ARTICLE IN PRESS

Available online at www.sciencedirect.com

ScienceDirect

Indagationes Mathematicae xxx (xxxx) xxx

indagationes mathematicae

www.elsevier.com/locate/indag

On the Shannon capacity of sums and products of graphs

Alexander Schrijver

Centrum Wiskunde & Informatica and University of Amsterdam, Netherlands

Received 30 June 2022; received in revised form 23 August 2022; accepted 25 August 2022

Communicated by M. Mandjes

Abstract

Let $\Theta(G)$ denote the Shannon capacity of a graph G. We give an elementary proof of the equivalence, for any graphs G and H, of the inequalities $\Theta(G \sqcup H) > \Theta(G) + \Theta(H)$ and $\Theta(G \boxtimes H) > \Theta(G)\Theta(H)$. This was shown independently by Wigderson and Zuiddam (2022) using Kadison–Dubois duality and the Axiom of choice.

Keywords: Graph; Stable set number; Shannon capacity

1. Introduction

Let G be a graph. (All graphs in this paper are undirected and simple.) A *stable set* in G is a set of pairwise nonadjacent vertices. The *stable set number* $\alpha(G)$ is the maximum cardinality of a stable set in G.

The *sum* $G + H$ of graphs G and H is the disjoint union of G and H. Trivially,

$$\alpha(G + H) = \alpha(G) + \alpha(H). \tag{1}$$

The *strong product* GH of G and H is the graph with vertex set $V(G) \times V(H)$ where distinct (u, v) and (u', v') in $V(G) \times V(H)$ are adjacent if and only if (i) u and u' are equal or adjacent in G and (ii) v and v' are equal or adjacent in H.

Since sum and strong product are associative, commutative, and distributive (up to isomorphism), this makes the set of graphs to a commutative semiring, with unit the one-vertex graph K_1. Sum and strong product are often denoted by $G \sqcup H$ and $G \boxtimes H$, but the semiring notation $G + H$ and GH is more efficient here.

E-mail address: lex@cwi.nl.

https://doi.org/10.1016/j.indag.2022.08.009
0019-3577/

Please cite this article as: A. Schrijver, On the Shannon capacity of sums and products of graphs, Indagationes Mathematicae (2022), https://doi.org/10.1016/j.indag.2022.08.009.

As the cartesian product of stable sets in G and H is a stable set in GH we have

$$\alpha(GH) \geq \alpha(G)\alpha(H), \tag{2}$$

but strict inequality may occur, even if $G = H$ (for instance for $G = H = C_5$, the five-cycle). This made Shannon [5] define what is now called the *Shannon capacity* $\Theta(G)$ of a graph G:

$$\Theta(G) := \sup_{k\in\mathbb{N}} \alpha(G^k)^{1/k} = \lim_{k\to\infty} \alpha(G^k)^{1/k}. \tag{3}$$

The second equality in (3) follows from (2) and Fekete's lemma [2]. (In fact, Shannon introduced $\log \Theta(G)$ as the 'zero-error capacity' of the 'channel' G.)

Inequality (2) implies

$$\Theta(GH) \geq \Theta(G)\Theta(H). \tag{4}$$

Haemers [3] (disproving a conjecture of Shannon [5]) gave examples of graphs G, H with strict inequality in (4). In fact, Haemers showed that the 'Schläfli graph' G satisfies $\Theta(G)\Theta(\overline{G}) < |V(G)| \leq \alpha(G\overline{G}) \leq \Theta(G\overline{G})$. Here $\overline{G}$ is the graph complementary to G.

On the other hand, for each graph G and $n \in \mathbb{N}$:

$$\Theta(G^n) = \Theta(G)^n, \tag{5}$$

as follows directly from definition (3).

The value of $\Theta(C_5)$ was for a long time an open question, until Lovász [4] introduced the upper bound $\vartheta(G)$ on $\Theta(G)$ yielding $\Theta(C_5) = \sqrt{5}$. Since, as Lovász proved, $\vartheta(GH) = \vartheta(G)\vartheta(H)$ for all G, H, the Haemers examples imply that $\Theta(G) < \vartheta(G)$ may occur.

As for the sum, Shannon showed that for all graphs G and H one has

$$\Theta(G + H) \geq \Theta(G) + \Theta(H). \tag{6}$$

(For completeness, we give a proof in Section 2 below.) Shannon conjectured that for all G, H equality holds in (6). This was disproved by Alon [1], by displaying graphs G and H with $\Theta(G + H) > \Theta(G) + \Theta(H)$. In fact, strict inequality holds for any G and H that satisfy $\Theta(GH) > \Theta(G)\Theta(H)$, as follows (using (5) and (6)) from

$$\begin{aligned}\Theta(G + H)^2 &= \Theta((G + H)^2) = \Theta(G^2 + 2GH + H^2) \geq \Theta(G^2) + 2\Theta(GH) + \Theta(H^2)\\ &= \Theta(G)^2 + 2\Theta(GH) + \Theta(H)^2 > \Theta(G)^2 + 2\Theta(G)\Theta(H) + \Theta(H)^2\\ &= (\Theta(G) + \Theta(H))^2.\end{aligned} \tag{7}$$

So Haemers' counterexamples G, H for products also work for sums.

In this paper we give an elementary proof of the fact that for all G, H:

$$\Theta(GH) > \Theta(G)\Theta(H) \quad\Longleftrightarrow\quad \Theta(G + H) > \Theta(G) + \Theta(H). \tag{8}$$

(see Section 3). This was proved (independently) by Wigderson and Zuiddam [6], using Strassen's theory of asymptotic spectra (based on Kadison–Dubois duality) and the Axiom of choice.

More strongly, consider any $n \in \mathbb{N}$ and graphs $G_1, \ldots, G_n$. Then for any polynomial $p \in \mathbb{N}[x_1, \ldots, x_n]$ one has

$$\Theta(p(G_1, \ldots, G_n)) \geq p(\Theta(G_1), \ldots, \Theta(G_n)). \tag{9}$$

(This follows from (6) and (4).) Now if equality holds in (9) for one polynomial p in which each of the variables $x_1, \ldots, x_n$ occurs, then equality holds in (9) for all polynomials p. For this result of Wigderson and Zuiddam [6] we also give an elementary proof in Section 4.

2. Shannon's inequality

For self-containedness of this paper, we give a proof of Shannon's inequality:

Theorem 1 (*Shannon [5]*). $\Theta(G+H) \geq \Theta(G)+\Theta(H)$.

Proof. For all $n, t \geq 1$, using (1) and (2):

$$\begin{aligned}\alpha((G+H)^n) &= \alpha(\sum_{k=0}^{n}\tbinom{n}{k}G^kH^{n-k}) = \sum_{k=0}^{n}\tbinom{n}{k}\alpha(G^kH^{n-k}) \\ &\geq \sum_{k=0}^{n}\tbinom{n}{k}\alpha(G^k)\alpha(H^{n-k}) \geq \sum_{k=0}^{n}\tbinom{n}{k}\alpha(G^t)^{\lfloor k/t\rfloor}\alpha(H^t)^{\lfloor (n-k)/t\rfloor} \\ &\geq \sum_{k=0}^{n}\tbinom{n}{k}\alpha(G^t)^{k/t}\alpha(G^t)^{-1}\alpha(H^t)^{(n-k)/t}\alpha(H^t)^{-1} \\ &= (\alpha(G^t)^{1/t}+\alpha(H^t)^{1/t})^n\alpha(G^t)^{-1}\alpha(H^t)^{-1}. \end{aligned} \tag{10}$$

So for each $t \geq 1$:

$$\begin{aligned}\Theta(G+H) &= \sup_{n\in\mathbb{N}}\alpha((G+H)^n)^{1/n} \geq \sup_{n\in\mathbb{N}}(\alpha(G^t)^{1/t}+\alpha(H^t)^{1/t})\alpha(G^t)^{-1/n}\alpha(H^t)^{-1/n} \\ &= \alpha(G^t)^{1/t}+\alpha(H^t)^{1/t}. \end{aligned} \tag{11}$$

So letting $t \to \infty$ gives the theorem. ∎

(Note that this proof also applies if α is replaced by any superadditive and supermultiplicative graph function.)

3. Equivalence of $\Theta(GH) > \Theta(G)\Theta(H)$ and $\Theta(G+H) > \Theta(G)+\Theta(H)$

Theorem 2. $\Theta(GH) > \Theta(G)\Theta(H)$ *if and only if* $\Theta(G+H) > \Theta(G)+\Theta(H)$.

Proof. Necessity follows from (7). To see sufficiency, assume $\Theta(GH) \leq \Theta(G)\Theta(H)$. Then for all $i, j \in \mathbb{N}$, using (4) and (5):

$$\begin{aligned}\Theta(G^iH^j)\Theta(G)^j\Theta(H)^i &= \Theta(G^iH^j)\Theta(G^j)\Theta(H^i) \leq \Theta((GH)^{i+j}) \\ &= \Theta(GH)^{i+j} \leq \Theta(G)^{i+j}\Theta(H)^{i+j}. \end{aligned} \tag{12}$$

So $\Theta(G^iH^j) \leq \Theta(G)^i\Theta(H)^j$. Hence for each n, using (1):

$$\begin{aligned}\alpha((G+H)^n) &= \alpha(\sum_{k=0}^{n}\tbinom{n}{k}G^kH^{n-k}) = \sum_{k=0}^{n}\tbinom{n}{k}\alpha(G^kH^{n-k}) \\ &\leq \sum_{k=0}^{n}\tbinom{n}{k}\Theta(G^kH^{n-k}) \leq \sum_{k=0}^{n}\tbinom{n}{k}\Theta(G)^k\Theta(H)^{n-k} = (\Theta(G)+\Theta(H))^n. \end{aligned} \tag{13}$$

Taking nth roots and letting $n \to \infty$ gives $\Theta(G+H) \leq \Theta(G)+\Theta(H)$. ∎

4. Extension to polynomials

We also give an elementary proof of the following extension of Theorem 2, that was shown by Wigderson and Zuiddam [6] using Kadison–Dubois duality and the Axiom of choice.

For given graphs $G_1, \ldots, G_n$, define

$$\mathcal{P} = \{p \in \mathbb{N}[x_1, \ldots, x_n] \mid \Theta(p(G_1, \ldots, G_n)) = p(\Theta(G_1), \ldots, \Theta(G_n))\}. \tag{14}$$

Theorem 3. *Let $G_1, \ldots, G_n$ be graphs with at least one vertex. Then $\mathcal{P} = \mathbb{N}[x_1, \ldots, x_n]$ if and only if $\mathcal{P}$ contains a polynomial in which all variables $x_1, \ldots, x_n$ occur.*

Proof. Necessity being trivial, we prove sufficiency. Let $\underline{G} := (G_1, \ldots, G_n)$ and $\Theta(\underline{G}) := (\Theta(G_1), \ldots, \Theta(G_n))$. So $p(\Theta(\underline{G})) \le \Theta(p(\underline{G}))$ for any polynomial $p \in \mathbb{N}[x_1, \ldots, x_n]$.

We first show that for $p, q \in \mathbb{N}[x_1, \ldots, x_n]$:

$$\text{if } p + q \in \mathcal{P}, \text{ then } p \in \mathcal{P}. \tag{15}$$

Indeed,

$$\begin{aligned}\Theta((p+q)(\underline{G})) &= (p+q)(\Theta(\underline{G})) = p(\Theta(\underline{G})) + q(\Theta(\underline{G}))\\ &\le \Theta(p(\underline{G})) + \Theta(q(\underline{G})) \le \Theta(p(\underline{G}) + q(\underline{G})) = \Theta((p+q)(\underline{G})).\end{aligned} \tag{16}$$

Hence we have equality throughout, implying $\Theta(p(\underline{G})) = p(\Theta(\underline{G}))$. This proves (15).

Similarly,

$$\text{if } pq \in \mathcal{P} \text{ and } q \ne 0, \text{ then } p \in \mathcal{P}. \tag{17}$$

Indeed,

$$\begin{aligned}\Theta((pq)(\underline{G})) &= (pq)(\Theta(\underline{G})) = p(\Theta(\underline{G}))q(\Theta(\underline{G}))\\ &\le \Theta(p(\underline{G}))\Theta(q(\underline{G})) \le \Theta(p(\underline{G})q(\underline{G})) = \Theta((pq)(\underline{G})).\end{aligned} \tag{18}$$

Hence we have equality throughout, implying $\Theta(p(\underline{G})) = p(\Theta(\underline{G}))$. This proves (17).

Moreover, for $p \in \mathbb{N}[x_1, \ldots, x_n]$ and $k \in \mathbb{N}$,

$$\text{if } p \in \mathcal{P} \text{ then } p^k \in \mathcal{P}. \tag{19}$$

Indeed, if $p \in \mathcal{P}$, then

$$\Theta(p^k(\underline{G})) = \Theta(p(\underline{G})^k) = (\Theta(p(\underline{G})))^k = (p(\Theta(\underline{G})))^k = (p^k(\Theta(\underline{G}))), \tag{20}$$

proving (19).

Now let $p \in \mathcal{P}$ with each $x_1, \ldots, x_n$ occurring in p. Then for some $k \in \mathbb{N}$, p^k contains as term a monomial q in which each variable occurs at least once. As $p^k \in \mathcal{P}$ by (19), we know by (15) that $q \in \mathcal{P}$. Now for each monomial μ in $\mathbb{N}[x_1, \ldots, x_n]$ there exists a large enough N such that μ is a divisor of q^N. So by (14), each monomial belongs to $\mathcal{P}$.

Now consider any polynomial $r = q_1 + \cdots + q_t$ in $\mathbb{N}[x_1, \ldots, x_n]$, where each q_i is a monomial. Then for each $i_1, \ldots, i_t \in \mathbb{N}$, $\mu := \prod_{j=1}^t q_j^{i_j}$ is a monomial, implying

$$\Theta(\prod_{j=1}^t q_j(\underline{G})^{i_j}) = \Theta(\mu(\underline{G})) = \mu(\Theta(\underline{G})) = \prod_{j=1}^t q_j(\Theta(\underline{G}))^{i_j}. \tag{21}$$

This implies, for each $k \in \mathbb{N}$, using the additivity (1) of the function α:

$$\alpha(r(\underline{G})^k) = \alpha((\sum_{j=1}^t q_j(\underline{G}))^k) = \alpha(\sum_{\substack{i_1,\ldots,i_t \in \mathbb{N}\\ i_1+\cdots+i_t=k}} \binom{k}{i_1,\ldots,i_t} \prod_{j=1}^t q_j(\underline{G})^{i_j})$$

$$= \sum_{\substack{i_1,\ldots,i_t\in\mathbb{N}\\ i_1+\cdots+i_t=k}} \binom{k}{i_1,\ldots,i_t} \alpha\Big(\prod_{j=1}^{t} q_j(\underline{G})^{i_j}\Big)$$

$$\leq \sum_{\substack{i_1,\ldots,i_n\in\mathbb{N}\\ i_1+\cdots+i_t=k}} \binom{k}{i_1,\ldots,i_t} \Theta\Big(\prod_{j=1}^{t} q_j(\underline{G})^{i_j}\Big)$$

$$= \sum_{\substack{i_1,\ldots,i_n\in\mathbb{N}\\ i_1+\cdots+i_t=k}} \binom{k}{i_1,\ldots,i_t} \prod_{j=1}^{t} q_j(\Theta(\underline{G}))^{i_j}$$

$$= \Big(\sum_{j=1}^{t} q_j(\Theta(\underline{G}))\Big)^k = (r(\Theta(\underline{G})))^k. \tag{22}$$

Taking kth roots and letting $k \to \infty$ we obtain $\Theta(r(\underline{G})) \leq r(\Theta(\underline{G}))$. So $r \in \mathcal{P}$. ■

Acknowledgments

We thank an anonymous referee and Monique Laurent for useful comments and suggestions.

References

[1] N. Alon, The Shannon capacity of a union, Combinatorica 18 (1998) 301–310.
[2] M. Fekete, Über die Verteilung der Wurzeln bei gewissen algebraischen Gleichungen mit ganzzahligen Koeffizienten, Math. Z. 17 (1923) 228–249.
[3] W. Haemers, On some problems of Lovász concerning the Shannon capacity of a graph, IEEE Trans. Inf. Theory IT 25 (1979) 231–232.
[4] L. Lovász, On the Shannon capacity of a graph, IEEE Trans. Inf. Theory IT 25 (1979) 1–7.
[5] C.E. Shannon, The zero error capacity of a noisy channel, IRE Trans. Inf. Theory IT 2 (1956) 8–19.
[6] A. Wigderson, J. Zuiddam, Asymptotic spectra: Theory, applications and extensions, 2022, manuscript.