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$9 \times 4 = 6 \times 6$: Understanding the Quantum Solution to Euler's Problem of 36 Officers

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Abstract.

The famous combinatorial problem of Euler concerns an arrangement of 36 officers from six different regiments in a 6×6 square array. Each regiment consists of six officers each belonging to one of six ranks. The problem, originating from Saint Petersburg, requires that each row and each column of the array contains only one officer of a given rank and given regiment. Euler observed that such a configuration does not exist. In recent work, we constructed a solution to a quantum version of this problem assuming that the officers correspond to superpositions of quantum states. In this paper, we explain the solution which is based on a partition of 36 officers into nine groups, each with four elements. The corresponding quantum states are locally equivalent to maximally entangled two-qubit states, hence each quantum officer is represented by a superposition of at most 4 classical states. The entire quantum combinatorial design involves 9 Bell bases in nine complementary 4-dimensional subspaces.

1. Introduction

Designs composed of elements of a finite set arranged with certain symmetry and balance have been known since antiquity. Nowadays, they are commonly referred to as *combinatorial designs* [1], mostly developed in the 18th century along with the general growth of combinatorics. Some of them, such as magic squares and Sudoku grids, offer mathematical recreations. However, the 20th century brought to light several applications of combinatorics to real-world problems. The theory of combinatorial designs



is applicable to the area of design of biological experiments, tournament scheduling, algorithm design and analysis, group testing, and cryptography [1].

The notion of combinatorial designs can be quantized. In a 1999 Ph.D. thesis of Zauner [2], it was proposed to analyze particular configurations of quantum states belonging to a finite-dimensional Hilbert space, $|\psi_j\rangle \in \mathcal{H}_N$, called *quantum designs*, which satisfy certain symmetry conditions. Observe that constructing a classical design involves working with a finite set of objects and their transformations under the discrete permutation group. In a quantum design, the set of states of size N is continuous and is governed by the unitary group $U(N)$.

Quantum combinatorial designs allow for various interpretations and offer several applications of quantum theory. For instance, *mutually unbiased bases* (MUBs) [3–7] and *symmetric informationally complete positive operator valued measures* (SIC POVMs) [2, 8–11] can provide schemes of quantum measurements with particularly appealing properties. A constellation of pure states of a simple system forms a *quantum t -design* [15] if the average of any polynomial function of order t over the constellation agrees with the average over the entire space of pure quantum states with respect to the natural unitarily invariant measure. In particular, MUBs and SICs form a special case of quantum 2-designs, while designs of a higher degree t are also known [16].

In the case of multipartite systems, examples of often analyzed structures include symmetric states with high geometric entanglement [12] and designs in composite spaces constructed out of iso-entangled states [13, 14]. Another related problem is the construction of *absolutely maximally entangled* (AME) states of a system composed of $N = 2k$ parties, such that the state is maximally entangled for any symmetric splitting into two parties of k subsystems each [17–20]. AME states are resources for various quantum information protocols like parallel quantum teleportation, quantum secret sharing [20], and quantum error-correcting codes [17]. In general, the construction of AME states for an arbitrary number of parties and local dimensions is an open problem. An updated list of existing AME states can be found in [21, 22]. Interestingly, such states do not exist for a system composed of four qubits [23]. Furthermore, for four subsystems with local dimension d , it is possible to construct an AME state based on a classical design of two orthogonal Latin squares of size d . However, this construction does not work for $d = 2$ as there are no orthogonal Latin squares of size two [24, 25].

Since the times of Euler [28] and Tarry [29] it is known that two orthogonal Latin squares of order $d = 6$ do not exist. Therefore, it was not known whether AME states for four subsystems do exist for $d = 6$ [30]. Our very recent result [31] can thus be considered as a quantum solution to a classically impossible problem – we constructed an explicit analytic example of a pair of *quantum orthogonal Latin squares* of size six.

The main aim of this work is to interpret and explain the solution in a possibly accessible way. This paper is organized as follows. In Section II, we recall some properties of classical orthogonal Latin squares. Section III summons the corresponding quantum object – quantum Latin squares, originally proposed by Musto and Vicary [32]. In Section IV, we describe orthogonal quantum Latin squares and present an exemplary configuration of size $d = 6$. In particular, we show that the construction is based on splitting the $6 \times 6 = 36$ dimensional Hilbert space into 9 subspaces of dimension 4 each. An explicit form of the constructed state, a short description of quantum entanglement

and generalized Bell states, and some further technical details are provided in [Appendix A – Appendix D](#).

2. Graeco-Latin squares and other combinatorial designs

As a simple example of a classical combinatorial design, we shall recall a single Latin square. A *Latin square* is a $d \times d$ array filled with d different symbols, each occurring exactly once in each row and exactly once in each column. Below, we present an example of a 3×3 Latin square.

A	C	B	\sim	0	2	1
B	A	C		1	0	2
C	B	A		2	1	0

The name “Latin square” might be traced back to the papers of Leonhard Euler, who used Latin characters as symbols. Obviously, any set of symbols can be used: above we replaced the alphabetic sequence A, B, C with the integer sequence 0, 1, 2. An inquisitive reader might see that an analogous construction can be provided for a grid of any size d . As an example, one of the constructions consists of placing the number $i + j \pmod{d}$ at the intersection of the i^{th} column with the j^{th} row.

To sum up, the general construction of a Latin square of any dimension is relatively simple. Perhaps this is why Euler became interested in constructing pairs of Latin squares that would be mutually independent in some sense. Two Latin squares of the same size d are said to be *orthogonal* if, by superposing symbols in each place, all possible pairs of symbols are present on the grid. A pair of *orthogonal Latin squares* (OLS) has traditionally been called a *Graeco-Latin square*, which refers to a popular way to represent such a pair by one Greek letter and one Latin. We present an example of a square of order 3 – note that all nine combinations of three Latin and three Greek letters are present in the grid.

βA	γC	αB	\sim	K♦	Q♠	A♣	\sim	10	22	01	\sim	(1)
γB	αA	βC		Q♣	A♦	K♠		21	00	12		
αC	βB	γA		A♠	K♣	Q♦		02	11	20		

Greek and Latin letters might be replaced by ranks and suits of cards or simply by pairs of numbers, as indicated above. Euler noticed that by replacing the pair (a, b) sitting at the position (i, j) of a Graeco-Latin square of size d with the number $X_{ij} = ad + b + 1$ the matrix X forms a magic square, so the sums of entries in each row and each column are equal. It is easy to check that in the simplest case $d = 3$ presented above this sum equals 15.

Constructing orthogonal Latin squares is a much harder problem than constructing a single Latin square. The problem can already be encountered in dimension $d = 2$.

Indeed, the desired arrangement of four pairs of Graeco-Latin characters (equivalently pairs of numbers or playing cards) is simply impossible on a 2×2 grid. As we have already seen, it is possible to construct such a combinatorial design for dimension $d = 3$. In fact, similar constructions can be found for any odd dimension d . Indeed, one can place a pair of numbers $(i + j, i + 2j) \pmod d$ at the intersection of the i^{th} column with the j^{th} row. One can check that, in each column and each row of such grid all numbers: $0, \dots, d - 1$ appear on both positions, furthermore, all pairs of numbers appear on the grid. With a little more effort, the construction of orthogonal Latin squares might be given for any dimension d which is a multiple of four [24]. In particular, for $d = 4$, it takes the form of an old puzzle involving a standard deck of cards: arrange all aces, kings, queens, and jacks in a 4×4 grid such that each row and each column contains all four suits as well as one of each face value. Such a construction predate Euler and was published by Jacques Ozanam in 1725.

A♠	K♥	Q♦	J♣
Q♣	J♦	A♥	K♠
J♥	Q♠	K♣	A♦
K♦	A♣	J♠	Q♥

 \sim

αA	βB	γC	δD
γD	δC	αB	βA
δB	γA	βD	αC
βC	αD	δA	γB

(2)

Let us summarize the rules defining a Graeco-Latin square of order d , also written as OLS(d), which consists of d^2 symbols, each described by its color and rank:

- A) all d^2 symbols are different,
- B) all d symbols in each row are of different colors and different ranks,
- C) all d symbols in each column are of different colors and different ranks.

Any such solution can be written as a table of length d^2 , containing two numbers encoding the position in the square, the color, and the rank. We present such a table for the case $d = 3$ corresponding to the last form in pattern (1), and label the rows and columns by indices running from 0 to $d - 1$,

row	column	rank	color
0	0	1	0
0	1	2	2
0	2	0	1
1	0	2	1
1	1	0	0
1	2	1	2
2	0	0	2
2	1	1	1
2	2	2	0

(3)

Using condition **A**, namely the uniqueness of the symbols, given the **rank** and the **color** of a symbol we can establish its position in the square, i.e. its row and column. Conditions **B–C** imply that by knowing the **column** and the **color** of a symbol we can uniquely find

its **rank** and the **row** in which it is located. Similarly, once the rank (or color) and row (or column) of a symbol are given, we can directly infer the two other missing features. Hence, all three conditions **A–C** are equivalent to a more general one.

(*) An OLS(d) can be represented by a table consisting of d^2 rows, each with four numbers (i, j, k, ℓ) as shown in Eq. (3). Then, there exist three invertible functions, which map a pair of two numbers connected with any two columns into the other two: $(k, \ell) = F_1(i, j)$ and $(j, \ell) = F_2(i, k)$ and $(k, j) = F_3(i, \ell)$.

It is easy to see that in the classical case analyzed so far these three invertible functions form permutations of size d^2 , which specify, e.g. which symbol should be placed in which position of the Graeco-Latin square. As with any permutation matrix, P is by construction orthogonal, its inverse is directly given by the transposition, $P^{-1} = P^T$.

The last array in Eq. (1) encodes the permutation matrix P_9 of order 9 below in the following way: for each pair (a, b) , the position of unity in a given column of P_9 is equal to $3a + b + 1$. In a similar way, one can see the correspondence to the card-like version of P_9 , where $(A, K, Q) \leftrightarrow (0, 1, 2)$ and $(\heartsuit, \clubsuit, \spadesuit) \leftrightarrow (0, 1, 2)$,

$$\left[\begin{array}{ccc|ccc|ccc}
 0 & 0 & 0 & 0 & A\heartsuit & 0 & 0 & 0 & 0 \\
 0 & 0 & A\clubsuit & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & A\spadesuit & 0 & 0 \\
 \hline
 K\heartsuit & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & K\clubsuit & 0 \\
 0 & 0 & 0 & 0 & 0 & K\spadesuit & 0 & 0 & 0 \\
 \hline
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Q\heartsuit \\
 0 & 0 & 0 & Q\clubsuit & 0 & 0 & 0 & 0 & 0 \\
 0 & Q\spadesuit & 0 & 0 & 0 & 0 & 0 & 0 & 0
 \end{array} \right] = \left[\begin{array}{ccc|ccc|ccc}
 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 \hline
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
 \hline
 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0
 \end{array} \right] = P_9. \tag{4}$$

This matrix provides a simple recipe, in which place of the Latin square a given card has to be put. To assure that no color and rank repeats in each row and column of the square the non-zero entries of the matrix P_9 satisfy the rules of a strong Sudoku: in each row, column, and 3×3 block there is a single entry equal to 1. Furthermore, all the locations of these entries in each block are different.

As already mentioned, there does not exist an OLS of dimension $d = 2$, as locating ace and king of spades at the diagonal of the square of size two excludes placing ace and king of hearts. At the same time, we presented such construction for dimensions $d = 3, 4$, and 5. The next open case is thus $d = 6$, which turned out to be special. This problem was circulating in St. Petersburg in the late 1700s and, according to folklore, Catherine the Great asked Euler to solve it for a military parade. This question is known as the thirty-six officers¹ problem and was introduced by Euler in the following way [28].

¹ It is sad to note that these Russian officers recently left their parade ground in Saint Petersburg, where they belong, and went a thousand miles South... However, explicit analytical results described in this work strongly suggest that the officers might eventually suffer a transition into a highly *entangled state* [26, 27].

Six different regiments have six officers, each one belonging to different ranks. Can these 36 officers be arranged in a square formation so that each row and column contains one officer of each rank and one of each regiment?

– Leonhard Euler

To start searching for a solution consider a simple chess problem: Place six rooks on a chessboard of size six, in such a way that no figure attacks any other. An exemplary solution is shown in Fig 1. Let us now take six pieces of five other figures and place them onto the board in an analogous way, so that the figures of the same kind do not attack each other, assuming the rook's rules and that all other figures are not there. Such a design, shown in Fig. 2a, forms a Latin square of order six.

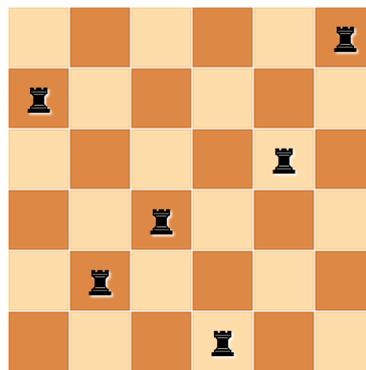


Figure 1. A chessboard of order six with six black rooks placed in such a way that no two rooks attack each other.

To look for Graeco-Latin squares of this size we need to color all the figures into six colors and try to arrange them in such a way that all colors in each row and each column are different. As Euler observed, such an arrangement does not exist – see Fig. 2b – but he was not able to prove it rigorously. A proper proof was established only 121 years later, in a rather lengthy manner by exhaustion, by Gaston Tarry [29], who analyzed 9408 separate cases. A three-page short proof of the nonexistence of a pair of orthogonal Latin squares of order six given in 1984 by Stinson [33] was based on modern results from finite vector spaces and graph theory.

On one hand, Euler was aware of methods for constructing Graeco-Latin squares in odd dimensions and a multiple of 4. On the other hand, he observed that Graeco-Latin squares of order 2 do not exist, and was unable to construct an arrangement of order 6. This resulted in one of his famous conjectures – see [34].

There is no Graeco-Latin square of any oddly even order d , i.e. $d \equiv 2 \pmod{4}$.

– Euler's Graeco-Latin square conjecture

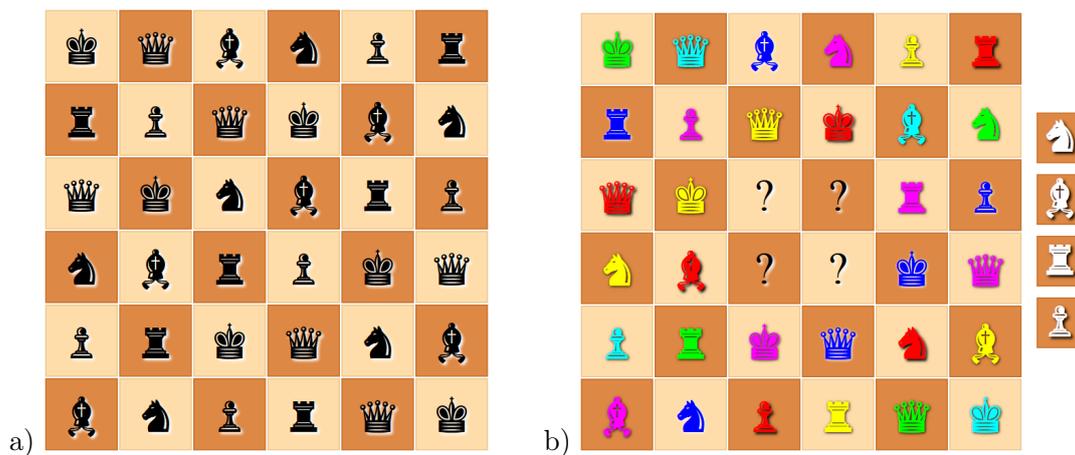


Figure 2. a) A single Latin square of order six composed of 36 black chess figures: none of the six rooks attacks any other rook (assuming that other chess pieces are removed from the board), and the same property holds also for other figures if they were replaced by rooks – see Fig. 1. To construct a Graeco–Latin square we have to paint the figures into six colors. b) The problem of 36 officers of Euler in a chess setup: check if you can place the remaining 4 figures, two of them in cyan and two in green, on the chessboard so that it contains different figures of different colors and additionally there is no repetition of figures or colors in any row and column. Even if you do not succeed in finding an exact solution to the problem, you will obtain a fair approximation to it – a permutation matrix analogous to the one investigated in Ref. [38].

Euler’s conjecture remained unsolved until 1959 when Raj Chandra Bose and Sharadchandra Shankar Shrikhande constructed a counterexample of order $d = 22$ [36]. In the same year, Ernest Tilden Parker found a lower-dimensional counterexample of order $d = 10$ using a computer search, which was one of the earliest combinatorics problems solved by means of a digital computer [25]. Later on, this serious mathematical result was popularized in a renowned novel by Perc [35], in which the $d = 10$ Graeco-Latin square plays a crucial role.

In April 1959, all three mathematicians: Parker, Bose, and Shrikhande, together generalized the results and proved Euler’s conjecture to be false for all $d \geq 10$ by constructing related arrangements [37]. The result of these “Euler’s spoilers” as they were dubbed, completed the answer to the question:

For which number d there exists a Graeco-Latin square of order d ?

Answer: It exists for all numbers d except two and six.

It is worth mentioning that Euler was not only interested in pairs of orthogonal Latin squares, which we called after him Graeco-Latin squares, but also by tuples of such squares which are pairwise orthogonal. A set of n Latin squares of the same order d in which all pairs are orthogonal is called a set of *mutually orthogonal Latin squares*

(MOLS). Below, we present an example of three mutually orthogonal Latin squares of order 4, where, in addition to Greek and Latin letters in (2), we include here also four Hebrew: $\aleph, \beth, \daleth, \gimel$.

$$\begin{array}{|c|c|c|c|} \hline \alpha A \aleph & \beta B \beth & \gamma C \daleth & \delta D \gimel \\ \hline \gamma D \beth & \delta C \aleph & \alpha B \gimel & \beta A \daleth \\ \hline \delta B \daleth & \gamma A \gimel & \beta D \aleph & \alpha C \beth \\ \hline \beta C \gimel & \alpha D \daleth & \delta A \beth & \gamma B \aleph \\ \hline \end{array} \sim \begin{array}{|c|c|c|c|} \hline 0 0 0 & 1 1 1 & 2 2 2 & 3 3 3 \\ \hline 2 3 1 & 3 2 0 & 1 1 3 & 2 0 2 \\ \hline 3 1 2 & 2 0 3 & 2 3 0 & 1 2 1 \\ \hline 2 2 3 & 1 3 2 & 3 0 1 & 2 1 0 \\ \hline \end{array} \tag{5}$$

As we have already seen, constructing even a single pair of OLS is not a simple task in some dimensions d . Up to this date, the following general question regarding MOLS remains open.

What is the maximum number of mutually orthogonal Latin squares of a given order d ?

There are some partial results regarding answers to the above question. Firstly, it might be observed that for each dimension d there are at most $d - 1$ MOLS [25]. Therefore the Graeco-Latin square (1) cannot be extended into three MOLS of order three, similarly, the set of three MOLS of order four (5) cannot be extended further. Such a maximal set of $d - 1$ MOLS is known to exist for all prime and prime power dimensions d . Furthermore, the aforementioned results concerning Graeco-Latin squares show that in any dimension apart from $d = 2, 6$ there are at least two MOLS. While in the case $d = 2$, the single existing Latin square saturates the upper bound, $d - 1 = 1$, the dimension $d = 2 \times 3 = 6$ is entirely different, as the upper bound implies that in this case there are no more than $d - 1 = 5$ MOLS.

3. Quantum Latin squares

Looking for a classical combinatorial design we analyze constellations formed of various objects belonging to a given discrete set of numbers, letters, cards, or chess figures. In quantum theory, a state is described by a vector $|\psi\rangle$ from a complex d -dimensional Hilbert space \mathcal{H}_d . We shall assume that such vectors are normalized, $\|\psi\|^2 = \langle\psi|\psi\rangle = 1$, and that two vectors that differ by a complex phase are identified to be the same, according to an equivalence class given by $|\psi\rangle \sim e^{i\alpha}|\psi\rangle$. Then, the set of quantum states is continuous, as any two states can be connected by a unitary transformation, $|\psi\rangle = U|\phi\rangle$. Two quantum states $|\psi\rangle$ and $|\phi\rangle$ are distinguishable if and only if they are orthogonal, $\langle\psi|\phi\rangle = 0$. A set of d different classical objects labeled by $i = 0, 1, \dots, d - 1$ corresponds to an orthogonal basis $|i\rangle \in \mathcal{H}_d$ of distinguishable states, $\langle i|j\rangle = \delta_{ij}$.

Apart from the classical states like $|0\rangle$ and $|1\rangle$ one can consider their arbitrary superpositions, say $|\Psi_{\theta,\chi}\rangle = \cos\theta|0\rangle + e^{i\chi}\sin\theta|1\rangle$. In the case of bipartite systems consisting of two subsystems A and B , one uses a composed Hilbert space defined by the tensor product, $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$. A separable state has a *product form*, $|\psi_{AB}\rangle = |\phi_A\rangle \otimes |\phi_B\rangle$, for brevity written as $|\phi_A, \phi_B\rangle$ or $|\phi_A \phi_B\rangle$. All other (non-product)

states are called *entangled* [39]. A key example is given by the celebrated *Bell state*, $|\phi_+\rangle := (|00\rangle + |11\rangle)/\sqrt{2}$ – for details, see [Appendix A](#).

A rule of thumb says that any good notion can be quantized. Following this reasoning, recently Musto and Vicary defined quantum Latin square (QLS) [32]. A QLS of size d consists of d^2 quantum states $|\psi_{ij}\rangle \in \mathcal{H}_d$ for $i, j = 0, \dots, d - 1$, such that each row and each column forms an orthogonal basis. Note that the classical condition: all objects in each row and each column are *different*, is now replaced by its natural quantum analog: all quantum states in each row and column are *orthogonal*.

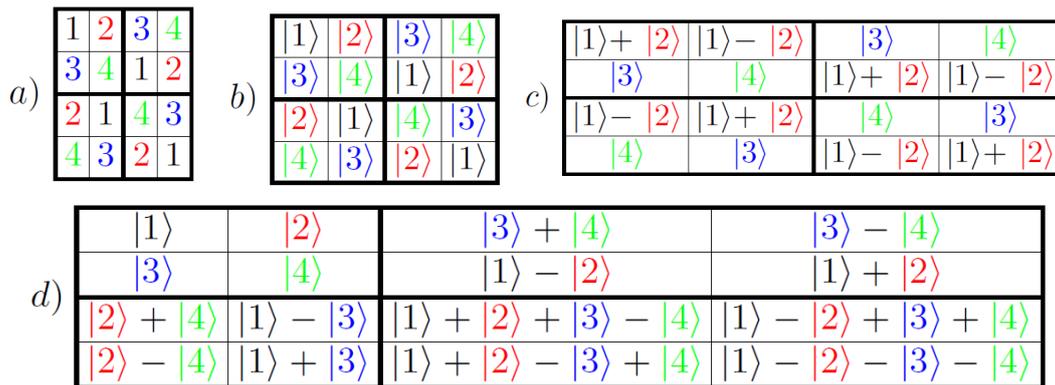


Figure 3. Examples of Latin squares of size 4 with the Sudoku property – each square of size 2 contains different classical elements or orthogonal quantum states: a) classical Latin square related to movement of chess knight; b) its direct quantum analog; c) apparently quantum Latin square with $C = 4 = d$ different states which can be rotated to the computational basis; d) genuinely quantum Latin square with $C = 16$ different states forming $4 \times 3 = 12$ different orthogonal bases in \mathcal{H}_4 . For legibility, quantum states are not normalized and we use here a notation where the kets are numbered from 1 to d .

Fig. 3a shows a classical Latin square of order $d = 4$, while Fig. 3b presents its direct quantum analog. The labels run here from 1 to $d = 4$. It is clear that all orthogonality rules are satisfied, but this object does not differ much from its classical ancestor. Observe that Fig. 3c looks "more quantum" as it contains superposition states, $|\pm\rangle = |1\rangle \pm |2\rangle$. However, this design is only *apparently quantum*, as it can be converted to a classical one by a unitary rotation. To characterize the degree of quantumness it is sufficient to count the number of different states in the pattern – this number is called the *cardinality* C of a QLS. Any design of size d with cardinality $C = d$ is apparently quantum, as it is equivalent with respect to a unitary rotation to a classical design. Hence all orthogonal bases corresponding to various rows and columns of the pattern are equal. On the other hand, if $C > d$ this is no longer the case and such a design is called *genuinely quantum*.

It is possible to show [40] that in dimensions $d = 2$ and $d = 3$ all quantum Latin squares satisfy the relation $C = d$, so they are apparently quantum. For $d = 4$ there exist QLS with cardinality greater than four, which are genuinely quantum and are not equivalent to any classical solution. Fig. 3d shows the quantum pattern with the maximal cardinality, $C = d^2 = 16$. It satisfies the rules of a quantum Sudoku [41], so it offers a

constellation of $3 \times 4 = 12$ different orthogonal measurements in \mathcal{H}_4 . Further examples of QLS(d) with maximal cardinality $C = d^2$ were constructed in [40] for an arbitrary dimension d .

4. Quantum orthogonal Latin squares

Quantization of a classical Graeco–Latin square of size d seems to be straightforward. In each field of the square it is enough to place a product state $|ik\rangle = |i\rangle \otimes |k\rangle$, where $i, k = 0, \dots, d-1$. In such a way all d^2 entries of a square form a product basis in $\mathcal{H}_d \otimes \mathcal{H}_d$. This construction, explicitly visualized for $d = 3$ in Eq. (6), corresponds directly to the classical design shown in Eq. (1).

$$\begin{array}{|c|c|c|}
 \hline
 |1\rangle \otimes |0\rangle & |2\rangle \otimes |2\rangle & |0\rangle \otimes |1\rangle \\
 \hline
 |2\rangle \otimes |1\rangle & |0\rangle \otimes |0\rangle & |1\rangle \otimes |2\rangle \\
 \hline
 |0\rangle \otimes |2\rangle & |1\rangle \otimes |1\rangle & |2\rangle \otimes |0\rangle \\
 \hline
 \end{array} \quad (6)$$

More formally, a pair of quantum orthogonal Latin squares (QOLS) of size d can be defined as a set of d^2 bipartite states, $|\psi_{ij}\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ which form a square,

$$\text{QOLS} := \begin{array}{|c|c|c|}
 \hline
 |\psi_{11}\rangle & \dots & |\psi_{1d}\rangle \\
 \hline
 \dots & \dots & \dots \\
 \hline
 |\psi_{d1}\rangle & \dots & |\psi_{dd}\rangle \\
 \hline
 \end{array}, \quad (7)$$

and satisfy certain constraints analogous to the classical conditions **A–C** above. As the bipartite states $|\psi_{ij}\rangle$, in general, can be entangled, it is not always possible to split such a pair of QOLS into two separate orthogonal quantum Latin squares.

Condition **A**, imposing that all the objects of the classical design are different, is easy to quantize:

A') All d^2 states are mutually orthogonal and they form a basis,

$$\langle \psi_{ij} | \psi_{k\ell} \rangle = \delta_{ik} \delta_{j\ell}. \quad (8)$$

Two further conditions **B–C** are more involved. Since in each row (or column) we analyze *only* ranks or *only* colors of the chess pieces in the design, it is natural to expect that the corresponding quantum constraints will deal with the *partial trace* (see [Appendix A](#)) of a symmetric combination of projectors. Investigating the QOLS shown in (6) we realize that the superposition of bipartite states along the second column forms a maximally entangled state, $|\gamma_2\rangle = (|22\rangle + |00\rangle + |11\rangle)/\sqrt{3}$. Thus, the partial trace over the second subsystem B (or the first subsystem A) is maximally mixed, $\text{Tr}_B |\gamma_2\rangle\langle\gamma_2| \propto \mathbb{I}_3$, as all elements in this column of the design are different. It is easy to see that the superpositions of three states in each column and each row have the same property as they are locally equivalent to the generalized Bell state, $|\gamma_2\rangle \in \mathcal{H}_3 \otimes \mathcal{H}_3$. Such a condition was initially advocated in [42] to become a part of a possible definition of QOLS. However,

it occurred [43,44] that these conditions, satisfied for superposition of product states, are not sufficient to assure the required properties of an AME state of four subsystems, with d levels reach. Necessary and sufficient conditions can be formulated in terms of the partial trace of sums of projectors in each row and column [45],

B') All rows of the square satisfy the conditions

$$\mathrm{Tr}_B \left(\sum_{k=0}^{d-1} |\psi_{ik}\rangle\langle\psi_{jk}| \right) = \delta_{ij} \mathbb{I}_d . \quad (9)$$

Analogous to dealing only with the colors of pieces in each row and forgetting their ranks, and dual conditions for the other partial trace Tr_A analogous to the requirement that all ranks in each row of the classical design are different.

C') All columns of the square satisfy the partial trace conditions

$$\mathrm{Tr}_B \left(\sum_{k=0}^{d-1} |\psi_{ki}\rangle\langle\psi_{kj}| \right) = \delta_{ij} \mathbb{I}_d , \quad (10)$$

and dual conditions for Tr_A .

These general conditions can be understood in analogy to their classical counterparts. Namely, if $i = j$, conditions **B'** and **C'** with the identity mean that each symbol occurs exactly once – the mixing of features (rank, suits, etc.) is perfect. For $i \neq j$, the vanishing value of the partial trace is assured by the fact that while taking two distinct columns, the property of being a Latin square means that no feature is repeated.

It is possible to show [45] that the above conditions, easy to verify and visualize, are equivalent to those used in the former paper [31] and in earlier works on AME states [17–20]. The above conditions are invariant with respect to local unitary operations $U_A \otimes U_B$, so they are by construction satisfied by apparently orthogonal quantum Latin squares obtained by a local unitary transformation of any classical solution. See [Appendix C](#) for a visual explanation of partial tracing.

Another option to analyze a QOLS consists in representing the square as a table with d^2 rows and 4 columns analogous to (3). However, this scheme is not yet defined well enough, as the states can be entangled. Placing into the first field of (2), for instance, $|\psi_{11}\rangle = (|A\spadesuit\rangle + |K\heartsuit\rangle)/\sqrt{2}$, we see that neither the color nor the rank of this object is well specified. However, it is possible to modify the classical condition (*) introducing the corresponding four-party AME state [31]

$$|\Psi_{ABCD}\rangle = \frac{1}{d} \sum_{i,j=0}^{d-1} |i\rangle_A |j\rangle_B |\psi_{ij}\rangle_{CD} = \frac{1}{d} \sum_{i,j,k,\ell=0}^{d-1} T_{ijkl} |i\rangle_A |j\rangle_B |k\rangle_C |\ell\rangle_D \quad (11)$$

This form of representing the square consisting of d^2 bi-partite states $|\psi_{ij}\rangle$ allows one to formulate a single general condition analogous to the classical one (*),

() Representing a quantum design (7) by a tensor T_{ijkl} with four indices, each running from 0 to $d-1$, introduced in (11), we say that it forms quantum**

orthogonal Latin squares if all three matrices obtained by various pairing of two indices, $X_{\mu,\nu} = T_{ij,kl}$, $Y_{\mu,\nu} = T_{ik,jl}$ and $Z_{\mu,\nu} = T_{il,kj}$, form unitary matrices of size d^2 .

Observe that in the classical case these matrices form three orthogonal permutation matrices of size d^2 , mentioned in condition (*), exemplified by Eq. (4) and its reorderings for $d = 3$. Given a matrix X with entries $X_{\mu,\nu} = X_{ij,kl}$ the matrix Y with entries $Y_{ij,kl} = X_{ik,jl}$, written, $Y = X^\Gamma$, is called partial transpose of X , while matrix Z with entries $Z_{ij,kl} = X_{il,kj}$, written, $Z = X^R$, is called reshuffling of X . More details concerning this notation can be found in the Supplementary Material of the previous work [31]. A tensor T satisfying such conditions is called perfect [46], while a unitary matrix U of size d^2 , which remains unitary under two corresponding exchanges of indices is called 2-unitary [47].

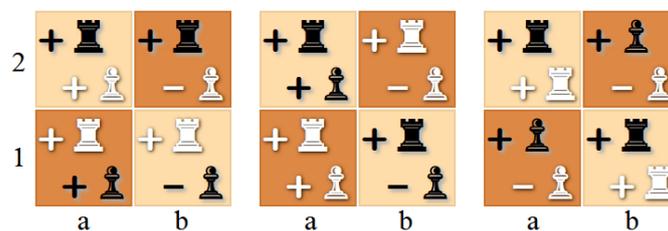


Figure 4. Exemplary configurations of four two-qubits entangled states represented on a chessboard of size 2. None of the above configurations satisfies the necessary conditions **A’-C’**). In the array on the left, the first and second columns do not satisfy **C’** condition while tracing over colors. The proof that the other two arrangements do not satisfy some conditions is left as an exercise to the reader.

Before presenting a solution to a QOLS for $d = 6$ let us return first to the simpler case $d = 2$. Fig. 4 shows potential designs, composed of a basis of four orthogonal Bell states. It is possible to demonstrate that the presented structures do not satisfy all necessary conditions **A’-C’**. The fact that there are no QOLS of order $d = 2$ is equivalent to the seminal result of Higuchi and Sudbery, who proved that there are no AME states for a system composed of four qubits [23].

5. Thirty-six entangled officers of Euler

As there are no solutions to the classical problem of 36 officers of Euler, we cannot look for a quantum design (7) taking the bi-partite states $|\psi_{ij}\rangle$ in a product form as it was done for $d = 3$ in (6). Instead, one has to allow the ranks and regiments to be entangled and visualize such superpositions of quantum states by several chess figures occupying the same field of the chessboard. For instance, in place of putting a classical red king ♔ into the chess field a1 and a blue queen ♚ into a2, we can locate a Bell state $|\text{♔}\rangle + |\text{♚}\rangle$ into the first field and an orthogonal state $|\text{♔}\rangle - |\text{♚}\rangle$ into the second one (for simplicity we shall now use non-normalized states). Entangled states, like $|\text{♚}\rangle + |\text{♔}\rangle$, can contain

arbitrary complex phases and the number of states in a superposition can be larger, so it is not possible to exhaust all possible configurations on a chessboard.

The original solution to the problem was found numerically [31, 45, 50] using a Sinkhorn-like algorithm, earlier applied in [48] to find dual unitary matrices, for which we relaxed one condition and required that only U and U^R are unitary [49]. Later on, we have found a purely analytic form of the solution [51] that we shall now present in some detail.

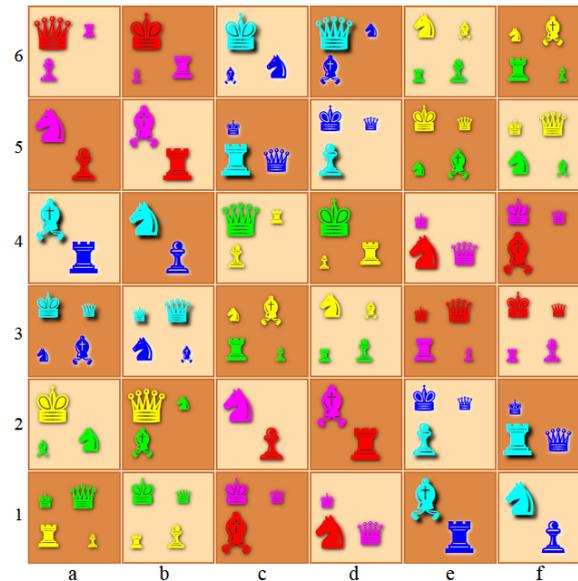


Figure 5. Chess-like illustration of a solution to the QOLS(6) problem corresponding to the matrix U . Identify nine squares of size two each consisting of 4 fields, each containing figures in only two colors. Note that such a coarse-grained structure forms a Latin square of size 3.

A simplified version of the solution to the problem of 36 entangled officers is visualized in Fig. 5 on a truncated chessboard of size 6×6 . Each field corresponds to one officer, or a quantum state $|\psi_{ij}\rangle \in \mathcal{H}_6 \otimes \mathcal{H}_6$, or a row of a 2-unitary matrix $U \in U(36)$. A chess field with two or more different figures in different colors represents an entangled state. Interestingly, there are at most four figures in each field, which means that the entanglement concerns only 2, 3, or 4 classical states out of 36.

The size of each figure represents its relative weight in the superposition, so the fields with two figures of the same size represent Bell states. The ratio of the larger to the smaller sizes in the fields hosting four figures is equal to the golden mean, $b/a = (1 + \sqrt{5})/2 = \varphi$, with $a = \frac{1}{2}\sqrt{1 - 1/\sqrt{5}}$ and $b = \frac{1}{2}\sqrt{1 + 1/\sqrt{5}}$. The third and the largest amplitude is $c = 1/\sqrt{2}$ and they all fulfill the Pythagorean relation: $a^2 + b^2 = c^2$.

However, a reader with a sharp eye will immediately identify a problem: the fields c2 and a5 contain a red pawn and a purple knight, so they look the same, although they should be distinguishable! The answer is contained in the complex phases of each

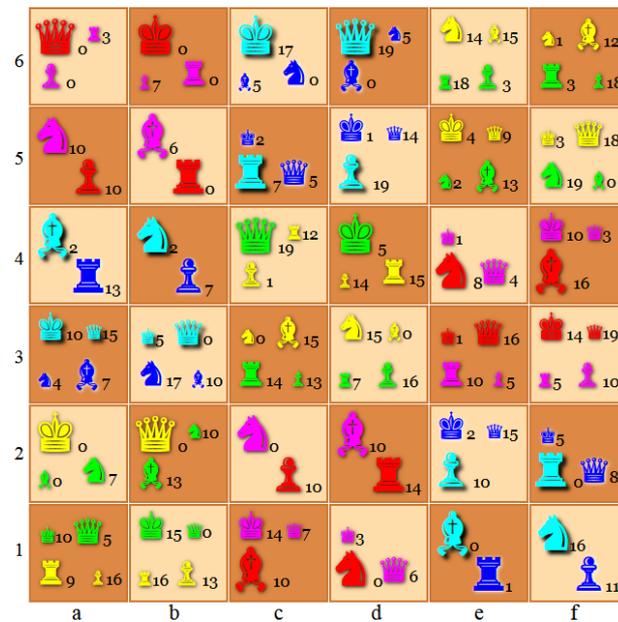


Figure 6. Number $k \in \{0, 1, \dots, 19\}$ decorating each figure denotes the complex phase, $\exp(2\pi ik/20)$, making the representation of the QOLS(6) complete. This single figure encodes the entire solution to the problem! See Appendix B for a detailed explanation.

state shown explicitly in Fig. 6. An integer number k labeling each figure denotes the complex phase ω^k , with $\omega = \exp(2\pi i/20)$ equal to the 20th root of unity. The phase ω plays an important geometric role as it is related to the regular pentagon and the golden number φ , see [31]. Since the phase $k = 10$ corresponds to -1 the chess field c2 represents the Bell state $|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle$. Note also that the other field a5 corresponds to another, orthogonal, Bell state equivalent to $|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle$, so both states are perfectly distinguishable, as they should be. Similar reasoning shows that all 36 states, uniquely determined by each chess field, are mutually orthogonal.

This implies that condition (8) of perfect distinguishability of all entries of the designs is satisfied. Hence, the matrix X of size 36 representing the design is unitary, so it will now be referred to as U . Its structure is shown in Fig. 7a, in which three colors represent three different amplitudes. Note that the matrix is sparse – each row contains at most 4 non-zero entries, as there are at most 4 figures in each chess field. Performing a suitable permutation of size 36 it is possible to bring the matrix into a block diagonal form: as shown in Fig. 7b, it consists of nine unitary blocks of size 4. In such a way entire space of size 36 can be decomposed into nine subspaces of size four.

One can also check that the two other necessary conditions (9) and (10) hold true. To check this, one can take into account the decomposition $36 = 9 \times 4$, as shown in Fig. 8, in which each four-dimensional base is labeled by a background color. Note that each row and each column of the design contains entries of three of six different bases which play a role in the diagonal part, $i = j$, of conditions, **B'** and **C'**. Furthermore, checking the off-diagonal constraints for $i \neq j$, one observes that by choosing any pair

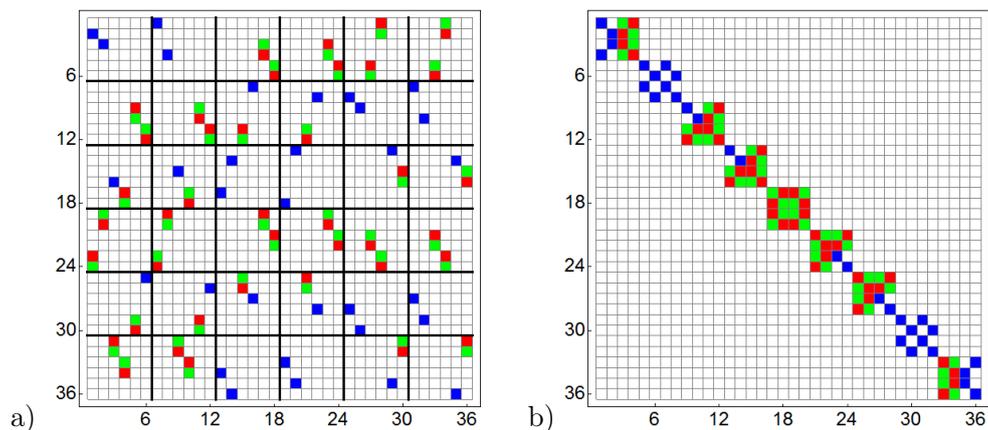


Figure 7. a) The structure of the unitary matrix U of size 36 which describes the quantum design: each row represents a single officer and it has at most 4 non-zero entries. The colors describe different amplitudes; red for a , green for b , and blue for c . Note that the color used here have nothing in common with the colors in other figures. b) The matrix U is permutation-equivalent to a block-diagonal form, which contains a direct sum of nine unitary blocks of order four. Compare this with [Appendix D](#).

of rows (or columns) of the square and comparing their corresponding entries, only two different configurations are possible. In the first case (example: the first and the second column) all pairs of fields have the same color of the background, so the corresponding states belong to the same 4D subspace. In the second case, (example: first and third column or any two rows), all pairs of squares have different colors, so the states belong to orthogonal subspaces.

As the matrix U and the set of 36 states $|\psi_{ij}\rangle$ determined by its rows and represented in a chessboard satisfy all conditions (8), (9), and (10), the design forms a constructive example of QOLS(6). It is also instructive to look at this solution analyzed from another perspective, choosing another basis. If a matrix U forms a valid solution to the problem also reordered matrices U^R and U^T exhibit the same property. The corresponding two (equivalent) solutions are shown in two chessboards presented in Fig. 9. Furthermore, it is possible to show that every state out of 36, living in a 4-dimensional subspace, is locally equivalent to the standard, two-qubit Bell state $|\phi_+\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$.

Looking at the structure of the representation of U^T , shown in Fig. 9b, we realize that kings are only entangled with queens or other kings, bishops with knights, and rooks with pawns. Using another vocabulary, all generals talk only to admirals, colonels to majors, while captains are only entangled with lieutenants. Furthermore, six colors split into three pairs of colors which produce a single state, hence the entire cohort of 36 officers splits into nine groups of size four, according to the title of this paper.

6. Concluding remarks

The famous combinatorial problem of 36 officers was posed by Euler, who claimed in 1779 that no solution exists. The first paper with proof of this statement, by Tarry [29],

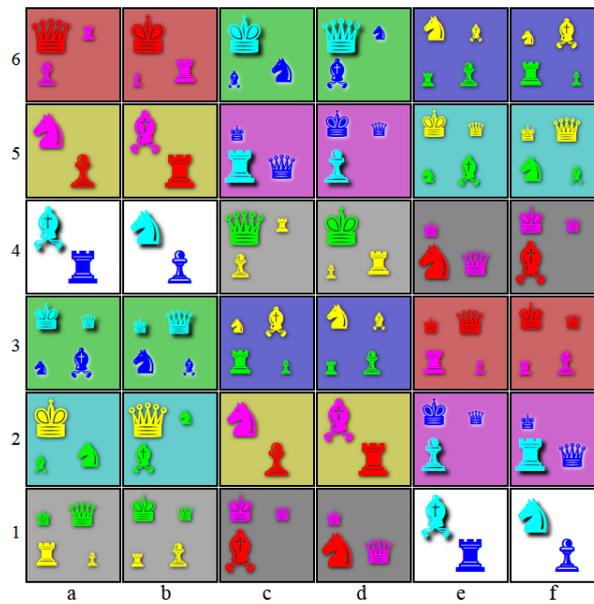


Figure 8. The same chessboard as in Fig. 5 with nine background colors representing each 4-dimensional subspace and corresponding to each block in Fig. 7b.

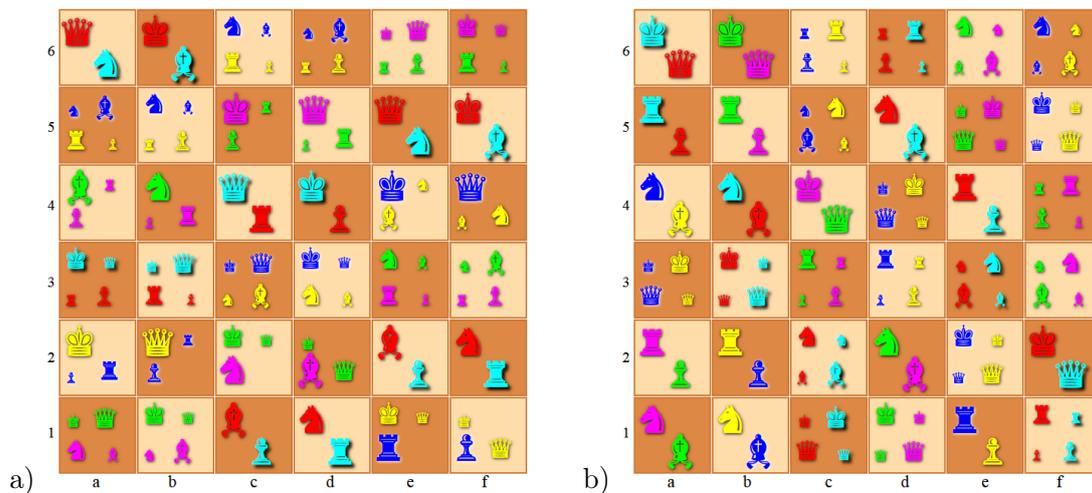


Figure 9. The same QOLS(6) solution now corresponding to a) the reshuffled matrix U^R and b) the partially transposed matrix U^T . Note that after the reorderings of the matrices the structure of each field and colors and coupled figures do change.

came only 121 years later, in 1900. After another 121 years, we have presented a solution to the quantum analog of the Euler problem [31], in which superpositions of officers and entanglement between their ranks and their units is allowed. This unexpected result implies constructive solutions to the related problems of the existence of absolutely

maximally entangled states of four subsystems with six levels each, a 2-unitary matrix \mathcal{U}_{36} of size 36 with maximal entangling power [47, 48] and a perfect tensor T_{ijkl} with four indices, each running from one to six [46]. Our results allowed us to construct original quantum error-correcting codes [17]: a pure code usually denoted as $((4, 1, 3))_6$, and a shortened code $((3, 6, 2))_6$, which allows encoding a 6-level state into a set of three such subsystems. Furthermore, it implies the existence of a quantum orthogonal array [42] of strength two with six rows and four columns: two classical and two quantum, written $\text{QOA}(6, 2+2, 2, 6)$.

Although we worked for more than a year to understand the fine mathematical structure of the solution found, it is clear that more effort is needed to provide its optimal explanation, accessible to a broad audience. In this paper we used a chess analogy, first advocated in [52].

Observe a similarity between our solution and the games played with "quantum chess" [53], in which a player can position a single piece in multiple locations on the board at once! In both problems the quantum rules allow one to create a superposition of states corresponding to various chess pieces at different locations.

Our solution to the quantum design of 36 entangled officers of Euler was initially obtained by an extensive numerical search. A posteriori, applying reverse engineering, we can now propose a general construction scheme of the unitary matrix $U \in U(36)$, which encompasses the entire solution:

i) Generate nine copies of Bell bases, represented by a simple sum of nine copies of Bell matrix (A.11), written $U' = \oplus_{i=1}^9 B_i$ with $B_i = B$.

ii) Rotate each basis locally and apply a diagonal matrix of phases, $B'_i = (W_i \otimes \tilde{W}_i) B_i D_i$, with local unitary matrices W_i , \tilde{W}_i and a diagonal unitary matrix D_i , to obtain a unitary block matrix $U'' = \oplus_{i=1}^9 B'_i$, similar to the one shown in Fig. 7b.

iii) Permute the entire matrix by suitable permutation matrices P_1 and P_2 of size 36 to get the form $U^\Gamma = P_1 U'' P_2$. This scheme allows one to obtain the partially transposed matrix presented in Fig. 9b, so to get the final matrix presented in Fig. 7a one needs to perform a partial transpose, $U = (U^\Gamma)^\Gamma$.

Unfortunately, this rather simple prescription is not a constructive one, as it does not specify explicitly 18 unitary matrices W_i and \tilde{W}_i of order two, diagonal matrices D_i , and additionally, two permutation matrices P_1 and P_2 of size 36.

Let us conclude this paper by posing a list of natural open questions:

a) Does there exist a real solution to the problem of entangled 36 officers of Euler? In another language, we ask if an orthogonal matrix of size 36 with a 2-unitary property can be found. In particular, one can ask if there exists a 2-unitary Hadamard matrix of size $d^2 = 36$?

b) Does there exist a QOLS(6) not equivalent to the form presented with respect to local unitary transformations?

c) Are there genuinely quantum OLS(d) for $d = 3, 4$ or 5 , which are not locally equivalent to a classical solution?

d) Can we change the title equation into e.g. $4 \times 16 = 8 \times 8$, and find a genuinely quantum solution for QOLS(8) based on 16 Bell bases of maximally entangled two qubit states, or a two-unitary matrix of size 64 ?

Acknowledgments.

It is a pleasure to thank Dan Garisto and Phillip Ball for posing several questions [52, 54] which inspired this work. We are also grateful to Albert Rico and Markus Grassl for numerous fruitful discussions and to Marco Enriquez for his encouragement and patience.

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Appendix A. Basic notions of the quantum theory

The purpose of this Appendix is to explain the nomenclature of quantum information from the main body of the paper. We shall assume full knowledge about quantum states, i.e. that they are pure. Quantum-mechanical description of the microworld uses vectors of length N (from a Hilbert space \mathcal{H}_N of dimension N) to describe quantum states with N levels, e.g.

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (\text{A.1})$$

represent two perfectly distinguishable states of a two-level system (qubit). Classically, these states are analogous to on/off states of bits in an ordinary computer.

In order to provide a measure for the distinguishability of two quantum states, we use a product of vectors

$$\langle \phi | \psi \rangle = \sum_i a_i^* b_i, \quad (\text{A.2})$$

where complex numbers a_i and b_i refer to coefficients of the quantum states $|\phi\rangle$ and $|\psi\rangle$

$$|\phi\rangle = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{pmatrix}, \quad |\psi\rangle = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{pmatrix}. \quad (\text{A.3})$$

If two states are orthogonal (their product is 0), then such states are perfectly distinguishable. However, if two states overlap (their product is non-zero) then these states cannot be distinguished with probability 1.

Up till now, to describe states we have implicitly used the special computational basis. Therefore, an alternative description of the state $|\phi\rangle$ from Eq. (A.3) is given by a combination of the basis states

$$|\phi\rangle = \sum_i a_i |i\rangle. \quad (\text{A.4})$$

An example of the computational basis is given by states in Eq. (A.1). Their orthogonality means that $\langle 0|1\rangle = 0$. Generally, all the basis states are orthogonal to each other

$$\langle i|j\rangle = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases} \quad (\text{A.5})$$

However, this choice of the basis is not the only one – any set of N vectors $\{|v_i\rangle\}_{i=1}^N$ orthogonal to each other will form a proper basis. For practical purposes, we usually require them to be normalized, $\langle v_i|v_i\rangle = 1$. A particularly useful example of another basis of a qubit system is formed by $\{|+\rangle, |-\rangle\}$

$$|\pm\rangle = \frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle). \quad (\text{A.6})$$

These states are orthogonal to each other ($\langle +|-\rangle = 0$), and such a basis is unitarily equivalent to the computational basis, as exemplified by matrices with columns given by appropriate states

$$(|+\rangle \quad |-\rangle) = H \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = H (|0\rangle \quad |1\rangle), \quad (\text{A.7})$$

where $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ denotes a Hadamard matrix.

In the case of a 4-level system, it is sometimes convenient to divide it into 2 parts of 2 levels each. Consequently, we name it a two-qubit system; such a system might exhibit non-classical correlations between its subsystems which we call *entanglement*. The states from the computational basis of a two-qubit system $\{|0\rangle, |1\rangle, |2\rangle, |3\rangle\}$ are not enough to reveal entanglement; however, famous *Bell states* possess the maximal degree of entanglement. These states $\{|\alpha_i\rangle\}_{i=1}^4$ form a basis of the two-qubit system with $\langle \alpha_i|\alpha_j\rangle = \delta_{ij}$;

$$\begin{aligned} |\alpha_1\rangle &= (|00\rangle + |11\rangle)/\sqrt{2}, \\ |\alpha_2\rangle &= (|01\rangle + |10\rangle)/\sqrt{2}, \\ |\alpha_3\rangle &= (|01\rangle - |10\rangle)/\sqrt{2}, \text{ and} \\ |\alpha_4\rangle &= (|00\rangle - |11\rangle)/\sqrt{2}. \end{aligned} \quad (\text{A.8})$$

An especially useful operation on matrices is the notion of the partial trace of a matrix M of dimension $d_A d_B$. It can be defined for $M = M_{d_A} \otimes M_{d_B}$ as

$$\text{Tr}_A(M_{d_B} \otimes M_{d_B}) = (\text{Tr} \otimes \mathbb{1}_{d_B})(M_{d_A} \otimes M_{d_B}) = \text{Tr} M_{d_A} \cdot M_{d_B} \quad (\text{A.9})$$

and

$$\text{Tr}_B(M_{d_A} \otimes M_{d_B}) = (\mathbb{1}_{d_A} \otimes \text{Tr})(M_{d_A} \otimes M_{d_B}) = \text{Tr} M_{d_B} \cdot M_{d_A}, \quad (\text{A.10})$$

where d_A and d_B are dimensions of appropriate subsystems represented by the matrix M_{d_A} or M_{d_B} . Since the partial trace is linear, from the above definition one can obtain the partial trace of any matrix $M = \sum_i M_{Ai} \otimes M_{Bi}$. The partial trace is widely used across quantum information; in particular, the partial trace of a maximally entangled state is maximally mixed. In other words, although the state is entirely determined, no information about it is accessible from a perspective of a single subsystem. For instance, the partial trace of any Bell state from Eq. (A.8) amounts to the identity matrix $\text{Tr}_A |\alpha_i\rangle \langle \alpha_i| = \mathbb{1}_2/2$.

The Bell basis, defined above, can be conveniently represented by a unitary matrix of order four,

$$B = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}, \quad (\text{A.11})$$

so that each Bell state reads, $|\alpha_i\rangle = B|i\rangle$ with $i = 1, 2, 3, 4$.

This is not the only basis composed of maximally entangled states; another example is given by the following vectors

$$\begin{aligned} |\beta_1\rangle &= (|00\rangle + |01\rangle + |10\rangle + |11\rangle)/2, \\ |\beta_2\rangle &= (|00\rangle + |01\rangle - |10\rangle + |11\rangle)/2, \\ |\beta_3\rangle &= (|00\rangle - |01\rangle + |10\rangle + |11\rangle)/2, \text{ and} \\ |\beta_4\rangle &= (-|00\rangle + |01\rangle + |10\rangle + |11\rangle)/2. \end{aligned} \quad (\text{A.12})$$

Finally, an especially important maximally entangled basis of a two-qubit system is given by $|\gamma_i\rangle$ obtained from the green-yellow block of size four plotted in Fig. 3 in the original paper [31]:

$$\begin{aligned} |\gamma_1\rangle &= a\omega |00\rangle + a\omega^{19} |01\rangle + b\omega^{14} |10\rangle + b\omega^{16} |11\rangle, \\ |\gamma_2\rangle &= a\omega |00\rangle + a\omega^3 |01\rangle + b\omega^{10} |10\rangle + b\omega^4 |11\rangle, \\ |\gamma_3\rangle &= b\omega^4 |00\rangle + b\omega^{18} |01\rangle + a\omega^3 |10\rangle + a\omega^9 |11\rangle, \text{ and} \\ |\gamma_4\rangle &= b\omega^2 |00\rangle + b\omega^8 |01\rangle + a\omega^5 |10\rangle + a\omega^{15} |11\rangle. \end{aligned} \quad (\text{A.13})$$

The coefficients read: $a = (5 + \sqrt{5})^{-1/2}$ and $b = ((5 + \sqrt{5})/20)^{-1/2}$, while phases ω^k are given by the twentieth root of unity $\omega = e^{i\pi/10}$, with $k \in \{0, 1, \dots, 19\}$. Interestingly, the number $b/a = (1 + \sqrt{5})/2 = \varphi$ is the golden ratio, which prompted us to name the state as the *golden* AME(4, 6) state. To show that these states form a basis, consider a product of two of these vectors, e.g. $|\gamma_2\rangle$ and $|\gamma_3\rangle$ using Eq. (A.2):

$$\begin{aligned} \langle \gamma_2 | \gamma_3 \rangle &= a\omega^{-1} b\omega^4 + a\omega^{-3} b\omega^{18} + b\omega^{-10} a\omega^3 + b\omega^{-4} a\omega^9 = ab(\omega^3 + \omega^{15} + \omega^{13} + \omega^5) = \\ &= ab(\omega^3 - \omega^5 - \omega^3 + \omega^5) = 0. \end{aligned} \quad (\text{A.14})$$

The orthogonality conditions are satisfied due to a proper choice of complex phases ω . Therefore, the states forming blocks of the golden AME state create a maximally entangled basis of the two-qubit system. Equivalently, we can say that the states $\{|\gamma_i\rangle\}$ are perfectly distinguishable.

Appendix B. Full form of the golden AME state

Explicit expressions for the row of matrix U , corresponding to the chessboard in Fig. 6 with

$$a = \frac{1}{2}\sqrt{1 - \frac{1}{\sqrt{5}}} < b = \frac{1}{2}\sqrt{1 + \frac{1}{\sqrt{5}}} < c = \frac{1}{\sqrt{2}} \quad \text{and} \quad \omega = \exp \frac{i\pi}{10}. \quad (\text{B.1})$$

Each non-zero element in matrix U can be encoded in the senary (base-six) numeral system $\{|j\rangle\}_{j=0}^5$, e.g. $|12\rangle = |1\rangle \otimes |2\rangle \leftrightarrow 9^{\text{th}}$ index, or $|55\rangle \leftrightarrow 36^{\text{th}}$ index. See Appendix A for a primer for the algebra of quantum states. On the other hand, one can encode matrix U using chess figures and colors due to the following scheme:

$$\text{♔} = 0, \quad \text{♚} = 1, \quad \text{♞} = 2, \quad \text{♜} = 3, \quad \text{♝} = 4, \quad \text{♛} = 5 \quad (\text{B.2})$$

and for colors:

$$\text{red} = 0, \quad \text{cyan} = 1, \quad \text{green} = 2, \quad \text{magenta} = 3, \quad \text{blue} = 4, \quad \text{yellow} = 5. \quad (\text{B.3})$$

Now $|12\rangle = |\text{♚}\rangle$ and $|55\rangle = |\text{♜}\rangle$.

All rows of U are presented below in the standard Dirac notation and chess notation, cf. Fig. 6. Note that for example $|\psi_{00}\rangle$ has three components, in accordance with the fact that the first row in Fig. 7 contains exactly three non-zero elements. Here, every row of a matrix U is treated as a quantum state $|\psi_{ij}\rangle$.

$$\begin{aligned} |\psi_{00}\rangle &= c|10\rangle + a\omega^3|43\rangle + b|53\rangle = c|\text{♚}\rangle + a\omega^3|\text{♝}\rangle + b|\text{♜}\rangle \\ |\psi_{01}\rangle &= c|00\rangle + b|43\rangle + a\omega^7|53\rangle = c|\text{♔}\rangle + b|\text{♝}\rangle + a\omega^7|\text{♜}\rangle \\ |\psi_{02}\rangle &= c\omega^{17}|01\rangle + b|24\rangle + a\omega^5|34\rangle = c\omega^{17}|\text{♚}\rangle + b|\text{♞}\rangle + a\omega^5|\text{♜}\rangle \\ |\psi_{03}\rangle &= c\omega^{19}|11\rangle + a\omega^5|24\rangle + b|34\rangle = c\omega^{19}|\text{♚}\rangle + a\omega^5|\text{♞}\rangle + b|\text{♜}\rangle \\ |\psi_{04}\rangle &= b\omega^{14}|25\rangle + a\omega^{15}|35\rangle + a\omega^{18}|42\rangle + b\omega^3|52\rangle = b\omega^{14}|\text{♞}\rangle + a\omega^{15}|\text{♜}\rangle + a\omega^{18}|\text{♝}\rangle + b\omega^3|\text{♜}\rangle \\ |\psi_{05}\rangle &= a\omega|25\rangle + b\omega^{12}|35\rangle + b\omega^3|42\rangle + a\omega^{18}|52\rangle = a\omega|\text{♞}\rangle + b\omega^{12}|\text{♞}\rangle + b\omega^3|\text{♝}\rangle + a\omega^{18}|\text{♜}\rangle \\ |\psi_{10}\rangle &= c\omega^{10}|23\rangle + c\omega^{10}|50\rangle = c\omega^{10}|\text{♞}\rangle + c\omega^{10}|\text{♜}\rangle \\ |\psi_{11}\rangle &= c\omega^6|33\rangle + c|40\rangle = c\omega^6|\text{♜}\rangle + c|\text{♝}\rangle \\ |\psi_{12}\rangle &= a\omega^2|04\rangle + b\omega^5|14\rangle + c\omega^7|41\rangle = a\omega^2|\text{♚}\rangle + b\omega^5|\text{♚}\rangle + c\omega^7|\text{♝}\rangle \\ |\psi_{13}\rangle &= b\omega|04\rangle + a\omega^{14}|14\rangle + c\omega^{19}|51\rangle = b\omega|\text{♚}\rangle + a\omega^{14}|\text{♚}\rangle + c\omega^{19}|\text{♞}\rangle \\ |\psi_{14}\rangle &= b\omega^4|05\rangle + a\omega^9|15\rangle + a\omega^2|22\rangle + b\omega^{13}|32\rangle = b\omega^4|\text{♚}\rangle + a\omega^9|\text{♚}\rangle + a\omega^2|\text{♞}\rangle + b\omega^{13}|\text{♜}\rangle \\ |\psi_{15}\rangle &= a\omega^3|05\rangle + b\omega^{18}|15\rangle + b\omega^{19}|22\rangle + a|32\rangle = a\omega^3|\text{♚}\rangle + b\omega^{18}|\text{♚}\rangle + b\omega^{19}|\text{♞}\rangle + a|\text{♜}\rangle \\ |\psi_{20}\rangle &= c\omega^2|31\rangle + c\omega^{13}|44\rangle = c\omega^2|\text{♞}\rangle + c\omega^{13}|\text{♝}\rangle \\ |\psi_{21}\rangle &= c\omega^2|21\rangle + c\omega^7|54\rangle = c\omega^2|\text{♞}\rangle + c\omega^7|\text{♞}\rangle \\ |\psi_{22}\rangle &= c\omega^{19}|12\rangle + a\omega^{12}|45\rangle + b\omega|55\rangle = c\omega^{19}|\text{♚}\rangle + a\omega^{12}|\text{♝}\rangle + b\omega|\text{♞}\rangle \\ |\psi_{23}\rangle &= c\omega^5|02\rangle + b\omega^{15}|45\rangle + a\omega^{14}|55\rangle = c\omega^5|\text{♚}\rangle + b\omega^{15}|\text{♝}\rangle + a\omega^{14}|\text{♞}\rangle \\ |\psi_{24}\rangle &= a\omega|03\rangle + b\omega^4|13\rangle + c\omega^8|20\rangle = a\omega|\text{♚}\rangle + b\omega^4|\text{♚}\rangle + c\omega^8|\text{♞}\rangle \end{aligned}$$

$$\begin{aligned}
|\psi_{25}\rangle &= b\omega^{10}|03\rangle + a\omega^3|13\rangle + c\omega^{16}|30\rangle = b\omega^{10}|\text{♙}\rangle + a\omega^3|\text{♚}\rangle + c\omega^{16}|\text{♗}\rangle \\
|\psi_{30}\rangle &= b\omega^{10}|01\rangle + a\omega^{15}|11\rangle + a\omega^4|24\rangle + b\omega^7|34\rangle = b\omega^{10}|\text{♘}\rangle + a\omega^{15}|\text{♙}\rangle + a\omega^4|\text{♞}\rangle + b\omega^7|\text{♜}\rangle \\
|\psi_{31}\rangle &= a\omega^5|01\rangle + b|11\rangle + b\omega^{17}|24\rangle + a\omega^{10}|34\rangle = a\omega^5|\text{♘}\rangle + b|\text{♘}\rangle + b\omega^{17}|\text{♞}\rangle + a\omega^{10}|\text{♜}\rangle \\
|\psi_{32}\rangle &= a|25\rangle + b\omega^{15}|35\rangle + b\omega^{14}|42\rangle + a\omega^{13}|52\rangle = a|\text{♞}\rangle + b\omega^{15}|\text{♗}\rangle + b\omega^{14}|\text{♝}\rangle + a\omega^{13}|\text{♖}\rangle \\
|\psi_{33}\rangle &= b\omega^{15}|25\rangle + a|35\rangle + a\omega^7|42\rangle + b\omega^{16}|52\rangle = b\omega^{15}|\text{♞}\rangle + a|\text{♞}\rangle + a\omega^7|\text{♝}\rangle + b\omega^{16}|\text{♖}\rangle \\
|\psi_{34}\rangle &= a\omega|00\rangle + b\omega^{16}|10\rangle + b\omega^{10}|43\rangle + a\omega^5|53\rangle = a\omega|\text{♙}\rangle + b\omega^{16}|\text{♚}\rangle + b\omega^{10}|\text{♛}\rangle + a\omega^5|\text{♜}\rangle \\
|\psi_{35}\rangle &= b\omega^{14}|00\rangle + a\omega^{19}|10\rangle + a\omega^5|43\rangle + b\omega^{10}|53\rangle = b\omega^{14}|\text{♙}\rangle + a\omega^{19}|\text{♚}\rangle + a\omega^5|\text{♛}\rangle + b\omega^{10}|\text{♜}\rangle \\
|\psi_{40}\rangle &= c|05\rangle + b\omega^7|22\rangle + a|32\rangle = c|\text{♞}\rangle + b\omega^7|\text{♖}\rangle + a|\text{♗}\rangle \\
|\psi_{41}\rangle &= c|15\rangle + a\omega^{10}|22\rangle + b\omega^{13}|32\rangle = c|\text{♞}\rangle + a\omega^{10}|\text{♖}\rangle + b\omega^{13}|\text{♗}\rangle \\
|\psi_{42}\rangle &= c|23\rangle + c\omega^{10}|50\rangle = c|\text{♞}\rangle + c\omega^{10}|\text{♗}\rangle \\
|\psi_{43}\rangle &= c\omega^{10}|33\rangle + c\omega^{14}|40\rangle = c\omega^{10}|\text{♞}\rangle + c\omega^{14}|\text{♛}\rangle \\
|\psi_{44}\rangle &= b\omega^2|04\rangle + a\omega^{15}|14\rangle + c\omega^{10}|51\rangle = b\omega^2|\text{♘}\rangle + a\omega^{15}|\text{♙}\rangle + c\omega^{10}|\text{♞}\rangle \\
|\psi_{45}\rangle &= a\omega^5|04\rangle + b\omega^8|14\rangle + c|41\rangle = a\omega^5|\text{♘}\rangle + b\omega^8|\text{♙}\rangle + c|\text{♛}\rangle \\
|\psi_{50}\rangle &= a\omega^{10}|02\rangle + b\omega^5|12\rangle + b\omega^9|45\rangle + a\omega^{16}|55\rangle = a\omega^{10}|\text{♖}\rangle + b\omega^5|\text{♖}\rangle + b\omega^9|\text{♝}\rangle + a\omega^{16}|\text{♞}\rangle \\
|\psi_{51}\rangle &= b\omega^{15}|02\rangle + a|12\rangle + a\omega^{16}|45\rangle + b\omega^{13}|55\rangle = b\omega^{15}|\text{♖}\rangle + a|\text{♖}\rangle + a\omega^{16}|\text{♝}\rangle + b\omega^{13}|\text{♞}\rangle \\
|\psi_{52}\rangle &= b\omega^{14}|03\rangle + a\omega^7|13\rangle + c\omega^{10}|30\rangle = b\omega^{14}|\text{♙}\rangle + a\omega^7|\text{♚}\rangle + c\omega^{10}|\text{♗}\rangle \\
|\psi_{53}\rangle &= a\omega^3|03\rangle + b\omega^6|13\rangle + c|20\rangle = a\omega^3|\text{♙}\rangle + b\omega^6|\text{♚}\rangle + c|\text{♞}\rangle \\
|\psi_{54}\rangle &= c|31\rangle + c\omega|44\rangle = c|\text{♘}\rangle + c\omega|\text{♙}\rangle \\
|\psi_{55}\rangle &= c\omega^{16}|21\rangle + c\omega^{11}|54\rangle = c\omega^{16}|\text{♞}\rangle + c\omega^{11}|\text{♞}\rangle
\end{aligned}$$

Simple chess generator and other scripts related to the golden AME(4,6) state can be found on Github [55]. Note that the matrix analyzed in the original paper [31] and visualized therein by 36 cards corresponds to the partially transposed matrix U^Γ . See also Appendix D. It is interesting to observe that an AME state of a heterogenic system $2 \times 3 \times 3 \times 3$ found in [22] has a slightly similar structure with coefficients given by roots of an algebraic equation.

Appendix C. Visual explanation of quantum conditions for OLS

The colors in Fig. 5 and Fig. 6 are especially attuned. Note that they form two triplets of primary colors: **red**, **green**, **blue** and their appropriate complements: **cyan**, **magenta**, and **yellow**. Such a configuration is very helpful in translating the formulas **B'** and **C'** into a visual explanation, because we can easily infer – without cumbersome calculations – which combinations of the chess figures compensate, exactly as it is in the case of the color domain.

Consider for example formula **B'** and the first two rows in Fig. 6. Since a chess figure of a given color simply encodes the position of the non-zero element in the unitary matrix U , (see Appendix B), it is easy to understand that the only contributions to the partial trace over the second subsystem will occur when two figures of the same color appear in both rows. Namely, the **red** queen, king, pawn, and rook will combine with their **cyan** counterparts. To confirm this formally one must consider full formulas including phases and amplitudes:

$$c^2 w^{-10} |\text{♙}\rangle \langle \text{♙}| + c^2 |\text{♚}\rangle \langle \text{♚}| = -\frac{1}{2} |\text{♙}\rangle \langle \text{♙}| + \frac{1}{2} |\text{♚}\rangle \langle \text{♚}| \quad (\text{C.1})$$

and

$$c^2 |\text{♘}\rangle \langle \text{♘}| + c^2 w^{10} |\text{♙}\rangle \langle \text{♙}| = \frac{1}{2} |\text{♘}\rangle \langle \text{♘}| - \frac{1}{2} |\text{♙}\rangle \langle \text{♙}|. \quad (\text{C.2})$$

After tracing out the colors one obtains

$$-\frac{1}{2}|\text{♔}\rangle\langle\text{♚}| + \frac{1}{2}|\text{♕}\rangle\langle\text{♗}| + \frac{1}{2}|\text{♖}\rangle\langle\text{♙}| - \frac{1}{2}|\text{♘}\rangle\langle\text{♜}| = 0. \quad (\text{C.3})$$

As a more complicated example, let us consider the last two rows in Fig. 6 to see how the yellowfigures coincide with the **blue** ones. We have

$$acw^{-16}|\text{♔}\rangle\langle\text{♙}| + acw^{-16}|\text{♕}\rangle\langle\text{♗}| + bcw^{-9}|\text{♖}\rangle\langle\text{♘}| + bcw^{-13}|\text{♘}\rangle\langle\text{♙}| \quad (\text{C.4})$$

and

$$acw^{-6}|\text{♙}\rangle\langle\text{♙}| + acw^{14}|\text{♗}\rangle\langle\text{♗}| + bcw|\text{♗}\rangle\langle\text{♗}| + bcw^{-3}|\text{♙}\rangle\langle\text{♙}|. \quad (\text{C.5})$$

Again, after tracing out the color, we see that all terms in (C.4) and (C.5) can be coupled so that the coefficients form antipodal complex numbers on the two circles of radii ac and bc , and, when combined, they all vanish. One can repeat this examination for all possible complementary colors in every possible pair of rows.

For the partial trace over the first subsystem in **B'**, the interpretation is slightly different, but one can just exchange the chess figures with colors and the explanation will remain the same.

Finally, exactly the same reasoning is applicable for formulas in **C'**. However, this time we must consider pairs of figures instead of colors! Take, for example, the two first rows in Fig. 6 and the pair of (**♙**, **♗**). We have

$$c^2|\text{♙}\rangle\langle\text{♗}| + b^2w^{-10}|\text{♙}\rangle\langle\text{♗}| + a^2w^{-10}|\text{♙}\rangle\langle\text{♗}| = \dots \quad (\text{C.6})$$

Remembering the Pythagorean relation for the amplitudes, we see that after tracing out the figures, the above formula gives zero

$$\dots = c^2 - b^2 - a^2 = 0. \quad (\text{C.7})$$

As in the previous case, we can check all appropriately coupled figures in all different columns of the array in Fig. 6. Note that the pair (**♗**, **♙**) \neq (**♙**, **♗**), thus we must consider all six possibilities individually: (**♔**, **♗**), (**♕**, **♗**), (**♖**, **♙**), (**♘**, **♙**), (**♗**, **♙**), and (**♙**, **♗**).

Having gained a little practice, one can immediately predict which elements in Fig. 5 or Fig. 6 compensate in partial tracing. However, this does not directly apply to other variants of the state, especially those presented in Fig. 9. That is one of the reasons for which we consider such a particular representation of the AME(4, 6) state in the form of matrix U .

Appendix D. Block diagonal form of AME and Bell bases

The matrix U^Γ , visualized by the chessboard shown in Fig. 9b, can be rearranged to a block diagonal form U_{block}^Γ where nine 4×4 blocks form maximally entangled Bell bases,

see Fig. D1b. Two permutation matrices that bring U^Γ to $P_1 U^\Gamma P_2 = U_{\text{block}}^\Gamma$, shown in Fig. D2, can be written as two vectors consisting of 36 integers,

$$\begin{aligned}
 P_1 &= [6, 2, 36, 24, 13, 29, 22, 10, 32, 27, 1, 17, 31, 26, 3, 19, 23, 9, \\
 &\quad 18, 5, 12, 33, 28, 16, 11, 34, 25, 15, 20, 7, 14, 30, 8, 4, 35, 21], \\
 P_2 &= [3, 4, 9, 10, 7, 8, 1, 2, 27, 28, 33, 34, 16, 15, 22, 21, 11, 5, 12, \\
 &\quad 6, 25, 26, 31, 32, 13, 14, 19, 20, 17, 18, 23, 24, 29, 35, 30, 36],
 \end{aligned} \tag{D.1}$$

where non-zero element indicates the position of unity in appropriate column of P_j for $j \in \{1, 2\}$.

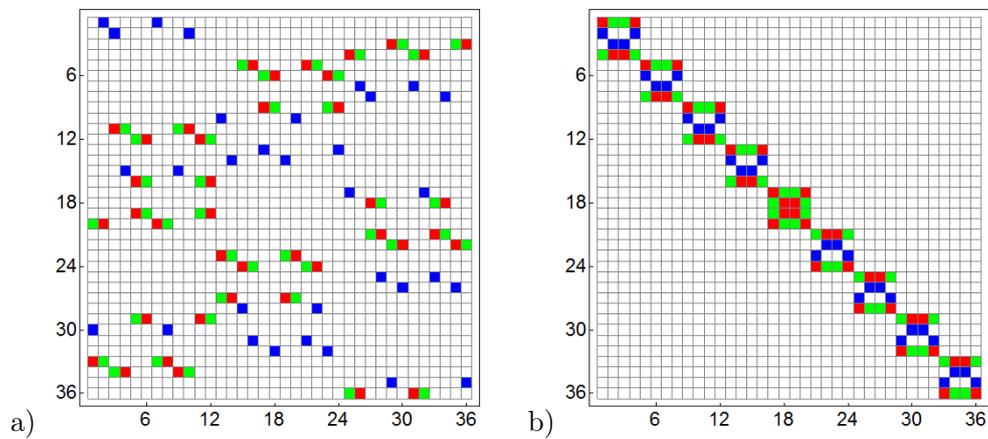


Figure D1. a) Partially transposed matrix U^Γ corresponding to the chessboard shown in Fig. 9 was already published in [31] under the name \mathcal{U}_{36} . b) Its block diagonal form U_{block}^Γ , where each 4×4 block forms a (maximally entangled) Bell basis for two qubits. Analogously to Fig. 7, we present only absolute values where red corresponds to a , green to b and blue to c amplitude, respectively. Note that the block diagonal matrix U_{block}^Γ no longer fulfills the properties of an AME state.

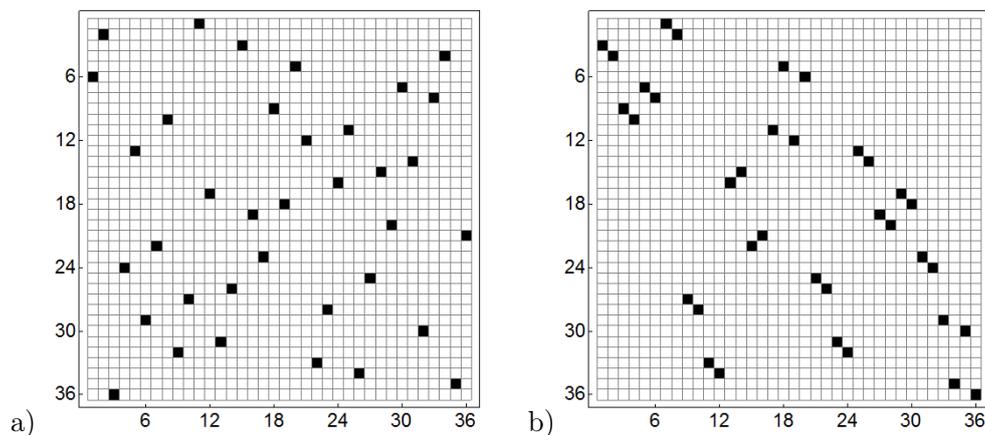


Figure D2. a) Left and b) right permutation matrices P_1 and P_2 defined in (D.1), which bring U^Γ to the block diagonal form presented in Fig. D1b.

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