

---

# CONSIDERATIONS ON THE THEORY OF TRAINING MODELS WITH DIFFERENTIAL PRIVACY

---

**Marten van Dijk<sup>1,2,3</sup> and Phuong Ha Nguyen<sup>4</sup>**

<sup>1</sup> CWI Amsterdam, The Netherlands

<sup>2</sup> Department of Computer Science, Vrije Universiteit Amsterdam, The Netherlands

<sup>3</sup> Department of Electrical and Computer Engineering, University of Connecticut, CT, USA

<sup>4</sup> eBay, CA, USA

marten.van.dijk@cwi.nl and phuongha.ntu@gmail.com

## ABSTRACT

In federated learning collaborative learning takes place by a set of clients who each want to remain in control of how their local training data is used, in particular, how can each client's local training data remain private? Differential privacy is one method to limit privacy leakage. We provide a general overview of its framework and provable properties, adopt the more recent hypothesis based definition called Gaussian DP or  $f$ -DP, and discuss Differentially Private Stochastic Gradient Descent (DP-SGD). We stay at a meta level and attempt intuitive explanations and insights *in this book chapter*.

**Keywords** Stochastic Gradient Descent (SGD) · DP-SGD · Differential Privacy (DP) · Gaussian DP

## 1 Introduction

Privacy leakage is a big problem in the big-data era. Solving a learning task based on big data intrinsically means that only through a collaborative effort sufficient data is available for training a global model with sufficient clean accuracy (utility). Federated learning is a framework where a learning task is solved by a loose federation of participating devices/clients which are coordinated by a central server [42, 8, 3, 33, 40, 9, 58, 29, 60, 10, 34, 36, 37, 39, 12, 30]. Clients, who use own local data to participate in a learning task by training a global model, want to have privacy guarantees for their local proprietary data. For this reason DP-SGD [1] was introduced as it adapts distributed Stochastic Gradient Descent (SGD)[55] with Differential Privacy (DP)[19, 15, 21, 18].

The optimization problem for training many Machine Learning (ML) models using a training set  $\{\xi_i\}_{i=1}^m$  of  $m$  samples can be formulated as a finite-sum minimization problem as follows

$$\min_{w \in \mathbb{R}^d} \left\{ F(w) = \frac{1}{m} \sum_{i=1}^m f(w; \xi_i) \right\}. \quad (1)$$

The objective is to minimize a loss function with respect to model parameters  $w$ . This problem is known as empirical risk minimization and it covers a wide range of convex and non-convex problems from the ML domain, including, but not limited to, logistic regression, multi-kernel learning, conditional random fields and neural networks.

We want to solve (1) in a distributed setting where many clients have their own local data sets and the finite-sum minimization problem is over the collection of all local data sets. A widely accepted approach is to repeatedly use the SGD [50, 46, 47] recursion

$$w_{t+1} = w_t - \eta_t \nabla f(w_t; \xi), \quad (2)$$

where  $w_t$  represents the model after the  $t$ -th iteration;  $w_t$  is used in computing the gradient of  $f(w_t; \xi)$ , where  $\xi$  is a data sample randomly selected from the data set  $\{\xi_i\}_{i=1}^m$  which comprises the union of all local data sets.

---

<sup>†</sup> This is a book chapter

This approach allows each client to perform local SGD recursions for the data samples  $\xi$  that belong to the client’s local training data set. The updates as a result of the SGD recursion (2) are sent to a centralized server who aggregates all received updates and maintains a global model. The server regularly broadcasts its most recent global model so that clients can use it in their local SGD computations. This allows each client to use what has been learned from the local data sets at the other clients. This leads to good accuracy of the final global model.

Each client is doing SGD recursions for a batch of local data. These recursions together represent a local round and at the end of the local round a local model update (in the form of an aggregate of computed gradients during the round) is transmitted to the server. The server in turn adds the received local update to its global model – and once the server receives new updates from (a significant portion of) all clients, the global model is broadcast to each of the clients. When considering privacy, we are concerned about how much information these local updates reveal about the used local data sets. Each client wants to keep its local data set as private as possible with respect to the outside world which observes round communication (the outside world includes all other clients as well).

Rather than reducing the amount of round communication such that less sensitive information is leaked, differential privacy [19, 15, 21, 18] offers a solution in which each client-to-server communication is obfuscated by noise. If the magnitude of the added noise is not too much, then a good accuracy of the global model can still be achieved albeit at the price of more overall SGD iterations needed for convergence. On the other hand, only if the magnitude of the added noise is large enough, then good differential privacy guarantees can be given. This leads to a friction between desired differential privacy and desired utility/accuracy.

Section 2 starts discussing DP-SGD [1], which implements differentially private mini-batch SGD. Section 3 explains differential privacy with various (divergence based) measures and properties. Section 4 continues detailing the state-of-the-art hypothesis testing based differential privacy, called  $f$ -DP [13], applied to DP-SGD. We conclude with open questions in Section 5.

## 2 Differential Private SGD (DP-SGD)

We analyse the Gaussian based differential privacy method, called DP-SGD, of [1] in a distributed setting with many clients and a central aggregating server. A slightly generalized description of DP-SGD is depicted in Algorithm 1. The main goal of DP-SGD is to hide whether the collection of transmitted round updates  $\bar{U}$  corresponds to data set  $d$  versus a neighboring data set  $d'$ ; sets  $d$  and  $d'$  are called neighbors if they differ in exactly one element. In order to accomplish this, DP-SGD introduces noise, which we will see comes in two flavors *clipping noise* and *Gaussian noise*.

### 2.1 Clipping

Rather than using the gradient  $a_h = \nabla f(w, \xi_h)$  itself, DP-SGD uses its clipped version  $[\nabla f(w, \xi_h)]_C$  where

$$[x]_C = x / \max\{1, \|x\|/C\}.$$

We call this the *individual clipping* approach since each computed gradient is individually clipped. Clipping is needed because in general we cannot assume a bound  $C$  on the gradients (for example, the bounded gradient assumption is in conflict with strong convexity [46]), yet the added gradients in update  $U$  need to be bounded by some constant  $C$  in order for the DP analysis of [1] to go through. The reason is that clipping introduces a bound on how much  $U = \sum_{h=1}^m [a_h]_C$  gets affected if the differentiating sample between  $d$  and  $d'$  is used in its computation. Clipping forces a small distance between an update  $U$  that does not use the differentiating sample and an update  $U'$  that computes the same gradients as  $U$  except for one of its gradient computations which uses the differentiating sample. This means that if Gaussian noise is added to  $U$  and  $U'$ , respectively, then the smaller the distance between  $U$  and  $U'$ , the harder it is to figure out whether the actually observed noised update originates from  $d$  or  $d'$ . This leads to a differential privacy guarantee.

Suppose that  $a_h$  influences another gradient computation, e.g.,  $a_{h+1}$ . Then, if the differentiating sample is used in the computation of  $a_h$ , this affects not only  $a_h$  but also  $a_{h+1}$ . Even though both  $a_h$  and  $a_{h+1}$  will be clipped, this increases the distance between  $U$  and  $U'$ , hence, this weakens the differential privacy. For this reason, the different gradient computations  $a_h$  in  $U$  should be independent of one another. In particular, we do not want to implement classical SGD where the computation of  $a_h$  updates the local model  $w$  which is used in the next gradient computation  $a_{h+1}$ . This is the reason for implementing mini-batch SGD where each gradient  $a_h$  is computed for the same  $w$ .

Clipping introduces clipping noise defined as the difference between the clipped gradient  $[a_h]_C$  and the original gradient  $a_h$ . This affects the rate of convergence. Notice that in general, once convergence sets in, individual gradients tend to get closer to zero. This means that their norms get smaller  $\leq C$ . This shows that the clipping noise in updates  $U$  becomes very small close to zero.

---

**Algorithm 1** Differential Private SGD

---

```
1: procedure DP-SGD
2:    $N =$  size training data set  $d = \{\xi_i\}_{i=1}^N$ 
3:    $E =$  total number of epochs
4:   diminishing step size sequence  $\{\eta_i\}$ 
5:
6:   initialize  $w$  as the default initial model
7:   Interrupt Service Routine (ISR): Whenever a new global model  $\hat{w}$  is received, computation is interrupted
   and an ISR is called that replaces  $w \leftarrow \hat{w}$  after which computation is resumed
8:
9:   for  $e \in \{1, \dots, E\}$  do
10:     $\{S_b\}_{b=1}^{N/m} \leftarrow \text{Sample}_{e,m}$  with  $S_b \subseteq \{1, \dots, N\}$ ,  $|S_b| = m$ 
11:    for  $b \in \{1, \dots, \frac{N}{m}\}$  do
12:      Start of round  $(e-1)\frac{N}{m} + b$ :
13:      for  $h \in S_b$  do
14:         $a_h = \nabla_w f(w; \xi_h)$ 
15:      end for
16:       $U = \sum_{h=1}^m [a_h]_C$ 
17:       $\bar{U} \leftarrow U + \mathcal{N}(0, (2C\sigma)^2 \mathbf{I})$ 
18:      Transmit  $\bar{U}/m$  to central server
19:      Locally update  $w \leftarrow w - \eta_{(e-1)\frac{N}{m} + b} \cdot \bar{U}/m$ 
20:    end for
21:  end for
22: end procedure
```

---

## 2.2 Mini-Batch SGD

DP-SGD is constraint to a mini-batch SGD approach where before the start of the  $b$ -th local round in epoch  $e$  a random min-batch  $S_b$  of sample size  $|S_b| = m$  is selected out of a local data set  $d$  of size  $|d| = N$ : In the description of Algorithm 1 the sampling is done by a sampling procedure  $\text{Sample}_{e,m}$  before the start of epoch  $e$  for all rounds together. DP-SGD implements *subsampling* which chooses a uniformly random subset  $S_b \subseteq d$  of size  $m$ .

The inner loop computes  $m$  gradients  $a_h = \nabla_w f(w; \xi_h)$ . Since there are  $N/m$  rounds within an epoch, each epoch has (indeed) a total gradient complexity of  $N = |d|$ . We notice that each gradient is computed based on  $w$  which is the last received global model from the server through the interrupt service routine. In the original DP-SGD, a client waits at the start of a round till it receives the global model which includes the aggregated updates of all previous rounds from all clients. The formulation in Algorithm 1 allows for asynchronous behavior, including dropped (or reordering of) messages from the server which can lead to a client missing out on receiving global model versions. More importantly, the server may decide to broadcast global models at a lower rate than the rate(s) at which clients compute and communicate their noised round updates. This allows clients with different compute speeds/resources. Also, the rate at which round updates are computed is not restricted by the throughput of broadcast messages from the server to clients (of course, it remains restricted by the network throughput from the clients through aggregation nodes to the server). This implies that parameter  $m$  can potentially be chosen from the whole range  $\{1, \dots, N\}$  including very small  $m$  leading to many round updates per epoch or large  $m$  leading to only a couple round updates per epoch. We will later discuss the effect of  $m$  on convergence and accuracy and DP guarantee.

We notice that too much asynchronous behavior will hurt convergence of the mini-batch SGD approach and may lead to worse accuracy of the final global model. For this reason, before starting a round, a client can check into what extent the recently received global model deviates from the locally kept model. If this gets too far apart or if the last received global model happened too many rounds ago, then the client will want to wait till a new global model is received and the interrupt service routine is triggered. This implements the necessary synchronous behavior with respect to convergence and accuracy.

## 2.3 Gaussian Noise

The clipped gradients  $[a_h]_C$  are summed together in round update  $U$ . At the end of each local round the round update  $U$  is obfuscated by adding Gaussian noise

$$\mathcal{N}(0, (2C\sigma)^2)$$

to each of  $U$ 's vector entries. The resulting noised round update  $\bar{U}$  divided by the mini-batch size  $m$  is transmitted to the server.

For neighboring data sets  $d$  and  $d'$ , we have that the *sensitivity* measured as the Euclidean distance between  $U$  based on  $d$  and  $U'$  based on  $d'$  (see also Section 2.1) is at most  $2C$ . An adversary trying to distinguish whether the observed update is from  $d$  or  $d'$  needs to figure out whether the observation is from

$$U + \mathcal{N}(0, (2C\sigma)^2 \mathbf{I}) \quad \text{or} \quad U' + \mathcal{N}(0, (2C\sigma)^2 \mathbf{I}).$$

Since  $\|U - U'\| \leq 2C$ , this is at best (for the adversary) equivalent to hypothesis testing between  $\mathcal{N}(0, (2C\sigma)^2)$  and  $\mathcal{N}(2C, (2C\sigma)^2)$ . After dividing by  $2C$ , this is equivalent to hypothesis testing between  $\mathcal{N}(0, \sigma^2)$  and  $\mathcal{N}(1, \sigma^2)$ . We see that any differential privacy guarantee for the round update is characterized by  $\sigma$ .

The attentive reader may notice that the original DP-SGD adds  $\mathcal{N}(0, (C\sigma)^2 \mathbf{I})$ , a factor 2 less. This is because its DP analysis and proof assume a slightly different subsampling method. In the original DP-SGD we have that each round selects a random mini-batch of *exactly*  $m$  samples; this leads to the factor 2 since  $U$  and  $U'$  will differ in one gradient, hence,  $U - U'$  cancels all gradients except for one in  $U$  and one in  $U'$ , both contributing at most  $C$  to the norm  $\|U - U'\|$ , hence, the factor 2.

However, the software package Opacus [48] implements the sampling of DP-SGD differently: Mini-batches do not have a fixed size, they have a probabilistic size. For each sample  $\xi \in d$ , we flip a coin and with probability  $m/N$  we add  $\xi$  to the mini-batch. This means that the *expected* mini-batch size is equal to  $m$ . As a result, the DP analysis of [1] holds true and the factor 2 can be eliminated. The reason is that now (in the DP analysis)  $U'$  has all the gradients of  $U$  together with one extra gradient based on the single differentiating sample between  $d$  and  $d'$ . This implies that all gradients in  $U - U'$  cancel except for the one based on the differentiating sample, hence,  $\|U - U'\| \leq C$ .

In the above argument, we *assume* that the adversary does not learn the actually used mini-batch size otherwise we will again need the factor 2 (see also Section 4.5). The observed scaled noised update  $\bar{U}/m$  scales in expectation with the expected norm of a single computed gradient times the used mini-batch size divided by the expected mini-batch size  $m$ . This shows how  $\bar{U}/m$  depends on the used mini-batch size where, for large  $m$  and  $N$ , it seems reasonable to assume that the adversary cannot gain significant knowledge about the used mini-batch size from  $\bar{U}/m$ . We conclude that a probabilistic mini-batch size is a DP technique that offers a factor 2 gain. This chapter summarizes the  $f$ -DP framework explained for sampling with fixed mini-batch size leading to the extra factor 2 (the probabilistic approach can be added as a complimentary technique).

## 2.4 Aggregation at the Server

The server maintains a global model, which we denote by  $\hat{w}$ . The server adds to  $\hat{w}$  the received scaled noised round update  $\bar{U}/m$  after multiplying with the round step size\* for round  $b$  of epoch  $e$ ,

$$\eta_{(e-1)\frac{N}{m}+b}$$

(the same as the local model update of  $w$  by the client). This allows a diminishing step size sequence. Notice that dividing by the mini-batch size  $m$  corresponds to  $U$  representing a mini-batch computation in mini-batch SGD.

Each client will select its own DP posture with own selected parameters  $m$ ,  $C$ ,  $\sigma$ , and own data set  $d$  with its own size  $N$ . It makes sense for the server to collect the noised round updates from various clients during consecutive time windows and broadcast updated global models at the end of each window. Rather than adding all the received  $\bar{U}$  within a time window to the global model  $\hat{w}$  (after multiplying with the appropriate client-specific step sizes and dividing by the appropriate client-specific mini-batch sizes), the server will add a mix of the various local updates. The mix is according to some weighing vector giving more weight to those clients whom the server judges having 'better' training data sets for the learning task at hand. In federated learning the server will ask for each time window a random subset of clients to participate in the training. In the above context it makes sense to have the step sizes be diminishing from time window to time window rather than have these be client specific.

## 2.5 Interrupt Service Routine

The interrupt service routine will replace the locally kept model  $w$  by a received global model  $\hat{w}$ . This may happen in the middle of a round. We notice that  $\hat{w}$  depends on previously transmitted noised round updates by the client and other clients. We will discuss how each of these previous noised round updates have a DP guarantee. By the so-called

---

\*The client transmits  $(b, e, \bar{U})$  to the server and the server knows an a-priori agreed (with the client) round step size sequence. In practice, the client will only transmit a sparsification or lossy compression of  $\bar{U}$  where small entries are discarded.

post-processing lemma, these previously transmitted noised round updates can participate in the current computation of a round update  $U$  through its dependency on the global model  $\hat{w}$  (through the gradients in  $U$ ) without weakening the DP guarantee for  $\bar{U}$  (which includes Gaussian noise on top of  $U$ ).

Similarly, the client locally updates model  $w$  with  $\bar{U}$  at the end of a round. In next rounds this implies that  $w$  still only depends on previously transmitted noised round updates by the client and other clients, and again by the post-processing lemma the DP guarantees of future noised round updates do not diminish. As soon as a new global model  $\hat{w}$  is received by the interrupt service routine it will overwrite  $w$ , that is, the current local model is discarded. This is justified because the newly received global model includes the client’s own previously communicated noised updates  $\bar{U}$  (if the corresponding messages were not dropped and did not suffer too much latency), hence, the information of its own local updates is incorporated in the newly received  $\hat{w}$ .

## 2.6 DP Principles and Utility

The strength of the resulting DP guarantee depends on how much utility we are okay with sacrificing. The differential privacy guarantee is discussed in Section 4 and turns out to be approximately equivalent to differentiating between samples from  $\mathcal{N}(0, \sigma^2)$  and  $\mathcal{N}(\sqrt{E}, \sigma^2)$  (we will also discuss group privacy where  $d$  and  $d'$  differ in a group of  $g$  samples, which will have another  $\sqrt{g}$  dependency). This shows that  $\sigma$  should be large enough in order to make hypothesis testing between the two normal distributions unreliable. The reason why we have this DP guarantee is because the principle of using Gaussian noise *bootstraps* DP for each round, the principle of subsampling in the form of random mini-batches of size  $m$  *amplifies* DP (because only with probability  $m/N$  a round uses the differentiating sample and can leak privacy in the first place), and the principle of *composition* of DP guarantees for each round over multiple epochs.

Utility is measured in terms of the (test) accuracy of the final global model and secondary metrics are convergence rate, round complexity  $(N/m) \cdot E$  calculated as the total number of rounds per client (communication is costly), total gradient complexity  $E \cdot N$  calculated as the total number of computed gradients per client, information dispersal characterized by the delay or latency of what is learned from local data sets which is calculated as the number  $m$  of gradient computations between consecutive round communications to the server, and client’s memory usage<sup>†</sup>.

The final accuracy depends on the amount of clipping noise and Gaussian noise, and also depends on the amount of delay (information dispersal) introduced by  $m$ : Once convergence sets in, we argued that clipping noise will be small and close to zero. However, each round update  $U$  has noise sampled from  $\mathcal{N}(0, (2C\sigma)^2\mathbf{I})$  added to itself. If this noise is small relative to the norm of  $U$ , then we expect accuracy not to suffer too much. When convergence progresses, the gradients in  $U$  get closer to zero and therefore the norm of  $U$  gets smaller, which means that the Gaussian noise relative to the norm of  $U$  becomes larger. Since the DP guarantee depends on  $\sigma$  but not on  $C$ , we will want to implement a form of (differential private) adaptive clipping where  $C$  is reduced when convergence progresses (we notice that the DP analysis of DP-SGD holds for clipping constants  $C$  that vary from round to round). This will allow us to contain the Gaussian noise relative to the norm of  $U$  when convergence sets in. Experimentation is needed to fine-tune the parameters  $m$ , (adaptive)  $C$ , and  $\sigma$ . Despite fine tuning, we remark that the added clipping and Gaussian noise for differential privacy results in convergence to a final global model with smaller (test) accuracy (than what otherwise, without DP, can be achieved).

The  $\approx G_{\sqrt{gE}/\sigma}$ -DP guarantee for group privacy of Section 4 does not reflect the role of the batch size  $|S_b| = m$ . This is implicitly captured in  $\sigma$ . The following thought experiment shows how: Suppose we increase  $m$  to  $am$ , a factor  $a$  larger. Then the norm of updates  $U$  will become a factor  $a$  larger. As a result, with respect to convergence to the final global model, we should be able to cope with a factor  $a$  larger Gaussian noise. That is, by keeping the relative amount of noise with respect to the norm of  $U$  constant, the new updates corresponding to batch size  $am$  can be noised with

$$a \cdot \mathcal{N}(0, (2C\sigma)^2\mathbf{I}) = \mathcal{N}(0, (2C\sigma \cdot a)^2\mathbf{I}).$$

In fact the communicated averaged noised round update  $\bar{U}/(am)$  has noise

$$a \cdot \mathcal{N}(0, (2C\sigma)^2\mathbf{I})/(am) = \mathcal{N}(0, (2C\sigma/m)^2\mathbf{I}),$$

the same as the original communicated averaged noised round update (before the thought experiment). This shows that we can use the factor  $a$  for increasing (1) the clipping constant  $C$  (which reduces the clipping noise which is prevalent at the start of DP-SGD so that convergence can more easily start) and/or increasing (2) the standard deviation  $\sigma$  (which improves the  $G_{\sqrt{gE}/\sigma}$ -DP guarantee as it gets closer to  $G_0$  for larger  $\sigma$ ); the resulting new clipping constant  $C'$  and standard deviation  $\sigma'$  satisfy  $2C'\sigma' = 2C\sigma \cdot a$ . The disadvantage of increasing the batch size with

<sup>†</sup>The mini-batch computation  $U = \sum_{h=1}^m [a_h]_C$  needs to keep track of all  $m$  gradients  $a_h = \nabla_w f(w; \xi_h)$ , while the unclipped original mini-batch SGD can keep track of  $\sum_{h=1}^m a_h = \nabla_w \sum_{h=1}^m f(w; \xi_h)$ , a single gradient computing thread.

a factor  $a$  is a multiplicative factor  $a$  increased delay, i.e., the number of gradient computations between successive round communications to the server is multiplied by a factor  $a$ , and this reduces information dispersal and may hurt convergence of the global model. Here, we note that mini-batch SGD is rather robust with respect to large delays, but experiments need to show into what extent  $m$  can be increased without affecting the accuracy of the final global model too much.

Hyperparameter search depends on the used data set. Either we adopt a hyperparameter setting from another similar learning task, or we search for hyperparameters based on the client data sets. In practice, in order to find good parameters  $m$ ,  $C$ , and  $\sigma$ , we basically do a grid search by (1) fixing some standard settings (from similar learning tasks) for sample size  $m$ , e.g., 16, 32, 64, 128 and 256 etc., (2) fixing some standard settings (from similar learning tasks) for clipping constant  $C$ , e.g., 0.001, 0.01, 0.1, etc., and then (3) trying some reasonable settings for  $\sigma$  (based on the client data sets). If the grid search indeed uses client data sets, then we need to make sure that the additional privacy leakage due to the search is small. This is discussed in Appendix D of [1], see also [27].

## 2.7 Normalization

In practice we will also want to use data normalization [53] as a pre-processing step. This requires computing the mean and variance over all data samples from  $d$ . This makes normalized data samples depend on all samples in  $d$ . For this reason we need differential private data normalization. That is, a differential private noisy mean and noisy variance is revealed a-priori. This leads to some privacy leakage. The advantage is that we can now re-write  $\mathcal{A}$  as an algorithm that takes as input  $w$ , the original data samples  $\{\xi_h\}_{h \in S_b}$  together with the revealed noisy mean and noisy variance.  $\mathcal{A}$  first normalizes each data sample after which it starts to compute gradients etc. In the  $f$ -DP framework, privacy leakage is now characterized as a trade-off function of the differential private data normalization pre-processing composed with the trade-off function corresponding to the DP analysis of DP-SGD (which does not consider data normalization).

We notice that batch normalization is not compatible with the DP analysis of DP-SGD (since this introduces dependencies among the clipped gradients in  $U$  and the upper bound of  $2C$  on the sensitivity does not hold). On the other hand layer normalization as well as group and instance normalization are compatible (because these only concern single gradient computations).

As a final remark, our discussion assumes that we already know how to represent data samples by extracting features. We can use Principal Component Analysis (PCA) for dimensionality reduction, that is, learning a set of features which we want to use to represent data samples. PCA can be made differentially private [7] in that the resulting feature extraction method (feature transform) has a DP guarantee with respect to the data samples that were used for computing the transform. DP-SGD can be seen as a post-processing after PCA, which is used to represent the local training data samples for which DP-SGD achieves a DP guarantee. In practice, we often already know how to represent the data for our learning task and we already know which function  $f(w; \xi)$  to use, i.e., which neural network topology and loss function to use (due to the success of transfer learning we can adopt data representations and  $f$  from other learning tasks).

## 3 Differential Privacy

In order to prevent data leakage from inference attacks in machine learning [38] such as the deep leakage from gradients attack [63, 62, 23] or the membership inference attack [51, 45, 52] a range of privacy-preserving methods have been proposed. Privacy-preserving solutions for federated learning are Local Differential Privacy (LDP) solutions [1, 2, 44, 54, 28, 14] and Central Differential Privacy (CDP) solutions [44, 25, 41, 49, 61]. In LDP, the noise for achieving differential privacy is computed locally at each client and is added to the updates before sending to the server – in this chapter we only consider LDP. In CDP, a *trusted server* (aka trusted third party) aggregates received client updates into a global model; in order to achieve differential privacy the server adds noise to the global model before communicating it to the clients.

Differential privacy [19, 15, 21, 18], see [16] for an excellent textbook, defines privacy guarantees for algorithms on databases, in our case a client’s sequence of mini-batch gradient computations on his/her training data set. The guarantee quantifies into what extent the output of a client (the collection of updates communicated to the server) can be used to differentiate among two adjacent training data sets  $d$  and  $d'$  (i.e., where one set has one extra element compared to the other set).

### 3.1 Characteristics of a Differential Privacy Measure

In DP-SGD, the client wants to keep its local training data set as private as possible. Each noised round update  $\bar{U}$  leaks privacy. Let us define round mechanism  $\mathcal{M}_b$  as the round computation that outputs  $\bar{U}$  for round  $b$ . The input of  $\mathcal{M}_b$  is data set  $d$  together with an updated local model  $w$ . We have the following recursion

$$\bar{U}_b \leftarrow \mathcal{M}_b(w_b; d),$$

where  $w_b$  is a function of received global model updates which themselves depend on other client's round updates in combination with own previously transmitted round updates  $\bar{U}_1, \dots, \bar{U}_{b-1}$ . To express this dependency, we use the notation

$$w_b \leftarrow W(\bar{U}_1, \dots, \bar{U}_{b-1}),$$

where  $W$  receives the global models of the server (and in essence reflects the interrupt service routine). We define the overall mechanism  $\mathcal{M}$  as the composition of all round mechanisms  $\mathcal{M}_b$ , i.e.,

$$\{\bar{U}_b\} \leftarrow \mathcal{M}(d) \text{ with } \bar{U}_b \leftarrow \mathcal{M}_b(W(\bar{U}_1, \dots, \bar{U}_{b-1}); d).$$

When defining a DP measure, we will want to be able to *compose* the DP guarantees for the different round mechanisms  $\mathcal{M}_b$ : If we can prove that  $\mathcal{M}_b(\text{aux}; \cdot)$  has a certain DP guarantee, denoted by  $\text{DP}_b$ , for *all*  $\text{aux}$ , then the composition  $\mathcal{M}$  of all round mechanisms  $\mathcal{M}_b$  should have a composed DP guarantee

$$\text{DP}_1 \otimes \text{DP}_2 \otimes \dots \otimes \text{DP}_{(N/m) \cdot E}$$

for some composition tensor  $\otimes$  over DP measures.

Once a DP guarantee for mechanism  $\mathcal{M}$  is proven, we do not want it to weaken due to *post-processing* of the output of  $\mathcal{M}$ . In particular, the central server uses the output of  $\mathcal{M}$  for keeping track of and computing a final global model for the learning task at hand. This final model should still have the same (or stronger) differential privacy posture. Let us denote the post-processing by a procedure  $P$ . If  $\mathcal{M}$  has DP guarantee  $\text{DP}$ , then we want  $P \circ \mathcal{M}$  to also have DP guarantee  $\text{DP}$  (this is called the post-processing lemma),

$$[\text{DP for } \mathcal{M}] \Rightarrow [\text{DP for } P \circ \mathcal{M}].$$

We want our DP measure to be compatible with *subsampling*: We want to be able to show that if a round mechanism  $\mathcal{M}_b$  has guarantee  $\text{DP}$  without subsampling, then  $\mathcal{M}_b \circ \text{Sample}_m$  has an 'easy' to characterize amplified guarantee  $\text{DP}'$ , ' $\text{DP}' \geq \text{DP}$ '.

Finally, we want a differential privacy measure which fits our intuition, in particular, how privacy should be characterized and in what circumstances an attacker can learn private information from observed mechanism outputs. Differential privacy measures are about the difficulty of distinguishing whether the observed output  $o$  is from the distribution  $\mathcal{M}(d)$  or from the distribution  $\mathcal{M}(d')$ , where  $d$  and  $d'$  are neighboring data sets in that they have all but one differentiating sample in common. The DP guarantee measures in to what extent

$$\Pr[o \sim \mathcal{M}(d)] \text{ and } \Pr[o \sim \mathcal{M}(d')]$$

are alike for *all* neighboring  $d$  and  $d'$ . Here, we want to reflect the intuition that for more likely observations  $o$  the two probabilities should be close together while for unlikely observations  $o$  we care less whether the two probabilities are close. This reflects how we think about the adversary: Only in rare unlikely cases, a lot or all privacy may leak, while in the common case there is very little privacy leakage. In cryptology we would want to interpret 'rare' as a negligible probability in some security parameter and in the common case we want the two probabilities/distributions to be 'statistically close' with their distance negligible in some security parameter. Such strong guarantees cannot be extracted from DP analysis where we control privacy leakage in exchange for utility/accuracy; we cannot make privacy leakage negligible.

The DP measure is characterized in terms of probabilities and statistics. This is referred to as static security or information theoretical security and allows an adversary with unbounded computational resources in order to differentiate between the hypothesis  $o \sim \mathcal{M}(d)$  and hypothesis  $o \sim \mathcal{M}(d')$ . For completeness, in cryptology we also have the notion of computational security meaning that the difficulty of differentiating the two hypotheses can be reduced to solving a computational hard problem (and, since the brightest mathematicians and computer scientists have not been able to find an algorithm which solves this problem efficiently with practical computational resources, we believe that the attacker cannot solve this problem in feasible time). Computational security allows one to obtain security guarantees where the attacker's advantage or success is negligible in some security parameter.

The above expresses individual privacy. We can generalize towards group privacy by considering data sets  $d$  and  $d'$  that differ in at most  $g$  samples. In this case we say that a mechanism has a DP guarantee with respect to a group of  $g$  samples.

### 3.2 $(\epsilon, \delta)$ -Differential Privacy

A randomized mechanism  $\mathcal{M} : D \rightarrow R$  is  $(\epsilon, \delta)$ -DP (Differentially Private) [18] if for any adjacent  $d$  and  $d'$  in  $D$  and for any subset  $S \subseteq R$  of outputs,

$$\Pr[\mathcal{M}(d) \in S] \leq e^\epsilon \cdot \Pr[\mathcal{M}(d') \in S] + \delta, \quad (3)$$

where the probabilities are taken over the coin flips of mechanism  $\mathcal{M}$ .

Historically, differential privacy was introduced [18] and first defined as  $\epsilon$ -DP [19] which is  $(\epsilon, \delta)$ -DP with  $\delta = 0$ . In order to achieve  $\epsilon$ -DP even an unlikely set  $S$  of outputs needs to satisfy (3) for  $\delta = 0$ . This means that the tail distributions of  $\Pr[\mathcal{M}(d) \in S]$  and  $\Pr[\mathcal{M}(d') \in S]$  cannot differ more than a factor  $e^\epsilon$ . This is a much too strong DP requirement, since the probability to observe an output that corresponds to unlikely tail events is already very small to begin with. Therefore,  $\delta$  was introduced so that tail distributions with probability  $\leq \delta$  do not need to be close together within a factor  $e^\epsilon$ . This allows one to achieve the more relaxed  $(\epsilon, \delta)$ -DP guarantee where an  $\epsilon$ -DP guarantee cannot be proven.

The privacy loss incurred by observing an output  $o$  is given by

$$L_{\mathcal{M}(d)\|\mathcal{M}(d')}^o = \ln \left( \frac{\Pr[\mathcal{M}(d) = o]}{\Pr[\mathcal{M}(d') = o]} \right). \quad (4)$$

As explained in [21]  $(\epsilon, \delta)$ -DP ensures that for all adjacent  $d$  and  $d'$  the absolute value of privacy loss will be bounded by  $\epsilon$  with probability at least  $1 - \delta$  (with probability at most  $\delta$ , observation  $o$  is part of the tail);  $(\epsilon, \delta)$ -DP allows a  $\delta$  probability of ‘catastrophic privacy failure’ and from a cryptographic perspective we want this negligibly small. However, when using differential privacy in machine learning we typically use  $\delta = 1/N$  (or  $1/(10N)$ ) inversely proportional with the data set size  $N$  (this seems to correspond well with the intuition when a local update should cause an unlikely/tail observation due to the nature of the specific batch of local data samples that was used in the computation of the local update). Concerning parameter  $\epsilon$ , the larger  $\epsilon$  the more certain the adversary is about which of  $d$  or  $d'$  caused observation  $o$ .

Compared to  $(\epsilon, 0)$ -DP, the relaxation by  $\delta$  allows an improved and asymptotically tight analysis of the cumulative privacy loss incurred by composition of multiple differentially private mechanisms; [17] states an advanced composition theorem (a factor half improvement over [20]): For all  $\epsilon, \delta, \delta' \geq 0$ , the class of  $(\epsilon, \delta')$ -DP mechanisms satisfies

$$(\sqrt{2k \ln(1/\delta)} \cdot \epsilon + k\epsilon(e^\epsilon - 1)/2, k\delta' + \delta)\text{-DP}$$

under  $k$ -fold adaptive composition. This means that only for  $k \leq (1 - \delta)/\delta'$  the privacy failure probability remains bounded to something smaller than 1.

For group privacy, the literature shows  $(g\epsilon, ge^{g-1}\delta)$ -DP for groups of size  $g$ . Here, we see an exponential dependency in  $g$  due to the  $ge^{g-1}$  term in the privacy failure probability. This means that only for very small  $\delta$ , the failure probability remains bounded to something smaller than 1.

We conclude that  $k$ -fold composition and group privacy for group size  $g$  only lead to useful bounds for relatively small  $k$  and  $g$ . If we restrict ourselves to a subclass of mechanisms, then we may be able to prove practical DP bounds for composition and group privacy for much larger and practical  $k$  and  $g$ . We will define such subclasses by imposing properties on the privacy loss.

### 3.3 Divergence Based DP Measures

In order to get better trade-offs for composition and group privacy we want to weigh the tail distribution of unlikely observations in such a way that more unlikely observations are allowed to leak even more privacy. So, rather than weighing all unlikely observations equally likely, which results in the privacy failure probability  $\delta$ , we want to be more careful. This will allow improved DP bounds for composition and group privacy.

The first idea is to treat the loss function (4) as a random variable  $Z$  and note that in a  $k$ -fold composition we observe  $k$  drawings of random variable  $Z$ . Due to the law of large numbers, the average of these drawings will be concentrated around the mean of the loss function. This leads to the notion of Concentrated Differential Privacy (CDP) first introduced in [17] by framing the loss function as a subgaussian random variable after subtracting its mean. This was re-interpreted and relaxed by using Renyi entropy in [4] and its authors followed up with the notion zero-CDP (zCDP) in [5]: A mechanism  $\mathcal{M}$  is  $\rho$ -zCDP if, for all  $\alpha > 1$ , the Renyi divergence

$$D_\alpha(\mathcal{M}(d)\|\mathcal{M}(d')) = \frac{\ln(\mathbb{E}_{o \sim \mathcal{M}(d)}[e^{(1-\alpha)Z}])}{1-\alpha} \text{ with } Z = L_{\mathcal{M}(d)\|\mathcal{M}(d')}^o$$

satisfies

$$D_\alpha(\mathcal{M}(d)\|\mathcal{M}(d')) \leq \rho\alpha. \quad (5)$$

This DP guarantee requires the tail of  $Z$  to be subgaussian, i.e.,  $\Pr[Z > t + \rho] < e^{-t^2/(4\rho)}$  for all  $t \geq 0$  (the tail behaves like  $Z \sim \mathcal{N}(\rho, 2\rho)$ ). If the loss function satisfies this property for a collection of  $k$  mechanisms (each of the mechanisms is  $\rho$ -zCDP), then their  $k$ -fold adaptive composition is  $k\rho$ -zCDP. If a mechanism is  $\rho$ -zCDP for individual privacy, then it is  $g^2\rho$ -zCDP for groups of size  $g$ . This shows that if we can prove that our DP principles lead to a subgaussian tail of the loss function  $Z$ , then we obtain interpretable DP guarantees even for large  $k$  and  $g$ .

After the introduction of  $\rho$ -zCDP, Renyi DP (RDP) was introduced by [43];  $(\omega, \tau)$ -RDP requires

$$D_\alpha(\mathcal{M}(d)\|\mathcal{M}(d')) \leq \tau \text{ for all } \alpha \in (1, \omega).$$

Here,  $\alpha = 1$  bounds the geometric mean of  $e^Z$ ,  $\alpha = 2$  bounds the arithmetic mean of  $e^Z$ ,  $\alpha = 3$  bounds the quadratic mean of  $e^Z$ , etc., and  $\alpha = \infty$  bounds the maximum value of  $e^Z$  which is equivalent to  $(\tau, 0)$ -DP. RDP also leads to simple computable composition and group privacy. The advantage of zCDP over RDP is that it covers all  $\alpha$  at once: Larger  $\alpha$  put more weight on the tail of  $Z$ , also the mean gets larger. This means that  $\tau$  in the RDP definition should increase with  $\alpha$  and this is realized by zCDP by setting  $\tau = \rho\alpha$  for all  $\alpha \in (1, \infty)$ .

The above discussion leads naturally to the definition of  $(\rho, \omega)$ -tCDP [6]: A mechanism is  $\omega$ -truncated  $\rho$ -CDP if it satisfies (5) only for  $\alpha \in (1, \omega)$ . tCDP requires  $Z$  to be subgaussian near the origin (like zCDP), i.e.,  $\Pr[Z > t + \rho] < e^{-t^2/(4\rho)}$  for all  $0 \leq t \leq 2\rho(\omega - 1)$ , but only subexponential in  $Z$ 's tail, i.e., we get the weaker subexponential tail bound  $\Pr[Z > t + \rho] \leq e^{(\omega-1)^2\rho}e^{-(\omega-1)t}$ . This relaxes zCDP while still obtaining interpretable DP guarantees for composition and group privacy, and also subsampling.

The main concern with each of the divergence based DP measures is a lack of transparency of how the attacker can best distinguish the hypotheses  $o \sim \mathcal{M}(d)$  and  $o \sim \mathcal{M}(d')$ . The next section introduces the  $f$ -DP framework which provides a hypothesis testing based approach. It introduces trade-off functions that capture all the information needed for fully characterizing privacy leakage; a trade-off function can be used to derive any divergence based DP guarantee like the ones discussed above (but not the other way around), see Appendix B in [13]. Rather than extracting a divergence based DP guarantee from a trade-off function for DP-SGD, we will keep the trade-off function itself as it turns out to have a simple form with an easy transparent interpretation.

## 4 Gaussian Differential Privacy

Dong et al. [13] introduced the state-of-the-art DP formulation based on hypothesis testing. From the attacker's perspective, it is natural to formulate the problem of distinguishing two neighboring data sets  $d$  and  $d'$  based on the output of a DP mechanism  $\mathcal{M}$  as a hypothesis testing problem:

$$H_0 : \text{the underlying data set is } d \quad \text{versus} \quad H_1 : \text{the underlying data set is } d'.$$

Here, neighboring means that either  $|d \setminus d'| = 1$  or  $|d' \setminus d| = 1$ . More precisely, in the context of mechanism  $\mathcal{M}$ ,  $\mathcal{M}(d)$  and  $\mathcal{M}(d')$  take as input representations  $r$  and  $r'$  of data sets  $d$  and  $d'$  which are 'neighbors.' The representations are mappings from a set of indices to data samples with the property that if  $r(i) \in d \cap d'$  or  $r'(i) \in d \cap d'$ , then  $r(i) = r'(i)$ . This means that the mapping from indices to data samples in  $d \cap d'$  is the same for the representation of  $d$  and the representation of  $d'$ . In other words the mapping from indices to data samples for  $d$  and  $d'$  only differ for indices corresponding to the differentiating data samples in  $(d \setminus d') \cup (d' \setminus d)$ . In this sense the two mappings (data set representations) are neighbors.

We define the Type I and Type II errors by

$$\alpha_\phi = \mathbb{E}_{o \sim \mathcal{M}(d)}[\phi(o)] \text{ and } \beta_\phi = 1 - \mathbb{E}_{o \sim \mathcal{M}(d')}[\phi(o)],$$

where  $\phi$  in  $[0, 1]$  denotes the rejection rule which takes the output of the DP mechanism as input. We flip a coin and reject the null hypothesis with probability  $\phi$ . The optimal trade-off between Type I and Type II errors is given by the trade-off function

$$T(\mathcal{M}(d), \mathcal{M}(d'))(\alpha) = \inf_{\phi} \{\beta_\phi : \alpha_\phi \leq \alpha\},$$

for  $\alpha \in [0, 1]$ , where the infimum is taken over all measurable rejection rules  $\phi$ . If the two hypotheses are fully indistinguishable, then this leads to the trade-off function  $1 - \alpha$ . We say a function  $f \in [0, 1] \rightarrow [0, 1]$  is a trade-off function if and only if it is convex, continuous, non-increasing, and  $0 \leq f(x) \leq 1 - x$  for  $x \in [0, 1]$ .

We define a mechanism  $\mathcal{M}$  to be  $f$ -DP if  $f$  is a trade-off function and

$$T(\mathcal{M}(d), \mathcal{M}(d')) \geq f$$

for all neighboring  $d$  and  $d'$ . Proposition 2.5 in [13] is an adaptation of a result in [59] and states that a mechanism is  $(\epsilon, \delta)$ -DP if and only if the mechanism is  $f_{\epsilon, \delta}$ -DP, where

$$f_{\epsilon, \delta}(\alpha) = \min\{0, 1 - \delta - e^\epsilon \alpha, (1 - \delta - \alpha)e^{-\epsilon}\}.$$

We see that  $f$ -DP has the  $(\epsilon, \delta)$ -DP formulation as a special case. It turns out that the original DP-SGD algorithm can be tightly analysed by using  $f$ -DP.

#### 4.1 Gaussian DP

In order to proceed, [13] first defines Gaussian DP as another special case of  $f$ -DP as follows: We define the trade-off function

$$G_\mu(\alpha) = T(\mathcal{N}(0, 1), \mathcal{N}(\mu, 1))(\alpha) = \Phi(\Phi^{-1}(1 - \alpha) - \mu),$$

where  $\Phi$  is the standard normal cumulative distribution of  $\mathcal{N}(0, 1)$ . We define a mechanism to be  $\mu$ -Gaussian DP if it is  $G_\mu$ -DP. Corollary 2.13 in [13] shows that a mechanism is  $\mu$ -Gaussian DP if and only if it is  $(\epsilon, \delta(\epsilon))$ -DP for all  $\epsilon \geq 0$ , where

$$\delta(\epsilon) = \Phi\left(-\frac{\epsilon}{\mu} + \frac{\mu}{2}\right) - e^\epsilon \Phi\left(-\frac{\epsilon}{\mu} - \frac{\mu}{2}\right). \quad (6)$$

Suppose that a mechanism  $\mathcal{M}(d)$  computes some function  $u(d) \in \mathbb{R}^n$  and adds Gaussian noise  $\mathcal{N}(0, (c\sigma)^2 \mathbf{I})$ , that is, the mechanism outputs  $o \sim u(d) + \mathcal{N}(0, (c\sigma)^2 \mathbf{I})$ . Suppose that  $c$  denotes the sensitivity of function  $u(\cdot)$ , that is,

$$\|u(d) - u(d')\| \leq c$$

for neighboring  $d$  and  $d'$ ; the mechanism corresponding to one round update in Algorithm 1 has *sensitivity*  $c = 2C$ . After projecting the observed  $o$  onto the line that connects  $u(d)$  and  $u(d')$  and after normalizing by dividing by  $c$ , we have that differentiating whether  $o$  corresponds to  $d$  or  $d'$  is in the best case for the adversary (i.e.,  $\|u(d) - u(d')\| = c$ ) equivalent to differentiating whether a received output is from  $\mathcal{N}(0, \sigma^2)$  or from  $\mathcal{N}(1, \sigma^2)$ . Or, equivalently, from  $\mathcal{N}(0, 1)$  or from  $\mathcal{N}(1/\sigma, 1)$ . This is how the Gaussian trade-off function  $G_{\sigma^{-1}}$  comes into the picture.

#### 4.2 Subsampling

Besides implementing Gaussian noise, DP-SGD also uses sub-sampling: For a data set  $d$  of  $N$  samples,  $\text{Sample}_m(d)$  selects a subset of size  $m$  from  $d$  uniformly at random. We define convex combinations

$$f_p(\alpha) = pf(\alpha) + (1 - p)(1 - \alpha)$$

with corresponding  $p$ -sampling operator

$$C_p(f) = \min\{f_p, f_p^{-1}\}^{**},$$

where the conjugate  $h^*$  of a function  $h$  is defined as

$$h^*(y) = \sup_x \{yx - h(x)\}$$

and the inverse  $h^{-1}$  of a trade-off function  $h$  is defined as

$$h^{-1}(\alpha) = \inf\{t \in [0, 1] \mid h(t) \leq \alpha\} \quad (7)$$

and is itself a trade-off function (as an example, we notice that  $G_\mu = G_\mu^{-1}$  and we say  $G_\mu$  is symmetric). Theorem 4.2 in [13] shows that if a mechanism  $\mathcal{M}$  on data sets of size  $N$  is  $f$ -DP, then the subsampled mechanism  $\mathcal{M} \circ \text{Sample}_m$  is  $C_{m/N}(f)$ -DP.

The intuition behind operator  $C_p$  is as follows. First,  $\text{Sample}_m(d)$  samples the differentiating element between  $d$  and  $d'$  with probability  $p$ . In this case the computations  $\mathcal{M} \circ \text{Sample}_m(d)$  and  $\mathcal{M} \circ \text{Sample}_m(d')$  are different and hypothesis testing is possible with trade-off function  $f(\alpha)$ . With probability  $1 - p$  no hypothesis testing is possible and we have trade-off function  $1 - \alpha$ . This leads to the convex combination  $f_p$ .

Second, we notice if  $h = T(\mathcal{M}(d), \mathcal{M}(d'))$ , then  $h^{-1} = T(\mathcal{M}(d'), \mathcal{M}(d))$ . Therefore, if  $\mathcal{M}$  is  $f$ -DP (which holds for all pairs of neighboring data sets, in particular, for the pairs  $(d, d')$  and  $(d', d)$ ), then both  $h \geq f$  and  $h^{-1} \geq f$  and we have a symmetric upper bound  $\min\{h, h^{-1}\} \geq f$ . Since  $f$  is a trade-off function,  $f$  is convex and we can compute a tighter upper bound:  $f$  is at most the largest convex function  $\leq \min\{h, h^{-1}\}$ , which is equal to the double conjugate  $\min\{h, h^{-1}\}^{**}$ . From this we obtain the definition of operator  $C_p$ .

### 4.3 Composition

The tensor product  $f \otimes h$  for trade-off functions  $f = T(P, Q)$  and  $h = T(P', Q')$  is well-defined by

$$f \otimes h = T(P \times P', Q \times Q').$$

Let  $y_i \leftarrow \mathcal{M}_i(\text{aux}, d)$  with  $\text{aux} = (y_1, \dots, y_{i-1})$ . Theorem 3.2 in [13] shows that if  $\mathcal{M}_i(\text{aux}, \cdot)$  is  $f_i$ -DP for all  $\text{aux}$ , then the composed mechanism  $\mathcal{M}$ , which applies  $\mathcal{M}_i$  in sequential order from  $i = 1$  to  $i = T$ , is  $(f_1 \otimes \dots \otimes f_T)$ -DP. The tensor product is commutative.

As a special case Corollary 3.3 in [13] states that composition of multiple Gaussian operators  $G_{\mu_i}$  results in  $G_\mu$  where

$$\mu = \sqrt{\sum_i \mu_i^2}.$$

### 4.4 Tight Analysis DP-SGD

We are now able to formulate the differential privacy guarantee of original DP-SGD since it is a composition of subsampled Gaussian DP mechanisms. Theorem 5.1 in [13] states that DP-SGD as introduced in [1] is

$$C_{m/N}(G_{\sigma^{-1}})^{\otimes T}\text{-DP},$$

where  $T = (N/m) \cdot E$  is the total number of local rounds. Since each of the theorems and results from [13] enumerated above are exact, we have a tight analysis. This leads in [64] to a (tight) differential privacy accountant (using complex characteristic functions for each of the two hypotheses based on taking Fourier transforms), which can be used by a client to keep track of its current DP guarantee and to understand when to stop helping the server to learn a global model. Because the accountant is tight, it improves over the momentum accountant method of [1].

### 4.5 Strong Adversarial Model

We assume an adversary who knows the differentiating samples in  $d \setminus d'$  and  $d' \setminus d$ , but who a-priori (before mechanism  $\mathcal{M}$  is executed) may only know (besides say a 99% characterization of  $d \cap d'$ ) an estimate of the number of samples in the intersection of  $d$  and  $d'$ , i.e., the adversary knows  $|d \cap d'| + \text{noise}$  where the noise is large enough to yield a ‘sufficiently strong’ DP guarantee with respect to the size of the used data set ( $d$  or  $d'$ ). Since  $\mathcal{M}$  does not directly reveal the size of the used data set, we assume (as in prior literature) that the effect of  $N = |d| \neq N' = |d'|$  contributes at most a very small amount of privacy leakage, sufficiently small to be discarded in our DP analysis: That is, we may as well assume  $N = N'$  in our DP analysis.

In this setting of  $N = N'$  the DP analysis in prior work considers an adversary who can mimic mechanism  $\mathcal{M} \circ \text{Sample}_m$  in that it can replay into large extent how  $\text{Sample}_m$  samples the used data set ( $d$  or  $d'$ ): We say a round has  $k$  differentiating data samples if  $\text{Sample}_m$  sampled a subset of indices which contains exactly  $k$  indices of differentiating data samples from  $(d \setminus d') \cup (d' \setminus d)$ . The adversary knows how  $\text{Sample}_m$  operates and can derive a joint probability distribution  $\mathbb{P}$  of the number of differentiating data samples for each round within the sequence of rounds that define the series of epochs during which updates are computed. We consider two types of strong adversaries in our proofs when bounding trade-off functions:

Adversary  $\mathcal{A}_0$  does not know the exact instance drawn from  $\mathbb{P}$  but who is, in the DP proof, given the ability to realize for each round the trade-off function  $f_k(\alpha)$  that corresponds to hypothesis testing between  $\mathcal{M} \circ \text{Sample}_m(d)$  and  $\mathcal{M} \circ \text{Sample}_m(d')$  if  $\text{Sample}_m$  has selected  $k$  differentiating samples in that round. Adversary  $\mathcal{A}_0$  in the DP analysis that characterizes  $f_k(\alpha)$  is given knowledge about the mapping from indices to values in  $d$  or  $d'$ . Here (as discussed before), the mapping from indices to values in  $d \cap d'$  is the same for the mapping from indices to values in  $d$  and the mapping from indices to values in  $d'$ . Furthermore, the adversary can replay how  $\text{Sample}_m$  samples a subset of  $m$  indices from<sup>‡</sup>  $\{1, \dots, N = N'\}$ , and it knows all the randomness used by  $\mathcal{M}$  before  $\mathcal{M}$  adds Gaussian noise for differential privacy (this includes when and how the interrupt service routine overwrites the local model). This strong adversary represents a worst-case scenario for the ‘defender’ when analyzing the differential privacy of a single round. For DP-SGD this analysis for neighboring data sets leads to the argument of Section 4.2 where with probability  $p$  (i.e.,  $k = 1$ ) the adversary can achieve trade-off function  $f(\alpha)$  and with probability  $1 - p$  (i.e.,  $k = 0$ ) can achieve trade-off function  $1 - \alpha$  leading ultimately to operator  $C_p$ . This in turn leads to the trade-off function  $C_{m/N}(G_{\sigma^{-1}})^{\otimes T}$  with

<sup>‡</sup>By assuming  $N = N'$  in the DP analysis, knowledge of how  $\text{Sample}_m$  samples a subset of indices cannot be used to differentiate the hypotheses of  $d$  versus  $d'$  based on their sizes (since the index set corresponding to  $d$  is exactly the same as the index set corresponding to  $d'$ ).

$p = m/N$ , which is *tight for adversary*  $\mathcal{A}_0$ . We notice that adversary  $\mathcal{A}_0$  is used in DP analysis of current literature including the moment accountant method of [1] for analysing  $(\epsilon, \delta)$ -DP and analysis of divergence based DP measures.

In the DP analysis adversary  $\mathcal{A}_0$  is given knowledge about the number  $k$  of differentiating samples when analysing a single round. That is, it is given an instance of  $\mathbb{P}$  projected on a single round. We notice that in expectation the sensitivity (see Section 4.1) of a single round as observed by adversary  $\mathcal{A}_0$  for neighboring data sets is equal to  $(1-p) \cdot 0 + p \cdot 2C = (m/N) \cdot 2C$  and this gives rise to an ‘expected’ trade-off function  $G_{1/(\sigma N/m)}$ . Composition over  $c^2(N/m)^2$  rounds gives  $G_{1/\sigma}$ . This leads us to believe that  $C_{m/N}(G_{\sigma^{-1}})^{\otimes T}$  converges to  $G_{c \cdot h(\sigma)}$  for  $T = c^2(N/m)^2 \rightarrow \infty$  (or, equivalently,  $\sqrt{T} \cdot m/N = c$  with  $T \rightarrow \infty$  and  $N \rightarrow \infty$ ) where  $h(\sigma)$  is some function that only depends on  $\sigma$ . This intuition is confirmed by Corollary 5.4 in [13], and is also indirectly confirmed by [56] which shows that DP-SGD is  $(\epsilon, \delta)$ -DP for  $\sigma = \sqrt{2(\epsilon + \ln(1/\delta))/\epsilon}$  for a wide range of parameter settings  $N, m, T$  with  $T$  at most  $\approx \epsilon(N/m)^2/2$ , and which matches Corollary 5.4 in [13] in that the upper bound on  $T$  can at most be a constant factor  $\approx 8$  larger (without violating the corollary).

We define the second type of adversary  $\mathcal{A}_1$  (first introduced in [57]) as one who has knowledge about a full instance of  $\mathbb{P}$ , not just a projection of  $\mathbb{P}$  on a single round as for adversary  $\mathcal{A}_0$ . This allows a DP analysis that into some extent computes a convex combination of trade-off functions that each characterize all the rounds together as described by an instance of  $\mathbb{P}$ . This gives adversary  $\mathcal{A}_1$  more information and the resulting DP guarantees should be weaker compared to the analysis based on adversary  $\mathcal{A}_0$  (because adversary  $\mathcal{A}_1$  considers a more worst-case leakage scenario). Since each epoch has  $N/m$  rounds and since  $m/N$  is equal to the probability that  $\text{Sample}_m$  selects a differentiating sample in a round when considering neighboring data sets, we have that a single epoch of  $N/m$  rounds has in expectation exactly one round with one differentiating sample while all other rounds only use non-differentiating samples. This is a composition of  $G_{\sigma^{-1}}$  with  $N/m - 1$  times  $G_0$ . We have that the trade-off function for  $c^2 \cdot (N/m)$  rounds has in expectation a trade-off function  $G_{c/\sigma}$ . This shows convergence to some  $G_{c \cdot h(\sigma)}$  for  $T = c^2 \cdot (N/m) \rightarrow \infty$  rounds. This is indeed a weaker statement compared to the one for adversary  $\mathcal{A}_0$ . A detailed discussion about DP guarantees resulting from an analysis based on adversary  $\mathcal{A}_1$  is given in Section 4.7 (and turns out valuable as it adds more insight).

Clearly, a weaker (than  $\mathcal{A}_0$ ) adversary with less capability (less knowledge of the used randomness by  $\text{Sample}_m$  and  $\mathcal{M}$ ) achieves a trade-off function  $\geq C_{m/N}(G_{\sigma^{-1}})^{\otimes T}$  closer to  $1 - \alpha$ . It remains an open problem to characterize realistic weaker adversaries that lead to larger (lower bounds of) trade-off functions.

## 4.6 Group Privacy

Theorem 2.14 in [13] analyzes how privacy degrades if  $d$  and  $d'$  do not differ in just one sample, but differ in  $g$  samples. If a mechanism is  $f$ -DP, then it is

$$[1 - (1 - f)^{\circ g}]\text{-DP}$$

for groups of size  $g$  (where  $\circ g$  denotes the  $g$ -fold iterative composition of function  $1 - f$ , where 1 denotes the constant integer value 1 and not the identity function, i.e.,  $(1 - f)(\alpha) = 1 - f(\alpha)$ ). This is a tight statement in that *there exist*  $f$  such that the trade-off function for groups of size  $g$  cannot be bounded better. In particular, for  $f = G_\mu$  we have  $G_{g\mu}$ -DP for groups of size  $g$ .

The intuition behind the  $[1 - (1 - f)^{\circ g}]$ -DP result is that the adversary can create a sequence of data sets  $d_0 = d, d_1, \dots, d_{g-1}, d_g = d'$  such that each two consecutive data sets  $d_i$  and  $d_{i+1}$  are neighboring. We know that  $T(\mathcal{M}(d_i), \mathcal{M}(d_{i+1})) \geq f$ . For each rejection rule we may plot a point (in x and y coordinates)

$$(\mathbb{E}_{o \sim \mathcal{M}(d_i)}[\phi(o)], \mathbb{E}_{o \sim \mathcal{M}(d_{i+1})}[\phi(o)]).$$

Since  $f(\alpha)$  is a lower bound on the Type I vs Type II error curve, the resulting collection of points is upper bounded by the curve  $1 - f(\alpha)$ . We have that  $\alpha = \mathbb{E}_{o \sim \mathcal{M}(d_i)}[\phi(o)]$  is mapped to

$$\mathbb{E}_{o \sim \mathcal{M}(d_{i+1})}[\phi(o)] \leq 1 - f(\alpha) = (1 - f)(\alpha).$$

By transitivity, we have that  $\alpha = \mathbb{E}_{o \sim \mathcal{M}(d=d_0)}[\phi(o)]$  is mapped to

$$\mathbb{E}_{o \sim \mathcal{M}(d'=d_g)}[\phi(o)] \leq (1 - f)^{\circ g}(\alpha).$$

This yields the lower bound

$$T(\mathcal{M}(d), \mathcal{M}(d')) \geq 1 - (1 - f)^{\circ g}$$

on the Type I vs Type II error curve.

Let  $\phi[\alpha]$  denote a rejection rule that realizes the mapping from

$$\alpha = \mathbb{E}_{o \sim \mathcal{M}(d_i)}[\phi[\alpha](o)] \quad \text{to} \quad (1 - f)(\alpha) = \mathbb{E}_{o \sim \mathcal{M}(d_{i+1})}[\phi[\alpha](o)].$$

Then the mapping from  $(1 - f)^{\circ i}(\alpha) = \mathbb{E}_{o \sim \mathcal{M}(d_i)}[\phi(o)]$  to  $(1 - f)^{\circ(i+1)}(\alpha) = \mathbb{E}_{o \sim \mathcal{M}(d_{i+1})}[\phi(o)]$  is realized by  $\phi = \phi[(1 - f)^{\circ i}(\alpha)]$ . This shows that the lower bound  $1 - (1 - f)^{\circ g}$  is tight only if we can choose all  $\phi[(1 - f)^{\circ i}(\alpha)]$  equal to one another. This is not the case for DP-SGD for which it turns out that this lower bound is not tight at all; rather than a multiplicative factor  $g$  as in the mentioned  $G_{g\mu}$ -DP guarantee we have a  $\sqrt{g}$  dependency for adversary  $\mathcal{A}_1$  [57] (and this should also hold for the seemingly weaker adversary  $\mathcal{A}_0$ ). This is done by considering how, due to sub-sampling, the  $g$  differentiating samples are distributed across all the rounds within an epoch and how composition of trade-off functions across rounds yields the  $\sqrt{g}$  dependency.

#### 4.7 DP-SGD’s Trade-Off Function

Assuming adversary  $\mathcal{A}_1$ , recent work [57] shows that for  $g = 1$ , DP-SGD is  $h$ -DP for

$$h \approx G_{\sqrt{(1+1/\sqrt{2E})E}/\sigma}$$

and if DP-SGD is  $h$ -DP, then it is upper bounded by  $h \leq \bar{h}$  with

$$\bar{h} \approx G_{\sqrt{(1-1/\sqrt{2E})E}/\sigma}.$$

The approximations become tight if  $e^{-E}$  tends to zero. Notice that we do not need to compute the subsampling operator (which is only specific to adversary  $\mathcal{A}_0$ ) and composition tensor in order to get a good approximation. The approximation is easy to interpret as it behaves like  $G_{\sqrt{E}/\sigma}$  as opposed to general  $f$ -DP theory which has, cited from [13], “the disadvantage is that the expressions it yields are more unwieldy: they are computer evaluable, so usable in implementations, but do not admit simple closed form.”

We remind the reader that we can directly infer  $(\epsilon, \delta)$ -DP guarantees from (6); function  $\delta(\epsilon)$  turns out to be completely independent from the data set size  $N$ , hence, see Section 3.2, setting  $\delta(\epsilon) = 1/N$  favorably biases smaller data sets. Appendix B in [13] shows how to infer divergence based DP guarantees. In particular,  $G_{\sqrt{E}/\sigma}$ -DP implies  $(\omega, \frac{E}{2\sigma^2} \cdot \omega)$ -RDP (Renyi differential privacy) for any  $\omega > 1$ , hence, we have

$$\frac{E}{2\sigma^2}\text{-zCDP}.$$

As noted in [57] the resulting  $G_{\sqrt{E}/\sigma}$ -DP guarantee for individual privacy scales with the square root of the total number  $E$  of epochs, and does not depend on the explicit number of rounds executed within these epochs. In other words, even though more local updates can be observed by the adversary, this turns out not to lead to more privacy leakage. This is because each local update is based on a smaller batch of training data samples which in itself leads to less privacy leakage per local update. We remind the reader about our discussion in Section 2.6 where we show that the role of mini-batch size  $m$  (and, hence, the total number of rounds  $N/m$  per epoch) is implicitly captured in  $\sigma$ .

For group privacy with  $g \geq 1$ , [57] shows a similar approximate DP guarantee as for  $g = 1$  where everywhere  $E$  is substituted by  $gE$ ; we have that DP-SGD with sampling based on ‘shuffling’ is  $G_{\sqrt{gE}/\sigma}$ -DP for groups of size  $g$ . This leads to a  $\sqrt{gE}$  dependency and not a linear dependency in  $g$  (and notice that we obtain  $gE/(2\sigma^2)$ -zCDP with a linear dependency on  $g$  rather than  $g^2$ ).

## 5 Future Work

We are still in the midst of bringing DP-SGD to practice where we want to achieve good convergence to and accuracy of the final global model and where we have a strong DP guarantee (the trade-off function should be close to  $1 - \alpha$  which represents random guessing between the two hypotheses). Towards finding a good balance between utility and privacy, we discuss a couple future directions in next subsections.

### 5.1 Using Synthetic Data

One main problem is that local data is used for training models for various learning tasks. Each application of DP-SGD will leak privacy since the local data set is being re-used. One way to control and be in charge of the amount of privacy leakage is to have data samples in local client data expire according to some expiration date (per sample). This is problematic because in our current data economy, data is a valuable asset which we do not want to give a limited lifetime.

In order to cope with this problem, a client may decide to not use its own local data set in each of these DP-SGD instantiations. Instead, differential private GAN [26] modeling can be used to learn a distribution model based on a

local data set that generates synthetic data with a similar distribution. Due to the post-processing lemma, we can freely use the synthetic data in any optimization algorithm and FL approach. This circumvents multiple use of DP-SGD, but requires the design of differential private GAN which produces ‘high’ quality synthetic data. This is an open problem: GAN modeling is itself a learning task which can use the DP-SGD approach for the discriminator (which is very noise sensitive). Here, we use DP-SGD only once and as soon as a GAN model is learned, it can be published and transmitted to the central server who uses the GAN models from all clients to generate synthetic samples on which it trains a global model for a learning task of its choice. Of course, as a caveat, working with synthetic data may not lead to a global model with good test accuracy on real data. Notice that by using synthetic data we avoid the FL paradigm altogether since the large amounts of data distributed over clients is now compressed into (relatively short transmittable) representations that code GAN models.

In the same line of thinking, if differentially private GAN models do not lead to high quality synthetic data, then we will want to research other general methods for pre-processing local data that filter or hide features that are considered privacy sensitive. This brings us back to the basics of how a membership or inference attack is actually implemented in order to understand what type of information should be filtered out for making reconstruction of certain types of private data hard or unreliable.

## 5.2 Adaptive Strategies

We need to fine tune parameters and this can be done during DP-SGD’s execution: Consecutive segments of multiple rounds may work with their own  $m$ ,  $\sigma$ , and  $C$ . Into what extent does an adaptive approach work, where the current convergence rate and test accuracy (preferably based on public data at the server so as not to leak additional privacy) of the current global model is used to determine  $(m, \sigma, C)$  for the next segment?

In Section 2.6 we discussed the benefit of adaptive reducing the clipping constant  $C$  (based on prior rounds or based on using a DP approach within a round to collect information that influences the choice of the used  $C$  in that round). Similarly, since a smaller  $\sigma$  directly influences the amount of noise added to the global model and therefore influences its accuracy negatively, it makes sense to reduce  $\sigma$  once convergence has been achieved. After reducing  $\sigma$ , new convergence to an improved global model may start. The problem is that a lower  $\sigma$  leads to more privacy leakage. For this reason we want to lower  $\sigma$  to a smaller  $\hat{\sigma}$  only for e.g. the final epoch; this yields for adversary  $\mathcal{A}_1$  a trade-off function approximately equal to

$$G_{\sqrt{g(E-1)}/\sigma} \otimes G_{\sqrt{g}/\hat{\sigma}} = G_{\sqrt{g} \cdot \sqrt{(E-1)/\sigma^2 + 1/\hat{\sigma}^2}}.$$

We may decide to choose a significantly smaller  $\hat{\sigma} = \sigma/\sqrt{E+1}$  which yields  $G_{\sqrt{2gE}/\sigma}$ -DP, sacrificing a factor  $\sqrt{2}$  in the differential privacy guarantee. The significantly smaller  $\hat{\sigma}$  will likely improve the accuracy of the final global model during the last epoch. Notice that, when adapting  $C$  and  $\sigma$ , it also makes sense to fine tune  $m$  accordingly as the sensitivity to a lack of information dispersal may reduce for smaller  $C$  and  $\sigma$ .

We may also modify the noise distribution: DP-SGD selects noise  $N$  from a Gaussian distribution. before adding  $N$  to the round update, we may replace  $N$  by  $a \cdot \text{arsinh}(a^{-1} \cdot N)$  where  $\text{arsinh}(x) = \ln(x + \sqrt{x^2 + 1})$  as suggested for tCDP [6]. The result resembles the same Gaussian but with exponentially faster tail decay and this may help improving the convergence to and accuracy of the final global model. Here, we notice that for the same reason of faster tail decay, DP-SGD chooses to use Gaussian noise over Laplace noise.

Finally, DP-SGD can be placed in a larger algorithmic framework with DP guarantees for a more general clipping strategy (including ‘batch clipping’) which allows more general optimization algorithms (beyond mini-batch SGD), and more general sampling strategies (in particular a sampling strategy based on ‘shuffling’) [57].

It remains an open problem to unveil adaptive strategies possibly in a more general algorithmic framework that optimally balance utility and differential privacy. Here we prefer to discover adaptive strategies that proactively provide the DP guarantee based on changed parameter settings, i.e., we do not want to change parameters based solely on utility and discover later (by using a differential privacy accountant) that this has violated or is about to violate our privacy budget.

## 5.3 DP Proof: A Weaker Adversarial Model

Section 4.5 explains strong adversarial models used in the DP analysis under which the derived DP guarantee is tight. In practice, this is too strong. In general, we may assume a weaker adversary with less capability in terms of knowledge about the used randomness by  $\text{Sample}_m$  and  $\mathcal{M}$ . By explicitly stating the knowledge of a weaker adversary in combination with assumptions on the data set itself, we may be able to derive an  $f$ -DP guarantee with  $f(\alpha)$  closer to  $1 - \alpha$ . It remains an open problem to exploit such a line of thinking.

## 5.4 Computing Environment with Less Adversarial Capabilities

In order to impose restrictions on adversarial capabilities we may be able to use confidential computing techniques such as secure processor technology [11], homomorphic computing with secret sharing and/or secure Multi-Party Computation (MPC) [32], and possibly even hardware accelerated fully homomorphic encryption [24, 22]; for a survey see [31, 35]. These techniques hide round updates in encrypted form. Hence, only the final global model itself (if it is published) or querying the final global model (if it is kept private) can leak information about how local data sets shaped the final model. This means that CDP, see Section 3, is still needed. CDP has a better trade-off between privacy and utility compared to LDP as discussed in this chapter. However, confidential computing does not come for free: Either we need to assume a larger Trusted Computing Base (TCB) in the form of trusted hardware modules or processors at the clients, intermediate aggregators, and server or we need a Trusted Third Party (TTP). E.g., in the secure MPC solution of [32] the generation of Beaver triples is outsourced to a TTP otherwise impractical additional communication among clients and server is needed (for an oblivious transfer phase in MPC). We are still studying balanced and practical combinations of confidential computing techniques including the use of differential privacy.

## References

- [1] Martin Abadi, Andy Chu, Ian Goodfellow, H Brendan McMahan, Ilya Mironov, Kunal Talwar, and Li Zhang. Deep learning with differential privacy. In *Proceedings of the 2016 ACM SIGSAC Conference on Computer and Communications Security*, pages 308–318. ACM, 2016.
- [2] Abhishek Bhowmick, John Duchi, Julien Freudiger, Gaurav Kapoor, and Ryan Rogers. Protection against reconstruction and its applications in private federated learning, 2019.
- [3] Keith Bonawitz, Hubert Eichner, Wolfgang Grieskamp, Dzmitry Huba, Alex Ingerman, Vladimir Ivanov, Chloe Kiddon, Jakub Konecny, Stefano Mazzocchi, H Brendan McMahan, et al. Towards federated learning at scale: System design. *arXiv preprint arXiv:1902.01046*, 2019.
- [4] Mark Bun and Thomas Steinke. Concentrated differential privacy: Simplifications, extensions, and lower bounds. *arXiv*, 2016.
- [5] Mark Bun and Thomas Steinke. Concentrated differential privacy: Simplifications, extensions, and lower bounds. In Martin Hirt and Adam D. Smith, editors, *TCC*, volume 9985, pages 635–658, 2016.
- [6] Mark Bun, Cynthia Dwork, Guy N. Rothblum, and Thomas Steinke. Composable and versatile privacy via truncated CDP. In Ilias Diakonikolas, David Kempe, and Monika Henzinger, editors, *STOC*. ACM, 2018.
- [7] Kamalika Chaudhuri, Anand D Sarwate, and Kaushik Sinha. A near-optimal algorithm for differentially-private principal components. *Journal of Machine Learning Research*, 14, 2013.
- [8] Jianmin Chen, Rajat Monga, Samy Bengio, and Rafal Jozefowicz. Revisiting distributed synchronous sgd. *ICLR Workshop Track*, 2016.
- [9] Yang Chen, Xiaoyan Sun, and Yaochu Jin. Communication-efficient federated deep learning with asynchronous model update and temporally weighted aggregation. *arXiv preprint*, 2019. URL <https://arxiv.org/pdf/1903.07424.pdf>.
- [10] Yang Chen, Xiaoyan Sun, and Yaochu Jin. Communication-efficient federated deep learning with asynchronous model update and temporally weighted aggregation. *arXiv preprint*, 2019. URL <https://arxiv.org/pdf/1903.07424.pdf>.
- [11] Victor Costan and Srinivas Devadas. Intel SGX explained. *IACR Cryptol. ePrint Arch.*, page 86, 2016.
- [12] Christopher M De Sa, Ce Zhang, Kunle Olukotun, and Christopher Ré. Taming the wild: A unified analysis of hogwild-style algorithms. In *NIPS*, pages 2674–2682, 2015.
- [13] Jinshuo Dong, Aaron Roth, and Weijie Su. Gaussian differential privacy. *Journal of the Royal Statistical Society*, 2021.
- [14] John C. Duchi, Michael I. Jordan, and Martin J. Wainwright. Local privacy, data processing inequalities, and statistical minimax rates, 2014.
- [15] Cynthia Dwork. A firm foundation for private data analysis. *Communications of the ACM*, 54(1):86–95, 2011.
- [16] Cynthia Dwork and Aaron Roth. The algorithmic foundations of differential privacy. *Foundations and Trends® in Theoretical Computer Science*, 9(3–4):211–407, 2014.
- [17] Cynthia Dwork and Guy N Rothblum. Concentrated differential privacy. *arXiv preprint arXiv:1603.01887*, 2016.

- [18] Cynthia Dwork, Krishnaram Kenthapadi, Frank McSherry, Ilya Mironov, and Moni Naor. Our data, ourselves: Privacy via distributed noise generation. In *Annual International Conference on the Theory and Applications of Cryptographic Techniques*, pages 486–503. Springer, 2006.
- [19] Cynthia Dwork, Frank McSherry, Kobbi Nissim, and Adam Smith. Calibrating noise to sensitivity in private data analysis. In *Theory of cryptography conference*, pages 265–284. Springer, 2006.
- [20] Cynthia Dwork, Guy N. Rothblum, and Salil P. Vadhan. Boosting and differential privacy. In *51th Annual IEEE Symposium on Foundations of Computer Science, FOCS 2010, October 23-26, 2010, Las Vegas, Nevada, USA*, pages 51–60. IEEE Computer Society, 2010.
- [21] Cynthia Dwork, Aaron Roth, et al. The algorithmic foundations of differential privacy. *Foundations and Trends® in Theoretical Computer Science*, 9(3–4):211–407, 2014.
- [22] Axel Feldmann, Nikola Samardzic, Aleksandar Krastev, Sridhar Devadas, Ron Dreslinski, Karim Eldefrawy, Nicholas Genise, Chris Peikert, and Daniel Sanchez. F1: A fast and programmable accelerator for fully homomorphic encryption (extended version). *arXiv preprint arXiv:2109.05371*, 2021.
- [23] Jonas Geiping, Hartmut Bauermeister, Hannah Dröge, and Michael Moeller. Inverting gradients – how easy is it to break privacy in federated learning? In *NIPS*, 2020.
- [24] Craig Gentry. Fully homomorphic encryption using ideal lattices. In *STOC*, 2009.
- [25] Robin C. Geyer, Tassilo Klein, and Moin Nabi. Differentially private federated learning: A client level perspective, 2018.
- [26] Ian Goodfellow, Jean Pouget-Abadie, Mehdi Mirza, Bing Xu, David Warde-Farley, Sherjil Ozair, Aaron Courville, and Yoshua Bengio. Generative adversarial networks. *Communications of the ACM*, 63(11):139–144, 2020.
- [27] Anupam Gupta, Katrina Ligett, Frank McSherry, Aaron Roth, and Kunal Talwar. Differentially private approximation algorithms. *CoRR*, abs/0903.4510, 2009.
- [28] Meng Hao, Hongwei Li, Xizhao Luo, Guowen Xu, Haomiao Yang, and Sen Liu. Efficient and privacy-enhanced federated learning for industrial artificial intelligence. *IEEE Transactions on Industrial Informatics*, 16(10):6532–6542, 2020. doi: 10.1109/TII.2019.2945367.
- [29] Kevin Hsieh, Aaron Harlap, Nandita Vijaykumar, Dimitris Konomis, Gregory R. Ganger, Phillip B. Gibbons, and Onur Mutlu. Gaia: Geo-distributed machine learning approaching LAN speeds. *14th USENIX Symposium on Networked Systems Design and Implementation (NSDI 17)*, 2017.
- [30] Fei Wang Jie Xu, Wei Zhang. Asynchronous decentralized parallel stochastic gradient descent with differential privacy. *arXiv*, 2020.
- [31] Peter Kairouz, H Brendan McMahan, Brendan Avent, Aurélien Bellet, Mehdi Bennis, Arjun Nitin Bhagoji, Kallista Bonawitz, Zachary Charles, Graham Cormode, Rachel Cummings, et al. Advances and open problems in federated learning. *Foundations and Trends® in Machine Learning*, 14(1–2):1–210, 2021.
- [32] Brian Knott, Shobha Venkataraman, Awni Hannun, Shubho Sengupta, Mark Ibrahim, and Laurens van der Maaten. Crypten: Secure multi-party computation meets machine learning. *Advances in Neural Information Processing Systems*, 34:4961–4973, 2021.
- [33] Jakub Konečný, H Brendan McMahan, Daniel Ramage, and Peter Richtárik. Federated optimization: Distributed machine learning for on-device intelligence. *arXiv preprint arXiv:1610.02527*, 2016.
- [34] Tian Li, Anit Kumar Sahu, Manzil Zaheer, Maziar Sanjabi, Ameet Talwalkar, and Virginia Smith. Federated optimization for heterogeneous networks. *arXiv preprint*, 2019. URL <https://arxiv.org/pdf/1812.06127.pdf>.
- [35] Tian Li, Anit Kumar Sahu, Ameet Talwalkar, and Virginia Smith. Federated learning: Challenges, methods, and future directions. *IEEE Signal Processing Magazine*, 37(3):50–60, 2020.
- [36] Xiangru Lian, Yijun Huang, Yuncheng Li, and Ji Liu. Asynchronous parallel stochastic gradient for nonconvex optimization. In *Advances in Neural Information Processing Systems*, pages 2737–2745, 2015.
- [37] Xiangru Lian, Wei Zhang, Ce Zhang, and Ji Liu. Asynchronous decentralized parallel stochastic gradient descent. *arXiv preprint arXiv:1710.06952*, 2017.
- [38] Lingjuan Lyu, Han Yu, and Qiang Yang. Threats to federated learning: A survey, 2020.
- [39] Horia Mania, Xinghao Pan, Dimitris Papailiopoulos, Benjamin Recht, Kannan Ramchandran, and Michael I Jordan. Perturbed Iterate Analysis for Asynchronous Stochastic Optimization. *SIAM Journal on Optimization*, pages 2202–2229, 2015.

- [40] Brendan McMahan and Daniel Ramage. Federated learning: Collaborative machine learning without centralized training data, 2017. URL <https://ai.googleblog.com/2017/04/federated-learning-collaborative.html>. Last accessed 09/24/2019.
- [41] Brendan McMahan, Daniel Ramage, Kunal Talwar, and Li Zhang. Learning differentially private recurrent language models. In *International Conference on Learning Representations (ICLR)*, 2018. URL <https://openreview.net/pdf?id=BJ0hF1Z0b>.
- [42] H. Brendan McMahan, Eider Moore, Daniel Ramage, and Blaise Agüera y Arcas. Federated learning of deep networks using model averaging. *ICLR Workshop Track*, 2016.
- [43] Ilya Mironov. Rényi differential privacy. In *2017 IEEE 30th computer security foundations symposium (CSF)*, pages 263–275. IEEE, 2017.
- [44] Mohammad Naseri, Jamie Hayes, and Emiliano De Cristofaro. Toward robustness and privacy in federated learning: Experimenting with local and central differential privacy, 2021.
- [45] M. Nasr, R. Shokri, and A. Houmansadr. Comprehensive privacy analysis of deep learning: Passive and active white-box inference attacks against centralized and federated learning. In *2019 IEEE Symposium on Security and Privacy (SP)*, pages 739–753, 2019. doi: 10.1109/SP.2019.00065.
- [46] Lam Nguyen, Phuong Ha Nguyen, Marten Dijk, Peter Richtárik, Katya Scheinberg, and Martin Takáč. Sgd and hogwild! convergence without the bounded gradients assumption. In *International Conference on Machine Learning*, pages 3750–3758. PMLR, 2018.
- [47] Lam M. Nguyen, Phuong Ha Nguyen, Peter Richtárik, Katya Scheinberg, Martin Takáč, and Marten van Dijk. New convergence aspects of stochastic gradient algorithms. *Journal of Machine Learning Research*, 20(176): 1–49, 2019.
- [48] Opacus. Opacus pytorch library. Available from [opacus.ai](https://github.com/danmehring/opacus).
- [49] Nicolas Papernot, Shuang Song, Ilya Mironov, Ananth Raghunathan, Kunal Talwar, and Úlfar Erlingsson. Scalable private learning with pate. In *International conference on learning representations*, 2018.
- [50] Herbert Robbins and Sutton Monro. A stochastic approximation method. *The Annals of Mathematical Statistics*, 22(3):400–407, 1951.
- [51] Reza Shokri, Marco Stronati, Congzheng Song, and Vitaly Shmatikov. Membership inference attacks against machine learning models. In *2017 IEEE Symposium on Security and Privacy (SP)*, pages 3–18. IEEE, 2017.
- [52] L. Song, R. Shokri, and P. Mittal. Membership inference attacks against adversarially robust deep learning models. In *2019 IEEE Security and Privacy Workshops (SPW)*, pages 50–56, 2019. doi: 10.1109/SPW.2019.00021.
- [53] Valery V. Starovoitov and Yu. I. Golub. Data normalization in machine learning. *Informatics*, 2021.
- [54] Stacey Truex, Nathalie Baracaldo, Ali Anwar, Thomas Steinke, Heiko Ludwig, Rui Zhang, and Yi Zhou. A hybrid approach to privacy-preserving federated learning. In *Proceedings of the 12th ACM Workshop on Artificial Intelligence and Security*, pages 1–11, 2019.
- [55] Marten van Dijk, Nhung V Nguyen, Toan N Nguyen, Lam M Nguyen, Quoc Tran-Dinh, and Phuong Ha Nguyen. Hogwild! over distributed local data sets with linearly increasing mini-batch sizes. *arXiv preprint arXiv:2010.14763*, 2020.
- [56] Marten van Dijk, Nhung V. Nguyen, Toan N. Nguyen, Lam M. Nguyen, and Phuong Ha Nguyen. Differential private hogwild! over distributed local data sets. *CoRR*, abs/2102.09030, 2021.
- [57] Marten van Dijk, Phuong Ha Nguyen, Toan Nguyen Nguyen, and Lam Minh Nguyen. Generalizing dp-sgd with shuffling and batching clipping. *arXiv*, 2022.
- [58] Luping Wang, Wei Wang, and Bo Li. Cmf1: Mitigating communication overhead for federated learning. *IEEE International Conference on Distributed Computing Systems.*, 2019.
- [59] Larry Wasserman and Shuheng Zhou. A statistical framework for differential privacy. *Journal of the American Statistical Association*, 105(489):375–389, 2010.
- [60] Cong Xie, Sanmi Koyejo, and Indranil Gupta. Asynchronous federated optimization. *arXiv preprint*, 2019. URL <https://arxiv.org/pdf/1903.03934v1.pdf>.
- [61] Lei Yu, Ling Liu, Calton Pu, Mehmet Emre Gursoy, and Stacey Truex. Differentially private model publishing for deep learning. *2019 IEEE Symposium on Security and Privacy (SP)*, May 2019. doi: 10.1109/sp.2019.00019. URL <http://dx.doi.org/10.1109/SP.2019.00019>.

- [62] Bo Zhao, Konda Reddy Mopuri, and Hakan Bilen. idlg: Improved deep leakage from gradients, 2020.
- [63] Ligeng Zhu, Zhijian Liu, and Song Han. Deep leakage from gradients. *Neurips*, 2019.
- [64] Yuqing Zhu, Jinshuo Dong, and Yu-Xiang Wang. Optimal accounting of differential privacy via characteristic function. *arXiv preprint arXiv:2106.08567*, 2021.