots \cup K_N$ as stated. Let $f \in L_\infty(K; V)$ be mean zero and such that, for all $i \geqslant 2$ the set $B_i \subseteq K_i$ exists as stated. Furthermore, let $r > 0$ be such that $r < \lambda(K) - R$ and $r < \min\{\lambda(K_i) \colon 1 \leqslant i \leqslant N\} \setminus \{0\}$. We will assume that $\lambda(K_i) > 0$ for every $i = 1, \ldots, N$, since otherwise we can set $E_i = \varnothing$ and apply the induction hypothesis to $K \setminus K_i$, which then yields the result. For convenience, we set $\widetilde{K} := K_1 \cup \cdots \cup K_{N-1}$. We denote $v := ⨍_{K_N} f\, d\lambda$, and define

$$h_1 = f|_{\widetilde{K}} - ⨍_{\widetilde{K}} f\, d\lambda = f|_{\widetilde{K}} + \frac{\lambda(K_N)}{\lambda(\widetilde{K})} v, \qquad h_2 = f|_{K_N} - ⨍_{K_N} f\, d\lambda = f|_{K_N} - v,$$

so that $h_1 \in L_\infty(\widetilde{K}; V)$ and $h_2 \in L_\infty(K_N; V)$ are mean zero. For every $A \subset V$, $x \in V$, we have $\operatorname{Aff}(A + x) = \operatorname{Aff}(A) + x$. Therefore,

$$\operatorname{Aff}(\sigma(h_2)) = \operatorname{Aff}\Big(\sigma(f|_{K_N} - ⨍_{K_N} f\, d\lambda)\Big) = \operatorname{Aff}(\sigma(f|_{K_N})) - ⨍_{K_N} f\, d\lambda.$$

Thus, $W := \operatorname{Aff}(\sigma(h_2)) + ⨍_{K_N} f\, d\lambda = \operatorname{Aff}(\sigma(f|_{K_N})) \subset \operatorname{Aff}(\sigma(f))$, in particular, W is an affine subspace of $\operatorname{Aff}(\sigma(f))$.

Since $f|_{B_N} = h_2|_{B_N} + ⨍_{K_N} f\, d\lambda$, it follows that $\lambda(f|_{B_N}^{-1}(W)) = \lambda(B_N) > 0$. Therefore, by the assumption on B_N we must have the equality $W = \operatorname{Aff}(\sigma(f))$ (as W cannot be its proper subspace).

Now since f, h_2 are mean zero, we also have $\operatorname{Span}(\sigma(f)) = \operatorname{Aff}(\sigma(f))$ and $\operatorname{Span}(\sigma(h_2)) = \operatorname{Aff}(\sigma(h_2))$. Hence $\operatorname{Span}(\sigma(h_2)) = \operatorname{Span}(\sigma(f))$.

Now, by Lemma 5.2 we can find $\delta > 0$ such that, for $u \in \operatorname{Span}(\sigma(h_2))$, $\|u\| \leqslant \delta$, there is a compact set $\widetilde{E_N} \subseteq K_N$ such that $\lambda(K_N \setminus \widetilde{E_N}) < (1/4)\lambda(K_N) r$ and $\int_{E_2} (h_2 + u)\, d\lambda = 0$.

We may assume that $((\lambda(\widetilde{K})/\lambda(K_N))(1 - \delta') + 1)^{-1} \in \mathbb{Q}_2$ for some $0 < \delta' < \min\{1, r/4, \delta/(\|v\| + 1)\}$. We set $u = \delta' v$, so that $u \in \operatorname{Span}(\sigma(f)) = \operatorname{Span}(\sigma(h_2))$ and $\|u\| \leqslant \delta$. By the choice of δ, there exists a compact set $\widetilde{E_N} \subseteq K_N$ such that $\lambda(K_N \setminus \widetilde{E_N}) < (1/4)\lambda(K_N) r$ and $\int_{\widetilde{E_N}} (h_2 + u)\, d\lambda = 0$. Now, we set

$$\begin{aligned}\widetilde{r} &:= \lambda(\widetilde{K}) - \frac{\lambda(\widetilde{K})}{\lambda(K_N)} \lambda(\widetilde{E_N})(1 - \delta') = \lambda(\widetilde{K}) - \frac{\lambda(\widetilde{K})}{\lambda(K_N)} \big(\lambda(K_N) - \lambda(K_N \setminus \widetilde{E_N})\big)(1 - \delta') \\ &= \delta' \lambda(\widetilde{K}) + \frac{\lambda(\widetilde{K})}{\lambda(K_N)} \lambda(K_N \setminus \widetilde{E_N})(1 - \delta').\end{aligned}$$

So, $0 < \widetilde{r} \leqslant \delta' + (1/\lambda(K_N))\lambda(K_N \setminus \widetilde{E_N}) < r/4 + r/4 = r/2$. In particular, $\widetilde{r} < \lambda(K_i)$ for $i = 1, \ldots, N - 1$. We now apply the induction hypothesis to obtain a set $\widetilde{E} = \widetilde{E_1} \cup \cdots \cup \widetilde{E_{N-1}} \subseteq \widetilde{K}$, where $\widetilde{E_i} \subseteq K_i$ are compact, $\lambda(\widetilde{K} \setminus \widetilde{E}) = \widetilde{r}$, $\int_{\widetilde{E}} h_1\, d\lambda = 0$, and $\lambda(\widetilde{E_i})/\lambda(\widetilde{E}) \in \mathbb{Q}_2$ for $i = 1, \ldots, N - 1$.

Next,

$$\int_{\widetilde{E}\cup\widetilde{E_N}} f\,d\lambda = \int_{\widetilde{E}} h_1 - \frac{\lambda(K_N)}{\lambda(\widetilde{K})} v\,d\lambda + \int_{\widetilde{E_N}} (h_2+v)\,d\lambda$$
$$= -\frac{\lambda(K_N)}{\lambda(\widetilde{K})}\lambda(\widetilde{E})v + \lambda(\widetilde{E_N})v + \int_{\widetilde{E}} h_1\,d\lambda + \int_{\widetilde{E_N}} h_2\,d\lambda$$
$$= -\frac{\lambda(K_N)}{\lambda(\widetilde{K})}\lambda(\widetilde{E})v + \lambda(\widetilde{E_N})v - \lambda(\widetilde{E_N})u = \biggl(-\frac{\lambda(K_N)}{\lambda(\widetilde{K})}\lambda(\widetilde{E}) + \lambda(\widetilde{E_N}) - \lambda(\widetilde{E_N})\delta'\biggr)v$$
$$= \biggl(-\frac{\lambda(K_N)}{\lambda(\widetilde{K})}(\lambda(\widetilde{K})-\widetilde{r}) + \lambda(\widetilde{E_N})(1-\delta')\biggr)v = 0,$$
$$\frac{\lambda(\widetilde{E_N})}{\lambda(\widetilde{E}\cup\widetilde{E_N})} = \frac{\lambda(\widetilde{E_N})}{\lambda(\widetilde{K})-\widetilde{r}+\lambda(\widetilde{E_N})} = \frac{\lambda(\widetilde{E_N})}{(\lambda(\widetilde{K})/\lambda(K_N))\lambda(\widetilde{E_N})(1-\delta')+\lambda(\widetilde{E_N})}$$
$$= \frac{1}{(\lambda(\widetilde{K})/\lambda(K_N))(1-\delta')+1} \in \mathbb{Q}_2.$$

Now, for $i=1,\dots,N-1$, we have

$$\frac{\lambda(\widetilde{E_i})}{\lambda(\widetilde{E}\cup\widetilde{E_N})} = \frac{\lambda(\widetilde{E_i})}{\lambda(\widetilde{E})}\,\frac{\lambda(\widetilde{E})}{\lambda(\widetilde{E}\cup\widetilde{E_N})} \in \mathbb{Q}_2.$$

Finally, we have $\lambda(K\setminus(\widetilde{E}\cup\widetilde{E_N})) = \lambda(\widetilde{K}\setminus\widetilde{E}) + \lambda(K_N\setminus\widetilde{E_N}) \leqslant \widetilde{r}+r/4 < r$.

We set $E' = \widetilde{E}\cup\widetilde{E_N} = \bigcup_{i=1}^N \widetilde{E_i}$, so that $\lambda(E') > R$. Indeed, the scalar r was chosen to satisfy $0<r<\lambda(K)-R$, and hence $\lambda(K)-\lambda(E') = \lambda(K\setminus E') \leqslant r < \lambda(K)-R$. It suffices to shrink the sets $\widetilde{E_i}$ for $i=1,\dots,N$ by a fixed ratio, so that the measure of their union would be exactly R. Let us explain this construction.

Lemma 5.3 guarantees each set $\widetilde{E_i}$ contains a compact subset E_i such that

$$\lambda(E_i) = \frac{\lambda(\widetilde{E_i})R}{\lambda(E')}, \qquad ⨍_{E_i} f\,d\lambda = ⨍_{\widetilde{E_i}} f\,d\lambda$$

and for which $\inf\widetilde{E_i}, \sup\widetilde{E_i} \notin E_i$. Putting $E = \bigcup_{i=1}^N E_i$, we see that $f|_E$ is mean zero, $\lambda(E)=R$ and $\lambda(E_i)/\lambda(E) = \lambda(\widetilde{E_i})/\lambda(E') \in \mathbb{Q}_2$ for $i=1,\dots,N$. This proves the required result for N, completing the induction. Lemma 5.4 is proved.

§ 6. Solutions for the homological equation over the Cantor set

Let $q\in\mathbb{N}$, $r\in\mathbb{R}$, and let the set

$$\mathcal{C}(q,r) = \{1,\dots,q\}\times\{1,2\}^{\mathbb{N}},$$

be equipped with the Tikhonov topology and a product measure

$$\mu = \mu_1\times\mu_2^{\mathbb{N}}$$

such that $\mu_1(\{i\}) = r/q$, $i=1,\dots,q$; $\mu_2(\{j\}) = 1/2$, $j=1,2$.

The set $\mathcal{C}(q,r)$ is a Cantor type set with $\mu(\mathcal{C}(q,r)) = r$. Let V be a finite-dimensional vector space. Let $C(\mathcal{C}(q,r);V)$ be the Banach space of continuous V-valued functions on $\mathcal{C}(q,r)$.

Let p_0 be the mapping from $\mathcal{C}(q,r)$ onto $\{1,\dots,q\}$, defined by $p_0(i;i_1,i_2,\dots) = i$, and let $\mathcal{C}(q,r,i) = p_0^{-1}(i)$, $i = 1,\dots,q$. For brevity, we set

$$C_V = \frac{8\dim(V)^2}{\log 1.5}(S_V + 1).$$

Recall that the diameter of a subset X of V is defined by

$$\mathrm{Diam}(X) = \sup_{x,y\in X} \|x - y\|.$$

Theorem 6.1. *Let V be a finite-dimensional real normed vector space, and let $0 \neq f \in C(\mathcal{C}(q,r);V)$ be mean zero function. We set*

$$a = \frac{\max_i\{\mathrm{Diam}(f(\mathcal{C}(q,r,i)))\}}{\|f\|}.$$

Then there exist $g \in C(\mathcal{C}(q,r);V)$, $\|g\| \leqslant (S_V + a(1 + C_V))\|f\|$, and a measure preserving continuous invertible transformation T of $\mathcal{C}(q,r)$ such that $f = g\circ T - g$.

Moreover, the system of sets $\Gamma = \{\mathcal{C}(q,r,i),\ i = 1,\dots,q\}$, can be labelled so that

$$\Gamma = \{X_1,\dots,X_q\}, \qquad T(X_i) = X_{i+1}, \quad i < q, \qquad T(X_q) = X_1$$

and $\|g|_{X_1}\| \leqslant (1 + C_V)a\|f\|$.

Proof. For every $n \in \mathbb{N}$, we denote by p_n the mapping from $\mathcal{C}(q,r)$ onto $\{1,\dots,q\} \times \{1,2\}^n$ defined by

$$p_n(i;i_1,\dots,i_n,i_{n+1},\dots) = (i;i_1,\dots,i_n).$$

For $n \geqslant 0$, let

$$v_n\colon \{1,\dots,q\} \times \{1,2\}^n \to \{1,\dots,2^n q\}$$

be the function arranging the elements in $\{1,\dots,q\} \times \{1,2\}^n$ in lexicographical order. Next, for $i \in \{1,\dots,2^n q\}$, we denote

$$I_i^n = (p_n^{-1}(v_n^{-1}(i)).$$

The sets I_i^n, $i \in \{1,\dots,2^n q\}$, $n \in \mathbb{N}$ are clopen and form a base of the topology in $\mathcal{C}(q,r)$. Clearly, we have

$$\{I_i^0\colon i = 1,\dots,q\} = \{\mathcal{C}(q,r,i)\colon i = 1,\dots,q\}.$$

Let $f_n = \sum_{i=1}^{2^n q} \chi_{I_i^n} ⨍_{I_i^n} f\,d\mu$. Then $f_n \in C(\mathcal{C}(q,r);V)$, $\|f_n - f\| \to 0$ as $n \to \infty$. Hence there exists a sequence $(n_k)_{k\geqslant 1}$ of natural numbers such that, for $n \geqslant n_k$,

$$\|f_n - f\| \leqslant 2^{-k-2}C_V^{-1}a\|f\|.$$

Setting,

$$h_0 = f_0, \qquad h_1 = f_{n_1} - f_0, \quad \|h_1\| \leqslant a\|f\|,$$
$$h_k = f_{n_k} - f_{n_{k-1}} \quad \Longrightarrow \quad \|h_k\| \leqslant 2^{-k} C_V^{-1} a\|f\|, \qquad k > 1$$

we have

$$f = \sum_{k=0}^{\infty} h_k.$$

Let a_i be the value of h_0 taken on I_i^0 for $1 \leqslant i \leqslant q$. As $\int f d\mu = 0$ we have $\sum_{i=1}^q a_i = 0$, so that there is a permutation π of $\{1, \dots, q\}$ such that $\left\|\sum_{i=1}^m a_{\pi(i)}\right\| \leqslant S_d\|h_0\|$ for $0 \leqslant m \leqslant q$. Now, let T_0 be the measure preserving continuous cyclic transformation of $\mathcal{C}(q, r)$ sending $I_{\pi(i)}^0$ to $I_{\pi(i+1)}^0$ for $1 \leqslant i \leqslant q-1$ and sending $I_{\pi(q)}^0$ to $I_{\pi(1)}^0$. We now denote by $g_0 \colon \mathcal{C}(q, r) \to V$ the continuous function which assumes on $I_{\pi(l)}^0$ the value $\sum_{i=1}^{l-1} a_{\pi(i)}$ for $l = 2, \dots, q$ and which vanishes on the set $I_{\pi(1)}^0$. Hence $\|g_0\| \leqslant S_d\|f_0\| \leqslant S_d\|f\|$, and, for $l = 2, \dots, q$ and $t \in I_{\pi(l)}^0$, we have

$$g_0(T_0(t)) - g_0(t) = \sum_{i=1}^{l} a_{\pi(i)} - \sum_{i=1}^{l-1} a_{\pi(i)} = a_{\pi(l)} = f_0(t).$$

For $l = 1$ and $t \in I_{\pi(1)}$, we have

$$g_0(T_0(t)) - g_0(t) = \sum_{i=1}^{1} a_{\pi(i)} - 0 = a_{\pi(1)} = h_0(t).$$

Proceeding as in [8], for each $k \geqslant 0$, we denote $J_k = \{I_i^{n_k} \colon 1 \leqslant i \leqslant 2^{n_k} q\}$, and construct a sequence $\{T_k\}_{k=0}^{\infty}$ of continuous automorphisms T_k of $\mathcal{C}(q, r)$ and functions $\{g_k\}_{k=1}^{\infty}$, $g_k \in C(\mathcal{C}(q, r); V)$ satisfying:

(i) T_k is a cyclic permutation of the sets of J_k;

(ii) T_{k+1} extends T_k in the sense that if $I \in J_k$, $I' \in J_{k+1}$ and $I' \subseteq I$ then $T_{k+1}(I') \subseteq T_k(I)$;

(iii) $\|g_k\| \leqslant C_V\|h_k\|$;

(iv) g_k is constant on each $I \in J_k$;

(v) $h_k = g_k \circ T_k - g_k$ on $\mathcal{C}(q, r)$.

Now, suppose that transformations $T_0, \dots, T_k$ and functions $g_0, \dots, g_k$ with given properties are constructed. For convenience, we set $n = |J_k|$ and $m = |J_{k+1}|/|J_k|$. Let $I_1, \dots, I_n$ be the sets from J_k, enumerated so that $T_k(I_i) = I_{i+1}$ for $i < n$ and $T_k(I_n) = I_1$. This can be done since T_k is a cyclic permutation of the sets of J_k. Furthermore, for $i = 1, \dots, n$, for $j = 1, \dots, m$, let $I_{i,j}|$ be all sets from J_{k+1} that lie in I_i. Let $a_{i,j}$ be the value of the function h_{k+1} on $I_{i,j}$. Since

$$\int_{I_i} h_{k+1}\, d\lambda = \sum_{j=1}^{m} \int_{I_{i,j}} (f_{n_{k+1}} - f_{n_k})\, d\lambda = 0 \quad \forall\, I_i \in J_k,$$

it follows that $\sum_{j=1}^m a_{i,j} = 0$ for all $i = 1, \dots, n$. In addition, $\|a_{i,j}\| \leqslant \|h_{k+1}\|$ for all $i = 1, \dots, n$, $j = 1, \dots, m$. Therefore, by Theorem 3.3, there exist cyclic permutations $\pi_1, \dots, \pi_n$ of $\{1, \dots, m\}$ such that

$$\Big\|\sum_{i=1}^{l} a_{i,\pi_i(j)}\Big\| \leqslant M\|h_{k+1}\|$$

for $l = 1, \dots, n$ and $j = 1, \dots, m$, where $M = 8d^2/\log 1.5$. Consider the a measure preserving homeomorphism $T_{k+1}\colon \mathcal{C}(q,r) \to \mathcal{C}(q,r)$ defined by

$$T_{k+1}(I_{i,\pi_i(j)}) = I_{i+1,\pi_{i+1}(j)}, \qquad i = 1, \dots, n-1, \quad j = 1, \dots, m.$$

We set

$$b_j = \sum_{i=1}^{n} a_{i,\pi_i(j)}, \qquad j = 1, \dots, m.$$

Since $\sum_{j=1}^{m} b_j = \sum_{i=1}^{n}\sum_{j=1}^{m} a_{i,j} = 0$ and $\|b_j\| \leqslant M\|h_{k+1}\|$, there exists a cyclic permutation π_0 of $1, \dots, m$ such that

$$\Big\|\sum_{j=1}^{l} b_{\pi_0(j)}\Big\| \leqslant MS_V\|h_{k+1}\| \quad \forall\, l = 1, \dots, m.$$

We set

$$T_{k+1}(I_{n,\pi_n(\pi_0(j))}) = I_{1,\pi_1(\pi_0(j+1))} \quad \forall\, j = 1, \dots, m-1$$

and define

$$T_{k+1}(I_{n,\pi_n(\pi_0(m))}) = I_{1,\pi_1(\pi_0(1))}.$$

Next, we have

$$\Big\|\sum_{r=0}^{l} h_{k+1}(T_{k+1}^r(t))\Big\| = \Big\|\sum_{j=1}^{p-1} b_{\pi_0(j)} + \sum_{i=1}^{z} a_{i,\pi_i(\pi_0(p))}\Big\| \leqslant C_V\|h_{k+1}\|,$$

where $l+1 = (p-1)n + z$ for some $p \in \{1, \dots, m\}$ and $z \in \{1, \dots, n\}$, for every $t \in I_{1,\pi_1(\pi_0(1))}$ and every $l = 0, \dots, nm-1$.

Now, let us define the function g_{k+1} by $\sum_{r=0}^{l-1} h_{k+1}(T_{k+1}^r(t))$ on $T_{k+1}^l(I_{1,\pi_1(\pi_0(1))})$, where $t \in I_{1,\pi_1(\pi_0(1))}$ for $l = 1, \dots, nm-1$ and by $g_{k+1}(I_{1,\pi_1(\pi_0(1))}) = 0$. We have

$$\|g_{k+1}\| \leqslant C_V\|h_{k+1}\|.$$

Let $t \in I_{1,\pi_1(\pi_0(1))}$. If $0 < l < nm-1$, then

$$\begin{aligned} g_{k+1}(T_{k+1}(T_{k+1}^l(t))) - g_{k+1}(T_{k+1}^l(t)) &= \sum_{r=0}^{l} h_{k+1}(T_{k+1}^r(t)) - \sum_{r=0}^{l-1} h_{k+1}(T_{k+1}^r(t)) \\ &= h_{k+1}(T_{k+1}^l(t)), \end{aligned}$$

and further,

$$\begin{aligned} g_{k+1}(T_{k+1}(t)) - g_{k+1}(t) &= h_{k+1}(t) - 0 = h_{k+1}(t), \\ g_{k+1}(T_{k+1}(T_{k+1}^{nm-1}(t))) - g_{k+1}(T_{k+1}^{nm-1}(t)) &= 0 - \sum_{r=0}^{nm-2} h_{k+1}(T_{k+1}^r(t)) \\ &= h_{k+1}(T_{k+1}^{nm-1}(t)). \end{aligned}$$

Thus, for every $t \in \mathcal{C}(q,r)$,

$$g_{k+1}(T_{k+1}(t)) - g_{k+1}(t) = h_{k+1}(t).$$

This completes the construction of functions $\{g_k\}_{k=0}^{\infty}$ and transformations $\{T_k\}_{k=0}^{\infty}$ with the required properties.

The above T_{k+1} satisfies condition (ii). Hence the sequences T_k and g_k satisfy conditions (i)–(v). Observe that the inverse mappings T_k^{-1} also obeys condition (ii).

From condition (iii) it follows that the series $\sum_{k=0}^{\infty} g_k$ converges in $C(\mathcal{C}(q,r);V)$ to some function g satisfying

$$\|g\| \leqslant \|g_0\| + \|g_1\| + \sum_{k=2}^{\infty} \|g_k\| \leqslant S_V\|f\| + C_V a\|f\| + a\|f\| = (S_V + (1+C_V)a)\|f\|.$$

Next, (ii) implies that, for all $t \in \mathcal{C}(q,r)$, the sequence $T_k(t)$ converges. We next define $T(t) = \lim_{k\to\infty} T_k(t) \in \mathcal{C}(q,r)$, $T^{-1}(t) = \lim_{k\to\infty} T_k^{-1}(t)$.

If $n \in \mathbb{N}$, $I \in J_n$, then $T_n(I) = I' \in J_n$. From (ii), we have $T_m(I) = I'$ for $m > n$. Since I' is closed, $T(I) = I'$. Hence T permutes elements of J_n for every n. Since $\bigcup_n J_n$ is a base of the topology in $\mathcal{C}(q,r)$ and generates the σ-algebra of measurable sets, T is a continuous automorphism of $\mathcal{C}(q,r)$.

Now, for $k \geqslant 0$, we have

$$g_k(T(x)) - g_k(x) = g_k(T_k(x)) - g_k(x) = h_k(x).$$

Hence

$$g(T(x)) - g(x) = \sum_{k=0}^{\infty}(g_k(T(x)) - g_k(x)) = \sum_{k=0}^{\infty} h_k = f.$$

The last assertion of the theorem follows from the fact that T is T_0 on J_0 and since g_0 vanishes on $I^0_{\pi(1)}$. Theorem 6.1 is proved.

Proposition 6.1. *Let $q \in \mathbb{N}$, $r \in \mathbb{R}$, and $\{m_n\}$ be a sequence from $\mathbb{N}$. On the set $\mathcal{E} = \{1,\dots,q\} \times \prod_{n=1}^{\infty}\{1,\dots,2^{m_n}\}$ equipped with the product topology consider the product measure*

$$\nu = \nu_0 \times \prod_{n=1}^{\infty} \nu_n, \qquad \nu_0(\{i\}) = \frac{r}{q}, \quad \nu_n(\{j_n\}) = \frac{1}{2^{m_n}}, \quad 1 \leqslant i \leqslant q, \quad 1 \leqslant j_n \leqslant 2^{m_n}.$$

Then there exists a measure preserving homeomorphism $\varphi\colon \mathcal{C}(q,r) \to \mathcal{E}$ such that $\varphi(\mathcal{C}(q,r,i)) = \{i\} \times \prod_{n=1}^{\infty}\{1,\dots,2^{m_n}\}$, $i = 1,\dots,q$.

Proof. Let φ_0 be the identity mapping on $\{1,\dots,q\}$, and

$$\varphi_n\colon \{1,2\}^{m_n} \to \{1,\dots,2^{m_n}\}$$

be a bijection such that

$$\varphi_n(i_1,\dots,i_{2^{m_n}}) = 1 + \sum_{k=1}^{m_n} 2^{k-1}(i_k - 1), \qquad n \geqslant 1.$$

The compact set $\mathcal{C}(q,r)$ can be represented as

$$\mathcal{C}(q,r)=\{1,\dots,q\}\times\prod_{n=1}^{\infty}\{1,2\}^{m_n}.$$

Define the bijection $\varphi\colon\mathcal{C}(q,r)\to\mathcal{E}$ as the product $\varphi=\prod_{n=0}^{\infty}\varphi_n$. Every φ_n is measure preserving, and hence so is φ.

Let $k\in\mathbb{N}$, $x\in\{1,\dots,q\}\times\prod_{n=1}^{k}\{1,2\}^{m_n}$, $P(x)=x\times\prod_{n=k+1}^{\infty}\{1,2\}^{m_n}$. We have $\varphi(P(x))=\bigl(\prod_{n=0}^{k}\varphi_n\bigr)(x)\times\prod_{n=k+1}^{\infty}\{1,\dots,2^{m_n}\}$.

Recalling that the sets $P(x)$ form the base of the topology in $\mathcal{C}(q,r)$, and the sets $\varphi(P(x))$ form the base of the topology in $\mathcal{E}$, we conclude that φ is homeomorphism. This proves Proposition 6.1.

§ 7. Solutions to the homological equation. The affinely homogeneous setting

In this section, we show that the equation $f\in L_\infty(D;V)$ is solvable for any mean zero affinely homogeneous function $f=g\circ T-g$. The transformation T we construct here is not ergodic. We will circumvent this ergodicity issue in the next section.

Note that if a function f is affinely homogeneous, then, for any $v\in V$ and any measurable subset $D'\subseteq D$, $f|_{D'}+v$ is affinely homogeneous as well. Moreover, note that the conditions of Lemma 5.4 on the function $f\in L_\infty(D;V)$ are satisfied if f is mean zero and affinely homogeneous.

Let $D\subset[0,1]$ be a measurable set, $f\in L_\infty(D;V)$ be a mean zero function, $q\in\mathbb{N}$, $R\in(0,\lambda(D))$, $\mathcal{F}\colon\mathcal{C}(q,R)\to D$. The system $(q,\mathcal{F},R)$ is said to be *a Cantor tower* for f if $\mathcal{F}$ is a measure preserving continuous injection, the function $f|_{\mathcal{F}(\mathcal{C}(q,R))}$ is continuous, and $\int_{\mathcal{F}(\mathcal{C}(q,R))}f\,d\lambda=0$.

Proposition 7.1. *Let V be a finite-dimensional real normed space, let $D\subset[0,1]$ be a measurable set, let $f\in L_\infty(D;V)$ be a mean zero affinely homogeneous function, and let $R\in(0,\lambda(D))$.*

(i) *For every $q\in\mathbb{N}$, there exists a Cantor tower $(q,\mathcal{F},R)$ for f;*

(ii) *For every $\varepsilon>0$, there exists a Cantor tower $(q,\mathcal{F},R)$ for f such that*

$$\operatorname{Diam}(f(\mathcal{F}(\mathcal{C}(q,R,i))))<\varepsilon,\qquad i=1,\dots,q. \tag{7.1}$$

Proof. Assertion (ii) differs from (i) because q in (ii) is not given in advance, and should to be determined so as to satisfy (7.1).

Once the sought-for q in (ii) is found, the construction of the Cantor tower is the same for both cases (i) and (ii).

By Lemma 5.2, we know that there exists a compact set $K'\subset D$, with $\lambda(K')>R$ such that f is continuous and mean zero on K'.

For every $\varepsilon>0$ there are points $x_0=\inf K'<x_1<\dots<x_n=\sup K'$ such that $\operatorname{Diam}(f([x_{i-1},x_i]\cap K'))<\varepsilon$, $i=1,\dots,n$. Let $\{K_1',\dots,K_m'\}$ be a subfamily of $\{[x_{i-1},x_i]\cap K'\colon i=1,\dots,n\}$, consisting of all sets of non-zero measure.

By Lemma 5.4, there exist compact sets $K_1''\subset K_1',\dots,K_m''\subset K_m'$ such that $\int_K f\,d\lambda=0$, $\lambda(K)>R$, where $K=K_1''\cup\dots\cup K_m''$, and $\lambda(K_i'')/\lambda(K)\in\mathbb{Q}_2$, $i=1,\dots,m$.

Hence the compact set K admits a splitting $\{K_1, \dots, K_q\}$ inscribed in the splitting $\{K_1'', \dots, K_m''\}$ so that $\lambda(K_1) = \dots = \lambda(K_q)$, where q is the common denominator of the ratios $\{\lambda(K_i'')/\lambda(K) \colon i = 1, \dots, m\}$, and $\sup K_i \leqslant \inf K_{i+1}$ for $i = 1, \dots, m$.

Thus, for case (ii), we have found the number q and have constructed the compact sets $K_1, \dots, K_q$ so that $\operatorname{Diam}(f(K_i)) < \varepsilon$, $i = 1, \dots, q$.

In case (i), we set $K = K'$, and, for q given in advance, we choose points $x_0 = \inf K < x_1 < \dots < x_q = \sup K$ such that

$$\lambda([x_{i-1}, x_i] \cap K) = \frac{\lambda(K)}{q}, \qquad i = 1, \dots, q.$$

In this case, we set

$$K_i = [x_{i-1}, x_i] \cap K, \qquad i = 1, \dots, q.$$

Now, we only need to build a Cantor tower $(q, \mathcal{F}, R)$ such that $\mathcal{F}(\mathcal{C}(q, R, i)) \subset K_i$, $i = 1, \dots, q$.

Let us fix a strictly decreasing sequence

$$R_0 = \lambda(K) > R_1 > \dots > R_n > \cdots, \qquad \lim_n R_n = R.$$

We will build a sequence (m_n) of positive integers and sets K_a, $a \in \mathcal{E}_n$, where

$$\mathcal{E}_n = \{1, \dots, q\} \times \prod_{i=1}^{n} \{1, \dots, 2^{m_n}\}, \qquad n \geqslant 0.$$

Below, throughout this proof, we write $|\mathcal{E}_n| = \operatorname{Card}(\mathcal{E}_n)$ for brevity.

For $m \leqslant n$, we define the projection $p_{n,m} \colon \mathcal{E}_n \to \mathcal{E}_m$ by

$$p_{n,m}(i_0; i_1, \dots, i_m, \dots, i_n) = (i_0; i_1, \dots, i_m).$$

We also set

$$C_n = \bigcup_{a \in \mathcal{E}_n} K_a, \qquad C = \bigcap_{n=0}^{\infty} C_n.$$

Clearly, $C_0 = K$.

The sets K_a should satisfy the following conditions:

1) for $a \in \mathcal{E}_n$, the set K_a is a compact subset of $[0, 1]$; for $a_1, a_2 \in \mathcal{E}_n$, $a_1 \neq a_2$, we have either $\sup K_{a_1} \leqslant \inf K_{a_2}$ or $\sup K_{a_2} \leqslant \inf K_{a_1}$;

2) if $a \in \mathcal{E}_{n-1}$ and $b \in \mathcal{E}_n$ are such that $p_{n,n-1}(b) = a$, then $K_b \subseteq K_a$ and $\operatorname{Diam}(K_b) \leqslant (1/2) \operatorname{Diam}(K_a)$;

3) $\lambda(K_a) = M_n := R_n/|\mathcal{E}_n|$ for all $a \in \mathcal{E}_n$;

4) $\int_{C_n} f \, d\lambda = 0$ for all $n \geqslant 0$;

5) the sets $K_a \cap C_{n+1}$ and $K_b \cap C_{n+1}$ are disjoint for $a, b \in \mathcal{E}_n$ whenever $a \neq b$;

6) $\lambda(K_a \cap C_n) = R_n/|\mathcal{E}_k|$ for any $k < n$, $a \in \mathcal{E}_k$.

The construction of a sequence (m_n) and compact sets K_a is by induction on n.

If $n = 0$, the set $\mathcal{E}_0 = \{1, \dots, q\}$ and the compacts $K_1, \dots, K_q$ are already constructed.

Let $n \geqslant 0$ and assume that the set $\{m_1, \dots, m_n\}$ (when $n = 0$ this set is empty) and the compacts K_a, $a \in \mathcal{E}_k$, $k \leqslant n$ are found. We now define m_{n+1} and K_a for all $a \in \mathcal{E}_{n+1}$.

Given a fixed $a \in \mathcal{E}_n$, we set

$$K_a^L := K_a \cap \left[\inf K_a, \frac{\inf K_a + \sup K_a}{2}\right], \quad K_a^R := K_a \cap \left[\frac{\inf K_a + \sup K_a}{2}, \sup K_a\right].$$

Note that $\operatorname{Diam}(K_a^L) \leqslant (1/2)\operatorname{Diam}(K_a)$ and $\operatorname{Diam}(K_a^R) \leqslant (1/2)\operatorname{Diam}(K_a)$. We also set

$$h_a = f|_{K_a} - \frac{1}{\lambda(K_a)} \int_{K_a} f\, d\lambda.$$

We have $h_a \in L_\infty(K_a; V)$, $\int_{K_a} h_a\, d\lambda = 0$. Let us now show that Lemma 5.4 applies to the set $K_a = K_a^L \cup K_a^R$, the mean zero function h_a, and the number s $R_{n+1}/|\mathcal{E}_n|$. First, note that $0 < R_{n+1}/|\mathcal{E}_n| < R_n/|\mathcal{E}_n| = \lambda(K_a)$ and $\lambda(K_a^L \cap K_a^R) = 0$. Further, h_a is mean zero and, as f is affinely homogeneous, and hence h_a is also affinely homogeneous. This shows that we can indeed apply Lemma 5.4 to obtain a subset

$$\widetilde{K_a} = \widetilde{K_a^L} \cup \widetilde{K_a^R} \subset K_a \cap (\inf K_a, \sup K_a)$$

(we emphasize the importance of the preceding inclusion for the validity of condition 5) above!) with $\widetilde{K_a^L} \subset K_a^L$ and $\widetilde{K_a^R} \subset K_a^R$ both compact and of positive measure, so that $\lambda(\widetilde{K_a}) = R_{n+1}/|\mathcal{E}_n|$ and $\int_{\widetilde{K}_a} h_a\, d\lambda = 0$ and so that $\lambda(\widetilde{K_a^L})/\lambda(\widetilde{K_a}) = p_a/2^{q_a}$ for some integer $p_a \geqslant 0$ and positive integer q_a.

Now, we set

$$m_{n+1} = 1 + \sum_{a \in \mathcal{E}_n} q_a, \qquad k_a = 2^{m_{n+1} - q_a} p_a.$$

We now select points

$$x_a^0 < x_a^1 < \cdots < x_a^{k_a} = \frac{\inf K_a + \sup K_a}{2} < \cdots < x_a^{2^{m_{n+1}}}$$

in K_a such that, for $1 \leqslant i \leqslant 2^{m_{n+1}}$ the sets

$$K_a^i := \widetilde{K_a} \cap [x_a^{i-1}, x_a^i]$$

all have equal measure

$$\lambda(K_a^i) = \frac{\lambda(\widetilde{K_a})}{2^{m_{n+1}}} = \frac{R_{n+1}}{|\mathcal{E}_n| 2^{m_{n+1}}}$$

and, in addition,

$$K_a^i \subset \widetilde{K_a^L} \quad \forall\, i \leqslant k_a, \qquad K_a^i \subset \widetilde{K_a^R} \quad \forall\, k_a < i \leqslant 2^{m_{n+1}}.$$

Now if $b = a \times i \in \mathcal{E}_{n+1}$, $1 \leqslant i \leqslant 2^{m_{n+1}}$, we define $K_b = K_a^i$.

By construction, conditions 1)–3) are met for K_c, $c \in \mathcal{E}_{n+1}$, and condition 5) is satisfied for $a, b \in \mathcal{E}_n$.

Now, let us verify condition 4). Indeed, we have

$$\begin{aligned}\int_{C_{n+1}} f\,d\lambda &= \sum_{a\in\mathcal{E}_n}\int_{\widetilde{K_a}} f\,d\lambda = \sum_{a\in\mathcal{E}_n}\frac{\lambda(\widetilde{K_a})}{\lambda(K_a)}\int_{K_a} f\,d\lambda\,d\lambda\\ &= \sum_{a\in\mathcal{E}_n}\frac{R_{n+1}}{|\mathcal{E}_n|\lambda(K_a)}\int_{K_a} f\,d\lambda\,d\lambda = \frac{R_{n+1}}{|\mathcal{E}_n|M_n}\sum_{a\in\mathcal{E}_n}\int_{K_a} f\,d\lambda\,d\lambda\\ &= \frac{R_{n+1}}{|\mathcal{E}_n|M_n}\int_{C_n} f\,d\lambda\,d\lambda = 0.\end{aligned}$$

Now, let us verify condition 6). To this end, we note that the number of compact sets K_x, $x\in\mathcal{E}_{n+1}$, contained inside K_a, $a\in\mathcal{E}_n$, is equal to $|\mathcal{E}_{n+1}|/|\mathcal{E}_n|$. Hence, for $k<n$, $a\in\mathcal{E}_k$, the number of K_x, $x\in\mathcal{E}_n$, contained inside K_a is

$$\frac{|\mathcal{E}_n|}{|\mathcal{E}_{n-1}|}\,\frac{|\mathcal{E}_{n-1}|}{|\mathcal{E}_{n-2}|}\cdots\frac{|\mathcal{E}_{k+1}|}{|\mathcal{E}_k|} = \frac{|\mathcal{E}_n|}{|\mathcal{E}_k|}.$$

Therefore, we have

$$\lambda(K_a\cap C_n) = \frac{|\mathcal{E}_n|}{|\mathcal{E}_k|}\,\frac{R_n}{|\mathcal{E}_n|} = \frac{R_n}{|\mathcal{E}_k|}.$$

This completes the construction of the compact sets K_a.

We now claim that

$$\lambda(C) = R, \qquad \int_C f\,d\lambda = 0.$$

Indeed, we have $C_{n+1}\subset C_n$, $n\geqslant 0$, $\lambda(C)=\lim_n\lambda(C_n)=\lim_n R_n = R$, and $\left\|\int_C f\,d\lambda\right\|\leqslant\left\|\int_{C_n} f\,d\lambda\right\|+\lambda(C_n\setminus C)\|f\|_\infty = \lambda(C_n\setminus C)\|f\|_\infty = (R_n-R)\|f\|_\infty\to 0$ as $n\to\infty$.

Further, by Proposition 6.1, we may identify $\mathcal{C}(q,R)$ with $\mathcal{E}_\infty=\{1,\dots,q\}\times\prod_{i=1}^\infty\{1,\dots,2^{m_n}\}$ with the above measure ν.

For every $n\geqslant 0$, we define the mapping $p_n\colon\mathcal{E}_\infty\to\mathcal{E}_n$ by

$$p_n(i_0;i_1,\dots,i_n,i_{n+1},\dots) = (i_0;i_1,\dots,i_n).$$

For every $a\in\mathcal{E}_\infty$, we set $\mathcal{F}(a)=\bigcap_{n=0}^\infty K_{p_n(a)}$. Combining the equality $p_{n+1,n}\circ p_{n+1}=p_n$ and condition 2), we infer that $|\mathcal{F}(a)|=\operatorname{Card}(\mathcal{F}(a))=1$. Therefore, the mapping $\mathcal{F}\colon\mathcal{E}_\infty\to C$ is correctly defined.

Let $a,b\in\mathcal{E}_\infty$, $a\neq b$. Then, there exists n such that $p_n(a)\neq p_n(b)$. By 5), we have $\mathcal{F}(a)\neq\mathcal{F}(b)$, that is, the mapping $\mathcal{F}$ is injective.

Let $x\in C$. It follows from the construction of C that, for every n, there exists a unique $a_n\in\mathcal{E}_n$ such that $x\in K_{a_n}$. Using conditions 1) and 2), we find that $p_{n,n-1}(a_n)=a_{n-1}$, $n>1$. Hence, there exists $a\in\mathcal{E}_\infty$ such that $p_n(a)=a_n$. This guarantees $\mathcal{F}(a)=x$, and so the mapping $\mathcal{F}$ is surjective.

Let $\{a_{(n)}\}\subset\mathcal{E}_\infty$ converge to $a\in\mathcal{E}_\infty$. This means that, for every n, there exists an index k_n such that $p_n(a_{(m)})=p_n(a)$ when $m>k_n$. That is, $\mathcal{F}(a_{(m)})\in K_{p_n(a)}$. By 2), we have $|\mathcal{F}(a_{(m)})-\mathcal{F}(a)|\leqslant 1/2^n$. Hence $\mathcal{F}$ is continuous.

Since $\mathcal{E}_\infty$ is compact, the mapping $\mathcal{F}^{-1}$ is continuous.

Using 6), we see that

$$\lambda(K_a \cap C) = \frac{R}{|\mathcal{E}_n|} \quad \forall\, n \geqslant 0, \quad a \in \mathcal{E}_n.$$

However, $K_a \cap C = \mathcal{F}(p_n^{-1}(a))$, $\mu(p_n^{-1}(a)) = R/|\mathcal{E}_n|$, that is

$$\lambda(\mathcal{F}(p_n^{-1}(a))) = \mu(p_n^{-1}(a)).$$

Taking into account that the sets $p_n^{-1}(a)$, $n \geqslant 0$, $a \in \mathcal{E}_n$, generate the σ-algebra of measurable subsets in $\mathcal{E}_\infty$, we conclude that the mapping $\mathcal{F}$ is measure preserving. This proves Proposition 7.1.

Proposition 7.2. *Let V be a finite-dimensional real normed space, let $D \subseteq [0,1]$ be of positive measure, and let $f \in L_\infty(D;V)$ be mean zero and affinely homogeneous function. Then, for any $\varepsilon > 0$ and $R \in (0, \lambda(D))$, there exist a measurable set $C \subset D$, $\lambda(C) = R$, $g \in L_\infty(C;V)$ and a* mod 0 *automorphism T of C such that $\|g\| \leqslant (S_V + \varepsilon)\|f\|_\infty$ and $f = g \circ T - g$.*

Proof. By Proposition 7.1 (ii), for f there exists a Cantor tower $(q, \mathcal{F}, R)$ such that

$$\operatorname{Diam}(f(\mathcal{F}(\mathcal{C}(q,R,i)))) < \frac{\varepsilon}{(1 + C_V)}, \qquad i = 1, \dots, q.$$

An application of Theorem 6.1 completes the proof.

Theorem 7.1. *Let V be a finite-dimensional real normed space. Let $D \subseteq [0,1]$ be a set of positive measure and let $f \in L_\infty(D;V)$ be mean zero and affinely homogeneous. Then, for any $\varepsilon > 0$, there exist $g \in L_\infty(D;V)$ and a* mod 0 *automorphism T of D such that $\|g\| \leqslant (S_V + \varepsilon)\|f\|_\infty$ and $f = g \circ T - g$.*

Proof. By Zorn's lemma, there exists a maximal family $\{K_i\}_{i \in I}$ of pairwise disjoint compact subsets of D with positive measure such that there exist $g_i \in L_\infty(K_i;V)$, $\|g_i\|_\infty \leqslant (S_V + \varepsilon)\|f\|_\infty$ and a mod 0 automorphism T_i of K_i such that $f|_{K_i} = g_i \circ T_i - g_i$. Clearly, the set of indices I is at most countable.

It suffices to show that $\lambda\bigl(D \setminus \bigcup_{i \in I} K_i\bigr) = 0$. Indeed, in this case we can define g and T so that $g|_{K_i} = g_i$, $T|_{K_i} = T_i$ for any $i \in I$.

Suppose that the set $D_0 := D \setminus \bigcup_{i \in I} K_i$ has non-zero measure. By Proposition 7.2, there exists a compact subset K_0 in D_0 such that $f|_{K_0} = g_0 \circ T_0 - g_0$ for some function $g_0 \in L_\infty(K_0;V)$, $\|g_0\|_\infty \leqslant (S_V + \varepsilon)\|f\|_\infty$, and there exists a mod 0 automorpishm T_0 of K_0. But this contradicts the assumption concerning the maximality of the family $\{K_i\}_{i \in I}$. Hence $\lambda(D_0) = 0$. Theorem 7.1 is proved.

§ 8. Proof of main results for general mean zero functions

We begin this section with two lemmas, which are based on classical results.

Lemma 8.1. *Given $x \in \mathbb{R}^n$ with $1, x_1, \dots, x_n$ rationally independent, let $\varepsilon > 0$. Then, for any given non-zero vector $v \in \mathbb{R}^n$, there are integers $q \geqslant 1, p_1, \dots, p_n \in \mathbb{Z}$, such that the vector $w \in \mathbb{R}^n$ with $w_l = p_l/q - x_l$ satisfies $\|w\|_\infty < \varepsilon/q$ and $(w, v) > 0$.*

Proof. Let us denote $\alpha_l = \operatorname{sign}(v_l)\varepsilon/2$.

Since $1, x_1, \dots, x_n$ are rationally independent, by [22], Theorem 442, we can find integers $q \geqslant 1$ and $p_1, \dots, p_n \in \mathbb{Z}$ such that

$$|qx_l - p_l + \alpha_l| < \frac{\varepsilon}{2}, \qquad l = 1, \dots, n.$$

Now, since $|qx_l - p_l| < \varepsilon/2 + |\alpha_l| = \varepsilon$, for the vector $w \in \mathbb{R}^n$ given by $w_l = p_l/q - x_l$ we have $\|w\|_\infty < \varepsilon/q$. Moreover, $|\alpha_l| = \varepsilon/2$, and so $\operatorname{sign}(w_l) = \operatorname{sign}(p_l - x_l q) = \operatorname{sign}(\alpha_l) = \operatorname{sign}(v_l)$. Further, since x_l is irrational for all l, we have $|w_l| = |p_l/q - x_l| > 0$. Now, $v \neq 0$, and so $(w, v) > 0$. Lemma 8.1 is proved.

Lemma 8.2. *Let $\{T_n\}$ be a sequence of* mod 0 *automorphisms of* $[0,1]$ *and let $T_n \to T$, $T_n^{-1} \to S$ in measure. Then T and S are also* mod 0 *automorphisms of* $[0,1]$, *and $S = T^{-1}$ almost everywhere.*

Proof. The fact that T and S are measure preserving follows from [12], Proposition 9.9.10. The fact that the equality $S = T^{-1}$ holds almost everywhere follows from [12], Corollary 9.9.11. Lemma 8.2 is proved.

The following lemma plays a crucial role in the proof of our main result.

Lemma 8.3. *Let V be a finite-dimensional real normed space, $f \in L_\infty([0,1]; V)$ be mean zero, and $\varepsilon > 0$. There exist a sequence $(q_i)_{i\geqslant 1}$ in $\mathbb{N}\setminus\{1\}, \mathbb{N}$ with $q_i \geqslant 2$, a partition $\{A_{i,j} \colon i \geqslant 1, 1 \leqslant j \leqslant q_i\}$ of* $[0,1]$ *into sets of positive measure, and a* mod 0 *automorphism T on* $[0,1]$ *such that*:

1) $T(A_{i,j}) = A_{i,j+1}\ \forall\, i \geqslant 1,\ 1 \leqslant j < q_i$;
2) $\|h\|_\infty < \varepsilon$;
3) $\bigl\|\sum_{j=1}^{l-1} f \circ T^{j-1}\bigr\|_{L_\infty(A_{i,1};V)} < S_V\|f\|_\infty + \varepsilon,\ l = 1, \dots, q_i$,

where $A := \bigcup_{i=1}^\infty A_{i,1} \to V$ and $h\colon A \to V$ by $h|_{A_{i,1}} = \sum_{j=1}^{q_i} f \circ T^{j-1}$, h is mean zero, and $\|h\|_\infty < \varepsilon$.

Proof. By Theorem 4.3, we can find a subset $D \subseteq [0,1]$ of positive measure on which f is mean zero, and find an integer $n \geqslant 1$, and a partition $\{D_1, \dots, D_n\}$ of D such that $f|_{D_l}$ is affinely homogeneous for $l = 1, \dots, n$.

Let $\varepsilon > 0$. We will set

$$\varepsilon' = \frac{1}{2}\min\biggl\{1,\ \frac{\varepsilon}{3},\ \min_i(\lambda(D_i))\biggr\} > 0.$$

Each function $f_{(l)} = f|_{D_l} - ⨍_{D_l} f\,d\lambda$ is mean zero and affinely homogeneous on D_l, $l = 1, \dots, n$.

By Proposition 7.1, (ii), for every $l = 1, \dots, n$, for the function $f_{(l)}$, there exists a Cantor tower $(q_{(l)}, \mathcal{F}_{(l)}, \frac{1}{2}\lambda(D_l))$ satisfying

$$K_l := \mathcal{F}_{(l)}\biggl(\mathcal{C}\biggl(q_{(l)}, \frac{1}{2}\lambda(D_l)\biggr)\biggr) \subset D_l,$$

$$K_{l,m} := \mathcal{F}_{(l)}\biggl(\mathcal{C}\biggl(q_{(l)}, \frac{1}{2}\lambda(D_l), m\biggr)\biggr),$$

$$\operatorname{Diam}(f(K_{l,m})) < \frac{\varepsilon'}{(1 + C_V)(\|f\|_\infty + 1)}, \qquad m = 1, \dots, q_{(l)}$$

(recall that $C_V = (8\dim(V)^2/\log 1.5)(S_V + 1))$.

We have $\lambda(K_l) = (1/2)\lambda(D_l)$, $⨍_{K_l} f\,d\lambda = ⨍_{D_l} f\,d\lambda$ for any l. Setting $K = \bigcup_{l=1}^n K_l$, we find that

$$\int_K f\,d\lambda = \sum_{l=1}^n \int_{K_l} f\,d\lambda = \sum_{l=1}^n \frac{\lambda(K_l)}{\lambda(D_l)} \int_{D_l} f\,d\lambda = \frac{1}{2}\int_D f\,d\lambda = 0.$$

Hence $\{K_{l,1}, \dots, K_{l,q_{(l)}}\}$ is a partition of K_l such that $\lambda(K_{l,m}) = \lambda(K_l)/q_{(l)}$ for all l, m.

Let $x \in \mathbb{R}^n$ be the vector given by $x_l = \lambda(K_l)/(q_{(l)}\lambda(K)) > 0$. We can find a maximal subset $\mathcal{J} \subseteq \{1, \dots, n\}$ such that $\{1\} \cup \{x_j \colon j \in \mathcal{J}\}$ are rationally independent. Then $b_l x_l = a_{l,0} + \sum_{j\in\mathcal{J}} a_{l,j}x_j$ for $l = 1, \dots, n$ for some integers $a_{l,j}$ and non-zero integers b_l. We set $M = 2\big|\prod_{l=1}^n b_l\big| \max\{|a_{l,j}| \colon j \in \mathcal{J} \cup \{0\}, l = 1, \dots, n\}$. We also define $q_{(0)} = \max_l q_{(l)}$, $\rho = \min_l \lambda(K_l) > 0$, $x_0 = \min_l x_l > 0$, and

$$N = nM^2 \max\left\{\frac{q_{(0)}}{\rho},\, 2nq_{(0)},\, \frac{2}{x_0},\, \frac{nq_{(0)}\|f\|_\infty}{\varepsilon'}\right\}.$$

If $\mathcal{J}$ is empty, we set $\widetilde{q_i} = 1$ for $i \in \mathbb{N}$. Now suppose that $\mathcal{J}$ is non-empty. Let $\mathbb{R}^{\mathcal{J}}$ be the vector space of functions $\mathcal{J} \to \mathbb{R}$ equipped with the Euclidean norm, and let $\mathbb{S}(\mathcal{J})$ be the set of all unit vectors in $\mathbb{R}^{\mathcal{J}}$. For every $v \in \mathbb{S}(\mathcal{J})$, using Lemma 8.1 we can find integers $\widetilde{q_v} \geqslant 1$ and $\widetilde{p_{v,j}} \in \mathbb{Z}$ for $j \in \mathcal{J}$ such that, for the vector $\widetilde{w_v} \in \mathbb{R}^{\mathcal{J}}$ with $(\widetilde{w_v})_j = \widetilde{p_{v,j}}/\widetilde{q_v} - x_j$, we have $\|\widetilde{w_v}\|_\infty < 1/\widetilde{q_v}N$, and so $(\widetilde{w_v}, v) > 0$. Next, there is a sequence $(\xi_i)_{i\geqslant 1}$ in $\mathbb{S}(\mathcal{J})$ such that $\{\widetilde{w_{\xi_i}} \colon i \in \mathbb{N}\}$ is dense in $\{\widetilde{w_v} \colon v \in \mathbb{S}(\mathcal{J})\}$. Now, by the density, for $v \in \mathbb{S}(\mathcal{J})$, we can find $i \geqslant 1$ such that $\|\widetilde{w_{\xi_i}} - \widetilde{w_v}\|_2 < (\widetilde{w_v}, v)$, and hence

$$(\widetilde{w_{\xi_i}}, v) = (\widetilde{w_v}, v) + (\widetilde{w_{\xi_i}} - \widetilde{w_v}, v) \geqslant (\widetilde{w_v}, v) - \|\widetilde{w_{\xi_i}} - \widetilde{w_v}\|_2 > 0.$$

For $i \geqslant 1$, we set $\widetilde{q_i} := \widetilde{q_{\xi_i}}$ and $\widetilde{p_{i,j}} := \widetilde{p_{\xi_i,j}}$ for $j \in \mathcal{J}$ and define $\widetilde{w_i} := \widetilde{w_{\xi_i}}$. Note that

$$\|\widetilde{w_i}\|_\infty = \|\widetilde{w_{\xi_i}}\|_\infty \leqslant \frac{1}{\widetilde{q_{\xi_i}}N} = \frac{1}{\widetilde{q_i}N}.$$

By the above, for every non-zero $v \in \mathbb{R}^{\mathcal{J}}$, we can find $i \geqslant 1$ such that $(\widetilde{w_i}, v) > 0$.

Regardless on whether $\mathcal{J}$ is empty or non-empty, we now fix $i \geqslant 1$ and define $q_i = M\widetilde{q_i} \geqslant 2$. For $l = 1, \dots, n$, we set

$$p_{i,l} = \frac{q_i}{b_l\widetilde{q_i}}\left(a_{l,0}\widetilde{q_i} + \sum_{j\in\mathcal{J}} a_{l,j}\widetilde{p_{i,j}}\right) = \frac{q_i}{b_l}\left(a_{l,0} + \sum_{j\in\mathcal{J}} a_{l,j}\frac{\widetilde{p_{i,j}}}{\widetilde{q_i}}\right).$$

This number is integer, since $q_i/(b_l\widetilde{q_i}) = M/b_l \in \mathbb{N}$. Let the vector $w_i \in \mathbb{R}^n$ be defined by

$$(w_i)_l := \frac{p_{i,l}}{q_i} - x_l = \sum_{j\in\mathcal{J}} \frac{a_{l,j}}{b_l}\left(\frac{\widetilde{p_{i,j}}}{\widetilde{q_i}} - x_j\right).$$

If $\mathcal{J}$ is empty, then we set $w_i = 0$. If $\mathcal{J}$ is non-empty, we have

$$\|w_i\|_\infty \leqslant \sum_{j\in\mathcal{J}} M\,\|\widetilde{w_i}\|_\infty \leqslant \frac{nM}{\widetilde{q_i}N} = \frac{1}{q_i}\,\frac{nM^2}{N}.$$

For $i \geqslant 1$, we now choose $c_i > 0$ so that $\sum_{i=1}^{\infty} c_i = (1/3)\lambda(K)$. For $1 \leqslant l \leqslant n$ and $1 \leqslant m \leqslant q_{(l)}$, we then have

$$\sum_{i=1}^{\infty} c_i \frac{p_{i,l}}{q_i} = \sum_{i=1}^{\infty} c_i \left(\frac{\lambda(K_l)}{q_{(l)}\lambda(K)} + (w_i)_l \right) \leqslant \sum_{i=1}^{\infty} \frac{c_i}{\lambda(K)} \left(\frac{\lambda(K_l)}{q_{(l)}} + \frac{\rho}{q_{(0)}} \right)$$
$$\leqslant 2\frac{\lambda(K_l)}{q_{(l)}} \sum_{i=1}^{\infty} \frac{c_i}{\lambda(K)} < \lambda(K_{l,m}).$$

Now, for $i \geqslant 1$, let $\{I_{l,m}^{(i)} \colon 1 \leqslant l \leqslant n,\ 1 \leqslant m \leqslant q_{(l)}\}$ be a partition of $\{1, \dots, q_i\}$ such that $I_{l,m}^{(i)}$ is of cardinality $|I_{l,m}^{(i)}| = p_{i,l}$. We fix the bijections

$$\alpha_{l,m}^{(i)} \colon I_{l,m}^{(i)} \to \{1, \dots, p_{i,l}\}, \qquad \beta_{l,m}^{(i)} \colon I_{l,m}^{(i)} \times \{1,2\}^{\mathbb{N}} \to \{1, \dots, p_{i,l}\} \times \{1,2\}^{\mathbb{N}},$$

by setting

$$\beta_{l,m}^{(i)}(i_0; i_1, \dots) = (\alpha_{l,m}^{(i)}(i_0); i_1, \dots).$$

Note that such a partition actually exists. Indeed, $q_i = \sum_{l=1}^{n} q_{(l)} p_{i,l}$, because

$$\left| q_i - \sum_{l=1}^{n} q_{(l)} p_{i,l} \right| \leqslant \sum_{l=1}^{n} \left| q_i \frac{\lambda(K_l)}{\lambda(K)} - q_{(l)} p_{i,l} \right| \leqslant q_i \sum_{l=1}^{n} q_{(l)} \left| x_l - \frac{p_{i,l}}{q_i} \right|$$
$$\leqslant q_i n q_{(0)} \|w_i\|_\infty \leqslant \frac{n^2 M^2 \kappa}{N} \leqslant \frac{1}{2}$$

is integer. Also, $p_{i,l}$ is an integer such that $p_{i,l} \geqslant q_i x_l - |q_i x_l - p_{i,l}| > x_l - q_i \|w_i\|_\infty \geqslant x_l - x_0/2 > 0$. Hence $p_{i,l} \in \mathbb{N}$.

Let l $1 \leqslant l \leqslant n$ and $1 \leqslant m \leqslant q_{(l)}$. By Proposition 7.1 (i), we can find a Cantor tower $(p_{1,l}, \mathcal{F}_{l,m}^{(1)}, c_1 p_{1,l}/q_1)$ for the function $f|_{K_{l,m}} - ⨍_{K_{l,m}} f\, d\lambda$. We have

$$E_{l,m}^{(1)} := \mathcal{F}_{l,m}^{(1)}\left(\mathcal{C}\left(p_{1,l}, c_1 \frac{p_{1,l}}{q_1} \right) \right) \subset K_{l,m}, \qquad \lambda(E_{l,m}^{(1)}) = c_1 \frac{p_{1,l}}{q_1},$$
$$⨍_{E_{l,m}^{(1)}} f\, d\lambda = ⨍_{K_{l,m}} f\, d\lambda.$$

We have $\lambda(K_{l,m} \setminus E_{l,m}^{(1)}) > \sum_{i=2}^{\infty} c_i p_{i,l}/q_i > c_2 p_{2,l}/q_2$ and hence, arguing as above, we find a measure preserving homeomorphism

$$\mathcal{F}_{l,m}^{(2)} \colon \mathcal{C}\left(p_{2,l}, c_2 \frac{p_{2,l}}{q_2} \right) \to E_{l,m}^{(2)} \subseteq K_{l,m} \setminus E_{l,m}^{(1)},$$

such that

$$⨍_{E_{l,m}^{(2)}} f\, d\lambda = ⨍_{K_{l,m} \setminus E_{l,m}^{(1)}} f\, d\lambda = ⨍_{K_{l,m}} f\, d\lambda.$$

Similarly, for every $i \geqslant 2$, we can find a measure preserving homeomorphism

$$\mathcal{F}_{l,m}^{(i)} \colon \mathcal{C}\left(p_{i,l}, c_i \frac{p_{i,l}}{q_i} \right) \to E_{l,m}^{(i)} \subseteq K_{l,m} \setminus (E_{l,m}^{(1)} \cup \dots \cup E_{l,m}^{(i-1)}),$$

such that $⨍_{E_{l,m}^{(i)}} f\, d\lambda = ⨍_{K_{l,m}} f\, d\lambda$.

Now, once the disjoint sets $E^{(i)}_{l,m}$ for $i \geqslant 1$, $1 \leqslant l \leqslant n$ and $1 \leqslant m \leqslant q_{(l)}$ are constructed, we can set $E^{(i)} = \bigcup_{l=1}^{n} \bigcup_{m=1}^{q_{(l)}} E^{(i)}_{l,m}$ for $i \geqslant 1$. For $i \geqslant 1$, these sets are also pairwise disjoint and are of measure

$$\lambda(E^{(i)}) = \sum_{l=1}^{n} \sum_{m=1}^{q_{(l)}} c_i \frac{p_{i,l}}{q_i} = \frac{c_i}{q_i} \sum_{l=1}^{n} q_{(l)} p_{i,l} = c_i, \qquad i \geqslant 1.$$

So, for every i, by gluing the homeomorphisms $\mathcal{F}^{(i)}_{l,m}$, we obtain a measure preserving homeomorphism $\mathcal{F}_i \colon \mathcal{C}(q_i, c_i) \to E^{(i)}$ defined by $\mathcal{F}_i|_{I^{(i)}_{l,m} \times \{1,2\}^{\mathbb{N}}} = \mathcal{F}^{(i)}_{l,m} \circ \beta^{(i)}_{l,m}$.

For the sets $\widetilde{A_{i,j}} \colon = \mathcal{F}_i(\mathcal{C}(q_i, c_i, j))$, $j = 1, \ldots, q_i$, we have $\widetilde{A_{i,j}} \subseteq E^{(i)}_{l,m} \subseteq K_{l,m}$, where l, m are such that $j \in I^{(i)}_{l,m}$. This means that, for $j = 1, \ldots, q_i$, we have $\operatorname{Diam}(f(\widetilde{A_{i,j}})) < (1 + C_V)^{-1}(\|f\|_\infty + 1)^{-1}\varepsilon'$ because this holds for $\operatorname{Diam}(f(K_{l,m}))$ for all $1 \leqslant l \leqslant n$ and $1 \leqslant m \leqslant q_{(l)}$. Next, since $E^{(i)} \subseteq K$, the restriction $f|_{E^{(i)}}$ is continuous. We also have

$$\begin{aligned}
\int_{E^{(i)}} f\, d\lambda &= \sum_{l=1}^{n} \sum_{m=1}^{q_{(l)}} \int_{E^{(i)}_{l,m}} f\, d\lambda = \sum_{l=1}^{n} \sum_{m=1}^{q_{(l)}} \frac{\lambda(E^{(i)}_{l,m})}{\lambda(K_{l,m})} \int_{K_{l,m}} f\, d\lambda \\
&= c_i \sum_{l=1}^{n} \frac{p_{i,l}}{q_i} \frac{q_{(l)}}{\lambda(K_l)} \int_{K_l} f\, d\lambda = c_i \sum_{l=1}^{n} \left(\frac{p_{i,l}}{q_i} - \frac{\lambda(K_l)}{q_{(l)}\lambda(K)} \right) \frac{q_{(l)}}{\lambda(K_l)} \int_{K_l} f\, d\lambda \\
&= \lambda(E^{(i)}) \sum_{l=1}^{n} (w_i)_l q_{(l)} \fint_{K_l} f\, d\lambda.
\end{aligned}$$

Therefore, $\int_{E^{(i)}} f\, d\lambda = 0$ whenever $w_i = 0$. In general, we have

$$\begin{aligned}
\left\| \fint_{E^{(i)}} f\, d\lambda \right\| &\leqslant \sum_{l=1}^{n} \left\| (w_i)_l q_{(l)} \fint_{K_l} f\, d\lambda \right\| \leqslant n q_{(0)} \|w_i\|_\infty \|f\|_\infty \\
&\leqslant \frac{1}{q_i} \frac{n^2 M^2 q_{(0)} \|f\|_\infty}{N} \leqslant \frac{\varepsilon'}{q_i}.
\end{aligned}$$

Next, define the function $f^{(i)} \colon E^{(i)} \to V$ by

$$f^{(i)} := f|_{E^{(i)}} - \fint_{E^{(i)}} f\, d\lambda;$$

this function is mean zero and continuous. Moreover, for $j \in \{1, \ldots, q_i\}$, we have

$$\operatorname{Diam}(f^{(i)}(\widetilde{A_{i,j}})) = \operatorname{Diam}\big(f(\widetilde{A_{i,j}})\big) < \frac{\varepsilon'}{(1 + C_V)(\|f\|_\infty + 1)}.$$

So, by Theorem 6.1, if $f^{(i)} \neq 0$, there exist a cyclic permutation $\{A_{i,j}\}_{j=1}^{n}$ of the sets $\{\widetilde{A_{i,j}}\}_{j=1}^{n}$ and a mod 0 automorphism $T^{(i)}$ of $E^{(i)}$ such that $T^{(i)}(A_{i,j}) = A_{i,j+1}$, for $j = 1, \ldots, q_i - 1$ and $T^{(i)}(A_{i,q_i}) = A_{i,1}$, there exists a function $g^{(i)} \in L_\infty(E^{(i)}; V)$ such that $\|g^{(i)}\|_\infty \leqslant S_V \|f^{(i)}\|_\infty + \varepsilon'$, $\|g^{(i)}|_{A_{i,1}}\|_\infty \leqslant \varepsilon'$, and $f = g^{(i)} \circ T^{(i)} - g^{(i)}$.

For $f^{(i)} = 0$, this can also be done by simply putting $g^{(i)} = 0$ and taking $T^{(i)}$ in the given form.

Next, we define the a transformation $T\colon \bigcup_{i\geqslant 1} E^{(i)} \to \bigcup_{i\geqslant 1} E^{(i)}$ by $T|_{A_{i,j}} := T^{(i)}|_{A_{i,j}}$. We also define $A = \bigcup_{i\geqslant 1} A_{i,1}$ and $h\colon A \to V$, where $h|_{A_{i,1}} = \sum_{j=1}^{q_i} f\circ T^{j-1}$. We then have

$$
\begin{aligned}
\|h|_{A_{i,1}}\|_\infty &\leqslant \biggl\|\sum_{j=1}^{q_i} f^{(i)}\circ T^{j-1}\biggr\|_{L_\infty(A_{i,1};V)} + q_i\biggl\|⨍_{E^{(i)}} f\,d\lambda\biggr\| \\
&\leqslant \biggl\|\sum_{j=1}^{q_i} f^{(i)}\circ (T^{(i)})^{j-1}\biggr\|_{L_\infty(A_{i,1};V)} + \varepsilon' \leqslant \|g^{(i)}\circ (T^{(i)})^{q_i} - g^{(i)}\|_{L_\infty(A_{i,1};V)} + \varepsilon' \\
&\leqslant \|g^{(i)}\circ (T^{(i)})^{q_i}\|_{L_\infty(A_{i,1};V)} + \|g^{(i)}\|_{L_\infty(A_{i,1};V)} + \varepsilon' \leqslant 2\|g^{(i)}|_{A_{i,1}}\|_\infty + \varepsilon' \leqslant 3\varepsilon',
\end{aligned}
$$

and hence $\|h\|_\infty < \varepsilon$. Next, for $l = 1, \dots, q_i$, we have

$$
\begin{aligned}
\biggl\|\sum_{j=1}^{l-1} f\circ T^{j-1}\biggr\|_{L_\infty(A_{i,1};V)} &\leqslant \biggl\|\sum_{j=1}^{l-1} f^{(i)}\circ T^{j-1}\biggr\|_{L_\infty(A_{i,1};V)} + q_i\biggl\|⨍_{E^{(i)}} f\,d\lambda\biggr\| \\
&\leqslant \|g^{(i)}\circ T^l - g^{(i)}\|_{L_\infty(A_{i,1};V)} + \varepsilon' \leqslant \|g^{(i)}\circ T^l\|_{L_\infty(A_{i,1};V)} + \|g^{(i)}\|_{L_\infty(A_{i,1};V)} + \varepsilon' \\
&\leqslant \|g^{(i)}\|_\infty + \|g^{(i)}|_{A_{i,1}}\|_\infty + \varepsilon' \leqslant (S_V\|f^{(i)}\|_\infty + \varepsilon') + \varepsilon' + \varepsilon' \\
&\leqslant S_V\|f\|_\infty + 3\varepsilon' < S_V\|f\|_\infty + \varepsilon.
\end{aligned}
$$

We will now show that there exists a subset $A' \subseteq A$ of positive measure such that $h|_{A'}$ is mean zero. If $\mathcal{J}$ is empty, then $w_i = 0$ for $i \geqslant 1$, and so we have

$$
\int_A h\,d\lambda = \sum_{i=1}^\infty \int_{A_{i,1}} h\,d\lambda = \sum_{i=1}^\infty \int_{E^{(i)}} f\,d\lambda = 0.
$$

Therefore, we can take $A' = A$. Now assume that $\mathcal{J}$ is non-empty. Let $u \in V$ and define $v \in \mathbb{R}^{\mathcal{J}}$ by

$$
v_j = \sum_{l=1}^n \frac{a_{l,j}}{b_l}\, q_{(l)}\Bigl(u, ⨍_{K_l} f\,d\lambda\Bigr).
$$

For $i \geqslant 1$, we have

$$
\begin{aligned}
\Bigl(u, ⨍_{A_{i,1}} h\,d\lambda\Bigr) &= \frac{q_i}{\lambda(E^{(i)})}\Bigl(u, \int_{E^{(i)}} f\,d\lambda\Bigr) = q_i\Bigl(u, \sum_{l=1}^n (w_i)_l q_{(l)} ⨍_{K_l} f\,d\lambda\Bigr) \\
&= q_i \sum_{l=1}^n (w_i)_l q_{(l)}\Bigl(u, ⨍_{K_l} f\,d\lambda\Bigr) = q_i \sum_{l=1}^n \Bigl(\sum_{j\in\mathcal{J}} \frac{a_{l,j}}{b_l}(\widetilde{w_i})_j\Bigr) q_{(l)}\Bigl(u, ⨍_{K_l} f\,d\lambda\Bigr) \\
&= q_i \sum_{j\in\mathcal{J}} (\widetilde{w_i})_j \sum_{l=1}^n \frac{a_{l,j}}{b_l}\, q_{(l)}\Bigl(u, ⨍_{K_l} f\,d\lambda\Bigr) = q_i(\widetilde{w_i}, v).
\end{aligned}
$$

Now, for $v = 0$, this expression is zero for all i, that is, u is orthogonal to the subspace spanned by $\bigl\{⨍_{A_{i,1}} h\,d\lambda\colon i \geqslant 1\bigr\}$. If $v \neq 0$, then, by the above, there exists $i \geqslant 1$ such that

$$
\Bigl(u, ⨍_{A_{i,1}} h\,d\lambda\Bigr) = q_i(\widetilde{w_i}, v) > 0.
$$

This means that $0 \in \operatorname{Conv}\big(\{ ⨍_{A_{i,1}} h\,d\lambda \colon i \geqslant 1\}\big)$. Indeed, suppose on the contrary that $0 \notin \operatorname{Conv}\big(\{ ⨍_{A_{i,1}} h\,d\lambda \colon i \geqslant 1\}\big)$. By the Hahn–Banach theorem, there exists $u \in \mathbb{R}^{\mathcal{J}}$ such that $\big(u, \int_{A_{i,1}} h\,d\lambda\big) < 0$ for all i. But this contradicts the fact that, for every non-zero v, there exists $\widetilde{w_i}$ such that $(\widetilde{w_i}, v) > 0$. Hence $0 \in \operatorname{Conv}\big(\{ ⨍_{A_{i,1}} h\,d\lambda \colon i \geqslant 1\}\big)$, and therefore, by Theorem 2.3, there is a subset $A' \subseteq A$ of positive measure on which h is mean zero.

We now set $A'_{i,j} := T^{j-1}(A_{i,1} \cap A')$ for $i \geqslant 1$ and $1 \leqslant j \leqslant q_i$. We also set $A'_0 := \bigcup_{i\geqslant 1} \bigcup_{j=1}^{q_i} A'_{i,j}$, and define $T' \colon A'_0 \to A'_0$ as $T'|_{A'_{i,j}} = T|_{A'_{i,j}}$ for $j < q_i$ and as $T'|_{A'_{i,q_i}} = T|^{-q_i+1}_{A'_{i,q_i}}$. Finally, we put $h' = h|_{A'_0}$. Now the partition $\{A'_{i,j} \colon i \geqslant 1, 1 \leqslant j \leqslant q_i\}$ of A'_0, the function h' on A', and the transformation T' of A'_0 have all the properties required in the lemma (save for the fact that $A'_0 \neq [0,1]$). However, by Zorn's lemma, we can iterate this argument to obtain a partition of the entire interval $[0,1]$, thereby completing the proof. Lemma 8.3 is proved.

Remark 8.1. There is a relation between the partition $\{A_{i,j} \colon i \geqslant 1, 1 \leqslant j \leqslant q_i\}$ constructed in Lemma 8.3 and the collection of disjoint sets $\{I_{i,j} \colon 1 \leqslant j \leqslant w, 1 \leqslant i \leqslant h_j\}$ constructed in Lemma 12.4 of [5]. The sets $\{I_{i,j}\}$ (referred to as W-TUB(ε, M, h, w)) are used together with a certain transformation τ that maps $I_{i,j}$ to $I_{i+1,j}$ for $i = 1, \dots, h_j - 1$. This difference is that $\{I_{i,j}\}$ is finite, and $\{A_{i,j}\}$ is countably infinite. Also, to construct the collection $\{I_{i,j}\}$, the function f has to take infinitely many values, while the partition $\{A_{i,j}\}$ can always be constructed. Furthermore, the sets $\{I_{i,j}\}$ do not partition the entire interval like the sets $\{A_{i,j}\}$, though the function f is still mean zero on their union. Some bounds that hold for the W-TUB are $\big|\sum_{i=0}^{k} f(\tau^i(x))\big| \leqslant M\|f\|_\infty$ for $x \in I_{1,j}$, $k < h_j$ and $\big|\sum_{i=0}^{h_j-1} f(\tau^i(x))\big| < \varepsilon$ for $x \in I_{1,j}$. These conditions are similar to those given in Lemma 8.3.

Now, we are fully prepared to start the proof of our main result. We explain the main idea of the proof. Intuitively, in order to solve the equation for the function f, we build another bounded mean zero function $\widetilde{h}^{(1)}$ on a smaller domain. We then solve the equation for the function $\widetilde{h}^{(1)}$, and from this, we obtain a solution for the function f itself. However, we solve the equation for the function $\widetilde{h}^{(1)}$ by building yet another bounded mean zero function $\widetilde{h}^{(2)}$, on an even smaller domain, and solving the equation for this function. It follows inductively that we first have to build an entire sequence of bounded mean zero functions $(\widetilde{h}^{(k)})_{k\geqslant 0}$ on nested domains $A^{(0)} \supseteq A^{(1)} \supseteq \cdots$. Once this is done, we can in fact solve the equation for all these functions simultaneously. In particular, we find a solution for the function f. By adding coordinate functions to the function $\widetilde{h}^{(k)}$ in every step of the construction, we ensure that the final transformation is also ergodic.

Let now prove the following result, which implies both Theorems 1.2 and 1.3.

Theorem 8.1. *Let V be a finite dimensional normed real vector space. Let $f \in L_\infty([0,1]; V)$ be a mean zero function and let $\varepsilon > 0$. Then there exist a function $g \in L_\infty([0,1]; V)$ and an ergodic* mod 0 *automorphism T of $[0,1]$ such that $\|g\|_\infty \leqslant (S_V + \varepsilon)\|f\|_\infty$ and $f = g \circ T - g$.*

Furthermore, there exists a subset X of $[0,1]$ of positive measure such that $\big\|\sum_{j=0}^{k} f \circ T^j\big\|_{L_\infty(X;V)} \leqslant (S_V + \varepsilon)\|f\|_\infty$ for all $k \geqslant 0$.

Proof. Let $(V, \|\cdot\|_V)$, f and ε be as in the theorem. We set $d = \dim V$. Given $k \geqslant 0$, we let $V_k = V \times \mathbb{R}^k$ denote the $(d+k)$-dimensional vector space with norm $\|(v,w)\|_{V_k} = \|v\|_V + \|w\|_1$. Let $\{D_l\}_{l\geqslant 1}$ be the sequence of all sets of the form

$$\left\{\bigcup_{i=1}^{N}(a_i,b_i)\colon N\in\mathbb{N},\ a_i,b_i\in\mathbb{Q},\ 0\leqslant a_i\leqslant b_i\leqslant 1\right\}.$$

We define the corresponding mean zero functions $Z_l^{(0)}\colon A^{(0)} \to \mathbb{R}$ by

$$Z_l^{(0)} = \chi_{D_l} - \lambda(D_l).$$

We can assume that $f \neq 0$, since otherwise the required result is trivial. First, we set $\varepsilon' = \varepsilon\|f\|_\infty/(2(S_V+1)) > 0$, and for $k \geqslant 0$, we define

$$\varepsilon_k' = \frac{\varepsilon'}{2^{k+2}(d+k+1)} > 0. \tag{8.1}$$

We also put $\widetilde{h}^{(0)} := f$ and $A^{(0)} := [0,1]$. Since $\widetilde{h}^{(0)}$ is mean zero, by Lemma 8.3 there exist a sequence $(q_i^{(0)})_{i\geqslant 1}$ in $\mathbb{N}\setminus\{1\}$ such that $(q_i^{(0)})_{i\geqslant 2}$, a partition $\{A_{i,j}^{(0)} : i \geqslant 1,\ 1 \leqslant j \leqslant q_i^{(0)}\}$ of $A^{(0)}$, and a mod 0 automorphism $T^{(0)}$ of $A^{(0)}$ such that $T^{(0)}(A_{i,j}^{(0)}) = A_{i,j+1}^{(0)}$ for $j = 1,\dots,q_i^{(1)} - 1$ and $T^{(0)}(A_{i,q_i^{(0)}}^{(0)}) = A_{i,1}^{(0)}$. Moreover, this can be done in such a way that if we denote $A^{(1)} = \bigcup_{i\geqslant 1} A_{i,1}^{(0)}$ and define the function $h^{(1)}\colon A^{(1)} \to V_0$ by $h^{(1)}|_{A_{i,1}^{(0)}} = \sum_{j=1}^{q_i^{(0)}} \widetilde{h}^{(0)} \circ (T^{(0)})^{j-1}$, then $h^{(1)}$ is mean zero and $\|h^{(1)}\|_\infty \leqslant \varepsilon_1'$. Furthermore, for $i \geqslant 1$ and $1 \leqslant l \leqslant q_i$, Lemma 8.3 gives the bound $\left\|\sum_{j=1}^{l-1} \widetilde{h}^{(0)} \circ (T^{(0)})^{j-1}\right\|_{L_\infty(A_{i,1}^{(0)};V)} \leqslant S_V\|\widetilde{h}^{(0)}\|_\infty + \varepsilon_1'$.

For $l \geqslant 1$, define the function $Z_l^{(1)}\colon A^{(1)} \to \mathbb{R}$ by $Z_l^{(1)}|_{A_{i,1}^{(0)}} = \sum_{j=1}^{q_i^{(0)}} Z_l^{(0)} \circ (T^{(0)})^{j-1}$. Note that

$$\begin{aligned}\|Z_l^{(1)}\|_1 &= \sum_{i\geqslant 1}\int_{A_{i,1}^{(0)}} |Z_l^{(1)}|\,d\lambda \leqslant \sum_{i\geqslant 1}\sum_{j=1}^{q_i^{(0)}}\int_{A_{i,1}^{(0)}} |Z_l^{(0)}|\circ (T^{(0)})^{j-1}\,d\lambda\\ &= \sum_{i\geqslant 1}\sum_{j=1}^{q_i^{(0)}}\int_{A_{i,j}^{(0)}} |Z_l^{(0)}|\,d\lambda = \int_{A^{(0)}} |Z_l^{(0)}|\,d\lambda = \|Z_l^{(0)}\|_1,\end{aligned}$$

which implies that $Z_l^{(1)} \in L_1(A^{(1)};\mathbb{R})$. Moreover, in the actual fact

$$\begin{aligned}\int_{A^{(1)}} Z_l^{(1)}\,d\lambda &= \sum_{i\geqslant 1}\int_{A_{i,1}^{(0)}} Z_l^{(1)}\,d\lambda = \sum_{i\geqslant 1}\sum_{j=1}^{q_i^{(0)}}\int_{A_{i,1}^{(0)}} Z_l^{(0)}\circ (T^{(0)})^{j-1}\,d\lambda\\ &= \sum_{i\geqslant 1}\sum_{j=1}^{q_i^{(0)}}\int_{A_{i,j}^{(0)}} Z_l^{(0)}\,d\lambda = \int_{A^{(0)}} Z_l^{(0)}\,d\lambda = 0,\end{aligned}$$

and hence $Z_l^{(1)}$ is mean zero. As the bounded functions are dense in $L_1(A^{(1)};\mathbb{R})$, we can find a $\widehat{Z}_1^{(1)} \in L_\infty(A^{(1)};\mathbb{R})$ such that $\|Z_1^{(1)} - \widehat{Z}_1^{(1)}\|_1 \leqslant \varepsilon_1'/2$. As $Z_1^{(1)}$ is mean zero, we have $\left|\int_{A^{(1)}} \widehat{Z}_1^{(1)}\, d\lambda\right| \leqslant \|Z_1^{(1)} - \widehat{Z}_1^{(1)}\|_1 \leqslant \varepsilon_1'/2$. We now define a mean zero function $\widetilde{Z}_1^{(1)}$ in $L_\infty(A^{(1)};\mathbb{R})$ by $\widetilde{Z}_1^{(1)} = \widehat{Z}_1^{(1)} - ⨍_{A^{(1)}} \widehat{Z}_1^{(1)}\, d\lambda$. Hence

$$\|Z_1^{(1)} - \widetilde{Z}_1^{(1)}\|_1 \leqslant \|Z_1^{(1)} - \widehat{Z}_1^{(1)}\|_1 + \left|\int_{A^{(1)}} \widehat{Z}_1^{(1)}\, d\lambda\right| \leqslant \frac{\varepsilon_1'}{2} + \frac{\varepsilon_1'}{2} = \varepsilon_1'.$$

We now define $\widetilde{h}^{(1)} \in L_\infty([0,1];V_1)$ by $\widetilde{h}^{(1)} = (h^{(1)},\, \varepsilon_1'\widetilde{Z}_1^{(1)}/(\|\widetilde{Z}_1^{(1)}\|_\infty + 1))$. Hence

$$\|\widetilde{h}^{(1)}\|_\infty = \|h^{(1)}\|_\infty + \frac{\varepsilon_1'\|\widetilde{Z}_1^{(1)}\|_\infty}{\|\widetilde{Z}_1^{(1)}\|_\infty + 1} \leqslant 2\varepsilon_1'.$$

As $\widetilde{h}^{(1)}$ is mean zero, we can apply to this function the same construction as for $\widetilde{h}^{(0)}$.

Continuing this process by induction on $k \geqslant 0$, we can find:

- a sequence $(q_i^{(k)})_{i\geqslant 1}$ in $\mathbb{N}$ with $q_i^{(k)} \geqslant 2$;
- a partition $\{A_{i,j}^{(k)} : i \geqslant 1, 1 \leqslant j \leqslant q_i^{(k)}\}$ of $A^{(k)}$ of sets of positive measure;
- a mod 0 automorphism $T^{(k)}\colon A^{(k)} \to A^{(k)}$ defined by $T^{(k)}(A_{i,j}^{(k)}) = A_{i,j+1}^{(k)}$ for $j < q_i^{(k)}$ and $T^{(k)}(A_{i,q_i}^{(k)}) = A_{i,1}^{(k)}$;
- a set $A^{(k+1)} = \bigcup_{i\geqslant 1} A_{i,1}^{(k)}$;
- a mean zero function $h^{(k+1)}\colon A^{(k+1)} \to V_k$ given by

$$h^{(k+1)}|_{A_{i,1}^{(k)}} = \sum_{j=1}^{q_i^{(k)}} \widetilde{h}^{(k)} \circ (T^{(k)})^{j-1}; \tag{8.2}$$

- for $l \geqslant 1$, the mean zero functions $Z_l^{(k+1)} \in L_1(A^{(k+1)},\mathbb{R})$ given by

$$Z_l^{(k+1)}|_{A_{i,1}^{(k)}} = \sum_{j=1}^{q_i^{(k)}} Z_l^{(k)} \circ (T^{(k)})^{j-1}; \tag{8.3}$$

- a mean zero function $\widetilde{Z}_{k+1}^{(k+1)} \in L_\infty(A^{(k+1)};\mathbb{R})$ such that

$$\|Z_{k+1}^{(k+1)} - \widetilde{Z}_{k+1}^{(k+1)}\|_1 \leqslant \varepsilon_{k+1}'; \tag{8.4}$$

- a mean zero function $\widetilde{h}^{(k+1)} \in L_\infty(A^{(k+1)};V_{k+1})$ given by

$$\widetilde{h}^{(k+1)} = \left(h^{(k+1)}, \frac{\varepsilon_k' \widetilde{Z}_{k+1}^{(k+1)}}{\|\widetilde{Z}_{k+1}^{(k+1)}\|_\infty + 1}\right). \tag{8.5}$$

Note thats, for $k \geqslant 0$, the construction shows that

$$\|h^{(k+1)}\|_\infty \leqslant \varepsilon_{k+1}', \qquad \|\widetilde{h}^{(k+1)}\|_\infty \leqslant 2\varepsilon_{k+1}',$$

and, for $i \geqslant 1$ and $j = 1, \dots, q_i^{(k)}$,

$$\Big\| \sum_{l=1}^{j-1} \widetilde{h}^{(k)} \circ (T^{(k)})^{l-1} \Big\|_{L_\infty(A_{i,1}^{(k)}; S_{V_k})} \leqslant S_{V_k} \|\widetilde{h}^{(k)}\|_\infty + \varepsilon'_k. \tag{8.6}$$

We will define the transformation T, the function g, and the set X. For $k \geqslant 0$, let the mapping $P^{(k)} \colon A^{(k)} \to A^{(k+1)}$ be defined by

$$P^{(k)}|_{A_{i,j}^{(k)}} := (T^{(k)})^{1-j}; \tag{8.7}$$

this mapping "projects" points from $A^{(k)}$ to points from $A^{(k+1)}$. Let us construct the mod 0 automorphisms $T_k \colon [0,1] \to [0,1]$ for $k \geqslant 0$ as follows. We set $T_k|_{A_{i,j}^{(k)}} := T^{(k)}$ for $i \geqslant 1$ and $1 \leqslant j < q_i^{(k)}$, define $T_k|_{A^{(k)}_{i,q_i^{(k)}}} := P^{(k)}|_{A^{(k)}_{i,q_i^{(k)}}}$ for $i \geqslant 1$. We also set $T_k|_{[0,1]\setminus A^{(k)}} = \mathrm{Id}_{[0,1]\setminus A^{(k)}}$. For $k_0 \geqslant 0$, consider the transformations $R_{k_0} \colon A^{(k_0)} \to A^{(k_0)}$ defined by

$$R_{k_0} := \lim_{N\to\infty} T_N \circ T_{N-1} \circ \dots \circ T_{k_0}|_{A^{(k_0)}}, \tag{8.8}$$

where the convergence is with respect to the measure topology. Indeed the limit exists since $A^{(k+1)} \subseteq A^{(k)}$ for $k \geqslant 1$, $\lim_{k\to\infty} \lambda(A^{(k)}) = 0$, and $T_{k'}|_{[0,1]\setminus A^{(k)}} = \mathrm{Id}_{[0,1]\setminus A^{(k)}}$ for all $k' \geqslant k$. Similarly, the limit of the inverses exists. Now, since T_k for $k \geqslant 0$ are mod 0 automorphisms, it follows from Lemma 8.2 that R_{k_0} is a mod 0 automorphism. We now define the final transformation by $T := R_0$.

Next, we define $g_k \colon A^{(k)} \to V_k$ by

$$g_k|_{A_{i,j}^{(k)}} := \Big(\sum_{l=1}^{j-1} \widetilde{h}^{(k)} \circ (T^{(k)})^{l-1} \Big) \circ P^{(k)}. \tag{8.9}$$

Note that he function g_k vanishes t on $A^{(k+1)} = \bigcup_{i\geqslant 1} A_{i,1}^{(k)}$. Given integers $k_1 \geqslant k_2$, we define the coordinate projections $p_{k_1,k_2} \colon V_{k_1} \to V_{k_2}$ by

$$p_{k_1,k_2}(v, w_1, \dots, w_{k_1}) := (v, w_1, \dots, w_{k_2}). \tag{8.10}$$

We now define, for $k_0 \geqslant 0$, the functions $r_{k_0} \colon A^{(k_0)} \to V_{k_0}$ by

$$r_{k_0} := \sum_{j=k_0}^{\infty} p_{j,k_0} \circ g_j \circ P^{(j-1)} \circ \dots \circ P^{(k_0)}. \tag{8.11}$$

Let us show that these series converge. Indeed, for $k \geqslant 0$, we have $S_{V_k} \leqslant \dim V_k = d + k$ [15], and hence, for $k_0 \geqslant 0$, we have

$$\begin{aligned}
\sum_{j=k_0}^{\infty} \|p_{j,k_0} \circ g_j\|_\infty &\leqslant \sum_{k=0}^{\infty} \|p_{k,k_0} \circ g_k\|_\infty \leqslant \sum_{k=0}^{\infty} \|g_k\|_\infty \\
&\leqslant \|g_0\|_\infty + \sum_{k\geqslant 1} \max_{\substack{i\geqslant 1\\ 1\leqslant j\leqslant q_i^{(k)}}} \|g_k|_{A_{i,j}^{(k)}}\|_\infty \overset{(8.6)}{\leqslant} \|g_0\|_\infty + \sum_{k\geqslant 1} \big(S_{V_k} \|\widetilde{h}^{(k)}\|_\infty + \varepsilon'_k \big) \\
&\leqslant \|g_0\|_\infty + \sum_{k\geqslant 1} (2S_{V_k} + 1)\varepsilon'_k \overset{(8.1)}{\leqslant} \|g_0\|_\infty + \sum_{k\geqslant 1} \frac{\varepsilon'}{2^{k+1}} \leqslant \|g_0\|_\infty + \varepsilon' < \infty.
\end{aligned}$$

This shows that series (8.11) converges absolutely. Hence $r_{k_0} \in L_\infty(A^{(k_0)}; V_{k_0})$ is well-defined.

We now set $g := r_0$ and define $X := A^{(1)}$. Let us now prove the theorem.

1) Let us first estimate $\|g\|_\infty$. By the above,

$$\|g\|_\infty \leqslant \sum_{k\geqslant 0} \|p_{k,0} \circ g_k\|_\infty \leqslant \|g_0\|_\infty + \varepsilon' \leqslant \left(\max_{\substack{i\geqslant 1 \\ 1\leqslant j\leqslant q_i^{(0)}}} \|g_0|_{A_{i,j}^{(0)}} \right) + \varepsilon'$$

$$\overset{(8.6)}{\leqslant} (S_{V_0}\|\widetilde{h}^{(0)}\|_\infty + \varepsilon_0') + \varepsilon' \leqslant S_V\|f\|_\infty + 2\varepsilon' \leqslant (S_V + \varepsilon)\|f\|_\infty,$$

which gives us an estimate for $\|g\|_\infty$.

2) Let us now show that $\widetilde{h}^{(k_0)} = r_{k_0} \circ R_{k_0} - r_{k_0}$ for all $k_0 \geqslant 0$. In particular, this will imply that $f = g \circ T - g$.

For $x \in A^{(k_0)}$ and $k \geqslant k_0$, we set

$$x_k := P^{(k-1)} \circ \cdots \circ P^{(k_0)}(x) \in A^{(k)}. \tag{8.12}$$

Let $x_{k_0} = x$. Next, for $k \geqslant k_0$, we define

$$B^{(k)} := \bigcup_{i\geqslant 1} A^{(k)}_{i,q_i^{(k)}}.$$

If $x_k \in B^{(k)}$ for some $k \geqslant k_0$, then $T_k(x_k) = x_{k+1}$. Therefore, in the case $x_k \in B^{(k)}$ for all $k \geqslant k_0$, we have $R_{k_0}(x) \in \bigcap_{k\geqslant 1} A^{(k)}$. Since R_{k_0} is measure preserving and $\lim_{k\to\infty} \lambda(A^{(k)}) = 0$, we find that, for almost all $x \in A^{(k_0)}$, there exists $N \geqslant k_0$ such that $x_N \notin B^{(N)}$. Let $N(x)$ be the smallest integer (not smaller than k_0) with this property. We will assume that $N(x)$ is finite for all $x \in A^{(k_0)}$. For $k = k_0, \dots, N(x) - 1$, we have $x_k \in B^{(k)}$, and so $T_k(x_k) = x_{k+1}$. Hence

$$x_{N(x)} = T_{N(x)-1} \circ \cdots \circ T_{k_0}(x). \tag{8.13}$$

By (8.12) and from the definition of $N(x)$, we have $x_{N(x)} \in A^{(N(x))} \setminus B^{(N(x))}$, and, therefore, $x_{N(x)} \in A^{(N(x))}_{i,j}$ for some $i \geqslant 1$ and $1 \leqslant j \leqslant q_i^{(N(x))} - 1$. Hence

$$T_{N(x)}(x_{N(x)}) = T^{(N(x))}(x_{N(x)}) \in A^{(N(x))}_{i,j+1} \tag{8.14}$$

and, therefore,

$$\begin{aligned} P^{(N(x))} \circ T_{N(x)}(x_{N(x)}) &= (T^{(N(x))})^{1-(j+1)}(T^{(N(x))}(x_{N(x)})) \\ &= (T^{(N(x))})^{1-j}(x_{N(x)}) = P^{(N(x))}(x_{N(x)}). \end{aligned} \tag{8.15}$$

Next, by (8.14) we have $T_{N(x)}(x_{N(x)}) \in A^{(N(x))} \setminus A^{(N(x)+1)}$, and hence

$$\begin{aligned} R_{k_0}(x) &\overset{(8.8)}{=} \lim_{M\to\infty} T_M \circ \cdots \circ T_{k_0}(x) \overset{(8.13)}{=} \lim_{M\to\infty} T_M \circ \cdots \circ T_{N(x)}(x_{N(x)}) \\ &= T_{N(x)}(x_{N(x)}) \in A^{(N(x))}. \end{aligned} \tag{8.16}$$

An appeal to (8.15) and (8.16) shows that

$$P^{(N(x))} \circ R_{k_0}(x) = P^{(N(x))} \circ T_{N(x)}(x_{N(x)}) = P^{(N(x))}(x_{N(x)}). \tag{8.17}$$

Note that by definition, for $k \geqslant k_0$, $P^{(k)}$ is identical on $A^{(k+1)}$. Hence from (8.16) we have $R_{k_0}(x) \in A^{(N(x))} \subseteq \cdots \subseteq A^{(k_0)}$, for $k = k_0, \ldots, N(x) - 1$, we have

$$P^{(k)} \circ \cdots \circ P^{(k_0)} \circ R_{k_0}(x) = R_{k_0}(x). \tag{8.18}$$

Hence

$$\begin{aligned} P^{(N(x))} \circ \cdots \circ P^{(k_0)} \circ R_{k_0}(x) &\overset{(8.18)}{=} P^{(N(x))} \circ R_{k_0}(x) \overset{(8.17)}{=} P^{(N(x))}(x_{N(x)}) \\ &\overset{(8.12)}{=} P^{(N(x))} \circ \cdots \circ P^{(k_0)}(x). \end{aligned}$$

As a result, for $M \geqslant N(x)$, we obtain

$$P^{(M)} \circ \cdots \circ P^{(k_0)} \circ R_{k_0}(x) = P^{(M)} \circ \cdots \circ P^{(k_0)}(x). \tag{8.19}$$

Next, we have

$$\begin{aligned} (r_{k_0} \circ R_{k_0} - r_{k_0})(x) &\overset{(8.11)}{=} \sum_{k=k_0}^{\infty} p_{k,k_0} \circ g_k \circ P^{(k-1)} \circ \cdots \circ P^{(k_0)} \circ R_{k_0}(x) \\ &\quad - \sum_{k=1}^{\infty} p_{k,k_0} \circ g_k \circ P^{(k-1)} \circ \cdots \circ P^{(k_0)}(x) \\ &\overset{(8.19)}{=} \sum_{k=k_0}^{N(x)} p_{k,k_0} \circ g_k \circ P^{(k-1)} \circ \cdots \circ P^{(k_0)} \circ R_{k_0}(x) \\ &\quad - \sum_{k=k_0}^{N(x)} p_{k,k_0} \circ g_k \circ P^{(k-1)} \circ \cdots \circ P^{(k_0)}(x) \\ &\overset{(8.12),(8.18)}{=} \sum_{k=k_0}^{N(x)} p_{k,k_0} \circ g_k \circ R_{k_0}(x) - \sum_{k=k_0}^{N(x)} p_{k,k_0} \circ g_k(x_k). \end{aligned}$$

As a result,

$$(r_{k_0} \circ R_{k_0} - r_{k_0})(x) = \sum_{k=k_0}^{N(x)} p_{k,k_0} \circ (g_k \circ R_{k_0}(x) - g_k(x_k)). \tag{8.20}$$

Let us now consider the summands on the right to show that this expression is $\widetilde{h}^{(k_0)}(x)$.

By the definition of $N(x)$, $x_{N(x)} \in A_{i,j}^{(N(x))}$ for some $i \geqslant 1$ and $1 \leqslant j \leqslant q_i^{N(x)} - 1$, and by (8.14) we have $T_{N(x)}(x_{N(x)}) \in A_{i,j+1}^{(N(x))}$. Hence, by definition of $g_{N(x)}$,

$$\begin{aligned} g_{N(x)} \circ T_{N(x)}(x_{N(x)}) - g_{N(x)}(x_{N(x)}) \\ &\overset{(8.9)}{=} \Big(\sum_{l=1}^{j} \widetilde{h}^{(N(x))} \circ (T^{(N(x))})^{l-1} \circ P^{(N(x))} \circ T_{N(x)}(x_{N(x)})\Big) \\ &\qquad - \Big(\sum_{l=1}^{j-1} \widetilde{h}^{(N(x))} \circ (T^{(N(x))})^{l-1} \circ P^{(N(x))}(x_{N(x)})\Big) \\ &\overset{(8.15)}{=} \Big(\sum_{l=1}^{j} \widetilde{h}^{(N(x))} \circ (T^{(N(x))})^{l-1} \circ P^{(N(x))}(x_{N(x)})\Big) \\ &\qquad - \Big(\sum_{l=1}^{j-1} \widetilde{h}^{(N(x))} \circ (T^{(N(x))})^{l-1} \circ P^{(N(x))}(x_{N(x)})\Big) \\ &= \widetilde{h}^{(N(x))} \circ (T^{N(x)})^{j-1} \circ P^{(N(x))}(x_{N(x)}) \\ &\overset{(8.7)}{=} \widetilde{h}^{(N(x))} \circ (T^{N(x)})^{j-1} \circ (T^{N(x)})^{1-j}(x_{N(x)}) = \widetilde{h}^{(N(x))}(x_{N(x)}). \end{aligned}$$

This together with (8.16) gives

$$\widetilde{h}^{(N(x))}(x_{N(x)}) = g_{N(x)} \circ R_{k_0}(x) - g_{N(x)}(x_{N(x)}). \tag{8.21}$$

Let now $k_0 \leqslant k \leqslant N(x) - 1$ be fixed. By the definition of $N(x)$, $x_k \in B^{(k)}$, and hence, for some $i \geqslant 1$, we have $x_k \in A_{i,q_i^{(k)}}^{(k)}$. We also have $P^{(k)}(x_k) \in A_{i,1}^{(k)}$ by definition of

$$\begin{aligned} g_k(x_k) &\overset{(8.9)}{=} \sum_{j=1}^{q_i^{(k)}-1} \widetilde{h}^{(k)} \circ (T^{(k)})^{j-1} \circ P^{(k)}(x_k) \\ &= \Big(\sum_{j=1}^{q_i^{(k)}} \widetilde{h}^{(k)} \circ (T^{(k)})^{j-1} \circ P^{(k)}(x_k)\Big) - \big(\widetilde{h}^{(k)} \circ (T^{(k)})^{q_i^{(k)}-1} \circ P^{(k)}(x_k)\big) \\ &\overset{(8.2)}{=} h^{(k+1)}(P^{(k)}(x_k)) - \big(\widetilde{h}^{(k)} \circ (T^{(k)})^{q_i^{(k)}-1} \circ P^{(k)}(x_k)\big) \\ &\overset{(8.7)}{=} h^{(k+1)}(P^{(k)}(x_k)) - \big(\widetilde{h}^{(k)} \circ (T^{(k)})^{q_i^{(k)}-1} \circ (T^{(k)})^{1-q_i^{(k)}}(x_k)\big) \\ &= h^{(k+1)}(P^{(k)}(x_k)) - \widetilde{h}^{(k)}(x_k) \overset{(8.5),(8.12)}{=} p_{k+1,k} \circ \widetilde{h}^{(k+1)}(x_{k+1}) - \widetilde{h}^{(k)}(x_k). \end{aligned}$$

Note that, for $k \geqslant k_0$, by definition of g_k we have $g_k|_{A^{(k+1)}} = 0$. Next, $R_{k_0}(x) \in A^{(N(x))} \subseteq \cdots \subseteq A^{(k_0)}$ by (8.16), and hence, there exists $k = k_0, \ldots, N(x) - 1$ such that $g_k(R_{k_0}(x)) = 0$. Hence, for $k = k_0, \ldots, N(x) - 1$,

$$g_k \circ R_{k_0}(x) - g_k(x_k) = -g_k(x_k) = \widetilde{h}^{(k)}(x_k) - p_{k+1,k} \circ \widetilde{h}^{(k+1)}(x_{k+1}). \tag{8.22}$$

Finally, we obtain

$$(r_{k_0} \circ R_{k_0} - r_{k_0})(x) \overset{(8.20)}{=} \sum_{k=k_0}^{N(x)} p_{k,k_0} \circ [g_k \circ R_{k_0}(x) - g_k(x_k)]$$

$$\overset{(8.21)}{=} \left(\sum_{k=k_0}^{N(x)-1} p_{k,k_0} \circ [g_k \circ R_{k_0}(x) - g_k(x_k)] \right) + p_{N(x),k_0} \circ \widetilde{h}^{N(x)}(x_{N(x)})$$

$$\overset{(8.22)}{=} \left(\sum_{k=k_0}^{N(x)-1} p_{k,k_0} \circ \left[\widetilde{h}^{(k)}(x_k) - p_{k+1,k} \circ \widetilde{h}^{(k+1)}(x_{k+1})\right] \right) + p_{N(x),k_0} \circ \widetilde{h}^{N(x)}(x_{N(x)})$$

$$\overset{(8.10)}{=} \left(\sum_{k=k_0}^{N(x)-1} p_{k,k_0}(\widetilde{h}^{(k)}(x_k)) - p_{k+1,k_0}(\widetilde{h}^{(k+1)}(x_{k+1})) \right) + p_{N(x),k_0}(\widetilde{h}^{N(x)}(x_{N(x)}))$$

$$= p_{k_0,k_0}(\widetilde{h}^{(k_0)}(x_{k_0})) \overset{(8.10)}{=} \widetilde{h}^{(k_0)}(x_{k_0}) \overset{(8.12)}{=} \widetilde{h}^{(k_0)}(x).$$

This shows that $\widetilde{h}^{(k_0)} = r_{k_0} \circ R_{k_0} - r_{k_0}$ for $k_0 \geqslant 0$. Since $f = \widetilde{h}^{(0)}$, $g = r_0$ and $T = R_0$ this gives, in particular, that $f = g \circ T - g$.

3) Let us prove that T is ergodic. We fix $k \geqslant 0$ and $i \geqslant 1$. We first show that

$$R_k^{q_i^{(k)}}|_{A_{i,1}^{(k)}} = R_{k+1}|_{A_{i,1}^{(k)}}. \tag{8.23}$$

Let $x \in A_{i,j}^{(k)}$ for some $j = 1, \dots, q_i^{(k)} - 1$. Then, by definition

$$T_k(x) = T^{(k)}(x) \in A_{i,j+1}^{(k)}. \tag{8.24}$$

In particular, $T_k(x) \notin A^{(k+1)} \supseteq A^{(k+2)} \supseteq \cdots$. For $M \geqslant k+1$, T_M is identical on $A^{(M-1)} \setminus A^{(M)}$, and so, for $M \geqslant k$, we have

$$T_M \circ \cdots \circ T_k(x) = T_k(x). \tag{8.25}$$

Hence

$$R_k(x) \overset{(8.8)}{=} \lim_{M\to\infty} T_M \circ \cdots \circ T_k(x) \overset{(8.25)}{=} T_k(x) \overset{(8.24)}{=} T^{(k)}(x) \in A_{i,j+1}^{(k)}.$$

Now if $y \in A_{i,1}^{(k)}$, then it follows inductively that, for $j = 1, \dots, q_i^{(k)}$,

$$R_k^{j-1}(y) = (T^{(k)})^{j-1}(y) \in A_{i,j}^{(k)}. \tag{8.26}$$

We put $z := R_k^{q_i^{(k)}-1}(y)$. Since $z \in A_{i,q_i^{(k)}}^{(k)}$, we have, by definition of T_k and $P^{(k)}$,

$$T_k(z) = P^{(k)}(z) = (T^{(k)})^{1-q_i^{(k)}}(z) \overset{(8.26)}{=} y \in A_{i,1}^{(k)} \subseteq A^{(k+1)}. \tag{8.27}$$

As a result,

$$R_k^{q_i^{(k)}}(y) = R_k(z) \overset{(8.8)}{=} \lim_{M\to\infty} T_M \circ \cdots \circ T_k(z)$$

$$= \left(\lim_{M\to\infty} T_M \circ \cdots \circ T_{k+1}\right)\Big|_{A^{(k+1)}} \circ T_k(z) \overset{(8.8)}{=} R_{k+1} \circ T_k(z) \overset{(8.27)}{=} R_{k+1}(y).$$

Therefore,

$$R_k^{q_i^{(k)}}|_{A_{i,1}} = R_{k+1}|_{A_{i,1}}. \tag{8.28}$$

Note also that, for $j = 1, \dots, q_i^{(k)}$, it follows from (8.26) that

$$R_k^{j-1}|_{A_{i,1}^{(k)}} = (T^{(k)})^{j-1}|_{A_{i,1}^{(k)}}, \qquad R_k^{j-1}(A_{i,1}^{(k)}) = A_{i,j}^{(k)}. \tag{8.29}$$

Let $k \geqslant 0$ and let $F \subseteq A^{(k)}$ be an R_k-invariant set of positive measure. Since

$$R_{k+1}(F \cap A^{(k+1)}) = \bigcup_{i \geqslant 1} R_{k+1}(F \cap A_{i,1}^{(k)}) \overset{(8.28)}{=} \bigcup_{i \geqslant 1} R_k^{q_i^{(k)}}(F \cap A_{i,1}^{(k)}) \subseteq F,$$

and since $R_{k+1}(A^{(k+1)}) \subseteq A^{(k+1)}$, by definition of the map R_{k+1}, it follows that $R_{k+1}(F \cap A^{(k+1)}) \subseteq F \cap A^{(k+1)}$, which means that $F \cap A^{(k+1)}$ is R_{k+1}-invariant.

Now, we fix a T-invariant set $D \subseteq [0,1]$ of positive measure. Let us show that $\lambda(D) = 1$. By what has been established, it follows by induction on $k \geqslant 0$ that $D \cap A^{(k)}$ is R_k-invariant. Now we fix $k \geqslant 1$. Since $D \cap A^{(k-1)}$ is R_{k-1}-invariant, we have, for $i \geqslant 1$ and $j = 1, \dots, q_i^{(k-1)}$,

$$\begin{aligned} R_{k-1}^{j-1}(D \cap A_{i,1}^{(k-1)}) &= R_{k-1}^{j-1}(D \cap A^{(k-1)}) \cap R_{k-1}^{j-1}(A_{i,1}^{(k-1)}) \\ &\overset{(8.29)}{=} (D \cap A^{(k-1)}) \cap A_{i,j}^{(k-1)} = D \cap A_{i,j}^{(k-1)}. \end{aligned} \tag{8.30}$$

Now, for $l \geqslant 1$, we have

$$\begin{aligned} \int_{D \cap A^{(k)}} Z_l^{(k)}\, d\lambda &= \sum_{i=1}^{\infty} \int_{D \cap A_{i,1}^{(k-1)}} Z_l^{(k)}\, d\lambda \\ &\overset{(8.3)}{=} \sum_{i=1}^{\infty} \int_{D \cap A_{i,1}^{(k-1)}} \sum_{j=1}^{q_i^{(k-1)}} Z_l^{(k-1)} \circ (T^{(k-1)})^{j-1}\, d\lambda \\ &\overset{(8.29)}{=} \sum_{i=1}^{\infty} \sum_{j=1}^{q_i^{(k-1)}} \int_{D \cap A_{i,1}^{(k-1)}} Z_l^{(k-1)} \circ R_{k-1}^{j-1}\, d\lambda \\ &\overset{(8.30)}{=} \sum_{i=1}^{\infty} \sum_{j=1}^{q_i^{(k-1)}} \int_{D \cap A_{i,j}^{(k-1)}} Z_l^{(k-1)}\, d\lambda = \int_{D \cap A^{(k-1)}} Z_l^{(k-1)}\, d\lambda. \end{aligned}$$

Hence, for $k \geqslant 0$ and $l \geqslant 1$, we have

$$\int_{D \cap A^{(k)}} Z_l^{(k)}\, d\lambda = \int_{D \cap A^{(0)}} Z_l^{(0)}\, d\lambda = \lambda(D \cap D_l) - \lambda(D)\lambda(D_l).$$

On step 2) of the proof, for $k \geqslant 0$, we have shown that $\widetilde{h}^{(k)} = r_k \circ R_k - r_k$. Now by (8.5), for $k \geqslant 1$, the function $\widetilde{Z}_k^{(k)}$ can be written as an R_k-coboundary, since $(\varepsilon_k'/\|\widetilde{Z}_k^{(k)}\|_\infty + 1)\widetilde{Z}_k^{(k)} = Y_k \circ R_k - Y_k$, where Y_k is the last coordinate function of r_k. Hence, since $D\cap A^{(k)}$ is R_k-invariant, we have $\int_{D\cap A^{(k)}} \widetilde{Z}_k^{(k)}\,d\lambda = 0$. Since $\|Z_k^{(k)} - \widetilde{Z}_k^{(k)}\|_1 \leqslant \varepsilon_k'$ (see (8.4)), this implies that, for $k \geqslant 1$,

$$|\lambda(D\cap D_k) - \lambda(D)\lambda(D_k)| \leqslant \varepsilon_k'.$$

We claim that $\lambda(D) = \lambda(D)^2$. Let $\rho > 0$. By regularity of the Lebesgue measure, there exists an open set U such that $D \subseteq U$ and $\lambda(U\setminus D) < \rho$. Now U can be written as a countable union of disjoint open intervals, that is,

$$U = \bigcup_{i=1}^{\infty}(a_i, b_i),$$

where $a_i, b_i \in \mathbb{Q}$. Hence, there is an integer number $l \geqslant 1$ such that $D_l \subseteq U$ and $\lambda(U\setminus D_l) \leqslant \rho$. We can moreover choose l large enough so that $\varepsilon_l' \leqslant \rho$. From the bounds on $\lambda(U\setminus D)$ and $\lambda(U\setminus D_l)$, we find, for the symmetric difference $D\Delta D_l$, that $\lambda(D\Delta D_l) \leqslant 2\rho$. Hence $|\lambda(D) - \lambda(D\cap D_l)| \leqslant 2\rho$, $|\lambda(D_l) - \lambda(D)| \leqslant 2\rho$, and so,

$$\begin{aligned}|\lambda(D) - \lambda(D)^2| &\leqslant |\lambda(D) - \lambda(D\cap D_l)| + |\lambda(D\cap D_l) - \lambda(D)\lambda(D_l)|\\ &\quad + |\lambda(D)\lambda(D_l) - \lambda(D)^2| \leqslant 2\rho + \varepsilon_l' + 2\rho\lambda(D) \leqslant 5\rho.\end{aligned}$$

As $\rho > 0$ was arbitrary, we have $\lambda(D) = \lambda(D)^2$. This gives us $\lambda(D) = 1$, proving the ergodicity of T.

4) Let now prove the required result for the set X. As $X = A^{(1)}$, it is clear that X has positive measure. Now, to obtain the required bound, we note that the function g_0 is such that $g_0|_{A_{i,1}^{(0)}} = 0$ for $i \geqslant 1$. Since $X = A^{(1)} = \bigcup_{i\geqslant 1} A_{i,1}^{(0)}$, this means that $g_0|_X = 0$. Proceeding as in the proof of the inequality $\|g\|_\infty \leqslant (S_V + \varepsilon/2)\|f\|_\infty$, we have

$$\|g\|_{L_\infty(X;V)} \leqslant \sum_{k\geqslant 1}\|g_k\|_\infty \leqslant \varepsilon' \leqslant \frac{\varepsilon}{2}\,\|f\|_\infty.$$

Hence, for $k = 0, 1, \dots,$

$$\begin{aligned}\Big\|\sum_{j=0}^{k} f\circ T^j\Big\|_{L_\infty(X;V)} &\leqslant \|g\circ T^{k+1} - g\|_{L_\infty(X;V)}\\ &\leqslant \|g\circ T^{k+1}\|_\infty + \|g\|_{L_\infty(X;V)} \leqslant (S_V + \varepsilon)\|f\|_\infty,\end{aligned}$$

proving the required result.

Lemma 8.4. *Let $f\colon [0,1] \to \mathbb{R}^d$ with components $f_1, \dots, f_d$, and let $P_i\colon \mathbb{R}^d \to \mathbb{R}$ the projection onto ith coordinate. Then:*

(i) *$f^{-1}(X_1 \times \dots \times X_d) = \bigcap_{i=1}^{d} f_i^{-1}(X_i)$ for any $X_1, \dots, X_d \subset \mathbb{R}$;*

(ii) *if f is a measurable function, then $\sigma(f) \subset \sigma(f_1) \times \dots \times \sigma(f_d)$;*

(iii) *if $f \in L_\infty([0,1];\mathbb{R}^d)$, then $\sigma(f)$ is compact in $\mathbb{R}^d$ and $\sigma(f_i) \subset P_i(\sigma(f))$, $i = 1, \dots, d$;*

(iv) *$f \in L_\infty([0,1];\mathbb{R}^d) \Leftrightarrow f_1, \dots, f_d \in L_\infty[0,1]$;*

(v) *if $f \in L_\infty([0,1];\mathbb{R}^d)$ and if a norm on $\mathbb{R}^d$ is such that $|P_i(\,\cdot\,)| \leqslant \|\cdot\|$, then $\|f_i\|_\infty \leqslant \|f\|_\infty$, $i = 1, \dots, d$.*

Proof. First of all, let us observe that $f_1, \dots, f_d$ are measurable if and only if so is f (see [12], Lemma 2.12.5).

(i) Indeed, $t \in f^{-1}(X_1 \times \cdots \times X_d) \Leftrightarrow f(t) \in X_1 \times \cdots \times X_d \Leftrightarrow f_i(t) \in X_i$, $i = 1, \dots, d \Leftrightarrow t \in \bigcap_{i=1}^d f_i^{-1}(X_i)$.

(ii) Let $v \in \sigma(f)$, and let U_i be neighbourhoods of $P_i(v)$ in $\mathbb{R}$ for each i. Then $U := U_1 \times \cdots \times U_d$ is a neighbourhood of v. By (i), $f^{-1}(U) \subset f_i^{-1}(U_i)$, $i = 1, \dots, d$. Therefore, $\lambda(f_i^{-1}(U_i)) > 0$, that is, $P_i(v) \in \sigma(f_i)$, $i = 1, \dots, d$. Hence,

$$\sigma(f) \subset \sigma(f_1) \times \cdots \times \sigma(f_d).$$

(iii) Since $f \in L_\infty([0,1]; \mathbb{R}^d)$, it follows that $\sigma(f)$ is bounded in $\mathbb{R}^d$. So, it remains to prove that $\sigma(f)$ is closed. Assume that $\sigma(f) \ni v_n \to v$. Then, for every neighbourhood U of the point v, there exists an index n for which $v_n \in U$. In this case, $\lambda(f^{-1}(U)) > 0$. Hence, $v \in \sigma(f)$. In other words, $\sigma(f)$ is compact.

Let $1 \leqslant i \leqslant d$, $t \in \sigma(f_i)$. By (i), we have

$$\lambda\biggl(f^{-1}\biggl(\mathbb{R}^{i-1} \times \biggl[t - \frac{1}{n}, t + \frac{1}{n}\biggr] \times \mathbb{R}^{d-i}\biggr)\biggr) = \lambda\biggl(f_i^{-1}\biggl(\biggl[t - \frac{1}{n}, t + \frac{1}{n}\biggr]\biggr)\biggr) > 0$$

for every $n \in \mathbb{N}$. So, we have

$$K_n := \biggl(\mathbb{R}^{i-1} \times \biggl[t - \frac{1}{n}, t + \frac{1}{n}\biggr] \times \mathbb{R}^{d-i}\biggr) \cap \sigma(f)$$

is a non-empty compact set in $\mathbb{R}^d$ for every $n \in \mathbb{N}$. Observing that $\{K_n\}_{n=1}^\infty$ is a centred system of compacts, we infer that

$$\sigma(f) \cap (\mathbb{R}^{i-1} \times \{t\} \times \mathbb{R}^{d-i}) = \bigcap_{n=1}^\infty K_n \neq \varnothing.$$

In particular, $t \in P_i(\sigma(f))$, and, therefore, $\sigma(f_i) \subset P_i(\sigma(f))$, $i = 1, \dots, d$.

(iv) This follows from a combination of (ii) and (iii).

(v) There exists $r \in \sigma(f_i)$ such that $\|f_i\|_\infty = |r|$. By (iii), we know that $r = P_i(v)$ for some $v \in \sigma(f)$. Then $\|f_i\|_\infty = |r| = |P_i(v)| \leqslant \|v\| \leqslant \sup\{\|w\| \colon w \in \sigma(f)\} = \|f\|_\infty$. Lemma 8.4 is proved.

Proof of Corollary 1.1. Let $f \in L_\infty([0,1])$ be a complex-valued mean zero function, $f_1 := \operatorname{Re}(f)$, $f_2 := \operatorname{Im}(f) \in L_\infty[0,1]$. Then $\widetilde{f} := (f_1, f_2) \in L_\infty([0,1]; \mathbb{R}^2)$ (see Lemma 8.4 (iv)), where $\mathbb{R}^2$ is equipped with the Euclidean norm $\|\cdot\|$.

Theorem 1.2 guarantees that there exist $\widetilde{g} \in L_\infty([0,1]; \mathbb{R}^2)$ and an ergodic mod 0 automorphism T of $[0,1]$ such that

$$\widetilde{f} = \widetilde{g} \circ T - \widetilde{g}, \qquad \|\widetilde{g}\|_\infty \leqslant (S_{\mathbb{R}^2} + \varepsilon)\|\widetilde{f}\|_\infty = \|f\|_\infty.$$

Let $\widetilde{g} = (g_1, g_2)$, then $g := g_1 + ig_2 \in L_\infty[0,1]$ (see Lemma 8.4, (iv)), $\|g\|_\infty = \|\widetilde{g}\|_\infty$ and

$$f = g \circ T - g, \qquad \|g\|_\infty \leqslant \biggl(\frac{\sqrt{5}}{2} + \varepsilon\biggr)\|f\|_\infty,$$

since $S_{\mathbb{R}^2} = \sqrt{5}/2$ (see [16], Theorem 2, and [17]). Corollary is proved.

Another interesting extension of Theorem 1.1 may be stated for an arbitrary finite collection of real valued mean zero functions.

Theorem 8.2. *Let $f_1,\dots,f_n \in L_\infty[0,1]$ be mean zero real-valued functions. Then, for any $\varepsilon > 0$, there exists an ergodic $\operatorname{mod}0$ automorphism T and real-valued functions $g_1,\dots,g_n \in L_\infty[0,1]$ with $\|g_i\|_\infty \leqslant (n+\varepsilon)\|f_i\|_\infty$ such that $f_i = g_i \circ T - g_i$, $i=1,\dots,n$.*

Proof. Without loss of generality, we may assume that $\|f_i\|_\infty \neq 0$, $i=1,\dots,n$. Consider the norm on $\mathbb{R}^n$

$$\|v\| = \max_i(|v_i|), \qquad v=(v_1,\dots,v_n),$$

and define the function

$$\widetilde{f} = (\widetilde{f_1},\dots,\widetilde{f_n})\colon [0,1] \to \mathbb{R}^n,$$

where $\widetilde{f_i} = f_i/\|f_i\|_\infty$, $i=1,\dots,n$. By Lemma 8.4 (iv), (ii),

$$\widetilde{f} \in L_\infty([0,1];\mathbb{R}^n), \qquad \|\widetilde{f}\|_\infty \leqslant 1.$$

It is straightforward that $\widetilde{f}$ is a mean zero function.

By Theorem 1.2, there exists a $\widetilde{g} \in L_\infty([0,1];\mathbb{R}^n)$ and an ergodic $\operatorname{mod}0$ automorphism $\operatorname{mod}0$ T of $[0,1]$ such that

$$\widetilde{f} = \widetilde{g}\circ T - \widetilde{g}, \qquad \|\widetilde{g}\|_\infty \leqslant n+\varepsilon,$$

since $S_{\mathbb{R}^n} \leqslant n$ (see [15]).

Let $\widetilde{g} = (\widetilde{g_1},\dots,\widetilde{g_n})$, then $\widetilde{g_1},\dots,\widetilde{g_n} \in L_\infty[0,1]$ and $\|\widetilde{g_i}\|_\infty \leqslant \|\widetilde{g}\|_\infty$, $i=1,\dots,d$ (Lemma 8.4 (iv), (v)). Therefore,

$$\|\widetilde{g_i}\|_\infty \leqslant n+\varepsilon, \quad \widetilde{f_i} = \widetilde{g_i}\circ T - \widetilde{g_i}, \qquad i=1,\dots,n.$$

It remains to set $g_i = \|f_i\|_\infty \widetilde{g_i}$, $i=1,\dots,n$. Theorem 8.2 is proved.

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