TRANSCENDENCE IN FIELDS OF POSITIVE CHARACTERISTIC

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VOORWOORD

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CONTENTS

GENERAL INTRODUCTION AND SUMMARY

In the theory of transcendental numbers one starts with a field K with a subfield k and one studies properties of those elements of K which are transcendental over k. In complex transcendental number theory, the most common case, one takes for K the field $\mathbb{C}$ of complex numbers and for the subfield k its prime field, i.e. the field $\mathbb{Q}$ of rational numbers. Of the various properties enjoyed by $\mathbb{C}$ we emphasize the following two:

(i) the valuation of $\mathbb{C}$ is *archimedean,*

(ii) the characteristic of $\mathbb{C}$ is *zero.*

In p-adic transcendental number theory the situation has changed with respect to property (i): here one takes for K an algebraically closed, with respect to its valuation complete field $\mathbb{C}_p$, which is an extension of the field $\mathbb{Q}_p$ of p-adic numbers. For k one takes again the prime field $\mathbb{Q}$.

In this thesis we move a step further from the classical case; not only will our field K be provided with a non-archimedean valuation, but moreover, its characteristic will be positive.

Now new difficulties arise, which did not occur in the change from the complex to the p-adic case. We will illustrate this by an example.

One of the most famous theorems of classical transcendental number theory is the theorem of Gelfond and Schneider, which says that if α and β are non-zero algebraic numbers, $\alpha \neq 1$, β not rational, then α^β is transcendental. This is in fact a theorem on the exponential function and its inverse, the logarithm, for α^β is defined as $\exp(\beta \log \alpha)$. If one sets out to prove this theorem in the p-adic case the *definition* of α^β presents no difficulties. The exponential function is again defined by the power series $\sum_{n=0}^{\infty} \frac{z^n}{n!}$, the only difference being that in the p-adic case this series is not convergent for all z. But in our case of positive characteristic this definition loses its meaning and it is not at all clear what we must regard as the equivalent of α^β.

In this thesis k will be the field $\mathbb{F}_q(X)$ of rational functions in one variable over a finite field $\mathbb{F}_q$ and K will be an algebraically closed, complete extension of k, called Φ. L. Carlitz indicated in 1935 a function ψ, which might be regarded as the equivalent of the exponential function and L.I. Wade proved in 1941 the Gelfond-Schneider theorem for this function.

In chapter I we start with the construction of Φ and a study of the Carlitz-ψ-function, which we introduce in a way different from Carlitz'. Further we define the operators Δ_k for linear functions and we introduce the class of functions J_n, which may be regarded as analogues of Bessel functions. The main section, section 5, of the first chapter is devoted to analysis on Φ. Mainly we follow the work of U. Güntzer (1966), but the introduction of the concept of hooking-radius so fundamental in the study of the occurrence and location of zeros, is a different one. The Maximum Modulus Theorem and the Product Formula for Entire Functions are both needed for the Siegel-Schneider method in chapter IV.

Chapter II gives a survey of known results on transcendence in Φ.

In chapter III we introduce the concept of transcendence measure in Φ and we give an analogue of P.L. Cijsouw's result on series for which a certain gap-condition is fulfilled. Moreover, with the same method, we generalize a result of S.M. Spencer (1952).

In chapter IV we define the class of E-functions and we prove that if $\alpha,\beta \in \Phi$, $\alpha \neq 0$ and $\beta \notin \mathbb{F}_q(X)$ and if $f_1,f_2,\dots,f_n$ are E-functions such that $\Delta_k f_\nu$, $k \in \mathbb{N}$, $1 \leq \nu \leq n$ are polynomials in $f_1,f_2,\dots,f_n$ satisfying certain conditions, then at least one of the 2n+1 elements $\beta,f_1(\alpha),f_2(\alpha),\dots,f_n(\alpha),f_1(\alpha\beta),f_2(\alpha\beta),\dots,f_n(\alpha\beta)$ is transcendental over $\mathbb{F}_q(X)$. This theorem contains, among others, the Wade analogue of the Gelfond-Schneider theorem.

0. NOTATIONS AND PRELIMINARIES

In this thesis we adopt the following notations:

$\emptyset$	The empty set.
$A\backslash B$	The set of elements which are contained in the set A but not in the set B.
$f: A \to B$	A function f which adjoins to every element of the set A an element of the set B; A is called the domain of f.
$f\|V$	The restriction of f to a subset V of the domain of f.
$g\circ f$	The composition of the functions $f: A \to B$ and $g: B \to C$.
$\mathbb{N}$	The set of natural numbers.
$\mathbb{N}^0$	$\mathbb{N} \cup \{0\}$.
$\mathbb{Z}$	The ring of rational integers.
$\mathbb{Q}$	The field of rational numbers.
$\mathbb{R}$	The field of real numbers.
$\mathbb{C}$	The field of complex numbers.
$\mathbb{F}_q$	The finite field of q elements, where $q = p^n$ for a certain $n \in \mathbb{N}$ and a prime $p \in \mathbb{N}$.
K^*	The multiplicative group formed by the non-zero elements of the field K.
$\mathcal{R}[t_1,t_2,\ldots,t_n]$	The ring of polynomials in the n variables $t_1,t_2,\ldots,t_n$ over a commutative ring $\mathcal{R}$ with identity.
$K(t)$	The field of rational functions in t with coefficients in a field K.
$\square$	The end of a proof.

As usual an empty sum has to be taken equal to zero and an empty product equal to one.

For convenience of the reader we formulate some standard notions and theorems, used throughout this thesis.

0.1. <u>DEFINITION</u>. Let $\mathcal{R}$ be a commutative ring with identity and let $P,Q \in \mathcal{R}[t]$. Then P is called a *divisor* of Q, notation $P|Q$, if there exists an $R \in \mathcal{R}[t]$ such that $Q = PR$.

P is called *irreducible* if P is not a unit and has no divisors in $\mathcal{R}[t]$ other than units and associates of P.

P is called *monic* if the leading coefficient of P is the identity of $\mathcal{R}$.

P is called *primitive* if its coefficients have no common divisor in $\mathcal{R}$ (other than units).

0.2. DEFINITION. Let K_1 and K_2 be fields with a common subfield k. A monomorphism $\sigma\colon K_1 \to K_2$ for which

$$\sigma(\alpha) = \alpha, \qquad \alpha \in k$$

is called a k-*monomorphism*.

0.3. THEOREM. *Let $\mathcal{R}$ be a commutative ring with identity. Every symmetric polynomial* P *from* $\mathcal{R}[t_1,t_2,\ldots,t_n]$ *of degree* m *can be written uniquely in the form*

$$\sum c_{\lambda_1\ldots\lambda_n}\, \sigma_1^{\lambda_1}\sigma_2^{\lambda_2}\ldots\sigma_n^{\lambda_n}, \qquad c_{\lambda_1\ldots\lambda_n} \in \mathcal{R}$$

with

$$\lambda_1 + 2\lambda_2 +\ldots+ n\lambda_n \le m,$$

where $\sigma_1,\sigma_2,\ldots,\sigma_n$ *are the elementary symmetric functions of* $t_1,t_2,\ldots,t_n$.

PROOF. See e.g. VAN DER WAERDEN (1960), §29. □

0.4. COROLLARY. *Let $\mathcal{R}$ be a commutative ring with identity. Let* $P \in \mathcal{R}[t_1,t_2,\ldots,t_n]$ *be a symmetric polynomial. Let* $\beta_1,\beta_2,\ldots,\beta_n$ *be the zeros of a monic polynomial from* $\mathcal{R}[t]$. *Then*

$$P(\beta_1,\beta_2,\ldots,\beta_n) \in \mathcal{R}.$$

0.5. THEOREM. *Let $\mathcal{R}$ be a commutative ring with identity. If the polynomial* P *from theorem* 0.3 *is homogeneous of degree* k *in each* t_i, $1 \le i \le n$, *then, in the notation of theorem* 0.3, *we have*

$$\lambda_1 + \lambda_2 +\ldots+ \lambda_n \le k.$$

PROOF. See O. PERRON, Satz 69. □

0.6. COROLLARY. *Let* R *be a commutative ring with identity, let* $Q \in R[t]$ *be of degree* $N \geq 1$ *and let* $\beta_1,\beta_2,\ldots,\beta_N$ *denote the zeros of* Q. *Put*

$$Q(t) = A \prod_{i=1}^{N} (t-\beta_i), \qquad A \in R$$

and

$$D := A^{2N-2} \prod_{1\leq i<j\leq N} (\beta_i-\beta_j)^2.$$

Then $D \in R$.

PROOF. $\prod_{1\leq i<j\leq N} (\beta_i-\beta_j)^2$ is a homogeneous symmetric polynomial in $\beta_1,\beta_2,\ldots,\beta_N$ of total degree $N(N-1)$ and of degree $2(N-1)$ in β_i, $1 \leq i \leq N$. If $\sigma_1,\sigma_2,\ldots,\sigma_N$ denote the elementary symmetric functions of $\beta_1,\beta_2,\ldots,\beta_N$, then it follows from the theorems 0.3 and 0.5 that

$$\prod_{1\leq i<j\leq N} (\beta_i-\beta_j)^2 = \sum C_{\lambda_1\ldots\lambda_N} \sigma_1^{\lambda_1}\ldots\sigma_N^{\lambda_N}$$

with $C_{\lambda_1\ldots\lambda_N} \in R$ and $\lambda_1 + \lambda_2 +\ldots+ \lambda_N \leq 2(N-1)$. Since $A\sigma_i \in R$ it follows that $D \in R$. $\square$

For an introduction to finite fields we refer to I.T. ADAMSON (1964), Ch.IV. We shall frequently use the following

0.7. PROPERTY. For every finite field $\mathbb{F}_q$ one has

(0.7.1) $\quad \prod_{c\in\mathbb{F}_q^*} (t-c) = t^{q-1} - 1;$

(0.7.2) $\quad c^q = c, \qquad c \in \mathbb{F}_q.$

Finally we shall recall some notions and properties in algebraic extensions of a field.

0.8. DEFINITION. Let k,K be fields with $k \subset K$. Then $\alpha \in K$ is called *algebraic over* k if there exists a non-trivial polynomial $P \in k[t]$ such that $P(\alpha) = 0$.

If $\alpha \in K$ is not algebraic over k, then α is called *transcendental over* k.

0.9. THEOREM. *Let* k, K *be fields,* $k \subset K$ *and let* $\alpha \in K$ *be algebraic over* k. *Then there is one and (apart from an arbitrary unit factor) only one irreducible polynomial* $P \in k[t]$ *such that* $P(\alpha) = 0$. *There is exactly one such polynomial which is monic.*

PROOF. See O. ZARISKI and P. SAMUEL (1958), Ch.II §2, Cor.th.1. □

0.10. DEFINITION. Let k, K be fields, $k \subset K$, and let $\alpha \in K$ be algebraic over k. Then the degree of an irreducible polynomial $P \in k[t]$ for which $P(\alpha) = 0$ is called the *degree* of α (*with respect to* k).

0.11. DEFINITION. Let k be a field. Let $P \in k[t]$ be given by

$$P(t) = a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0, \quad a_i \in k.$$

The *derivative* P' of P is defined by

$$P'(t) := na_n t^{n-1} + (n-1)a_{n-1} t^{n-2} + \ldots + a_1,$$

where

$$na_n := \sum_{\nu=1}^{n} a_n.$$

0.12. DEFINITION. Let k, K be fields, $k \subset K$ and let $\alpha \in K$ be algebraic over k of degree n. The unique, monic, irreducible polynomial $P \in k[t]$ of degree n for which $P(\alpha) = 0$ is called the *minimal polynomial of* α *over* k.

An irreducible polynomial $P \in k[t]$ is called *separable* if $P' \neq 0$. An arbitrary polynomial $P \in k[t]$ is called separable if all its irreducible factors are separable.

The element $\alpha \in K$ is called *separable algebraic over* k if the minimal polynomial of α over k is separable.

The field K is called a (*separable*) *algebraic extension* of k if every element of K is (*separable*) *algebraic over* k.

0.13. THEOREM. *Let* k *be a field of characteristic* $p \neq 0$. *An irreducible polynomial* $P \in k[t]$ *is not separable if and only if it has the form*

$$P(t) = a_0 + a_1 t^p + a_2 t^{2p} + \ldots + a_n t^{np}, \quad n \geq 1,\ a_i \in k,\ a_0 \neq 0,\ a_n \neq 0.$$

<u>PROOF</u>. See I.T. ADAMSON (1964), Ch.I, th.5.3 or O. ZARISKI and P. SAMUEL P. SAMUEL (1958), Ch.II §5. □

0.14. <u>COROLLARY</u>. *Let* k,K *be fields of characteristic* $p \neq 0$, $k \subset K$. *If* $\alpha \in K$ *is algebraic over* k, *then there exists an* $e \in \mathbb{N}^0$ *such that* α^{p^e} *is separable algebraic over* k. *Moreover, for every* $n \in \mathbb{N}$ *with* $n > e$ *the element* α^{p^n} *is separable algebraic over* k.

CHAPTER I

INTRODUCTION

1. THE FIELD Φ

Let $\mathbb{F}_q$ be the finite field of q elements where q is a positive power of the prime number p. We denote the ring of polynomials with coefficients in $\mathbb{F}_q$ by $\mathbb{F}_q[X]$ and its quotient field by $\mathbb{F}_q(X)$.

For all non-zero elements of $\mathbb{F}_q[X]$ we define the (logarithmic) non-archimedean valuation dg by

$$\mathrm{dg}\, E := \text{degree of } E;$$

furthermore we put

$$\mathrm{dg}\, 0 := -\infty.$$

Hence for all non-zero elements $E \in \mathbb{F}_q[X]$ the valuation is a non-negative integer.

For the elements of $\mathbb{F}_q(X)$ we define the valuation as follows: if $E \neq 0$ and $F \neq 0$ are two elements of $\mathbb{F}_q[X]$, then

$$\mathrm{dg}\left(\frac{E}{F}\right) := \mathrm{dg}\, E - \mathrm{dg}\, F.$$

Clearly, if $\frac{E}{F} = \frac{E'}{F'}$, then $\mathrm{dg}\left(\frac{E}{F}\right) = \mathrm{dg}\left(\frac{E'}{F'}\right)$.

1.1. <u>THEOREM</u>. *The valuation* dg *of* $\mathbb{F}_q(X)$ *determines a Hausdorff topology on* $\mathbb{F}_q(X)$ *and for each* $\alpha \in \mathbb{F}_q(X)$ *a fundamental system of neighbourhoods of* α *is given by*

$$\{U(\alpha,n) \mid n = 1,2,\ldots\},$$

where

$$U(\alpha,n) = \{\beta \in \mathbb{F}_q(X) \mid dg(\alpha-\beta) < -n\}.$$

PROOF. See E. WEISS (1963), prop. 1-1-2 or E. ARTIN (1967), Ch. I th.4. □

1.2. DEFINITION. A sequence $\{\alpha_k\}_{k=1}^{\infty}$ of elements of $\mathbb{F}_q(X)$ is said to be *convergent* (in $\mathbb{F}_q(X)$) if an element $\alpha \in \mathbb{F}_q(X)$ exists such that the following condition is satisfied: for all $n \in \mathbb{N}$ there is a $k_0 \in \mathbb{N}$ such that for $k > k_0$

$$dg(\alpha-\alpha_k) < -n.$$

The sequence $\{\alpha_k\}_{k=1}^{\infty}$ is called a *Cauchy-sequence* if it satisfies the following condition: given any $n \in \mathbb{N}$, a $k_0 \in \mathbb{N}$ exists such that for each $k > k_0$, $\ell > k_0$

$$dg(\alpha_k-\alpha_\ell) < -n.$$

1.3. THEOREM. *Let* K *be a valued field. Then a unique valued field* L *exists such that*

(i) K *is a subfield of* L,

(ii) *the valuation on* L *restricted to* K *coincides with the valuation on* K,

(iii) *every Cauchy-sequence in* L *is convergent,*

(iv) K *is dense in* L.

PROOF. See E. WEISS (1963), th. 1-7-1 or E. ARTIN (1967), Ch. I §6. □

The valued field L is called the *completion* of the valued field K. A valued field is called *complete* if it coincides with its completion, i.e. when every Cauchy-sequence in it is convergent.

The completion of the field $\mathbb{F}_q(X)$ with its valuation dg will in the sequel be denoted by F, the valuation on F will also be denoted by dg. Note that $\{dg\ \alpha \mid \alpha \in F\} = \mathbb{Z} \cup \{-\infty\}$.

The next step is that we go over to the *algebraic closure* Ω of F. (For a definition of algebraic closure, see B.L. VAN DER WAERDEN (1960), §62.) To define a valuation on Ω, which coincides with dg on F we first consider finite extensions of F.

Let E be a finite extension of a field K of degree $[E:K] = n$.

We shall define the norm of an element of E with respect to K and we shall mention some properties which we shall need in the future. For a detailed exposition we refer to the book of O. ZARISKI & P. SAMUEL, Ch. II §10.

Let $\omega_1, \omega_2, \ldots, \omega_n$ be a basis for E over K, then for every $\alpha \in E$ and $i \in \{1,2,\ldots,n\}$ there exist $a_{ij} \in K$ such that

$$\alpha\omega_i = \sum_{j=1}^{n} a_{ij}\omega_j.$$

The $n \times n$-matrix $(a_{ij})_{i,j}$ will be denoted by (a) and the $n \times n$-unit-matrix by (e). The so-called *field polynomial of* α

$$\det(t(e)-(a))$$

is a monic polynomial of degree n in t which does not depend on the choice of the basis. It has the form

$$t^n + b_{n-1}\, t^{n-1} + \ldots + b_1 t + b_0,$$

where $b_i \in K$, $i = 0,1,\ldots,n-1$ and

$$b_0 = (-1)^n \det(a).$$

We define the *norm* $N_{E\to K}(\alpha)$ *of* $\alpha \in E$ *with respect to* K by

$$N_{E\to K}(\alpha) := \det(a) = (-1)^n b_0.$$

Hence $N_{E\to K}(\alpha)$ is an element of K. Furthermore we have

$$N_{E\to K}(b) = b^n, \qquad b \in K,$$

$$N_{E\to K}(\alpha\beta) = N_{E\to K}(\alpha).N_{E\to K}(\beta), \qquad \alpha,\beta \in E.$$

Finally, if L is a finite extension of E, then

$$N_{L\to K}(\beta) = N_{E\to K}(N_{L\to E}(\beta)), \qquad \beta \in L.$$

1.4. THEOREM. *Let* K *be a field complete with respect to a (logarithmic) non-archimedean valuation* dg *and let* E *be a finite extension of* K. *Then there exists a unique extension of the valuation* dg *on* K *to* E, *which will be denoted by* dg_E. *For all* $\alpha \in E$ *we have*

$$dg_E(\alpha) = \frac{dg(N_{E\to K}(\alpha))}{[E:K]} .$$

The field E *is complete with respect to this valuation* dg_E.

PROOF. See E. WEISS (1966), th.2-2-10 or E. ARTIN (1967), Ch. I, th.7. □

In view of theorem 1.4 we define $dg_\Omega\colon \Omega \to \mathbb{R} \cup \{-\infty\}$ by

$$dg_\Omega(\alpha) := dg_{F(\alpha)}(\alpha),$$

where $dg_{F(\alpha)}$ is the unique valuation of the finite extension $F(\alpha)$ of F, which extends dg. Then dg_Ω is a valuation of Ω.

1.5. PROPERTIES OF Ω. With $\mathbb{F}_q$, the field Ω has characteristic p. (Recall that q is a power of p.) Hence

(1.5.1) $$(u+v)^{p^n} = u^{p^n} + v^{p^n}, \quad n \in \mathbb{N}^0;\ u,v \in \Omega.$$

The valuation dg_Ω is non-archimedean. Therefore we have for all $u,v \in \Omega$

(1.5.2) $$dg_\Omega(uv) = dg_\Omega(u) + dg_\Omega(v)$$

and

(1.5.3) $$dg_\Omega(u+v) \leq \max\ (dg_\Omega(u), dg_\Omega(v)).$$

If $dg_\Omega(u) \neq dg_\Omega(v)$, we even have

$$dg_\Omega(u+v) = \max\ (dg_\Omega(u), dg_\Omega(v)).$$

The following example shows that the valued field Ω with dg_Ω as its valuation is not complete. Define the sequence $\{\alpha_n\}_{n=0}^{\infty}$ by

$$\alpha_n := \sum_{\nu=0}^{n} X^{-q^\nu+1/q^\nu}.$$

Since Ω is algebraically closed, $\alpha_n \in \Omega$. We have

$$dg_\Omega(\alpha_{n+1}-\alpha_n) = -q^{n+1} + \frac{1}{q^{n+1}}, \qquad n \in \mathbb{N}^0 .$$

Hence by (1.5.3) $\{\alpha_n\}_{n=0}^{\infty}$ is a Cauchy sequence in Ω. Suppose that the sequence is convergent. Call its limit α. Then according to corollary 0.14, there exists an $e \in \mathbb{N}$, such that α^{q^e} is separable algebraic over F.

It follows from the theorem of KRASNER (see e.g. E. ARTIN (1967), Ch. II th.8) that for n chosen sufficiently large

$$F(\alpha^{q^e}) \subset F(\alpha_n^{q^e})$$

and therefore

$$\alpha^{q^e} - \alpha_n^{q^e} \in F(\alpha_n^{q^e}).$$

Hence $\alpha^{q^e} - \alpha_n^{q^e}$ is algebraic over F of degree μ_n, say, and

$$\mu_n \mid q^{n-e} .$$

From the definition of dg_Ω we see that $\mu_n\, dg_\Omega(\alpha^{q^e}-\alpha_n^{q^e})$ equals the valuation of an element of F and hence

$$\mu_n\, dg_\Omega(\alpha^{q^e}-\alpha_n^{q^e}) \in \mathbb{Z} .$$

On the other hand we have

$$dg_\Omega(\alpha^{q^e}-\alpha_n^{q^e}) = -q^{n+1+e} + \frac{1}{q^{n+1-e}} .$$

Thus $q^{n+1-e} \mid \mu_n$, which contradicts $\mu_n \mid q^{n-e}$.

Our final step is that we form the completion Φ of Ω with respect to dg_{Ω}. (See theorem 1.3.) That this is the last step in the process of forming algebraic closures and completions follows from

1.6. THEOREM. *Φ is algebraically closed.*

PROOF. See E. ARTIN (1967), Ch. II, th.12. □

1.7. RECAPITULATION. Starting with $\mathbb{F}_q$ we have obtained a field Φ with a (logarithmic) non-archimedean valuation dg, i.e.

$$dg(uv) = dg\,u + dgv, \qquad u,v \in \Phi, \tag{1.7.1}$$

$$dg(u+v) \leq \max\,(dg\,u, dg\,v), \qquad u,v \in \Phi, \tag{1.7.2}$$

and if dg u $\neq$ dg v, then

$$dg(u+v) = \max\,(dg\,u, dg\,v).$$

From (1.7.2) it follows that

$$\{dg\,\alpha \mid \alpha \in \Phi\} = \mathbb{Q} \cup \{-\infty\}. \tag{1.7.3}$$

The field Φ is algebraically closed and complete with respect to the valuation dg. It contains the field $\mathbb{F}_q(X)$ and the valuation dg on Φ restricted to $\mathbb{F}_q(X)$ coincides with the valuation dg on $\mathbb{F}_q(X)$. Furthermore Φ has characteristic p; hence

$$(u+v)^{p^n} = u^{p^n} + v^{p^n}, \qquad n \in \mathbb{N}^0,\ u,v \in \Phi. \tag{1.7.4}$$

In view of the completeness of Φ and the fact that the valuation dg is non-archimedean, a series $\sum_{n=1}^{\infty} \alpha_n$, $\alpha_n \in \Phi$ is convergent if and only if $\lim_{n\to\infty} dg\,\alpha_n = -\infty$.

In this thesis the role played by the field Φ can be compared with that of $\mathbb{C}$ in the classical case; $\mathbb{F}_q[X]$ and $\mathbb{F}_q(X)$ take the part of $\mathbb{Z}$ and $\mathbb{Q}$ respectively.

1.8. THEOREM. *The field Φ is not locally compact.*

PROOF. Suppose Φ is locally compact. Then it follows from a theorem which can be found e.g. in N. BOURBAKI (1964), Chap. VI §5 no. 1, prop. 2, that the valuation of Φ is discrete. But this contradicts (1.7.3). □

2. THE FUNCTIONS ψ_k AND ψ

2.1. DEFINITION. We define the elements F_k, L_k ($k \in \mathbb{N}^0$) of $\mathbb{F}_q[X]$ as follows

$$F_k := \prod_{j=0}^{k-1} (X^{q^k} - X^{q^j}), \qquad k = 1,2,\ldots\ ;$$

$$F_0 := 1,$$

$$L_k := \prod_{j=1}^{k} (X^{q^j} - X), \qquad k = 1,2,\ldots\ ;$$

$$L_0 := 1.$$

2.2. REMARK. For $k \geq 1$ we have the relations

(a) $$F_k = (X^{q^k} - X) F_{k-1}^q,$$

(b) $$L_k = (X^{q^k} - X) L_{k-1}.$$

Furthermore, we note that for $k \geq 0$

$$\mathrm{dg}\ F_k = kq^k,$$

$$\mathrm{dg}\ L_k = \frac{q}{q-1}(q^k - 1).$$

In the following we shall see that F_k can be compared with $k!$ in the classical case.

2.3. DEFINITION. For $k \in \mathbb{N}^0$ the polynomial $\psi_k \in \mathbb{F}_q[X][t]$ is defined by

$$\psi_k(t) := \prod_{\substack{E\in\mathbb{F}_q[X] \\ \mathrm{dg}E<k}} (t-E).$$

Moreover, we put

$$\psi_{-1}(t) := 1.$$

N.B. $\psi_0(t) = t$.

The polynomials ψ_k were introduced by L. CARLITZ (1935). In the following we shall mention some of his results, which we shall need in this thesis.

2.4. THEOREM. (Carlitz) *The polynomial* ψ_k, $k \in \mathbb{N}^0$ *has the following representation*

$$\psi_k(t) = \sum_{j=0}^{k} (-1)^j \frac{F_k}{L_j^{q^{k-j}} F_{k-j}} t^{q^{k-j}}. \tag{2.4.1}$$

Furthermore, the function ψ_k *has the properties:*

$$\psi_k(t+u) = \psi_k(t) + \psi_k(u), \quad t,u \in \Phi, \tag{2.4.2}$$

$$\psi_k(ct) = c\psi_k(t), \quad c \in \mathbb{F}_q, \ t \in \Phi, \tag{2.4.3}$$

$$\psi_k(Xt) - X\psi_k(t) = (X^{q^k}-X)\psi_{k-1}^q(t), \quad t \in \Phi, \tag{2.4.4}$$

$$\psi_k(X^k) = F_k. \tag{2.4.5}$$

PROOF. For $k = 0$ the theorem is trivial.
Suppose the formulae are correct for $k = 0,1,\ldots,\kappa$. From the definition of $\psi_{\kappa+1}$ we get

$$\begin{aligned}\psi_{\kappa+1}(t) &= \prod_{\mathrm{dg}E<\kappa+1} (t-E) = \\ &= \Big(\prod_{\mathrm{dg}E<\kappa} (t-E)\Big) \prod_{c\in\mathbb{F}_q^*} \prod_{\mathrm{dg}E<\kappa} (t-cX^\kappa-E) \\ &= \psi_\kappa(t) \prod_{c\in\mathbb{F}_q^*} \psi_\kappa(t-cX^\kappa).\end{aligned}$$

From (2.4.2), (2.4.3) and (2.4.5) for $k = \kappa$ we have

$$\psi_\kappa(t-cX^\kappa) = \psi_\kappa(t) - cF_\kappa.$$

Since

$$\prod_{c\in\mathbb{F}_q^*} (t-c) = t^{q-1} - 1,$$

we have

$$\psi_{\kappa+1}(t) = \psi_\kappa(t)\ \{\psi_\kappa^{q-1}(t) - F_\kappa^{q-1}\} =$$

$$= \psi_\kappa^q(t) - F_\kappa^{q-1}\psi_\kappa(t).$$

Now using (2.4.1) for $k = \kappa$ and remark 2.2a,b, we obtain formula (2.4.1) for $k = \kappa + 1$ by a straightforward computation. Using (1.6.3), the formulae (2.4.2) and (2.4.3) for $k = \kappa + 1$ follow immediately from (2.4.1).

It only remains to prove (2.4.4) and (2.4.5) for $k = \kappa + 1$. Using remark 2.2(a), it follows from (2.4.1) for $k = \kappa + 1$ that

$$\psi_{\kappa+1}(Xt) - X\psi_{\kappa+1}(t) = \sum_{j=0}^{\kappa} (-1)^j \frac{F_{\kappa+1}}{L_j^{q^{\kappa+1-j}} F_{\kappa+1-j}} (X^{q^{\kappa+1-j}} - X)\, t^{q^{\kappa+1-j}}$$

$$= \sum_{j=0}^{\kappa} (-1)^j \frac{(X^{q^{\kappa+1}} - X) F_\kappa^q}{L_j^{q^{\kappa+1-j}} F_{\kappa-j}^q}\, t^{q^{\kappa+1-j}}$$

$$= (X^{q^{\kappa+1}} - X)\psi_\kappa^q(t).$$

Substituting $t = X^\kappa$ in this formula gives

$$\psi_{\kappa+1}(X^{\kappa+1}) = (X^{q^{\kappa+1}} - X)\psi_\kappa^q(X^\kappa) = (X^{q^{\kappa+1}} - X)F_\kappa^q = F_{\kappa+1}. \quad \square$$

2.5. THEOREM. *For* $A \in \mathbb{F}_q[X]$ *and* $k \in \mathbb{N}^0$ *we have*

$$\frac{\psi_k(A)}{F_k} \in \mathbb{F}_q[X].$$

PROOF. If

$$A = a_m X^m + a_{m-1}X^{m-1} + \ldots + a_1 X + a_0, \qquad a_i \in \mathbb{F}_q,\ i = 0,1,\ldots,m,$$

we have from formulae (2.4.2) and (2.4.3)

$$\psi_k(A) = \sum_{i=0}^{m} a_i \, \psi_k(X^i).$$

Hence it is sufficient to prove that

$$(2.5.1) \qquad \frac{\psi_k(X^i)}{F_k} \in \mathbb{F}_q[X], \qquad i,k \in \mathbb{N}^0.$$

First we remark that for $i \in \mathbb{N}^0$

$$(2.5.2) \qquad \frac{\psi_0(X^i)}{F_0} = X^i \in \mathbb{F}_q[X].$$

Furthermore we have by the definition of ψ_k

$$(2.5.3) \qquad \psi_k(X^i) = 0, \qquad k \in \mathbb{N};\ i = 0,1,\ldots,k-1.$$

Hence (2.5.1) is satisfied for $k \in \mathbb{N}^0$, $i = 0$ and $i \in \mathbb{N}^0$, $k = 0$.

Suppose we have proved (2.5.1) for $k \in \mathbb{N}^0$ and $i = 0,1,\ldots,\nu-1$. From relation (2.4.4) and remark 2.2a we have for $k \in \mathbb{N}$

$$(2.5.4) \qquad \frac{\psi_k(X^\nu)}{F_k} = X\,\frac{\psi_k(X^{\nu-1})}{F_k} + \left(\frac{\psi_{k-1}(X^{\nu-1})}{F_{k-1}}\right)^q .$$

Now (2.5.1) for $k \in \mathbb{N}$, $i = \nu$ follows from (2.5.4) by the induction hypothesis. $\square$

2.6. <u>REMARK</u>. It is easily verified that for $A \in \mathbb{F}_q[X]$, $dg\,A \geq k$ we have

$$dg\,\frac{\psi_k(A)}{F_k} = (dgA-k)q^k.$$

2.7. <u>REMARK</u>. The polynomial $\frac{\psi_k}{F_k}$ bears some resemblance to the polynomial $\binom{z}{k} = \frac{z(z-1)\ldots(z-k+1)}{k!}$ in the real case; apart from theorem 2.5 we mention relation (2.5.4) and the relation

$$\frac{\psi_k(X^m)}{F_k} = \frac{\psi_k(X^m)}{\psi_k(X^k)} = \prod_{dgE<k} \frac{X^m-E}{X^k-E}.$$

2.8. DEFINITION. The *Carlitz-ψ-function* $\psi\colon \Phi \to \Phi$ is defined by

$$\psi(t) := \sum_{j=0}^{\infty} (-1)^j \frac{t^{q^j}}{F_j} .$$

(Note that in view of $\operatorname{dg} F_j = jq^j$, the sequence converges for every $t \in \Phi$.)

Let $u \in \Phi$ be a solution of the equation

$$(2.8.1) \qquad t^{q-1} = X^q - X.$$

This number u will be fixed in the sequel.

For $c \in \mathbb{F}_q^*$ we have $c^{q-1} = 1$, hence cu is also a solution of the equation above. Since (2.8.1) has exactly $q-1$ solutions, the complete set solutions of (2.8.1) is given by $\{cu \mid c \in \mathbb{F}_q^*\}$. Furthermore

$$\operatorname{dg} cu = \operatorname{dg} u = \frac{q}{q-1} .$$

2.9. LEMMA. *The sequence* $\{\frac{u^{q^k}}{L_k}\}_{k\in\mathbb{N}^0}$ *is convergent in* Φ.

PROOF. From the definition of u and remark 2.2 it follows that

$$\frac{u^{q^{k+1}}}{L_{k+1}} - \frac{u^{q^k}}{L_k} = \frac{u^{q^k}}{L_k}\left(\frac{(X^q-X)^{q^k}}{X^{q^{k+1}}-X} - 1\right)$$
$$= -\frac{u^{q^k}}{L_k}\left(\frac{X^{q^k}-X}{X^{q^{k+1}}-X}\right)$$

and that

$$(2.9.1) \qquad \operatorname{dg} \frac{u^{q^k}}{L_k} = \frac{q}{q-1} .$$

Hence for arbitrary $j \in \mathbb{N}$ we have

$$\operatorname{dg}\left(\frac{u^{q^{k+j}}}{L_{k+j}} - \frac{u^{q^k}}{L_k}\right) \le \max_{0\le\nu<j} \operatorname{dg}\left(\frac{u^{q^{k+\nu+1}}}{L_{k+\nu+1}} - \frac{u^{q^{k+\nu}}}{L_{k+\nu}}\right) = \frac{q}{q-1} - q^k(q-1).$$

So the sequence is a Cauchy-sequence. Since Φ is complete, it is convergent. □

2.10. DEFINITION. The element $\xi \in \Phi$ is defined by

$$(2.10.1) \qquad \xi := \lim_{k\to\infty} \frac{u^{q^k}}{L_k} .$$

Note that it follows from (2.9.1) that $\mathrm{dg}\ \xi = \frac{q}{q-1}$.

2.11. THEOREM. *The function* ψ *has the following properties*:

(a) *for every* $t,v \in \Phi$

$$\psi(t+v) = \psi(t) + \psi(v),$$

(b) *for every* $t \in \Phi$, $c \in \mathbb{F}_q$

$$\psi(ct) = c\psi(t),$$

(c) *for every* $t \in \Phi$

$$\psi(Xt) = X\psi(t) - \psi^q(t),$$

(d) *for every* $t \in \Phi$

$$\psi(\xi t) = \lim_{k\to\infty} (-1)^k\ u(X^q-X)^{\frac{q^k-1}{q-1}} \frac{\psi_k(t)}{F_k} .$$

PROOF. The properties (a) and (b) follow immediately from the definition of ψ.

(c) From definition 2.8 and remark 2.2a we have

$$X\psi(t) - \psi^q(t) = \sum_{j=0}^{\infty} (-1)^j \frac{Xt^{q^j}}{F_j} - \sum_{j=1}^{\infty} (-1)^{j-1} \frac{t^{q^j}(X^{q^j}-X)}{F_j}$$

$$= Xt + \sum_{j=1}^{\infty} (-1)^j \frac{t^{q^j}X^{q^j}}{F_j} = \psi(Xt).$$

(d) Let $t \in \Phi$, t fixed. From the definitions 2.8 and 2.10 and property 2.11a it follows that for every $N \in \mathbb{N}$ there exists a $k_0 \in \mathbb{N}$, $k_0 = k_0(N,t)$, such that

$$(2.11.1) \qquad \mathrm{dg}\left(\psi(t\xi)-\psi\left(t\ \frac{u^{q^k}}{L_k}\right)\right) < -N,\ k > k_0.$$

We write

$$\psi\left(t\,\frac{u^{q^k}}{L_k}\right) = S_1(t) + S_2(t),$$

where

$$S_1(t) := \sum_{j=0}^{k} \frac{(-1)^j}{F_j}\, t^{q^j}\, \frac{u^{q^{k+j}}}{L_k^{q^j}}$$

and

$$S_2(t) := \sum_{j=k+1}^{\infty} \frac{(-1)^j}{F_j}\, t^{q^j}\, \frac{u^{q^{k+j}}}{L_k^{q^j}}\, .$$

From (2.8.1) it follows that

$$u^{q^{j+k}} = u(X^q-X)^{\frac{q^{j+k}-1}{q-1}}\, .$$

Therefore by (2.4.1) we get

$$(2.11.2)\qquad S_1(t)-(-1)^k u(X^q-X)^{\frac{q^k-1}{q-1}}\,\frac{\psi_k(t)}{F_k} = \sum_{j=0}^{k} \frac{(-1)^j}{F_j}\, ut^{q^j}\alpha_{kj}\, ,$$

where

$$\alpha_{kj} := \frac{(X^q-X)^{\frac{q^{j+k}-1}{q-1}}}{L_k^{q^j}} - \frac{(X^q-X)^{\frac{q^k-1}{q-1}}}{L_{k-j}^{q^j}}\, , \qquad j = 0,1,\ldots,k.$$

Note that $\alpha_{k0} = 0$. For $j \geq 1$ we have from remark 2.2

$$\alpha_{kj} = \frac{(X^q-X)^{\frac{q^k-1}{q-1}}}{L_{k-j}^{q^j}} \left\{\prod_{\nu=0}^{j-1}\left(1 - \frac{X^{q^{k+\nu}}-X^{q^j}}{X^{q^{k+\nu+1}}-X^{q^j}}\right) - 1\right\}\, .$$

Hence for $j = 1,2,\ldots,k$ we have from remark 2.2

$$\mathrm{dg}\ \alpha_{kj} \leq \frac{q^k-1}{q-1}.q - \frac{q^{j+1}}{q-1}\,(q^{k-j}-1) + q^k(1-q).$$

Therefore

$$(2.11.3)\qquad \mathrm{dg}\left(\sum_{j=0}^{k} \frac{(-1)^j}{F_j}\, ut^{q^j}\alpha_{kj}\right) \leq \max_{1\leq j\leq k}\left(q^j(\mathrm{dg}t+\frac{q}{q-1}-j) + q^k(1-q)\right)$$

$$\leq q^{[\mathrm{dg}t]+3} + q^k(1-q).$$

From (2.11.2) and (2.11.3) we conclude that for k large enough

$$(2.11.4) \qquad dg\left(S_1(t)-(-1)^k u(X^q-X)^{\frac{q^k-1}{q-1}}\frac{\psi_k(t)}{F_k}\right) < -N.$$

From remark 2.2 we get for $k > [dgt] + 3$

$$dg\ S_2(t) \le \max_{j\ge k+1} q^j(dgt-j+q^k dgu-dgL_k)$$
$$= \max_{j\ge k+1} q^j(dgt-j+\frac{q}{q-1}) = q^{k+1}(dgt-k+\frac{1}{q-1}).$$

Hence for k large enough

$$(2.11.5) \qquad dg\ S_2(t) < -N.$$

Now it follows from (2.11.1), (2.11.4) and (2.11.5) that for k large enough

$$dg\left(\psi(t\xi)-(-1)^k u(X^q-X)^{\frac{q^k-1}{q-1}}\frac{\psi_k(t)}{F_k}\right) < -N. \quad \square$$

2.12. THEOREM. *The set of zeros of* ψ *is given by*

$$\{E\xi \mid E \in \mathbb{F}_q[X]\}.$$

PROOF. From property 2.11d and definition 2.3 it follows that $\psi(E\xi) = 0$ for all $E \in \mathbb{F}_q[X]$.

Now let α be a zero of ψ, $\alpha \neq 0$. Let $k_1 \in \mathbb{N}^0$ be such that

$$k_1 \le dg\ \alpha\xi^{-1} < k_1 + 1 \qquad \text{if } dg\ \alpha\xi^{-1} \ge 0,$$
$$k_1 = 0 \qquad \text{if } dg\ \alpha\xi^{-1} < 0.$$

It follows from definition 2.3 that for $k > k_1$

$$(2.12.1) \qquad \begin{aligned} dg\ \psi_k(\alpha\xi^{-1}) &= \sum_{dgE<k_1} dg(\alpha\xi^{-1}-E) + \sum_{k_1\le dgE<k} dg(\alpha\xi^{-1}-E) \\ &= c + (k-1)q^k - \frac{q^k}{q-1} + \sum_{dgE=k_1} dg(\alpha\xi^{-1}-E), \end{aligned}$$

where

$$c := \{dg(\alpha\xi^{-1}) - k_1 q + \frac{q}{q-1}\}\ q^{k_1}.$$

Let $N \in \mathbb{N}$. According to property 2.11d and the assumption that α is a zero of ψ there exists a $k_0 \in \mathbb{N}$, $k_0 = k_0(N)$, such that

$$dg\left(u(X^q-X)^{\frac{q^k-1}{q-1}} \frac{\psi_k(\alpha\xi^{-1})}{F_k}\right) < -N, \qquad k > k_0.$$

Hence for $k > k_0$

$$\text{(2.12.2)} \qquad dg\ \psi_k(\alpha\xi^{-1}) < (k-1)q^k - \frac{q^k}{q-1} - N.$$

The relations (2.11.1) and (2.11.2) give

$$\sum_{dgE=k_1} dg(\alpha\xi^{-1}-E) < -c-N.$$

Hence

$$\sum_{dgE=k_1} dg(\alpha\xi^{-1}-E) = -\infty.$$

Thus there is an $E \in \mathbb{F}_q[X]$ such that $\alpha\xi^{-1} = E$. $\square$

2.13. <u>THEOREM</u>. *The function ψ has the following property: for every $M \in \mathbb{F}_q[X]$*

$$\text{(2.13.1)} \qquad \psi(Mt) = \sum_{j=0}^{dgM} (-1)^j \frac{\psi_j(M)}{F_j} \psi^{q^j}(t).$$

<u>PROOF</u>. For $M = 1$ the relation is trivial. Suppose (2.13.1) is correct for $M = 1, X, \ldots, X^{m-1}$. Then from property 2.11c and the induction hypothesis we get

$$\begin{aligned}\psi(X^m t) &= X \sum_{j=0}^{m-1} (-1)^j \frac{\psi_j(X^{m-1})}{F_j} \psi^{q^j}(t) - \sum_{j=1}^{m} (-1)^{j-1} \frac{\psi_{j-1}^q(X^{m-1})}{F_{j-1}^q} \psi^{q^j}(t) \\ &= X.X^{m-1}\psi(t) + \sum_{j=1}^{m} (-1)^j \left(X \frac{\psi_j(X^{m-1})}{F_j} + \frac{\psi_{j-1}^q(X^{m-1})}{F_{j-1}^q}\right)\psi^{q^j}(t) + \\ &\qquad - (-1)^m X \frac{\psi_m(X^{m-1})}{F_m} \psi^{q^m}(t).\end{aligned}$$

Hence by (2.5.4) and (2.5.3) we have

$$\psi(X^m t) = X^m\psi(t) + \sum_{j=1}^{m} (-1)^j \frac{\psi_j(X^m)}{F_j} \psi^{q^j}(t),$$

which gives, with (2.5.2),

$$\psi(X^m t) = \sum_{j=0}^{m} (-1)^j \frac{\psi_j(X^m)}{F_j} \psi^{q^j}(t).$$

In view of (2.4.2), (2.4.3) and theorem 2.11a,b formula (2.13.1) follows now for arbitrary $M \in \mathbb{F}_q[X]$. □

2.14. <u>THEOREM</u>. *The function* ψ *defines a bijection from*

$$V = \{t \in \Phi \mid \mathrm{dg}\, t < \frac{q}{q-1}\}$$

onto itself.

2.15. <u>DEFINITION</u>. The function $\lambda\colon V \to V$ is defined as the inverse of $\psi|V$.

2.16. <u>THEOREM</u>. *For* $t \in V$ *we have*

$$\lambda(t) = \sum_{j=0}^{\infty} \frac{t^{q^j}}{L_j}.$$

<u>Proof of the theorems 2.14 and 2.16</u>

(i) Let $t \in V$. From the definition of ψ it follows that

$$\mathrm{dg}\, \psi(t) \le \max_{k\ge 0} q^k(\mathrm{dg}\,t-k) < \max_{k\ge 0} q^k(\frac{q}{q-1} - k) = \frac{q}{q-1},$$

which means $\psi(t) \in V$.

(ii) Suppose $t_1, t_2 \in V$ and $\psi(t_1) = \psi(t_2)$. Then in view of theorem 2.12 there exists an $E \in \mathbb{F}_q[X]$ such that

$$t_1 - t_2 = E\xi.$$

By the assumption $t_1, t_2 \in V$ we have

$$\mathrm{dg}(t_1-t_2) < \frac{q}{q-1}.$$

On the other hand

$$dg(t_1-t_2) = dg\ E + dg\ \xi = dg\ E + \frac{q}{q-1}\ .$$

Therefore $E = 0$ and $t_1 = t_2$. Hence ψ is injective on V.
(iii) Finally we have to prove that for every $\alpha \in V$ there exists a $\beta \in V$ such that $\psi(\beta) = \alpha$.

Let $\alpha \in V$. Since $\psi(0) = 0$ we may suppose that $\alpha \neq 0$. Consider the series

$$\sum_{n=0}^{\infty} \frac{\alpha^{q^n}}{L_n}\ .$$

Since $\alpha \in V\backslash\{0\}$ there exists an $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$ such that

$$dg\ \alpha = \frac{q}{q-1} - \varepsilon.$$

Now

$$dg\ \frac{\alpha^{q^n}}{L_n} = \frac{q^{n+1}}{q-1} - \varepsilon q^n - q\cdot \frac{q^n-1}{q-1} = \frac{q}{q-1} - \varepsilon q^n.$$

This shows that the general term goes to zero, hence the series is convergent. Let β be its sum. Clearly, $\beta \in V$. We shall prove that $\psi(\beta) = \alpha$.

Define

$$\beta_n := \sum_{k=0}^{n} \frac{\alpha^{q^k}}{L_k}\ .$$

Remark that

$$dg(\beta-\beta_n) = \frac{q}{q-1} - \varepsilon q^{n+1}$$

and that

$$\psi(\beta) = \psi(\beta_n) + \psi(\beta-\beta_n), \qquad n \in \mathbb{N}.$$

Furthermore

$$\psi(\beta_n) = \sum_{k=0}^{n} \psi\left(\frac{\alpha^{q^k}}{L_k}\right) = \sum_{k=0}^{n} \sum_{j=0}^{\infty} \frac{(-1)^j}{F_j} \frac{\alpha^{q^{k+j}}}{L_k^{q^j}} =$$
$$= \sum_{\nu=0}^{\infty} \sum_{k=0}^{\min(n,\nu)} \frac{(-1)^{\nu-k}}{F_{\nu-k} L_k^{q^{\nu-k}}}\, \alpha^{q^\nu}\ .$$

Hence by theorem 2.4 it follows that

$$\psi(\beta_n) = \sum_{\nu=0}^{n} (-1)^\nu \frac{\psi_\nu(1)}{F_\nu} \alpha^{q^\nu} + \gamma_n ,$$

where

$$\gamma_n := \sum_{\nu=n+1}^{\infty} \sum_{k=0}^{n} \frac{(-1)^{\nu-k}}{F_{\nu-k} L_k^{q^{\nu-k}}} \alpha^{q^\nu} .$$

Since $\psi_\nu(1) = 0$ for $\nu \geq 1$, we have

$$\psi(\beta_n) = \alpha + \gamma_n, \qquad n = 1,2,\ldots$$

and therefore

$$\psi(\beta) - \alpha = \psi(\beta-\beta_n) + \gamma_n, \qquad n = 1,2,\ldots .$$

Now we estimate dg γ_n:

$$\begin{aligned} \mathrm{dg}\ \gamma_n &\leq \max_{\nu\geq n+1}\left[\max_{0\leq k\leq n}\left(q^\nu \mathrm{dg}\alpha-(\nu-k)q^{\nu-k}-q^{\nu-k}.q.\frac{q^k-1}{q-1}\right)\right] \\ &= \max_{\nu\geq n+1}\left[q^\nu\left(\frac{q}{q-1}-\varepsilon\right)-q^{\nu-n}\left(\nu-n+q.\frac{q^n-1}{q-1}\right)\right] \\ &= -\varepsilon q^{n+1} + \frac{q}{q-1} . \end{aligned}$$

Hence for all $n \in \mathbb{N}$ we have

$$\mathrm{dg}(\psi(\beta)-\alpha) \leq \max(\mathrm{dg}\psi(\beta-\beta_n),\mathrm{dg}\gamma_n) \leq \frac{q}{q-1} - \varepsilon q^{n+1},$$

which means

$$\mathrm{dg}(\psi(\beta)-\alpha) = -\infty,$$

i.e. $\psi(\beta) = \alpha$. □

<u>REMARK</u>. The function λ was already introduced by L. CARLITZ (1935).

2.17. THEOREM. *The function $\psi: \Phi \to \Phi$ is surjective.*

PROOF. Let $v \in \Phi$. If $\mathrm{dg}\, v < \frac{q}{q-1}$ it follows from theorem 2.14 that v is in the range of ψ. The proof proceeds by induction on $\mathrm{dg}\, v$.

Let $v \in \Phi$, $\mathrm{dg}\, v \geq \frac{q}{q-1}$ and let $m \in \mathbb{N}$ be defined by

$$m + \frac{1}{q-1} \leq \mathrm{dg}\, v < m + \frac{q}{q-1} .$$

Suppose for all $t \in \Phi$ with $\mathrm{dg}\, t < m + \frac{1}{q-1}$ there exists a $t^* \in \Phi$ such that

$$\psi(t^*) = t.$$

Since Φ is algebraically closed, Φ contains every solution of the equation in t

$$(2.17.1) \qquad Xt - t^q = v.$$

For a solution t of (2.17.1) we have

$$\mathrm{dg}\, t \leq \mathrm{dg}\, v - 1.$$

Therefore

$$\mathrm{dg}\, t < m + \frac{1}{q-1}$$

and according to the induction hypothesis there exists a $t^* \in \Phi$ with $\psi(t^*) = t$. Put

$$v^* := Xt^*;$$

then according to theorem 2.11c

$$\psi(v^*) = \psi(Xt^*) = X\psi(t^*) - \psi^q(t^*) = Xt - t^q = v. \qquad \square$$

REMARK. It follows from work of D.R. HAYES (1974) and H.W. LENSTRA Jr. (private communication) that the Carlitz-ψ-function can be compared with the exponential function in the classical case.

3. LINEAR FUNCTIONS AND THE Δ-OPERATOR

3.1. DEFINITION. Let $V \subset \Phi$ be such that

$$t, v \in V \Rightarrow t + v \in V$$

and

$$t \in V, c \in \mathbb{F}_q \Rightarrow ct \in V.$$

A function $f\colon V \to \Phi$ is called *linear on* V if

$$(3.1.1) \qquad f(t+v) = f(t) + f(v), \qquad t, v \in V$$

and

$$(3.1.2) \qquad f(ct) = cf(t), \qquad t \in V, c \in \mathbb{F}_q.$$

EXAMPLES. It follows from the theorems 2.4, 2.11 and 2.16 that the functions ψ and ψ_k are linear on Φ and that the function λ is linear on $V = \{t \in \Phi \mid \operatorname{dg} t < \frac{q}{q-1}\}$.

3.2. THEOREM. *Let* $\{a_n\}_{n=0}^{\infty}$ *be a sequence of elements of* Φ. *Put*

$$R := -\limsup_{n\to\infty} \frac{\operatorname{dg} a_n}{n}.$$

Then the series $\sum_{n=0}^{\infty} a_n t^n$ *converges for all* $t \in \Phi$ *with* $\operatorname{dg} t < R$ *and diverges for all* $t \in \Phi$ *with* $\operatorname{dg} t > R$.

PROOF. Assume $R \in \mathbb{R}$.

(i) Let $t \in \Phi$ be such that $\operatorname{dg} t < R$. Choose $\rho \in \mathbb{R}$ such that

$$-R < \rho < -\operatorname{dg} t.$$

There exists an $n_0 \in \mathbb{N}$ such that for $n > n_0$

$$\frac{\operatorname{dg} a_n}{n} < \rho.$$

Hence for $n > n_0$

$$dg(a_n t^n) = dg\ a_n + n\ dg\ t < n(\rho + dgt).$$

Since from the choice of ρ we know that $\rho + dg\ t < 0$, we may conclude that

$$\lim_{n\to\infty} dg(a_n t^n) = -\infty.$$

This suffices to prove that $\sum_{n=0}^{\infty} a_n t^n$ converges.

(ii) Let $t \in \Phi$ be such that $dg\ t > R$ and let $\rho \in \mathbb{R}$ be such that $-dg\ t < \rho < -R$. Then there exists an increasing sequence $\{n_k\}_{k=1}^{\infty}$ such that

$$\frac{dg\ a_{n_k}}{n_k} > \rho, \qquad k \geq 1$$

and hence

$$dg(a_{n_k} t^{n_k}) > n_k(\rho + dgt) > 0.$$

This means that $\sum_{n=0}^{\infty} a_n t^n$ diverges.

The cases $R = \pm\infty$ are left to the reader. $\square$

3.3. REMARKS.

a) A series of the form $\sum_{n=0}^{\infty} a_n t^n$, $a_n \in \Phi$ is called a *power series* and R its *radius of convergence*.

b) Since Φ is a complete metric space, the notions of limit, continuity, differentiability and derivative of a function are defined in the obvious way. See J. DIEUDONNÉ (1969), 3.11; 3.13; 8.1.

c) If the function $f: U \to \Phi$ $(U \subset \Phi)$ has a power series expansion $\sum_{n=0}^{\infty} a_n t^n$ with radius of convergence $R > -\infty$, then this expansion is unique.

3.4. THEOREM. *Let the function* f *be defined by the power series* $\sum_{n=0}^{\infty} a_n t^n$, $a_n \in \Phi$ *with radius of convergence* R. *Then* f *is differentiable on* $\{t \in \Phi \mid dg\ t < R\}$ *and*

$$f'(t) = \sum_{n=1}^{\infty} na_n t^{n-1},$$

where $na_n := \sum_{i=1}^{n} a_n$. *The power series* $\sum_{n=1}^{\infty} na_n t^{n-1}$ *has radius of convergence* $\geq R$.

PROOF. The proof is left to the reader. □

3.5. THEOREM. *Let* f *be defined by* $f(t) := \sum_{n=0}^{\infty} a_n t^n$, $a_n \in \Phi$ *with radius of convergence* $R > -\infty$. *If* f *is linear on* $\{t \in \Phi \mid dg\ t < R\}$, *then*

$$f(t) = \sum_{k=0}^{\infty} a_{q^k} t^{q^k}.$$

PROOF. Denote $V = \{t \in \Phi \mid dg\ t < R\}$. From relation (3.1.2) it follows that $a_0 = 0$. Using relation (3.1.1) we conclude from the definition of differentiability that $f'(t) = a_1$ on V. Therefore it follows from theorem 3.4 and remark 3.3c that

$$na_n = 0, \qquad n = 2,3,\ldots .$$

Hence

$$a_n = 0, \qquad p \nmid n,$$

i.e.

$$f(t) = a_1 t + \sum_{j=1}^{\infty} a_{jp}\, t^{jp}.$$

So we have proved the relation

$$f(t) = a_1 t + a_p t^p + \ldots + a_{p^{\kappa-1}} t^{p^{\kappa-1}} + \sum_{j=1}^{\infty} a_{jp^{\kappa}} t^{jp^{\kappa}}, \qquad \kappa \in \mathbb{N} \tag{3.5.1}$$

for $\kappa = 1$.
Suppose (3.5.1) is correct for $\kappa = 1,2,\ldots,k$. Define

$$V_k := \{t \in V \mid p^k dg\ t < R\}$$

and

$$g_k(t) := \sum_{j=1}^{\infty} a_{jp^k} t^j, \qquad t \in V_k.$$

Let $t_1, t_2 \in V_k$ and let v_1 resp. v_2 be solutions of

$$t^{p^k} - t_1 = 0, \qquad t^{p^k} - t_2 = 0$$

respectively. Then

$$dg(v_1+v_2) \le p^k \max(dgt_1, dgt_2) < R$$

and using (3.1.1) we find

$$\begin{aligned} g_k(t_1+t_2) &= \sum_{j=1}^{\infty} a_{jp^k}(v_1^{p^k}+v_2^{p^k})^j = \sum_{j=1}^{\infty} a_{jp^k}(v_1+v_2)^{jp^k} \\ &= f(v_1+v_2)-a_1(v_1+v_2)-a_p(v_1+v_2)^p -\ldots- a_{p^{k-1}}(v_1+v_2)^{p^{k-1}} \\ &= f(v_1)+f(v_2)-a_1v_1-a_2v_2 -\ldots- a_{p^{k-1}}v_1^{p^{k-1}} -a_{p^{k-1}}v_2^{p^{k-1}} \\ &= g_k(t_1) + g_k(t_2). \end{aligned}$$

Therefore $g_k'(t) = a_{p^k}$ on V_k. On the other hand it follows from theorem 3.4 that

$$g_k'(t) = \sum_{j=1}^{\infty} ja_{jp^k}t^{j-1} ,$$

hence

$$a_{jp^k} = 0, \qquad p \nmid j.$$

Thus

$$f(t) = a_1t + a_pt^p +\ldots+ a_{p^k}t^{p^k} + \sum_{j=1}^{\infty} a_{jp^{k+1}}t^{jp^{k+1}} .$$

So we have showed by induction that

$$f(t) = \sum_{k=0}^{\infty} a_{p^k}t^{p^k}. \tag{3.5.1}$$

If $q = p$ we have proved our theorem.

From relations (3.1.2) and (3.5.1) we conclude that

$$a_{p^k}c(c^{p^k-1}-1) = 0, \qquad k \in \mathbb{N}^0,\ c \in \mathbb{F}_q. \tag{3.5.2}$$

Recall that $q = p^n (n \in \mathbb{N})$. Hence for $k \in \mathbb{N}$ there are $\ell \in \mathbb{N}^0$, $m \in \mathbb{N}$ such that

$$k = \ell n + m, \qquad 1 \le m \le n.$$

Using relations (0.7.1) and (0.7.2), relation (3.5.2) gives

$$a_{p^k} c \prod_{d\in \mathbb{F}^*_{p^m}} (c-d) = a_{p^k} c(c^{p^m-1}-1) = a_{p^k} c(c^{p^{k-1}}-1) = 0,$$

$$k \in \mathbb{N},\ c \in \mathbb{F}_q.$$

Therefore

(3.5.3) either $c \in \mathbb{F}_{p^m}$ or $a_{p^k} = 0$, $k \in \mathbb{N}$.

If $1 \le m < n$, then $\mathbb{F}_q \backslash \mathbb{F}_{p^m} \neq \emptyset$. Hence we conclude from (3.5.2) and (3.5.3) that $a_{p^k} = 0$ unless p^k is a power of q. □

3.6. <u>DEFINITION</u>. Let $V(r) \subset \Phi$ denote the set $\{t \mid dg\ t < r\}$ and let $f\colon V(r) \to \Phi$. Then we define the functions $\Delta_n f\colon V(r-n) \to \Phi$, $n = 0,1,2,\ldots$ by

$$\Delta_0 f := f,$$

$$\Delta_1 f(t) := \Delta f(t) := f(Xt) - Xf(t),$$

$$\vdots$$

$$\Delta_n f(t) := \Delta_{n-1} f(Xt) - X^{q^{n-1}} \Delta_{n-1} f(t).$$

For $n = 0,1,2,\ldots$ the *operators* Δ_n are defined above by their action on functions $f\colon V(r) \to \Phi$.

Note that $\Delta(\Delta f)$ need not be equal to $\Delta_2 f$, etc.

3.7. <u>THEOREM</u>. *When* f *is linear on* $V(r)$, *so is* $\Delta_n f$ *on* $V(r-n)$, $n \in \mathbb{N}$.

<u>PROOF</u>. Trivial. □

3.8. <u>THEOREM</u>. *The following relations hold*:

(3.8.1) $$\Delta_n \frac{X^{q^h} t^{q^k}}{F_k} = \frac{X^{q^h} t^{q^k}}{F^{q^n}_{k-n}}, \quad n = 0,1,\ldots,k;\ h \in \mathbb{N}^0,$$

$$\Delta_n \frac{X^{q^h} t^{q^k}}{F_k} = 0, \quad n > k;\ h \in \mathbb{N}^0,$$

(3.8.2) $$\Delta_n \psi(t) = (-1)^n \psi^{q^n}(t), \quad n = 0,1,2,\ldots$$

and

$$(3.8.3)\qquad \Delta_n \frac{\psi_k(t)}{F_k} = \left(\frac{\psi_{k-n}(t)}{F_{k-n}}\right)^{q^n}, \qquad n = 0,1,\dots,k,$$

$$\Delta_n \frac{\psi_k(t)}{F_k} = 0, \qquad n > k.$$

<u>PROOF</u>. The proof proceeds by induction on n and uses relation 2.2a, theorem 2.11c and relation (2.4.4) respectively. □

Note: The relations (3.8.2) and (3.8.3) were already given by L. CARLITZ (1935) in §5 and §3 respectively.

3.9. <u>LEMMA</u>. *Let* $g \in \Phi[t]$ *be a linear polynomial of degree* q^n. *Then for every* $t,v \in \Phi$ *we have*

$$(3.9.1)\qquad g(tv) = \sum_{j=0}^{n} \frac{\psi_j(v)}{F_j} \Delta_j g(t).$$

<u>PROOF</u>. (See also L. CARLITZ (1935), th.3.1).
For $n = 0$ the assertion is evident.

Suppose (3.9.1) has been proved for $n = 0,1,\dots,N-1$. We shall prove it for $n = N$. By linearity, $g(t)$ is necessarily of the form

$$g(t) = \sum_{k=0}^{N} a_k \frac{t^{q^k}}{F_k}.$$

From definition 3.6 and relation (3.8.1) we obtain

$$(3.9.2)\qquad \Delta_j g(t) = \sum_{k=j}^{N} a_k \frac{t^{q^k}}{F_{k-j}^{q^j}}, \qquad j = 0,1,\dots,N.$$

Hence from the induction hypothesis we have for $t,v \in \Phi$

$$g(tv) = \sum_{j=0}^{N-1} \frac{\psi_j(v)}{F_j}\left(\Delta_j g(t) - a_N \frac{t^{q^N}}{F_{N-j}^{q^j}}\right) + a_N \frac{t^{q^N} v^{q^N}}{F_N}$$

$$= \sum_{j=0}^{N} \frac{\psi_j(v)}{F_j} \Delta_j g(t) + a_N t^{q^N}\left(\frac{v^{q^N}}{F_N} - \sum_{j=0}^{N} \frac{\psi_j(v)}{F_j F_{N-j}^{q^j}}\right).$$

It remains to prove that

$$(3.9.3)\qquad \frac{v^{q^N}}{F_N} = \sum_{j=0}^{N} \frac{\psi_j(v)}{F_j F_{N-j}^{q^j}}, \qquad v \in \Phi.$$

Since the polynomial $\psi_j \in \mathbb{F}_q[X][v]$ is linear on Φ of degree q^j for $j = 0,1,\dots,N$, we can put v^{q^N}/F_N in the form

$$\frac{v^{q^N}}{F_N} = \sum_{j=0}^{N} b_j \frac{\psi_j(v)}{F_j}.$$

From theorem 3.8 we obtain for $i = 0,1,\dots,N$

$$(3.9.4)\qquad \Delta_i\left(\frac{v^{q^N}}{F_N}\right) = \sum_{j=i}^{N} b_j \left(\frac{\psi_{j-i}(v)}{F_{j-i}}\right)^{q^i}.$$

On the other hand

$$(3.9.5)\qquad \Delta_i\left(\frac{v^{q^N}}{F_N}\right) = \frac{v^{q^N}}{F_{N-i}^{q^i}}, \qquad i = 0,1,\dots,N.$$

Since $\psi_k(1) = 0$ for $k > 0$ and $\psi_0(1) = 1$, the relations (3.9.4) and (3.9.5) for $v = 1$ imply

$$b_i = \frac{1}{F_{N-i}^{q^i}}, \qquad i = 0,1,\dots,N.$$

Hence (3.9.3) is proved and the induction step is completed. □

3.10. THEOREM. (Expansion Formula). *Let* $f\colon \Phi \to \Phi$ *be a linear function defined by a power series with radius of convergence* R:

$$f(t) = \sum_{n=0}^{\infty} a_n t^{q^n}, \qquad a_n \in \Phi.$$

Let $M \in \mathbb{F}_q[X]$ *with* $\deg M = m$. *Then for every* $t \in \Phi$ *with* $\deg t + m < R$ *we have*

$$(3.10.1)\qquad f(Mt) = \sum_{j=0}^{m} \frac{\psi_j(M)}{F_j} \Delta_j f(t).$$

PROOF. Consider for $n > m$ the linear polynomials

$$f_n(t) = \sum_{k=0}^{n} a_k t^{q^k}.$$

For $t \in \Phi$ with $\mathrm{dg}\, t < R$ we have

$$f(t) = \lim_{n\to\infty} f_n(t).$$

For $t \in \Phi$ with $\mathrm{dg}\, t + m < R$ we have

$$\Delta_j f(t) = \lim_{n\to\infty} \Delta_j f_n(t), \qquad j = 1,2,\ldots,m.$$

Now using lemma 3.9 with $g = f_n$ and $v = M$, we get

$$f(Mt) = \lim_{n\to\infty} f_n(Mt) = \lim_{n\to\infty} \sum_{k=0}^{m} \frac{\psi_k(M)}{F_k} \Delta_k f_n(t) = \sum_{k=0}^{m} \frac{\psi_k(M)}{F_k} \Delta_k f(t). \quad \square$$

3.11. <u>COROLLARY</u> (= theorem 2.13). *Let* $M \in \mathbb{F}_q[X]$ *with* $\mathrm{dg}\, M = m$. *Then for all* $t \in \Phi$

$$\psi(Mt) = \sum_{k=0}^{m} (-1)^k \frac{\psi_k(M)}{F_k} \psi^{q^k}(t).$$

<u>PROOF</u>. Since ψ is an entire linear function (3.10.1) is valid for all $t \in \Phi$. Now the expression for $\psi(Mt)$ follows by using theorem 3.8 in (3.10.1). $\square$

3.12. <u>LEMMA</u>. *Let* $f\colon \Phi \to \Phi$ *be an entire, linear function. Then for every* $k \in \mathbb{N}$

$$(3.12.1) \qquad \Delta_k f^q(t) = (\Delta_k f(t))^q + (X^{q^k} - X)(\Delta_{k-1} f(t))^q.$$

<u>PROOF</u>. For $k = 1$ we have

$$\Delta f^q(t) = f^q(Xt) - Xf^q(t) = (f(Xt) - Xf(t))^q + (X^q - X) f^q(t),$$

which proves (3.12.1) for $k = 1$.

Now suppose that (3.12.1) has been proved for $k = 1,\ldots,\kappa-1$. Then we have

$$\begin{aligned}
\Delta_\kappa f^q(t) &= \Delta_{\kappa-1} f^q(Xt) - X^{q^{\kappa-1}} \Delta_{\kappa-1} f^q(t) \\
&= (\Delta_{\kappa-1} f(Xt))^q + (X^{q^{\kappa-1}} - X)(\Delta_{\kappa-2} f(Xt))^q + \\
&\quad - X^{q^{\kappa-1}} \{(\Delta_{\kappa-1} f(t))^q + (X^{q^{\kappa-1}} - X)(\Delta_{\kappa-2} f(t))^q\} =
\end{aligned}$$

$$= \{\Delta_{\kappa-1} f(Xt) - X^{q^{\kappa-1}} \Delta_{\kappa-1} f(t)\}^q + X^{q^{\kappa}} (\Delta_{\kappa-1} f(t))^q$$

$$+ (X^{q^{\kappa-1}} X)\{\Delta_{\kappa-2} f(Xt) - X^{q^{\kappa-2}} \Delta_{\kappa-2} f(t)\}^q - X^{q^{\kappa-1}} (\Delta_{\kappa-1} f(t))^q$$

$$= (\Delta_{\kappa} f(t))^q + (X^{q^{\kappa}} - X)(\Delta_{\kappa-1} f(t))^q. \quad \square$$

4. THE FUNCTIONS J_n

In 1960 L. CARLITZ introduced a class of functions which have formal resemblance with classical cylinder functions.

4.1. DEFINITION. For $n \in \mathbb{N}^0$ the function $J_n : \Phi \to \Phi$ is defined by

$$(4.1.1) \qquad J_n(t) := \sum_{k=0}^{\infty} (-1)^k \frac{t^{q^{n+k}}}{F_{n+k} F_k^{q^n}} .$$

For $n \in \mathbb{N}$ we define the function $J_{-n} : \Phi \to \Phi$ by

$$(4.1.2) \qquad J_{-n}(t) := \sum_{k=0}^{\infty} (-1)^{n+k} \frac{t^{q^k}}{F_k F_{n+k}^{q^{-n}}} .$$

REMARK. $F_{n+k}^{q^{-n}}$ is uniquely determined.

If we put $F_{-n}^{-1} = 0$, $n \in \mathbb{N}$, then for all $n \in \mathbb{Z}$ the function J_n can be defined by formula (4.1.1).

4.2. THEOREM [L. CARLITZ (1960), formulae (5.3), (5.9), (5.13) and (5.14)]. *Let $n \in \mathbb{Z}$. The function J_n as defined above is an entire, linear function, which has the properties*:

(i) $\qquad \{J_{-n}(t)\}^{q^n} = (-1)^n J_n(t),$

(ii) $\qquad \Delta_k J_n(t) = J_{n-k}^{q^k}(t), \qquad k = 1,2,\dots,$

(iii) $\qquad J_{n+1}(t) - (X^{q^n} - X) J_n(t) + J_{n-1}^q(t) = 0,$

(iv) $\qquad J_n(X^2 t) - (X^{q^n} + X) J_n(Xt) + X^{q^n+1} J_n(t) = - J_n^q(t).$

PROOF. The formulae can be computed directly from the definition of J_n, using (1.8.3). $\square$

4.3. REMARK. From the definition of Δ_2 we see that (iv) can also be written as

$$\text{(iva)} \qquad \Delta_2 J_n(t) - (X^{q^n} - X^q)\Delta J_n(t) + J_n^q(t) = 0.$$

4.4. THEOREM. *For all* $n \in \mathbb{N}^0$, $k \in \mathbb{N}$ *we have*

$$\Delta_k J_n(t) = P_k(J_n(t), \Delta J_n(t)),$$

where P_k *is a linear polynomial in* $\mathbb{F}_q[X][t_1,t_2]$ *of total degree* $q^{[k/2]}$. *The valuation of the coefficients of* P_k *is less than* q^{n+k-1}.

PROOF. For $k = 1$ the theorem is obvious. For $k = 2$ the assertion follows immediately from remark 4.3.

Now suppose that the assertion has been proved for $k = 1,2,\ldots,\kappa-1; \kappa \geq 3$. Then it follows from theorem 4.2(ii) and (iii) that

$$\begin{aligned}
\Delta_\kappa J_n(t) &= J_{n-\kappa}^{q^\kappa}(t) = (J_{n-\kappa}^q(t))^{q^{\kappa-1}} \\
&= \{(X^{q^{n-\kappa+1}} - X)J_{n-\kappa+1}(t) - J_{n-k+2}(t)\}^{q^{\kappa-1}} \\
&= (X^{q^n} - X^{q^{\kappa-1}})J_{n-\kappa+1}^{q^{\kappa-1}}(t) - (J_{n-\kappa+2}^{q^{\kappa-2}}(t))^q \\
&= (X^{q^n} - X^{q^{\kappa-1}})\Delta_{\kappa-1}J_n(t) - (\Delta_{\kappa-2}J_n(t))^q.
\end{aligned}$$

Hence by the induction hypothesis for $k = \kappa - 1$, $\kappa - 2$ we have

$$\text{(4.4.1)} \qquad \Delta_\kappa J_n(t) = (X^{q^n} - X^{q^{\kappa-1}})P_{\kappa-1}(J_n(t), \Delta J_n(t)) - P_{\kappa-2}^q(J_n(t), \Delta J_n(t))$$

and therefore

$$\Delta_\kappa J_n(t) = P_\kappa(J_n(t), \Delta J_n(t)).$$

It follows from (4.4.1) and the induction hypothesis that the degree of P_κ is equal to $q^{[\kappa/2]}$ and that the valuation of the coefficients of P_κ is at most $q^{n+\kappa-1}$. $\square$

The rest of this section will not be used in the following chapters. The function J_n is a solution of the equation

$$f(X^2t) - (X^{q^n}+X)f(Xt) + X^{q^n+1}f(t) = -f^q(t),$$

with $n \in \mathbb{Z}$. We are interested in all solutions of this equation which are of the form

$$f(t) = \sum_{\nu=-h}^{\infty} a_\nu t^{q^\nu}, \quad h \in \mathbb{Z},\ a_\nu \in \Phi,\ a_{-h} \neq 0.$$

It turns out that for $n \in \mathbb{Z}$ there is essentially only one such solution of the equation; see L. CARLITZ (1960). However, the equation above can be slightly generalized. Recall that q is a power of p, say p^m and that the field Φ has characteristic p. Hence for those $r \in \mathbb{Q}$ such that $rm \in \mathbb{Z}$, the element $X^{q^r} \in \Phi$ is uniquely defined.

4.6. <u>DEFINITION</u>. Let $q = p^m$. Let $r \in \mathbb{Q}$ be such that $rm \in \mathbb{Z}$. For $r > -1$ we define the element $F_r \in \mathbb{F}_q[X]$ by

$$F_r := \begin{cases} \displaystyle\prod_{\substack{j\in\mathbb{Z} \\ 0\le j<r}} (X^{p^{mr}} - X^{p^{mj}}) & \text{if } r > 0, \\ 1 & \text{if } -1 < r \le 0. \end{cases}$$

For $r \le -1$ we put

$$\frac{1}{F_r} := \prod_{\substack{j\in\mathbb{Z} \\ r\le j<0}} (X^{p^{mr}} - X^{p^{mj}}).$$

4.7. <u>REMARK</u>. For $r \in \mathbb{N}^0$ definition 4.6 equals definition 2.1 of this thesis; furthermore $F_r^{-1} = 0$ for $-r \in \mathbb{N}$. For q,r as in definition 4.6 we have

(4.7.1) $$F_r = (X^{p^{mr}} - X)F_{r-1}^{p^m}.$$

4.8. <u>DEFINITION</u>. Let $q = p^m$. Let $r \in \mathbb{Q}$ be such that $rm \in \mathbb{Z}$. We define the function $J_r : \Phi \to \Phi$ by

$$J_r(t) := \sum_{k=0}^{\infty} (-1)^k \frac{t^{q^{r+k}}}{F_{r+k}F_k^{q^r}}, \quad t \in \Phi.$$

(The series is convergent for all $t \in \Phi$.)

4.9. THEOREM. *The function* J_r *from definition* 4.8 *has the properties*:

(i) $$J_{r+1}(t) - (X^{q^r}-X)J_r(t) + J_{r-1}^q(t) = 0,$$

(ii) $$J_r(X^2t) - (X^{q^r}+X)J_r(Xt) + X^{q^r+1}J_r(t) = - J_r^q(t).$$

PROOF. Analogous to the proof of theorem 4.2. □

5. ANALYSIS ON Φ

5.1. DEFINITION. Let $R \in \mathbb{R} \cup \{+\infty\}$ and $U = \{t \in \Phi \mid dg\ t < R\}$. A function $f\colon U \to \Phi$ is called *analytic on* U if there exists a power series $\sum_{i=0}^{\infty} a_i t^i$, $a_i \in \Phi$ with radius of convergence $\geq R$ such that

$$f(t) = \sum_{i=0}^{\infty} a_i t^i, \quad t \in U.$$

If $R = +\infty$ then f is called *an entire function*.

5.2. REMARK. Let f be analytic on $U = \{t \in \Phi \mid dg\ t < R\}$. Suppose that the power series $\sum_{i=0}^{\infty} a_i t^i$, which represents f on U, has radius of convergence R. Then f has no analytic continuation outside U in the classical sense, see J. DE GROOT (1942), L.I. WADE (1946). Recently PH. ROBBA (1973) and J. TATE (1971) have given different methods for analytic continuation of functions over a complete non-archimedean valued field. For an exposé in the p-adic case we refer to the book of Y. AMICE (1975).

In the following chapters we shall need some results from the theory of functions $f\colon \Phi \to \Phi$. Since there are fundamental differences between Φ and $\mathbb{C}$ (Φ has characteristic p, the valuation of Φ is non-archimedean, Φ is not locally compact), we may also expect great differences between this theory and the classical theory of complex functions of one variable. Surprisingly some fundamental classical theorems have analogues in the theory of functions based on Φ. So we have e.g. a maximum modulus theorem and a product formula for entire functions. (See theorem 5.16 and corollary 5.24 respectively.)

We shall give complete proofs of the theorems needed later on. For a more general treatment we refer to the works of U. GÜNTZER (1966), M. LAZARD (1962) and A.F. MONNA (1970). The first results in non-archimedean analysis are contained in the thesis of W. SCHÖBE (1930). For a discussion of SCHNIRELMAN's proof of the maximum-modulus principle we refer to his own work (1938) or to W.W. ADAMS (1966, appendix), who gives an exposition for the p-adic case.

5.3. DEFINITION. Let $\Phi[[t]]$ be the set of formal power series with coefficients in Φ. For each $r \in \mathbb{R}$ the subset $\mathcal{P}_r$ of $\Phi[[t]]$ is defined as follows.

Let $f \in \Phi[[t]]$, $f(t) = \sum_{i=0}^{\infty} a_i t^i$. Then $f \in \mathcal{P}_r$ if and only if

$$(5.3.1) \qquad \lim_{i\to\infty} (\mathrm{dg}\ a_i + ir) = -\infty. \text{ *)}$$

For such r we put

$$M_r(f) := \max_{i\geq 0} (\mathrm{dg} a_i + ir).$$

Further we define

$$\|f\|_r := q^{M_r(f)}, \qquad f \in \mathcal{P}_r.$$

5.4. LEMMA. *$\mathcal{P}_r$ is a Φ-Banach space with norm $\|\cdot\|_r$.*

PROOF. Clearly, $\mathcal{P}_r$ is a vector space over Φ and

$$\|f+g\|_r \leq \|f\|_r + \|g\|_r.$$

Finally, let $\{f_k\}_{k=1}^{\infty}$, $f_k(t) = \sum_{i=0}^{\infty} a_{ki} t^i$ be a Cauchy sequence in $\mathcal{P}_r$. Then the proof of the completeness can be given by standard arguments in the following steps:

(i) for each i, $\lim_{k\to\infty} a_{ki} =: a_i$ exists in Φ,

(ii) f, defined by $f(t) := \sum_{i=0}^{\infty} a_i t^i$ belongs to $\mathcal{P}_r$,

(iii) $\lim_{k\to\infty} f_k = f$ in the norm topology of $\mathcal{P}_r$. □

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*) This implies that for every $t \in \Phi$ with dg $t = r$ the series $\sum_{i=0}^{\infty} a_i t^i$ converges.

5.5. REMARK. From the proof of lemma 5.4 we see that $\{f_k\}_{k=1}^{\infty}$, $f_k(t) = \sum_{i=0}^{\infty} a_{ki}t^i$ is a convergent sequence in P_r if and only if for every $t \in \Phi$ with $\mathrm{dg}\, t \le r$ the sequence of elements $\{f_k(t)\}_{k=1}^{\infty}$ is convergent in Φ.

5.6. REMARK. When $f \in P_r$, then the radius of convergence R of f is not smaller than r.

When $f \in P_r$, then $f \in P_\rho$ for all $\rho \le r$ and for all $\rho \le r$ we have

$$\sup_{\mathrm{dg}t=\rho} \mathrm{dg}\, f(t) \le M_\rho(f).$$

If there is only one $i \in \mathbb{N}^0$ such that

$$\mathrm{dg}\, a_i + i\rho = M_\rho(f), \tag{5.6.1}$$

then we even have for all $t \in \Phi$ with $\mathrm{dg}\, t = \rho$

$$\mathrm{dg}\, f(t) = M_\rho(f). \tag{5.6.2}$$

Those $\rho \le r$ for which there exists more than one $i \in \mathbb{N}^0$ such that (5.6.1) is valid, will play a special role in the theory, since they are connected with the occurence and the location of the zeros of f.

5.7. DEFINITION. Let $r \in \mathbb{R}$, $f \in P_r$, $f(t) = \sum_{i=h}^{\infty} a_i t^i$, $a_h \ne 0$. If for $\rho \in \mathbb{R}$, $\rho \le r$, there exist $i,j \ge h$, $i \ne j$, such that

$$\mathrm{dg}\, a_i + i\rho = \mathrm{dg}\, a_j + j\rho = M_\rho(f),$$

then ρ is called a *hooking-radius* of f.

5.8. LEMMA. *Let $r \in \mathbb{R}$, $f \in P_r$, $f(t) = \sum_{i=h}^{\infty} a_i t^i$, $a_h \ne 0$. The number of hooking-radii of f in $(-\infty,r]$ is finite.*

PROOF. Because of (5.3.1) there exists an n_0 such that

$$i > n_0 \Rightarrow \mathrm{dg}\, a_i + ir < \mathrm{dg}\, a_h + hr. \tag{5.8.1}$$

Hence for all $i > n_0$ and $\rho \le r$

$$\mathrm{dg}\, a_i + i\rho < \mathrm{dg}\, a_h + h\rho \le M_\rho(f). \tag{5.8.2}$$

Since for $i \neq j$, $h \le i,j \le n_0$ there is at most one $\rho \le r$ with $dg\ a_i + i\rho = dg\ a_j + j\rho$, the number of hooking-radii of f in $(-\infty,r]$ is at most $\binom{n_0-h+1}{2}$. □

5.9. REMARK. In 5.11 we shall introduce a kind of Newton polygon to describe the behaviour of $M_\rho(f)$. The hooking-radii will be the angular points of this polygon. Note that because of (5.8.1) the indices $i > n_0$ can be neglected in arguments on $M_\rho(f)$.

5.10. DEFINITION. Let $r \in \mathbb{R}$, $f \in P_r$, $f(t) = \sum_{i=h}^{\infty} a_i t^i$, $a_h \neq 0$. Let $R_1, R_2, \ldots, R_\ell$ be the (possibly empty) sequence of hooking-radii of f in $(-\infty,r]$ in increasing order. Define

$$i_0 := h$$

and

$$i_k := \max_{i \ge h} \{i \mid dg\ a_i + iR_k = M_{R_k}(f)\}, \qquad k = 1,2,\ldots,\ell.$$

5.11. THEOREM. *In the notation of definition* 5.10 *we have*

(i) $\quad i_0 < i_1 < \ldots < i_\ell.$

(ii) *If* $\{R_1, R_2, \ldots, R_\ell\} = \emptyset$:

$$\max_{i \ge h} \{i \mid dg\ a_i + i\rho = M_\rho(f)\} = \min_{i \ge h} \{i \mid dg\ a_i + i\rho = M_\rho(f)\} = i_0, \qquad -\infty < \rho \le r.$$

(iii) *If* $\{R_1, R_2, \ldots, R_\ell\} \neq \emptyset$:

$$\max_{i \ge h} \{i \mid dg\ a_i + i\rho = M_\rho(f)\} = \begin{cases} i_0, & -\infty < \rho < R_1, \\ i_k, & R_k \le \rho < R_{k+1},\ k = 1,2,\ldots,\ell-1, \\ i_\ell, & R_\ell \le \rho \le r. \end{cases}$$

and

$$\min_{i \ge h} \{i \mid dg\ a_i + i\rho = M_\rho(f)\} = \begin{cases} i_0, & -\infty < \rho \le R_1, \\ i_k, & R_k < \rho \le R_{k+1},\ k = 1,2,\ldots,\ell-1, \\ i_\ell, & R_\ell < \rho \le r. \end{cases}$$

PROOF. Let $1 \le k \le \ell$ and $h \le i < i_k$. Since

$$dg\ a_i + iR_k \le dg\ a_{i_k} + i_k R_k,$$

one has for $\rho \in (R_k, r]$

$$(5.11.1) \qquad dg\ a_i + i\rho < dg\ a_{i_k} + i_k\rho \le M_\rho(f).$$

In particular, by $R_k < R_{k+1} \le r$ and for $k = 0$ trivially,

$$(5.11.2) \qquad \min_{i \ge h} \{i \mid dg\ a_i + iR_{k+1} = M_{R_{k+1}}(f)\} \ge i_k, \qquad k = 0,1,2,\dots,\ell-1.$$

It follows, by definition 5.10, that $i_{k+1} > i_k$ for $k = 0,1,\dots,\ell-1$. This proves (i).

By means of continuity arguments it is easily seen that assertion (ii) and the assertions of (iii) for $-\infty < \rho < R_1$ and $-\infty < \rho \le R_1$ respectively are obvious.

Now we consider the case that there are one or more hooking-radii. Let $n_0 \ge h$ be such that (5.8.1) is valid. From the maximality in the definition of i_k we see that

$$(5.11.3) \qquad dg\ a_{i_k} + i_k\rho > \max_{i_k < i \le n_0} (dga_i + i\rho)\ , \quad \rho = R_k, \qquad k = 1,2,\dots,\ell.$$

Let $1 \le k \le \ell$ and suppose that the inequality in (5.11.3) holds for all $\rho \in (R_k, r]$. Then it follows from (5.8.2) that

$$dg\ a_{i_k} + i_k\rho > dg\ a_i + i\rho, \qquad i > i_k,\ R_k < \rho \le r.$$

On the other hand (5.11.1) tells us that

$$dg\ a_{i_k} + i_k\rho > dg\ a_i + i\rho, \qquad h \le i < i_k,\ R_k < \rho \le r.$$

Hence $(R_k, r]$ does not contain a hooking-radius of f, i.e. $k = \ell$ and i_ℓ is the unique i for which

$$dg\ a_i + i\rho = M_\rho(f), \qquad R_\ell < \rho \le r.$$

We see that for $1 \le k \le \ell-1$ the inequality of (5.11.3) does not hold for all $\rho \in (R_k, r]$, i.e. there exists a $\rho \in (R_k, r]$ such that

$$\text{(5.11.4)} \qquad \mathrm{dg}\, a_{i_k} + i_k\rho \le \max_{i_k < i \le n_0} (\mathrm{dg} a_i + i\rho).$$

Since both sides of this inequality are continuous functions of ρ, the smallest number ρ for which (5.11.4) is valid is a point where the equality holds. Since

$$\mathrm{dg}\, a_i + i\rho < M_\rho(f)$$

for $h \le i < i_k$ by (5.11.1) and for $i > n_0$ by (5.8.2), this point must be the smallest hooking-radius of f in $(R_k, r]$, i.e. R_{k+1}. Moreover we have

$$\min_{i \ge h} \{i \mid \mathrm{dg}\, a_i + iR_{k+1} = M_{R_{k+1}}(f)\} = i_k, \qquad k = 1,\dots,\ell-1.$$

Furthermore we conclude that for $k = 1,\dots,\ell-1$ and $R_k < \rho < R_{k+1}$

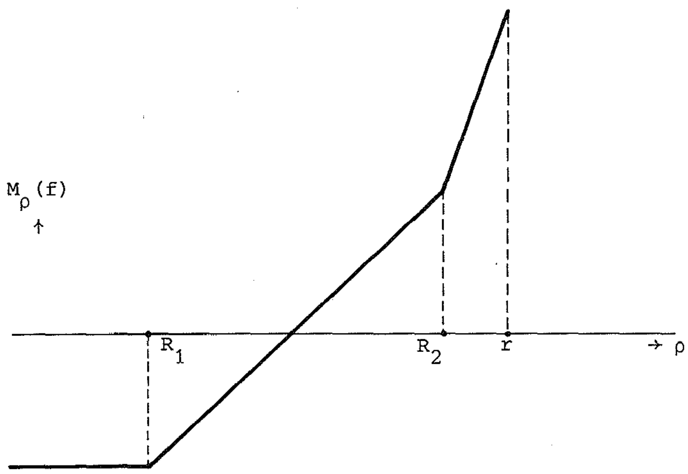
$$\mathrm{dg}\, a_{i_k} + i_k\rho > \max_{i_k < i \le n_0} (\mathrm{dg} a_i + i\rho).$$

Since $\mathrm{dg}\, a_i + i\rho < M_\rho(f)$ for $h \le i < i_k$ by (5.11.1) and for $i > n_0$ by (5.8.2), i_k is the unique i such that

$$\mathrm{dg}\, a_i + i\rho = M_\rho(f), \qquad R_k < \rho < R_{k+1}, \qquad 1 \le k \le \ell-1.$$

This completes the proof. $\square$

The following figure illustrates the curve for $M_\rho(f)$, $\rho \le r$. Here $h = 0$, $\ell = 2$, $R_2 < r$, $i_1 = 1$, $i_2 = 3$. This figure also explains the term "hooking-radius".



5.12. COROLLARY. *In the notation of definition* 5.10 *we have*

(5.12.1) $$R_k = \min_{i>i_{k-1}} \frac{\operatorname{dg} a_{i_{k-1}} - \operatorname{dg} a_i}{i-i_{k-1}}, \qquad k = 1,2,\ldots,\ell.$$

PROOF. From theorem 5.11 we have

$$\min_{i\geq h} \{i \mid \operatorname{dg} a_i + iR_k = M_{R_k}(f)\} = i_{k-1}, \qquad k = 1,2,\ldots,\ell.$$

Hence

$$\operatorname{dg} a_{i_{k-1}} - \operatorname{dg} a_i \geq (i-i_{k-1})R_k, \qquad i \geq i_{k-1},$$

from which we obtain

(5.12.2) $$\frac{\operatorname{dg} a_{i_{k-1}} - \operatorname{dg} a_i}{i-i_{k-1}} \geq R_k, \qquad i > i_{k-1}.$$

Moreover it follows from theorem 5.11 that

(5.12.3) $$\operatorname{dg} a_{i_k} + i_k R_k = M_{R_k}(f) = \operatorname{dg} a_{i_{k-1}} + i_{k-1} R_k, \qquad k = 1,2,\ldots,\ell.$$

Now formula (5.12.1) follows from (5.12.2) and (5.12.3). □

5.13. <u>DEFINITION</u>. Let $r \in \mathbb{R}$, $f \in P_r$, $f(t) = \sum_{i=h}^{\infty} a_i t^i$, $a_h \neq 0$. For $\rho \leq r$ we define

$$(5.13.1) \qquad d(f,\rho) := \max_{i \geq h} \{i \mid dg\, a_i + i\rho = M_\rho(f)\} - \min_{i \geq h} \{i \mid dg\, a_i + i\rho = M_\rho(f)\}.$$

5.14. <u>COROLLARY</u>. *In the notation of the definitions* 5.10 *and* 5.13 *we have*

$$d(f,\rho) = \begin{cases} 0 & \text{if } \rho \neq R_k, \quad k = 1,2,\ldots,\ell, \\ i_k - i_{k-1} & \text{if } \rho = R_k, \quad k = 1,2,\ldots,\ell. \end{cases}$$

<u>PROOF</u>. Obvious from theorem 5.11. □

5.15. <u>REMARK</u>. Let $r \in \mathbb{R}$, $f \in P_r$. If f has no hooking-radii in $(-\infty,r]$, then for all $t \in \Phi$ with $dg\, t = \rho \leq r$ we have

$$dg\, f(t) = M_\rho(f).$$

If $R_1 < R_2 < \ldots < R_\ell \leq r$ are the hooking-radii of f in $(-\infty,r]$, then for $t \in \Phi$ with $dg\, t = \rho \leq r$ we have

$$(5.15.1) \qquad dg\, f(t) = M_\rho(f), \qquad \rho \neq R_1,R_2,\ldots,R_\ell$$

and

$$(5.15.2) \qquad dg\, f(t) \leq M_\rho(f), \qquad \rho = R_1,R_2,\ldots,R_\ell.$$

But we can prove more.

5.16. <u>THEOREM</u>. (Maximum Modulus Principle). *Let* $r \in \mathbb{Q}$ *), $f \in P_r$. *Then*

$$\sup_{dgt \leq r} dg\, f(t) = \sup_{dgt = r} dg\, f(t) = M_r(f).$$

---

*) In view of (1.7.3) ($dgt \in \mathbb{Q}$ for $t \in \Phi^*$) we restrict r to $\mathbb{Q}$.

For the proof of theorem 5.16 we need two lemmas. Note that if r is not a hooking-radius of f, then theorem 5.16 is an immediate consequence of remark 5.15 and theorem 5.11. ($M_\rho(f)$ is a monotonic function of ρ on $(-\infty,r]$.)

5.17. LEMMA. *Let* $r \in \mathbb{Q}$ *and* $f \in P_r$. *Then*

$$\sup_{dgt<r} dg\, f(t) = \sup_{dgt\leq r} dg\, f(t) = M_r(f).$$

PROOF. According to lemma 5.8 f has at most a finite number of hooking-radii in $(-\infty,r]$. Hence there is a $\rho < r$ such that f has no hooking-radii in $[\rho,r)$.

Since $\{dg\, t \mid t \in \Phi\} = \mathbb{Q}$ we can choose an infinite sequence of points $t_\nu \in \Phi$, $\nu \in \mathbb{N}$, such that

$$\rho < dg\, t_1 < dg\, t_2 < \dots$$

and

$$\lim_{\nu\to\infty} dg\, t_\nu = r. \tag{5.17.1}$$

If we denote $\rho_\nu := dg\, t_\nu$, $\nu \in \mathbb{N}$, then from remark 5.15 we have

$$dg\, f(t_\nu) = M_{\rho_\nu}(f).$$

From (5.17.1) and the continuity of $M_\rho(f)$ as a function of ρ we conclude that

$$\lim_{\nu\to\infty} dg\, f(t_\nu) = \lim_{\nu\to\infty} M_{\rho_\nu}(f) = M_r(f).$$

Hence

$$\sup_{dgt<r} dg\, f(t) \geq M_r(f). \tag{5.17.2}$$

On the other hand we have from remark 5.15

$$\sup_{dgt\leq r} dg\, f(t) \leq M_r(f). \tag{5.17.3}$$

Now the lemma follows from (5.17.2) and (5.17.3). $\square$

5.18. LEMMA. *Let* $r \in \mathbb{R}$, $f \in \mathcal{P}_r$. *Then for every* $t_0 \in \Phi$ *with* $dg\ t_0 \leq r$ *the function* g, *defined by*

$$g(t) = f(t+t_0), \qquad t \in \Phi,\ dg\ t \leq r,$$

is also an element of $\mathcal{P}_r$.

PROOF. Denote $f(t) = \sum_{i=0}^{\infty} a_i t^i$ and define a sequence of polynomials $\{g_\nu\}_{\nu=1}^{\infty}$ in $\mathcal{P}_r$ by

$$g_\nu(t) := \sum_{i=0}^{\nu} a_i (t+t_0)^i.$$

For all $t \in \Phi$ with $dg\ t \leq r$ and $\mu < \nu$ we have

$$dg(g_\nu(t)-g_\mu(t)) \leq \max_{\mu<i\leq\nu} \{dg\ a_i + i\ dg(t+t_0)\}$$

$$\leq \max_{\mu<i\leq\nu} (dga_i+ir)$$

and therefore

$$\sup_{dgt\leq r} dg(g_\nu(t)-g_\mu(t)) \leq \max_{\mu<i\leq\nu} (dga_i+ir).$$

Hence, in view of lemma 5.17, we have

$$M_r(g_\nu-g_\mu) \leq \max_{\mu<i\leq\nu} (dga_i+ir).$$

Since $f \in \mathcal{P}_r$, this means that $\{g_\nu\}_{\nu=1}^{\infty}$ is a Cauchy sequence in $\mathcal{P}_r$ with the norm topology from lemma 5.4 and hence a convergent sequence with limit, say g. In view of remark 5.5 we have for every $t \in \Phi$ with $dg\ t \leq r$

$$g(t) = \lim_{\nu\to\infty} g_\nu(t) = \sum_{i=0}^{\infty} a_i (t+t_0)^i = f(t+t_0). \qquad \square$$

Proof of theorem 5.16. Let $t_0 \in \Phi$, $dg\ t_0 = r$. According to lemma 5.18 the function g, defined by

$$(5.16.1) \qquad g(t) = f(t+t_0), \qquad t \in \Phi,\ dg\ t \leq r,$$

belongs to $\mathcal{P}_r$. Hence

(5.16.2) $\sup_{dgt<r} dg\, g(t) \le \sup_{dgt=r} dg\, f(t) \le \sup_{dgt\le r} dg\, f(t).$

On the other hand it follows from lemma 5.17 and (5.16.1) that

(5.16.3) $\sup_{dgt<r} dg\, g(t) = \sup_{dgt\le r} dg\, g(t) = \sup_{dgt\le r} dg\, f(t) = M_r(f).$

Now the theorem follows from (5.16.2) and (5.16.3). □

5.19. LEMMA. *Let* $g \in \Phi[t]$ be given by

$$g(t) := a_0 + a_1 t + \ldots + a_n t^n,\ a_0 \neq 0,\ a_n \neq 0,\ n > 0.$$

Let $R_1 < R_2 < \ldots < R_\ell$ *be the hooking-radii of* g *in* $(-\infty,\infty)$. *Then* g *has* $d(g,R_k)$ *zeros* $\beta \in \Phi$ *with* $dg\ \beta = R_k$, $1 \le k \le \ell$, *multiple zeros counted according to their multiplicity. There are no other zeros of* g, *i.e.*

$$\sum_{k=1}^{\ell} d(g,R_k) = n.$$

PROOF. Since Φ is algebraically closed, g has exactly n zeros in Φ. Denote them by $\beta_1,\beta_2,\ldots,\beta_n$.

In view of $dg\ g(\beta_i) = -\infty$, it follows from remark 5.15 that

$$dg\ \beta_i \in \{R_1,R_2,\ldots,R_\ell\}, \qquad i = 1,2,\ldots,n.$$

Hence, if $\mu_j \in \mathbb{N}^0$ denotes the number of zeros β with $dg\ \beta = R_j$, $j = 1,2,\ldots,\ell$, then

$$\mu_1 + \mu_2 + \ldots + \mu_\ell = n.$$

From

$$g(t) = a_n \prod_{i=1}^{n} (t-\beta_i)$$

we infer that

(5.19.1) $$dg\ g(t) = dg\ a_n + \sum_{i=1}^{n} dg(t-\beta_i).$$

Now take a number k from the set $\{1,2,\ldots,\ell\}$. Let $t \in \Phi$ be such that

$R_k < dg\ t < R_{k+1}$ if $k \neq \ell$ and $R_k < dg\ t$ if $k = \ell$. Then it follows from (5.19.1) that

$$dg\ g(t) = dg\ a_n + (\mu_1+\mu_2+\ldots+\mu_k)dg\ t + \sum_{j=k+1}^{\ell} \mu_j R_j .$$

Now $dg\ g(t) = M_\rho(g)$ where $\rho = dg\ t$. (See (5.15.1).) Hence for $k = 1,2,\ldots,\ell$ and $\rho \in \mathbb{Q}$ such that $R_k < \rho < R_{k+1}$ if $k \neq \ell$ and $R_k < \rho$ if $k = \ell$, we have

$$(5.19.2) \qquad M_\rho(g) = dg\ a_n + (\mu_1+\mu_2+\ldots+\mu_k)\rho + \sum_{j=k+1}^{\ell} \mu_j R_j .$$

Taking $\rho \to R_k + 0$, it follows from (5.19.2) and the continuity of M_ρ, that

$$(5.19.3) \qquad M_{R_k}(g) = dg\ a_n + (\mu_1+\mu_2+\ldots+\mu_k)R_k + \sum_{j=k+1}^{\ell} \mu_j R_j, \qquad 1 \le k \le \ell .$$

From this it follows by subtraction that for $1 \le k < \ell$

$$M_{R_{k+1}}(g) - M_{R_k}(g) = (\mu_1+\mu_2+\ldots+\mu_k)(R_{k+1}-R_k).$$

By theorem 5.11

$$M_{R_{k+1}}(g) - M_{R_k}(g) = dg\ a_{i_k} + i_k R_{k+1} - (dga_{i_k} + i_k R_k) = i_k(R_{k+1}-R_k)$$

and so, in view of $R_{k+1} - R_k \neq 0$, we have

$$(5.19.4) \qquad i_k = \mu_1 + \mu_2 + \ldots + \mu_k, \qquad 1 \le k < \ell .$$

For $k = \ell$ we have from (5.19.2) and theorem 5.11

$$dg\ a_{i_\ell} + i_\ell \rho = dg\ a_n + (\mu_1+\mu_2+\ldots+\mu_\ell)\rho, \qquad \rho > R_\ell .$$

Hence

$$(5.19.5) \qquad i_\ell = \mu_1 + \mu_2 + \ldots + \mu_\ell .$$

The lemma now follows immediately from (5.19.4), (5.19.5) and corollary 5.14. □

5.20 THEOREM. *Let* $r \in \mathbb{R}$, $f \in P_r$, $f(t) = \sum_{i=h}^{\infty} a_i t^i$, $a_h \neq 0$. *Then* f *has a zero* β, $\beta \in \Phi$, $\beta \neq 0$ *with* $\mathrm{dg}\,\beta = \rho \leq r$ *if and only if* ρ *is a hooking-radius of* f.

PROOF. Suppose that ρ is not a hooking-radius of f. Then it follows from (5.15.1) that $\mathrm{dg}\, f(t) = M_\rho(f) \neq -\infty$ for $t \in \Phi$, $\mathrm{dg}\, t = \rho$. Hence t cannot be a zero of f.

Suppose now that R_k is a hooking-radius of f in $(-\infty, r]$. Let $\{n_\nu\}_{\nu=1}^{\infty}$ be the increasing sequence of natural numbers such that

$$
\begin{aligned}
& n_1 > n_0, \text{ where } n_0 \text{ is defined by (5.8.1)},\\
& a_{n_\nu} \neq 0,\; \nu = 1,2,\ldots,\\
& a_k = 0 \quad \text{for } k > n_0,\; k \notin \{n_\nu\}_{\nu=1}^{\infty},
\end{aligned}
$$

i.e. the a_{n_ν} are the non-zero coefficients in $\sum_{i=h}^{\infty} a_i t^i$ with index greater than n_0. For $\nu \in \mathbb{N}$ we define

$$
(5.20.1) \qquad P_\nu(t) := \sum_{i=h}^{n_\nu} a_i t^i .
$$

In view of $n_1 > n_0$, it follows from the definition 5.7 of the hooking-radii that P_ν and f have the same set of hooking-radii $R_1, R_2, \ldots, R_\ell$ in $(-\infty, r]$. Also the numbers i_k, $k = 1,2,\ldots,\ell$ coincide for P_ν and f. We obtain from lemma 5.19 and corollary 5.14 that P_ν has just

$$
d = d_k = d(P_\nu, R_k) = d(f, R_k) = i_k - i_{k-1}
$$

zeros $\beta_1^{(\nu)}, \beta_2^{(\nu)}, \ldots, \beta_\alpha^{(\nu)}$ in Φ with $\mathrm{dg}\,\beta_j^{(\nu)} = R_k$, $j = 1,2,\ldots,d$ and just i_{k-1} zeros β in Φ with $\mathrm{dg}\,\beta < R_k$. ($i_0 := h$.)

From

$$
P_\nu(t) = a_{n_\nu} \prod_{j=1}^{d} (t - \beta_j^{(\nu)}) \prod_{\substack{P_\nu(\beta)=0 \\ \mathrm{dg}\beta \neq R_k}} (t - \beta)
$$

it follows that

$$\mathrm{dg}\, P_\nu(t) = \mathrm{dg}\, a_{n_\nu} + \sum_{j=1}^{d} \mathrm{dg}(t-\beta_j^{(\nu)}) + i_{k-1}R_k + \sum_{\substack{P_\nu(\beta)=0 \\ \mathrm{dg}\beta>R_k}} \mathrm{dg}\, \beta$$

for every $t \in \Phi$ with $\mathrm{dg}\, t = R_k$. From theorem 5.11, (5.19.3) and from (5.19.4) or (5.19.5) we infer that

$$\mathrm{dg}\, a_{i_k} - \mathrm{dg}\, a_{n_\nu} = \sum_{\substack{P_\nu(\beta)=0 \\ \mathrm{dg}\beta>R_k}} \mathrm{dg}\, \beta.$$

Hence we have

$$\text{(5.20.2)} \qquad \mathrm{dg}\, P_\nu(t) = \sum_{j=1}^{d} \mathrm{dg}(t-\beta_j^{(\nu)}) + c_k, \qquad t \in \Phi,\ \mathrm{dg}\, t = R_k,$$

where c_k is an abbreviation for $\mathrm{dg}\, a_{i_k} + i_{k-1} R_k$; note that c_k is independent of ν.

Now we construct inductively a sequence $\{\beta_\nu\}_{\nu=1}^{\infty}$ in the following way. We choose β_1 arbitrarily from the set $\{\beta_1^{(1)}, \beta_2^{(1)}, \ldots, \beta_d^{(1)}\}$. Then we take β_2 from the set $\{\beta_1^{(2)}, \beta_2^{(2)}, \ldots, \beta_d^{(2)}\}$ in such a way that

$$\mathrm{dg}(\beta_2-\beta_1) = \min_{1\le j\le d} \mathrm{dg}(\beta_j^{(2)}-\beta_1).$$

In general, when $\beta_1, \beta_2, \ldots, \beta_{\nu-1}$ are determined, we take $\beta_\nu \in \{\beta_1^{(\nu)}, \beta_2^{(\nu)}, \ldots, \beta_d^{(\nu)}\}$ such that

$$\text{(5.20.3)} \qquad \mathrm{dg}(\beta_\nu-\beta_{\nu-1}) = \min_{1\le j\le d} \mathrm{dg}(\beta_j^{(\nu)}-\beta_{\nu-1}), \qquad \nu = 2,3,\ldots .$$

Clearly

$$P_\nu(\beta_\nu) = 0, \qquad \nu = 1,2,\ldots,$$

$$\mathrm{dg}\, \beta_\nu = R_k, \qquad \nu = 1,2,\ldots .$$

From (5.20.3) we derive that

$$\mathrm{dg}(\beta_\nu-\beta_{\nu-1}) \le \frac{1}{d_k} \sum_{j=1}^{d} \mathrm{dg}(\beta_j^{(\nu)}-\beta_{\nu-1})$$

and then from (5.20.2) with $t = \beta_{\nu-1}$

$$dg(\beta_\nu - \beta_{\nu-1}) \leq \frac{1}{d_k} dg\, P_\nu(\beta_{\nu-1}) - \frac{1}{d_k} c_k.$$

The polynomials P_ν were constructed in such a way that

$$P_\nu(t) = P_{\nu-1}(t) + a_{n_\nu} t^{n_\nu}, \qquad \nu = 2,3,\ldots;$$

hence

$$P_\nu(\beta_{\nu-1}) = P_{\nu-1}(\beta_{\nu-1}) + a_{n_\nu} \beta_{\nu-1}^{n_\nu} = a_{n_\nu} \beta_{\nu-1}^{n_\nu}.$$

So we come to the conclusion that

$$dg(\beta_\nu - \beta_{\nu-1}) \leq \frac{1}{d_k} (dga_{n_\nu} + n_\nu R_k) - \frac{1}{d_k} c_k$$

and since

$$\lim_{\nu\to\infty} dg\, a_{n_\nu} + n_\nu R_k = -\infty,$$

because $R_k \leq r$, we see that $\{\beta_\nu\}_{\nu=1}^{\infty}$ is a Cauchy-sequence.

Define

$$\beta := \lim_{\nu\to\infty} \beta_\nu.$$

Clearly $dg\, \beta = R_k$. Finally

$$dg\, f(\beta) = \lim_{\nu\to\infty} dg\, f(\beta_\nu) = \lim_{\nu\to\infty} dg\{P_\nu(\beta_\nu) + \sum_{i>n_\nu} a_i \beta_\nu^i\} = -\infty,$$

i.e.

$$f(\beta) = 0. \qquad \square$$

5.21. <u>COROLLARY</u> (SCHÖBE). *An entire function $f: \Phi \to \Phi$ which has no zeros in Φ is a non-zero constant.*

<u>PROOF</u>. Since f has no zeros in Φ we have

$$f(t) = \sum_{i=0}^{\infty} a_i t^i, \qquad a_0 \neq 0.$$

From theorem 5.20 we see that f has no hooking-radii in $(-\infty,\infty)$. Hence by theorem 5.11(ii) we have

$$\mathrm{dg}\ a_i + i\rho < M_\rho(f) = \mathrm{dg}\ a_0, \qquad i \in \mathbb{N},\ \rho \in \mathbb{R}.$$

This can only hold for all $\rho \in \mathbb{R}$ if

$$\mathrm{dg}\ a_i = -\infty, \qquad i \in \mathbb{N},$$

which means that $f(t) = a_0$. □

5.22. LEMMA. *Let* $r \in \mathbb{R}$, $f \in P_r$, $f(t) = \sum_{i=h}^{\infty} a_i t^i$, $a_h \neq 0$. *Let* $\beta \neq 0$ *be a zero of* f *with* $\mathrm{dg}\ \beta = \rho \leq r$. *Then there exists a* $g \in P_r$ *such that*

$$f(t) = (t-\beta)g(t)$$

and

$$d(f,\rho) = d(g,\rho) + 1.$$

PROOF. Since $f \in P_r$, $\mathrm{dg}\ \beta \leq r$ and $\beta \neq 0$, we can define

$$b_j := \frac{1}{\beta^{j+1}} \sum_{i>j} a_i\beta^i, \qquad j \geq h. \tag{5.22.1}$$

Next we show that if we put

$$g(t) := \sum_{j=h}^{\infty} b_j t^j, \tag{5.22.2}$$

then $g \in P_r$. Indeed, for $j = h, h+1, \ldots$ we have from (5.22.1)

$$\mathrm{dg}\ b_j + j\rho \leq \max_{i>j} (\mathrm{dg} a_i + i\rho) - \rho. \tag{5.22.3}$$

Hence, as $\rho \leq r$,

$$\mathrm{dg}\ b_j + jr \leq \max_{i>j} (\mathrm{dg} a_i + ir) - r$$

and since

$$\lim_{i\to\infty} (\mathrm{dg}\ a_i + ir) = -\infty,$$

we conclude that $g \in \mathcal{P}_r$. From (5.22.1), (5.22.2) and $f(\beta) = 0$ we see that

$$g(t)(t-\beta) = \sum_{j=h}^{\infty} b_j t^{j+1} - \sum_{j=h}^{\infty} \beta b_j t^j$$

$$= \sum_{j=h+1}^{\infty} (b_{j-1} - \beta b_j) t^j - \beta b_h t^h = f(t).$$

This proves the first assertion of the lemma.

By the Maximum Modulus Principle, theorem 5.16, we have

$$M_\rho(f) = \sup_{\mathrm{dg} t = \rho} (\mathrm{dg}\ g(t) + \mathrm{dg}(t-\beta)),$$

from which it follows immediately that

$$M_\rho(f) \le M_\rho(g) + \rho.$$

On the other hand we derive from (5.22.3) that

$$M_\rho(g) \le M_\rho(f) - \rho.$$

Hence

(5.22.4) $\quad M_\rho(g) = M_\rho(f) - \rho.$

From theorem 5.20 we observe that $\rho = \mathrm{dg}\ \beta$ is a hooking-radius of f, say R_k. From theorem 5.11 we observe that

(5.22.5) $\quad \max_{i \ge h} \{i \mid \mathrm{dg}\ a_i + iR_k = M_{R_k}(f)\} = i_k$

and

(5.22.6) $\quad \min_{i \ge h} \{i \mid \mathrm{dg}\ a_i + iR_k = M_{R_k}(f)\} = i_{k-1}$ *).

Hence from (5.22.1) and (5.22.5) we obtain

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*) where $i_0 := h$.

$$(5.22.7)\qquad \mathrm{dg}\, b_j + jR_k = \mathrm{dg}\Big(\sum_{i>j} a_i\beta^i\Big) - R_k < M_{R_k}(f) - R_k, \qquad j \geq i_k$$

and

$$(5.22.8)\qquad \mathrm{dg}\, b_{i_k-1} + (i_k-1)R_k = M_{R_k}(f) - R_k.$$

Since $f(\beta) = 0$ we can rewrite (5.22.1) as

$$b_j = -\frac{1}{\beta^{j+1}} \sum_{i\leq j} a_i\beta^i, \qquad j \geq h,$$

from which it follows, using (5.22.6), that

$$(5.22.9)\qquad \mathrm{dg}\, b_j + jR_k < M_{R_k}(f) - R_k, \qquad j < i_{k-1}$$

and

$$(5.22.10)\qquad \mathrm{dg}\, b_{i_{k-1}} + i_{k-1} R_k = M_{R_k}(f) - R_k.$$

From (5.22.7),...,(5.22.10) and corollary 5.14 we obtain

$$d(g,R_k) = d(f,R_k) -1. \qquad \square$$

5.23. <u>THEOREM</u> (SCHÖBE). *Let* $r \in \mathbb{R}$, $f \in \mathcal{P}_r$, $f(t) = \sum_{i=h}^{\infty} a_i t^i$, $a_h \neq 0$. *For* $\rho \leq r$ *let* $d(f,\rho)$ *be defined by* (5.13.1). *If* $R_1 < R_2 < \ldots < R_\ell$ *are the hooking-radii of* f *in* $(-\infty,r]$, *then* f *has a zero of order* h *in* 0 *and* $d(f,R_k)$ *zeros* β *with* $\mathrm{dg}\,\beta = R_k$, $k = 1,2,\ldots,\ell$, *with multiple zeros counted according to their multiplicity* *). *These are the only zeros of* f *in* $\{t \in \Phi \mid \mathrm{dg}\, t \leq r\}$.

<u>PROOF</u>. In view of theorem 5.20 we only have to prove that f has $d(f,R_k)$ zeros in $\{t \in \Phi \mid \mathrm{dg}\, t = R_k\}$, $k = 1,2,\ldots,\ell$. From theorem 5.20 we observe that f has at least one zero β with $\mathrm{dg}\,\beta = R_k$, $1 \leq k \leq \ell$. According to lemma 5.22 there is a $g \in \mathcal{P}_r$, $g(t) = \sum_{i=h}^{\infty} b_i t^i$, such that

$$f(t) = (t-\beta)g(t)$$

and

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*) In view of the previous lemma it is obvious what must be understood by the order of a zero.

$$d(g,R_k) = d(f,R_k) - 1.$$

If $d(g,R_k) = 0$, then it follows from (5.13.1) that there is only one $i \geq h$ such that

$$dg\, b_i + iR_k = M_{R_k}(g).$$

Thus R_k is not a hooking-radius of g and therefore g has no zeros in $\{t \mid dg\, t = R_k\}$. Hence in this case f has $d(f,R_k) = 1$ zero in $\{t \in \Phi \mid dg\, t = R_k\}$.

In case $d(g,R_k) > 0$ it follows from (5.13.1) that R_k is a hooking-radius of g. Then we apply the argument above with g instead of f. Now it is obvious how we proceed and that the process stops after $d(f,R_k)$ steps. □

5.24. COROLLARY (Product Formula for Entire Functions). *Let* $f\colon \Phi \to \Phi$ *be an entire function,* $f(t) = \sum_{i=h}^{\infty} a_i t^i$, $a_h \neq 0$. *Let* $\mathcal{R}$ *denote the set of hooking-radii of* f *in* $(-\infty,\infty)$. *(*$\mathcal{R}$ *can be empty, finite or infinite.) For* $R \in \mathcal{R}$, *let* $\beta_{R,1}, \beta_{R,2}, \ldots, \beta_{R,d(f,R)}$ *denote the zeros of* f *with valuation* R. *Then for all* $t \in \Phi$ *we have*

$$(5.24.1) \qquad f(t) = a_h t^h \prod_{R\in\mathcal{R}} \prod_{i=1}^{d(f,R)} \left(1 - \frac{t}{\beta_{R,i}}\right).$$

PROOF. If f has no zeros, the theorem is a special case, with $h = 0$, of corollary 5.21. If f has a finite number of zeros, the theorem follows easily from lemma 5.22 and corollary 5.21.

Now we suppose that f has an infinite number of hooking-radii in $(-\infty,\infty)$. Let $\{R_k\}_{k=1}^{\infty}$ be the increasing sequence of hooking-radii of f. According to theorem 5.23 and lemma 5.22 we can define a sequence of entire functions g_n by

$$(5.24.2) \qquad f(t) = a_h t^h \prod_{k=1}^{n} \prod_{i=1}^{d(f,R_k)} \left(1 - \frac{t}{\beta_{R_k,i}}\right) g_n(t).$$

Clearly g_n has no zeros in $(-\infty,R_n]$ and we can write

$$(5.24.3) \qquad g_n(t) = 1 + \sum_{i=1}^{\infty} b_{ni} t^i, \qquad b_{ni} \in \Phi.$$

From theorem 5.20 we conclude that g_n has no hooking-radii in $(-\infty,R_n]$ and therefore, by theorem 5.11, (5.24.4) $\quad \mathrm{dg}\, b_{ni} + iR_n < 0, \qquad i \geq 1.$

Now let $r \in \mathbb{R}$ be arbitrary but fixed. From (5.24.3) we get

$$M_r(g_n-1) = \max_{i\geq 1} (\mathrm{dg} b_{ni}+ir) \leq \max_{i\geq 1} (\mathrm{dg} b_{ni}+iR_n) + \max_{i\geq 1} i(r-R_n).$$

Since $\{R_n\}_{n=1}^{\infty}$ is an infinite, increasing sequence we infer from (5.24.4) that

$$\lim_{n\to\infty} M_r(g_n-1) = -\infty,$$

i.e. the sequence $\{g_n\}_{n=1}^{\infty}$ in $\mathcal{P}_r$ is convergent to the identity function $1 \in \mathcal{P}_r$. Hence (5.24.1) is valid for $t \in \Phi$ with $\mathrm{dg}\, t \leq r$. But since r was chosen arbitrarily we have proved (5.24.1) for all $t \in \Phi$. $\square$

The following corollary is equivalent to theorem 2.12, but its proof is different.

5.25. <u>COROLLARY</u>. *The function* ψ, *given by*

$$\psi(t) := \sum_{j=0}^{\infty} (-1)^j \frac{t^{q^j}}{F_j}, \qquad t \in \Phi$$

has a zero of order 1 *in* 0 *and* $q^k - q^{k-1}$ *zeros* $\beta \in \Phi$ *with* $\mathrm{dg}\, \beta = k + \frac{1}{q-1}$, $k \in \mathbb{N}$. *Moreover, if* $\alpha \in \Phi$ *is any zero of* ψ *with* $\mathrm{dg}\, \alpha = \frac{q}{q-1}$, *then*

$$\psi(t) = t \prod_{\substack{E\in\mathbb{F}_q[X] \\ E \neq 0}} \left(1 - \frac{t}{E\alpha}\right).$$

<u>PROOF</u>. From corollary 5.12 and definition 5.10 we have

$$i_0 = 1;$$

$$R_1 = \min_{j>0} \frac{-\mathrm{dg}F_0+\mathrm{dg}F_j}{q^j-1} = 1 + \frac{1}{q-1};$$

$$i_1 = \max_{j>0} \{q^j \mid -\mathrm{dg}\, F_j + q^j \cdot \frac{q}{q-1} = M_{R_1}(\psi)\} = q;$$

$$d(\psi,R_1) = q - 1,$$

and inductively for $k > 1$

$$R_k = \min_{j\geq k} \frac{-dgF_{k-1}+dgF_j}{q^j-q^{k-1}} = k + \frac{1}{q-1} ;$$

$$i_k = \max_{j\geq k} \{q^j \mid -\operatorname{dg} F_j + q^j(k+\frac{1}{q-1}) = M_{R_k}(\psi)\} = q^k;$$

$$d(\psi,R_k) = q^k - q^{k-1}.$$

According to theorem 5.23 ψ has exactly $q^k - q^{k-1}$ zeros β with $\operatorname{dg}\beta = k + \frac{1}{q-1}$, $k \in \mathbb{N}$.

Let α be a zero of ψ, then it follows from theorem 2.11a,b,c that $\psi(E\alpha) = 0$ for all $E \in \mathbb{F}_q[X]$.

Now let $\alpha \neq 0$ be a zero of ψ such that $\operatorname{dg}\alpha$ is minimal, i.e. $\operatorname{dg}\alpha = \frac{q}{q-1}$. Since the number of polynomials in $\mathbb{F}_q[X]$ of degree less than k equals q^k, we conclude that the set of zeros of ψ is exactly $\{E\alpha \mid E \in \mathbb{F}_q[X]\}$. The last assertion now follows from (5.24.1). □

5.26. <u>COROLLARY</u>. *The functions* $J_n (n\in\mathbb{N}^0)$, *defined in* (4.1.1) *by*

$$J_n(t) := \sum_{k=0}^{\infty} (-1)^k \frac{t^{q^{n+k}}}{F_{n+k}F_k^{q^n}} ,$$

have a zero of order q^n *in* $t = 0$ *and have* $q^{k+1} - q^k$ *different zeros* β *with* $\operatorname{dg}\beta = n + 2k + \frac{2}{q-1}$, *each of order* q^n.

<u>PROOF</u>. From corollary 5.12 and definition 5.10 we have

$$i_0 = q^n;$$

$$R_1 = \min_{k>0} \frac{-\operatorname{dg}(F_nF_0^{q^n}) + \operatorname{dg}(F_{n+k}F_k^{q^n})}{q^{n+k}-q^n} = n + 2 + \frac{2}{q-1} ;$$

$$i_1 = \max_{k>0} \left\{q^{n+k} \mid -\operatorname{dg}\left(F_{n+k}F_k^{q^n}\right)+\left(n+\frac{2q}{q-1}\right)q^{n+k} = M_{R_1}(J_n)\right\}$$

$$= \max_{k>0} \left\{q^{n+k} \mid q^{n+k}\left(-2k+\frac{2q}{q-1}\right) = M_{R_1}(J_n)\right\} = q^{n+1}$$

and inductively

$$R_k = n + 2k + \frac{2}{q-1} ;$$

$$i_k = q^{n+k}.$$

Now it follows from theorem 5.23 that J_n has a zero of order q^n in $t = 0$ and that J_n has $q^{n+k} - q^{n+k-1}$ zeros β with

$$dg\ \beta = n + 2k + \frac{2}{q-1}, \qquad k \in \mathbb{N}.$$

Besides, it follows that J_n has no other zeros.

From theorem 4.2(i) we see that every zero of J_n is a zero of J_{-n}, moreover that every zero of J_n has multiplicity at least q^n.

Let β be a zero of J_n with $dg\ \beta = n + 2 + \frac{2}{q-1}$. Then it follows from the linearity of J_n that $c\beta$, $c \in \mathbb{F}_q^*$ is also a zero of J_n and $dg(c\beta) = dg\ \beta$. Hence J_n has at least q-1 different zeros β with $dg\ \beta = n + 2 + \frac{2}{q-1}$ and multiplicity $\geq q^n$. Since $d(J_n,R_1) = q^{n+1} - q^n$, we conclude that J_n has exactly q-1 different zeros β with $dg\ \beta = n + 2 + \frac{2}{q-1}$, each of multiplicity q^n.

Suppose we have proved that J_n has exactly $q^k - q^{k-1}$ different zeros β with $dg\ \beta = n + 2k + \frac{2}{q-1}$, each of multiplicity q^n, $k = 1,2,\ldots,\kappa$. Then the number of different zeros β with $dg\ \beta \leq n + 2\kappa + \frac{2}{q-1}$ equals q^κ. Let β^* be a zero of J_n with $dg\ \beta^* = n + 2(\kappa+1) + \frac{2}{q-1}$. Then for every zero β with $dg\ \beta < dg\ \beta^*$ it follows, from the linearity of J_n, that $c\beta^* + \beta (c \in \mathbb{F}_q^*)$ is a zero of J_n and $dg(c\beta^*+\beta) = dg\ \beta^*$. Hence J_n has at least $(q-1)q^\kappa$ different zeros β with $dg\ \beta = n + 2(\kappa+1) + \frac{2}{q-1}$, each of multiplicity $\geq q^n$. Since $d(J_n,R_{\kappa+1}) = q^{n+\kappa+1} - q^{n+\kappa}$, we conclude that J_n has exactly $q^{\kappa+1} - q^\kappa$ different zeros β with $dg\ \beta = n + 2(\kappa+1) + \frac{2}{q-1}$, each of multiplicity q^n. □

<u>FINAL REMARK</u>. The supremum in the Maximum Modulus Principle (theorem 5.16) is actually attained and is therefore a maximum. To prove this we may suppose that $r = 0$ and that

$$f(t) = \sum_{i=0}^{\infty} a_i t^i, \qquad a_0 \neq 0.$$

Let n_0 denote the smallest natural number such that $dg\ a_i < dg\ a_0$, $i > n_0$ (see 5.8.1). If we define

$$g(t) := \sum_{i=0}^{n_0} a_i t^i,$$

then

$$M_0(g) = M_0(f).$$

Now we define inductively the following sequence of elements of Φ: $t_0 = 1$; for $i = 1,2,\ldots,n_0$ the element t_i is a solution of the equation

$$t^q - t + t_{i-1} = 0.$$

(This is possible since Φ is algebraically closed.) Then

$$\mathrm{dg}\ t_i = 0, \qquad 0 \le i \le n_0$$

and

$$\mathrm{dg}(t_i - t_j) = 0, \qquad i \neq j,\ 0 \le i,\ j \le n_0.$$

The system of equations

$$\sum_{i=0}^{n_0} a_i t_j^i = g(t_j), \qquad j = 0,1,\ldots,n_0$$

in $a_0, a_1, \ldots, a_{n_0}$ is solvable and

$$\mathrm{dg}\ a_i \le \max_{0 \le j \le n_0} \mathrm{dg}\ g(t_j), \qquad i = 0,1,\ldots,n_0.$$

So according to theorem 5.16

$$M_0(g) = \max_{0 \le i \le n_0} \mathrm{dg}\ a_i \le \max_{0 \le j \le n_0} \mathrm{dg}\ g(t_j) \le \sup_{\mathrm{dg} t = 0} \mathrm{dg}\ g(t) = M_0(g).$$

Hence there exists a $t^* \in \Phi$ with $\mathrm{dg}\ t^* = 0$ such that

$$\mathrm{dg}\ g(t^*) = M_0(g).$$

Since

$$\mathrm{dg}\ f(t^*) = \mathrm{dg}(g(t^*) + \sum_{i > n_0} a_i t^{*i}) = \mathrm{dg}\ g(t^*)$$

and since $M_0(f) = M_0(g)$, we have proved our assertion.

CHAPTER II

TRANSCENDENCE IN Φ

In the first section of this chapter we shall mention some properties of elements of Φ which are algebraic over $\mathbb{F}_q(X)$. In the second section we shall give a survey of known results on transcendence in the field Φ. For instance, we mention analogues of the following three classical theorems:

(i) the theorem of Liouville on the approximation of algebraic numbers by rational numbers (M. MAHLER, 1949),

(ii) the theorem on transcendence of the values of the exponential function in non-zero algebraic points (L.I. WADE, 1941),

(iii) the Gelfond-Schneider theorem (L.I. WADE, 1946).

6. PRELIMINARIES

In this section k is always a subfield of Φ.

6.1. DEFINITION. An element $E \in \mathbb{F}_q[X]$ is called a *monic element of* $\mathbb{F}_q[X]$ if E is a monic polynomial over $\mathbb{F}_q$.

The elements $A_1,A_2,\dots,A_n \in \mathbb{F}_q[X]$ are called *relatively prime* if they do not have a common divisor in $\mathbb{F}_q[X]$ other than units.
Notation: $(A_1,A_2,\dots,A_n) = 1$.

The *least common multiple of the* n *elements* $B_1,B_2,\dots,B_n \in \mathbb{F}_q[X]\backslash\{0\}$ is an element $B \in \mathbb{F}_q[X]$ for which

(i) $\frac{B}{B_i} \in \mathbb{F}_q[X]$, $i = 1,2,\dots,n$,

(ii) dg B is minimal,

(iii) B is monic.

It follows that B is uniquely determined.

Let $\alpha \in \Phi$ be algebraic over $\mathbb{F}_q(X)$ of degree n. From theorem 0.9 it is obvious that there exists a unique, irreducible polynomial $Q \in \mathbb{F}_q[X][t]$

of degree n with the properties:

(i) $Q(\alpha) = 0$,

(ii) Q is a primitive polynomial over $\mathbb{F}_q[X]$,

(iii) the leading coefficient of Q is monic.

6.2. DEFINITION. Let $\alpha \in \Phi$ be algebraic over $\mathbb{F}_q(X)$ of degree n. The unique, irreducible, primitive polynomial $Q \in \mathbb{F}_q[X][t]$ of degree n with monic leading coefficient for which $Q(\alpha) = 0$ is called the *minimal polynomial of* α *over* $\mathbb{F}_q[X]$.

The element α is called *integral algebraic over* $\mathbb{F}_q(X)$ or an *algebraic integer of* Φ if the minimal polynomial of α over $\mathbb{F}_q[X]$ has leading coefficient 1.

N.B. In the following chapters by "minimal polynomial of α" we shall always mean the minimal polynomial of α over $\mathbb{F}_q[X]$.

6.3. DEFINITION. Let $\alpha \in \Phi$ be algebraic. Every $E \in \mathbb{F}_q[X]\backslash\{0\}$, for which $E\alpha$ is an algebraic integer, is called a *denominator of* α.

6.4. LEMMA. (WADE 1941). *Let* $P \in \mathbb{F}_q(X)[t]$ *be a polynomial of degree* $n \geq 1$ (in t). *Then there exists a linear polynomial* $Q \in \mathbb{F}_q[X][t]$ *of degree* q^n (in t) *such that* P *divides* Q.

PROOF. By the Euclidean algorithm we have

$$(6.4.1) \qquad t^{q^i} = \sum_{j=0}^{n-1} b_j^{(i)} t^j + R_i(t)P(t), \qquad i = 0,1,\ldots,n,$$

with $R_i \in \mathbb{F}_q(X)[t]$, $b_j^{(i)} \in \mathbb{F}_q(X)$. Note that if $m \in \mathbb{N}^0$ is defined by $q^m \leq n-1 < q^{m+1}$, then $R_i = 0$ and

$$b_{q^i}^{(i)} = 1, \qquad i = 0,1,\ldots,m.$$

Furthermore R_i has degree $q^i - n$, $i = m+1,\ldots,n$.

If we eliminate $1,t,\ldots,t^{n-1}$ succesively in the right hand side of (6.4.1), we obtain

$$b_0 t + b_1 t^q + \ldots + b_n t^{q^n} = R(t)P(t),$$

where $b_i \in \mathbb{F}_q(X)$ and $R \in \mathbb{F}_q(X)[t]$. From the elimination process it follows that not all the b_i can be zero. Let

$$\nu := \max_{1\le i\le n} \{i \mid b_i \neq 0\}$$

and let $C \in \mathbb{F}_q[X]\backslash\{0\}$ be such that $Cb_0,\ldots,Cb_\nu \in \mathbb{F}_q[X]$. The polynomial Q, defined by

$$Q(t) := (Cb_0)^{q^{n-\nu}} t^{q^{n-\nu}} +\ldots+ (Cb_\nu)^{q^{n-\nu}} t^{q^n},$$

satisfies the conditions of the lemma. □

6.5. LEMMA. *Let $\alpha \in \Phi$ be separable algebraic over $k \subset \Phi$ and let $P \in k[t]$ be its minimal polynomial. Then the zeros of P are all different.*

PROOF. See O. ZARISKI and P. SAMUEL (1958), Ch.II,§5 def.3, cor.2. □

6.6. DEFINITION. Let $\alpha \in \Phi$ be algebraic over $k \subset \Phi$. The different zeros of the minimal polynomial of α are called the *conjugated elements of α over k*.

6.7. THEOREM. *Let $\alpha_1,\alpha_2,\ldots,\alpha_m \in \Phi$ be separable algebraic over $k \subset \Phi$. Then $k(\alpha_1,\alpha_2,\ldots,\alpha_m)$ is a separable algebraic extension of k.*

PROOF. See O. ZARISKI and P. SAMUEL (1958) Ch.II th.10 or I. ADAMSON, th.13.7. □

6.8. THEOREM. *Let $\alpha \in \Phi$ be separable algebraic over $k \subset \Phi$ of degree n and let $\alpha_1 = \alpha,\alpha_2,\ldots,\alpha_n$ be the conjugated elements of α over k. Then there exist exactly n distinct monomorphisms $\sigma_i\colon k(\alpha) \hookrightarrow \Phi$, $i = 1,\ldots,n$ under which k is invariant. These k-monomorphisms can be given by*

$$\sigma_i(\alpha) = \alpha_i, \qquad i = 1,2,\ldots,n.$$

PROOF. See O. ZARISKI and P. SAMUEL (1958), Ch.II, th.16 or I. ADAMSON, th.15.4. □

6.9. LEMMA. *Let $\alpha \in \Phi$ be algebraic over $k \subset \Phi$ of degree n. For $\beta \in k(\alpha)$ let $P \in k[t]$ denote the monic, irreducible polynomial with $P(\beta) = 0$, given by*

$$P(t) := t^m + b_{m-1} t^{m-1} +\ldots+ b_1 t + b_0.$$

Then

$$N_{k(\alpha)\to k}(\beta) = (-1)^n b_0^{n/m}.$$

PROOF. See O. ZARISKI and P. SAMUEL, Ch.II,§10 or P. RIBENBOIM, part II, 5A. □

6.10. LEMMA. *Let* $\alpha \in \Phi$ *be separable algebraic over* $k \subset \Phi$ *of degree* n *and let* $\sigma_1,\sigma_2,\ldots,\sigma_n$ *be the* n k-*monomorphisms* $k(\alpha) \hookrightarrow \Phi$. *Then for every* $\beta \in k(\alpha)$:

$$N_{k(\alpha)\to k}(\beta) = \prod_{j=1}^{n} \sigma_j(\beta).$$

PROOF. See O. ZARISKI and P. SAMUEL, Ch.II,§10 or P. RIBENBOIM, part II, 5A. □

6.11. REMARK. Let K be a finite, separable algebraic extension of $\mathbb{F}_q(X)$. Then there exists a $\theta \in K$ such that $K = \mathbb{F}_q(X)(\theta)$ (see O. ZARISKI and P. SAMUEL Ch.II,th.19.) It follows from lemma 6.9 that for all $\beta \in K$

$$N_{K\to\mathbb{F}_q(X)} \in \mathbb{F}_q(X).$$

Moreover, if β is an algebraic integer of K, then

$$N_{K\to\mathbb{F}_q(X)} \in \mathbb{F}_q[X].$$

Hence, if $\beta \neq 0$ is an algebraic integer of K, then

$$dg(N_{K\to\mathbb{F}_q(X)}(\beta)) \in \mathbb{N}^0.$$

In 1946 L.I. WADE proved an analogon of the classical Gelfond-Schneider theorem. The proof of Wade's theorem starts with the construction of an auxiliary function. This leads to the problem of solving a system of r homogeneous, linear equations in s variables ($r<s$) with coefficients in a given separable algebraic extension of the groundfield $\mathbb{F}_q(X)$. In the classical case we know, by Siegel's lemma (see e.g. Th. SCHNEIDER (1957), HILFSSATZ 31), that there is a solution with absolute value not too large. In the following we shall give a proof of an analogue of Siegel's lemma.

6.12. LEMMA. *Let* $m,n \in \mathbb{N}$ *with* $m < n$. *The system of* m *homogeneous, linear equations in the* n *unknowns* X_i, $i = 1,2,\ldots,n$,

$$(6.12.1) \qquad \sum_{i=1}^{n} A_{ki}\, X_i = 0, \qquad k = 1,2,\ldots,m,$$

where $A_{ki} \in \mathbb{F}_q[X]$ *and*

$$\max_{\substack{1\le i\le n\\ 1\le k\le m}} \operatorname{dg} A_{ki} \le a \qquad (a\ge 0),$$

has a non-trivial solution $C_1, C_2, \ldots, C_n$ *with*

$$C_i \in \mathbb{F}_q[X], \qquad i = 1,\ldots,n,$$

such that

$$\operatorname{dg} C_i \le \frac{am}{n-m}.$$

PROOF. Define $y_k \in \mathbb{F}_q[X][t_1,\ldots,t_n]$ by

$$y_k(t_1,\ldots,t_n) := \sum_{i=1}^{n} A_{ki}\, t_i, \qquad k = 1,2,\ldots,m.$$

For $X_i \in \mathbb{F}_q[X]$, $i = 1,2,\ldots,n$, we have

$$(6.12.2) \qquad Y_k := y_k(X_1,\ldots,X_n) \in \mathbb{F}_q[X], \qquad k = 1,2,\ldots,m.$$

Let $\ell \in \mathbb{N}$ be arbitrary. The "cube" $\{(\xi_1,\ldots,\xi_n) \mid \xi_i \in \Phi,\ \operatorname{dg} \xi_i < \ell\}$ contains $q^{\ell n}$ lattice points $(X_1,\ldots,X_n)$. (The notion of lattice point in Φ^n means an n-tuple $(X_1,\ldots,X_n)$ of elements $X_i \in \mathbb{F}_q[X]$, $i = 1,\ldots,n$.) For these lattice points $(X_1,\ldots,X_n)$ we have

$$(6.12.3) \qquad \operatorname{dg} Y_k < \max_{1\le i\le n} \operatorname{dg} A_{ki} + \ell \le a + \ell, \qquad k = 1,2,\ldots,m.$$

Hence every lattice point $\{(X_1,\ldots,X_n) \mid \operatorname{dg} X_i < \ell,\ i = 1,\ldots,n\}$ corresponds, via (6.12.2), with one of the $q^{(a+\ell)m}$ lattice points of the cube $\{(\eta_1,\ldots,\eta_m) \mid \eta_i \in \Phi,\ \operatorname{dg} \eta_i < a + \ell\}$.

Now let ℓ be the smallest number such that the number of lattice points $\{(Y_1,\ldots,Y_m) \mid \operatorname{dg} Y_i < a + \ell\}$ is less than the number of lattice points $\{(X_1,\ldots,X_n) \mid \operatorname{dg} X_i < \ell\}$;

$$\ell := \left[\frac{am}{n-m} + 1\right].$$

Then according to the Box Principle of Dirichlet there are at least two different lattice points $(C_1^{(1)},\dots,C_n^{(1)})$ and $(C_1^{(2)},\dots,C_n^{(2)})$ which correspond with the same lattice point $(Y_1,\dots,Y_m)$. Hence $(C_1,\dots,C_n)$ with $C_i = C_i^{(1)} - C_i^{(2)}$, $i = 1,2,\dots,n$, is a solution of (6.12.1) and

$$\operatorname{dg} C_i \le \max(\operatorname{dg} C_i^{(1)}, \operatorname{dg} C_i^{(2)}) < \left[\frac{am}{n-m} + 1\right].$$

Since $C_i \in \mathbb{F}_q[X]$, we conclude

$$\operatorname{dg} C_i \le \frac{am}{n-m}, \qquad i = 1,2,\dots,n. \quad \square$$

6.13. LEMMA. *Let* K *be a finite, separable algebraic extension of degree* h *of* $\mathbb{F}_q(X)$. *Then there exists a basis* $\beta_1,\beta_2,\dots,\beta_h$ *of algebraic integers of* K *such that every algebraic integer* $\xi \in K$ *can be written uniquely as*

$$\xi = \sum_{i=1}^{h} A_i\, \beta_i, \qquad A_i \in \mathbb{F}_q[X].$$

PROOF. See for instance O. ZARISKI and P. SAMUEL (1958), Ch.V,§4, Cor. 2. $\square$

6.14. DEFINITION. Let $\alpha \in \Phi$ be algebraic over $\mathbb{F}_q(X)$ of degree n and let $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_n \in \Phi$ be the roots of the minimal polynomial of α. Then we define

$$d^*(\alpha) := \max(\operatorname{dg}\alpha_1, \operatorname{dg}\alpha_2, \dots, \operatorname{dg}\alpha_n; 0).$$

REMARK. Let K be a finite, separable algebraic extension of $\mathbb{F}_q(X)$ of degree h and let $\sigma_1,\sigma_2,\dots,\sigma_h$ denote the distinct $\mathbb{F}_q(X)$-monomorphisms $K \hookrightarrow \Phi$. If $P \in \mathbb{F}_q[X][t]$ is the minimal polynomial of $\beta \in K$, then

$$P(\sigma_j(\beta)) = \sigma_j(P(\beta)) = 0$$

and

$$\prod_{j=1}^{h} (t - \sigma_j(\beta))$$

is a polynomial with coefficients in $\mathbb{F}_q(X)$. Hence the set of zeros of P equals the set $\{\sigma_1(\beta), \sigma_2(\beta), \dots, \sigma_n(\beta)\}$. Therefore in this case we have

$$d^*(\beta) = \max\{\mathrm{dg}\ \sigma_1(\beta), \mathrm{dg}\ \sigma_2(\beta), \dots, \mathrm{dg}\ \sigma_h(\beta); 0\}.$$

6.15. LEMMA. *If* α *and* β *are algebraic over* $\mathbb{F}_q(X)$, *then*

(6.15.1) $\quad d^*(\alpha+\beta) \le \max(d^*(\alpha), d^*(\beta))$

and

(6.15.2) $\quad d^*(\alpha\beta) \le d^*(\alpha) + d^*(\beta)$.

PROOF. Let $\alpha_1 = \alpha, \alpha_2, \dots, \alpha_n$ and $\beta_1 = \beta, \beta_2, \dots, \beta_m$ denote the zeros of the minimal polynomials of α and β, respectively. Then the coefficients of

$$\prod_{\substack{i=1,\dots,n\\ j=1,\dots,m}} (t-\alpha_i-\beta_j)$$

are elements of $\mathbb{F}_q(X)$. The minimal polynomial of $\alpha + \beta$ is a divisor of this polynomial. Hence the zeros of this minimal polynomial belong to the set $\{\alpha_i+\beta_j \mid i = 1,\dots,n;\ j = 1,\dots,m\}$. Therefore

$$\begin{aligned} d^*(\alpha+\beta) &\le \max_{i,j}(\mathrm{dg}(\alpha_i+\beta_j); 0) \le \max_{i,j}(\max(\mathrm{dg}\alpha_i, \mathrm{dg}\beta_j); 0) \\ &\le \max(d^*(\alpha), d^*(\beta)). \end{aligned}$$

Relation (6.15.2) is proved analogously by considering the polynomial

$$\prod_{\substack{i=1,\dots,n\\ j=1,\dots,m}} (t-\alpha_i\beta_j). \quad \square$$

6.16. LEMMA. (WADE 1946) *Let* K *be a finite, separable algebraic extension of degree* h *of* $\mathbb{F}_q(X)$. *Let* $r,s \in \mathbb{N}$, $r < s$. *Then the system of* r *homogeneous, linear equations in the* s *unknowns*

(6.16.1) $\quad \sum_{i=1}^{s} \alpha_{ki} X_i = 0, \qquad k = 1,2,\dots,r,$

where the α_{ki} *are algebraic integers in* K *and*

$$a := \max_{\substack{1\le i\le s\\ 1\le k\le r}} d^*(\alpha_{ki}),$$

has a non-trivial solution $(\xi_1,\xi_2,\dots,\xi_s)$ *in algebraic integers* ξ_i *of* K *with*

$$d^*(\xi_i) < \frac{cs+ar}{s-r}, \qquad i = 1,2,\dots,s.$$

Here c denotes a positive constant which depends only on the field K.

PROOF. Let $\beta_1,\beta_2,\dots,\beta_h$ be a basis of algebraic integers of K as mentioned in lemma 6.13. Since $\alpha_{ki}\,\beta_j$, $k = 1,\dots,r$; $i = 1,\dots,s$; $j = 1,\dots,h$ are algebraic integers of K, we can write

$$(6.16.2) \qquad \alpha_{ki}\,\beta_j = \sum_{\nu=1}^{h} A_{kij\nu}\,\beta_\nu$$

with $A_{kij\nu} \in \mathbb{F}_q[X]$. Now consider the rh homogeneous, linear equations in the sh unknowns X_{ij}, $1 \le i \le s$; $1 \le j \le h$

$$(6.16.3) \qquad \sum_{i=1}^{s}\sum_{j=1}^{h} A_{kij\nu}\,X_{ij} = 0, \qquad k = 1,\dots,r;\ \nu = 1,\dots,h.$$

Since $rh < sh$ and $A_{kij\nu} \in \mathbb{F}_q[X]$, we can now apply lemma 6.12. To this end we need an upper bound for $\mathrm{dg}\,A_{kij\nu}$.

Let $\sigma_1,\dots,\sigma_h$ denote the h distinct $\mathbb{F}_q(X)$-monomorphisms $K \hookrightarrow \Phi$; then for $1 \le k \le r$, $1 \le i \le s$, $1 \le j \le h$ we have

$$\sigma_\mu(\alpha_{ki}\beta_j) = \sum_{\nu=1}^{h} A_{kij\nu}\,\sigma_\mu(\beta_\nu), \qquad \mu = 1,\dots,h.$$

Since $\{\beta_1,\dots,\beta_h\}$ is a basis, we have

$$\det(\sigma_\mu(\beta_\nu))_{\mu,\nu} \neq 0.$$

Hence we can express $A_{kij\nu}$ as a linear combination of the elements $\sigma_1(\alpha_{ki}\beta_j),\dots,\sigma_h(\alpha_{ki}\beta_j)$ with coefficients which only depend on the field K. Therefore

$$\mathrm{dg}\,A_{kij\nu} < c_1 + \max_{k,i,j,\mu} \mathrm{dg}\,\sigma_\mu(\alpha_{ki}\beta_j)$$

$$< c_2 + \max_{k,i} d^*(\alpha_{ki}) = c_2 + a,$$

where c_1, c_2 are positive constants depending only on K.

According to lemma 6.12 the system (6.16.3) has a non-trivial solution in polynomials $C_{ij} \in \mathbb{F}_q[X]$, $i = 1,\dots,s$; $j = 1,\dots,h$ such that

$$(6.16.4) \qquad \mathrm{dg}\, C_{ij} < \frac{(c_2+a)rh}{sh-rh}\,.$$

Now we define

$$(6.16.5) \qquad \xi_i := \sum_{j=1}^{h} C_{ij}\,\beta_j, \qquad i = 1,\dots,s.$$

Then the ξ_i are algebraic integers of K, not all zero, and from (6.16.5) and (6.16.2) we have

$$\sum_{i=1}^{s} \alpha_{ki}\,\xi_i = \sum_{\nu=1}^{h} \sum_{i=1}^{s} \sum_{j=1}^{h} A_{kij\nu}\, C_{ij}\, \beta_\nu.$$

But since $\sum_{i=1}^{s} \sum_{j=1}^{h} A_{kij\nu}\, C_{ij} = 0, \quad k = 1,\dots,r;\ \nu = 1,\dots,h,$

the s-tuple $(\xi_1,\dots,\xi_s)$ is a non-trivial solution of (6.16.1). Furthermore it follows from (6.16.5) and (6.16.4) that

$$d^*(\xi_i) \le \max_{i,j} (\mathrm{dg}\, C_{ij} + d^*(\beta_j)) < \frac{(a+c_2)r}{s-r} + c_3 < \frac{ar+cs}{s-r}\,,$$

where the positive constant c depends only on K. □

7. SUMMARY OF KNOWN RESULTS ON TRANSCENDENCE IN Φ

As already mentioned in chapter I, the functions $\psi, \lambda: \Phi \to \Phi$ and the quantity $\xi \in \Phi$ were introduced by L. CARLITZ in 1935. In 1941 L.I. WADE proved the transcendence over $\mathbb{F}_q(X)$ of $\psi(\alpha)$ for every non-zero algebraic element $\alpha \in \Phi$. From $\psi(\xi) = 0$ it follows that ξ is transcendental over $\mathbb{F}_q(X)$ and since $\lambda: \{t \in \Phi \mid \mathrm{dg}\, t < \frac{q}{q-1}\} \to \Phi$ is defined as the inverse of ψ we also immediately see that $\lambda(\alpha)$ is transcendental over $\mathbb{F}_q(X)$ for every non-zero algebraic $\alpha \in \Phi$ with $\mathrm{dg}\, \alpha < \frac{q}{q-1}$.

In the same article Wade remarked that he was not able to prove the transcendence of

$$\sum_{j=0}^{\infty} c_j \frac{\alpha^{q^j}}{F_j}, \qquad c_j \in \mathbb{F}_q, \qquad j = 1,2,\dots,$$

where an infinite number of c_j is non-zero and where α is an arbitrary algebraic element of Φ. However, the transcendence in a special case, namely for $\alpha \in \mathbb{F}_q[X]\backslash\{0\}$, follows from the following theorem which Wade proved in the same article.

7.1. THEOREM. (WADE (1941). *Let the sequence* $\{B_k\}_{k=0}^{\infty}$ *satisfy the conditions*:

(i) $B_k \in \mathbb{F}_q[X]$, $k = 0,1,2,\ldots,$

(ii) *infinitely many of the* B_k *are non-zero,*

(iii) *there exist a* $k_0 \in \mathbb{N}$ *and a sequence* $\{c_k\}_{k=k_0}^{\infty}$ *of real numbers with* $\lim_{k\to\infty} c_k = \infty$ *such that*

$$\text{(7.1.1)} \qquad \mathrm{dg}\, B_k \le k(q-1)q^{k-1} - c_k\, q^{k-1}, \qquad k > k_0.$$

Then

$$\sum_{k=0}^{\infty} \frac{B_k}{F_k}$$

is transcendental over $\mathbb{F}_q(X)$.

All proofs in Wade's article follow the same line. To illustrate this method we shall prove theorem 7.1.

Proof of theorem 7.1. Suppose $\gamma = \sum_{k=0}^{\infty} \frac{B_k}{F_k}$ is algebraic over $\mathbb{F}_q(X)$ of degree n. According to lemma 6.4, γ is a zero of a linear polynomial f of degree q^n:

$$f(t) := \sum_{j=\ell}^{n} A_j\, t^{q^j}, \qquad A_j \in \mathbb{F}_q[X], \qquad j = \ell,\ldots,n;\ A_\ell \ne 0;$$

i.e.

$$\text{(7.1.2)} \qquad 0 = \sum_{j=\ell}^{n} A_j \sum_{k=0}^{\infty} \frac{B_k^{q^j}}{F_k^{q^j}} = \sum_{i=\ell}^{\infty} \frac{D_i}{F_i},$$

where

$$D_i := \sum_{j=\ell}^{\min(n,i)} \frac{A_j B_{i-j}^{q^j} F_i}{F_{i-j}^{q^j}}.$$

From remark 2.2(a) we see that $D_i \in \mathbb{F}_q[X]$.

For $m \ge \ell$ a "multiplier" $M_m \in \mathbb{F}_q[X]$ will be defined in such a way that

$$M_m \sum_{i=\ell}^{\infty} \frac{D_i}{F_i}$$

can be split up into two parts

$$I := \sum_{i=\ell}^{m} \frac{M_m D_i}{F_i}$$

and

$$Q := \sum_{i=m+1}^{\infty} \frac{M_m D_i}{F_i} ,$$

such that

(i) $I \in \mathbb{F}_q[X]$,

(ii) every sum of Q has valuation less than zero if m is chosen large enough.

In our case, (7.1.2), F_m will do as such a multiplier. Using (7.1.2) we have

(7.1.3) $\qquad I + Q = 0.$

From $I \in \mathbb{F}_q[X]$ we have either dg $I \geq 0$ or $I = 0$. But from (7.1.1) we can deduce that dg $Q < 0$ and in view of (7.1.3) we conclude that $I = 0$. It now remains to prove that for m chosen sufficiently large this leads to a contradiction. We have

$$\sum_{i=\ell}^{m} \frac{F_m}{F_i} D_i = 0, \qquad m \geq m_0 - 1.$$

This yields

$$D_m + \frac{F_m}{F_{m-1}} \sum_{i=\ell}^{m-1} \frac{F_{m-1}}{F_i} D_i = 0, \qquad m \geq m_0 - 1$$

and hence $D_m = 0$, $m \geq m_0$. Recalling the definition of D_m we have

(7.1.4) $\qquad \displaystyle\sum_{j=\ell}^{n} A_j \frac{B_{m-j}^{q^j}}{F_{m-j}^{q^j}} = 0, \qquad m \geq m_0.$

We proceed by induction. From remark 2.2a it follows that

$$\sum_{j=\ell+1}^{n} A_j\, B_{m_0-j}^{q^j} \frac{F_{m_0-\ell-1}^{q^{\ell+1}}}{F_{m_0-j}^{q^j}} \in \mathbb{F}_q[X].$$

Hence by (7.1.4)

$$A_\ell\, B_{m_0-\ell}^{q^\ell} \frac{F_{m_0-\ell-1}^{q^{\ell+1}}}{F_{m_0-\ell}^{q^\ell}} \in \mathbb{F}_q[X].$$

Suppose that

$$(7.1.5)\qquad A_\ell^{\frac{q^{\kappa+1}-1}{q-1}}\, B_{m_0+\kappa-\ell}^{q^\ell} \frac{F_{m_0-\ell-1}^{q^{\kappa+\ell+1}}}{F_{m_0+\kappa-\ell}^{q^\ell}} \in \mathbb{F}_q[X], \qquad \kappa = 0,1,\dots,k-1.$$

Then it follows from (7.1.4) with $m = m_0 + k$ that

$$A_\ell^{\frac{q^{k+1}-1}{q-1}} \frac{B_{m_0+k-\ell}^{q^\ell}}{F_{m_0+k-\ell}^{q^\ell}} F_{m_0-\ell-1}^{q^{k+\ell+1}} +$$

$$+ \sum_{\nu=1}^{\min(k,n-\ell)} A_{\ell+\nu}\, A_\ell^{\frac{q^\nu-1}{q-1}-1} \left(A_\ell^{\frac{q^{k-\nu+1}-1}{q-1}} \frac{B_{m_0+k-\ell-\nu}^{q^\ell}}{F_{m_0+k-\ell-\nu}^{q^\ell}} F_{m_0-\ell-1}^{q^{k-\nu+\ell+1}} \right)^{q^\nu} +$$

$$+ \sum_{\nu=\min(k,n-\ell)+1}^{n-\ell} A_{\ell+\nu}\, A_\ell^{\frac{q^{k+1}-1}{q-1}-1} B_{m_0+k-\ell-\nu}^{q^{\ell+\nu}} \frac{F_{m_0-\ell-1}^{q^{k+\ell+1}}}{F_{m_0+k-\ell-\nu}^{q^{\ell+\nu}}} = 0,$$

which, by the induction hypothesis, yields (7.1.5) with $\kappa = k$. Since $\{B_k\}_{k=1}^{\infty}$ contains infinitely many non-zero elements, we have infinitely often

$$\frac{q^{k+1}-1}{q-1} \operatorname{dg} A_\ell + q^\ell \operatorname{dg} B_{m_0+k-\ell} - (k+1)q^{m_0+k} \geq 0,$$

which for large k contradicts (7.1.1). □

The transcendence of the special element $\sum_{k=1}^{\infty} \frac{1}{X^{q^k}-X} = \sum_{k=1}^{\infty} \frac{F_{k-1}^q}{F_k}$ does not follow from theorem 7.1, but using its special character and chosing the right multiplier, Wade proved its transcendence in theorem 4.1 of his article from 1941. By the same method he proved in 1943/44 the following three transcendence results for certain elements of Φ.

7.2. THEOREM. *For* $n \in \mathbb{N}$ *the element*

$$\sum_{k=0}^{\infty} \frac{1}{L_k^n}$$

is transcendental over $\mathbb{F}_q(X)$.

PROOF. See WADE (1943), §4. □

7.3. THEOREM. *Let* $G \in \mathbb{F}_q[X]$, $\mathrm{dg}\, G > 0$ *and* $n \in \mathbb{N}$, $n > 1$. *Then*

$$\sum_{k=0}^{\infty} \frac{1}{G^{n^k}}$$

is algebraic over $\mathbb{F}_q(X)$ *if* $n = p^s$, $s \in \mathbb{N}$ *and transcendental otherwise.*

PROOF. See WADE (1944), th.1. □

7.4. THEOREM. *Let* $G \in \mathbb{F}_q[X]$, $\mathrm{dg}\, G > 0$ *and* $n \in \mathbb{N}$, $n > 1$. *Then*

$$\sum_{k=0}^{\infty} \frac{1}{G^{k^n}}$$

is transcendental over $\mathbb{F}_q(X)$.

PROOF. See WADE (1944), th.2. □

The theorems 7.1 and 7.3 were generalized by S.M. SPENCER jr, (1952). His proofs are based on the principle sketched in the proof of theorem 7.1.

Spencer's generalisation of th.7.1 consists of replacing the sequence $\{F_k\}_{k=0}^{\infty}$ by a sequence $\{G_k\}_{k=0}^{\infty}$ of elements of $\mathbb{F}_q[X]$ which satisfy the following two conditions:

(i) $$\frac{G_{k+1}}{G_k^q} \in \mathbb{F}_q[X], \qquad k \geq 0,$$

(ii) $$\lim_{k\to\infty} \frac{\mathrm{dg}\, G_k}{q^k} = \infty.$$

See SPENCER (1952), theorem 4.

The generalisation of theorem 7.3 reads:

7.5. THEOREM. *Let the sequence* $\{G_k\}_{k=0}^{\infty}$ *satisfy the two conditions:*

(i) $\qquad G_k \in \mathbb{F}_q[X], \qquad k \geq 0$

and for some k_0, $\operatorname{dg} G_{k_0} > 0$,

(ii) $\qquad \frac{G_{k+1}}{G_k} \in \mathbb{F}_q[X], \qquad k \geq 0.$

Let $\{e_k\}_{k=0}^{\infty}$, $e_k \in \mathbb{N}$ *satisfy*

(iii) $\qquad e_k \mid e_{k+1}, \qquad k \geq 0,$

(iv) $\qquad p \nmid \frac{e_{k+1}}{e_k}, \qquad k \geq 0.$

Then $\sum_{k=0}^{\infty} \frac{1}{G_k^{e_k}}$ *is transcendental over* $\mathbb{F}_q(X)$.

PROOF. See SPENCER (1952), th.7. Compare the case $G_k = G$ and $e_k = n^k$ with theorem 7.3. □

Furthermore we mention that in the same paper by Spencer the following result is proved.

7.6. THEOREM. *Let the entire function* $f\colon \Phi \to \Phi$ *be given by*

$$f(t) := \sum_{n=0}^{\infty} b_n t^n, \qquad b_n \in \mathbb{F}_q(X)$$

and $b_n \neq 0$ *for infinitely many* n. *Let* G_n *denote a denominator for* $b_0, b_1, \ldots, b_n$ *of smallest valuation. Let* $\alpha \in \Phi\backslash\{0\}$ *be algebraic and* $\operatorname{dg} \alpha \leq 0$.

If there exist an increasing sequence $n_1, n_2, \ldots$ *of natural numbers and an increasing sequence* $k_1, k_2, \ldots$ *of positive real numbers with* $\lim_{i\to\infty} k_i = \infty$, *such that*

$$(7.6.1)\begin{cases} \text{(i)}\ \operatorname{dg} b_\nu < -k_i \operatorname{dg} G_{n_i}, & i = 1,2,\ldots;\ \nu \geq n_i, \\ \text{(ii)}\ \sum_{\nu=n_i+1}^{\infty} b_\nu \alpha^\nu \neq 0, & i = 1,2,\ldots, \end{cases}$$

then $f(\alpha)$ *is transcendental over* $\mathbb{F}_q(X)$.

PROOF. See S.M. SPENCER (1952), th.1 or section 9 of this thesis. In Spencer's article the theorem is proved only in the case that f is defined on $\bar{F}$, but the proof also works in case f is defined for all $t \in \Phi$. □

N.B. Spencer does not mention the condition $dg\ \alpha \le 0$ but it is not clear how his proof works without it.

In 1946 L.I. WADE proved an analogue of the Gelfond-Schneider theorem using the Siegel-Schneider method. We shall formulate this theorem and give a sketch of the proof. In 1971 and 1973 the same method was used to obtain transcendence results for a wider class of functions. See J.M. GEIJSEL (1971,1973) or chapter IV.

7.7. <u>THEOREM</u>. (WADE 1946) *Let* $\alpha,\beta \in \Phi$. *If* $\alpha \neq 0$, $dg\ \alpha < \frac{q}{q-1}$ *and* $\beta \notin \mathbb{F}_q(X)$, *then at least one of the three quantities* $\alpha,\beta,\psi(\beta\lambda(\alpha))$ *is transcendental over* $\mathbb{F}_q(X)$.

<u>PROOF</u>. Suppose α,β and $\psi(\beta\lambda(\alpha))$ are algebraic over $\mathbb{F}_q(X)$. For some $e \in \mathbb{N}^0$ the elements $\alpha^{q^e},\beta^{q^e},\psi^{q^e}(\beta\lambda(\alpha))$ generate a separable algebraic extension K of $\mathbb{F}_q(X)$.

Let $\Gamma \in \mathbb{F}_q[X]$ be such that $\Gamma\alpha^{q^e},\Gamma\beta^{q^e}$ and $\Gamma\psi^{q^e}(\beta\lambda(\alpha))$ are algebraic integers of K.

The proof, that the assumption on α,β and $\psi(\beta\lambda(\alpha))$ leads to a contradiction, consists of three steps.
Step I: construction of an auxiliary function L with many prescribed zeros.
Step II: proof with the aid of the Maximum Modulus Theorem that L has infinitely many distinct zeros of a certain type.
Step III: Application of the Product Formula for Entire Functions from which the desired contradiction follows.

I. The natural numbers k, ℓ with $\ell > 3k$ will be chosen later. Set $m := k + \ell - 1$. Define the entire function $L\colon \Phi \to \Phi$ by

$$L(t) := \sum_{j=0}^{q^{2\ell}-1} \sum_{i=0}^{q^{2k}-1} X_{ij}\, t^{jq^e}\, \psi^{iq^e}(\lambda(\alpha)t),$$

where the algebraic integers X_{ij} of K will be determined in such a way that $L(A+\beta B) = 0$ for all $A,B \in \mathbb{F}_q[X]$ with $dg\ A < m, dg\ B < m$. The condition

$$\Gamma^{q^{2\ell}+q^{2k+m}}\, L(A+\beta B) = 0, \quad dg\ A,\ dg\ B < m$$

on L implies a system of at most q^{2m} linear equations in the $q^{2(k+\ell)}$ variables X_{ij} with integral algebraic coefficients (apply th.2.11(a), th.2.13 and th.2.5). Using that

$$\operatorname{dg} \frac{\psi_\mu(A)}{F_\mu} = (\operatorname{dg} A-\mu)q^\mu \le q^{\operatorname{dg}A-1}$$

(see remark 2.6) we find that the valuation of these coefficients and also of their conjugates is less than $q^{2\ell+e}(m+c_1)$, where the rational constant $c_1 > 0$ does not depend on k and ℓ. According to lemma 6.16 we can determine the X_{ij} in such a way that not all of them are zero and that

$$\text{(7.7.1)} \qquad \operatorname{dg} X_{ij} < (m+c_2)q^{2\ell+e},$$

where $c_2 > 0$ is independent of k and ℓ.

From now on we suppose that the X_{ij} are fixed accordingly.

II. For $\mu \ge m$ we define

$$\mathcal{B}(\mu) := \{A + \beta B \mid A,B \in \mathbb{F}_q[X]; \text{ A and B not both zero; } \operatorname{dg} A < \mu, \operatorname{dg} B < \mu\}.$$

Let $\mathcal{B} := \bigcup_{\mu=m}^{\infty} \mathcal{B}(\mu)$. The second step now consists of proving by induction that L vanishes on $\mathcal{B}$. We have constructed L such that $L(t) = 0$ for $t \in \mathcal{B}(m)$. So it is sufficient to prove that

$$(t \in \mathcal{B}(\mu) \Rightarrow L(t) = 0) \Rightarrow (t \in \mathcal{B}(\mu+1) \Rightarrow L(t) = 0).$$

Since $\beta \notin \mathbb{F}_q(X)$, all the $A + \beta B$ are different. Hence the number of elements of $\mathcal{B}(\mu)$ is $q^{2\mu}-1$.

Let $t_0 \in \mathcal{B}(\mu+1)\setminus\mathcal{B}(\mu)$. If ℓ is chosen large enough, then $\operatorname{dg} t_0 \le \mu + d^*(\beta) < 2\mu$. By assumption

$$L(t) \prod_{a\in\mathcal{B}(\mu)} (t-a)^{-1}$$

is an entire function. Hence we can apply the Maximum Modulus Principle (th.5.16) and obtain

$$\operatorname{dg} L(t_0) - \sum_{a\in\mathcal{B}(\mu)} \operatorname{dg}(t_0-a) \le \max_{\operatorname{dg}t=2\mu} \operatorname{dg} L(t) - 2\mu(q^{2\mu}-1).$$

From the definitions of L and ψ and inequality (7.7.1) it follows that

$$(7.7.2) \qquad \max_{dgt=2\mu} dg\, L(t) < (2\mu+m+c_2)q^{2\ell+e} + c_3 q^{2k+e+2\mu},$$

where $c_3 > 0$ is independent of k and ℓ. Now put

$$\eta := \mu - k + 1,$$

then $\eta \geq \ell$ and

$$dg\, L(t_0) \leq q^{2\eta+e}\{\mu(3-q^{2k-e-2}+\frac{1}{q^{2\eta+e}}) + c_2 + c_3 q^{4k} + d^*(\beta)q^{2k}\}.$$

From the choice of t_0 and the definitions of L and Γ it follows that

$$\Gamma^{q^{2\eta}+q^{2k+\mu}} L(t_0)$$

is an algebraic integer of K. Therefore its norm is an element of $\mathbb{F}_q[X]$ with

$$(7.7.3) \qquad dg\, N_{K\to\mathbb{F}_q(X)}(\Gamma^{q^{2\eta}+q^{2k+\mu}} L(t_0)) \leq hq^{2\eta+e}\{\mu(4-q^{2k-e-2})+ c_4 q^{4k}\},$$

where $c_4 > 0$ and $h := [K\colon \mathbb{F}_q(X)]$. Now first choose k such that $4-q^{2k-e-2} < 0$. Then take ℓ so large that

(i) $d^*(\beta) < \ell$ (this was required in the calculation above),
(ii) $\ell > 3k$ (as was assumed throughout the proof),
(iii) $\mu(4-q^{2k-e-2}) \leq m(4-q^{2k-e-2}) = (k+\ell-1)(4-q^{2k-e-2}) < -c_4 q^{4k}$.

III. Now k and ℓ are fixed. According to the Product Formula for Entire Functions, corollary 5.24, we have

$$L(t) = \gamma t^{\rho} \prod_{a\in\mathcal{B}(\mu)} (1-\frac{t}{a}) \prod_{b\in\mathcal{R}^*\setminus\mathcal{B}(\mu)} (1-\frac{t}{b}),$$

where $\rho \in \mathbb{N}^0$, $\gamma \in \Phi, \gamma \neq 0$, $\mathcal{R}^* = \mathcal{R}\setminus\{0\}$ and $\mathcal{R}$ denotes the set of zeros of L. Comparing the maximal value on $\{t \mid dg\, t = 2\mu\}$ and the value in $t = 0$ of the last product, the Maximum Modulus Principle yields

$$(7.7.4) \qquad \max_{dgt=2\mu} dg \prod_{b\in\mathcal{R}^*\setminus\mathcal{B}(\mu)} (1-\frac{t}{b}) \geq 0.$$

Further we write

$$\prod_{a\in\mathcal{B}(\mu)} (1-\frac{t}{a}) = \frac{\prod_{a\in\mathcal{B}(\mu)} (a-t)}{\prod_{a\in\mathcal{B}(\mu)} a} .$$

Then it follows from (7.7.4) that

(7.7.5) $$\max_{dgt=2\mu} dg\ L(t) \geq dg\ \gamma + 2\mu\rho + 2\mu(q^{2\mu}-1)-(\mu+d^*(\beta))(q^{2\mu}-1).$$

For μ large enough (7.7.2) and (7.7.5) are contradictory. □

In 1949 K. MAHLER proved an analogue of the well-known theorem of Liouville on the approximation of algebraic numbers by irrational numbers for certain function fields. His proof also works for our field Φ. Therefore we have, in our notation,

7.8. THEOREM. (MAHLER) *If* $\alpha \in \Phi$ *is algebraic over* $\mathbb{F}_q(X)$ *of degree* $n \geq 2$, *then there exists a* $c \in \mathbb{R}$ *such that for all pairs* $P,Q \in \mathbb{F}_q[X]$ *with* $Q \neq 0$ *we have*

$$dg(\alpha - \frac{P}{Q}) \geq c - n\ dg\ Q.$$

PROOF. See MAHLER (1949), th.1. □

In case the characteristic of the function field is 0, Mahler's theorem does not give the best possible result [see B.P. GILL (1930)]. Mahler gave an example from which it follows that in case the ground field has characteristic p, theorem 7.8 is sharpest.

7.9. THEOREM. *Let* $\alpha \in \Phi$ *be the element*

$$\alpha := \sum_{i=0}^{\infty} \frac{1}{X^{p^i}} ,$$

then α *is algebraic over* $\mathbb{F}_q(X)$ *of degree* $p \geq 2$ *and there exist an infinite sequence of relatively prime polynomials* $A_m, B_m \in \mathbb{F}_q[X]$ *with* $B_m \neq 0$ *such that*

$$dg\ (\alpha - \frac{A_m}{B_m}) = -\ p\ dg\ B_m,$$

where $\lim_{m\to\infty} \mathrm{dg}\, B_m = \infty$.

PROOF. See MAHLER (1949), th.2. Note that α is a root of the equation $t^p - t + \frac{1}{X} = 0$. □

7.10. REMARK. In the same paper Mahler raised the question whether the result of theorem 7.8 still gives the best possible result for elements α of the form

$$(7.10.1) \qquad \alpha = \sum_{i=-m}^{\infty} a_i X^{-i}, \qquad m \in \mathbb{Z},\ a_i \in \mathbb{F}_q,$$

which are algebraic over $\mathbb{F}_q(X)$ of degree at least 2 and at most p-1.

Recently L.E. BAUM and M.M. SWEET (1976) proved the following statement:

"There exists a unique element α of the form (7.10.1) with q = 2 that satisfies the irreducible equation

$$t^{2^n+1} + Xt + 1 = 0, \qquad n \geq 1.$$

For this α there exists an infinite sequence $A_m, B_m \in \mathbb{F}_q[X]$ such that $(A_m, B_m) = 1$, $B_m \neq 0$, $\lim_{m\to\infty} \mathrm{dg}\, B_m = \infty$ and such that

$$\mathrm{dg}(\alpha - \frac{A_m}{B_m}) = -1 - (2^n+1)\mathrm{dg}\, B_m".$$

This contradicts an earlier assertion of J.V. ARMITAGE (1968) to the effect that a Thue-Siegel-Roth theorem should hold for algebraic elements in Φ which are not contained in a cyclic extension of $\mathbb{F}_q(X)$ of degree p^n ($n \in \mathbb{N}$). Armitage's assertion was earlier showed to be false by C.F. OSGOOD (1975).

Theorem 7.8 enables us to construct a new type of transcendental elements of Φ; this will be done in Chapter III.

Finally we mention that P. BUNDSCHUH in 1974 gave an analogue of Mahler's classification of transcendental numbers in S-, T- and U-numbers and that he introduces a notion of transcendence measure in Φ. (See Séminaire Delange-Pisot-Poitou 1974/75, §3.)

CHAPTER III

ON THE TRANSCENDENCE OF CERTAIN POWER SERIES OF ALGEBRAIC ELEMENTS OF Φ

8. LIOUVILLE NUMBERS

As already mentioned in chapter II, section 7, Mahler's analogon of the theorem of Liouville (see th.7.8) enables one to construct transcendental elements of Φ.

8.1. DEFINITION. An element $\eta \in \Phi$ is called a *Liouville number* if for every $m \in \mathbb{N}^0$ there exist elements $A_m, B_m \in \mathbb{F}_q[X]$, with $(A_m, B_m) = 1$, dg $B_m > 0$ and $A_m/B_m \neq \eta$ such that

$$(8.1.1) \qquad \mathrm{dg}\left(\eta - \frac{A_m}{B_m}\right) < - m \,\mathrm{dg}\, B_m.$$

8.2. THEOREM. *Every Liouville number $\eta \in \Phi$ is transcendental over $\mathbb{F}_q(X)$.*

PROOF. Suppose η is algebraic over $\mathbb{F}_q(X)$ of degree n. If $n = 1$, then there exist $A, B \in \mathbb{F}_q[X]$ with $(A,B) = 1$ such that $\eta = \frac{A}{B}$. For all $C,D \in \mathbb{F}_q[X]$ $\frac{C}{D} \neq \frac{A}{B}$ and dg $D >$ dg B we have

$$(8.2.1) \qquad \mathrm{dg}\left(\eta - \frac{C}{D}\right) \geq - \mathrm{dg}\, D - \mathrm{dg}\, B \geq - 2\mathrm{dg}\, D.$$

For $m > 2$ the relations (8.1.1) and (8.2.1) are contradictory.

Now suppose $n \geq 2$. According to theorem 7.8 there exists a $c \in \mathbb{R}$ such that for all pairs $P,Q \in \mathbb{F}_q[X]$ with $Q \neq 0$

$$\mathrm{dg}\left(\eta - \frac{P}{Q}\right) > c - n \,\mathrm{dg}\, Q > - m \,\mathrm{dg}\, Q$$

for m sufficiently large. This contradicts (8.1.1). $\square$

8.3. EXAMPLES. (i) Let $\alpha \in \Phi$ be defined by

$$\alpha := \sum_{j=1}^{\infty} \frac{c_j}{X^{j!}},$$

where $c_j \in \mathbb{F}_q$, $c_j \neq 0$ for infinitely many j. For $m \in \mathbb{N}$ we define

$$\mu := \max_{1 \le j \le m} \{j \mid c_j \neq 0\},$$

$$A_m := X^{\mu!} \sum_{j=1}^{\mu} \frac{c_j}{X^{j!}}$$

and

$$B_m := X^{\mu!}.$$

Then A_m, $B_m \in \mathbb{F}_q[X]$, $(A_m, B_m) = 1$, $\mathrm{dg}\, B_m = \mu! > 0$ and

$$\mathrm{dg}\left(\alpha - \frac{A_m}{B_m}\right) \le -(m+1)! \le -(m+1)\mathrm{dg}\, B_m.$$

Hence α is a Liouville number.

(ii) Let $\alpha \in \Phi$ be defined by

$$\alpha := \sum_{j=0}^{\infty} \frac{c_j}{F_{q^j}},$$

where $c_j \in \mathbb{F}_q$, $c_j \neq 0$ for infinitely many j. For $m \in \mathbb{N}^0$ we define

$$\mu := \max_{0 \le j \le m} \{j \mid c_j \neq 0\},$$

$$A_m := F_{q^\mu} \sum_{j=0}^{\mu} \frac{c_j}{F_{q^j}}$$

and

$$B_m := F_{q^\mu}.$$

Then A_m, $B_m \in \mathbb{F}_q[X]$, $(A_m, B_m) = 1$, $\mathrm{dg}\, B_m = q^\mu \cdot q^{q^\mu} > 0$ and

$$\mathrm{dg}\left(\alpha - \frac{A_m}{B_m}\right) \le -q^{m+1} \cdot q^{q^{m+1}} < -m\, \mathrm{dg}\, B_m.$$

Hence α is a Liouville number.

9. TRANSCENDENTAL VALUES OF GAP-SERIES

In 1972 P.L. Cijsouw proved that if a certain gap-condition for a power series S with algebraic coefficients is fullfilled, then S assumes transcendental values for non-zero algebraic arguments. For details and a proof we refer to CIJSOUW (1972), th.1.11 or CIJSOUW & TIJDEMAN (1973). In this section we shall give an analogue of Cijsouw's theorem for the field Φ.

9.1. <u>DEFINITION</u>. Let $P \in \Phi[t]$ be given by

$$P(t) := a_n t^n + a_{n-1} t^{n-1} + \ldots + a_1 t + a_0.$$

Then the *height of the polynomial* P, notation $H(P)$, is defined as the maximum of the valuations of the coefficients of P, i.e.

$$H(P) := \max_{0 \le i \le n} \operatorname{dg} a_i.$$

If $\alpha \in \Phi$ is algebraic over $\mathbb{F}_q(X)$, then the *height of* α, notation $h(\alpha)$, is defined as the height of the minimal polynomial of α over $\mathbb{F}_q[X]$.

In the next two lemmas we shall give a lower and an upper bound for $h(\alpha)$ in terms of suitable characteristics of α.

9.2. <u>LEMMA</u>. *Let* $\alpha \in \Phi$ *be algebraic over* $\mathbb{F}_q(X)$, *then*

(9.2.1) $\quad \operatorname{dg} \alpha \le h(\alpha).$

<u>PROOF</u>. Since $h(\alpha) \ge 0$, we restrict ourselves to the case $\operatorname{dg} \alpha \ge 0$. Let $P \in \mathbb{F}_q[X][t]$, given by

$$P(t) := A_n t^n + A_{n-1} t^{n-1} + \ldots + A_1 t + A_0,$$

be the minimal polynomial of α. Then

$$A_n \alpha^n = - A_0 - A_1 \alpha - \ldots - A_{n-1} \alpha^{n-1}.$$

Hence, using $\mathrm{dg}\,\alpha \geq 0$, we obtain

$$n\,\mathrm{dg}\,\alpha \leq n\,\mathrm{dg}\,\alpha + \mathrm{dg}\,A_n \leq \max_{0\leq i\leq n-1} (i\mathrm{dg}\alpha+\mathrm{dg}A_i)$$

$$\leq (n-1)\mathrm{dg}\,\alpha + h(\alpha),$$

from which the inequality (9.2.1) follows. $\square$

9.3. <u>LEMMA</u>. *Let* α *be algebraic over* $\mathbb{F}_q(X)$ *of degree* n *and let* M *be a denominator for* α. *Then*

$$h(\alpha) \leq n(\mathrm{dg}\,M + d^*(\alpha)).$$

<u>PROOF</u>. Let $Q \in \mathbb{F}_q[X][t]$ be the minimal polynomial for α, given by

$$Q(t) := A_n t^n + A_{n-1}t^{n-1} +\ldots+ A_1 t + A_0.$$

Let $\alpha_1 = \alpha,\alpha_2,\ldots,\alpha_n$ be the conjugates of α, then

$$Q(t) = A_n \prod_{i=1}^{n} (t-\alpha_i).$$

Now A_j/A_n, $j = 0,1,\ldots,n-1$ are the elementary symmetric polynomials in $\alpha_1,\alpha_2,\ldots,\alpha_n$, disregarding the sign. Hence

$$\mathrm{dg}\,\frac{A_j}{A_n} \leq \max_{\substack{1\leq i_\nu\leq n\\ 1\leq\nu\leq n-j}} \mathrm{dg}(\alpha_{i_1}\alpha_{i_2}\ldots\alpha_{i_j}) \leq n\,d^*(\alpha), \qquad j = 0,1,\ldots,n-1. \tag{9.3.1}$$

Since $M\alpha$ is an algebraic integer, there exists a polynomial $P \in \mathbb{F}_q[X][t]$, given by

$$P(t) := (Mt)^n + B_{n-1}(Mt)^{n-1} +\ldots+ B_1(Mt) + B_0,$$

for which $P(\alpha) = 0$. Since Q is the minimal polynomial of α, P must be a multiple (in $\mathbb{F}_q[X][t]$) of Q and therefore

$$CA_n = M^n$$

for some $C \in \mathbb{F}_q[X]$, $C \neq 0$. Hence

$$\text{(9.3.2)} \qquad \mathrm{dg}\, A_n \le \mathrm{dg}\, A_n + \mathrm{dg}\, C = n\, \mathrm{dg}\, M.$$

Now the lemma follows from (9.3.1) and (9.3.2). $\square$

9.4. LEMMA. *Let* $P_1, P_2 \in \mathbb{F}_q[X][t]$ *be polynomials of degree* N_1, N_2 *in* t *and height* H_1, H_2 *respectively. If there exists an element* $\omega \in \Phi$ *such that*

$$\text{(9.4.1)} \qquad \max\,(\mathrm{dg}P_1(\omega), \mathrm{dg}P_2(\omega)) < -\,(N_1H_2+N_2H_1),$$

then P_1 *and* P_2 *have a common zero.*

PROOF. Let

$$P_1(t) := A_{N_1}t^{N_1} + A_{N_1-1}t^{N_1-1} + \ldots + A_1t + A_0, \quad A_{N_1} \ne 0,$$

$$P_2(t) := B_{N_2}t^{N_2} + B_{N_2-1}t^{N_2-1} + \ldots + B_1t + B_0, \quad B_{N_2} \ne 0$$

and let det R be the *resultant* of P_1 and P_2:

$$\text{(9.4.2)} \qquad R = \begin{array}{c} \left(\begin{array}{cccccccc} A_{N_1} & \cdots & A_1 & A_0 & 0 & \cdots & & 0 \\ 0 & A_{N_1} & \cdots & & A_1 & A_0 & & \vdots \\ \vdots & & \ddots & & & & \ddots & 0 \\ 0 & \cdots & 0 & A_{N_1} & & \cdots & A_1 & A_0 \\ B_{N_2} & \cdots & B_1 & B_0 & 0 & \cdots & & 0 \\ 0 & B_{N_2} & \cdots & & B_1 & B_0 & & \vdots \\ \vdots & & \ddots & & & & \ddots & 0 \\ 0 & \cdots & 0 & B_{N_2} & & \cdots & B_1 & B_0 \end{array}\right) \end{array} \begin{array}{l} \left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\} N_2 \text{ rows} \\ \left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\} N_1 \text{ rows} \end{array}.$$

Then it is well-known, see e.g. VAN DER WAERDEN §30, that det R = 0 if and only if P_1 and P_2 have a common zero. The coefficients of P_1 and P_2 are elements of $\mathbb{F}_q[X]$ and hence det $R \in \mathbb{F}_q[X]$, i.e. det R = 0 or dg(det R) ≥ 0. So if we show that the condition (9.4.1) implies dg(det R) < 0, the lemma will be proved.

First suppose $dg\ \omega \le 0$. Multiply the i^{th} column of R by $\omega^{N_1+N_2-i}$ and add the result to the last column, $i = 1,2,\dots,N_1+N_2-1$. Then divide the the result by

$$P(\omega) := \begin{cases} P_1(\omega) & \text{if} \quad dg\ P_1(\omega) \ge dg\ P_2(\omega)\ , \\ P_2(\omega) & \text{if} \quad dg\ P_1(\omega) < dg\ P_2(\omega)\ . \end{cases}$$

So we obtain

$$(9.4.3) \qquad R = P(\omega)R',$$

where R' is a matrix that is obtained from R by replacing the last column by a new one in which all elements have valuation at most zero. Every term in the expansion of det R' is the product of one element of Φ with valuation at most zero, at most N_2 elements from the set $\{A_0, A_1, \dots, A_{N_1}\}$ and at most N_1 elements from the set $\{B_0, B_1, \dots, B_{N_2}\}$. Hence from (9.4.3) and (9.4.1) we obtain

$$dg(\det R) \le dg\ P(\omega) + N_1H_2 + N_2H_1 < 0.$$

This proves the lemma in case $dg\ \omega \le 0$.

Now suppose $dg\ \omega > 0$. Define the polynomials P_j^* by

$$P_j^*(t) := t^{N_j}\ P_j(t^{-1}), \qquad j = 1,2.$$

Then P_1^* and P_2^* are of degree $M_1 \le N_1$, $M_2 \le N_2$ and height H_1, H_2 respectively. Since $dg\ \omega > 0$, we have

$$dg\ P_j^*(\omega^{-1}) = dg\ P_j(\omega) - N_j\ dg\ \omega \le dg\ P_j(\omega)$$

and therefore

$$\max\ (dgP_1^*(\omega^{-1}),\ dgP_2^*(\omega^{-1})) < -\ (N_1H_2+N_2H_1) < -\ (M_1H_2+M_2H_1).$$

Since $dg(\omega^{-1}) < 0$, we have the case considered previously and we conclude that P_1^* and P_2^* have a common zero, say γ. Since $A_{N_1} \ne 0$ it follows that

$\gamma \neq 0$. Now γ^{-1} is a common zero of P_1 and P_2. $\square$

9.5. <u>LEMMA</u>. *Let P_1 and P_2 be a polynomials in $\Phi[t]$ of height H_1 and H_2 respectively. Then the product P_1P_2 has height $H_1 + H_2$.*

<u>PROOF</u>. Write $P_1(t) = A_Nt^N + A_{N-1}t^{N-1} + \ldots + A_1t + A_0$, $A_N \neq 0$. Define n_1 by

$$\begin{aligned} &\mathrm{dg}\, A_{n_1} = H_1, \\ &\mathrm{dg}\, A_n < H_1, \qquad n = 0,1,\ldots,n_1-1. \end{aligned}$$

Define in a similar way n_2 for P_2. Then the coefficient of $t^{n_1+n_2}$ in P_1P_2 has degree $H_1 + H_2$. Since it is clear that in P_1P_2 no coefficients with a degree greater than $H_1 + H_2$ occur, the lemma is proved. $\square$

9.6. <u>LEMMA</u>. *Let $P \in \mathbb{F}_q[X][t]$ have degree $N \geq 1$ and height H. Let $\alpha \in \Phi$ be algebraic of degree n and height h. Then either $P(\alpha) = 0$ or*

$$\mathrm{dg}\, P(\alpha) \geq -(hN+nH). \tag{9.6.1}$$

<u>PROOF</u>. First we supppose that α is separable. Let Q denote its minimal polynomial and let $\sigma_1,\sigma_2,\ldots,\sigma_n$ be the n $\mathbb{F}_q(X)$-monomorphisms $\mathbb{F}_q(X)(\alpha) \hookrightarrow \Phi$. Hence the zeros of Q are $\sigma_j(\alpha)$, $j = 1,2,\ldots,n$. Now if (9.6.1) were not true, we would have

$$\max\{\mathrm{dg}\, P(\alpha),\ \mathrm{dg}\, Q(\alpha)\} = \mathrm{dg}\, P(\alpha) < -(hN+nH).$$

Then lemma 9.4 says that P and Q have a common zero, i.e. for some $j \in \{1,2,\ldots,n\}$

$$0 = P(\sigma_j(\alpha)) = \sigma_j(P(\alpha))$$

and hence $P(\alpha) = 0$.

Now let α be non-separable. Take $e \in \mathbb{N}$ such that α^{p^e} is separable. If $Q \in \Phi[t]$, we denote by Q^* the polynomial obtained from Q by raising the coefficients of Q to the power p^e. Clearly, Q and Q^* are of the same degree and $H^* = p^eH$, with the obvious meaning for H and H^*. Now let $Q \in \mathbb{F}_q[X][t]$ be the minimal polynomial of α. Then $Q^*(\alpha^{p^e}) = 0$. Hence the minimal polynomial of α^{p^e} divides Q^*. In view of lemma 9.5 the height of α^{p^e} does not exceed $p^e h$.

Suppose $P(\alpha) \neq 0$. Then we have

$$P^*(\alpha^{p^e}) \neq 0.$$

Applying the part of the lemma already proved on P^* and α^{p^e}, we find that

$$\text{(9.6.2)} \qquad \operatorname{dg} P^*(\alpha^{p^e}) \geq -(p^e hN + np^e H).$$

The lemma now follows from (9.6.2) and

$$p^e \operatorname{dg} P(\alpha) = \operatorname{dg} P^*(\alpha^{p^e}). \quad \square$$

Now we are ready to prove the analogue of Cijsouw's theorem mentioned in the beginning of this section.

9.7. <u>THEOREM</u>. *Let* $\{\alpha_k\}_{k=0}^{\infty}$ *be a sequence of non-zero algebraic elements of* Φ. *Denote*

$$a_k := \max_{0 \leq i \leq k} d^*(\alpha_i)$$

and

$$d_k := [\mathbb{F}_q(X)(\alpha_0, \alpha_1, \ldots, \alpha_k) : \mathbb{F}_q(X)].$$

Let M_k *be a denominator for* $\alpha_0, \alpha_1, \ldots, \alpha_k$. *Finally suppose that the power series*

$$S(t) := \sum_{k=0}^{\infty} \alpha_k t^{n_k},$$

where $\{n_k\}_{k=0}^{\infty}$ *is an increasing sequence of non-negative integers, has radius of convergence* $R > -\infty$.

Then, if

$$\text{(9.7.1)} \qquad \lim_{k\to\infty} \frac{(n_k + \operatorname{dg} M_k + a_k)\, d_k}{n_{k+1}} = 0,$$

$S(\theta)$ *is transcendental over* $\mathbb{F}_q(X)$ *for every non-zero algebraic* $\theta \in \Phi$ *with* $\mathrm{dg}\,\theta < R$.

PROOF. *Let* $\theta \neq 0$ be algebraic, $\mathrm{dg}\,\theta < R$ and let n denote the degree of θ. M is a denominator of θ. Put

$$S_k(\theta) := \sum_{i=0}^{k} \alpha_i \theta^{n_i}$$

and

$$r_k(\theta) := S(\theta) - S_k(\theta), \qquad k \in \mathbb{N}^0.$$

Now $S_k(\theta) \in \mathbb{F}_q(X)(\alpha_0, \alpha_1, \ldots, \alpha_k, \theta)$ and therefore $S_k(\theta)$ is algebraic over $\mathbb{F}_q(X)$ of degree $s_k \le nd_k$. Denote its height by h_k. Since $M_k M^{n_k}$ is a denominator for $S_k(\theta)$, we obtain from lemma 9.3 and from lemma 6.15

$$h_k \le nd_k\{\mathrm{dg}(M_k M^{n_k}) + d^*(S_k(\theta))\}$$

$$\le nd_k\{\mathrm{dg}\,M_k + n_k\,\mathrm{dg}\,M + a_k + n_k d^*(\theta)\}.$$

Let $P \in \mathbb{F}_q[X][t]$ be an arbitrary but fixed polynomial of degree $N \ge 1$ and height H. Let $\beta_1, \beta_2, \ldots, \beta_m$ be the different zeros of P in Φ and suppose $m \ge 2$. Then, by the convergence of $\{S_k(\theta)\}_{k=1}^{\infty}$, there exists a κ_1 such that for $k > \kappa_1$

$$\mathrm{dg}(S_k(\theta) - S_{k+1}(\theta)) < \min_{\substack{1 \le i,j \le m \\ i \neq j}} \mathrm{dg}(\beta_i - \beta_j).$$

Hence for $k > \kappa_1$

$$P(S_k(\theta)) = 0 \Rightarrow P(S_{k+1}(\theta)) \neq 0.$$

Clearly, this also holds if P has one zero of multiplicity N. Consequently there exists an infinite subsequence $\{k_j\}_{j=1}^{\infty}$ of the sequence of natural numbers such that

$$P(S_{k_j}(\theta)) \neq 0, \qquad j = 1, 2, \ldots .$$

Now it follows from lemma 9.6 that

$$\mathrm{dg}\, P(S_{k_j}(\theta)) \geq -(h_{k_j}N+s_{k_j}H)$$

$$\geq -nd_{k_j}\{(\mathrm{dg}M_{k_j}+n_{k_j}\mathrm{dg}M+a_{k_j}+n_{k_j}d^*(\theta))N+H\}.$$

Hence

$$\text{(9.7.2)} \qquad \mathrm{dg}\, P(S_{k_j}(\theta)) \geq -c_1 d_{k_j}(\mathrm{dg}M_{k_j}+a_{k_j}+n_{k_j}),$$

where $c_1 > 0$ is independent of j.

We now estimate $r_k(\theta)$ as follows. Choose $\rho \in \mathbb{R}$ with $\mathrm{dg}\,\theta < \rho < R$. Then since

$$\limsup_{k\to\infty} \frac{\mathrm{dg}\alpha_k}{n_k} = -R,$$

we have for $k > \kappa_2$ the inequality $\mathrm{dg}\,\alpha_k < -\rho n_k$ and hence

$$\text{(9.7.3)} \qquad \mathrm{dg}\, r_k(\theta) \leq \max_{i\geq k+1} n_i(\mathrm{dg}\theta-\rho) = n_{k+1}(\mathrm{dg}\theta-\rho).$$

Put

$$P(t) = B_N t^N + B_{N-1}t^{N-1} + \ldots + B_1 t + B_0$$

and suppose that $r_k(\theta) \neq 0$. Then we may write

$$P(S(\theta)) - P(S_k(\theta)) = r_k(\theta) \sum_{i=1}^{N} B_i \frac{S^i(\theta)-S_k^i(\theta)}{S(\theta)-S_k(\theta)}.$$

From (9.7.3) it follows that for $k > \kappa_2$ we have

$$\mathrm{dg}\{P(S(\theta)) - P(S_k(\theta))\} \leq n_{k+1}(\mathrm{dg}\theta-\rho) + H +$$

$$+ \max_{1\leq i\leq N} \mathrm{dg}\{S^{i-1}(\theta) + S^{i-2}(\theta)S_k(\theta) + \ldots + S_k^{i-1}(\theta)\}.$$

Since for k sufficiently large

$$\max_{1\leq i\leq N} \max_{0\leq j\leq i-1} \mathrm{dg}\, S^{i-1-j}(\theta)\, S_k^j(\theta) \leq (N-1)\max(\mathrm{dg}S(\theta),0),$$

we certainly have

$$(9.7.4) \qquad dg\{P(S(\theta)) - P(S_k(\theta))\} \le - c_2\, n_{k+1}, \qquad k > \kappa_3,$$

where $c_2 > 0$ is independent of k. Clearly, this inequality also holds for the case that $r_k(\theta) = 0$. The inequalities (9.7.2) and (9.7.4) yield for $k_j > \kappa_3$

$$dg \frac{P(S(\theta))-P(S_{k_j}(\theta))}{P(S_{k_j}(\theta))} \le - n_{k_j+1}\left[c_2 - c_1 d_{k_j} \frac{(dgM_{k_j}+a_{k_j}+n_{k_j})}{n_{k_j+1}}\right].$$

Using condition (9.7.1), we infer that there exists a $\kappa_4 > \kappa_3$ such that

$$dg \frac{P(S(\theta))-P(S_{k_j}(\theta))}{P(S_{k_j}(\theta))} < 0, \qquad k_j > \kappa_4.$$

Hence for $k_j > \kappa_4$

$$dg\ P(S(\theta)) = dg\left[P(S_{k_j}(\theta))\left\{1 + \frac{P(S(\theta))-P(S_{k_j}(\theta))}{P(S_{k_j}(\theta))}\right\}\right] = dg\ P(S_{k_j}(\theta)),$$

from which we conclude that $P(S(\theta)) \neq 0$. Since P is chosen arbitrarily, we have proved the theorem. □

9.8. <u>REMARKS</u>. (i) A power series $\sum_{k=0}^{\infty} \alpha_k t^{n_k}$ is called a *gap series*, when $\lim_{k\to\infty} n_k/n_{k+1} = 0$. Thus we infer from the previous theorem that the sum of the gap series

$$\sum_{k=0}^{\infty} c_k\, \theta^{n_k}, \quad c_k \in \mathbb{F}_q^*, \qquad k = 0,1,\ldots$$

is transcendental over $\mathbb{F}_q(X)$ for every non-zero algebraic θ from Φ with $dg\ \theta < 0$.

(ii) In case R is finite, $S(\theta)$ need not be transcendental for algebraic θ with $dg\ \theta = R$. For instance, take $n_k = k!$, $\alpha_k = X^{k!}/X^k$. Then $R = -1$, the conditions of theorem 9.7 are satisfied and we obtain

$$S(X^{-1}) = \sum_{k=0}^{\infty} \frac{1}{X^k} = (1-X)^{-1}.$$

The following example shows that $S(\theta)$ can be transcendental for an algebraic θ with $dg\ \theta = R$; L.I. WADE (1941) proved the transcendence of

$\sum_{k=1}^{\infty} (X^{q^k}-X)^{-1}$, whereas $X^q \sum_{k=1}^{\infty} (X^{q^k}-X)^{-1}$ can be seen as the value for $\theta = X^{-1}$ of the gap series

$$S(t) = \sum_{k=1}^{\infty} \frac{X^{q^{k!}} t^{q^{k!}}}{X^{q^k}-X}$$

with radius of convergence $R = -1$.

(iii) If the elements α_k, $k \in \mathbb{N}^0$ belong to a fixed, separable, finite extension of $\mathbb{F}_q(X)$, then the condition in theorem 9.7 can be weakened to

$$\lim_{k\to\infty} \frac{n_k + dgM_k + a_k}{n_{k+1}} = 0.$$

(iv) The element

$$\theta = \sum_{k=1}^{\infty} \frac{c_k}{X^{k!}}$$

of example 8.3 is a Liouville number, which can be seen as a certain value of the gap series

$$S(t) = \sum_{k=1}^{\infty} c_k t^{k!},$$

which converges for $t \in \Phi$ with $dg\ t < 0$. Here $a_k = 0$, $d_k = 1$, $M_k = 1$ for $k = 1,2,\ldots$ and condition (9.7.1) is satisfied. Now it follows from theorem 9.7 that $S(X^{-1})$ is transcendental.

With the method used in the proof of theorem 9.7 we can generalize theorem 7.6 to

9.9. THEOREM. *Let* K *be a finite, separable algebraic extension of* $\mathbb{F}_q(X)$. *Let the entire function* $S: \Phi \to \Phi$ *be given by*

$$S(t) := \sum_{n=0}^{\infty} \alpha_n t^n, \qquad \alpha_n \in K.$$

Let M_n *denote a denominator for* $\alpha_0, \alpha_1, \ldots, \alpha_n$ *with minimal valuation. Let* $\theta \in \Phi\backslash\{0\}$ *be algebraic.*

If there exists a positive, real constant c *such that*

$$\text{(9.9.1)} \qquad d^*(\alpha_n) + nd^*(\theta) < c\ dg\ M_n,$$

and increasing sequences $\{n_k\}_{k=1}^{\infty}$, $n_k \in \mathbb{N}$ *and* $\{\lambda_k\}_{k=1}^{\infty}$, $\lambda_k \in \mathbb{R}, \lambda_k > 0$ *with* $\lim_{k\to\infty} \lambda_k = \infty$ *such that*

$$(9.9.2)\begin{cases} \text{(i)}\ \mathrm{dg}\ \alpha_n + n\ \mathrm{dg}\theta < -\lambda_k\ \mathrm{dg}\ M_{n_k}, & k = 1,2,\ldots;\ n > n_k, \\ \\ \text{(ii)}\ \sum\limits_{n=n_k+1}^{\infty} \alpha_n \theta^n \neq 0, & k = 1,2,\ldots, \end{cases}$$

then $S(\theta)$ *is transcendental over* $\mathbb{F}_q(X)$.

<u>PROOF</u>. Since S is entire, we have

$$(9.9.3) \qquad \limsup_{n\to\infty} \frac{\mathrm{dg}\ \alpha_n}{n} = -\infty.$$

If $\alpha_n \neq 0$, we have

$$N_{K\to\mathbb{F}_q(X)}(M_n\alpha_n) \in \mathbb{F}_q[X] \setminus \{0\}.$$

Put $h := [K: \mathbb{F}_q(X)]$ and let $\sigma_1, \sigma_2, \ldots, \sigma_h$ denote the h $\mathbb{F}_q(X)$-monomorphisms $K \hookrightarrow \Phi$. Then, using (9.9.1) and lemma 6.10, we have

$$0 \le \mathrm{dg}\ N_{K\to\mathbb{F}_q(X)}(M_n\alpha_n) = \prod_{\rho=1}^{h} \mathrm{dg}(\sigma_\rho(M_n\alpha_n)) \le$$

$$\le h\ \mathrm{dg}\ M_n + \mathrm{dg}\ \alpha_n + (h-1)d^*(\alpha_n) < (h+c(h-1))\mathrm{dg}\ M_n + \mathrm{dg}\ \alpha_n.$$

Hence by (9.9.3) there exists an n_0 such that

$$(9.9.4) \qquad \alpha_n \neq 0,\ n > n_0 \Rightarrow \frac{\mathrm{dg}\ M_n}{n} > 1.$$

First we remark that we may suppose that

$$(9.9.5) \qquad \alpha_{n_k} \neq 0, \qquad k = 1,2,\ldots .$$

For suppose that (9.9.5) does not hold a priori. It may occur that we can take subsequences

$$\{n_{k_j}\}_{j=1}^{\infty} \quad \text{and} \quad \{\lambda_{k_j}\}_{j=1}^{\infty}$$

such that not only (9.9.2) but also (9.9.5) holds for these subsequences. Then we continue after the appropriate relabelling. But such subsequences need not exist, due to the fact that for some k_0

$$\alpha_{n_k} = 0, \qquad k > k_0.$$

Then we proceed as follows. From the sequence $\{n_k\}_{k=1}^{\infty}$ we skip $n_1, n_2, \ldots, n_{k_0}$ and those n_k for which

$$\alpha_{n_{k-1}+1} = \alpha_{n_{k-1}+2} = \ldots = \alpha_{n_k-1} = \alpha_{n_k} = 0.$$

The remaining sequence of indices we denote again by $\{n_k\}_{k=1}^{\infty}$. Note that in view of (9.9.2)(ii) this sequence $\{n_k\}_{k=1}^{\infty}$ is infinite. From $\{\lambda_k\}_{k=1}^{\infty}$ we take the corresponding subsequence and call it $\{\lambda_k\}_{k=1}^{\infty}$ again. Now define

$$m_k := \max\{n \mid n_{k-1} < n < n_k,\ \alpha_n \neq 0\}, \qquad k = 1,2,\ldots .$$

Then M_{n_k} is a denominator for α_{m_k}, in fact

$$\mathrm{dg}\, M_{m_k} = \mathrm{dg}\, M_{n_k},$$

in view of the minimality condition of $\mathrm{dg}\, M_n$.
Finally

$$\sum_{n=m_k+1}^{\infty} \alpha_n \theta^n = \sum_{n=n_{k+1}}^{\infty} \alpha_n \theta^n.$$

Hence (9.9.2) holds for the sequence $\{m_k\}_{k=1}^{\infty}$, whereas moreover $\alpha_{m_k} \neq 0$.

After these preliminaries we now start with the actual proof. Let $\theta \neq 0$ be algebraic of degree s and let M be a denominator for θ. Put

$$S_k(\theta) := \sum_{i=0}^{n_k} \alpha_i \theta^i$$

and

$$r_k(\theta) := S(\theta) - S_k(\theta), \qquad k \in \mathbb{N}^0.$$

Then $S_k(\theta) \in K(\theta)$. Denote the height of $S_k(\theta)$ by h_k. According to lemma 9.3, lemma 6.15 and the inequality (9.9.4), we have

(9.9.6) $\qquad h_k \le c_1\{n_k + dg\, M_{n_k}\},$

where c_1 is a positive, real constant, independent of k.

Let $P \in \mathbb{F}_q[X][t]$ be an arbitrary but fixed polynomial of degree $N \ge 1$ and height H. Let $\beta_1,\beta_2,\ldots,\beta_m$ be the distinct zeros of P in Φ and suppose that $m \ge 2$. From the convergence of $\sum_{n=0}^{\infty} \alpha_n\theta^n$ it follows that for $k > k_1$ and $\nu \in \mathbb{N}$ we have

$$dg(S_{k+\nu}(\theta) - S_k(\theta)) < \min_{\substack{1\le i,j\le m\\ i\ne j}} dg(\beta_i-\beta_j).$$

On the other hand we see from (9.9.2)(ii) that for every $k \in \mathbb{N}^0$ there exists a $\nu(k) \in \mathbb{N}$ such that

$$-\infty < dg(S_{k+\nu(k)}(\theta)-S_k(\theta)).$$

Hence

(9.9.7) $\qquad P(S_k(\theta)) = 0 \Rightarrow P(S_{k+\nu(k)}(\theta)) \ne 0.$

Due to (9.9.2)(ii) this is also true in case P has but one zero, of order N. Relation (9.9.7) yields the existence of a sequence $\{k_j\}_{j=1}^{\infty}$ such that

(9.9.8) $\qquad P(S_{k_j}(\theta)) \ne 0, \qquad j = 1,2,\ldots\ .$

Now it follows from lemma 9.6 and from (9.9.6) that

$$dg\, P(S_{k_j}(\theta)) \ge -(h_{k_j}N+s_{k_j}H) \ge -c_2(n_{k_j}+dg\, M_{n_{k_j}}),$$

where $c_2 > 0$ is independent of j.

According to (9.9.2)(i), we have

$$dg\, r_k(\theta) < -\lambda_k\, dg\, M_{n_k}.$$

Hence for k sufficiently large

$$\text{(9.9.10)} \quad dg(P(S(\theta)) - P(S_k(\theta))) \le -\lambda_k \, dg\, M_{n_k} + H + (N-1)\max(dgS(\theta),0)$$

$$\le -c_3 \lambda_k \, dg\, M_{n_k},$$

where $c_3 > 0$ is independent of k.

In view of (9.9.8) and the inequalities (9.9.9) and (9.9.10), we have

$$dg \frac{P(S(\theta))-P(S_{k_j}(\theta))}{P(S_{k_j}(\theta))} \le -n_{k_j} \left\{(c_3\lambda_{k_j}-c_2) \frac{dg\, M_{n_{k_j}}}{n_{k_j}} - c_2\right\}.$$

Using (9.9.4) and

$$\lim_{j\to\infty} \lambda_{k_j} = \infty ,$$

we see that for j sufficiently large

$$dg \frac{P(S(\theta))-P(S_{k_j}(\theta))}{P(S_{k_j}(\theta))} < 0.$$

Hence $P(S(\theta)) \ne 0$. Since P was chosen arbitrarily, we have proved the theorem. □

10. TRANSCENDENCE MEASURES

Let $\alpha \in \Phi$ be transcendental over $\mathbb{F}_q(X)$. Then for all non-trivial $P \in \mathbb{F}_q[X][t]$ we have $P(\alpha) \ne 0$. Since the collection $\mathcal{C}(N,H)$ of all non-trivial $P \in \mathbb{F}_q[X][t]$ with degree at most N and height at most H is finite, we have

$$\min_{P\in\mathcal{C}(N,H)} dg\, P(\alpha) > -\infty.$$

Hence there exists an $f\colon \mathbb{N} \times \mathbb{N}^0 \to \mathbb{R}$ such that $dg\, P(\alpha) > f(N,H)$ for all $P \in \mathcal{C}(N,H)$.

10.1. <u>DEFINITION</u>. Let $\alpha \in \Phi$ be transcendental over $\mathbb{F}_q(X)$. A function $f\colon \mathbb{N} \times \mathbb{N}^0 \to \mathbb{R}$ such that

$$dg\, P(\alpha) \ge f(N,H)$$

for all non-trivial $P \in \mathbb{F}_q[X][t]$ of degree at most N and height at most H, is called a *transcendence measure of* α.

In this section we shall give an upper bound for the transcendence measures of all those transcendental $\alpha \in \Phi$ which occur as the limit of some sequence $\{\alpha_j\}_{j=1}^{\infty}$, where all the α_j lie in a fixed, finite, separable algebraic extension of $\mathbb{F}_q(X)$, see theorem 10.6. Lemma 10.2 and theorem 10.3 may be considered as analogues of well known classical results, generally called after Siegel.

10.2. LEMMA. *Let*

$$\sum_{i=1}^{s} a_{ki}\, x_i, \qquad k = 1,2,\ldots,r,$$

with $a_{ki} \in \mathbb{F}_q(X)$ *be a system of* r *linear forms in the* s *variables* $x_1, x_2, \ldots, x_s$ *and with* $r < s$. *Let* $a \in \mathbb{Z}$ *be such that*

$$\max_{\substack{1\le i\le s\\ 1\le k\le r}} \mathrm{dg}\, a_{ki} \le a.$$

Then for all $c \in \mathbb{N}$ *there exist* $C_1, C_2, \ldots, C_s \in \mathbb{F}_q[X]$, *not all zero, such that*

$$\mathrm{dg}\, C_i < c$$

and

$$\mathrm{dg}\,\Big(\sum_{i=1}^{s} a_{ki} C_i\Big) \le a + \Big(1-\frac{s}{r}\Big)c, \qquad k = 1,2,\ldots,r.$$

PROOF. Let $M \in \mathbb{F}_q[X]$ be such that $Ma_{ki} \in \mathbb{F}_q[X]$, $k = 1,2,\ldots,r$; $i = 1,2,\ldots,s$.
The cube $K_0 := \{(t_1,t_2,\ldots,t_s) \mid t_i \in \Phi, \mathrm{dg}\, t_i < c,\ i = 1,2,\ldots,s\}$ contains q^{sc} lattice points $(X_1,X_2,\ldots,X_s)$ with $X_i \in \mathbb{F}_q[X]$, $i = 1,2,\ldots,s$.
If for such lattice points we denote

$$Y_k := Y_k(X_1,\ldots,X_s) := \sum_{i=1}^{s} Ma_{ki}X_i, \qquad k = 1,2,\ldots,r$$

and if $m := \mathrm{dg}\, M$, then $Y_k \in \mathbb{F}_q[X]$ and

$$\mathrm{dg}\ Y_k < m + a + c, \quad k = 1,2,\ldots,r.$$

Hence every lattice point $(X_1,\ldots,X_s)$ of K_0 corresponds with one of the $q^{r(m+a+c)}$ lattice points of the cube

$$K := \{(t_1,t_2,\ldots,t_r) \mid t_i \in \Phi,\ \mathrm{dg}\ t_i < m + a + c\}.$$

Now choose $n \in \mathbb{N}$ such that

$$(10.2.1) \qquad c\,\frac{s}{r} - 1 \le n < c\,\frac{s}{r}\ .$$

We shall distribute the lattice points of the cube K over q^{rn} "cells" in the following way. For every $E \in \mathbb{F}_q[X]$ with dg $E < n$ we consider the set

$$A_E := \{t \in \Phi \mid \mathrm{dg}(t - EX^{m+a+c-n}) < m + a + c - n\}$$

Suppose that $A_{E_1} \cap A_{E_2} \neq \emptyset$, then it follows by subtraction that $\mathrm{dg}(E_1-E_2) < 0$, i.e. $E_1 = E_2$. Hence the sets A_E are disjoint. Furthermore we note that every $G \in \mathbb{F}_q[X]$ with dg $G < m + a + c$ belongs to one of these A_E. Therefore every lattice point of K belongs to just one of the q^{rn} cells of the form

$$\{(t_1,t_2,\ldots,t_r) \mid t_k \in A_{E_k},\ \mathrm{dg}\ E_k < n, \quad k = 1,2,\ldots,r\}.$$

From the construction above we infer that every lattice point $(X_1,X_2,\ldots,X_s)$ of K_0 corresponds with a cell of K. It follows from (10.2.1) that

$$q^{nr} < q^{cs},$$

i.e. the number of cells in K is less than the number of lattice points in K_0. Hence there are at least two different lattice points $(X_1^{(1)},X_2^{(1)},\ldots,X_s^{(1)})$, $(X_1^{(2)},X_2^{(2)},\ldots,X_s^{(2)})$ in K_0 which correspond with the same cell of K, i.e. there exist $E_1,E_2,\ldots,E_r \in \mathbb{F}_q[X]$ with dg $E_k < n$, $k = 1,\ldots,r$, such that

$$dg(Y_k(X_1^{(j)},X_2^{(j)},\ldots,X_s^{(j)}) - X^{m+a+c-n}E_k) < m + a + c - n,$$

$$k = 1,2,\ldots,r;\ j = 1,2.$$

If we put $C_i = X_i^{(1)} - X_i^{(2)}$, $i = 1,\ldots,s$, then $C_i \in \mathbb{F}_q[X]$, not all of them are zero, $dg\ C_i < c$ and

$$dg(\sum_{i=1}^{s} a_{ki}C_i) = -m + dg(\sum_{i=1}^{s} Ma_{ki}C_i) \leq$$

$$\leq a + c - n - 1 \leq a + c(1-\frac{s}{r}).\ \square$$

10.3. <u>THEOREM</u>. *Let* K *be a finite, separable extension of* $\mathbb{F}_q(X)$ *of degree* n. *Let*

$$\sum_{i=1}^{s} \alpha_{ki}x_i, \qquad k = 1,2,\ldots,r$$

with $\alpha_{ki} \in K$ *be a system of* r *linear forms in the* s *variables* $x_1,x_2,\ldots,x_s$ *and let* $nr < s$. *Let* $a \in \mathbb{Z}$ *be such that*

$$\max_{\substack{1\leq i\leq s\\ 1\leq k\leq r}} d^*(\alpha_{ki}) \leq a.$$

Then for every $c \in \mathbb{N}$ *there exist* $C_1,C_2,\ldots,C_s \in \mathbb{F}_q[X]$, *not all of them zero, such that*

$$dg\ C_i < c, \qquad i = 1,2,\ldots,s$$

and

$$dg(\sum_{i=1}^{s} \alpha_{ki}C_i) \leq a + b + (1-\frac{s}{rn})c,$$

where b *is a non-negative constant which depends only on* K. *More explicitly, if* $K = \mathbb{F}_q(X)(\theta)$, *then we may take* $b = (n-1)h(\theta) + n(n-1)d^*(\theta)$.

To prove this theorem we need two lemmas which are interesting in themselves. Lemma 10.4 is an analogue of a lemma of N.I. FEL'DMAN (1951; lemma 2, p.54), which is also proved by K. MAHLER (1960) and P. CIJSOUW (1972; lemma 2.7). Lemma 10.5 is an analogue of a result of R. GÜTING (1961; theorem 4).

10.4. <u>LEMMA</u>. *Let* $P \in \Phi[t]$ *be given by*

$$P(t) = a_N t^N + a_{N-1}t^{N-1} + \ldots + a_1 t + a_0 = a_N \prod_{i=1}^{N} (t-\beta_i),$$

$$\beta_i, a_i \in \Phi,\ a_N \neq 0,\ N \geq 1.$$

Then

$$H(P) = \mathrm{dg}\, a_N + \sum_{i=1}^{N} \max(\mathrm{dg}\beta_i, 0). \tag{10.4.1}$$

<u>PROOF</u>. Let $R_1, R_2, \ldots, R_\ell$ be the hooking-radii of P in increasing order. Put $R_0 := -\infty$, $R_{\ell+1} := +\infty$ and define $m \in \{0,1,\ldots,\ell\}$ by $R_m \leq 0 < R_{m+1}$. From theorem 5.11 we see that

$$M_0(P) = \max_{0\leq i\leq N} \mathrm{dg}\, a_i = \mathrm{dg}\, a_{i_m}$$

and hence that

$$H(P) = \mathrm{dg}\, a_{i_m}. \tag{10.4.2}$$

Now take a $t_0 \in \Phi$ such that $0 < \rho_0 := \mathrm{dg}\, t_0 < R_{m+1}$. Since ρ_0 is not a hooking-radius, we have

$$\mathrm{dg}\, P(t_0) = M_{\rho_0}(P). \tag{10.4.3}$$

Again from theorem 5.11 we see that

$$M_{\rho_0}(P) = \mathrm{dg}\, a_{i_m} + i_m\rho_0. \tag{10.4.4}$$

On the other hand it is clear that

$$\mathrm{dg}\, P(t_0) = \mathrm{dg}\, a_N + \sum_{i=1}^{N} \max(\mathrm{dg}\beta_i, 0) + v\rho_0, \tag{10.4.5}$$

where v denotes the number of zeros of P with non-positive valuation. But from lemma 5.19 and corollary 5.14 we have $v = i_m$. Combining (10.4.2), (10.4.3), (10.4.4) and (10.4.5) gives the desired

$$H(P) = \mathrm{dg}\ a_N + \sum_{i=1}^{N} \max(\mathrm{dg}\beta_i, 0). \quad \square$$

10.5. LEMMA. *Let* $Q \in \mathbb{F}_q[X][t]$ *be separable of degree* $N \geq 1$ *and height* H. *Let* $\beta_1, \beta_2, \ldots, \beta_N$ *denote the zeros of* Q. *Let* $\mathcal{N}$ *be an arbitrary non-empty subset of*

$$\Delta := \{(i,j) \mid 1 \leq i \leq N,\ 1 \leq j \leq N,\ i < j\}.$$

Then

$$(10.5.1) \qquad \sum_{\mathcal{N}} \mathrm{dg}(\beta_i - \beta_j) \geq -(N-1)H.$$

PROOF. Put

$$Q(t) = A \prod_{i=1}^{N} (t - \beta_i).$$

Then the discriminant of Q, defined by

$$D := A^{2N-2} \prod_{1 \leq i < j \leq N} (\beta_i - \beta_j)^2,$$

is an element of $\mathbb{F}_q[X]$, see Corollary 0.6. Since Q is separable, the zeros of Q are distinct and thus $D \neq 0$. Therefore

$$\mathrm{dg}\ D = (2N-2)\mathrm{dg}\ A + 2 \sum_{1 \leq i < j \leq N} \mathrm{dg}(\beta_i - \beta_j) \geq 0.$$

Hence

$$\sum_{\mathcal{N}} \mathrm{dg}(\beta_i - \beta_j) \geq -(N-1)\mathrm{dg}\ A - \sum_{\Delta \setminus \mathcal{N}} \mathrm{dg}(\beta_i - \beta_j).$$

We may suppose that $\beta_1, \beta_2, \ldots, \beta_N$ are arranged in such a way that $\mathrm{dg}\ \beta_1 \leq \mathrm{dg}\ \beta_2 \leq \ldots \leq \mathrm{dg}\ \beta_N$. Then

$$\sum_{(i,j)\in\Delta\setminus N} dg(\beta_i-\beta_j) \le \sum_{(i,j)\in\Delta\setminus N} dg\ \beta_j \le \sum_{(i,j)\in\Delta\setminus N} \max(0,dg\beta_j)$$

$$\le \sum_{j=1}^{N} (j-1)\ \max(0,dg\beta_j)$$

$$\le (N-1) \sum_{j=1}^{N} \max(0,dg\beta_j).$$

Thus

$$\sum_{N} dg(\beta_i-\beta_j) \ge -\ (N-1)(dgA+\sum_{j=1}^{N} \max(0,dg\beta_j)),$$

which, by lemma 10.4, yields

$$\sum_{N} dg(\beta_i-\beta_j) \ge -\ (N-1)H. \quad \square$$

Proof of theorem 10.3. Since K is a finite, separable extension of $\mathbb{F}_q(X)$, there exists a primitive element $\beta \in K$, i.e. $K = \mathbb{F}_q(X)(\beta)$. (See O. ZARISKI and P. SAMUEL (1958), Ch.II, §9 th.19.) We have

(10.3.1) $$\alpha_{ki} = \sum_{j=0}^{n-1} a_{kij}\ \beta^j, \quad a_{kij} \in \mathbb{F}_q(X),\quad k = 1,2,\dots,r;\ i = 1,2,\dots,s.$$

Let $\sigma_1,\sigma_2,\dots,\sigma_n$ denote the n $\mathbb{F}_q(X)$-monomorphisms $K \hookrightarrow \Phi$. For every $k \in \{1,2,\dots,r\}$ and $i \in \{1,2,\dots,s\}$ we solve the system of equations

$$\sigma_\nu(\alpha_{ki}) = \sum_{j=0}^{n-1} a_{kij}\ \sigma_\nu(\beta^j), \qquad \nu = 1,2,\dots,n$$

in a_{kij}, $j = 0,1,\dots,n-1$. Since $\det(\sigma_\nu(\beta^j))_{\nu,j} \ne 0$, we obtain from Cramer's rule

$$dg\ a_{kij} \le \max_{1\le\nu\le n} dg\ \sigma_\nu(\alpha_{ki}) + (n-1) \max_{\substack{1\le\nu\le n\\ 0\le j\le n-1}} dg\ \sigma_\nu(\beta^j)$$

$$-\ dg\ \det(\sigma_\nu(\beta^j))_{\nu,j}$$

$$\le a + (n-1)^2 d^*(\beta) - dg \prod_{1\le\nu<\mu\le n} (\sigma_\nu(\beta)-\sigma_\mu(\beta)).$$

Since the roots $\sigma_\nu(\beta)$, $\nu = 1,\dots,n$ of the minimal polynomial of β are distinct, we have according to lemma 10.5

$$\sum_{1\le\nu<\mu\le n} dg(\sigma_\nu(\beta)-\sigma_\mu(\beta)) \ge -(n-1)h(\beta).$$

If we define

$$b_0 := (n-1)h(\beta)+(n-1)^2 d^*(\beta),$$

then

$$dg\ a_{kij} \le a + b_0.$$

Now we consider the following rn linear forms in the s variables $x_1,x_2,\dots,x_s$:

$$\sum_{i=1}^{s} a_{kij}\,x_i, \qquad k = 1,2,\dots,r;\ j = 0,1,\dots,n-1.$$

It follows from lemma 10.2 that there exist $C_1,\dots,C_s$ in $\mathbb{F}_q[X]$, not all of them zero, such that

$$dg\ C_i < c$$

and

$$(10.3.2)\qquad dg\left(\sum_{i=1}^{s} a_{kij}C_i\right) \le a + b_0 + (1-\frac{s}{rn})c.$$

From (10.3.1) and (10.3.2) we obtain

$$dg\left(\sum_{i=1}^{s} \alpha_{ki}C_i\right) \le a + b_0 + (1-\frac{s}{rn})c + (n-1)d^*(\beta). \quad \square$$

10.6. <u>THEOREM</u>. *Let* $\alpha \in \Phi$ *be transcendental over* $\mathbb{F}_q(X)$. *Suppose that* $\alpha = \lim_{j\to\infty} \alpha_j$, *where all the* α_j *are contained in a fixed, finite, separable algebraic extension* K *of* $\mathbb{F}_q(X)$. *Then a transcendence measure for* α *cannot be better than* $-c_0NH + c_1N$, *where* c_0,c_1 *are suitable positive constants which depend only on* α.

PROOF. We may suppose that $H \geq 1$. Choose $\theta \in K$ such that $K = \mathbb{F}_q(X)(\theta)$ and put $n := [K: \mathbb{F}_q(X)]$. Since $\alpha = \lim_{j\to\infty} \alpha_j$, there exists an α_j such that

$$(10.6.1) \qquad \mathrm{dg}\ \alpha_j = \mathrm{dg}\ \alpha$$

and

$$(10.6.2) \qquad \mathrm{dg}(\alpha-\alpha_j) < -NH - H.$$

We consider the linear form

$$x_0 + \alpha_j x_1 + \ldots + \alpha_j^N x_N$$

in the $N + 1$ variables $x_0, x_1, \ldots, x_N$. If $N \geq n$ we can apply theorem 10.3 and it follows that there exist $C_0, C_1, \ldots, C_N \in \mathbb{F}_q[X]$, not all zero, such that

$$\mathrm{dg}\ C_i < H$$

and such that

$$(10.6.3) \qquad \mathrm{dg}(C_0+\alpha_j C_1+\ldots+\alpha_j^N C_N) \leq N \max(\mathrm{dg}\alpha_j, 0) + b + (1-\frac{N+1}{n})H,$$

where b is a non-negative constant depending only on K, i.e. on α. From (10.6.1) and (10.6.2) we infer that

$$\begin{aligned} &\mathrm{dg}\{(\alpha-\alpha_j)C_1 + \ldots + (\alpha^N-\alpha_j^N)C_N\} \leq \\ &\qquad \leq \mathrm{dg}(\alpha-\alpha_j) + \max_{1\leq\nu\leq N} \{\mathrm{dg}\ \frac{\alpha^\nu-\alpha_j^\nu}{\alpha-\alpha_j} + \mathrm{dg}\ C_\nu\} \\ &\qquad < -NH + (N-1)\max(\mathrm{dg}\alpha, 0). \end{aligned}$$

Hence, using (10.6.3), we obtain

$$(10.6.4) \qquad \mathrm{dg}(C_0+\alpha C_1+\ldots+\alpha^N C_N) \leq -(\frac{N+1}{n}-1)H + N\{\max(\mathrm{dg}\alpha, 0) + \frac{b}{N}\},$$

which proves our assertion. $\square$

10.7. REMARK. All elements of Φ which are up till now known to be transcendental, satisfy the condition of theorem 10.6. In section 9 we already mentioned that the element

$$\omega := \sum_{k=1}^{\infty} \frac{1}{X^{q^k}-X}$$

is transcendental over $\mathbb{F}_q(X)$. (See L.I. WADE (1941), theorem 4.1.) We see that $\omega \in F$ and from theorem 10.6 we infer that a transcendence measure for ω cannot be better than $-NH$. In 1974 P. BUNDSCHUH proved that there exist positive constants c_1, c_2, depending only on q, such that

$$\mathrm{dg}\ P(\omega) \geq -c_1 q^{3N} - c_2 N q^{2N} H$$

for every non-trivial $P \in \mathbb{F}_q[X][t]$ of degree at most N and height at most H. (See Séminaire Delange-Pisot-Poitou 1974/75, §3 th.2.) Recently P. BUNDSCHUH has also given transcendence measures for $\psi(1)$ and $\sum_{k=0}^{\infty} L_k^{-s}$, $s \in \mathbb{N}$.

11. A TRANSCENDENCE MEASURE FOR CERTAIN LIOUVILLE NUMBERS

It follows from example 8.4.1 as well as from theorem 9.7 that

$$c_0 + \sum_{k=1}^{\infty} c_k X^{-k!}, \qquad c_k \in \mathbb{F}_q^*,\ k \in \mathbb{N}^0,$$

is transcendental over $\mathbb{F}_q(X)$. In the following theorem we derive a transcendence measure for these Liouville numbers.

11.1. THEOREM. *Let*

$$(11.1.1) \qquad \alpha := c_0 + \sum_{k=1}^{\infty} c_k X^{-k!}, \qquad c_k \in \mathbb{F}_q^*,\ k \in \mathbb{N}^0.$$

Then for every polynomial $Q \in \mathbb{F}_q[X][t]$ *of degree* $N \geq 1$ *and height* H *one has*

$$(11.1.2) \qquad \mathrm{dg}\ Q(\alpha) > -51\{N^{N-1} + NH \log^2 2H\}.$$

PROOF. (i) First we suppose that Q is irreducible. Put

$$\alpha_k := c_0 + \sum_{i=1}^{k} c_i X^{-i!}.$$

Then α_k is algebraic over $\mathbb{F}_q(X)$ of degree 1 and height $h(\alpha_k) = k!$. According to lemma 9.6 we have either $Q(\alpha_k) = 0$ or

$$(11.1.3) \qquad dg\ Q(\alpha_k) \geq -(H+Nk!).$$

Since all c_k in (11.1.1) are non-zero, we have

$$dg(\alpha-\alpha_k) = -(k+1)!$$

and

$$dg\ \alpha = dg\ \alpha_k = 0.$$

Hence

$$(11.1.4) \qquad dg(Q(\alpha)-Q(\alpha_k)) \leq dg(\alpha-\alpha_k) + H + \max_{1\leq i\leq N} dg\left(\frac{\alpha^i-\alpha_k^i}{\alpha-\alpha_k}\right) \leq -(k+1)! + H.$$

Now we define

$$(11.1.5) \qquad \kappa := \min\{k \in \mathbb{N} \mid k! > \max((N-1)!, 2H)\}.$$

Then for all $k \geq \kappa$ such that $Q(\alpha_k) \neq 0$ it follows from (11.1.3), (11.1.4) and the triangle-inequality in its sharpened form that

$$(11.1.6) \qquad dg\ Q(\alpha) = dg\ Q(\alpha_k) \geq -(H+Nk!).$$

Suppose that $Q(\alpha_k) = 0$. Since Q is irreducible and since α_k is algebraic of degree 1, this is only possible if $N = 1$. Put $Q(t) = A_1t + A_0$, then it follows that

$$Q(\alpha_{k+1}) = A_1(\alpha_{k+1}-\alpha_k) = A_1\ c_{k+1}\ X^{-(k+1)!},$$

i.e. $\qquad Q(\alpha_{k+1}) \neq 0.$

Hence at least one of the numbers $Q(\alpha_\kappa)$ and $Q(\alpha_{\kappa+1})$ is different from zero and so, in view of (11.1.6), we have

$$(11.1.7) \qquad dg\ Q(\alpha) \geq -(H+N(\kappa+1)!).$$

Now we give an upper bound for $(\kappa+1)!$ in terms of N and H. First we suppose that

$$(11.1.8) \qquad (N-1)! > 2H.$$

Then $\kappa = N$ if $N \geq 2$ and $\kappa = 2$ if $N = 1$. Hence (11.1.7) and (11.1.8) give

$$(11.1.9) \qquad \mathrm{dg}\ Q(\alpha) \geq -\left(\frac{(N-1)!}{2} + N \max((N+1)!,6)\right) \geq -9N^{N-1}.$$

Secondly, if

$$(11.1.10) \qquad (N-1)! \leq 2H,$$

we have $\kappa \geq 3$. Hence

$$(\kappa+1)! < 25(\kappa-1)!\ \log^2(\kappa-1)!.$$

It follows from (11.1.5) and (11.1.10) that

$$(\kappa-1)! \leq 2H.$$

Now (11.1.7) yields

$$(11.1.11) \quad \mathrm{dg}\ Q(\alpha) \geq -(H+50NH\log^2 2H) \geq -51NH\log^2 2H.$$

Finally (11.1.2) follows from (11.1.9) and (11.1.11).

(ii) Now let Q be a reducible polynomial of degree $N \geq 1$ and height H and let

$$Q = Q_1^{\mu_1} Q_2^{\mu_2} \cdots Q_m^{\mu_m}$$

be a decomposition of Q in irreducible factors $Q_1, Q_2, \ldots, Q_m \in \mathbb{F}_q[X][t]$. Denote the degree and the height of Q_i by N_i and H_i respectively, $i = 1,2,\ldots,m$. Remark that $N_i \geq 1$, $i = 1,2,\ldots,m$ and that

$$(11.1.12) \quad N = \mu_1 N_1 + \mu_2 N_2 + \ldots + \mu_m N_m.$$

By lemma 9.5 we have

(11.1.13) $H = \mu_1 H_1 + \mu_2 H_2 + \ldots + \mu_m H_m.$

From part (i) of the proof we have

$$\mathrm{dg}\, Q_i(\alpha) \geq -51\{N_i^{N_i-1} + N_i H_i \log^2 2H_i\}, \qquad i = 1,2,\ldots,m;$$

hence

(11.1.14) $$\mathrm{dg}\, Q(\alpha) \geq -51 \sum_{i=1}^{m} \mu_i \{N_i^{N_i-1} + N_i H_i \log^2 2H_i\}$$

$$\geq -51 \sum_{i=1}^{m} (\mu_i N_i)^{\mu_i N_i - 1} - 51N \sum_{i=1}^{m} \mu_i H_i \log^2 2H_i.$$

Since

$$(n+m)^{n+m-1} \geq n^{n-1} + m^{m-1}, \qquad n,m \in \mathbb{N}$$

and since

$$(n+m)\log^2 2(n+m) \geq n \log^2 2n + m \log^2 2m, \qquad n,m \in \mathbb{N}^0,$$

relations (11.1.12), (11.1.13) and (11.1.14) give

$$\mathrm{dg}\, Q(\alpha) \geq -51\{N^{N-1} + NH \log^2 2H\}. \quad \square$$

11.2. THEOREM. *The function* $f\colon \mathbb{N} \times \mathbb{N}^0 \to \mathbb{R}$ given by

$$f(N,H) = -51\{N^{N-1} + NH \log^2 2H\}$$

is a transcendence measure for the element

$$\alpha := c_0 + \sum_{k=1}^{\infty} c_k X^{-k!}, \qquad c_k \in \mathbb{F}_q^*.$$

PROOF. Obvious from the previous theorem. $\square$

CHAPTER IV

ON THE TRANSCENDENCE OF CERTAIN VALUES TAKEN BY $\mathcal{E}$-FUNCTIONS

12. A GENERALISATION OF WADE'S ANALOGUE OF THE GELFOND-SCHNEIDER THEOREM

12.1. DEFINITION. A linear function $f\colon \Phi \to \Phi$, given by

$$f(t) := \sum_{k=0}^{\infty} \alpha_k \frac{t^{q^k}}{F_k},$$

is called an $\mathcal{E}$-*function* if

(i) there exists a finite, separable extension K of $\mathbb{F}_q(X)$ such that $\alpha_k \in K$, $k = 0,1,2,\ldots,$

(ii) there exists a $c \in \mathbb{R}$, $c > 0$, such that

$$d^*(\alpha_k) < cq^k, \qquad k = 0,1,2,\ldots .$$

The above definition of an $\mathcal{E}$-function differs from the classical one, which, in addition, contains a condition on the denominators of the coefficients α_k. (See for instance Th. SCHNEIDER (1957), p.112.)

13.2. REMARKS.

(i) An $\mathcal{E}$-function is an entire function.

(ii) The functions ψ and J_n, $n \in \mathbb{Z}$, are $\mathcal{E}$-functions.

(iii) Linear polynomials with separable algebraic coefficients are $\mathcal{E}$-functions. (See theorem 3.5.)

(iv) If f and g are $\mathcal{E}$-functions, then

$$f + g,\ \Delta_r f (r \geq 1), \quad f^{q^r} (r \geq 1)$$

are $\mathcal{E}$-functions.

(v) If P is a linear polynomial with separable algebraic coefficients in Φ and f is an $\mathcal{E}$-function, then $P \circ f$ is an $\mathcal{E}$-function.

In the proof of theorem 7.7 we have given an exposition of Siegel's method in the field Φ. We shall now use this method to prove the following

12.3. THEOREM. *Let* $f_1,\dots,f_n$ *be* $\bar{E}$*-functions, not all polynomials and none of them identically zero. Suppose that for* $1 \le \nu \le n$ *and* $r \in \mathbb{N}$ *we have*

$$\Delta_r f_\nu(t) = R_{\nu r}(f_1(t), f_2(t), \dots, f_n(t)) \tag{12.3.1}$$

where $R_{\nu r}(t_1,t_2,\dots,t_n)$ *is of the form*

$$R_{\nu r}(t_1,t_2,\dots,t_n) = \sum_{0 \le q^{j_1}+\dots+q^{j_n} \le q^r} A_{\nu r j_1 \dots j_n}\, t_1^{q^{j_1}} t_2^{q^{j_2}} \dots t_n^{q^{j_n}}, \tag{12.3.2}$$

with

$$A_{\nu r j_1 \dots j_n} \in \mathbb{F}_q[X]$$

and for some $c_0 \in \mathbb{R}$, $c_0 > 0$,

$$\max_{0 \le q^{j_1}+q^{j_2}+\dots+q^{j_n} \le q^r} \operatorname{dg} A_{\nu r j_1 \dots j_n} \le c_0 q^r. \tag{12.3.3}$$

Then, if $\alpha,\beta \in \Phi$, $\alpha \neq 0$ *and* $\beta \notin \mathbb{F}_q(X)$, *at least one of the* $2n+1$ *elements* β, $f_1(\alpha), f_2(\alpha),\dots,f_n(\alpha)$, $f_1(\alpha\beta)$, $f_2(\alpha\beta),\dots,f_n(\alpha\beta)$ *is transcendental over* $\mathbb{F}_q(X)$.

Before giving the proof we list some special cases as corollaries.

12.4. COROLLARY. *The analogue of the theorem of Gelfond-Schneider (theorem 7.7).*

PROOF. Take $n = 1$, $f_1 = \psi$, $\beta \in \Phi\backslash\mathbb{F}_q(X)$, $\alpha^* \in \Phi\backslash\{0\}$ with $\operatorname{dg}\alpha^* < \frac{q}{q-1}$ and $\alpha = \lambda(\alpha^*)$. From (3.8.2) we see that

$$R_{1r}(t) = (-1)^r t^{q^r}, \qquad r \in \mathbb{N}.$$

Then it follows from the above theorem that at least one of the elements α^*, β, $\psi(\beta\lambda(\alpha^*))$ is transcendental over $\mathbb{F}_q(X)$. □

12.5. COROLLARY. *Let* $\xi \in \Phi$ *be defined by* (2.10.1). *If* β *is algebraic over* $\mathbb{F}_q(X)$ *of degree* ≥ 2, *then* $\psi(\beta\xi)$ *is transcendental over* $\mathbb{F}_q(X)$.

PROOF. Let $f_1 = \psi$, $\alpha = \xi$. Then, since $\beta \notin \mathbb{F}_q(X)$, it follows from theorem 12.3 that at least one of the elements β, $\psi(\beta\xi)$, $\psi(\xi)$ is transcendental over $\mathbb{F}_q(X)$. From theorem 2.12 it follows that $\psi(\xi) = 0$. Hence, since β is algebraic over $\mathbb{F}_q(X)$, we conclude that $\psi(\beta\xi)$ is transcendental over $\mathbb{F}_q(X)$. □

If $\beta \in \mathbb{F}_q(X)$ the opposite of the above assertion is true, as shown by the following

12.6. LEMMA. *If $\beta \in \mathbb{F}_q(X)$, then $\psi(\beta\xi)$ is algebraic over $\mathbb{F}_q(X)$.*

PROOF. For $\beta \in \mathbb{F}_q[X]$ the assertion above is obvious from theorem 2.12. Now put $\beta = \frac{A}{B}$, $A,B \in \mathbb{F}_q[X]$, $B \neq 0$. Then it follows from the theorems 2.12 and 2.13 that

$$0 = \psi\left(B\,\frac{A}{B}\xi\right) = \sum_{j=0}^{\mathrm{dg}B} (-1)^j \frac{\psi_j(B)}{F_j}\, \psi^{q^j}\left(\frac{A}{B}\xi\right),$$

i.e. $\psi\left(\frac{A}{B}\xi\right)$ is algebraic over $\mathbb{F}_q(X)$. □

12.7. COROLLARY. (GEIJSEL, 1971). *Let $\alpha \in \Phi\backslash\{0\}$, $\beta \in \Phi\backslash\mathbb{F}_q(X)$ and $n \in \mathbb{Z}$. Then at least one of the five elements $\beta, J_n(\alpha)$, $J_n(\alpha\beta), \Delta J_n(\alpha)$, $\Delta J_n(\alpha\beta)$ is transcendental over $\mathbb{F}_q(X)$.*

PROOF. First we suppose that $n \geq 0$. Apply theorem 12.3 with $f_1 = J_n$ and $f_2 = \Delta J_n$. According to theorem 4.4, the conditions (12.3.1), (12.3.2) and (12.3.3) are satisfied for $\Delta_r f_1$ for all $r \in \mathbb{N}$. From lemma 3.12 and theorem 4.2(ii) we see that

$$\begin{aligned}\Delta_r f_2 &= \Delta_r J_{n-1}^q = (\Delta_r J_{n-1})^q + (X^{q^r} - X)(\Delta_{r-1} J_{n-1})^q \\ &= J_{n-1-r}^{q^{r+1}} + (X^{q^r} - X) J_{n-r}^{q^r} = \Delta_{r+1} J_n + (X^{q^r} - X)\Delta_r J_n.\end{aligned}$$

It follows again from theorem 4.4 that the three conditions from theorem 12.3 are also satisfied for $\Delta_r f_2$. This proves the corollary for $n \geq 0$.

Now let $n < 0$. Suppose β, $J_n(\alpha)$, $J_n(\alpha\beta)$, $\Delta J_n(\alpha)$, $\Delta J_n(\alpha\beta)$ are algebraic over $\mathbb{F}_q(X)$. Then it follows from theorem 4.2(i) that the elements β, $J_{-n}(\alpha)$, $J_{-n}(\alpha\beta)$, $\Delta J_{-n}(\alpha)$, $\Delta J_{-n}(\alpha\beta)$ are all algebraic, which we have just shown not to be true. □

Proof of theorem 12.3. Put

$$(12.3.4)\qquad f_\nu(t) = \sum_{k=0}^{\infty} \alpha_{\nu k} \frac{t^{q^k}}{F_k}, \qquad 1 \le \nu \le n.$$

Suppose β, $f_1(\alpha),\dots,f_n(\alpha),f_1(\alpha\beta),\dots,f_n(\alpha\beta)$ are algebraic over $\mathbb{F}_q(X)$. Then, for some $e \in \mathbb{N}$,

$$\beta^{q^e}, f_1^{q^e}(\alpha),\dots,f_n^{q^e}(\alpha), f_1^{q^e}(\alpha\beta),\dots,f_n^{q^e}(\alpha\beta)$$

are separable over $\mathbb{F}_q(X)$. Let K be a finite, separable algebraic extension of $\mathbb{F}_q(X)$ of degree h which contains all these elements and the $\alpha_{\nu k}$, $\nu = 1,\dots,n$; $k = 0,1,2,\dots$. Let $\Gamma \in \mathbb{F}_q[X]$ be such that

$$\Gamma\beta^{q^e}, \Gamma f_\nu^{q^e}(\alpha), \Gamma f_\nu^{q^e}(\alpha\beta), \qquad \nu = 1,\dots,n$$

are algebraic integers of K. The natural numbers κ,λ with

$$\lambda > 3\kappa$$

will be chosen later. Put

$$m := \kappa + \lambda - 1$$

and put

$$L(t) := \sum_{\nu=1}^{n} \sum_{j=0}^{q^{2\lambda}-1} \sum_{i=0}^{q^{2\kappa}-1} X_{ij\nu}\, t^{jq^e} f_\nu^{iq^e}(\alpha t),$$

where the $X_{ij\nu}$ will be determined non-trivially and in such a way that $L(A+\beta B) = 0$ for all $A,B \in \mathbb{F}_q[X]$ with $\mathrm{dg}\, A < m$, $\mathrm{dg}\, B < m$. Moreover the $X_{ij\nu}$ will be algebraic integers in K such that $d^*(X_{ij\nu})$ is not too large with respect to λ and κ. We have

$$(12.3.5)\qquad L(A+\beta B) = \sum_{\nu=1}^{n} \sum_{j=0}^{q^{2\lambda}-1} \sum_{i=0}^{q^{2\kappa}-1} X_{ij\nu}\, (A+\beta B)^{jq^e} f_\nu^{iq^e}(\alpha A+\alpha\beta B).$$

By the linearity of the f_ν we have

$$f_\nu(\alpha A+\alpha\beta B) = f_\nu(\alpha A) + f_\nu(\alpha\beta B).$$

The expansion formula (3.10.1) gives

$$f_\nu(\alpha A) = \sum_{\mu=0}^{dgA} \frac{\psi_\mu(A)}{F_\mu} \Delta_\mu f_\nu(\alpha)$$

and hence, by condition (12.3.1),

$$f_\nu^{q^e}(\alpha A) =$$

$$\sum_{\mu=0}^{dgA} \left(\frac{\psi_\mu(A)}{F_\mu}\right)^{q^e} \sum_{0 \le q^{j_1}+\ldots+q^{j_n} \le q^\mu} A^{q^e}_{\nu\mu j_1 \ldots j_n} f_1^{q^{e+j_1}}(\alpha) \ldots f_n^{q^{e+j_n}}(\alpha).$$

From this formula we see that $f_\nu^{q^e}(\alpha A)$ lies in K, i.e. is separable. In fact it is a polynomial in $f_1^{q^e}(\alpha),\ldots,f_n^{q^e}(\alpha)$ of total degree not exceeding $q^{e+dgA} < q^{m+e}$.

By theorem 2.5 we have

$$\frac{\psi_\mu(A)}{F_\mu} \in \mathbb{F}_q[X]$$

and hence $f_\nu^{q^e}(\alpha A) \in \mathbb{F}_q[X][f_1^{q^e}(\alpha),\ldots,f_n^{q^e}(\alpha)]$.From condition (12.3.3) and from remark 2.6 it follows that

$$dg\, f_\nu^{q^e}(\alpha A) \le q^e\{(dgA)q^{dgA} + c_0 q^{dgA}\} + q^{dgA} \max(dgf_1^{q^e}(\alpha),\ldots,dgf_n^{q^e}(\alpha)).$$

Now apply the h $\mathbb{F}_q(X)$-monomorphisms of K. Then we see that

$$(12.3.6) \qquad d^*(f_\nu^{q^e}(\alpha A)) \le q^{m+e}(m+c_0) + q^m \max_{1\le\nu\le n} d^*(f_\nu^{q^e}(\alpha)), \qquad \nu = 1,2,\ldots,n.$$

Similarly we have

$$(12.3.7) \qquad d^*(f_\nu^{q^e}(\alpha\beta B)) \le q^{m+e}(m+c_0) + q^m \max_{1\le\nu\le n} d^*(f_\nu^{q^e}(\alpha\beta)), \quad \nu = 1,2,\ldots,n.$$

We observe that the coefficients of the $X_{ij\nu}$ in (12.3.5) are polynomials in

β^{q^e} of degree not exceeding $q^{2\lambda}$

and in

$f_1^{q^e}(\alpha),\ldots,f_n^{q^e}(\alpha)$, $f_1^{q^e}(\alpha\beta),\ldots,f_n^{q^e}(\alpha\beta)$ of total degree not exceeding $q^{m+2\kappa}$

with coefficients in $\mathbb{F}_q[X]$. Hence, since

$$(12.3.8) \qquad q^{2\lambda} + q^{2\kappa+m} \le q^{2\lambda+1} ,$$

the condition

$$(12.3.9) \qquad \Gamma^{q^{2\lambda+1}} L(A+\beta B) = 0, \qquad A,B \in \mathbb{F}_q[X],\ \mathrm{dg}\, A < m,\ \mathrm{dg}\, B < m$$

implies a system of q^{2m} homogeneous, linear equations, say

$$\sum_{i,j,\nu} D_{ij\nu k}\, X_{ij\nu} = 0, \qquad k = 1,2,\dots,q^{2m},$$

in $nq^{2\lambda+2\kappa}$ unknowns $X_{ij\nu}$ with integral algebraic coefficients $D_{ij\nu k}$.
From (12.3.5), (12.3.6) and (12.3.7) we infer that

$$d^*(D_{ij\nu k}) \le q^{2\lambda+1}\ \mathrm{dg}\, \Gamma + (q^{2\lambda}-1)q^e(m+d^*(\beta)) +$$

$$+ (q^{2\kappa}-1)q^{m+e}[m+c_0+ \max_{1\le\nu\le n} \{d^*(f_\nu(\alpha)),d^*(f_\nu(\alpha\beta))\}].$$

Using (12.3.8), this yields

$$d^*(D_{ij\nu k}) \le q^{2\lambda+e}(2m+c_1),$$

where c_1 is a positive constant independent of κ and λ. According to lemma 6.16 with $r = q^{2m}$, $s = nq^{2\kappa+2\lambda}$ and

$$a = q^{2\lambda+e}(2m+c_1),$$

there exist algebraic integers $X_{ij\nu}$ in K, not all zero, such that condition (12.3.9) is satisfied and such that

$$(12.3.10) \qquad d^*(X_{ij\nu}) < q^{2\lambda+e}(m+c_2),$$

where $c_2 \ge 0$ is independent of λ and κ.

From now on we suppose that the $X_{ij\nu}$ are fixed accordingly.

For $\mu \ge m$ we define

$$\mathcal{B}(\mu) := \{A + \beta B \mid A,B \in \mathbb{F}_q[X];\ A \text{ and } B \text{ not both zero};$$
$$\operatorname{dg} A < \mu,\ \operatorname{dg} B < \mu\}.$$

Let $\mathcal{B} = \cup_{\mu=m}^{\infty} \mathcal{B}(\mu)$. The second step of the proof now consists of proving that L vanishes on $\mathcal{B}$. We have constructed L such that $L(t) = 0$ for $t \in \mathcal{B}(m)$. So it is sufficient to prove that for every $\mu \geq m$

$$(t\in\mathcal{B}(\mu) \Rightarrow L(t) = 0) \Rightarrow (t\in\mathcal{B}(\mu+1) \Rightarrow L(t) = 0).$$

Since $\beta \notin \mathbb{F}_q(X)$, the number of elements of $\mathcal{B}(\mu)$ is $q^{2\mu}-1$.

Let $t_0 \in \mathcal{B}(\mu+1)\setminus\mathcal{B}(\mu)$. If λ is chosen large enough, then

$$\operatorname{dg} t_0 \leq \mu + d^*(\beta) < 2\mu.$$

By the induction hypothesis and by lemma 5.22

$$L(t) \prod_{a\in\mathcal{B}(\mu)} (t-a)^{-1}$$

is an entire function. Hence we can apply the Maximum Modulus Principle (th.5.16) and we obtain

$$\operatorname{dg} L(t_0) - \sum_{a\in\mathcal{B}(\mu)} \operatorname{dg}(t_0-a) \leq \sup_{\operatorname{dg}t=2\mu} \operatorname{dg} L(t) - 2\mu(q^{2\mu}-1).$$

Therefore

$$\operatorname{dg} L(t_0) \leq \sup_{\operatorname{dg}t=2\mu} \operatorname{dg} L(t) - (\mu-d^*(\beta))(q^{2\mu}-1). \tag{12.3.11}$$

From the definition of L and inequality (12.3.10) we see that

$$\sup_{\operatorname{dg}t=2\mu} \operatorname{dg} L(t) \leq q^{2\lambda+e}(m+c_2) + 2\mu q^{2\lambda+e} +$$
$$+ q^{2\kappa+e} \max_{1\leq\nu\leq n} \sup_{\operatorname{dg}t=2\mu} \operatorname{dg} f_\nu(\alpha t).$$

From (12.3.4) and definition 12.1 we have

$$\sup_{\mathrm{dg}\,t=2\mu} \mathrm{dg}\, f_\nu(\alpha t) \le \max_{k\ge 0} (\mathrm{dg}\,\alpha_{\nu k}+2\mu q^k+q^k \mathrm{dg}\,\alpha-kq^k)$$

$$\le \max_{k\ge 0} q^k(c^{(\nu)}+2\mu+\mathrm{dg}\,\alpha-k) \le c_3^{(\nu)}\, q^{2\mu},$$

where $c^{(\nu)}$ and $c_3^{(\nu)}$ are positive constants independent of κ and λ. Hence

$$\sup_{\mathrm{dg}\,t=2\mu} \mathrm{dg}\, L(t) \le (2\mu+m+c_2)q^{2\lambda+e} + c_3 q^{2\mu+2\kappa+e}, \tag{12.3.12}$$

where $c_3 := \max_{1\le\nu\le n} c_3^{(\nu)}$.

Now put

$$\eta := \mu - \kappa + 1.$$

Then $\eta \ge \lambda$ and it follows from (12.3.11) and (12.3.12) that

$$\mathrm{dg}\, L(t_0) \le q^{2\eta+e}\,[\mu(4-q^{2\kappa-e-2}) + c_2 + c_3 q^{4\kappa} + d^*(\beta)q^{2\kappa}]. \tag{12.3.13}$$

From the choice of t_0 and the definitions of L and Γ it follows that

$$\Gamma^{q^{2\eta+1}} L(t_0)$$

is an algebraic integer of K and therefore its norm is an element of $\mathbb{F}_q[X]$. Since K is a finite, separable extension of $\mathbb{F}_q(X)$ of degree h, we have by lemma 6.10

$$N_{K\to\mathbb{F}_q(X)}(L(t_0)) = \prod_{\rho=1}^{h} \sigma_\rho(L(t_0)),$$

where $\sigma_1,\dots,\sigma_h$ are the h $\mathbb{F}_q(X)$-monomorphisms $K \hookrightarrow \Phi$. Furthermore

$$\sigma_\rho(L(t_0)) = \sum_{\nu=1}^{n}\sum_{j=0}^{q^{2\lambda}-1}\sum_{i=0}^{q^{2\kappa}-1} \sigma_\rho(X_{ij\nu})\,\sigma_\rho(t_0^{q^e})^j \left(\sum_{k=0}^{\infty}\sigma_\rho(\alpha_{\nu k}^{q^e})\frac{\sigma_\rho(t_0^{q^e})^{q^k i}}{F_k^{q^e}}\right).$$

Analogously to the derivation of (12.3.13) we derive

$$\mathrm{dg}\,\sigma_\rho(L(t_0)) \le q^{2\eta+e}[\mu(4-q^{2\kappa-e-2}) + c_2 + c_3 q^{4\kappa} + d^*(\beta)q^{2\kappa}].$$

Hence

$$(12.3.14)\quad dg\, N_{K\to\mathbb{F}_q(X)}\,(\Gamma^{q^{2\eta+1}} L(t_0)) \le h\, q^{2\eta+e}\{\mu(4-q^{2\kappa-e-2}) + c_4 q^{4\kappa}\},$$

where $c_4 > 0$ is independent of κ and λ. If κ is chosen such that

$$4 - q^{2\kappa-e-2} < 0$$

and then λ is chosen such that $d^*(\beta) < m$ and such that

$$m(4-q^{2\kappa-e-2}) + c_4 q^{4\kappa} < 0,$$

it follows from (12.3.14) that $L(t_0) = 0$. Hence we have proved that L vanishes on $\mathcal{B}(\mu+1)$.

Now κ and λ are fixed such that L vanishes on $\mathcal{B}$. According to the Product Formula for Entire Functions (Corollary 5.24), we have for every fixed μ $(\mu \ge m)$

$$L(t) = \gamma t^{\rho} \prod_{a\in\mathcal{B}(\mu)} (1-\frac{t}{a}) \prod_{b\in\mathcal{R}^*\setminus\mathcal{B}(\mu)} (1-\frac{t}{b}),$$

where $\rho \in \mathbb{N}^0$, $\gamma \in \Phi^*$, $\mathcal{R}^* = \mathcal{R}\setminus\{0\}$ and where $\mathcal{R}$ denotes the set of zeros of L. We now apply the Maximum Modulus Principle on

$$\prod_{b\in\mathcal{R}^*\setminus\mathcal{B}(\mu)} (1-\frac{t}{b}).$$

Comparing the maximal value on $\{t \in \Phi \mid dg\ t = 2\mu\}$ and the value in $t = 0$, the Maximum Modulus Principle (theorem 5.16) yields

$$(12.3.15)\quad \sup_{dgt=2\mu} dg \prod_{b\in\mathcal{R}^*\setminus\mathcal{B}(\mu)} (1-\frac{t}{b}) \ge 0.$$

Further we write

$$\prod_{a\in\mathcal{B}(\mu)} (1-\frac{t}{a}) = \frac{\prod_{a\in\mathcal{B}(\mu)} (a-t)}{\prod_{a\in\mathcal{B}(\mu)} a}\quad .$$

Then it follows from (12.3.15) that

$$(12.3.16)\qquad \sup_{dg\,t=2\mu} dg\, L(t) \geq dg\,\gamma + 2\mu\rho + 2\mu(q^{2\mu}-1) + - (\mu+d^*(\beta))(q^{2\mu}-1).$$

For μ large enough (12.3.12) and (12.3.16) are contradictory. □

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LIST OF DEFINITIONS

INDEX OF SPECIAL SYMBOLS

Symbol:	Defined on page:
Δ, Δ_n	1.24
λ	1.16
Φ	1.6
$\Phi[[t]]$	1.32
ψ	1.11
ψ_k	1.8
ξ	1.12
Ω	1.2
$\emptyset$	0.1
$\square$	0.1

SAMENVATTING

Zij Φ een algebraïsch gesloten lichaam dat niet-archimedisch gevalueerd is, met betrekking tot deze valuatie gesloten is en dat het lichaam $\mathbb{F}_q(X)$ der rationale funkties in één variabele over een eindig lichaam $\mathbb{F}_q$ omvat.

Dit proefschrift is gewijd aan het onderzoek naar transcendentie over $\mathbb{F}_q(X)$ van elementen van Φ. Als een van de resultaten noemen we:

als $\alpha, \beta \in \Phi$, $\alpha \neq 0$ en $\beta \notin \mathbb{F}_q(X)$ en als $f_1, f_2, \ldots, f_n$ $\bar{E}$-funkties zijn, die aan zekere voorwaarden voldoen, dan is minstens één van de 2n+1 elementen $\beta, f_1(\alpha), \ldots, f_n(\alpha), f_1(\alpha\beta), \ldots, f_n(\alpha\beta)$ transcendent over $\mathbb{F}_q(X)$.

In het eerste gedeelte van het proefschrift worden de analytische hulpmiddelen ontwikkeld, die in het tweede gedeelte bij het transcendentieonderzoek worden gebruikt.

STELLINGEN

behorende bij het proefschrift

TRANSCENDENCE IN FIELDS OF POSITIVE CHARACTERISTIC

I. Laten $z_1, z_2, \dots, z_n$ complexe getallen zijn met

$$0 < |z_i| \le 1, \qquad i = 1,2,\dots,n.$$

en

$$0 = |1-z_1| \le |1-z_2| \le \dots \le |1-z_n|.$$

Zij $m \in \mathbb{Z}$, $m \ge -1$. Stel $B_1, B_2, \dots, B_n$ zijn polynomen met complexe coëfficiënten van de graad respectievelijk $k_1, k_2, \dots, k_n$. Zij $k = k_1 + k_2 + \dots + k_n + n$. Dan bestaat er een geheel getal $\nu \in [m+1, m+k]$ zodanig dat

$$|B_1(\nu)z_1^\nu + B_2(\nu)z_2^\nu + \dots + B_n(\nu)z_n^\nu| \ge$$

$$\ge \frac{1}{4}\left(\frac{k-1}{8e(m+k)}\right)^{k-1} \min_{j=1,\dots,n} |B_1(0) + B_2(0) + \dots + B_j(0)|.$$

J.M. Geijsel, *On generalized sums of powers of complex numbers*, Math. Centre Report ZW 1968-013, Amsterdam, 1968.

II. In 1966 gaven P.J. Sally en M.H. Tableson met behulp van de Haarintegraal een representatie van complexwaardige Besselfuncties op een lokaal compact, niet-discreet, totaal onsamenhangend, niet-archimedisch gewaardeerd lichaam. De Carlitz-Besselfuncties J_n (gedefinieerd in [2] en in definitie 4.1 van dit proefschrift), beschouwd op de completering van het niet-archimedisch gewaardeerde lichaam $\mathbb{F}_{p^n}(X)$, zijn afbeeldingen van een lokaal compact lichaam in zichzelf. Voor deze Carlitz-Besselfuncties is geen Haarintegraalrepresentatie te geven.

[1] P.J. Sally en M.H. Tableson, *Special functions on locally compact fields*, Acta Math. 116 (1966), 279-309.

[2] L. Carlitz, *Some special functions over* GF(q,x), Duke Math. J. 27 (1960), 139-158.

III. Zij K een niet-archimedisch gevalueerd lichaam van karakteristiek p dat $\mathbb{F}_{p^m}$ omvat. De functie $f: K \to K$ wordt gegeven door de machtreeks

$$f(t) = \sum_{i=h}^{\infty} a_i t^i, \qquad a_i \in K,\ h \in \mathbb{N} \cup \{0\}, a_h \neq 0.$$

Indien er een $n \in \mathbb{N}$ bestaat zodanig dat f^n lineair is, d.w.z.

$$f^n(t+u) = f^n(t) + f^n(u), \qquad t,u \in K,$$
$$f^n(ct) = cf^n(t), \qquad c \in \mathbb{F}_{p^m}, t \in K,$$

dan heeft f de vorm

$$f(t) = \sum_{k=0}^{\infty} b_k t^{p^{m(r+k)}}, \qquad b_k \in K,\ b_0 \neq 0,$$

met $r \in \mathbb{Q}$, $r \geq 0$, $mr \in \mathbb{Z}$.

IV. Zij, met de notaties uit dit proefschrift, $\alpha \in \Phi$, $\alpha \neq 0$, α geheel algebraïsch over $\mathbb{F}_p(X)$. Dan is niet noodzakelijk dg $\alpha \geq 0$.

V. Laten $m,n,s,i_1,i_2,\ldots,i_t$ natuurlijke getallen zijn met $n \geq 2$, $m \geq s$. De gehele getallen $k_1,k_2,\ldots,k_t$ voldoen aan $k_1 > k_2 > \ldots > k_t \geq 0$. Als

$$i_1 + i_2 + \ldots + i_t = n^s$$

en

$$i_1 n^{k_1} + i_2 n^{k_2} + \ldots + i_t n^{k_t} = n^m,$$

dan geldt

$$i_1 k_1 n^{k_1} + i_2 k_2 n^{k_2} + \ldots + i_t k_t n^{k_t} \geq (m-s) n^m.$$

VI. Laten m,n,r natuurlijke getallen zijn met $n \geq 2$, $m \geq r-1 \geq 1$. De gehele getallen $k_1,k_2,\ldots,k_r$ voldoen aan $k_1 \geq k_2 \geq \ldots \geq k_r \geq 0$. Als

$$n^{k_1} + n^{k_2} + \ldots + n^{k_r} < n^m + n^{m-1} + \ldots + n^{m-r+1},$$

dan is

$$n^{k_1} + n^{k_2} + \ldots + n^{k_r} \le n^m + n^{m-1} + \ldots + n^{m-r+2} + n^{m-r}.$$

Als

$$n^{k_1} + n^{k_2} + \ldots + n^{k_r} > n^m + n^{m-1} + \ldots + n^{m-r+1},$$

dan is

$$n^{k_1} + n^{k_2} + \ldots + n^{k_r} \ge n^m + n^{m-1} + \ldots + n^{m-r+3} + 2n^{m-r+2}.$$

VII. Apostols stelling " Een Dirichlet-karakter is dan en slechts dan primitief als al zijn Gauss-sommen separabel zijn" ([1], stelling 1), volgt op eenvoudige wijze uit de formule

$$\tau_m(\chi) = \mu\left(\frac{m}{m'}\right) \chi'\left(\frac{m}{m'}\right) \tau_{m'}(\chi'), \qquad \text{(zie [2], pag.148),}$$

waarbij χ' mod m' het karakter χ mod m induceert.

[1] T.M. Apostol, *Euler's ϕ-function and separable Gauss sums,* Proc. Amer. Math. Soc., 24 (1970), 482-485.

[2] H. Davenport, *Multiplicative Number Theory,* Lectures in advanced mathematics, vol. 1, Chicago, 1967.

VIII. Men kan zich afvragen of het niet tot de taak van de redactie van een wetenschappelijk tijdschrift behoort op enigerlei wijze coördinerend op te treden wanneer zij binnen drie maanden tijds twee artikelen krijgt aangeboden waarin de auteurs geheel onafhankelijk van elkaar, een zeker twintig jaar oud probleem op vrijwel dezelfde wijze oplossen.

M. Waldschmidt, *Solutions du Huitième Problème de Schneider,* Journal of Number Theory, 5 (1973), 191-202. (received March 11, 1971; revised May 3, 1971).

W. Dale Brownawell, *The algebraic independence of certain numbers related by the exponential function,* Journal of Number Theory, 6 (1974), 22-31. (received June 1, 1971).

IX. Er bestaat kans op blijvend oogletsel wanneer bij bewusteloosheid contactlenzen niet tijdig worden verwijderd. Oogartsen en contactlensspecialisten attenderen hun cliënten hierop in onvoldoende mate.