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CHARACTERIZATION OF INCENTIVE COMPATIBLE SINGLE-PARAMETER MECHANISMS REVISITED

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ABSTRACT

We reexamine the characterization of incentive compatible single-parameter mechanisms introduced in Archer & Tardos (2001). We argue that the claimed uniqueness result, called 'Myerson's Lemma' was not well established. We provide an elementary proof of uniqueness that unifies the presentation for two classes of allocation functions used in the literature and show that the general case is a consequence of a little known result from the theory of real functions. We also clarify that our proof of uniqueness is more elementary than the previous one. Finally, by generalizing our characterization result to more dimensions, we provide alternative proofs of revenue equivalence results for multiunit auctions and combinatorial auctions.

Keywords: Incentive compatibility, single-parameter mechanisms, Myerson's lemma, auctions, revenue equivalence.

JEL Classification Numbers: D44, D82.

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1. INTRODUCTION

WE are concerned here with a characterization result of a specific class of incentive compatible direct selling mechanisms. For the sake of this article such a *mechanism* consists of an *allocation rule* that assigns some good or goods to the participants, called agents, and a *payment rule* that determines how much each agent needs to pay. Assuming that each agent has a *private valuation* of the good or goods, these decisions are taken in response to a vector of *bids* made by the agents. These bids may differ from agents' true valuations. Recall that a mechanism is (*dominant strategy*) *incentive compatible* (alternatively called *truthful*), if no agent is better off when providing false information regardless of reports of the other agents or, more precisely, when submitting a bid different from his/her valuation regardless of reports of the other agents.

Given a class of mechanisms one of the main problems is to characterize their incentive compatibility in terms of an appropriate payment rule. Several such results were established in the literature, starting with the one in Green & Laffont (1977) concerning Groves mechanisms, originally proposed in Groves (1973). One of the earliest characterization results was given in Myerson (1981), who considered single object auctions in an imperfect information setting. In Milgrom (2004) such characterizations are called 'Myerson's Lemma'. This terminology was adopted in Roughgarden (2016), Chapter 3 of which, titled 'Myerson's Lemma', is concerned with a characterization of incentive compatible single-parameter mechanisms, which were studied in Archer & Tardos (2001).

As we explain below, both in this article and in Roughgarden's book such a characterization result is actually not proved. Most (but not all) of the claims are rigorously established in Nisan (2007) in the context of randomized single-parameter mechanisms.

Given that this purported characterization of incentive compatible single-parameter mechanisms is frequently referred to in the literature (see e.g. Hartline & Karlin (2007) and Babaioff (2016)), we find it justified to review these claims. We will provide an elementary proof of the characterization result for two classes of allocation functions considered in Roughgarden (2016) and subsequently provide a proof of the original claim of Archer & Tardos (2001) by appealing to more advanced results from the theory of real functions. We conclude by comparing our proof to the one given in Krishna (2002) and

Börgers (2015).

2. PRELIMINARIES

We follow here the terminology of Roughgarden (2016) that is slightly different than the one originally used in Archer & Tardos (2001). In particular Roughgarden (2016) refers to a single-parameter mechanism and an allocation rule, while Archer & Tardos (2001) refer to a one-parameter mechanism and load.

Each *single-parameter mechanism* concerns sale of some 'stuff' to bidders and assumes

- a set of agents $\{1,\ldots,n\}$,
- for every agent i, a value $v_i \geq 0$ which specifies i's *private valuation* "per unit of stuff" that he or she acquires.

In the auction the agents simultaneously submit their *bids*, which are their reported valuations "per unit of stuff". The auctioneer receives the bids and determines how much 'stuff' each agent receives and against which price. So in contrast to the single-item auctions each agent i receives a possibly fractional amount $a_i \geq 0$ of an object (here 'a stuff') he or she is interested in.

An *allocation* is a vector $\mathbf{a} = (a_1, \ldots, a_n)$, where each $a_i \geq 0$ specifies the amount allocated to agent i. A *payment* is a vector $\mathbf{p} = (p_1, \ldots, p_n)$, where each $p_i \geq 0$ specifies the amount agent i has to pay.

Each single parameter mechanism consists of an *allocation rule*

$$\mathbf{a} : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$$

and a *payment rule*

$$\mathbf{p} : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}.$$

Given a vector of bids $\mathbf{b} = (b_1, \ldots, b_n)$ such a mechanism selects an allocation $\mathbf{a}(\mathbf{b}) = (a_1(\mathbf{b}), \ldots, a_n(\mathbf{b}))$ and a vector of payments $\mathbf{p}(\mathbf{b}) = (p_1(\mathbf{b}), \ldots, p_n(\mathbf{b}))$.

We assume that the *utility* of agent i is defined by

$$u_i(\mathbf{b}) = v_i a_i(\mathbf{b}) - p_i(\mathbf{b}).$$

We then say that a single-parameter mechanism is *incentive compatible* if for each agent i truthful bidding, i.e., bidding v_i, yields the best outcome regardless of bids of the other agents. More formally, it means that for all agents i

$$u_i(v_i, \mathbf{b}_{-i}) \geq u_i(b_i, \mathbf{b}_{-i}),$$

for all bids b_i of agent i and all vectors of bids $\mathbf{b}_{-i}$ of other agents, or equivalently—ignoring the parameters $\mathbf{b}_{-i}$—that for all $y \geq 0$

$$v_i a_i(v_i) - p_i(v_i) \geq v_i a_i(y) - p_i(y).$$

3. A CHARACTERIZATION RESULT

We say that a function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *monotonically non-decreasing*, in short *monotone*, if

$$0 \leq x \leq y \to f(x) \leq f(y).$$

We say that an allocation rule $\mathbf{a}$ is *monotone* if for every agent i and every vector of bids $\mathbf{b}_{-i}$ of other agents the function $a_i(\cdot, \mathbf{b}_{-i})$ is monotone.

The following result is stated in Archer & Tardos (2001), Nisan (2007), and Roughgarden (2016). In Nisan (2007) it is formulated as a result about randomized single-parameter mechanisms but the proofs are the same for the deterministic mechanisms considered here.

Theorem 1.

(i) If a mechanism $(\mathbf{a}, \mathbf{p})$ is incentive compatible then the allocation rule $\mathbf{a}$ is monotone.

(ii) If the allocation rule $\mathbf{a}$ is monotone then for some payment rule $\mathbf{p}$ the mechanism $(\mathbf{a}, \mathbf{p})$ is incentive compatible.

(iii) If the allocation rule $\mathbf{a}$ is monotone, then all payment rules $\mathbf{p}$ for which the mechanism $(\mathbf{a}, \mathbf{p})$ is incentive compatible differ by a constant.

There are some technically irrelevant differences between these three references. In Archer & Tardos (2001), instead of allocations, loads are considered, with the consequence that the loads are monotonically non-increasing, though the authors also state that the results equally apply to the set up that uses allocations. Following Nisan (2007) and Roughgarden (2016), we will use allocations. It leads to an analysis of monotonically non-decreasing functions. Further, in the last two references it is assumed that the payment rule yields 0 payment when bids are equal to 0, which makes it possible to drop in (iii) the qualification 'up to a constant'. To make the discussion applicable to arbitrary payment rules we do not adopt this assumption.

Item (i) is established in Archer & Tardos (2001) by appealing to the first derivative, so under some assumptions about the load function. However, a short argument given in Nisan (2007) and reproduced in Roughgarden (2016) shows that no assumptions are needed.

In turn, item (ii) is proved in Archer & Tardos (2001) 'by picture'. A rigorous proof is given in Nisan (2007), while in Roughgarden (2016) only a 'proof by picture' is provided for piecewise constant allocation rule and it is mentioned that "the same argument works more generally for monotone allocation rules that are not piecewise constant".

Finally, in Archer & Tardos (2001) item (iii) is claimed for arbitrary monotone loads and allocation rules. But in the paper only a short proof sketch is given that ends with a claim that "To prove that all truthful payment schemes take form (2), even when ω_i [the load rule] is not smooth, we follow essentially the same reasoning as in the [earlier given] calculus derivation." However, this derivation refers to load rules that are assumed to be smooth (actually only twice differentiable, so that integration by parts can be applied), while the characterization result is claimed for all monotone allocation functions.

In Nisan (2007) item (iii) is established by reducing in the last step the expression $\int_0^x zf'(z)dz$ to $xf(x) - \int_0^x f(z)dz$. We quote (adjusting the notation): "[...] we have that $p(x) = \int_0^x zf'(z)dz$, and integrating by parts completes the proof. (This seems to require the differentiability of f, but as f is monotone this holds almost everywhere, which suffices since we immediately integrate.)" (Recall that a property holds *almost everywhere* if it holds everywhere except at a set of measure 0, i.e., a set that can be covered by a countable union of intervals the total length of which is arbitrarily small.) A minor point is that the initial part of the proof is incomplete as it only deals with the right-hand derivative instead of the derivative.

Finally, in Roughgarden (2016) about item (iii) it is only stated without proof "We reiterate that these payments formulas [for the above two classes of allocation functions] give *the only possible* payment rule that has a chance of extending the given allocation rule $\mathbf{x}$ into a DSIC [i.e., incentive compatible] mechanism." Also here the formula (in the adjusted notation) $p(x) = \int_0^x zf'(z)dz$ is derived by discussing only the right-hand derivative.

In our view these arguments are incomplete as they do not take into account some restrictions that need to be imposed on the use of integrals and application of integration by parts. Note that, except in the final discussion, Riemann integration is assumed throughout.

Remark 2. To start with, integration by parts can fail for simple monotone functions, for example those considered in Roughgarden (2016). Indeed, let for $q > 0$

$$H_q(x) := \begin{cases} 0 & \text{if } 0 \leq x \leq q \\ 1 & \text{if } x > q \end{cases}$$

be an elementary step function with a single step at $x = q$.

Take now $f = H_q$ with $q = 1/2$. Then $f' = 0$ for $x \neq 1/2$ and f' is undefined for $x = 1/2$. Consequently (defining $f'(1/2)$ arbitrarily)

$$\int_0^1 zf'(z)dz = 0 \neq 1/2 = xf(x)\big|_0^1 - \int_0^1 f(z)dz. \tag{1}$$

Further, integration by parts can fail even if we insist on continuity. Indeed, take for f the Cantor function, see, e.g., (Tao, 2011, pages 170-171). It is monotone, continuous and almost everywhere differentiable on $[0,1]$, with $f(0) = 0$, $f(1) = 1$ and f' equal to 0 whenever defined. Additionally, (1) holds for f, as well.

Finally, there exists a monotone and everywhere differentiable function f for which the above integral $\int_0^1 zf'(z)dz$ does not exist. Indeed, as observed in Goffman (1977), there exists a monotone and everywhere differentiable function $f : [0,1] \to \mathbb{R}_{\geq 0}$ such that the integral $\int_0^1 f'(z)dz$ does not exist. By a result of Lebesgue (see, e.g., Bressoud (2008)) a bounded function defined on a bounded and closed interval is Riemann integrable iff it is continuous almost everywhere. But f' is continuous almost everywhere on $[0,1]$ iff the function $g(x) := xf'(x)$ is, so the claim follows. □

These points of concern motivate our subsequent considerations. To keep the paper self-contained we reprove items (i) and (ii), given that the proofs are very short.

4. AN ANALYSIS

Our analysis can be carried out without any reference to mechanisms by reasoning about functions on reals. We first rewrite the incentive compatibility condition as

$$p_i(y) - p_i(v_i) \geq v_i(a_i(y) - a_i(v_i)),$$

which from now on we analyze as the following condition on two functions $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$:

$$\forall x, y : g(y) - g(x) \geq x(f(y) - f(x)). \tag{2}$$

We are interested in solutions in g given f. We begin with the following obvious observation.

Note 3. *The inequality* (2) *is equivalent to*

$$\forall x, y : y(f(y) - f(x)) \geq g(y) - g(x) \geq x(f(y) - f(x)). \tag{3}$$

Proof. By interchanging in (2) x and y we get the additional inequality $y(f(y) - f(x)) \geq g(y) - g(x)$. □

Corollary 4 (Nisan (2007); Roughgarden (2016))**.** *Suppose* (2) *holds. Then the function f is monotone.*

Proof. Assume $0 \leq x < y$. By Note 3 (3) holds. The inequalities in (3) imply $(y - x)(f(y) - f(x)) \geq 0$, so $f(x) \leq f(y)$. □

This establishes item (i) of Theorem 1. To investigate items (ii) and (iii) we study existence and uniqueness of solutions of (2) in g. The following result establishes item (ii). The proof is from Nisan (2007).

Lemma 5. *Suppose f is monotone. Then* (2) *holds for*

$$g(x) = C + xf(x) - \int_0^x f(z)dz, \tag{4}$$

where C is some constant.

Because f is monotone g is well defined (see, e.g., Rudin (1976)).

Proof. By plugging the definition of g in (2) we get after some simplifications

$$\int_0^x f(z)dz - \int_0^y f(z)dz \geq (x - y)f(y), \tag{5}$$

which needs to be proved. Two cases arise.

Case 1. $x \geq y$.

Then

$$\int_0^x f(z)dz - \int_0^y f(z)dz = \int_y^x f(z)dz \geq (x-y)f(y),$$

where the last step follows by bounding the integral from below, since by the monotonicity of f, we have $f(y) \leq f(z)$ for $z \in [y,x]$.

Case 2. $y > x$.

Then

$$\int_0^x f(z)dz - \int_0^y f(z)dz = -\int_x^y f(z)dz \geq (x-y)f(y),$$

where the last step follows by bounding the integral from above, since by the monotonicity of f, we have $f(z) \leq f(y)$ for $z \in [x,y]$.

So (5) holds, which concludes the proof. □

To deal with uniqueness let us first consider the case for which the argument given in Nisan (2007) can be justified.

Lemma 6. *Suppose f is everywhere differentiable. Then any two solutions g of (2) differ by a constant.*

Proof. Suppose that (2) holds. By Note 3 (3) holds. Given an arbitrary $x \geq 0$ we first use it with $y = x+h$, where $h > 0$. Dividing by h we then obtain

$$\frac{(x+h)(f(x+h)-f(x))}{h} \geq \frac{g(x+h)-g(x)}{h} \geq \frac{x(f(x+h)-f(x))}{h}.$$

By the assumption about f

$$\lim_{h\to 0^+} \frac{(x+h)(f(x+h)-f(x))}{h} = \lim_{h\to 0^+} \frac{x(f(x+h)-f(x))}{h} = xf'(x),$$

so

$$\lim_{h\to 0^+} \frac{g(x+h)-g(x)}{h} = xf'(x). \tag{6}$$

Next, we use (3) with $x = y+h$, where $h < 0$. Dividing by h we then obtain

$$\frac{y(f(y)-f(y+h))}{h} \leq \frac{g(y)-g(y+h)}{h} \leq \frac{(y+h)(f(y)-f(y+h))}{h},$$

so replacing y by x and multiplying by -1 we get

$$\frac{x(f(x+h)-f(x))}{h} \geq \frac{g(x+h)-g(x)}{h} \geq \frac{(x+h)(f(x+h)-f(x))}{h}.$$

By the assumption about f and x

$$\lim_{h\to 0^-} \frac{(x+h)(f(x+h)-f(x))}{h} = \lim_{h\to 0^-} \frac{x(f(x+h)-f(x))}{h} = xf'(x),$$

so

$$\lim_{h\to 0^-} \frac{g(x+h)-g(x)}{h} = xf'(x). \tag{7}$$

We conclude from (6) and (7) that $g'(x)$ exists and

$$g'(x) = xf'(x). \tag{8}$$

Hence all solutions g to (2) have the same derivative and consequently differ by a constant. □

Remark 7. The above proof coincides with the one given in Nisan (2007), except on two points. First, only (6) is established there. This allows one only to conclude that the right derivative of g in x exists; to establish that $g'(x)$ exists also (7) is needed. More importantly, Nisan argued that all solutions g to (2) are of the form (4) given in Lemma 5. Under the assumption that f is everywhere differentiable this additional claim is a direct consequence of Lemmas 5 and 6.

Nisan's argument for this point involved integration and integration by parts. To justify it we need to assume that f' is continuous. Then by (8) also g' is continuous, which allows us to use the Fundamental Theorem of Calculus. It yields that for some constant C

$$g(x) = C + \int_0^x g'(z)dz.$$

Further, integration by parts of $\int_0^x zf'(z)dz$ is then also justified since f is everywhere differentiable and f' is integrable (see, e.g., Rudin (1976)). Then $\int_0^x zf'(z)dz$ exists and by integration by parts

$$\int_0^x zf'(z)dz = xf(x) - \int_0^x f(z)dz,$$

so (8) and the last two equalities imply that g is indeed of the form (4) given in Lemma 5. □

In the remainder of this section we do not use integration or the existence of solutions in the form (4), but proceed directly from (2). This allows us to sidestep the associated complications and show that the requirement of f being everywhere differentiable of Lemma 6 can be substantially weakened and, appealing to strong results from the theory of real functions, can even be removed altogether.

For $x = 0$ continuity (differentiability) means right continuity (differentiability), which we will not mention or treat separately.

We first need an auxiliary result.

Lemma 8. *Let g_1 and g_2 be two solutions of* (2) *and let $G = g_1 - g_2$.*

(i) G is continuous.

(ii) If f is continuous at x, then G is differentiable at x and $G'(x) = 0$.

Proof. (i) By Note 3 (3) holds for g_1 and g_2. By using it with $y = x + h$ for g_1 and for g_2 we obtain

$$0 \leq |G(x+h) - G(x)| \leq h(f(x+h) - f(x)). \tag{9}$$

We have $h(f(x+h) - f(x)) \leq hf(x+h)$ for $h > 0$ and $\leq -hf(x)$ for $h < 0$. But by Corollary 4 f is monotone, hence $\lim_{h \to 0} |G(x+h) - G(x)| = 0$, which establishes the claim.

(ii) Take some $x \geq 0$. By (9) for $h \neq 0$

$$0 \leq \left| \frac{G(x+h) - G(x)}{h} \right| \leq |f(x+h) - f(x)|,$$

which implies the claim. □

Note that the continuity of G holds for any f.

The following result with an elementary proof covers in a unified way item (iii) of Theorem 1 for two classes of allocation functions considered in Roughgarden (2016), piecewise constant and differentiable ones.

A function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is called *piecewise continuous* if it has at most a finite number of discontinuities in every bounded interval. Thus discontinuities can occur only at isolated points separated by open intervals of continuity. Piecewise constant and step functions are special cases. This definition is a straightforward generalization to $\mathbb{R}_{\geq 0}$ of the usual one for functions with a bounded domain.

Theorem 9. *Suppose f is piecewise continuous. Then any two solutions g of* (2) *differ by a constant.*

Proof. Let g_1 and g_2 be two solutions of (2) and let $G = g_1 - g_2$.

Let f be piecewise continuous with discontinuities $q_1 < q_2 < \dots$ and consider the intervals $I_0 = [0, q_1)$ ($\emptyset$ if $q_1 = 0$), $I_i = (q_i, q_{i+1})$ ($i \geq 1$), with $I_N = (q_N, \infty)$ if f has a finite number $N > 0$ of discontinuities and $I_0 = [0, \infty) = \mathbb{R}_{\geq 0}$ if $N = 0$.

If f has an infinite number of discontinuities, $\lim_{i \to \infty} q_i = \infty$ since there can be only finitely many of them in any bounded interval. Hence, the I_i and q_i together cover the whole of $\mathbb{R}_{\geq 0}$.

f is continuous on each I_i, so by Lemma 8(*ii*) G is constant on I_i, say $G = C_i$ on I_i. Since G is continuous everywhere by Lemma 8(*i*), $C_0 = G(q_1) = C_1 = G(q_2) = \dots$, so for some constant C, $G = g_1 - g_2 = C$. □

Theorem 9 can be generalized to a wider class of functions whose discontinuity sets may have limit points (accumulation points), at least to some degree. We give a simple example of a monotone function f, for which Theorem 9 does not apply but the stronger result presented below does.

Let

$$f(x) = \sum_{n=1}^{\infty} 2^{-n} H_{1-2^{-n}}(x),$$

where H_q is defined in Remark 2. It is not piecewise continuous, but has an infinite set of discontinuities $\{1/2, 3/4, \dots\}$ with a single limit point 1. Note that f happens to be continuous at $x = 1$, but this might also have been otherwise.

Functions like f and more complicated ones having discontinuity sets with limit points of limit points, etc. can, to some extent, be dealt with by adapting the proof of Theorem 9 and appealing to the well-known Bolzano-Weierstrass theorem (BW for short, see, e.g., Bressoud (2008)).

Given a set $S \subseteq \mathbb{R}_{\geq 0}$, we denote by $S^{(1)}$ the set of its limit points (which need not be in S) and define $S^{(n+1)} = (S^{(n)})^{(1)}$ for $n \geq 1$. A set S is called *first species* of type $n-1$ if $S^{(n)} = \emptyset$ and $S^{(m)} \neq \emptyset$ for $m < n$ (Bressoud, 2008). Such a set has limit points, limit points of limit points, etc., up to level $n-1$. A first species set of type 0 has no limit points.

Theorem 10. *Suppose the discontinuity set of f is first species of type $n \geq 0$. Then any two solutions g of* (2) *differ by a constant.*

Proof. Let g_1 and g_2 be two solutions of (2) and let $G = g_1 - g_2$. Let the discontinuity set of f be S and use induction with respect to the type of S.

If $n = 0$, $S^{(1)} = \emptyset$, so f can have only finitely many discontinuities in every bounded interval. Otherwise, there would be a limit point in some bounded and closed interval by BW. Hence, f is piecewise continuous and the case $n = 0$ corresponds to Theorem 9.

Assume the theorem holds for all $n \leq k$ for some $k > 0$ and consider S of type $k+1$. The elements of $S^{(k+1)}$ are the limit points of level $k+1$. Recall that these need not be elements of S. Since $S^{(k+2)} = \emptyset$, $S^{(k+1)}$ does not have limit points, so there can be only finitely many elements of $S^{(k+1)}$ in every bounded interval by BW as before.

Now let the elements of $S^{(k+1)}$ be $q_1 < q_2 < \dots$ and consider the intervals I_i ($i \geq 0$) as in the proof of Theorem 9. Together with the q_i, they cover the whole of $\mathbb{R}_{\geq 0}$ as in the previous proof.

However, the $S \cap I_i$ may still have q_i and/or q_{i+1} as limit points, which means the $S \cap I_i$ need not be of type $\leq k$ but can still be of type $k+1$, so the induction hypothesis cannot be applied to the I_i. Therefore, for fixed i and sufficiently small $\delta > 0$ consider a non-empty bounded and closed subinterval $J_i = J_i(\delta) = [q_i + \delta, q_{i+1} - \delta]$ of I_i (or, if the number of limit points is a finite number N, $J_N = J_N(\delta) = [q_N + \delta, \infty)$).

Now $S \cap J_i$ can no longer have q_i and/or q_{i+1} as limit points. Hence, it is of type $\leq k$ and the induction hypothesis applies to J_i, so G is constant on J_i, say $G = C_i$ on J_i. Since G is continuous everywhere by Lemma 8(i) and $\lim_{\delta \to 0} J_i = [q_i, q_{i+1}]$ (or, if the number of limits points is N, $\lim_{\delta \to 0} J_N = [q_N, \infty)$), we get $G(q_i) = C_i = G(q_{i+1})$. The final step of the proof is the same as in the proof of Theorem 9. □

Unfortunately, the above result does not cover all monotone functions $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$. Indeed, a monotone function may be discontinuous on the set $\mathbb{Q}_{\geq 0}$ of non-negative rational numbers, see, e.g., Rudin (1976), and $\mathbb{Q}_{\geq 0}$ is not first species, since $\mathbb{Q}_{\geq 0}^{(1)} = \mathbb{R}_{\geq 0}$ and $\mathbb{R}_{\geq 0}^{(1)} = \mathbb{R}_{\geq 0}$.

This limitation can be circumvented by appealing to a strong result of Goldowski and Tonelli. Recall first that a function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is differentiable *nearly everywhere* if it is differentiable except at a countable number of points. Note that *nearly everywhere* implies *almost everywhere*. We need

Theorem 11 (Goldowski (1928); Tonelli (1930-31); Saks (1937))**.** *Let* $G : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ *be a function such that*

- *G is continuous,*
- *G is differentiable nearly everywhere,*
- *$G' \geq 0$ almost everywhere.*

Then G is monotone.

This leads directly from Lemma 8 to the desired conclusion.

Theorem 12. *Any two solutions g of (2) differ by a constant.*

Proof. Let g_1 and g_2 be two solutions of (2) and let $G = g_1 - g_2$. By Lemma 8(i) G is continuous.

A monotone function is continuous nearly everywhere (see, e.g., Rudin (1976)). So by Lemma 8(ii) G is differentiable nearly everywhere and $G' = 0$ nearly everywhere. Hence by Theorem 11 both G and $-G$ are monotone, i.e., G is constant. □

The above theorem justifies item (iii) of Theorem 1. The following result summarizes the results of this section.

Theorem 13. *Inequality (2) holds iff f is monotone and for some constant C*

$$g(x) = C + xf(x) - \int_0^x f(z)\,dz.$$

Proof. By Corollary 4, Lemma 5, and Theorem 12. □

5. DISCUSSION

Results closely corresponding to our uniqueness result (Theorem 12) were also presented in Krishna (2002) (and its second edition Krishna (2009)) and Börgers (2015). The customary name of these results is Revenue Equivalence. Krishna considers in Chapter 5 a setup with a seller that has one indivisible object to sell and n potential buyers, while Börgers considers in Chapter 2 a setup in which there is just one potential buyer. In Krishna (2009) the equivalent of our function f is defined as an integral representing the probability that a buyer gets the object, while in Börgers (2015) f corresponds to the probability of selling the object to the buyer. However, a close inspection of the proofs of these Revenue Equivalence results reveals that they do not depend on the actual form of f.

Further, ignoring the differences in the setup, the corresponding proofs in both books are from the mathematical point of view essentially the same. As the arguments in the latter one are more detailed, we discuss them here, but using our notation.

The proof of the corresponding result (Proposition 2.2) in Börgers (2015) is not based on the equivalent of our function f but instead deals, in Lemma 2.2, with the function u (representing utility) defined by

$$u(x) := xf(x) - g(x),$$

and states that for all x for which u is differentiable,

$$u'(x) = f(x).$$

Lemma 2.2 also establishes that the function u is monotone and convex. Then in Lemma 2.3 it is shown that

$$u(x) = u(0) + \int_0^x f(z)dz,$$

which is equivalent to (4) by taking $C = u(0)$, so the uniqueness result (Lemma 2.4 (Revenue Equivalence)) corresponding to our Theorem 12, follows.

Lemma 2.3 is a direct consequence of two results from Royden & Fitzpatrick (2010), namely, that convexity implies absolute continuity (a notion we leave undefined here) and that every absolutely continuous function is equal to the integral of its derivative.

Note, however, that the latter result (Theorem 10 of Royden & Fitzpatrick (2010)) is the Fundamental Theorem of Calculus (FTC) for the Lebesgue integral, a fact not mentioned in Krishna (2002) and in Börgers (2015) deducible only indirectly from footnote 2 in Chapter 2. So the proofs of the uniqueness result (the Revenue Equivalence) presented in Krishna (2002) and Börgers (2015) *crucially* rely on the Lebesgue theory of integration. In contrast, our proof is much more elementary: it does not rely on *any* form of integration and appeals only to the notion of derivative. Only the existence result (Lemma 5) relies on the Riemann integral. Having said this, apart from its complications, use of the Lebesgue integral yields a very efficient proof.

Both Krishna and Börgers establish appropriate Revenue Equivalence results for other mechanisms. In particular, Börgers considers in Chapters 3 and

4 of Börgers (2015) Bayesian mechanisms and dominant mechanisms, each time for n buyers. In both cases he establishes the corresponding Revenue Equivalence result (Lemma 3.4 and Proposition 4.2) by explaining that the reasoning provi/ded in Chapter 2 can be repeated.

Given that in both setups the crucial inequalities are the counterparts of (2) (considered separately for each buyer), it follows that both results can be alternatively proved using our Theorem 12. We conclude that our more elementary approach can be applied to other mechanisms than the single-parameter mechanism considered in Section 2.

Finally, we show that a generalization of Theorem 13 allows one to provide alternative, more elementary proofs of Revenue Equivalence for two other types of auctions, considered in Krishna (2009) in Chapters 14 and 16. These auctions are concerned with multiple objects, leading to functions f and g having to deal with vectors.

We use the following notation. For functions $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ and $g : \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$ we introduce the functions $f_\mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ and $g_\mathbf{x} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by

$$f_\mathbf{x}(t) = f(t\mathbf{x}) \cdot \mathbf{x},$$

where $\cdot$ is the inner product (dot product), and

$$g_\mathbf{x}(t) = g(t\mathbf{x}).$$

We then say that $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ is *monotone* if for each $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$ the function $f_\mathbf{x}$ is monotone.

We now establish the following result which generalizes Theorem 13 to dimension $n > 1$. (Note that using the substitution $u(z) = zx$ we have $\int_0^x f(u)\,du = \int_0^1 f(zx)x\,dz$.)

Theorem 14. For two functions $f : \mathbb{R}^n_{\geq 0} \to \mathbb{R}^n_{\geq 0}$ and $g : \mathbb{R}^n_{\geq 0} \to \mathbb{R}_{\geq 0}$ the inequality

$$\forall \mathbf{x}, \mathbf{y} : g(\mathbf{y}) - g(\mathbf{x}) \geq (f(\mathbf{y}) - f(\mathbf{x})) \cdot \mathbf{x} \tag{10}$$

holds iff f is monotone and for some constant C

$$g(\mathbf{x}) = C + f(\mathbf{x}) \cdot \mathbf{x} - \int_0^1 f(z\mathbf{x}) \cdot \mathbf{x}\,dz. \tag{11}$$

Proof. First note that (10) holds iff

$$\forall \mathbf{x}, x, y : g_{\mathbf{x}}(y) - g_{\mathbf{x}}(x) \geq x(f_{\mathbf{x}}(y) - f_{\mathbf{x}}(x)), \tag{12}$$

since for each $\mathbf{x} \in \mathbb{R}^n_{\geq 0}$

$$\forall x, y : g_{\mathbf{x}}(y) - g_{\mathbf{x}}(x) = g(y\mathbf{x}) - g(x\mathbf{x})$$

and by linearity of the inner product

$$\forall x, y : x(f_{\mathbf{x}}(y) - f_{\mathbf{x}}(x)) = (f(y\mathbf{x}) - f(x\mathbf{x})) \cdot x\mathbf{x}.$$

By Theorem 9 (12) holds iff for all $\mathbf{x}$ the function $f_{\mathbf{x}}$ is monotone and for some constant $C_{\mathbf{x}}$

$$g_{\mathbf{x}}(x) = C_{\mathbf{x}} + x f_{\mathbf{x}}(x) - \int_0^x f_{\mathbf{x}}(z)\, dz. \tag{13}$$

But $C_{\mathbf{x}} = g_{\mathbf{x}}(0) = g(\mathbf{0})$, so the constant $C_{\mathbf{x}}$ does not depend on $\mathbf{x}$. Further, $g(\mathbf{x}) = g_{\mathbf{x}}(1)$, and $1 f_{\mathbf{x}}(1) = f(\mathbf{x}) \cdot \mathbf{x}$, so by putting $x = 1$ we see that(13) implies (11).

But also (11) implies (13), which can be seen by using (11) with $x\mathbf{x}$ instead of $\mathbf{x}$.

□

Let us return now to Krishna (2009). In Chapter 14 he studies multiunit auctions in which multiple identical objects are available. The relevant inequality (14.1) on page 204, capturing the expected payment in an equilibrium for a player, corresponds to (10). Theorem 14 then provides an alternative proof of his Proposition 14.1 stating that

> "The equilibrium payoff (and payment) functions of any bidder in any two multiunit auctions that have the same allocation rule differ at most by an additive constant."

Krishna's proof relies (implicitly) on the Lebesgue integral. We adopted from his proof the idea of reasoning about the functions $f_{\mathbf{x}}$ and $g_{\mathbf{x}}$. In Chapter 16 of his book he studies auctions in which one can bid for a set of nonidentical objects. They are called in the computer science literature combinatorial auctions. Krishna explains that "The proof is *identical* to that of Proposition 14.1." (italic used by the author). Consequently, our approach also yields an alternative proof of Revenue Equivalence for combinatorial auctions.

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