

# MOMENTUM-CONSERVING ROMS FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

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## Abstract.

Projection-based model order reduction of an ordinary differential equation (ODE) results in a projected ODE. Based on this ODE, an existing reduced-order model (ROM) for finite volume discretizations satisfies the underlying conservation law over arbitrarily chosen subdomains. However, this ROM does not satisfy the projected ODE exactly but introduces an additional perturbation term. In this work, we propose a novel ROM with the same subdomain conservation properties which also satisfies the perturbed ODE exactly.

We apply this ROM to the incompressible Navier-Stokes equations and show with regard to the mass equation how the novel ROM can be constructed to satisfy algebraic constraints.

Furthermore, we show that the resulting mass-conserving ROM allows us to derive kinetic energy conservation and consequently nonlinear stability, which was not possible for the existing ROM due to the presence of the perturbation term.

## 1 INTRODUCTION

For many applications such as optimization, uncertainty quantification and real-time simulation of large-scale systems, classical numerical methods such as finite elements or finite volumes are prohibitively expensive. To mitigate the computational costs, several techniques have been developed to reduce the complexity of the full order models (FOM) developed via such classical methods, yielding so-called reduced order models (ROM). Many of these techniques are projection-based, i.e., the FOM is projected onto a lower-dimensional subspace spanned by a reduced basis. Such a reduced basis can be obtained, e.g., via Proper Orthogonal Decomposition of FOM snapshots [1].

While ROMs have been developed successfully for elliptic and parabolic systems of partial differential equations, deriving stable ROMs for hyperbolic systems is challenging

[2]. Stability problems of these ROMs are often ascribed to lacking structure preservation. While many FOMs feature physical structures of the underlying systems, many model reduction techniques do not preserve such structures [2].

Therefore, many approaches to obtain stable reduced order models refine model reduction techniques such that they do preserve such physical structures [3]. Peng et al. [4] propose a POD-Galerkin ROM for Hamiltonian systems that preserve the systems' symplecticity. Chan [5] constructs ROMs for nonlinear conservation laws that exhibit a semi-discrete entropy dissipation. Sanderse [6] achieves nonlinear stability of a POD-Galerkin ROM for the incompressible Navier-Stokes equations by preserving the kinetic energy evolution and additionally conserves the global momentum.

Carlberg et al. [7] address another aspect of structure preservation specific to finite volume discretizations. By construction, these discretizations satisfy the underlying conservation law over all finite volumes. Projection-based ROMs, on the other hand, satisfy this conservation law generally over none of the finite volumes. While it is in most cases impossible to define ROMs that satisfy the conservation law over all finite volumes, Carlberg et al. propose ROMs that satisfy the conservation law at least over a few subdomains. These ROMs are formulated via a constrained optimization problem (COP) and are shown to be more accurate than ROMs without subdomain conservation. Furthermore, the COP framework has been extended by Schein et al. [8] to include other types of constraints.

In this work, we combine the concept of conservation over subdomain in [7] and the nonlinear stable model reduction in [6]. Unfortunately, the subdomain conservative ROM in [7] is not suitable for this purpose. As shown in [7], the solution to the COP can equivalently be expressed by an ODE. This ODE, however, differs from the conventional Galerkin ROM ODE by a perturbation term. This perturbation term impedes the derivation of nonlinear stability as described in [6]. Therefore, we propose a novel subdomain conservative ROM which does not introduce any perturbation terms.

We apply the novel ROM to the energy-conserving finite volume discretization of the incompressible Navier-Stokes equations proposed in [6]. Regarding the momentum equation, we can directly apply the novel ROM. Regarding the mass equation, however, we have to extend the novel ROM to treat algebraic constraints. The resulting mass-conserving ROM is shown to mimic the kinetic energy evolution of the FOM. This evolution states that the global kinetic energy does not increase over time, hence implying nonlinear stability of the ROM.

This article is structured as follows. First, we summarize the subdomain conservative ROM proposed in [7] in Section 2. Then, we derive our novel subdomain conservative ROM in Section 3, in the context of a general conservation law. In Section 4, we apply our ROM to the energy-conserving finite volume discretization of the incompressible Navier-Stokes equations proposed in [6] and show that the ROM preserves the kinetic energy evolution of the FOM. We summarize our findings in Section 5 and give an outlook on possible uses of the proposed ROMs in Section 6.

## 2 SUBDOMAIN CONSERVATION FOR GENERAL CONSERVATION LAWS

In this section, we summarize the concept of subdomain conservation and the subdomain conservative ROM described in [7]. For convenience only, we use a simplified notation and consider only scalar-valued conservation laws.

### 2.1 Subdomain conservation for finite volume discretizations

In the understanding of Carlberg et al. [7], a scalar-valued quantity  $u$  is conserved over a domain  $\omega$  if it satisfies the integral form of a conservation law over this domain

$$\frac{d}{dt} \int_{\omega} u \, dV + \int_{\partial\omega} g \cdot n \, dS = \int_{\omega} s \, dV. \quad (1)$$

where  $g = g(u)$  describes the flux across the boundary and  $s$  is a term describing sources and sinks.

For example, a quantity  $u$  satisfying the differential form of a conservation law

$$\frac{\partial}{\partial t} u + \nabla \cdot g = s \quad \text{in } \Omega, \quad (2)$$

is conservative over any subdomain of  $\Omega$ , because (2) implies (1) for all  $\omega \subset \Omega$ .

Let us now consider a finite volume discretization of (2) on a mesh  $\mathcal{M} = \{\Omega_j\}_{j=1}^{N_{\mathcal{M}}}$  that completely covers  $\Omega$ . Then, we can define the time-dependent state-vector  $\mathbf{u}_{\mathbf{h}} = [u_1 \dots u_j \dots u_{N_{\mathcal{M}}}]^T$  of cell averages

$$u_j = \frac{1}{|\Omega_j|} \int_{\Omega_j} u \, dV \quad j = 1, \dots, N_{\mathcal{M}}, \quad (3)$$

and the time-dependent vector  $\mathbf{f} = [-f_1^g + f_1^s, \dots, -f_j^g + f_j^s, \dots, -f_{N_{\mathcal{M}}}^g + f_{N_{\mathcal{M}}}^s]^T$  consisting of the net flux

$$f_j^g = \frac{1}{|\Omega_j|} \int_{\partial\Omega_j} g^{FV} \cdot n \, dS, \quad (4)$$

and the net sinks and sources

$$f_j^s = \frac{1}{|\Omega_j|} \int_{\Omega_j} s^{FV} \, dV, \quad (5)$$

of each cell, divided by the respective cell-volume. Here,  $g^{FV}$  and  $s^{FV}$  are approximations of  $g$  and  $s$  in (2).

Together, the state vector  $\mathbf{u}_{\mathbf{h}}(t)$  and the RHS vector  $\mathbf{f}(\mathbf{u}_{\mathbf{h}}, t)$  form the ordinary differential equation (ODE) system

$$\frac{d}{dt} \mathbf{u}_{\mathbf{h}} = \mathbf{f}(\mathbf{u}_{\mathbf{h}}, t), \quad (6)$$

which we can interpret as a system of conservation laws in integral form over all finite volume cells.

Even more, the vector  $\mathbf{u}_h$  represents a function on  $\Omega$  which is constant over each finite volume. This function is subdomain conservative over all finite volume cells with respect to the approximated fluxes  $g^{FV}$ , and sink and source terms  $s^{FV}$ . Therefore, we also denote  $\mathbf{u}_h$  itself as subdomain conservative over all finite volume cells.

In the following, it will be useful to express the ODE (6) equivalently as

$$\mathbf{r} \left( \frac{d}{dt} \mathbf{u}_h, \mathbf{u}_h, t \right) = 0, \quad (7)$$

with the residual  $\mathbf{r}(\mathbf{v}, \mathbf{w}, t) = \mathbf{v} - \mathbf{f}(\mathbf{w}, t)$ .

## 2.2 Subdomain conservation for Galerkin projection ROMs

Let us now investigate the subdomain conservation properties of a model ROM approximating the FOM (6). Given an orthonormal basis  $\Phi \in \mathbb{R}^{N_{\mathcal{M}} \times p}$ ,  $p \ll N_{\mathcal{M}}$ , we consider the ROM

$$\mathbf{u}_r(t) = \Phi \mathbf{a}(t) \approx \mathbf{u}_h(t), \quad (8)$$

that satisfies the Galerkin projection of the FOM ODE (6)

$$\Phi^T \frac{d}{dt} \mathbf{u}_r = \frac{d}{dt} \mathbf{a} = \Phi^T \mathbf{f}(\Phi \mathbf{a}, t) \Leftrightarrow \Phi^T \mathbf{r} \left( \Phi \frac{d}{dt} \mathbf{a}, \Phi \mathbf{a}, t \right) = 0. \quad (9)$$

As shown in [9], this ODE system, equipped with some initial condition  $\mathbf{a}(0) = \mathbf{a}_0$ , can be equivalently described by the optimization problem

$$\frac{d}{dt} \mathbf{a} = \arg \min_{\mathbf{b} \in \mathbb{R}^p} \|r(\Phi \mathbf{b}, \Phi \mathbf{a}, t)\|_2, \quad (10)$$

and the same initial condition. Hence, the ROM  $\mathbf{u}_r(t)$  minimizes the violation of the cell-wise conservation laws (6) but this minimum is not guaranteed to vanish. Consequently, the ROM  $\mathbf{u}_r(t)$  is not guaranteed to be subdomain conservative over any finite volume.

Hence, instead of aiming at subdomain conservation over all finite volumes, Carlberg et al. propose to require conservation over a set of a few subdomains each consisting of one or more finite volumes.

To this end, we consider a decomposed mesh  $\bar{\mathcal{M}}$  of  $N_{\bar{\mathcal{M}}}$  arbitrary subdomains  $\bar{\Omega}_k = \cup_{j \in S_k} \Omega_j$  of finite volumes  $\Omega_j$  where  $S_k \subset \{1, \dots, N_{\mathcal{M}}\}$ . These subdomains  $\bar{\Omega}_k$  can overlap and do not need to be connected. An example of such a decomposed mesh is depicted in Fig. 1. On this grid, we define, similarly to the finite volume quantities (3), (4) and (5), a time-dependent state vector  $\bar{\mathbf{u}}_h = [\bar{u}_1 \dots \bar{u}_k \dots \bar{u}_{N_{\bar{\mathcal{M}}}}]^T$  of subdomain averages

$$\bar{u}_k = \frac{1}{|\bar{\Omega}_k|} \int_{\bar{\Omega}_k} u \, dV = \frac{1}{|\bar{\Omega}_k|} \sum_{j \in S_k} \int_{\Omega_j} u \, dV = \frac{1}{|\bar{\Omega}_k|} \sum_{j \in S_k} |\Omega_j| u_j, \quad (11)$$

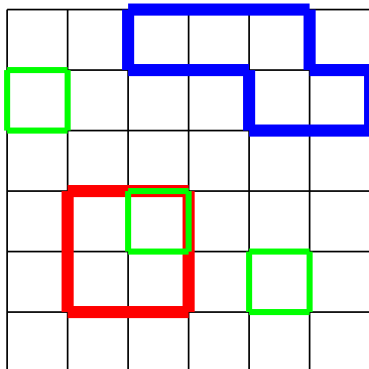


Figure 1: Example of three subdomains.

and analogously a time-dependent RHS vector  $\bar{\mathbf{f}}$ , consisting of the net flux and the net sinks and sources of each subdomain, divided by the respective subdomain volume. We can write  $\bar{\mathbf{u}}_{\mathbf{h}}$  and  $\bar{\mathbf{f}}$  in matrix vector notation as

$$\bar{\mathbf{u}}_{\mathbf{h}} = C^T \mathbf{u}_{\mathbf{h}} \quad \bar{\mathbf{f}} = C^T \mathbf{f}, \quad (12)$$

for some  $C \in \mathbb{R}^{N_{\mathcal{M}} \times N_{\bar{\mathcal{M}}}}$ .

As a result, subdomain conservation over all subdomains of  $\bar{\mathcal{M}}$  is described by the  $C^T$ -premultiplied ODE system

$$C^T \frac{d}{dt} \mathbf{u}_{\mathbf{h}} = C^T \mathbf{f} \quad \Leftrightarrow \quad C^T \mathbf{r} \left( \frac{d}{dt} \mathbf{u}_{\mathbf{h}}, \mathbf{u}_{\mathbf{h}}, t \right) = 0. \quad (13)$$

Based on the Galerkin projection ROM (9), we want to construct a ROM that also satisfies the subdomain conservation constraint (13). In the following section, we summarize the subdomain conservative ROM proposed in [7]. Our novel subdomain conservative ROM is presented in Section 3.

### 2.3 Existing approach for subdomain conservative ROMs: constrained optimization problem formulation

Carlberg et al. [7] suggest to use the interpretation of the ROM ODE (9) as minimization problem (10). Adding the conservation constraint (13) to this optimization problem, we get the constrained optimization problem

$$\text{minimize}_{\mathbf{b} \in \mathbb{R}^p} \quad \|r(\Phi \mathbf{b}, \Phi \mathbf{a}, t)\|_2 \quad (14)$$

$$\text{subject to} \quad C^T \mathbf{r}(\Phi \mathbf{b}, \Phi \mathbf{a}, t) = 0. \quad (15)$$

A sufficient condition for feasibility of this COP is given in [7, Proposition 5.1]. If the COP is feasible, the solution is equivalently expressed by the ODE

$$\frac{d}{dt} \mathbf{a} = \Phi^T \mathbf{f}(\Phi \mathbf{a}, t) + \mathbf{f}^*(\Phi \mathbf{a}, t) \quad (16)$$

with the perturbation term

$$\mathbf{f}^*(\Phi \mathbf{a}, t) = (C^T \Phi)^+ [C^T - C^T \Phi \Phi^T] \mathbf{f}(\Phi \mathbf{a}, t), \quad (17)$$

and the Moore-Penrose inverse  $(C^T \Phi)^+$  of  $C^T \Phi$ .

Except for the perturbation term, this ODE is equivalent to the conventional Galerkin ROM ODE (9). The perturbation term, however, lacks a clear physical motivation and impedes the preservation of the kinetic energy evolution as we will see in Section 4.4. Therefore, we propose a subdomain conservative ROM without any perturbation terms in the next section.

### 3 NOVEL APPROACH FOR SUBDOMAIN CONSERVATIVE ROMS: BASIS MODIFICATION

#### 3.1 Novel approach

The basic idea of our novel approach is to merge the ROM ODE (9) and the conservation constraint (13) in one linear system for  $\frac{d}{dt} \mathbf{u}_r$ ,

$$[\Phi \ C]^T \frac{d}{dt} \mathbf{u}_r = [\Phi \ C]^T \mathbf{f}(\mathbf{u}_r, t). \quad (18)$$

To satisfy this linear system, we define an orthogonal basis  $\tilde{\Phi} \in \mathbb{R}^{N_{\mathcal{M}} \times H}$ ,  $H := \text{rank}([\Phi \ C])$  that spans the same linear subspace as  $[\Phi \ C]$  and equip our novel ROM with this basis,

$$\mathbf{u}_r(t) = \tilde{\Phi} \tilde{\mathbf{a}}(t), \quad \tilde{\mathbf{a}}(t) \in \mathbb{R}^H. \quad (19)$$

Then, the Galerkin ROM ODE

$$\tilde{\Phi}^T \frac{d}{dt} \tilde{\Phi} \tilde{\mathbf{a}} = \frac{d}{dt} \tilde{\mathbf{a}} = \tilde{\Phi}^T f(\tilde{\Phi} \tilde{\mathbf{a}}, t), \quad (20)$$

satisfies the system (18). Hence, this ROM is subdomain conservative without introducing any perturbation terms.

#### 3.2 View on POD bases

A popular method to construct ROM bases is POD, which exhibits an optimality property: given a snapshot matrix  $X = [\mathbf{u}_h^0 \ \dots \ \mathbf{u}_h^{K-1}] \in \mathbb{R}^{N_{\mathcal{M}} \times K}$ , a POD basis  $\Phi_{\text{POD}} \in \mathbb{R}^{N_{\mathcal{M}} \times p}$  is known to minimize the sum of best approximation errors of the snapshots in  $X$  among all orthogonal matrices in  $\mathbb{R}^{N_{\mathcal{M}} \times p}$  [1].

On the other hand, the key element of our novel approach is the modification of the ROM basis in consideration of the predefined subdomains. In order that the Galerkin ROM ODE (20) includes the subdomain conservation constraint (13), the span of the basis  $\tilde{\Phi}$  must include the span of  $C$ . This requirement can be expressed as the equivalence

of  $C$  and the orthogonal projection of  $C$  onto  $\tilde{\Phi}$ ,  $C = \tilde{\Phi}\tilde{\Phi}^T C$ . Combining the minimization idea of POD and the subdomain conservation constraint, we find the constrained optimization problem for the ROM basis  $\tilde{\Phi}$

$$\text{minimize}_{\Xi \in \mathbb{R}^{N_{\mathcal{M}} \times q}} \sum_{j=0}^{K-1} \|\mathbf{u}_{\mathbf{h}}^j - \Xi \Xi^T \mathbf{u}_{\mathbf{h}}^j\|^2 \quad (21)$$

$$\text{subject to} \quad \Xi^T \Xi = I \quad (22)$$

$$\text{and} \quad C = \Xi \Xi^T C. \quad (23)$$

The orthogonal basis that spans the same subspace as  $[\Phi_{\text{POD}} \ C]$  does generally not solve this constrained optimization problem.

A solution to this constrained optimization problem is proposed in [10]. This solution is defined as

$$\tilde{\Phi} = [Q_1 \ Q_2 V], \quad (24)$$

where  $Q_1 \in \mathbb{R}^{N_{\mathcal{M}} \times H_C}$  and  $Q_2 \in \mathbb{R}^{N_{\mathcal{M}} \times (N_{\mathcal{M}} - H_C)}$  with  $H_C = \text{rank}(C)$  are obtained from the QR decomposition of  $C^T$ ,

$$C = [Q_1 \ Q_2] \begin{bmatrix} R_1 \\ 0 \end{bmatrix}, \quad R_1 \in \mathbb{R}^{H_C \times N_{\mathcal{M}}} \quad (25)$$

and  $V \in \mathbb{R}^{(N_{\mathcal{M}} - H_C) \times (q - H_C)}$  consists of the first  $q - H_C$  modes of the POD basis of  $Q_2^T X$ .

In view of Section 3.1,  $V$  can be interpreted as the initial basis  $\Phi$ . Observe that in contrast to  $\Phi_{\text{POD}}$ , the matrix  $V$  depends on  $C$ .

### 3.3 View on least-squares Petrov-Galerkin ROMs

In [7], Carlberg et al. discuss besides the Galerkin ROM also a least-squares Petrov-Galerkin (LSPG) ROM. The two ROMs differ in the order of performing the ROM approximation and the time discretization in their derivations. While the Galerkin ROM is obtained by introducing the ROM approximation on the time-continuous model and discretizing in time afterwards, the LSPG ROM is obtained by first discretizing in time and then performing the ROM approximation on the time-discrete model.

So far, we have introduced our novel approach only for the Galerkin ROM. In fact, our approach does not directly work in the context of the LSPG ROM. For the LSPG ROM trial and test basis are not the same. As a consequence, the choice of the basis (24) as trial basis does generally not imply that the span of the test basis includes the span of  $C$ . Hence, the subdomain conservation constraint (13) is generally not satisfied.

### 3.4 Subdomain conservation for invariants

We want to highlight a property of our novel ROM for the special case of invariants. Invariants are quantities that are constant over time, e.g., the global integral of a conservation quantity with periodic boundary conditions and in absence of external forces.

If constraints in the constraint matrix  $C$  describe such invariants, then the coefficients in  $\tilde{\mathbf{a}}(t)$  in the expansion (19) that correspond to those invariant constraints are constant. Hence, we do not need to integrate these coefficients via the ODE (20), but can compute their values from the initial conditions. This observation can be seen as a physical justification for a ROM Ansatz with a constant offset,  $\mathbf{u}_r(t) = \tilde{\Phi}\tilde{\mathbf{a}}(t) + \mathbf{u}_r^0$ .

## 4 MOMENTUM CONSERVATION OVER SUBDOMAINS FOR THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

### 4.1 Introduction to the FOM

We consider the energy-conserving finite volume discretization of the incompressible Navier-Stokes equations with periodic boundary conditions proposed in [6],

$$M_h \mathbf{V}_h(t) = 0, \quad (26)$$

$$\Omega_h \frac{d}{dt} \mathbf{V}_h(t) = \mathbf{F}_h^{\text{CD}}(\mathbf{V}_h(t), t) - G_h \mathbf{p}_h(t). \quad (27)$$

The vectors  $\mathbf{V}_h(t) \in \mathbb{R}^{N_V}$  and  $\mathbf{p}_h \in \mathbb{R}^{N_p}$  describe the velocity and the pressure, respectively. The matrices  $M_h \in \mathbb{R}^{N_p \times N_V}$  and  $G_h \in \mathbb{R}^{N_V \times N_p}$  are the discretizations of the divergence and the gradient operator, respectively, and satisfy the duality property  $M_h = -G_h^T$ . The matrix  $\Omega_h \in \mathbb{R}^{N_V \times N_V}$  is diagonal with the sizes of the finite volumes on its diagonal, so symmetric positive definite. The term  $\mathbf{F}_h^{\text{CD}}(\mathbf{V}_h(t), t) \in \mathbb{R}^{N_V}$  comprises convective, diffusive and body force contributions.

While the momentum equation (27) is an ODE, the mass equation (26) is an algebraic equation. The subdomain conservation ROM method described in Section 3, however, addresses conservation laws of ODE structure only. Therefore, we apply the subdomain conservation method only to the momentum equation and deal with the mass equation separately in Section 4.3.

### 4.2 Momentum conservation over subdomains

To construct a ROM that conserves momentum over a set of predefined subdomains represented by the matrix  $C \in \mathbb{R}^{N_V \times N_M}$ , we define the approximation of the velocity

$$\mathbf{V}_r(t) = \phi \mathbf{a}(t) \approx \mathbf{V}_r(t), \quad \phi \in \mathbb{R}^{N_V \times R_V}, \quad \mathbf{a}(t) \in \mathbb{R}^{R_V}, \quad R_V \ll N_V. \quad (28)$$

In contrast to Section 3, where we introduced the orthogonal basis  $\tilde{\Phi}$  with respect to the  $L^2$ -inner product, we construct the velocity basis  $\phi$  to be orthogonal with respect to the  $\Omega_h$ -weighted inner product, i.e.  $\phi^T \Omega_h \phi = I$ . As a result, the definition of  $\phi$ ,

$$\phi = [\tilde{Q}_1 \quad W], \quad (29)$$

differs slightly from the definition of  $\tilde{\Phi}$  (24). Here,  $\tilde{Q}_1 = \Omega_h^{-1/2} Q_1$  with the QR decomposition of  $\Omega_h^{-1/2} C$ ,

$$\Omega_h^{-1/2} C = [Q_1 \quad Q_2] \begin{bmatrix} R_1 \\ 0 \end{bmatrix} = Q_1 R_1, \quad (30)$$



and  $W$  consists of the first  $R_V - \text{rank}(C)$  columns of the  $\Omega_h$ -orthogonal POD basis  $\tilde{U}$  of a snapshot matrix  $X = [\mathbf{V}_h^0 \ \dots \ \mathbf{V}_h^{K-1}] \in \mathbb{R}^{N_V \times K}$  that is  $\Omega_h$ -orthogonal to  $\tilde{Q}_1$ . In detail:

We project the snapshot matrix  $X$

$$\text{onto the subspace } \Omega_h\text{-orthogonal to } \tilde{Q}_1: \quad \tilde{X} = [I - \tilde{Q}_1 \tilde{Q}_1^T \Omega_h] X. \quad (31)$$

$$\text{We transform to include the } \Omega_h\text{-inner product:} \quad \hat{X} = \Omega_h^{1/2} \tilde{X}. \quad (32)$$

$$\text{We compute the SVD of } \hat{X}: \quad \hat{X} = \hat{U} \hat{\Sigma} \hat{V}^T. \quad (33)$$

$$\text{We transform back:} \quad \tilde{U} = \Omega_h^{-1/2} \hat{U}. \quad (34)$$

$$\text{Finally, we truncate:} \quad W = [\tilde{U}]_{R_V - \text{rank}(C^T)}. \quad (35)$$

This construction is a generalization of the construction in [6, Appendix C].

By inserting the velocity approximation (28) into the momentum equation (27) and Galerkin-projecting, we find the ODE

$$\phi^T \Omega_h \frac{d}{dt} \phi \mathbf{a}(t) = \frac{d}{dt} \mathbf{a}(t) = \phi^T \mathbf{F}_h^{\text{CD}}(\phi \mathbf{a}(t), t) - \phi^T G_h \mathbf{p}_h(t). \quad (36)$$

The right-hand side of this ODE contains a pressure term. We will eliminate this pressure term in the next section as a by-product of enforcing mass conservation.

### 4.3 Mass conservation

Inserting the velocity approximation (28) into the mass equation (26), we find

$$M_h \phi \mathbf{a}(t) = 0. \quad (37)$$

To satisfy mass conservation, we observe that the ROM (28) satisfies the mass equation (37) for all  $\mathbf{a}(t) \in \mathbb{R}^{R_V}$ , if the ROM basis  $\phi$  is divergence-free, i.e.  $M_h \phi = 0$ .

To exploit this insight if  $\phi$  is not divergence-free, we decompose the basis into a divergence-free component  $\phi_0$  and a basis orthogonal to the space of divergence-free vectors,  $\phi_\perp$ . For this purpose, we compute the QR decomposition of  $(M_h \phi)^T$ ,

$$(M_h \phi)^T = [Q_1^M \ Q_2^M] \begin{bmatrix} R_1^M \\ 0 \end{bmatrix} = Q_1^M R_1^M \quad (38)$$

and decompose

$$\mathbf{a}(t) = Q_1^M \mathbf{a}_1(t) + Q_2^M \mathbf{a}_2(t). \quad (39)$$

Then, we find  $\phi \mathbf{a}(t) = \phi_0 \mathbf{a}_2(t) + \phi_\perp \mathbf{a}_1(t)$  with the divergence-free basis  $\phi_0 = \phi Q_2^M$  and the basis  $\phi_\perp = \phi Q_1^M$  orthogonal to the divergence-free subspace. Inserting the decomposition (39) in the mass equation (37), we find

$$M_h \phi \mathbf{a}(t) = (R_1^M)^T (Q_1^M)^T [Q_1^M \mathbf{a}_1(t) + Q_2^M \mathbf{a}_2(t)] = (R_1^M)^T \mathbf{a}_1(t). \quad (40)$$

Since  $(R_1^M)^T$  has full column rank, the velocity approximation (28) satisfies the mass equation (37), if and only if we set  $\mathbf{a}_1(t) = 0$ . As a result, the velocity approximation simplifies to  $V_r(t) = \phi Q_2^M \mathbf{a}_2(t) = \phi_0 \mathbf{a}_2(t)$ . The remaining coefficients  $\mathbf{a}_2(t)$  are computed via the  $(Q_2^M)^T$ -premultiplied ODE (36),

$$(Q_2^M)^T \frac{d}{dt} Q_2^M \mathbf{a}_2(t) = \frac{d}{dt} \mathbf{a}_2(t) = (Q_2^M)^T \phi^T \mathbf{F}_h^{\text{CD}}(\phi_0 \mathbf{a}_2(t), t). \quad (41)$$

The pressure term is omitted because  $(Q_2^M)^T \phi^T G_h = \phi_0^T G_h = (M_h \phi_0)^T = 0$ .

To solve this ODE, we propose to employ an energy-conserving Runge-Kutta method, e.g., [11].

#### 4.4 Kinetic energy conservation

As in [6], we define the kinetic energy of the ROM as

$$K_r(t) = \frac{1}{2} \mathbf{V}_r(t)^T \Omega_h \mathbf{V}_r(t) = \frac{1}{2} \|\mathbf{a}_2(t)\|_2^2. \quad (42)$$

Using the ODE (41), we calculate

$$\frac{d}{dt} K_r(t) = \mathbf{a}_2(t)^T \frac{d}{dt} \mathbf{a}_2(t) = \mathbf{a}_2(t)^T \phi_0^T \mathbf{F}_h^{\text{CD}}(\phi_0 \mathbf{a}_2(t), t), \quad (43)$$

where the pressure term is eliminated due to the mass equation (37).

As shown in [6], the right-hand side evaluates in absence of body forces to  $-\nu \|Q_r \mathbf{a}_2\|_2^2$  with some matrix  $Q_r \in \mathbb{R}^{N_V \times R_V}$ . Hence, the kinetic energy is constant in the inviscid limit and decreases otherwise. Consequently, the ROM is nonlinearly stable.

In contrast, for the ROM proposed in [7], the kinetic energy evolution is given by

$$\frac{d}{dt} K_r(t) = -\nu \|Q_r \mathbf{a}_2(t)\|_2^2 + \mathbf{a}_2(t)^T \mathbf{F}_h^{\text{CD},*}(\phi \mathbf{a}_2(t), t), \quad (44)$$

provided that we use the approach described in Section 4.3 to enforce mass conservation. Because of the perturbation term  $\mathbf{F}_h^{\text{CD},*}(\phi \mathbf{a}_2(t), t)$  corresponding to (17), it is unclear whether the global kinetic energy increases over time and whether nonlinear stability can be inferred.

## 5 CONCLUSION

In this article, we have proposed a novel approach to develop reduced order models (ROMs) for finite volume discretizations that preserve the underlying conservation law over arbitrary subdomains. The key element of this approach is the modification of the ROM basis in consideration of the predefined subdomains. This modification incorporates the subdomain conservation constraints into the test basis and thereby enforces their satisfaction. The main advantage of our proposed ROM is that it does not introduce perturbation terms and can be simply defined via the conventional Galerkin ROM ODE.

For the special case that conservation quantities are constant in time over the chosen subdomain, we have shown that the ROMs can be computed more efficiently by introducing a physically motivated constant offset.

Furthermore, we have applied the novel subdomain conservation approach to an energy-conserving finite volume discretization of the incompressible Navier-Stokes equations. To enforce the mass equation on the ROM level, we have proposed a method based on a QR decomposition (which can also be generalized to other algebraic constraints). From the mass conservation, we have inferred the kinetic energy evolution which implies nonlinear stability of the ROM. This makes it stand out from existing subdomain conservative ROMs that involve perturbation terms, for which such an argument probably does not hold.

## 6 OUTLOOK

In this paper we have not addressed the choice of subdomains and their effect on the accuracy of the resulting ROMs. We believe that subdomain conservation could improve the generalization accuracy of ROMs. In particular, we see a potential in choosing a small number of subdomains each consisting of a single finite volume. Consequently, the ROM would be simulated in the classical finite volume sense over these finite volumes, while the remaining finite volumes are simulated in the Galerkin ROM fashion. This way, we can decompose the simulation domain into areas where we have confidence in the reduced basis and areas where we foresee that a high-fidelity method is required.

This interpretation can be seen as a generalization of heterogeneously refined grids in classical numerical methods. Such heterogeneous grids have highly refined grid cells where they are believed to be necessary and coarse grid cells everywhere else. The size of the grid cells is either computed a priori based on prior knowledge or adaptively based on a posteriori error estimates. In a similar fashion we could determine the conservation subdomains based on prior knowledge, or based on the analysis of snapshot data of the FOM simulation.

An example of informative prior knowledge is a parametrized external force that acts locally in the domain. In the area where this force acts, the solution likely changes significantly over variation of the parameters. Hence, snapshot data of a given set of parametrizations might not generalize well to other parametrizations. Therefore, we suggest to compute the ROM via finite volumes in this area.

As an idea to determine conservation subdomains based on snapshot data, we suggest to compute in which finite volumes the conservation quantity changes the most over time. We interpret the conservation laws over these finite volumes as the most important ones to enforce and hence suggest to choose these finite volumes as conservation subdomains.

Apart from these ideas to improve the global accuracy, enforcing conservation over single finite volumes could be used to improve the ROM accuracy locally. This approach could be useful for applications that are characterized by a particular interest in simulation results only in a small portion of the computation domain, e.g., flow around an airfoil.

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