

Toric Eigenvalue Methods for Solving Sparse Polynomial Systems

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Abstract

We consider the problem of computing homogeneous coordinates of points in a zero-dimensional subscheme of a compact, complex toric variety X . Our starting point is a homogeneous ideal I in the Cox ring of X , which in practice might arise from homogenizing a sparse polynomial system. We prove a new eigenvalue theorem in the toric compact setting, which leads to a novel, robust numerical approach for solving this problem. Our method works in particular for systems having isolated solutions with arbitrary multiplicities. It depends on the multigraded regularity properties of I . We study these properties and provide bounds on the size of the matrices in our approach when I is a complete intersection.

Key words — solving polynomial systems, sparse polynomial systems, toric varieties, Cox rings, eigenvalue theorem, symbolic-numeric algorithm

AMS subject classifications — 14M25, 65H04, 65H10

1 Introduction

The problem of solving a system of polynomial equations is ubiquitous in both pure and applied mathematics and in several engineering disciplines. Here we will consider only the important case where the solution set is finite. In many applications, the coefficients of the equations come from (noisy) measurements and extremely short computation times are required. Moreover, it is often sufficient to have numerical approximations of the solutions to the system. This establishes a need for the development of robust, numerical algorithms for solving these problems in floating point arithmetic. Existing numerical methods include *homotopy continuation methods*, which solve the problem using continuous deformation techniques, and *algebraic methods*, which solve the system by performing numerical linear algebra computations. For an overview of these techniques and applications, see [20, 48, 51, 24, 15] and references therein. In this paper, our goal is to present a *robust*, yet *efficient* numerical algebraic method for solving polynomial equations and to develop the necessary theory. We now explain this in more detail.

When using numerical algorithms, often the best one can hope for is to find the exact solution to a problem with slightly perturbed input data. In our context, these input data are the coefficients of the polynomials defining the system. One of the intrinsic challenges in solving polynomial equations numerically comes from the fact that small perturbations of these coefficients may change the geometry of the solution set significantly. For instance, this perturbation can introduce new solutions “near infinity”. Moreover, it typically causes solutions with higher multiplicity to split up into several distinct solutions. To make matters worse, when there are more equations than unknowns, perturbing coefficients leads to a system with no solutions at all. By a *robust* numerical algorithm we mean one that can approximate the solutions of the original system, even in the presence of solutions at/near infinity or with higher multiplicities. Our strategy is to employ a *toric compactification* of the solution space to deal with solutions at infinity, and generalize previously developed methods for dealing with multiplicities in this setting. Our toric compactification takes the *polyhedral structure* of the equations into account. This has the usual beneficial effect on the efficiency of our numerical solver, as we clarify below.

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Previous work. The method in this paper belongs to the class of numerical eigenvalue methods for root finding, see e.g. [25, 50, 40, 52]. Such methods rely on different variants of the *eigenvalue theorem*. Classically, this theorem states the following. Consider polynomials $f_1, \dots, f_s \in R := \mathbb{C}[x_1, \dots, x_n]$ generating an ideal $I := \langle f_1, \dots, f_s \rangle$, such that $f_1 = \dots = f_s = 0$ has finitely many solutions in \mathbb{C}^n . We denote the set of solutions by $V(I) \subset \mathbb{C}^n$. The eigenvalues of the *multiplication map* $m_g : R/I \rightarrow R/I$, $h \mapsto m_g(h) := gh$, are given by $g(z), z \in V(I)$. Moreover, if we have a vector space basis of R/I given by $b_1, \dots, b_\delta \in R/I$ and I is radical, then the left eigenvectors of m_g are the row vectors $(b_1(z), \dots, b_\delta(z))$, for each solution $z \in V(I)$. The coordinates of the solutions can be recovered from these vectors. We review these classical results in more detail in Section 3.1 and we refer to [16] for a recent historical overview on this theorem.

The choice of basis b_i of the quotient ring influences the accuracy with which we can approximate the multiplication maps m_g [55]. As we illustrate in Example 1, classical affine methods to compute these maps, such as the Canny-Emiris sparse resultant matrix [11, 25], are not robust in the sense explained above. They might require the inversion of a near-singular matrix, resulting in large rounding errors. These methods consider an a priori fixed basis for the quotient ring, i.e. given by mixed cells [25, Sec. 5], which is often not the best choice for the specific system at hand, see the discussion in [55, Sect. 7].

Methods exploiting the sparse, polyhedral structure of the equations often lead to much more efficient algorithms than the classical, ‘total degree based’ approaches. In the context of homotopy solvers, this explains the great success of polyhedral homotopies [56, 33] as an alternative for the more classical total degree homotopies, see e.g. [10, Ch. 15]. In algebraic methods, exploiting these structures leads to smaller matrix constructions. Examples include the above mentioned sparse resultant matrices as opposed to the classical Macaulay resultant matrix [25, 26, 40], matrices in sparse Gröbner basis algorithms [5, 6] and matrices representing truncated normal forms [54].

Whereas polyhedral methods are used to increase the *efficiency* of both symbolic and numerical algorithms, a strategy to improve the *robustness* of numerical solvers is to *compactify the solution space*. Homotopy path tracking in (multi-)projective spaces has the advantage that there are no diverging paths [3, Sec. 5.1] and ill-conditioned matrices in normal form methods can be avoided by using homogeneous interpretations [54, Sec. 5 & 6]. The practical approach to compactification is to homogenize the input equations to a graded ring, naturally associated to the considered compact space. For sparse systems, the standard (multi-)homogeneous compactifications typically lead to a homogeneous system of equations defining spurious positive dimensional solution components on the boundary (or *at infinity*), which are often highly singular. This causes trouble for both homotopy continuation and algebraic algorithms. In this paper, we use toric compactifications for which in most cases, meaning for *general* choices of the coefficients, no spurious components are introduced on the boundary. The (multi-)homogeneous compactifications can be considered as special cases of the ones we construct.

The first steps towards a numerically robust algorithm for sparse polynomial systems were taken by the second author in [52]. To solve these systems, homogeneous polynomials in the *Cox ring* S of a compact toric variety X were considered. The algorithm computes homogeneous coordinates of the solutions from the eigenvalues of a *multiplication map* in certain (multi-)degrees α of S . However, it is required that the solutions of the system have multiplicity one and they belong to the simplicial part of X . Moreover, the actual value of α necessary for this approach to work was not determined. We point out that the recent Cox homotopy algorithm [21] uses homotopy continuation to solve systems in the Cox ring.

Contributions. In this work, we use eigenvalue computations to solve polynomial systems on the toric variety X . The equations are assumed to have isolated, possibly singular solutions, which need not belong to the simplicial part of X . We introduce and study a new notion of *regularity* for these zero-dimensional systems, similar to the classical Castelnuovo-Mumford regularity for projective space. As opposed to previous definitions that consider degrees in the *Picard group* $\text{Pic}(X)$, e.g. [38], we allow degrees in the

more general *class group* $\text{Cl}(X)$. This way we can work with smaller matrices in our eigenvalue algorithm.

Definition 1.1. Consider a homogeneous ideal $I \subset S$ defining a zero-dimensional subscheme $V_X(I) \subset X$ of degree δ^+ . Let $B \subset S$ be the irrelevant ideal of the Cox ring of X . The *regularity* of I is

$$\text{Reg}(I) = \{\alpha \in \text{Cl}(X) \mid \dim_{\mathbb{C}}(S/I)_\alpha = \delta^+, I_\alpha = (I : B^\infty)_\alpha, \text{no point in } V_X(I) \text{ is a basepoint of } S_\alpha\}.$$

A tuple $(\alpha, \alpha_0) \in \text{Cl}(X)^2$ is called a *regularity pair* if $\alpha, \alpha + \alpha_0 \in \text{Reg}(I)$ and no point in $V_X(I)$ is a basepoint of S_{α_0} .

In this definition, a point $p \in X$ is called a *basepoint* of S_α if all $f \in S_\alpha$ vanish at p . For a degree $\alpha \in \text{Reg}(I)$, the graded piece $(S/I)_\alpha$ carries all the geometric information of the zero-dimensional subscheme $V_X(I)$ defined by I . We formalize this (Theorem 4.1) and discuss other definitions of regularity in Subsection 4.1. Our first main result generalizes the classical affine eigenvalue theorem to the toric setting.

Theorem (Toric eigenvalue theorem; Theorem 3.1). *Let $I \subset S$ be a homogeneous ideal such that $V_X(I)$ is zero-dimensional of degree δ^+ . Let $\zeta_1, \dots, \zeta_\delta$ denote the points in $V_X(I)$ and let μ_i be the multiplicity of ζ_i (so that $\delta^+ = \mu_1 + \dots + \mu_\delta$). Let (α, α_0) be a regularity pair. For $g, h_0 \in S_{\alpha_0}$ such that the rational function $\phi = g/h_0$ is regular at $V_X(I)$, the map $M_\phi : (S/I)_{\alpha+\alpha_0} \rightarrow (S/I)_{\alpha+\alpha_0}$ representing ‘multiplication with ϕ ’ has eigenvalues $\phi(\zeta_i), i = 1, \dots, \delta$, where $\phi(\zeta_i)$ has algebraic multiplicity μ_i .*

Our proof presents an explicit description of the eigenstructure of the multiplication maps in terms of differential operators defining the multiplicity structure, adapting known ideas from the affine case [39, 41]. In Section 3.3, we present an algorithm based on the toric eigenvalue theorem to solve sparse systems. This algorithm uses tools from numerical linear algebra, such as QR with column pivoting and SVD, to improve the numerical stability of the computed multiplication maps. As a result, it computes bases for the quotient ring which might differ from the classical mixed cells as in [44, 25].

Our algorithm assumes that a regularity pair (α, α_0) is provided, which is our main motivation for studying the regularity of zero-dimensional ideals. We provide a general criterion for extending a degree $\alpha \in \text{Reg}(I)$ to a regularity pair via evaluation of the Hilbert function (Theorem 4.2). While the problem of explicitly describing regularity pairs of general zero-dimensional ideals seems out of reach at this moment, our second main result provides a conclusive answer for ideals coming from *square* systems. Such ideals can be generated by $n = \dim X$ polynomials. They correspond to the most ubiquitous case in practice. Geometrically, they define *complete intersections* on X .

Theorem (Regularity for complete intersections; Theorem 4.4). *Let $I = \langle f_1, \dots, f_n \rangle \subset S$ with $f_i \in S_{\alpha_i}$ such that $\alpha_i \in \text{Pic}(X)$ is basepoint free and $V_X(I)$ is zero-dimensional. For any nef $\alpha_0 \in \mathbb{Q}\text{Pic}(X)$ such that there is $h \in S_{\alpha_0}$ which does not vanish at any point of $V_X(I)$, the tuple (α, α_0) is a regularity pair, with $\alpha = \alpha_1 + \dots + \alpha_n$. In particular, (α, α_0) is a regularity pair for any basepoint free $\alpha_0 \in \text{Pic}(X)$.*

The dimension of the \mathbb{C} -vector space $S_{\alpha+\alpha_0}$, where (α, α_0) is a regularity pair, determines the size of the matrices involved in our eigenvalue algorithm, as described in Section 3.3. Therefore, we are interested in finding regularity pairs for which $\dim_{\mathbb{C}} S_{\alpha+\alpha_0}$ is as small as possible. By the previous theorem, in the case where I can be generated by $n = \dim X$ elements, this dimension is bounded by $\dim_{\mathbb{C}} S_{\alpha_0+\alpha_1+\dots+\alpha_n}$ for any basepoint free $\alpha_0 \in \text{Pic}(X)$. In practice, when our system comes from the homogenization of a sparse polynomial system, this bound corresponds to a matrix of size proportional to the Minkowski sum of the input polytopes, which, roughly speaking, agrees with the size of the matrices obtained by other methods based on the Canny-Emiris sparse resultant matrix as [25, 40]. However, this bound may be pessimistic, leading to unnecessarily big matrices. For this reason, in Section 4.3 we study sparse polynomial systems

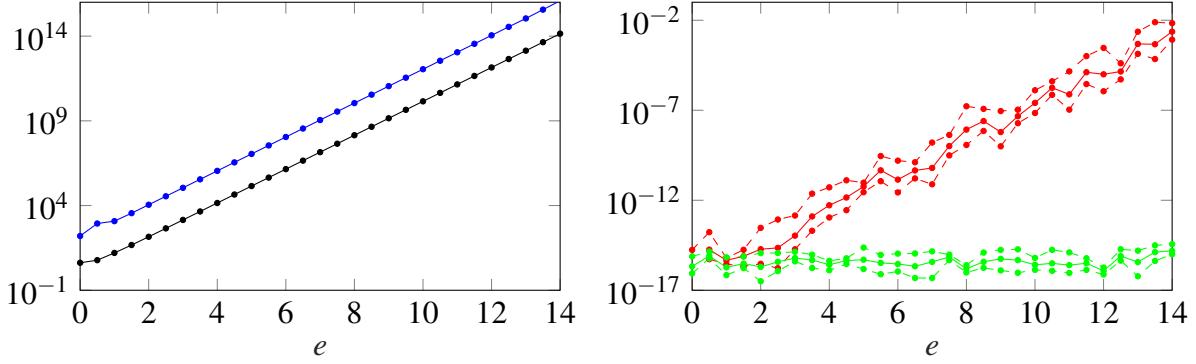


Figure 1: Left: norm of the largest solution (—●—) and condition number of the matrix inverted in the Canny-Emiris algorithm (—●—) as a function of e . Right: maximum, mean and minimum residual of all solutions computed by the Canny-Emiris algorithm (—●— and —●—) and by the algorithm based on this paper (—●— and —●—) in function of e .

with some extra structure and obtain regularity pairs related to smaller matrices. These systems include unmixed, classical homogeneous, weighted homogeneous and multihomogeneous square systems.

We conclude with an example that illustrates the effectiveness of our approach by comparing it to a classical sparse-resultant-based technique for an input system with a solution near infinity.

Example 1. Consider the system of equations on $(\mathbb{C}^*)^2$ given by $\hat{f}_1 = \hat{f}_2 = 0$, with

$$\begin{cases} \hat{f}_1 := -1 + t_1 + t_1^2 + t_2 + t_1 t_2, \\ \hat{f}_2 := -2 + 2t_1 + (5 - 2\epsilon)t_1^2 + 4t_2 + 5t_1 t_2. \end{cases}$$

The system involves a parameter ϵ , for which we will consider the real values $\epsilon \in [0, 1]$. For $\epsilon \in (0, 1]$, the system has 3 solutions in $(\mathbb{C}^*)^2$. As $\epsilon \rightarrow 0$, one out of the three solutions moves towards infinity. The norm of the coordinate vector in $(\mathbb{C}^*)^2$ of the largest solution is plotted in the left part of Figure 1. As we will see in Example 9, this diverging solution is moving to a torus invariant divisor on a Hirzebruch surface.

The solutions for $\epsilon \in (0, 1]$ can be computed via the eigenvectors of the Schur complement of a Canny-Emiris sparse resultant matrix, as in [20, Sec. 3.5.1]. We have done this for $\epsilon = 10^{-e}$, where $e = 0, 1/2, 1, 3/2, \dots, 14$. Computing the Schur complement requires the inversion of a square submatrix of the resultant matrix. We have plotted its condition number along with the solution norm on the left part of Figure 1. The drastic growth of this condition number causes big rounding errors on the coordinates of *all* solutions, not only the one that drifts off to infinity. This is shown in the right part of Figure 1, where we plot the *residual* of the numerically obtained approximate solutions. This is a measure for the relative backward error, see [53, Appendix C] for details. The same figure also shows the residuals for the approximate solutions obtained via the method presented in this paper. The results clearly illustrate that our method can deal perfectly with solutions drifting off to the boundary of the torus. We point out that, when solving this system using the state-of-the-art Julia homotopy package `HomotopyContinuation.jl` (v2.5.7) [9], only two solutions are computed for $e \geq 9$. The path leading to the largest solution is truncated prematurely, as it is assumed to diverge. This issue is addressed by a toric compactification for homotopy methods in [21]. \triangle

Outline. The paper is organized as follows. In Section 2, we review the notation and results on toric varieties that we need in the rest of the paper. In Section 3, we prove our toric eigenvalue theorem and detail our numerical algorithm for solving sparse polynomial systems. In Section 4, we investigate the regularity of zero-dimensional ideals and construct regularity pairs for complete intersections on toric varieties.

2 Preliminaries

In this section we recall some facts and introduce some notation related to toric varieties, Cox rings and divisors. The reader who is unfamiliar with concepts from toric geometry can find more details in [19, 28]. To avoid confusion, for an ideal I and a variety X we write $V_X(I)$ for the *subscheme* of X defined by I and $\text{Var}_X(I)$ for the *subvariety* of X defined by I . That is, $\text{Var}_X(I)$ is the reduced scheme associated to $V_X(I)$.

2.1 Toric geometry and the Cox construction

We write $T = (\mathbb{C}^*)^n = (\mathbb{C} \setminus \{0\})^n$ for the algebraic torus with character lattice $M = \text{Hom}_{\mathbb{Z}}(T, \mathbb{C}^*) \simeq \mathbb{Z}^n$ and cocharacter lattice $N = M^\vee = \text{Hom}_{\mathbb{Z}}(\mathbb{C}^*, T)$. A rational polyhedral fan Σ in $N \otimes_{\mathbb{Z}} \mathbb{R} = N_{\mathbb{R}} \simeq \mathbb{R}^n$ defines a normal toric variety X_Σ . This variety is complete (or equivalently, compact) if and only if the support of Σ is $N_{\mathbb{R}}$, in which case we also call Σ complete. In the rest of this paper, $X = X_\Sigma$ is a normal toric variety corresponding to a complete fan Σ , and we will sometimes write the subscript Σ to emphasize this correspondence.

The toric variety X_Σ admits an affine open covering given by the affine toric varieties $\{U_\sigma \mid \sigma \in \Sigma\}$ corresponding to the cones of σ . These affine varieties are defined as follows. For each cone $\sigma \in \Sigma$, the dual cone $\sigma^\vee \subset M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R} \simeq \mathbb{R}^n$ gives a saturated semigroup $\sigma^\vee \cap M$ whose associated \mathbb{C} -algebra $\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[U_\sigma]$ is the coordinate ring of the affine toric variety U_σ . The way these affine varieties are glued together to obtain X_Σ is encoded by the fan Σ , see [19, Ch. 3].

Let $\Sigma(d)$ be the set of d -dimensional cones of Σ . In particular, the rays of Σ are $\Sigma(1) = \{\rho_1, \dots, \rho_k\}$. They correspond to the torus invariant divisors D_1, \dots, D_k on X_Σ , which generate the free group $\text{Div}_T(X_\Sigma) \simeq \mathbb{Z}^k$. Each $\rho_i \in \Sigma(1)$ has a unique primitive ray generator $u_i \in N$. It is convenient to collect the u_i in a matrix

$$F = [u_1 \ \cdots \ u_k] \in \mathbb{Z}^{n \times k}.$$

The *divisor class group* $\text{Cl}(X_\Sigma)$ is isomorphic to $\text{Div}_T(X_\Sigma)/\text{im } F^\top$, so it is generated by the equivalence classes $[D_i]$, $i = 1, \dots, k$, see [19, Ch. 4].

In [14], Cox shows that X_Σ can be realized as a GIT quotient of a quasi-affine space by the action of an algebraic reductive group. The quotient is given by a surjective toric morphism

$$\pi : \mathbb{C}^k \setminus Z \rightarrow X_\Sigma, \tag{2.1}$$

where \mathbb{C}^k is the *total coordinate space* of X with coordinates labeled by $\Sigma(1)$ and the variety $Z = \text{Var}_{\mathbb{C}^k}(B)$ is the *base locus*, given by the zero set of the *irrelevant ideal* $B \subset S = \mathbb{C}[x_1, \dots, x_k]$. This is the square-free monomial ideal $B = \langle x^\delta \mid \sigma \in \Sigma \rangle$, where $x^\delta = \prod_{\rho_i \notin \sigma} x_i$. The map π is constant on the orbits of the action of an algebraic reductive subgroup $G \subset (\mathbb{C}^*)^k$, which acts on $\mathbb{C}^k \setminus Z$ by restricting the natural $(\mathbb{C}^*)^k$ -action.

For an element $\alpha = [\sum_{i=1}^k a_i D_i] \in \text{Cl}(X_\Sigma)$, we define the vector subspace

$$S_\alpha = \bigoplus_{F^\top m + a \geq 0} \mathbb{C} \cdot x^{F^\top m + a} \subset S,$$

where $a = (a_1, \dots, a_k) \in \mathbb{Z}^k$ and the sum ranges over all $m \in M$ satisfying $\langle u_i, m \rangle + a_i \geq 0$, $i = 1, \dots, k$. This definition is independent of the chosen representative for α . The action of G on \mathbb{C}^k induces an action of G on S : for $g \in G$, $f \in S$, $(g \cdot f)(x) = f(g^{-1} \cdot x)$, and an element $f \in S_\alpha$ defines an affine subvariety $\text{Var}_{\mathbb{C}^k}(f)$ that is stable under the action of G . From this observation, it follows that an element $f \in S_\alpha$ has a well-defined zero set on X_Σ , given by

$$\text{Var}_{X_\Sigma}(f) = \{\zeta \in X_\Sigma \mid f(x) = 0 \text{ for some } x \in \pi^{-1}(\zeta)\}.$$

This is why S is equipped with its *grading by the class group*: $S = \bigoplus_{\alpha \in \text{Cl}(X_\Sigma)} S_\alpha$. The ring S , together with this grading and its irrelevant ideal, is called the *Cox ring*, homogeneous coordinate ring or total coordinate ring of X_Σ . The homogeneous ideals of S , that is, the ideals generated by elements that are homogeneous with respect to the $\text{Cl}(X_\Sigma)$ -grading, define the closed subschemes of X_Σ , see [19, Ch. 5 & 6]. If X_Σ is smooth and we restrict to B -saturated homogeneous ideals, this correspondence is one-to-one [14, Cor. 3.8]. For a homogeneous ideal $I \subset S$, the corresponding subscheme is denoted by $V_{X_\Sigma}(I)$ and its associated variety is $\text{Var}_{X_\Sigma}(I)$.

Solving equations on X . Given homogeneous polynomials $f_1, \dots, f_s \in S$, such that the associated scheme $V_{X_\Sigma}(I)$ is zero-dimensional, the aim in this paper is to *solve* $f_1 = \dots = f_s = 0$ on X . We will now make this precise. For each $\zeta_i \in \text{Var}_X(I)$, we want to compute a point $z_i \in \mathbb{C}^k \setminus Z$ such that $\pi(z_i) = \zeta_i$. In this context, the point z_i is called a *set of homogeneous coordinates* for ζ_i .

2.2 Homogenization and dehomogenization

Homogenization. The \mathbb{C} -algebra $\mathbb{C}[M]$ over the lattice M is isomorphic to the ring $\mathbb{C}[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ of n -variate Laurent polynomials. We consider s elements $\hat{f}_1, \dots, \hat{f}_s$ of $\mathbb{C}[M]$. These elements define a system of relations $\hat{f}_1 = \dots = \hat{f}_s = 0$ on T , which extends to a system of relations on a toric compactification $X \supset T$. We will make this precise in this subsection.

For each \hat{f}_i , let $P_i \subset M_{\mathbb{R}} = \mathbb{R}^n$ be its Newton polytope, i.e., the convex hull in $M_{\mathbb{R}}$ of the characters appearing in \hat{f}_i with a nonzero coefficient. Let $P = P_1 + \dots + P_s$ be the Minkowski sum of all these polytopes. We assume that P is full-dimensional. The normal fan Σ_P of P defines a complete, normal toric variety $X = X_{\Sigma_P}$. We will use the same notation as in Section 2.1 for the rays, primitive ray generators, etc. To each of the polytopes P_i , we associate a torus invariant divisor $D_{P_i} \in \text{Div}_T(X)$ as follows. Let $a_i = (a_{i,1}, \dots, a_{i,k}) \in \mathbb{Z}^k$ be such that

$$a_{i,j} = \min_{\mathbb{Z}} c \text{ s.t. } P_i \subset \{m \in M_{\mathbb{R}} \mid \langle u_j, m \rangle + c \geq 0\}.$$

We set $D_{P_i} = \sum_{j=1}^k a_{i,j} D_j \in \text{Div}_T(X)$. With this construction, the D_{P_i} are *Cartier divisors*. The classes of all Cartier divisors in $\text{Cl}(X)$ form a group called the *Picard group* $\text{Pic}(X) \subset \text{Cl}(X)$. We have that $\alpha_i = [D_{P_i}] \in \text{Pic}(X)$ and, additionally, the divisors D_{P_i} are *basepoint free* (see Definition 2.1).

We start by ‘homogenizing’ the \hat{f}_i to the Cox ring S of X . For this, we observe that by construction

$$\hat{f}_i \in \bigoplus_{m \in P_i \cap M} \mathbb{C} \cdot t^m \simeq \bigoplus_{F^\top m + a_i \geq 0} \mathbb{C} \cdot t^m \simeq \bigoplus_{F^\top m + a_i \geq 0} \mathbb{C} \cdot x^{F^\top m + a_i} = S_{\alpha_i}.$$

This gives a canonical way of *homogenizing* \hat{f}_i :

$$\hat{f}_i = \sum_{F^\top m + a_i \geq 0} c_{m,i} t^m \mapsto f_i = \sum_{F^\top m + a_i \geq 0} c_{m,i} x^{F^\top m + a_i}. \quad (2.2)$$

The subvariety $\text{Var}_X(f_i) \subset X$ is the closure of $\text{Var}_T(\hat{f}_i)$ in X .

Example 2. Let $n = 2$, $\mathbb{C}[M] = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$ and consider the equations

$$\hat{f}_1 = t_1 - t_2^{-1} + t_2 + t_1^{-1}, \quad \hat{f}_2 = 2t_1 + t_2^{-1} - t_2 - t_1^{-1}.$$

There are no solutions of $\hat{f}_1 = \hat{f}_2 = 0$ in $(\mathbb{C}^*)^2$ (note that $\hat{f}_1 + \hat{f}_2$ is a unit in $\mathbb{C}[M]$). The Newton polygons P_1, P_2 are identical. The associated toric variety X is the *double pillow surface* (see [49, Sec. 3.3]). The

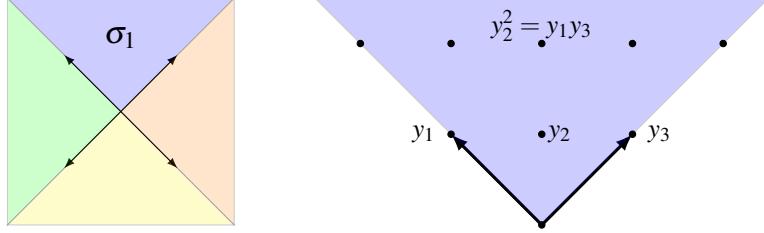


Figure 2: Illustration of the fan $\Sigma_{P_1+P_2}$ (left) of the toric variety from Example 2, and of the semigroup algebra $\mathbb{C}[U_{\sigma_1}] \simeq \mathbb{C}[y_1, y_2, y_3]/\langle y_2^2 - y_1 y_3 \rangle$ corresponding to the (dual cone of the) blue cone (right).

fan $\Sigma = \Sigma_{P_1+P_2}$ is depicted in Figure 2. We arrange the primitive ray generators of $\Sigma(1)$ in the matrix $F = [u_1 \ u_2 \ u_3 \ u_4] = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \end{bmatrix}$. Our equations homogenize to $f_1 = x_1^2 x_4^2 - x_3^2 x_4^2 + x_1^2 x_2^2 + x_2^2 x_3^2$, $f_2 = 2x_1^2 x_4^2 + x_3^2 x_4^2 - x_1^2 x_2^2 - x_2^2 x_3^2$ in the Cox ring $S = \mathbb{C}[x_1, x_2, x_3, x_4]$ of X . The degrees $\alpha_i = \deg(f_i)$ are $\alpha_1 = \alpha_2 = [\sum_{i=1}^4 D_i]$. The scheme $V_X(f_1, f_2)$ consists of two points, each with multiplicity two. These points correspond to the orbits of $z_1 = (0, 1, 1, 1), z_2 = (1, 1, 0, \sqrt{-1})$. \triangle

Dehomogenization. Recall that X_Σ is covered by the affine toric varieties $\{U_\sigma \mid \sigma \in \Sigma(n)\}$. The restriction of (2.1) to $\pi^{-1}(U_\sigma) = \mathbb{C}^k \setminus \text{Var}_{\mathbb{C}^k}(x^\hat{\sigma})$ identifies $\mathbb{C}[U_\sigma]$ with the ring of invariants $(S_{x^\hat{\sigma}})_0$, see the proof of [19, Thm. 5.1.10]. For each full-dimensional cone $\sigma \in \Sigma(n)$, we will define a \mathbb{C} -linear map $(\cdot)^\sigma : S \rightarrow (S_{x^\hat{\sigma}})_0 \simeq \mathbb{C}[U_\sigma]$, called *dehomogenization*. We do this by defining it on graded pieces S_α , and extending linearly. For $\alpha = [\sum_{i=1}^k a_i D_i]$ such that there exists $m_\sigma \in M$ with $\langle u_i, m_\sigma \rangle + a_i = 0$ for all i such that $\rho_i \in \sigma(1)$, we set

$$f \in S_\alpha \mapsto f^\sigma := \frac{f}{x^{\hat{\sigma}, \alpha}}, \quad \text{with} \quad x^{\hat{\sigma}, \alpha} := x^{F^\top m_\sigma + \alpha} \in S_\alpha.$$

Note that, although m_σ depends on the choice of representative $\sum_{i=1}^k a_i D_i$ for α , the monomial $x^{\hat{\sigma}, \alpha}$ does not. This is the only monomial of degree α such that $(x^{\hat{\sigma}, \alpha})^\sigma = 1$. For $\alpha \in \text{Cl}(X)$ for which no such m_σ exists, we set $(S_\alpha)^\sigma = 0$. Observe that, if $(S_\alpha)^\sigma \neq 0$, the restriction of the dehomogenization map to S_α is injective. In particular, this is the case for $\alpha \in \text{Pic}(X)$ [19, Thm. 4.2.8]. For each $\sigma \in \Sigma(n)$, dehomogenization $(\cdot)^\sigma : S \rightarrow \mathbb{C}[U_\sigma]$ is surjective. Moreover, for a homogeneous ideal $I \subset S$ we have

$$I^\sigma = (I_{x^\hat{\sigma}})_0 = \mathcal{J}(U_\sigma) \subset \mathbb{C}[U_\sigma], \quad (2.3)$$

where \mathcal{J} is the ideal sheaf associated to I .

Example 3 (Cont. Example 2). Let σ_1 be the blue cone in Figure 2. The ideal $I^{\sigma_1} = \langle f_1^{\sigma_1}, f_2^{\sigma_1} \rangle \subset \mathbb{C}[U_{\sigma_1}] = \mathbb{C}[y_1, y_2, y_3]/\langle y_2^2 - y_1 y_3 \rangle$ is $I^{\sigma_1} = \langle [y_2^2 - y_1 + y_3 + 1], [2y_2^2 + y_1 - y_3 - 1] \rangle$, where $[\cdot]$ denotes the residue class modulo $\langle y_2^2 - y_1 y_3 \rangle$. The ordering of the variables y_i is clarified in the right part of Figure 2. Only the solution corresponding to the orbit of z_1 is contained in U_{σ_1} , which explains that $\dim_{\mathbb{C}} \mathbb{C}[U_{\sigma_1}]/I^{\sigma_1} = 2$. \triangle

The following lemma points out a way of going back and forth between I and I^σ .

Lemma 2.1. Suppose that $f \in S$ is such that $f^\sigma \in I^\sigma \setminus \{0\}$, then there is $\ell \in \mathbb{N}$ such that $(x^\hat{\sigma})^\ell f \in I$. Conversely, for each $\hat{f} \in I^\sigma$, we can find a homogeneous $f \in I$ such that $f^\sigma = \hat{f}$.

Proof. The first statement follows immediately from $I^\sigma = (I_{x^\hat{\sigma}})_0$. For the second statement, note that \hat{f} can be written as $\hat{f} = f/(x^\hat{\sigma})^\ell$ with $f \in I_{\ell \deg(x^\hat{\sigma})}$. We have $x^{\hat{\sigma}, \ell \deg(x^\hat{\sigma})} = (x^\hat{\sigma})^\ell$ and thus $((x^\hat{\sigma})^\ell \hat{f})^\sigma = f^\sigma = \hat{f}$. \square

2.3 Divisors

It will be convenient to have a notation for subsets of the divisor class group that are of interest:

$$\begin{array}{c|l} \mathrm{Cl}(X) & \text{divisor class group of } X \\ \mathrm{Cl}(X)_+ & \text{divisor classes of } \textit{effective} \text{ divisors: } \{\alpha \in \mathrm{Cl}(X) \mid \alpha = [\sum_{i=1}^k a_i D_i], a_i \geq 0\} \end{array}$$

We will work with a subclass of these divisors given by the *basepoint free* divisors.

Definition 2.1. For a fixed degree $\alpha \in \mathrm{Cl}(X)$, a point $\zeta \in X$ is called a *basepoint* of S_α (or of α) if $\zeta \in V_X(f)$ for all $f \in S_\alpha$. The degree $\alpha \in \mathrm{Cl}(X)$ is called *basepoint free* if it has no basepoints. A torus invariant divisor $D \in \mathrm{Div}_T(X)$ is called *basepoint free* if $[D] \in \mathrm{Cl}(X)$ is basepoint free.

We introduce a similar notation for subsets of the Picard group, and the (in general larger) subgroup of $\mathrm{Cl}(X)$ consisting of the divisor classes of \mathbb{Q} -Cartier divisors on X . Note that, as X is a complete toric variety, nef Cartier divisors correspond to basepoint free Cartier divisors [19, Thm. 6.3.12] and nef \mathbb{Q} -Cartier divisors correspond to divisors of which a multiple is nef and Cartier [19, Lem. 9.2.1]. We make the following definitions.

$$\begin{array}{c|l} \mathrm{Pic}(X) & \text{Picard group of } X \\ \mathrm{Pic}^\circ(X) & \text{divisor classes of } \textit{nef Cartier} \text{ divisors:} \\ & \{\alpha \in \mathrm{Pic}(X) \mid \alpha \text{ is basepoint free}\} \\ \mathbb{Q}\mathrm{Pic}(X) & \text{divisor classes of } \mathbb{Q}\text{-Cartier divisors:} \\ & \{\alpha \in \mathrm{Cl}(X) \mid \ell\alpha \in \mathrm{Pic}(X) \text{ for some } \ell \in \mathbb{N}_{>0}\} \\ \mathbb{Q}\mathrm{Pic}^\circ(X) & \text{divisor classes of } \textit{nef } \mathbb{Q}\text{-Cartier divisors:} \\ & \{\alpha \in \mathrm{Cl}(X) \mid \ell\alpha \in \mathrm{Pic}^\circ(X) \text{ for some } \ell \in \mathbb{N}_{>0}\} \end{array} \quad \begin{array}{ccc} \mathrm{Pic}^\circ(X) & \subset & \mathrm{Pic}(X) \\ \cap & & \cap \\ \mathbb{Q}\mathrm{Pic}^\circ(X) & \subset & \mathbb{Q}\mathrm{Pic}(X) \\ \cap & & \cap \\ \mathrm{Cl}(X)_+ & \subset & \mathrm{Cl}(X) \end{array}$$

The diagram of inclusions on the right follows directly from these definitions. Note that if $\alpha \in \mathrm{Cl}(X) \setminus \mathrm{Cl}(X)_+$, we have that $S_\alpha = \{0\}$, and hence every $\zeta \in X$ is a basepoint of S_α .

Example 4. Consider again the double pillow surface X from Example 2. Its fan is depicted in Figure 2. One can check that $\mathbb{Q}\mathrm{Pic}(X) = \mathrm{Cl}(X)$ (which is true whenever X is simplicial), $\alpha_0 := [D_2 + D_4] \in \mathbb{Q}\mathrm{Pic}^\circ(X) \setminus \mathrm{Pic}^\circ(X)$, $2\alpha_0 \in \mathrm{Pic}^\circ(X)$, $-\alpha_0 \in \mathrm{Pic}(X) \setminus \mathrm{Cl}(X)_+$. Moreover, S_{α_0} is not basepoint free: $S_{\alpha_0} = \mathbb{C} \cdot x_2 x_3$ vanishes on $V_X(x_2) \cup V_X(x_3)$. \triangle

We generalize the concept of basepoint free divisors to subschemes of X .

Definition 2.2. Given a subscheme Y of X , we say that $\alpha \in S$ is *Y -basepoint free* if Y_{red} does not contain any basepoint of α , where Y_{red} is the variety (reduced scheme) associated to Y . Whenever Y is zero-dimensional, this condition is equivalent to the existence of $h \in S_\alpha$ such that $Y_{\mathrm{red}} \cap \mathrm{Var}_X(h) = \emptyset$.

3 A toric eigenvalue method

In this section we consider a homogeneous ideal $I = \langle f_1, \dots, f_s \rangle \subset S$ which defines a zero-dimensional, possibly *non-reduced* subscheme $V_X(I)$. That is, some of the points in the scheme have multiplicity greater than one.

We will give an explicit description of the toric eigenstructure of multiplication maps (see (3.8)) in the presence of non-reduced points. First, in Section 3.1 we recall what happens in the affine case. The approach is similar to [39, 41], in the specific case of a zero-dimensional subvariety of an affine toric variety. This fixes some notation and sets the stage for the homogeneous case. In Section 3.2, we characterize the

eigenstructure of multiplication maps by ‘gluing’ the affine constructions. The eigenvectors have a natural interpretation as elements of the dual of the Cox ring. Finally, in Section 3.3 we present an eigenvalue approach for solving $f_1 = \dots = f_s = 0$ on X , that is, for computing homogeneous coordinates of the points in $\text{Var}_X(I)$. Our algorithm also allows to compute the multiplicities of these points in $V_X(I)$.

Throughout the section, for a \mathbb{C} -vector space V , we denote $V^\vee = \text{Hom}_{\mathbb{C}}(V, \mathbb{C})$ for the dual vector space.

3.1 Non-reduced points on an affine toric variety

Let $I^\sigma \subset \mathbb{C}[U_\sigma]$ be an ideal in the coordinate ring of the normal affine toric variety U_σ coming from a full dimensional cone $\sigma \subset N_{\mathbb{R}}$. Let $\mathcal{A} = \{m_1, \dots, m_\ell\} \subset M$ be a set of characters such that $\sigma^\vee \cap M = \mathbb{N} \cdot \{m_1, \dots, m_\ell\}$. The ring $\mathbb{C}[U_\sigma]$ can be realized as a quotient ring $\mathbb{C}[U_\sigma] \simeq \mathbb{C}[y_1, \dots, y_\ell]/I_{\mathcal{A}}$, where the variables y_i correspond to the characters m_i and $I_{\mathcal{A}}$ is the toric ideal corresponding to the embedding of U_σ in \mathbb{C}^ℓ given by \mathcal{A} [19, Ch. 1]. Therefore, the ideal I^σ corresponds to an ideal in $\mathbb{C}[y_1, \dots, y_\ell]/I_{\mathcal{A}}$, which in turn corresponds to an ideal in $\mathbb{C}[y_1, \dots, y_\ell]$ containing $I_{\mathcal{A}}$. In other words, the subscheme of U_σ defined by I^σ is embedded in \mathbb{C}^ℓ via

$$\mathbb{C}[y_1, \dots, y_\ell] \rightarrow \mathbb{C}[y_1, \dots, y_\ell]/I_{\mathcal{A}} \simeq \mathbb{C}[U_\sigma] \rightarrow \mathbb{C}[U_\sigma]/I^\sigma.$$

We will think of $V_{U_\sigma}(I^\sigma)$ as a subscheme of \mathbb{C}^ℓ given by this embedding and denote the defining ideal by I_y^σ . That is,

$$V_{U_\sigma}(I^\sigma) = \text{Spec}(\mathbb{C}[y_1, \dots, y_\ell]/I_y^\sigma) = \text{Spec}(\mathbb{C}[U_\sigma]/I^\sigma)$$

and $I_{\mathcal{A}} \subset I_y^\sigma \subset \mathbb{C}[y_1, \dots, y_\ell]$. The reason why we embed $V_{U_\sigma}(I_\sigma)$ in \mathbb{C}^ℓ is that the multiplicity structure of non-reduced points in affine space have a nice, explicit description in terms of differential operators. We will now recall how this works. For every $\zeta \in \mathbb{C}^\ell$, let $\mathfrak{m}_\zeta \subset \mathbb{C}[y_1, \dots, y_\ell]$ be the corresponding maximal ideal. Assuming that $V_{U_\sigma}(I^\sigma)$ is zero-dimensional, defining δ_σ points $\{\zeta_1, \dots, \zeta_{\delta_\sigma}\} \subset \mathbb{C}^\ell$, the primary decomposition of I_y^σ is given by $I_y^\sigma = Q_1 \cap \dots \cap Q_{\delta_\sigma}$, where Q_i is \mathfrak{m}_{ζ_i} -primary. By the Chinese remainder theorem [35, Lem. 3.7.4] this gives

$$\mathbb{C}[y_1, \dots, y_\ell]/I_y^\sigma = \mathbb{C}[y_1, \dots, y_\ell]/Q_1 \oplus \dots \oplus \mathbb{C}[y_1, \dots, y_\ell]/Q_{\delta_\sigma}. \quad (3.1)$$

The multiplicities μ_i of the points ζ_i are given by $\mu_i = \dim_{\mathbb{C}}(\mathbb{C}[y_1, \dots, y_\ell]/Q_i)$. We denote $\delta_\sigma^+ = \mu_1 + \dots + \mu_{\delta_\sigma} = \dim_{\mathbb{C}}(\mathbb{C}[y_1, \dots, y_\ell]/I_y^\sigma)$. For an ℓ -tuple $a = (a_1, \dots, a_\ell) \in \mathbb{N}^\ell$ we define the \mathbb{C} -linear map $\partial_a : \mathbb{C}[y_1, \dots, y_\ell] \rightarrow \mathbb{C}[y_1, \dots, y_\ell]$ as

$$\partial_a(f) = \frac{1}{a_1! \cdots a_\ell!} \frac{\partial^{a_1 + \dots + a_\ell} f}{\partial y_1^{a_1} \cdots \partial y_\ell^{a_\ell}}$$

and $\mathcal{D} = \text{span}_{\mathbb{C}}(\partial_a, a \in \mathbb{N}^\ell)$. These operators allow for a very simple formulation of Leibniz’ rule, which says that for $\partial \in \mathcal{D}$,

$$\partial(fg) = \sum_{b \in \mathbb{N}^\ell} \partial_b(g)(s_b(\partial))(f) \quad \text{with} \quad s_b \left(\sum_a c_a \partial_a \right) = \sum_{a-b \geq 0} c_a \partial_{a-b}. \quad (3.2)$$

Definition 3.1. A \mathbb{C} -vector subspace $V \subset \mathcal{D}$ is *closed* if $\dim_{\mathbb{C}}(V) < \infty$ and for each $\partial \in V$ and each $b \in \mathbb{N}^\ell$, $s_b(\partial) \in V$.

Note that if $V \subset \mathcal{D}$ is closed, then $\partial_0 = \text{id} \in V$. For $\zeta \in \mathbb{C}^\ell$, let $\text{ev}_\zeta \in \text{Hom}_{\mathbb{C}}(\mathbb{C}[y_1, \dots, y_\ell], \mathbb{C}) = \mathbb{C}[y_1, \dots, y_\ell]^\vee$ be the linear functional defined by $\text{ev}_\zeta(f) = f(\zeta)$. It follows from [39, Thm. 2.6] that, for each $i \in \{1, \dots, \delta_\sigma\}$, there is a closed subspace $V_i \subset \mathcal{D}$ such that,

$$Q_i = \{f \in \mathbb{C}[y_1, \dots, y_\ell] \mid (\text{ev}_{\zeta_i} \circ \partial)(f) = \partial(f)(\zeta_i) = 0, \forall \partial \in V_i\}.$$

For each Q_i , we define the set of linear functionals Q_i^\perp as follows,

$$Q_i^\perp := \{v \in \mathbb{C}[y_1, \dots, y_\ell]^\vee \mid v(f) = 0, \forall f \in Q_i\} = \{\text{ev}_{\zeta_i} \circ \partial \mid \partial \in V_i\} =: \text{ev}_{\zeta_i} \circ V_i.$$

From basic linear algebra, it follows that

$$\text{ev}_{\zeta_i} \circ V_i = Q_i^\perp \simeq (\mathbb{C}[y_1, \dots, y_\ell]/Q_i)^\vee \subset (\mathbb{C}[y_1, \dots, y_\ell]/I_y^\sigma)^\vee.$$

In other words, we can interpret elements of $\text{ev}_{\zeta_i} \circ V_i$ as elements of $(\mathbb{C}[y_1, \dots, y_\ell]/I_y^\sigma)^\vee$, by setting $(\text{ev}_{\zeta_i} \circ \partial)(f + I_y^\sigma) = (\text{ev}_{\zeta_i} \circ \partial)(f)$ for $\partial \in V_i$.

For $g \in \mathbb{C}[y_1, \dots, y_\ell]$, the *multiplication map* $M_g : \mathbb{C}[y_1, \dots, y_\ell]/I_y^\sigma \rightarrow \mathbb{C}[y_1, \dots, y_\ell]/I_y^\sigma$ is defined as $M_g(f + I_y^\sigma) = fg + I_y^\sigma$. For a differential operator $\partial = \sum_a c_a \partial_a \in \mathcal{D}$ we define $\text{ord}(\partial) = \max_{c_a \neq 0} (a_1 + \dots + a_\ell)$. We denote by $(V_i)_{\leq d} = \{\partial \in V_i \mid \text{ord}(\partial) \leq d\}$ the subspace of differential operators in V_i of order bounded by d .

For giving explicit descriptions of the eigenstructure of multiplication maps, it is convenient to work with a special type of basis for the spaces V_i (see [41, Sec. 5]). Let $V \subset \mathcal{D}$ be a closed subspace. An ordered tuple $(\partial^1, \dots, \partial^\mu)$ with $\partial^j \in V, j = 1, \dots, \mu$ is called a *consistently ordered basis* for V if for every $d \geq 0$ there is j_d such that $\{\partial^1, \dots, \partial^{j_d}\}$ is a \mathbb{C} -vector space basis for $V_{\leq d}$. Note that a consistently ordered basis always exists for any closed subspace V , its first differential operator is always $\partial_0 = \text{id}$ and it is a \mathbb{C} -vector space basis for V . For $i = 1, \dots, \delta_\sigma$, let $(\partial^{i1}, \dots, \partial^{i\mu_i})$ be a consistently ordered basis for V_i . By Leibniz' rule, for all $f + I_y^\sigma \in \mathbb{C}[y_1, \dots, y_\ell]/I_y^\sigma$ we have

$$((\text{ev}_{\zeta_i} \circ \partial^{ij}) \circ M_g)(f + I_y^\sigma) = \text{ev}_{\zeta_i}(\partial^{ij}(fg)) = (\text{ev}_{\zeta_i} \circ \sum_{b \in \mathbb{N}^\ell} \partial_b(g)s_b(\partial^{ij}))(f + I_y^\sigma). \quad (3.3)$$

In particular, for $\partial^{i1} = \partial_0 = \text{id}$ we get $\text{ev}_{\zeta_i} \circ M_g = g(\zeta_i) \text{ev}_{\zeta_i}$, which shows the classical fact that the evaluation functionals ev_{ζ_i} are (left) eigenvectors of M_g with eigenvalues $g(\zeta_i)$. In general, by the property of being closed, $s_b(\partial^{ij})$ can be written as a \mathbb{C} -linear combination of $\partial^{i1}, \dots, \partial^{i\mu_i}$. For $b \neq 0$, by the property of being consistently ordered and by the fact that $\text{ord}(s_b(\partial)) < \text{ord}(\partial)$, $s_b(\partial^{ij})$ can be written as a \mathbb{C} -linear combination of $\partial^{i1}, \dots, \partial^{i,j-1}$ (in fact, we only need the differentials of order $< \text{ord}(\partial^{ij})$). Then, in matrix notation, (3.3) becomes

$$\begin{bmatrix} \text{ev}_{\zeta_i} \circ \partial^{i1} \\ \text{ev}_{\zeta_i} \circ \partial^{i2} \\ \vdots \\ \text{ev}_{\zeta_i} \circ \partial^{i\mu_i} \end{bmatrix} \circ M_g = \begin{bmatrix} g(\zeta_i) & & & \\ c_{i2}^{(1)} & g(\zeta_i) & & \\ \vdots & & \ddots & \\ c_{i\mu_i}^{(1)} & c_{i\mu_i}^{(2)} & \dots & g(\zeta_i) \end{bmatrix} \begin{bmatrix} \text{ev}_{\zeta_i} \circ \partial^{i1} \\ \text{ev}_{\zeta_i} \circ \partial^{i2} \\ \vdots \\ \text{ev}_{\zeta_i} \circ \partial^{i\mu_i} \end{bmatrix} \quad (3.4)$$

for some complex coefficients $c_{ij}^{(k)}$. This observation leads immediately to the classical eigenvalue theorem [18, Ch. 4, §2, Prop. 2.7] and it will play a key role in the proof of Theorem 3.1.

3.2 Toric eigenvalue theorem

In what follows, we fix a homogeneous ideal $I \subset S$ defining a zero-dimensional closed, possibly *non-reduced*, subscheme $V_X(I)$. As before, we write $\{\zeta_1, \dots, \zeta_\delta\} \subset X$ for $\text{Var}_X(I)$ and denote by μ_i the multiplicity of ζ_i . The number of solutions, counting these multiplicities, is $\delta^+ := \mu_1 + \dots + \mu_\delta \geq \delta$. Recall that, $I^\sigma = \mathcal{J}(U_\sigma) \subset \mathbb{C}[U_\sigma]$, where $(.)^\sigma$ is the *dehomogenization map*.

We can write $V_X(I)$ as a union of affine closed schemes Y_1, \dots, Y_δ , where Y_i corresponds to the point ζ_i in $V_X(I)$. For each $i = 1, \dots, \delta$, we denote by \mathcal{Q}_i the ideal sheaf of Y_i on X , that is, the ideal sheaf \mathcal{Q}_i

such that $\mathcal{O}_{Y_i} \simeq \mathcal{O}_X/\mathcal{Q}_i$. Note that, for any $\sigma \in \Sigma$, $\mathcal{Q}_i(U_\sigma) = H^0(U_\sigma, \mathcal{Q}_i)$ is the primary ideal of $\mathbb{C}[U_\sigma]$ corresponding to the point ζ_i in $V_{U_\sigma}(I^\sigma)$ if $\zeta_i \in U_\sigma$, and $\mathcal{Q}_i(U_\sigma) = \mathbb{C}[U_\sigma]$ otherwise. These primary ideals relate to the primary ideals Q_i in (3.1) as follows,

$$\mathbb{C}[U_\sigma]/I^\sigma = \bigoplus_{\zeta_i \in U_\sigma} \mathbb{C}[U_\sigma]/\mathcal{Q}_i(U_\sigma) \simeq \bigoplus_{\zeta_i \in U_\sigma} \mathbb{C}[y_1, \dots, y_\ell]/Q_i.$$

Additionally, $(\mathbb{C}[U_\sigma]/\mathcal{Q}_i(U_\sigma))^\vee \simeq \mathcal{Q}_i(U_\sigma)^\perp \simeq Q_i^\perp = \text{ev}_{\zeta_i} \circ V_i$, with

$$\mathcal{Q}_i(U_\sigma)^\perp := \{v \in \mathbb{C}[U_\sigma]^\vee \mid v(f^\sigma) = 0, \text{ for all } f^\sigma \in \mathcal{Q}_i(U_\sigma)\}$$

and with $V_i \subset \mathcal{D}$ a closed subspace of differential operators. This allows us to write elements of $\mathcal{Q}_i(U_\sigma)^\perp$ as $\text{ev}_{\zeta_i} \circ \partial$, with $\partial \in V_i$. Note that ev_{ζ_i} depends on the cone σ , even though we do not make it explicit in the notation. For an element $\alpha \in \text{Cl}(X)$ we denote by I_α^\perp the vector space

$$I_\alpha^\perp := \{v \in S_\alpha^\vee \mid v(f) = 0, \text{ for all } f \in I_\alpha\} \simeq (S/I)_\alpha^\vee.$$

Our first objective is to map the elements in $\mathcal{Q}_i(U_\sigma)^\perp$ to non-zero elements in I_α^\perp . When $\alpha \in \text{Pic}(X)$, we can do so by considering the dehomogenization of the elements in I_α . When $\alpha \in \text{Cl}(X)$, as $(S_\alpha)^\sigma$ might be zero, we need to lift the elements of I_α to a higher degree such that their image under the dehomogenization morphism is not trivially zero.

Lemma 3.1. *Consider $\sigma \in \Sigma(n)$ such that $\zeta_i \in U_\sigma$ and take any $\alpha, \gamma \in \text{Cl}(X)$ such that $(S_\gamma)^\sigma \neq 0$. For each $g \in S_{\gamma-\alpha}$ and $\text{ev}_{\zeta_i} \circ \partial \in \mathcal{Q}_i(U_\sigma)^\perp$, we have $(f \mapsto (\text{ev}_{\zeta_i} \circ \partial)((gf)^\sigma)) \in I_\alpha^\perp$.*

Proof. Suppose $\text{ev}_{\zeta_i} \circ \partial \in \mathcal{Q}_i(U_\sigma)^\perp$. For any element $f \in I_\alpha$, $gf \in I_\gamma$ and so $(gf)^\sigma \in I^\sigma \subset \mathcal{Q}_i(U_\sigma)$. Hence $(f \mapsto (\text{ev}_{\zeta_i} \circ \partial)((fg)^\sigma)) \in I_\alpha^\perp$. \square

Our next goal is to show that for polynomials whose degree α is in the regularity of I , ideal membership can be tested, after dehomogenizing, in the affine rings $\mathbb{C}[U_\sigma]$. As we explained before, we need to lift the elements in S_α by multiplying them with some $g \in S_{\gamma-\alpha}$ such that $(S_\gamma)^\sigma \neq 0$. We cannot choose any $g \in S_\gamma$ to do so, as it might happen that $gf \in I$ but $f \notin I$. For any $V_X(I)$ -basepoint free α , we can avoid this problem.

Proposition 3.1. *Consider a zero dimensional homogeneous ideal $I \subset S$ and a $V_X(I)$ -basepoint free degree $\alpha \in \text{Cl}(X)$. For any element $h \in S_\alpha$ such that h does not vanish at any point of $\text{Var}_X(I)$ and any full-dimensional cone $\sigma \in \Sigma(n)$, there exists $\gamma \in \text{Cl}(X)$ and $g_\sigma \in S_{\gamma-\alpha}$ such that $(g_\sigma h)^\sigma \neq 0$ and $(g_\sigma h)^\sigma - 1 \in I^\sigma$. Moreover, there is $\ell \in \mathbb{N}$ such that $(x^\hat{\sigma})^\ell (g_\sigma h - x^{\hat{\sigma}, \gamma}) \in I$.*

Proof. Let U_σ be the affine open set of X associated to the full-dimensional cone $\sigma \in \Sigma(n)$. Consider the ideal $\langle h \rangle + I$. As the dehomogenization map $(.)^\sigma$ is linear, we have $(\langle h \rangle + I)^\sigma = \langle h \rangle^\sigma + I^\sigma = ((\langle h \rangle + I)_{x^\hat{\sigma}})_0$. Since we assumed that h does not vanish at any of the points of $\text{Var}_X(I)$, by Hilbert's weak Nullstellensatz, we have $1 \in \langle h \rangle^\sigma + I^\sigma$. If $\text{Var}_X(I) \cap U_\sigma \neq \emptyset$, then $1 \notin I^\sigma$, and by Lemma 2.1 there is $\gamma \in \text{Cl}(X)$ and a non-zero $g_\sigma \in S_{\gamma-\alpha}$ such that $(g_\sigma h)^\sigma - 1 \in I^\sigma$. It is clear that $(g_\sigma h)^\sigma \neq 0$. If $1 \in I^\sigma$, we can take any $g_\sigma \in S_{\gamma-\alpha}$ such that $(g_\sigma h)^\sigma \neq 0$. Note that $(g_\sigma h)^\sigma - 1 = (g_\sigma h - x^{\hat{\sigma}, \gamma})^\sigma \in I^\sigma$, and hence by Lemma 2.1, $(x^\hat{\sigma})^\ell (g_\sigma h - x^{\hat{\sigma}, \gamma}) \in I$ for some $\ell \in \mathbb{N}$. \square

In what follows, we consider a $V_X(I)$ -basepoint free $\alpha \in \text{Cl}(X)$ and $h_\alpha \in S_\alpha$ such that $\text{Var}_X(h_\alpha) \cap \text{Var}_X(I) = \emptyset$. By Proposition 3.1, for each $\sigma \in \Sigma(n)$, there are $\gamma \in \text{Cl}(X)$ and $g_\sigma \in S_{\gamma-\alpha}$ such that $(h_\alpha g_\sigma)^\sigma - 1 \in I^\sigma$ and $(h_\alpha g_\sigma)^\sigma \neq 0$. We define the map

$$\eta_{\alpha, \sigma} : S_\alpha \rightarrow (S_{x^\hat{\sigma}})_0 \simeq \mathbb{C}[U_\sigma] \text{ such that } f \mapsto \eta_{\alpha, \sigma}(f) := (g_\sigma f)^\sigma. \quad (3.5)$$

Lemma 3.2. Consider $\alpha \in \text{Reg}(I)$. Then, $f \in I_\alpha$ if and only if $\eta_{\alpha,\sigma}(f) \in I^\sigma$, for all $\sigma \in \Sigma(n)$.

Proof. We assume $f \in I_\alpha \setminus \{0\}$, as the case $f = 0$ is trivial. It is clear that if $f \in I_\alpha$, then $\eta_{\alpha,\sigma}(f) \in I^\sigma$. Conversely, suppose that $\eta_{\alpha,\sigma}(f) = (g_\sigma \cdot f)^\sigma \in I^\sigma$, for all $\sigma \in \Sigma(n)$. As we observed in the paragraph above (2.3), as f is not zero and $(g_\sigma \cdot h_\alpha)^\sigma \neq 0$, then $(g_\sigma \cdot f)^\sigma \neq 0$. By Lemma 2.1, for each $\sigma \in \Sigma(n)$, there is $\ell \in \mathbb{N}$ such that $(x^{\hat{\sigma}})^\ell g_\sigma f \in I$. Moreover, by Proposition 3.1, there is $i \in \mathbb{N}$ such that $(x^{\hat{\sigma}})^i (g_\sigma h_\alpha - x^{\hat{\sigma},\gamma}) f \in I$. As $x^{\hat{\sigma},\gamma}$ divides $(x^{\hat{\sigma}})^j$ for big enough $j \in \mathbb{N}$, $(x^{\hat{\sigma}})^{i+\ell+j} f \in I$. Since $B = \langle x^{\hat{\sigma}} \mid \sigma \in \Sigma(n) \rangle$, we have that $f \in (I : B^\infty) = J$. As $f \in S_\alpha$ and $\alpha \in \text{Reg}(I)$, we conclude $f \in J_\alpha = I_\alpha$. \square

In our setting, we can solve the ideal membership problem by using the operators from Section 3.1.

Corollary 3.1. Fix $\alpha \in \text{Reg}(I)$ and for each $\sigma \in \Sigma$, consider the map $\eta_{\alpha,\sigma}$ from (3.5). For each $\zeta_i \in \text{Var}_X(I)$, consider $\sigma_i \in \Sigma(n)$ such that $\zeta_i \in U_{\sigma_i}$ and let $\{\text{ev}_{\zeta_i} \circ \partial^{i1}, \dots, \text{ev}_{\zeta_i} \circ \partial^{i\mu_i}\}$ be a basis for $\mathcal{Q}_i(U_{\sigma_i})^\perp$. For $f \in S_\alpha$, we have that $f \in I_\alpha$ if and only if

$$(\text{ev}_{\zeta_i} \circ \partial^{ij} \circ \eta_{\alpha,\sigma_i})(f) = 0, \quad i = 1, \dots, \delta, \quad j = 1, \dots, \mu_i. \quad (3.6)$$

Proof. As for Lemma 3.2, one implication is obvious. Note that the statement is independent of the choice of $\sigma_i \in \Sigma(n)$ such that $\zeta_i \in U_{\sigma_i}$ by the fact that the \mathcal{Q}_i are coherent sheaves. Suppose that f satisfies (3.6). We have that $(g_{\sigma_i} f)^{\sigma_i} \in I^{\sigma_i}$ if and only if $(g_{\sigma_i} f)^{\sigma_i} \in \mathcal{Q}_i(U_{\sigma_i})$ for all i such that $\zeta_i \in U_{\sigma_i}$, which is true for all $\sigma_i \in \Sigma$ by assumption. Therefore, by Lemma 3.2, we have $f \in I_\alpha$. \square

In what follows, for each $\zeta_i \in \text{Var}_X(I)$, we fix $\sigma_i \in \Sigma(n)$ such that $\zeta_i \in U_{\sigma_i}$. As in Section 3.1, let $\{\partial^{i1}, \dots, \partial^{i\mu_i}\}$ be a consistently ordered basis for V_i with $\mathcal{Q}_i(U_{\sigma_i})^\perp \simeq \text{ev}_{\zeta_i} \circ V_i$. Additionally, we fix a regularity pair (see Definition 1.1) $(\alpha, \alpha_0) \in \text{Cl}(X)^2$ and consider, for each σ_i , a triplet of homomorphisms $\eta_{\alpha_0, \sigma_i}$, η_{α, σ_i} , and $\eta_{\alpha + \alpha_0, \sigma_i}$ as in (3.5) such that for every $f_\alpha \in S_\alpha$ and $f_{\alpha_0} \in S_{\alpha_0}$ we have

$$\eta_{\alpha_0, \sigma_i}(f_\alpha) \eta_{\alpha, \sigma_i}(f_{\alpha_0}) = \eta_{\alpha + \alpha_0, \sigma_i}(f_\alpha f_{\alpha_0}). \quad (3.7)$$

This triplet exists because for every pair $h_\alpha \in S_\alpha$ and $h_{\alpha_0} \in S_{\alpha_0}$ such that h_α and h_{α_0} do not vanish anywhere on $\text{Var}_X(I)$, the polynomial $h_\alpha h_{\alpha_0} \in S_{\alpha + \alpha_0}$ does not vanish anywhere on $\text{Var}_X(I)$ either.

Remark 3.1. Whenever $\alpha \in \text{Pic}^\circ(X)$, as $(S_\alpha)^\sigma \neq 0$ for every $\sigma \in \Sigma(n)$, we can simplify $\eta_{\alpha, \sigma}$ and define it as $\eta_{\alpha, \sigma} = (\cdot)^\sigma$ for each $\sigma \in \Sigma(n)$. In this case, the proof of Lemma 3.2 follows in a similar way. In particular, if $(\alpha, \alpha_0) \in \text{Pic}^\circ(X)^2$, we can set $\eta_{\beta, \sigma} = (f \mapsto f^\sigma)$, for each $\beta \in \{\alpha, \alpha_0, \alpha + \alpha_0\}$.

For $\beta \in \{\alpha, \alpha + \alpha_0\}$, we introduce the notation $v_{ij, \beta} := (\text{ev}_{\zeta_i} \circ \partial^{ij} \circ \eta_{\beta, \sigma_i}) \in (S/I)_\beta^\vee$. Recall that, as $\beta \in \text{Reg}(I)$, $\dim_{\mathbb{C}}((S/I)_\beta) = \delta^+$. We define the map $\psi_\beta : (S/I)_\beta \rightarrow \mathbb{C}^{\delta^+}$ by

$$\psi_\beta(f + I_\beta) = (v_{ij, \beta}(f + I_\beta) \mid i = 1, \dots, \delta, j = 1, \dots, \mu_i).$$

By Corollary 3.1, as $\beta \in \text{Reg}(I)$, the map ψ_β is invertible. For each $g \in S_{\alpha_0}$, we define the *multiplication map* representing ‘multiplication with g ’ as

$$M_g : (S/I)_\alpha \rightarrow (S/I)_{\alpha + \alpha_0} \quad \text{such that} \quad M_g(f + I_\alpha) = fg + I_{\alpha + \alpha_0}. \quad (3.8)$$

Lemma 3.3. Let $(\alpha, \alpha_0) \in \text{Cl}(X)^2$ be a regularity pair. Consider $h_0 \in S_{\alpha_0}$ such that $\text{Var}_X(I) \cap \text{Var}_X(h_0) = \emptyset$. Then M_{h_0} is invertible.

Proof. Note that $v_{ij, \alpha+\alpha_0}(h_0 f + I_{\alpha+\alpha_0}) = (\text{ev}_{\zeta_i} \circ \partial^{ij} \circ \eta_{\alpha+\alpha_0, \sigma_i})(h_0 f)$ and $\eta_{\alpha+\alpha_0, \sigma_i}(h_0 f) = \eta_{\alpha_0, \sigma_i}(h_0) \eta_{\alpha, \sigma_i}(f)$, by (3.7). By Leibniz' rule we have $\partial^{ij}(\eta_{\alpha_0, \sigma_i}(h_0) \eta_{\alpha, \sigma_i}(f)) = \sum_{b \in \mathbb{N}^\ell} \partial_b(\eta_{\alpha_0, \sigma_i}(h_0)) \cdot s_b(\partial^{ij})(\eta_{\alpha, \sigma_i}(f))$. Using consistent ordering of the ∂^{ij} , it is an easy exercise to show that $\psi_{\alpha+\alpha_0} \circ M_{h_0} = \phi \circ \psi_\alpha$ where ϕ is represented by an invertible lower triangular matrix. The lemma follows, since $\alpha, \alpha+\alpha_0 \in \text{Reg}(I)$, implies that also ψ_α and $\psi_{\alpha+\alpha_0}$ are invertible. \square

Our next theorem characterizes the eigenvalues of the multiplication maps in terms of evaluations of rational functions on the solutions of the system.

Theorem 3.1 (Toric eigenvalue theorem). *Let $(\alpha, \alpha_0) \in \text{Cl}(X)^2$ be a regularity pair. For any $g \in S_{\alpha_0}$ and $h_0 \in S_{\alpha_0}$ such that $\text{Var}_X(I) \cap \text{Var}_X(h_0) = \emptyset$, consider the linear map $M_g \circ M_{h_0}^{-1} : (S/I)_{\alpha+\alpha_0} \rightarrow (S/I)_{\alpha+\alpha_0}$. For each ζ_i , we consider $\frac{g}{h_0}(\zeta_i) := \frac{\text{ev}_{\zeta_i}(\eta_{\alpha_0, \sigma_i}(g))}{\text{ev}_{\zeta_i}(\eta_{\alpha_0, \sigma_i}(h_0))}$. We have that*

$$\det(\lambda \text{id}_{(S/I)_{\alpha+\alpha_0}} - M_g \circ M_{h_0}^{-1}) = \prod_{i=1}^{\delta} \left(\lambda - \frac{g}{h_0}(\zeta_i) \right)^{\mu_i}.$$

Remark 3.2. Observe that the $\frac{g}{h_0}(\zeta_i)$ in the previous equation is independent of the choice of the cone σ_i and $\eta_{\alpha_0, \sigma_i}$ that we associated to ζ_i . When ζ_i belongs to the simplicial part of X , we can define evaluation of $\frac{f}{g}$ at ζ_i as $\frac{f(z_i)}{g(z_i)}$ for any $z_i \in \pi^{-1}(\zeta_i)$. This is well-defined because the evaluation of $\frac{f}{g}$ is invariant under the action of G and $\pi^{-1}(\zeta_i)$ consists of exactly one G -orbit. When ζ_i does not belong to the simplicial part, this evaluation can be defined as $\frac{f(z_i)}{g(z_i)}$ for z_i in the unique closed G -orbit contained in $\pi^{-1}(\zeta_i)$, see [19, Ch. 5].

Proof. Our strategy is to prove that there exist linear maps L_{h_0} and $L_g : \mathbb{C}^{\delta^+} \rightarrow \mathbb{C}^{\delta^+}$ such that $L_{h_0} \circ \psi_{\alpha+\alpha_0} \circ M_g = L_g \circ \psi_{\alpha+\alpha_0} \circ M_{h_0}$, the map L_{h_0} is invertible and

$$\det(\lambda \text{id}_{(S/I)_{\alpha+\alpha_0}} - L_{h_0}^{-1} \circ L_g) = \prod_{i=1}^{\delta} \left(\lambda - \frac{g}{h_0}(\zeta_i) \right)^{\mu_i}. \quad (3.9)$$

For $\sigma \in \Sigma(n)$, let $\tilde{h}_0^\sigma = \eta_{\alpha_0, \sigma}(h_0)$, $\tilde{g}^\sigma = \eta_{\alpha_0, \sigma}(g)$ and for any $f \in S_\alpha$, let $\tilde{f}^\sigma = \eta_{\alpha, \sigma}(f)$. We have that

$$v_{ij, \alpha+\alpha_0}(gf) = (\text{ev}_{\zeta_i} \circ \partial^{ij} \circ \eta_{\alpha+\alpha_0, \sigma_i})(gf) = (\text{ev}_{\zeta_i} \circ \partial^{ij})(\tilde{g}^\sigma \tilde{f}^\sigma).$$

Applying Leibniz' rule we find

$$\partial^{ij}(\tilde{h}_0^\sigma \tilde{g}^\sigma \tilde{f}^\sigma) = \sum_{b \in \mathbb{N}^\ell} \partial_b(\tilde{h}_0^\sigma) \cdot s_b(\partial^{ij})(\tilde{g}^\sigma \tilde{f}^\sigma) = \sum_{b \in \mathbb{N}^\ell} \partial_b(\tilde{g}^\sigma) \cdot s_b(\partial^{ij})(\tilde{h}_0^\sigma \tilde{f}^\sigma).$$

Composing with ev_{ζ_i} , by consistent ordering of the ∂^{ij} , as in (3.4) we get

$$\underbrace{\begin{bmatrix} \tilde{h}_0^\sigma(\zeta_i) & & & \\ c_{i2}^{(1)} & \tilde{h}_0^\sigma(\zeta_i) & & \\ \vdots & & \ddots & \\ c_{i\mu_i}^{(1)} & c_{i\mu_i}^{(2)} & \dots & \tilde{h}_0^\sigma(\zeta_i) \end{bmatrix}}_{L_{i,h_0}} \underbrace{\begin{bmatrix} (\text{ev}_{\zeta_i} \circ \partial^{i1})(\tilde{g}^\sigma \tilde{f}^\sigma) \\ (\text{ev}_{\zeta_i} \circ \partial^{i2})(\tilde{g}^\sigma \tilde{f}^\sigma) \\ \vdots \\ (\text{ev}_{\zeta_i} \circ \partial^{i\mu_i})(\tilde{g}^\sigma \tilde{f}^\sigma) \end{bmatrix}}_{L_{i,g}} = \underbrace{\begin{bmatrix} \tilde{g}^\sigma(\zeta_i) & & & \\ d_{i2}^{(1)} & \tilde{g}^\sigma(\zeta_i) & & \\ \vdots & & \ddots & \\ d_{i\mu_i}^{(1)} & d_{i\mu_i}^{(2)} & \dots & \tilde{g}^\sigma(\zeta_i) \end{bmatrix}}_{L_{i,g}} \underbrace{\begin{bmatrix} (\text{ev}_{\zeta_i} \circ \partial^{i1})(\tilde{h}_0^\sigma \tilde{f}^\sigma) \\ (\text{ev}_{\zeta_i} \circ \partial^{i2})(\tilde{h}_0^\sigma \tilde{f}^\sigma) \\ \vdots \\ (\text{ev}_{\zeta_i} \circ \partial^{i\mu_i})(\tilde{h}_0^\sigma \tilde{f}^\sigma) \end{bmatrix}}_{L_{i,g}}$$

for some complex coefficients $c_{ij}^{(k)}, d_{ij}^{(k)}$. Recall that $\tilde{g}^{\sigma_i} \tilde{f}^{\sigma_i} = \eta_{\alpha+\alpha_0}(gf)$ and $\tilde{h}_0^{\sigma_i} \tilde{f}^{\sigma_i} = \eta_{\alpha+\alpha_0}(h_0 f)$. Putting all the equations together for $i = 1, \dots, \delta$, we get

$$\begin{bmatrix} L_{1,h_0} & & & \\ & L_{2,h_0} & & \\ & & \ddots & \\ & & & L_{\delta,h_0} \end{bmatrix} \circ \psi_{\alpha+\alpha_0} \circ M_g = \begin{bmatrix} L_{1,g} & & & \\ & L_{2,g} & & \\ & & \ddots & \\ & & & L_{\delta,g} \end{bmatrix} \circ \psi_{\alpha+\alpha_0} \circ M_{h_0}, \quad (3.10)$$

which is the desired relation $L_{h_0} \circ \psi_{\alpha+\alpha_0} \circ M_g = L_g \circ \psi_{\alpha+\alpha_0} \circ M_{h_0}$. Indeed, by construction, $\tilde{h}_0^{\sigma_i}(\zeta_i) \neq 0, \forall i$, so L_{h_0} is invertible and (3.9) is satisfied. \square

Example 5 (Cont. Examples 2-3). Recall that $\alpha_1 = \alpha_2 = [\sum_{i=1}^4 D_i]$. As we will see (Corollary 4.1), $(\alpha, \alpha_0) \in \text{Pic}(X)$ is a regularity pair for I . By Remark 3.1, the maps $\eta_\alpha, \eta_{\alpha_0}, \eta_{\alpha+\alpha_0}$ coincide with dehomogenization. One can check that in the bases

$$\begin{aligned} \mathcal{B}_\alpha &= \{x_3^4 x_4^4 + I_\alpha, x_1 x_2 x_3^3 x_4^3 + I_\alpha, x_1 x_2^3 x_3^3 x_4 + I_\alpha, x_1^4 x_2^4 + I_\alpha\} \\ \mathcal{B}_{\alpha+\alpha_0} &= \{x_2^2 x_3^6 x_4^4 + I_{\alpha+\alpha_0}, x_1 x_2^3 x_3^5 x_4^3 + I_{\alpha+\alpha_0}, x_1 x_2^5 x_3^5 x_4 + I_{\alpha+\alpha_0}, x_1^4 x_2^6 x_3^2 + I_{\alpha+\alpha_0}\} \end{aligned}$$

of $(S/I)_\alpha$ and $(S/I)_{\alpha+\alpha_0}$ respectively, multiplication with $x_2^2 x_3^2, x_1 x_2 x_3 x_4 \in S_{\alpha_0}$ looks like this:

$$M_{x_2^2 x_3^2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_{x_1 x_2 x_3 x_4} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The solution $\zeta_1 = \pi(z_1) \in U_{\sigma_1}$ has local coordinates $(y_1, y_2, y_3) = (1, 0, 0)$, and a consistently ordered basis for Q_1^\perp is

$$\{\text{ev}_{\zeta_1} \circ \partial_{(0,0,0)}, \text{ev}_{\zeta_1} \circ \partial_{(0,1,0)}\} = \{\text{ev}_{\zeta_1}, \text{ev}_{\zeta_1} \circ \frac{\partial}{\partial y_2}\}.$$

composing these functionals with dehomogenization of degree $\alpha + \alpha_0$ and representing them in the basis $\mathcal{B}_{\alpha+\alpha_0}$ we get

$$\begin{bmatrix} v_{11, \alpha+\alpha_0} \\ v_{12, \alpha+\alpha_0} \end{bmatrix} = \begin{bmatrix} \text{ev}_{\zeta_1} \circ \eta_{\alpha+\alpha_0, \sigma_1} \\ \text{ev}_{\zeta_1} \circ \frac{\partial}{\partial y_2} \circ \eta_{\alpha+\alpha_0, \sigma_1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix},$$

which follows from $\eta_{\alpha+\alpha_0}(\mathcal{B}_{\alpha+\alpha_0}) = \{y_1, y_1 y_2, y_1^2 y_2, y_1 y_2^4\}$. For any $g \in S_{\alpha_0}$, the lower triangular matrix $L_{1,g}$ is given by

$$L_{1,g} = \begin{bmatrix} g^{\sigma_1}(\zeta_1) & 0 \\ \frac{\partial g^{\sigma_1}}{\partial y_2}(\zeta_1) & g^{\sigma_1}(\zeta_1) \end{bmatrix}, \quad \text{which gives} \quad L_{1,g} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad L_{1,h_0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for $g = x_1 x_2 x_3 x_4, h_0 = x_2^2 x_3^2$. This follows from $g^{\sigma_1} = y_2, h_0^{\sigma_1} = y_1$. For the rows of (3.10) corresponding to ζ_1 we get

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} M_{x_1 x_2 x_3 x_4} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix} M_{x_2^2 x_3^2}.$$

In order to complete this equation with the rows corresponding to ζ_2 , one has to work in the chart corresponding to either one of the orange or the yellow cone in Figure 2. \triangle

3.3 Computing coordinates of $\text{Var}_X(I)$

We now use the results from the previous sections to design a numerical algorithm for computing homogeneous coordinates of the points in $\text{Var}_X(I)$. We sketch the steps of the algorithm. More details on the numerical aspects of the strategy can be found in [53, Sec. 5.5.4]. Let $I = \langle f_1, \dots, f_s \rangle \subset S$ be such that $\text{Var}_X(I) = \{\zeta_1, \dots, \zeta_\delta\}$ where ζ_i has multiplicity μ_i and set $\delta^+ := \mu_1 + \dots + \mu_\delta$. We write $\alpha_i = \deg(f_i) \in \text{Cl}(X)_+$. For a regularity pair $(\alpha, \alpha_0) \in \text{Cl}(X)^2$ and some $h_0 \in S_{\alpha_0}$ for which $\text{Var}_X(I) \cap \text{Var}_X(h_0) = \emptyset$, we fix a basis for $(S/I)_\alpha$ and compute the matrices

$$M_{x^b/h_0} = M_{x_b} \circ M_{h_0}^{-1}, \quad \text{for all } x^b \in S_{\alpha_0}.$$

This can be done as follows [52, Prop. 5.5.5]. Consider the map

$$\text{Res} : S_{\alpha+\alpha_0-\alpha_i} \times \dots \times S_{\alpha+\alpha_0-\alpha_s} \rightarrow S_{\alpha+\alpha_0} \quad \text{given by} \quad (q_1, \dots, q_s) \mapsto q_1 f_1 + \dots + q_s f_s.$$

This map has the property that $\text{im Res} = I_{\alpha+\alpha_0}$, and hence a cokernel map $N : S_{\alpha+\alpha_0} \rightarrow \mathbb{C}^{\delta^+}$ satisfies $\ker N = \text{im Res} = I_{\alpha+\alpha_0}$. Such a cokernel map can be computed, for instance, using the SVD. We define the map $N_{h_0} : S_\alpha \rightarrow \mathbb{C}^{\delta^+}$ by setting $N_{h_0}(f) = N(h_0 f)$ and for $x^b \in S_{\alpha_0}$ we define the map $N_b : S_\alpha \rightarrow \mathbb{C}^{\delta^+}$ by $N_b(f) = N(x^b f)$. For any δ^+ -dimensional subspace W of S_α such that $(N_{h_0})|_W$ is invertible, we let $M_{x^b/h_0} = (N_{h_0})|_W^{-1} \circ (N_b)|_W$. It is crucial for the numerical stability of the algorithm to choose W such that $(N_{h_0})|_W$ is well-conditioned. This can be done, for instance, by using QR with column pivoting or SVD on the matrix N_{h_0} , see for instance [55, 54, 42]. Following [13], we compute a reordered Schur factorization of a random \mathbb{C} -linear combination $M_{h/h_0} = \sum_{x^b \in S_{\alpha_0}} c_b M_{x^b/h_0}$. Provided that the degree α_0 is ‘large enough’ (see below), taking this random combination helps to separate the eigenspaces corresponding to different roots, as they correspond to different eigenvalues $\frac{h}{h_0}(\zeta_i)$, see [54, Sec. 7]. The aforementioned Schur factorization of M_{h/h_0} gives a unitary matrix \mathbf{U} such that for each $x^b \in S_{\alpha_0}$,

$$\mathbf{U} M_{x^b/h_0} \mathbf{U}^H = \begin{bmatrix} \Delta_1^b & \times & \cdots & \times \\ 0 & \Delta_2^b & \cdots & \times \\ 0 & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & \Delta_\delta^b \end{bmatrix} \quad (\cdot^H \text{ denotes the Hermitian transpose})$$

is block upper triangular with the matrices $\Delta_i^b \in \mathbb{C}^{\mu_i \times \mu_i}$ on the diagonal. The matrices Δ_i^b are such that they only have one eigenvalue, which is x^b/h_0 evaluated at ζ_i . This eigenvalue can be computed as $\lambda_{b,i} := \text{Trace}(\Delta_i^b)/\mu_i$. In the reduced case, where $\mu_i = 1$ for all i , the eigenvalues can be read off the diagonal of $\mathbf{U} M_{x^b/h_0} \mathbf{U}^H$. Having computed these eigenvalues, a set of homogeneous coordinates of the points $\zeta_1, \dots, \zeta_\delta$ can be computed by solving the binomial systems of equations

$$\{x^b = \lambda_{b,i} \mid x^b \in S_{\alpha_0}\} \quad \text{for } i = 1, \dots, \delta^+,$$

provided that α_0 is ‘large enough’. What ‘large enough’ means in this context is specified in [53, Cor. 5.5.2]. To give some intuition, if $\zeta_i \in T \subset X$, it is necessary and sufficient that the lattice points in the polytope associated to α_0 affinely span the lattice M .¹ In theory, this restriction can be avoided by working with eigenvectors.

Remark 3.3 (Using eigenvectors instead of eigenvalues). *Our characterization of the eigenvectors in the proof of Theorem 3.1 gives an alternative method to compute homogeneous coordinates of the points in $\text{Var}_X(I)$, as in [2]. In analogy with what is observed in [53, Rmk. 4.3.4], the approach using eigenvalues gives more accurate results. However, we point out that the eigenvector approach might work for smaller degrees α_0 , i.e. degrees α_0 not satisfying the restrictions of [53, Cor. 5.5.2], see above.*

¹This is to ensure that $t \mapsto (t^m)_{F^\perp m+a_0 \geq 0}$ is injective on T , where $\alpha_0 = [\sum_{i=1}^k a_{0,i} D_i]$.

Example 6 (27 lines on a cubic surface). A classical result in intersection theory states that a general cubic surface in \mathbb{P}^3 contains 27 lines, see for instance [23, Sec. 6.2.1]. As detailed in [43, Sec. 4], these lines correspond to the solutions of the system defined by

$$\begin{aligned}\hat{f}_1 &= c_0t^3 + c_1t^2v + c_2tv^2 + c_3v^3 + c_4t^2 + c_5tv + c_6v^2 + c_7t + c_8v + c_9, \\ \hat{f}_2 &= c_0s^3 + c_1s^2u + c_2su^2 + c_3u^3 + c_{10}s^2 + c_{11}su + c_{12}u^2 + c_{16}s + c_{17}u + c_{19}, \\ \hat{f}_3 &= 3c_0st^2 + 2c_1stv + c_2sv^2 + c_1t^2u + 2c_2tuv + 3c_3uv^2 + 2c_4st + c_5sv + c_{10}t^2 \\ &\quad + c_5tu + c_{11}tv + 2c_6uv + c_{12}v^2 + c_7s + c_{13}t + c_8u + c_{14}v + c_{15}, \\ \hat{f}_4 &= 3c_0s^2t + c_1s^2v + 2c_1stu + 2c_2suv + c_2tu^2 + 3c_3u^2v + c_4s^2 + 2c_{10}st + c_5su \\ &\quad + c_{11}sv + c_{11}tu + c_6u^2 + 2c_{12}uv + c_{13}s + c_{16}t + c_{14}u + c_{17}v + c_{18},\end{aligned}$$

for general $c_i \in \mathbb{C}$, see [43, Eq. (14)]. The relations $\hat{f}_1 = \dots = \hat{f}_4 = 0$ on $(\mathbb{C}^*)^4$ extend naturally to a toric compactification $X = X_\Sigma \supset (\mathbb{C}^*)^4$, where X_Σ is the toric variety coming from the fan Σ that we will now describe. For $i = 1, \dots, 4$, let $P_i \subset \mathbb{R}^4$ be the Newton polytope of \hat{f}_i and define the convex polytope $P := P_1 + \dots + P_4 = \{m \in \mathbb{R}^4 \mid F^\top m + a \geq 0\}$, with

$$F = \begin{bmatrix} 0 & 0 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \end{bmatrix} = [u_1 \ u_2 \ u_3 \ u_4 \ u_5 \ u_6] \quad \text{and} \quad a = (0, 0, 6, 6, 0, 0)^\top.$$

The fan Σ is the normal fan of P . It has 6 rays, whose primitive generators u_i are the columns of F . The toric variety X_Σ is isomorphic to the multiprojective space $\mathbb{P}^2 \times \mathbb{P}^2$ and its class group is \mathbb{Z}^2 . We identify $[\sum_{i=1}^6 c_i D_i] = (c_1 + c_3 + c_5, c_2 + c_4 + c_6) \in \mathbb{Z}^2$. This way, $\hat{f}_1, \dots, \hat{f}_4$ correspond to homogeneous polynomials f_1, \dots, f_4 in the Cox ring of X_Σ of degrees $(0, 3), (3, 0), (1, 2)$ and $(2, 1)$ respectively. As we will show (Corollary 4.1), we have that $((6, 6), (1, 1))$ is a regularity pair for I .

The mixed volume $\text{MV}(P_1, P_2, P_3, P_4)$ is 45. The toric version of the BKK theorem, see [28, §5.5], tells us that the maximal number of isolated solutions of $f_1 = \dots = f_4 = 0$ on X is 45. However, we know from intersection theory that for generic parameter values c_0, \dots, c_{19} , there are only 27 solutions in $(\mathbb{C}^*)^4$. Solving a generic instance of our system using the aforementioned regularity pair in a proof of concept implementation, we find that there are in fact 45 isolated solutions on X (counting multiplicities), of which 18 are on the boundary $X \setminus (\mathbb{C}^*)^4$. The left part of Figure 3 shows the computed coordinates. The figure suggests clearly that there are indeed 27 solution in the torus, and 18 solutions that are on the intersection of the 3rd and 4th torus invariant prime divisors (corresponding to u_3 and u_4), which we will denote by $D_3, D_4 \subset X$. In fact, there are only 3 solutions on $D_3 \cap D_4$, each with multiplicity 6. These multiplicities become apparent when the \mathbf{U} matrix in the ordered Schur factorization of a generic linear combination of the M_{x^b/h_0} brings the matrices M_{x^b/h_0} into *block upper triangular* instead of *upper triangular* form, as described above. One of the matrices $\mathbf{U} M_{x^b/h_0} \mathbf{U}^H$ is shown in the right part of Figure 3. It is clear that this is a numerical approximation of a block upper triangular matrix with three 6×6 blocks on its diagonal. We now explicitly compute the three solutions on the boundary by solving the *face system* (see e.g. [33]) corresponding to u_3 and u_4 :

$$\begin{aligned}(\hat{f}_1)_{u_3, u_4}(s, u, t, v) &= c_0t^3 + c_1t^2v + c_2tv^2 + c_3v^3, \\ (\hat{f}_2)_{u_3, u_4}(s, u, t, v) &= c_0s^3 + c_1s^2u + c_2su^2 + c_3u^3, \\ (\hat{f}_3)_{u_3, u_4}(s, u, t, v) &= 3c_0st^2 + 2c_1stv + c_2sv^2 + c_1t^2u + 2c_2tuv + 3c_3uv^2, \\ (\hat{f}_4)_{u_3, u_4}(s, u, t, v) &= 3c_0s^2t + c_1s^2v + 2c_1stu + 2c_2suv + c_2tu^2 + 3c_3u^2v.\end{aligned}$$

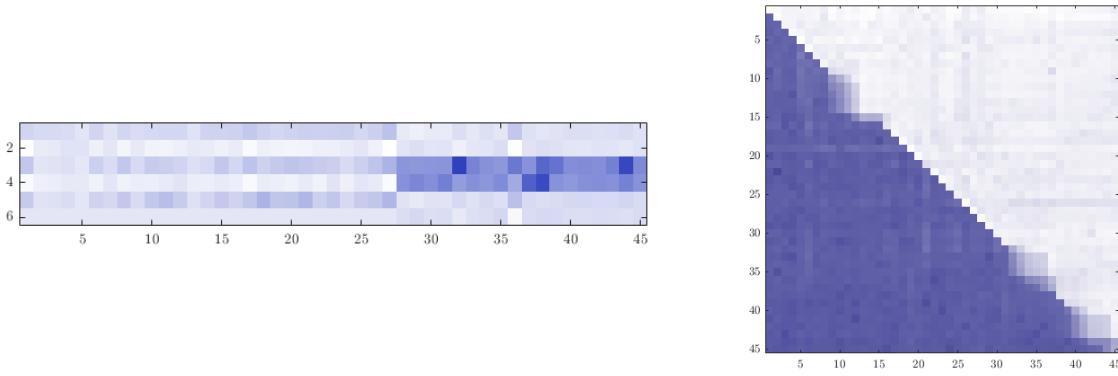


Figure 3: Left: absolute value of the computed homogeneous coordinates of 45 solutions. The i -th row corresponds to the i -th torus invariant prime divisor, associated to the ray generated by u_i , and the j -th column corresponds to the j -th computed solution. Dark colors correspond to small absolute values. Right: absolute values of the entries of a block upper triangularized homogeneous multiplication matrix M_{x^b/h_0} in Example 6.

One can see from these equations that $D_3 \cap D_4 \simeq \mathbb{P}^1 \times \mathbb{P}^1$, with coordinates $(s : u)$ and $(t : v)$ on the first and second copy of \mathbb{P}^1 respectively. The bidegrees of the equations are $(0, 3), (3, 0), (1, 2), (2, 1)$. We now interpret $(\hat{f}_1)_{u_3, u_4}$ as an equation on \mathbb{P}^1 and consider its three roots $(t_j^* : v_j^*)$, $j = 1, 2, 3$ (for which we can write down explicit expressions) and we define $\zeta_j = ((t_j^* : v_j^*), (t_j^* : v_j^*)) \in \mathbb{P}^1 \times \mathbb{P}^1$. It is clear that $(\hat{f}_1)_{u_3, u_4}(\zeta_j) = (\hat{f}_2)_{u_3, u_4}(\zeta_j) = 0$. If we substitute $s = t, u = v$ in $(\hat{f}_3)_{u_3, u_4}, (\hat{f}_4)_{u_3, u_4}$ we find

$$(\hat{f}_3)_{u_3, u_4}(t, v, t, v) = (\hat{f}_4)_{u_3, u_4}(t, v, t, v) = 3(\hat{f}_1)_{u_3, u_4}(s, u, t, v).$$

From this it is clear that also $(\hat{f}_3)_{u_3, u_4}(\zeta_j) = (\hat{f}_4)_{u_3, u_4}(\zeta_j) = 0$, $j = 1, \dots, 3$, and we have identified the three solutions on $D_3 \cap D_4$. \triangle

Remark 3.4 (Other fields). *Although our solving method was designed to work in floating point arithmetic over \mathbb{C} , it is expected to generalize for eigenvalue algorithms over other fields, as in [36] for p -adic numbers.*

4 Regularity for zero-dimensional ideals

An important property of the Cox ring is that every homogeneous ideal $I \subset S$ determines a closed subscheme of the toric variety X and every closed subscheme arises in this way, see [19, Prop. 6.A.6]. Unfortunately, this correspondence is not one-to-one: different homogeneous ideals may define the same closed subscheme.

Example 7 ([19, Ex. 5.3.11]). Let $X = \mathbb{P}(1, 1, 2)$ be the weighted projective plane with weights 1, 1, 2. Its \mathbb{Z} -graded Cox ring is $\mathbb{C}[x, y, z]$, where $\deg(x) = \deg(y) = 1$ and $\deg(z) = 2$. The irrelevant ideal is $B = \langle x, y, z \rangle$. Consider the homogeneous ideals $I_1 = \langle x^2, xy, y^2 \rangle$ and $I_2 = \langle x, y \rangle$. One can check that both these ideals are B -saturated and $V_X(I_1) = V_X(I_2)$, yet $(I_1)_d \neq (I_2)_d$ for any odd degree d . \triangle

Example 7 suggests that different homogeneous ideals defining the same subscheme of X do seem to agree at certain degrees. The (*Castelnuovo-Mumford*) regularity formalizes this intuition for $X = \mathbb{P}^n$. Roughly speaking, the regularity of an ideal is the set of degrees at which we can recover its geometric nature. For an introduction to this subject, see e.g. [22, Sec. 20.5].

Even though the Castelnuovo-Mumford regularity is well understood when our toric variety is \mathbb{P}^n , in general, the notion of regularity over the Cox ring is not uniquely defined. Equivalent definitions on \mathbb{P}^n , i.e. using local cohomology or Betti numbers, do not agree for other toric varieties. For simplicial toric varieties, in [38] the regularity is defined in terms of the vanishing of certain local cohomology modules. In contrast, in [46] the (resolution) regularity of the product of projective spaces is defined in terms of multigraded Betti numbers. Other works, e.g. [30, 47, 8], try to unify and/or generalize these approaches. In [31, 47], alternative definitions for fat points in multiprojective space are studied. For more general zero-dimensional closed subschemes of toric varieties, [45] describes the regularity (in the sense of [38]) of homogeneous ideals generated by regular sequences over the Cox ring. Alternatively, in [52], the second author of this paper defines the regularity of an ideal defining a zero-dimensional reduced subscheme in terms of its Hilbert function and the existence of *homogeneous Lagrange polynomials*. This definition of regularity only considers the ideal at a specific degree, in contrast to other definitions as [38, Def. 4.1] where the local cohomology modules of the ideal have to vanish at many different degrees.

In this work, we consider a notion of regularity (see Definition 1.1) which generalizes [52, Def. 4.3].² In contrast with previous approaches such as [38], we allow degrees outside of the Picard group:

$$\text{Reg}(I) := \{\alpha \in \text{Cl}(X) \mid \dim_{\mathbb{C}}(S/I)_\alpha = \delta^+, I_\alpha = (I : B^\infty)_\alpha, \alpha \text{ is } V_X(I)\text{-basepoint free}\}.$$

The aim of this section is to investigate properties of the regularity. In Section 4.1 we consider the general zero-dimensional case. In 4.2 we construct regularity pairs for complete intersections. For special systems, we improve this construction in 4.3.

4.1 Properties of regularity

In this subsection we prove that, at the degrees belonging to the regularity, the ideal contains all the information to recover the geometric nature of the associated closed subscheme. This fact is closely related to Lemma 3.2. Additionally, we show how to extend degrees in the regularity to regularity pairs.

Theorem 4.1. *Consider $I \subset S$ such that $Y = V_X(I)$ is zero-dimensional. If $\beta \in \text{Reg}(I)$, then $(S/I)_\beta \simeq H^0(Y, \mathcal{O}_Y)$.*

Proof. We recall the following exact sequence relating local cohomology to sheaf cohomology on toric varieties, [19, Thm. 9.5.7],

$$0 \rightarrow H_B^0(S/I)_\beta \rightarrow (S/I)_\beta \rightarrow H^0(X, \widetilde{(S/I)(\beta)}) \rightarrow H_B^1(S/I)_\beta \rightarrow 0, \quad (4.1)$$

where $\widetilde{(S/I)(\beta)}$ denotes the coherent sheaf associated to the S -module S/I shifted by β .

By definition, $H_B^0(S/I)_\beta = 0$ if and only if $I_\beta = (I : B^\infty)_\beta$. Hence, if $\beta \in \text{Reg}(I)$, $H_B^0(S/I)_\beta = 0$ and we have an injective map $(S/I)_\beta \rightarrow H^0(X, \widetilde{(S/I)(\beta)})$.

We prove that for $\beta \in \text{Reg}(I)$, $\widetilde{(S/I)(\beta)} \simeq \widetilde{S/I}$. As $\beta \in \text{Reg}(I)$, there is $h \in S_\beta$ such that h does not vanish at any of the points of Y_{red} . We consider the morphism of sheaves $\times h : \widetilde{S/I} \rightarrow \widetilde{(S/I)(\beta)}$ given by multiplication by h (which is a global section of $\mathcal{O}_X(\beta)$). We claim that this morphism is an isomorphism. This can be seen from the fact that it is an isomorphism on each chart of the affine covering $\{U_\sigma\}_{\sigma \in \Sigma(n)}$,

²We warn the reader that, even in the reduced case, the ideal J in [52] is not the same as $(I : B^\infty)$ in this work, see [53, Lem. 5.5.2].

given by the fan Σ associated to X . Following the notation used in the proof of Proposition 3.1, for each U_σ , the inverse of $H^0(U_\sigma, \widetilde{S/I}) \xrightarrow{\times h} H^0(U_\sigma, (\widetilde{S/I})(\beta))$ is

$$((S/I)_{x^\sigma})_\beta = H^0(U_\sigma, (\widetilde{S/I})(\beta)) \xrightarrow{\times \frac{g_\sigma}{x^\sigma \gamma}} ((S/I)_{x^\sigma})_0 = H^0(U_\sigma, \widetilde{S/I}),$$

where $\gamma \in \text{Cl}(X)$ is such that $(g_\sigma h)^\sigma = h \frac{g_\sigma}{x^\sigma \gamma}$.

By definition of $\text{Reg}(I)$, $\dim_{\mathbb{C}}((S/I)_\beta)$ is the number of solutions counting multiplicity, which is also $\dim_{\mathbb{C}}(H^0(Y, \mathcal{O}_Y))$. As $\widetilde{S/I} = i_* \mathcal{O}_Y$, where i is the immersion $Y \xrightarrow{i} X$, and $H^0(X, i_* \mathcal{O}_Y) \simeq H^0(Y, \mathcal{O}_Y)$, $(S/I)_\beta \rightarrow H^0(Y, \mathcal{O}_Y)$ is an injective map between equidimensional spaces. \square

Given a degree $\alpha \in \text{Reg}(I)$ we can extend it to a regularity pair (α, α_0) by checking that $\dim_{\mathbb{C}}((S/I)_{\alpha+\alpha_0}) = \delta^+ = \dim_{\mathbb{C}}((S/I)_\alpha)$. This is easier to verify than $I_{\alpha+\alpha_0} = (I : B^\infty)_{\alpha+\alpha_0}$.

Lemma 4.1. *Let $I \subset S$ be such that $V_X(I)$ is zero-dimensional. For any $\alpha_0 \in \text{Cl}(X)_+$ and $h_0 \in S_{\alpha_0}$ such that $\text{Var}_X(h_0) \cap \text{Var}_X(I) = \emptyset$, we have that h_0 is not a zero divisor in S/J where $J := (I : B^\infty)$.*

Proof. We prove this lemma by contra-positive. Let $J = Q_1 \cap \dots \cap Q_\ell$ be an irredundant primary decomposition. Since J is B -saturated, $\text{Var}_{\mathbb{C}^k}(Q_i) \not\subset \text{Var}_{\mathbb{C}^k}(B)$, for every $i = 1, \dots, \ell$. Consider $h_0 \in S_{\alpha_0}$ such that the image of h_0 in S/J is a zero divisor. Hence, $h_0 \notin J$ and there is a $f \notin J$ such that $h_0 f \in J$. Therefore, there is a primary ideal Q_i in the decomposition of J such that $f \notin Q_i$ but $h_0 f \in Q_i$, and so $h_0 \in \sqrt{Q_i}$. As $\text{Var}_{\mathbb{C}^k}(Q_i) \not\subset \text{Var}_{\mathbb{C}^k}(B)$, we have that $\text{Var}_X(h_0) \cap \text{Var}_X(I) \neq \emptyset$. \square

Theorem 4.2. *Let $I \subset S$ be a homogeneous ideal such that $V_X(I)$ is zero-dimensional of degree δ^+ and let $\alpha \in \text{Reg}(I)$. For each $V_X(I)$ -basepoint free $\alpha_0 \in \text{Cl}(X)_+$ such that $\dim_{\mathbb{C}}((S/I)_{\alpha+\alpha_0}) = \delta^+$, we have that (α, α_0) is a regularity pair of I .*

Proof. We only need to prove that $J_{\alpha+\alpha_0} = I_{\alpha+\alpha_0}$, or equivalently $(S/J)_{\alpha+\alpha_0} = (S/I)_{\alpha+\alpha_0}$. By definition of $J = (I : B^\infty)$ we have $I \subset J$, which implies $\dim_{\mathbb{C}}((S/J)_{\alpha+\alpha_0}) \leq \dim_{\mathbb{C}}((S/I)_{\alpha+\alpha_0})$, so it suffices to show the opposite inequality. By assumption, $\dim_{\mathbb{C}}(S/I)_{\alpha+\alpha_0} = \delta^+$. By Lemma 4.1, there is a non zero-divisor $h_0 \in S_{\alpha_0}$ in S/J such that the multiplication map $M_{h_0} : (S/J)_\alpha \rightarrow (S/J)_{\alpha+\alpha_0}$ (see (3.8)) is injective. We conclude that $\dim_{\mathbb{C}}(S/J)_{\alpha+\alpha_0} \geq \dim_{\mathbb{C}}(S/J)_\alpha = \delta^+ = \dim_{\mathbb{C}}((S/I)_{\alpha+\alpha_0})$. \square

4.2 Complete intersections

We say that the subscheme $Y = V_X(I)$ is a *complete intersection* if I can be generated by $\text{codim}_X(Y)$ many homogeneous elements. In particular, a zero-dimensional subscheme Y is a complete intersection if $Y = V_X(I)$ with $I = \langle f_1, \dots, f_n \rangle \subset S$. The associated system of equations $f_1 = \dots = f_n = 0$ is called *square*. In this subsection we describe the regularity $\text{Reg}(I)$ of a homogeneous ideal $I \subset S$ defining a zero-dimensional complete intersection subscheme of X . We assume that the generators f_1, \dots, f_n of I have degrees $\deg(f_i) = \alpha_i \in \text{Pic}^\circ(X)$. This is the case, for instance, when these polynomials arise from homogenization of Laurent polynomials, see Section 2.2. Our objective is to characterize regularity pairs in order to solve the square system $f_1 = \dots = f_n = 0$ on X using the algorithms in Section 3.

The main results of this subsection is Theorem 4.3, in which we establish a sufficient criterion for degrees to belong to the regularity. The strategy to prove this theorem is similar to the one in [29] for defining the resultant, to [17] for checking its vanishing, and to [40] for solving affine sparse systems. In the terminology of [7], what we do is construct a virtual resolution for S/I using the Koszul complex. The statement and proof of Theorem 4.3 involve some notions of homological algebra and sheaf cohomology, but its concrete consequences can be formulated without this language, see Theorem 4.4 and Corollary 4.2.

Theorem 4.3. Let $I = \langle f_1, \dots, f_n \rangle \subset S$ with $f_i \in S_{\alpha_i}$ such that $\alpha_i \in \text{Pic}^\circ(X)$ and $V_X(I)$ is a complete intersection. Let $\beta \in \text{Cl}(X)$ be $V_X(I)$ -basepoint free. We have that $\beta \in \text{Reg}(I)$ if

$$\text{for every } p > 0 \text{ and } \mathcal{J} \subset \{1, \dots, n\}, \quad H^p(X, \mathcal{O}_X(\beta - \sum_{i \in \mathcal{J}} \alpha_i)) = 0. \quad (4.2)$$

Proof. Let $Y = V_X(I)$ and denote its structure sheaf by \mathcal{O}_Y . We denote by $i_* \mathcal{O}_Y$ the push-forward of \mathcal{O}_Y through the immersion $Y \xrightarrow{i} X$. Consider the \mathcal{O}_X -module map $\phi : \bigoplus_{i=1}^n \mathcal{O}_X(-\alpha_i) \rightarrow \mathcal{O}_X$ such that, for every open $U_\sigma \subset X$, $\phi|_{U_\sigma} : (g_1, \dots, g_n) \mapsto \sum g_i \cdot f_i^\sigma$. Let $\mathcal{K}(f_1, \dots, f_n)_\bullet$ be the augmented Koszul complex of sheaves associated to ϕ ,

$$0 \rightarrow \mathcal{O}_X(-\sum_{i=1}^n \alpha_i) \rightarrow \cdots \rightarrow \bigoplus_{\substack{\mathcal{J} \subset \{1, \dots, n\} \\ \#\mathcal{J}=j}} \mathcal{O}_X(-\sum_{i \in \mathcal{J}} \alpha_i) \rightarrow \cdots \rightarrow \bigoplus_{i=1}^n \mathcal{O}_X(-\alpha_i) \rightarrow \mathcal{O}_X \rightarrow i_* \mathcal{O}_Y \rightarrow 0.$$

By construction, X is a normal toric variety, and so it is Cohen-Macaulay [19, Thm. 9.2.9]. As Y is a complete intersection and $\alpha_i \in \text{Pic}^\circ(X)$, the complex $\mathcal{K}(f_1, \dots, f_n)_\bullet$ is exact [22, Ex. 17.20].

We consider the sheaf $\mathcal{O}_X(\beta)$ and the twisted complex $\mathcal{K}(f_1, \dots, f_n)_\bullet \otimes \mathcal{O}_X(\beta)$, where we twist each sheaf by $\mathcal{O}_X(\beta)$. We will show that this twisted complex is exact. Recall that $\mathcal{K}(f_1, \dots, f_n)_\bullet \otimes \mathcal{O}_X(\beta)$ is exact if for every closed point $\zeta \in X$, the induced complex of stalks $(\mathcal{K}(f_1, \dots, f_n)_\bullet)_\zeta \otimes \mathcal{O}_X(\beta)_\zeta$ is exact [32, Pg. 64]. Consider a closed point $\zeta \in X$. By definition of the $\text{Tor}_i^{\mathcal{O}_{X,\zeta}}(-, -)$ functor, see e.g. [22, Sec. 6.2], the complex of stalks $(\mathcal{K}(f_1, \dots, f_n)_\bullet)_\zeta \otimes \mathcal{O}_X(\beta)_\zeta$ is exact if and only if $\text{Tor}_i^{\mathcal{O}_{X,\zeta}}((i_* \mathcal{O}_Y)_\zeta, \mathcal{O}_X(\beta)_\zeta) = 0$, for every $i > 0$. If either $(i_* \mathcal{O}_Y)_\zeta$ or $\mathcal{O}_X(\beta)_\zeta$ is a free $\mathcal{O}_{X,\zeta}$ -module, then these Tor modules vanish. If $\zeta \notin Y$, then $(i_* \mathcal{O}_Y)_\zeta = 0$, so $\text{Tor}_i^{\mathcal{O}_{X,\zeta}}(0, \mathcal{O}_X(\beta)_\zeta) = 0$. Consider now $\zeta \in Y$. By assumption, there is a global section h of $\mathcal{O}_X(\beta)$ that does not vanish at ζ . Therefore, any $(h/g)_\zeta \in \mathcal{O}_{X,\zeta}$ is invertible in $\mathcal{O}_{X,\zeta}$. The map $\mathcal{O}_{X,\zeta} \xrightarrow{\times h} \mathcal{O}_X(\beta)_\zeta$ is an isomorphism, so $\mathcal{O}_X(\beta)_\zeta \simeq \mathcal{O}_{X,\zeta}$ is a free $\mathcal{O}_{X,\zeta}$ -module, implying that $\text{Tor}_i^{\mathcal{O}_{X,\zeta}}((i_* \mathcal{O}_Y)_\zeta, \mathcal{O}_X(\beta)_\zeta) = 0$, for every $i > 0$.

Note that the previous argument also shows that $i_* \mathcal{O}_Y \otimes \mathcal{O}_X(\beta) \simeq i_* \mathcal{O}_Y$. Recall that whenever $\alpha \in \text{Pic}(X)$, $\mathcal{O}_X(\alpha) \otimes \mathcal{O}_X(\beta) = \mathcal{O}_X(\alpha + \beta)$, as $\mathcal{O}_X(\alpha)$ is a locally free sheaf. Hence, out of $(\mathcal{K}(f_1, \dots, f_n)_\bullet) \otimes \mathcal{O}_X(\beta)$, we obtain the following exact complex of sheaves,

$$0 \rightarrow \mathcal{O}_X(\beta - \sum_{i=1}^n \alpha_i) \rightarrow \cdots \rightarrow \bigoplus_{i=1}^n \mathcal{O}_X(\beta - \alpha_i) \rightarrow \mathcal{O}_X(\beta) \rightarrow i_* \mathcal{O}_Y \rightarrow 0. \quad (4.3)$$

As taking sheaf cohomology commutes with direct sums [32, Prop. III.2.9, Rmk. III.2.9.1], our hypothesis implies that, for $p > 0$,

$$H^p \left(X, \bigoplus_{\substack{\mathcal{J} \subset \{1, \dots, n\} \\ \#\mathcal{J}=j}} \mathcal{O}_X(\beta - \sum_{i \in \mathcal{J}} \alpha_i) \right) \simeq \bigoplus_{\substack{\mathcal{J} \subset \{1, \dots, n\} \\ \#\mathcal{J}=j}} H^p(X, \mathcal{O}_X(\beta - \sum_{i \in \mathcal{J}} \alpha_i)) = 0.$$

Since $Y = V_X(I)$ is a complete intersection and I is generated by n elements, Y is zero dimensional. As Y is a closed subscheme of X , by [19, Ex. 9.0.6], $H^p(X, i_* \mathcal{O}_Y) = H^p(Y, \mathcal{O}_Y)$. Moreover, as Y is zero dimensional, it is an affine scheme, so by Serre's criterion [32, Thm. III.3.7], $H^p(Y, \mathcal{O}_Y) = 0$ for $p > 0$. Therefore, every higher cohomology in (4.3) vanishes. By [29, Ch. 2, Lem. 2.4], taking global sections

of (4.3) preserves exactness. As $H^0(X, \mathcal{O}_X(\gamma)) = S_\gamma$ for any $\gamma \in \text{Cl}(X)$ by [19, Prop. 5.3.7], the following complex is exact:

$$0 \rightarrow S_{(\beta - \sum_{i=1}^n \alpha_i)} \rightarrow \cdots \rightarrow \bigoplus_{i=1}^n S_{(\beta - \alpha_i)} \rightarrow S_\beta \rightarrow (S/I)_\beta \rightarrow 0. \quad (4.4)$$

Here we used $(S/I)_\beta \simeq H^0(Y, \mathcal{O}_Y)$, which follows from the fact that the image of $\bigoplus_{i=1}^n H^0(X, \mathcal{O}_X(\beta - \alpha_i)) \rightarrow H^0(X, \mathcal{O}_X(\beta))$ is I_β . Hence, $\dim_{\mathbb{C}}((S/I)_\beta) = \dim_{\mathbb{C}}(H^0(Y, \mathcal{O}_Y)) = \delta^+$.

It remains to show that $I_\beta = (I : B^\infty)_\beta$. This follows from the exact sequence relating local and sheaf cohomology (4.1), together with the isomorphism $(S/I)_\beta \simeq H^0(Y, \mathcal{O}_Y)$. \square

The vanishing of these sheaf cohomologies can be computed in terms of the combinatorics of the associated polytopes; see [19, Ch. 9] for the classical approach or [1, Sec. III.3] for a newer and simpler one in the case of nef \mathbb{Q} -Cartier divisors. However, for our purpose, we can avoid these computations. The following classical vanishing theorems give a formula for the cohomologies in the relevant cases.

Proposition 4.1. *Consider a degree $\alpha \in \mathbb{Q}\text{Pic}^\circ(X)$ and its associated polytope $P = \{m \in M_{\mathbb{R}} \mid F^\top m + a \geq 0\}$ (see [19, Prop. 4.3.8]). Let $\text{Relint}(P)$ be the relative interior of the polytope P . We have*

Demazure vanishing [19, Thm 9.2.3],

- $H^0(X, \mathcal{O}_X(\alpha)) \simeq \bigoplus_{m \in P \cap M} \mathbb{C} \cdot x^{F^\top m + a}$,
- For every $i > 0$, $H^i(X, \mathcal{O}_X(\alpha)) \simeq 0$,

Batyrev-Borisov vanishing [19, Thm 9.2.7],

- $H^{\dim(P)}(X, \mathcal{O}_X(-\alpha)) \simeq \bigoplus_{m \in \text{Relint}(P) \cap M} \mathbb{C} \cdot x^{F^\top m + a}$,
- For every $i \neq \dim(P)$, $H^i(X, \mathcal{O}_X(-\alpha)) \simeq 0$.

Combining the previous results, we construct regularity pairs for any complete intersection.

Theorem 4.4. *Let $I = \langle f_1, \dots, f_n \rangle \subset S$ with $f_i \in S_{\alpha_i}$ such that $\alpha_i \in \text{Pic}^\circ(X)$ and $V_X(I)$ is a zero-dimensional. For any nef $V_X(I)$ -basepoint free $\alpha_0 \in \mathbb{Q}\text{Pic}^\circ(X)$, the degree $\beta = \sum_{i=1}^n \alpha_i + \alpha_0$ belongs to the regularity $\text{Reg}(I)$. In particular, $\sum_{i=1}^n \alpha_i \in \text{Reg}(I)$ and $(\sum_{i=1}^n \alpha_i, \alpha_0)$ is a regularity pair for I .*

Proof. Observe that, as $\sum_{i=1}^n \alpha_i \in \text{Pic}^\circ(X)$ is basepoint free and α_0 is $V_X(I)$ -basepoint free, $\beta = \sum_{i=1}^n \alpha_i + \alpha_0$ is $V_X(I)$ -basepoint free. Hence, by Theorem 4.3, it suffices to prove that $H^p(X, \mathcal{O}_X(\beta - \sum_{i \in \mathcal{J}} \alpha_i)) = 0$ for $p > 0$ and $\mathcal{J} \subset \{1, \dots, n\}$. First, note that for any $\mathcal{J} \subset \{1, \dots, n\}$ we have

$$\beta - \sum_{i \in \mathcal{J}} \alpha_i = \sum_{i \in \{1, \dots, n\} \setminus \mathcal{J}} \alpha_i + \alpha_0 \in \mathbb{Q}\text{Pic}^\circ(X).$$

By Demazure vanishing (see Proposition 4.1), $H^p(X, \mathcal{O}_X(\beta - \sum_{i \in \mathcal{J}} \alpha_i)) = 0$ for $p > 0$. The rest of the proof follows from Theorem 4.3. \square

Observe that, if $\alpha_0 \in \text{Pic}^\circ(X)$, then α_0 is $V_X(I)$ -basepoint free and we can simplify Theorem 4.4.

Corollary 4.1. *With the notation of Theorem 4.4, if $\alpha_0 \in \text{Pic}^\circ(X)$, $(\sum_{i=1}^n \alpha_i, \alpha_0)$ is a regularity pair.*

Corollary 4.1 implies [52, Thm. 4.2 & 4.3] and bounds the regularity of S/I in the sense of [38, Def. 3.1] (following their notation, if $\mathbb{N}\mathcal{C} \subseteq \text{Pic}^\circ(X)$, S/I is $(\sum_i \alpha_i)$ -regular).

Example 8 (Cont. of Example 6). The pair $((6, 6), (1, 1))$ is a regularity pair for I as $(6, 6)$ corresponds to the sum of the degrees of the polynomials f_1, \dots, f_4 and $(1, 1) \in \text{Pic}^\circ(X_\Sigma)$. \triangle

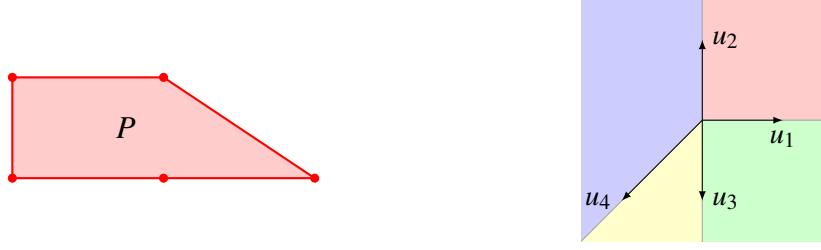


Figure 4: Polytope P and its normal fan Σ .

Theorem 4.4 corrects [52, Conj. 1], which states that Theorem 4.4 holds for $\alpha_0 \in \text{Cl}(X)_+$ with the same restrictions. As we show in the following example, the conjecture can fail to be true when we consider effective Cartier divisors which are not numerically effective.

Example 9 (Counter-example to [52, Conj. 1]). Consider the (smooth) Hirzebruch surface X_Σ associated to the polytope P shown in Figure 4 together with its normal fan Σ , see [19, Ex. 3.1.16]. Consider the sparse polynomials $\hat{f}_1, \hat{f}_2 \in \mathbb{C}[M]$ from Example 1 ($\varepsilon = 1$) with Newton polytope $P_1 = P_2 = P$:

$$\begin{cases} \hat{f}_1 := -1 + t_1 + t_1^2 + t_2 + t_1 t_2, \\ \hat{f}_2 := -2 + 2t_1 + 3t_1^2 + 4t_2 + 5t_1 t_2. \end{cases}$$

We homogenize \hat{f}_1, \hat{f}_2 to obtain $f_1, f_2 \in S_{\alpha_1} = S_{\alpha_2}$ where $\alpha_1 = \alpha_2 = [D_3 + 2D_4]$ (here D_i corresponds to u_i in Figure 4). We consider the ideal $I := \langle f_1, f_2 \rangle \subset S$. The closed subscheme associated to S/I is a zero-dimensional variety consisting of 3 points in $T \subset X$. In the coordinates (t_1, t_2) on T , we have

$$V_X(I) = \left\{ (-2, 1), \left(\frac{1}{\sqrt{2}}, \frac{-3}{\sqrt{2}} + 2 \right), \left(\frac{-1}{\sqrt{2}}, \frac{3}{\sqrt{2}} + 2 \right) \right\} \subset T.$$

Consider the degrees $\alpha := [2(D_3 + 2D_4)]$ and $\alpha_0 := [2D_3]$. As $\alpha = \alpha_1 + \alpha_2$, by Corollary 4.1, $\alpha \in \text{Reg}(I)$ and so $\dim((S/I)_\alpha) = 3$. Since each point in D_3 is a basepoint of $S_{k \cdot \alpha_0}$, for every $k > 0$, by [19, Lem. 9.2.1], we have $\alpha_0 \notin \mathbb{Q}\text{Pic}^\circ(X)$. Hence, α_0 does not satisfy the conditions of Theorem 4.4. However, α_0 is $V_X(I)$ -basepoint free, so α_0 satisfies the conditions of [52, Conj. 1]. As $\dim_{\mathbb{C}}((S/I)_{\alpha+\alpha_0}) = 4$, we have $\alpha + \alpha_0 \notin \text{Reg}(I)$ and the conjecture fails. \triangle

Combining Theorems 4.1 and 4.2, we can characterize the degrees at which [52, Conj. 1] holds.

Corollary 4.2. *With the notation of Theorem 4.4, consider a $V_X(I)$ -basepoint free degree $\alpha_0 \in \text{Cl}(X)_+$. We have that $(\sum_{i=1}^n \alpha_i, \alpha_0)$ is a regularity pair if and only $\dim_{\mathbb{C}}((S/I)_{\alpha_0+\alpha_1+\dots+\alpha_n}) = \delta^+$.*

Example 10 (Cont. Example 9). For $\alpha := [2(D_3 + 2D_4)] \in \text{Pic}^\circ(X)$ and $\alpha_0 := [D_3] \in \text{Cl}(X)_+$, we check that $\dim_{\mathbb{C}}((S/I)_{\alpha+\alpha_0}) = 3$. By Corollary 4.2, we conclude that $([2(D_3 + 2D_4)], [D_3])$ is a regularity pair. Note that $\alpha_0 \notin \mathbb{Q}\text{Pic}^\circ(X) \supset \text{Pic}^\circ(X)$. \triangle

As we showed in Example 9, the assumption on the Hilbert function of S/I in Corollary 4.2 cannot be dropped. Nevertheless, using Lemma 4.1, we can prove the following bound on $\dim_{\mathbb{C}}((S/I)_\beta)$, which generalizes [38, Prop. 6.7].

Lemma 4.2. *Let $I \subset S$ be such that $V_X(I)$ is zero-dimensional. For any $V_X(I)$ -basepoint free $\beta \in \text{Cl}(X)$ such that $I_\beta = (I : B^\infty)_\beta$, we have $\dim_{\mathbb{C}}((S/I)_\beta) \leq \delta^+$.*

Proof. By assumption, $H_B^0(S/I)_\beta = 0$ as $I_\beta = (I : B^\infty)_\beta$ and following the same argument as in Theorem 4.1, we also have $H^0(X, \widetilde{(S/I)(\beta)}) = H^0(Y, \mathcal{O}_Y)$. Hence, using Equation (4.1), we have an injective map $(S/I)_\beta \rightarrow H^0(X, \mathcal{O}_Y)$. The lemma follows as we defined $\delta^+ := \dim_{\mathbb{C}}(H^0(X, \mathcal{O}_Y))$. \square

4.3 Further improvements

The size of some of the matrices involved in the eigenvalue algorithm from Section 3.3 is given by the dimension of $S_{\alpha+\alpha_0}$, where (α, α_0) is a regularity pair for I . We are therefore interested in finding regularity pairs (α, α_0) for which $\dim_{\mathbb{C}} S_{\alpha+\alpha_0}$ is as small as possible. In Theorem 4.3, we showed how we can construct regularity pairs by looking at the vanishing of some sheaf cohomologies. In this subsection, we explain how this vanishing depends on the combinatorics of the polytopes related to X . We illustrate this relation by studying the regularity of unmixed, classical homogeneous, weighted homogeneous and multihomogeneous square systems. These improvements allow us to speed up the computations in Example 6 by a factor of 25. Moreover, the regularity pairs that we construct lead to matrices of roughly the same size as the ones considered in other (affine) algebraic approaches, such as Gröbner bases or sparse resultants, see for instance [4, Sec. 8] and [27, Thm. 12].

A straightforward consequence of Batyrev-Borisov vanishing (Proposition 4.1) is that, whenever the polytope of $\alpha \in \mathbb{Q}\text{Pic}^\circ(X)$ is *hollow* (no interior lattice points), the associated reflexive sheaf $\mathcal{O}_X(-\alpha)$ has no cohomologies (in the terminology of [1], it is *immaculate*). We can rephrase this condition using the concept of *codegree* of a polytope. This is the smallest $c \in \mathbb{N}$ such that $c \cdot P$ contains a lattice point in its relative interior $\text{Relint}(c \cdot P)$.

An important class of sparse polynomial systems consists of the so-called *unmixed sparse systems*. These are systems in which each of the Newton polytopes is a dilation of some lattice polytope P .

Theorem 4.5 (Unmixed sparse systems). *Let $\alpha_0 \in \text{Pic}^\circ(X)$ be a nef Cartier divisor such that its associated lattice polytope P is full dimensional. Consider an ideal $I = \langle f_1, \dots, f_n \rangle \subset S$ with $f_j \in S_{d_j \cdot \alpha_0} = H^0(X, \mathcal{O}_X(d_j \cdot \alpha_0))$, $d_j \in \mathbb{N}$ such that $V_X(I)$ is a complete intersection. Let c be the codegree of P . Then, for every $t < c$, $(\sum_j d_j - t) \alpha_0 \in \text{Reg}(I)$ and $((\sum_j d_j - c) \alpha_0, \alpha_0)$ is a regularity pair.*

Proof. As P is a lattice polytope, $\alpha_0 \in \text{Pic}^\circ(X)$, and, by [19, Prop. 6.1.1], for every $k \geq 0$, $k \alpha_0 \in \text{Pic}^\circ(X)$. Fix an integer $t < c$. By Theorem 4.3, we only need to prove that $H^p(X, \mathcal{O}_X((\sum_{j \in \mathcal{J}} d_j - t) \alpha_0)) = 0$ for $p > 0$ and $\mathcal{J} \subset \{1, \dots, n\}$. If $\sum_{j \in \mathcal{J}} d_j \geq t$, these cohomologies vanish by Demazure vanishing. If $\sum_{j \in \mathcal{J}} d_j < t$, by Batyrev-Borisov vanishing, $H^p(X, \mathcal{O}_X((\sum_{j \in \mathcal{J}} d_j - t) \alpha_0)) = 0$, for all $p \neq \dim(P)$. If $p = \dim(P)$, then the dimension of the p -th sheaf cohomology agrees with the number of lattice points in the interior of $(t - \sum_{j \in \mathcal{J}} d_j) \cdot P$, that is, $\dim_{\mathbb{C}}(H^p(X, \mathcal{O}_X((\sum_{j \in \mathcal{J}} d_j - t) \alpha_0))) = \#(\text{Relint}((t - \sum_{j \in \mathcal{J}} d_j) \cdot P) \cap M)$. As t is strictly smaller than the codegree of the polytope P , for every \mathcal{J} , $(t - \sum_{j \in \mathcal{J}} d_j) \cdot P$ has no interior points, so $H^p(X, \mathcal{O}_X((\sum_{j \in \mathcal{J}} d_j - t) \alpha_0))$ vanishes. \square

Note that if P is a hollow polytope, Theorem 4.5 improves the pairs obtained in Corollary 4.1. As a special case of Theorem 4.5, we recover the Macaulay bound [37], which bounds the Castelnuovo-Mumford regularity for (classical) homogeneous square systems over \mathbb{P}^n .

Corollary 4.3 (Macaulay bound). *Consider the smooth toric variety \mathbb{P}^n and let $S = \mathbb{C}[x_0, x_1, \dots, x_n]$ be its Cox ring. The class group of this variety is isomorphic to \mathbb{Z} and S is the standard \mathbb{Z} -graded polynomial ring. Consider the ideal $I = \langle f_1, \dots, f_n \rangle$ generated by homogeneous polynomials such that $\deg(f_i) = d_i$. If $V_X(I)$ is zero dimensional, then $(\sum_i d_i - n, 1)$ is a regularity pair for I .*

Proof. The class group of \mathbb{P}^n is generated by the class of a basepoint free divisor D representing a hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$. The polytope associated to D is the standard n -simplex $\Delta \in \mathbb{R}^n$, see [19, Ex. 5.4.2]. Hence, we can think of each f_i as a global section in $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d_i D))$. As the codegree of Δ is $(n+1)$, Theorem 4.5 tells us that $((\sum_i d_i - n)[D], [D])$ is a regularity pair. \square

Theorem 4.5 can be extended to non-lattice polytopes related to nef \mathbb{Q} -Cartier divisors. We illustrate such an extension in the important case of polynomial systems defined over a weighted projective

space. This is a non-smooth simplicial toric variety [19, Sec. 2.0]. In what follows, we consider the weighted projective space $X = \mathbb{P}(q_0, q_1, \dots, q_n)$, where $\gcd(q_0, \dots, \hat{q}_i, \dots, q_n) = 1$, for every i , and we fix $\ell := \text{lcm}(q_0, \dots, q_n)$. By [19, Ex. 4.2.11], the Class group of X is generated by a nef $\alpha_0 \in \mathbb{Q}\text{Pic}^\circ(X)$ whose associated polytope is $P := \{(a_0, \dots, a_n) \in \mathbb{R}^{n+1} \mid \sum_i q_i a_i = 1, q_i \geq 0\}$ and its Picard group is generated by $\ell \alpha_0 \in \text{Pic}^\circ(X)$.

Theorem 4.6. *Let $X = \mathbb{P}(q_0, q_1, \dots, q_n)$ and fix $d_1, \dots, d_n \in \mathbb{N}$. Consider $f_1, \dots, f_n \in S$ such that $f_i \in S_{\ell d_i} \alpha_0$. Consider $d_{\text{reg}} > \ell(\sum_{i=1}^n d_i) - \sum_{j=0}^n q_j$ such that $d_{\text{reg}} \alpha_0$ is $V_X(I)$ -basepoint free. If $I := \langle f_1, \dots, f_n \rangle$ is a complete intersection, then for any $V_X(I)$ -basepoint free $d_0 \alpha_0 \in \text{Cl}(X)_+$ such that $d_0 > 0$, we have that $(d_{\text{reg}} \alpha_0, d_0 \alpha_0)$ is a regularity pair for I .*

Proof. We consider the polytope P associated to $\alpha_0 \in \text{Cl}_+(X)$. Its codegree is $c = \sum_i q_i$ as this is the first dilation of P for which $(1, \dots, 1) \in \text{Relint}(c \cdot P)$. Using Proposition 4.1 as in the proof of Theorem 4.5, the theorem follows. \square

A natural generalization of the unmixed case is the case where the Newton polytopes of our polynomials are products of simpler polytopes. We can extend our approach to this case using the well-known Künneth formula.

Proposition 4.2 (Künneth formula). *Let $P \in \mathbb{R}^n, Q \in \mathbb{R}^m$ be two full-dimensional polytopes and consider their Cartesian product $P \times Q \subset \mathbb{R}^{n+m}$. Then, $X_P \times X_Q \simeq X_{P+Q}$ [19, Prop. 2.4.9] and $\text{Cl}(X_P) \oplus \text{Cl}(X_Q) \simeq \text{Cl}(X_{P+Q})$ [19, Ex. 4.1.2]. Moreover, given $\alpha + \beta \in \text{Cl}(X_{P+Q})$, such that $\alpha \in \text{Cl}(X_P)$ and $\beta \in \text{Cl}(X_Q)$, we can write the sheaf cohomologies of the coherent sheaf $\mathcal{O}_{X_{P+Q}}(\alpha + \beta)$ in terms of the cohomologies of $\mathcal{O}_{X_P}(\alpha)$ and $\mathcal{O}_{X_Q}(\beta)$; for every r we have*

$$H^r(X_{P+Q}, \mathcal{O}_{X_{P+Q}}(\alpha + \beta)) \simeq \bigoplus_{i+j=r} H^i(X_P, \mathcal{O}_{X_P}(\alpha)) \otimes H^j(X_Q, \mathcal{O}_{X_Q}(\beta)).$$

Proof. The statement follows immediately from the fact that toric varieties coming from polytopes are separated, see [19, Thm. 3.1.5], and [34, Prop. 9.2.4]. \square

We illustrate our approach for toric varieties of this kind by studying the regularity pairs of the multi-homogeneous system considered in Example 6.

Example 11 (Cont. of Example 6). Let $\Delta \subset \mathbb{R}^2$ be the standard 2-dimensional simplex in \mathbb{R}^2 . The codegree of Δ is 3. As we observed in the proof of Corollary 4.3, the toric variety X_Δ is \mathbb{P}^2 . Therefore, the toric variety $X = X_\Delta \times X_\Delta$ associated to the polytope $\Delta \times \Delta \subset \mathbb{R}^4$ is $\mathbb{P}^2 \times \mathbb{P}^2$. This variety is smooth $\text{Cl}(X) = \text{Pic}(X) = \text{Cl}(X_\Delta) \times \text{Cl}(X_\Delta)$ is generated by $(\beta, 0)$ and $(0, \beta)$, where $\beta \in \text{Cl}(X_\Delta)$ is the class of the divisor associated to Δ in X_Δ . We use the natural identification $\text{Cl}(X) \simeq \mathbb{Z}^2$ given by $(\beta, 0) \sim (1, 0)$ and $(0, \beta) \sim (0, 1)$. As in Corollary 4.3, by Proposition 4.1, if $i \notin \{0, 2\}$, then $H^i(X_\Delta, \mathcal{O}_\Delta(a)) = 0$. Hence, we rewrite the previous cohomology using Proposition 4.2 as

$$H^r(X, \mathcal{O}_X(a, b)) = \bigoplus_{\substack{i, j \in \{0, 2\} \\ i+j=r}} H^i(X_\Delta, \mathcal{O}_{X_\Delta}(a)) \otimes H^j(X_\Delta, \mathcal{O}_{X_\Delta}(b)).$$

This shows that $H^1(X, \mathcal{O}_X(a, b)) = H^3(X, \mathcal{O}_X(a, b)) = 0$. Hence, we observe that the only possible non-zero higher sheaf cohomologies are,

$$\begin{aligned} H^2(X, \mathcal{O}_X(a, b)) &= H^0(X_\Delta, \mathcal{O}_{X_\Delta}(a)) \otimes H^2(X_\Delta, \mathcal{O}_{X_\Delta}(b)) \oplus H^2(X_\Delta, \mathcal{O}_{X_\Delta}(a)) \otimes H^0(X_\Delta, \mathcal{O}_{X_\Delta}(b)) \\ H^4(X, \mathcal{O}_X(a, b)) &= H^2(X_\Delta, \mathcal{O}_{X_\Delta}(a)) \otimes H^2(X_\Delta, \mathcal{O}_{X_\Delta}(b)) \end{aligned}$$

Following the same argument as in Corollary 4.3, $H^4(X, \mathcal{O}_X(a, b)) \neq 0$ if and only if $a \leq -3$ and $b \leq -3$ (blue area in Figure 5a) and $H^2(X, \mathcal{O}_X(a, b)) \neq 0$ if and only if $a \leq -3$ and $b \geq 0$, or $a \geq 0$ and $b \leq -3$ (yellow and green areas in Figure 5a). Figure 5a shows the values (a, b) for which some higher cohomology does not vanish, summarizing this analysis.

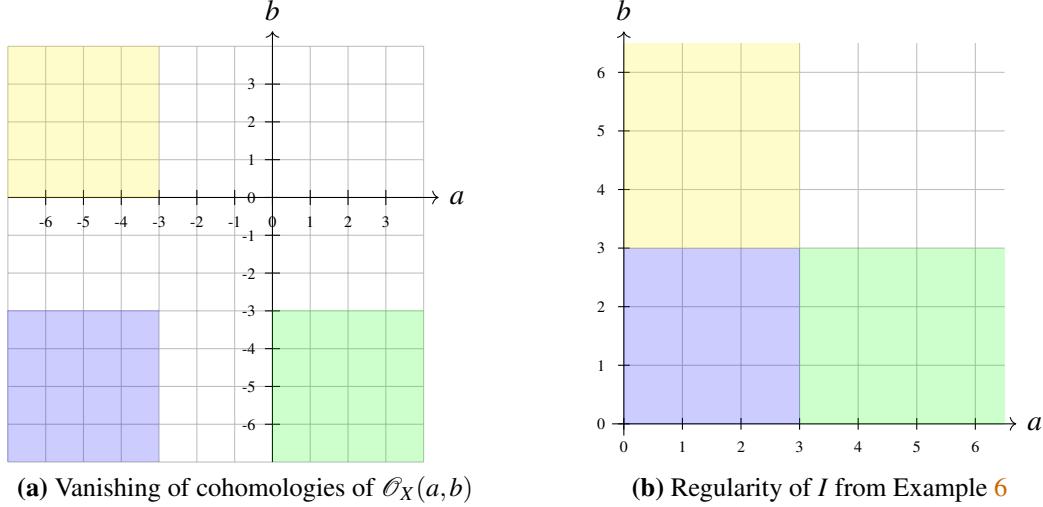


Figure 5: Figure 5a shows the vanishing of the higher cohomologies of $\mathcal{O}_X(a, b)$. Figure 5b shows the regularity of I from Example 6. The white area corresponds to degrees $(a, b) \in \text{Reg}(I)$. The blue area corresponds to the degrees (a, b) at which $H^4(X, \mathcal{O}_X(a-6, b-6)) \neq 0$, the green area corresponds to $H^2(X, \mathcal{O}_X(a-3, b-6)) \neq 0$ and the yellow area corresponds to $H^2(X, \mathcal{O}_X(a-6, b-3)) \neq 0$.

Now we will apply Theorem 4.3. If $a, b \in \mathbb{N}$, then the degree $(a, b) \in \text{Cl}(X)$ is basepoint free, hence we only need to find the pairs (a, b) such that the cohomologies in (4.2) vanish. In Figure 5b we show the values for $(a, b) \in \mathbb{N}^2$ for which all these cohomologies vanish. In Example 6, we used the fact that by Theorem 4.1, $((6, 6), (1, 1))$ is a regularity pair. Our previous analysis shows that $((4, 4), (1, 1))$ is a regularity pair as well. Using this new regularity pair reduces the size of the matrix of the map Res from Section 3.3 from 1296×2256 to 441×552 , which causes a speed-up by a factor ≈ 25 . \triangle

The analysis from the previous example can be generalized to recover the *multihomogeneous Macaulay bound*, see [5, Sec. 4].

Proposition 4.3 (Multihomogeneous Macaulay bound). *Let $X := \mathbb{P}^{n_1} \times \dots \times \mathbb{P}^{n_s}$ and consider $n := n_1 + \dots + n_s$ homogeneous polynomials f_1, \dots, f_n such that $f_i \in S_{d_i}$, for each i , where $d_i \in \text{Cl}_+(X) \simeq \mathbb{N}^s$ and $S = \bigoplus_{(a_1, \dots, a_s) \in \mathbb{N}^s} \bigotimes_{i=1}^s \mathbb{C}[x_{i,0}, \dots, x_{i,n_i}]_{a_i}$ is the Cox ring of X , [19, Ex. 2.4.8]. If $V_X(\langle f_1, \dots, f_n \rangle)$ is a complete intersection, for any $b \in \text{Cl}_+(X) \simeq \mathbb{N}^s$, $(\sum_{j=1}^n d_j - (n_1, \dots, n_s), b)$ is a regularity pair.*

In the case of complete intersection over $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2}$, more can be said about the vanishing of the higher cohomologies; see [12].

In this section, we showed how to improve the bounds from Corollary 4.1 by combining Theorem 4.3 with classical vanishing theorems. As mentioned before, in other cases not covered by Proposition 4.1, the sheaf cohomologies can be computed explicitly [1].

We emphasize that the results of Sections 4.2 and 4.3 do not cover the *overdetermined/non-square* case, where the number of equations exceeds n . It remains an open problem to characterize regularity pairs for such systems.

Acknowledgments We are grateful to Marta Panizzut and Sascha Timme for bringing Example 6 to our attention. We thank Laurent Busé, Yairon Cid-Ruiz, Marco Ramponi, Joaquín Jacinto Rodriguez, Mesut Şahin, Ivan Soprunov, Pierre-Jean Spaenlehauer, Josué Tonelli Cueto and Elias Tsigaridas for answering many questions regarding algebraic geometry, homological algebra and useful references. We thank Peter Bürgisser, David Cox, Alicia Dickenstein and Teresa Krick for their comments to improve a previous version of this paper. Part of this work was done during the visit of the second author to TU Berlin for the occasion of the MATH+ Thematic Einstein Semester on Algebraic Geometry, Varieties, Polyhedra, Computation. We thank the organizers of this nice semester for making this collaboration possible.

The first author was funded by the ERC under the European’s Horizon 2020 research and innovation programme (grant agreement No 787840). The second author was supported by the Research Council KU Leuven, C1-project (Numerical Linear Algebra and Polynomial Computations), and by the Fund for Scientific Research–Flanders (Belgium), G.0828.14N (Multivariate polynomial and rational interpolation and approximation), and EOS Project no 30468160.

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