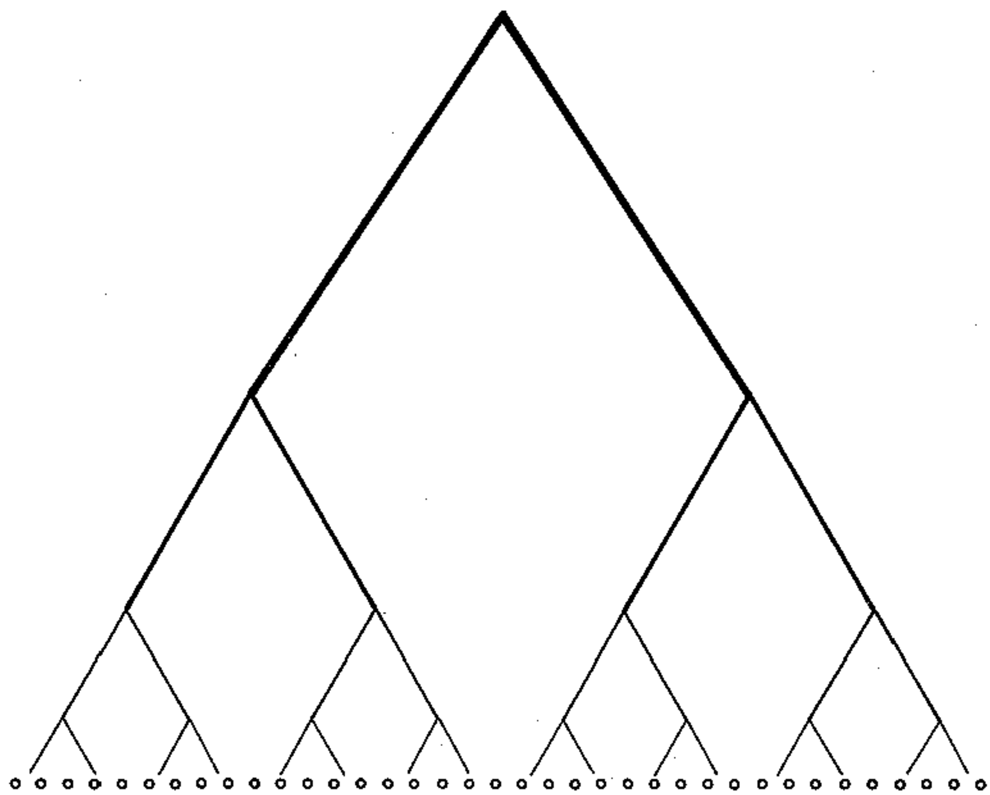
HOMOGENEOUS ZERO-DIMENSIONAL ABSOLUTE BOREL SETS



"Als ein Beispiel einer perfecten Punctmenge, die in keinem noch so kleinen Intervall überall dicht ist, führe ich den Inbegriff aller reellen Zahlen an, die in der Formel:

$$z = \frac{c_1}{3} + \frac{c_2}{3^2} + \cdots + \frac{c_\nu}{3^\nu} + \cdots$$

enthalten sind, wo die Coefficienten c_ν nach Belieben die beiden Werthe o und 2 anzunehmen haben und die Reihe sowohl aus einer endlichen, wie aus einer unendlichen Anzahl von Gliedern bestehen kann."

A.J.M. VAN ENGELEN

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ACADEMISCH PROEFSCHRIFT

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Prof. dr. A.B. Paalman-de Miranda

voor Christia
voor mijn moeder

CONTENTS

PREFACE

This dissertation was written while I was holding a position at the Mathematical Institute of the University of Amsterdam. My main debt is to my supervisors, J. van Mill and A.B. Paalman-de Miranda, for their contribution to my work, their constant encouragement, and their careful reading of the manuscript.
Thanks are also due to A.W. Miller, who pointed my attention to a paper of J. Steel that led me to the Wadge hierarchy and the results of Chapters 4 and 5 of this thesis; and to A. Louveau, for some helpful comments.

The text on the cover of this dissertation is the original definition of the Cantor set, taken from the paper 'Über unendliche, lineare Punktmannichfaltigkeiten' by G. Cantor, in Mathematische Annalen 21 (1883), p.590.

INTRODUCTION

All spaces under discussion are separable and metrizable.

This monograph is an investigation into the internal topological structure of zero-dimensional absolute Borel sets. The problem of finding topological characterizations of zero-dimensional absolute Borel sets was first stated explicitly in 1928, by Alexandroff and Urysohn [1]. Before this, Brouwer [4] and Sierpiński [50] had obtained the following characterizations of, respectively, the Cantor set C, and the space of rational numbers Q:

- up to homeomorphism, C is the only (non-empty) zero-dimensional compact space without isolated points;
- up to homeomorphism, Q is the only (non-empty) countable space without isolated points.

Alexandroff and Urysohn added to these: characterizations of $C\setminus\{p\}$ (i.e. the Cantor set minus one point), the space of irrationals P, and the product of the rationals with the Cantor set $Q\times C$:

- up to homeomorphism, $C\setminus\{p\}$ is the only zero-dimensional locally compact, non-compact space without isolated points;
- up to homeomorphism, P is the only (non-empty) zero-dimensional topologically complete space which is nowhere compact;
- up to homeomorphism, $Q\times C$ is the only (non-empty) zero-dimensional σ-compact space which is nowhere compact and nowhere countable.

These three spaces, together with the Cantor set, were called *irreducible Borel sets* by Alexandroff and Urysohn. More precisely, they defined a zero-dimensional space to be irreducible if it is of one of the following types:

1. compact, dense in itself;
2. locally compact, non-compact, dense in itself;
3. σ-compact, nowhere compact, nowhere countable;
4. topologically complete, nowhere compact.

Thus, from the above characterizations they concluded that of each of the types 1 through 4 there is exactly one space, up to homeomorphism, and that this space is homogeneous. Although Alexandroff and Urysohn also defined irreducibility for arbitrary Borel sets, this notion seems too general to allow characterizations of all such zero-dimensional spaces; in particular, there are many *non-homogeneous* spaces which are irreducible in that sense.

Let us now have a look at the above results against the framework of the Borel hierarchy in the Cantor set. The above spaces are all Borel sets of class 1, i.e. they are σ-compact or complete ($\equiv$ topologically complete). And from the characterizations it is easily deduced that, apart from the discrete spaces, they are the *only* homogeneous zero-dimensional absolute Borel sets of class 1; this can be read from the following diagram, bearing in mind that a homogeneous space which is not locally compact (not complete, not σ-compact) is also nowhere compact (nowhere complete, nowhere σ-compact):

- compact
  - isolated points: *finite spaces*
  - no isolated points: $\boldsymbol{C}$

- non-compact
  - locally compact
    - isolated points: *infinite discrete space*
    - no isolated points: $\boldsymbol{C}\setminus\{p\}$
  - nowhere compact
    - complete: $\boldsymbol{P}$
    - nowhere complete
      - nowhere σ-compact ....
      - σ-compact
        - countable: $\boldsymbol{Q}$
        - nowhere countable: $\boldsymbol{Q}\times\boldsymbol{C}$

In Chapter 2 of this monograph, proofs of all these old characterizations will be given.

We now turn our attention to zero-dimensional homogeneous absolute Borel sets which are of ambiguous class 2, but not of class 1; in other words, we consider homogeneous spaces which are both an $F_{\sigma\delta}$ and a $G_{\delta\sigma}$ in the Cantor set, but which are neither σ-compact nor complete. Of these homogeneous Borel sets of exactly ambiguous class 2, the first to be characterized was $\boldsymbol{Q} \times \boldsymbol{P}$, by van Mill [38], 53 years (!) after Alexandroff and Urysohn's paper. If we embed $\boldsymbol{Q} \times \boldsymbol{P}$ densely in $\boldsymbol{C}$, then its complement will be a nowhere complete, nowhere σ-compact Borel set of ambiguous class 2; there are exactly two such complements that are homogeneous, and one of them, denoted by $\boldsymbol{S}$, was also characterized by van Mill [39], the other, denoted by $\boldsymbol{T}$, was characterized by van Douwen [7].

In this monograph, we continue the investigation into the structure of the zero-dimensional absolute Borel sets of ambiguous class 2. In Chapter 3, we will prove that there are precisely ω_1 homogeneous elements in this class, and we will obtain internal, topological characterizations of each of them. To describe these spaces, we will define topological properties that are closely related to the so-called "small Borel classes" (see Kuratowski [28] §33.IV).

Then, in Chapter 4, we consider Borel subsets of the Cantor set that are not of ambiguous class 2; we will find characterizations of all homogeneous spaces at these levels, thus obtaining a complete picture of the class of homogeneous zero-dimensional absolute Borel sets. Whereas the properties describing the spaces at the lower levels are rather perspicuous, those needed in this chapter are in general not so easy to grasp. The characterizations we give involve the concept of the Wadge class of a space, a concept which is related to game theory; we heavily rely on results of Steel [56] and Louveau [32].

A somewhat unexpected corollary to our classification of the *homogeneous* Borel sets in the Cantor set is, that non-trivial *rigid* Borel sets in $\boldsymbol{C}$ do not exist; this answers a question of van Douwen [8]. The proof is given in Chapter 5.

Remark: Originally, my research was aimed at finding an internal, topological characterization of the countable infinite product of rationals, $\boldsymbol{Q}^{\omega}$. After I had succeeded in this, I learned of Steel's results which could easily be used to prove my theorem; my subsequent investigation into the theory of Wadge classes led to the results of Chapters 4 and 5. In an appendix to this monograph, I have nevertheless included my proof of the characterization of Q^{ω}, in the first place because it is purely topological and does not involve the deep results from game theory used in Chapter 4, and in the second place since the techniques used seem to be interesting in their own right (see e.g. van Mill [41]).

CHAPTER 1: PRELIMINARIES

In this chapter, we establish some notational conventions, and we state various results that will be used throughout this monograph, often without further reference.

1.1 As far as standard notions from general topology are concerned, we mostly follow Engelking [15].

1.2 $A \approx B$ means that A and B are homeomorphic; $h: A \approx B$ means that $h: A \to B$ is a homeomorphism.

1.3 An ordinal number is the set of all smaller ordinal numbers. If α is an ordinal, and $\alpha = \beta+1$, then α is a *successor ordinal*, and we define $\alpha-1 = \beta$. If $\alpha \neq 0$ and α is not a successor, then α is a *limit ordinal*; we write $\lim(\alpha)$.
The first infinite ordinal is denoted by ω, $N = \omega\backslash 1 = \omega\backslash\{0\}$, and ω_1 is the first uncountable ordinal.
If α is an ordinal, then α can be represented as $\gamma+n$, where $n < \omega$, and $\gamma = 0$ or γ is a limit; α is said to be *even*, resp. *odd*, if n is even, resp. odd.
The *cofinality* of a limit ordinal α is the least ordinal γ such that $\alpha = \sup\{\alpha_\beta : \beta < \gamma\}$ for certain ordinals $\alpha_\beta < \alpha$.
The cardinality of a set A is denoted by $|A|$; we identify cardinal numbers with *initial* ordinals, where α is initial if $|\beta| < |\alpha|$ for all ordinals $\beta < \alpha$.

1.4 The countable infinite product of a space X is denoted by X^ω; for $n < \omega$, $\pi_n: X^\omega \to X$ is the projection onto the n^{th} coordinate (starting with

the 0^{th} coordinate), and $\pi_{\leq n}: X^\omega \to X^{n+1}$ is the projection onto the first n+1 coordinates. Usually, X will be ω or $2 = \{0,1\}$.
By M, we denote the set $\omega^{<\omega}$ of all finite sequences of elements from ω, including the empty sequence $\emptyset$ (i.e. $M = \{\pi_{\leq n}(\sigma): \sigma \in \omega^\omega, n < \omega\} \cup \{\emptyset\}$).
Fix $s = (i_0,\ldots,i_{k-1}) \in M$, $k \in N$; then

- $|s| = k$, $|\emptyset| = 0$;
- $\nu(s) = i_0+i_1+\ldots+i_{k-1} + |s|$, $\nu(\emptyset) = 0$;
- $s\hat{\ }i = (i_0,\ldots,i_{k-1},i)$, $\emptyset\hat{\ }i = (i)$;
- $f(s) = i_{k-1}$;
- $s|\ell = (i_0,\ldots i_{\ell-1})$ if $1 \leq \ell \leq k$, $s|0 = \emptyset$;
- $\hat{s} = s|k-1$ (thus, $s = \hat{s}\hat{\ }f(s)$).

Furthermore, if $s,t \in M$, then we write $s \leq t$ (resp. $s < t$) if $s = t|k$ for some $k \leq |t|$ (resp. $k < |t|$).
Finally, if $s \in M$, and $\sigma = (i_n)_{n<\omega} \in \omega^\omega$, then for $k < \omega$, $\sigma|k = \pi_{\leq k-1}(\sigma)$ if $k > 0$, $\sigma|0 = \emptyset$; and $s < \sigma$ if $s = \sigma|n$ for some $n < \omega$.

1.5 All spaces in this monograph are separable and metrizable. Metrics are denoted by d, and assumed to be bounded by 1; almost invariably, the choice of metric is irrelevant, otherwise it is obvious.
If $f,g: X \to Y$ are continuous fuctions, and X is compact, then $d(f,g) = \sup\{d(f(x),g(x)): x \in X\}$.

1.6 A subset of a space X is *clopen* if it is both open and closed in X; thus, a *zero-dimensional* space is a space with a (countable) basis consisting of clopen sets.

1.7 If Y is a subset of a space X, then $\mathcal{U}$ is a *covering* of Y if $\cup\mathcal{U} = Y$. $\mathcal{U}$ is said to be *open* (*closed*, *clopen*) if it consists of open (closed, clopen) subsets of Y; if the elements of $\mathcal{U}$ are actually open (closed, clopen) in X, then we say that $\mathcal{U}$ is a covering of Y by open (closed, clopen) subsets of X.
$\mathcal{U}$ is said to be *disjoint* if it consists of pairwise disjoint sets; if we say that $\mathcal{U} = \{U_i: i \in I\}$ is disjoint, then this includes the statement that $U_i \neq U_j$ if $i \neq j$.

1.8 Let P be a topological property. A space X is *nowhere* P if no non-empty open subset of X is P, and if in addition X is non-empty. We also exclude $\emptyset$ from the spaces said to be 'dense in itself' or 'without isolated points' (hence, those are the spaces which are 'nowhere of cardinality 1').

1.8.1 *Let* P *be closed-hereditary, let* X *be non-empty and zero-dimensional, and let* $\mathcal{B}$ *be a basis for* X *consisting of clopen sets. Then* X *is nowhere* P *if and only if no non-empty* $B \in \mathcal{B}$ *is* P.

Proof: "Only if" is trivial. Conversely, if U is a non-empty open subset of X, then U contains a non-empty $B \in \mathcal{B}$; since B is closed in U, and P is closed-hereditary, U is not P. □

If Q is also a topological property, then we say that "X is $P \cup Q$" if X can be written as the union of two subspaces, one of which is P, the other Q.

1.9 A space X is *rigid* if the identity is the only autohomeomorphism of X. X is *homogeneous* if for each $x,y \in X$, there exists a homeomorphism $h: X \to X$ such that $h(x) = y$.

X is *homogeneous with respect to dense copies of* A if for all dense subspaces A_0, A_1 of X such that $A_0 \approx A \approx A_1$, there exists a homeomorphism $h: X \to X$ such that $h[A_0] = A_1$.

X is *strongly homogeneous* if $U \approx X$ for each non-empty clopen subspace U of X.

Of course, the notion of strong homogeneity is only of interest in the realm of zero-dimensional spaces.

1.9.1 *A strongly homogeneous zero-dimensional space* X *is homogeneous.*

Proof: Let $x,y \in X$. If $\{x\}$ or $\{y\}$ is open, then $|X| = 1$, and we are done. Otherwise, there exist strictly decreasing sequences $\langle U_n : n < \omega \rangle$, $\langle V_n : n < \omega \rangle$ of clopen subsets of X such that $\bigcap_{n=0}^{\infty} U_n = \{x\}$, $\bigcap_{n=0}^{\infty} V_n = \{y\}$; of course we may assume that $U_0 = X = V_0$. Now for $n < \omega$, put $U'_n = U_n \setminus U_{n+1}$, $V'_n = V_n \setminus V_{n+1}$; then U'_n, V'_n are non-empty clopen subsets of X, and hence there is a homeomorphism $h_n: U'_n \to V'_n$. Now define $h: X \to Y$ by $h|U'_n = h_n$ for each $n < \omega$, and $h(x) = y$. It is easily verified that h is a homeomorphism. □

1.9.2 *Let* X *be a non-compact zero-dimensional strongly homogeneous space; then* $U \approx X$ *for each non-empty open subspace of* X, *and* $X \approx \omega \times X$.

Proof: Let U be a non-empty open subspace of X. Then U is not compact (otherwise, U is clopen in X, so $U \approx X$; but X is non-compact), so there is an infinite disjoint covering $\{U_n : n < \omega\}$ of U by clopen subsets of X. Then each $U_n \approx X$, so $U \approx \omega \times X$. □

1.10 The reader is assumed to be familiar with the main facts about absolute Borel sets, which can be found in Kuratowski [28]. However, in this monograph, it turns out that it is often convenient to use the *notation* from Moschovakis [43]:

Let X be a space. We inductively define the Borel classes $\Sigma^0_\alpha, \Pi^0_\alpha$ in X, for $\alpha \in [1,\omega_1)$, by:
A *is a* Σ^0_1*-set in* X *if* A *is open in* X;
A *is a* Π^0_1*-set in* X *if* A *is closed in* X.
If $\alpha > 1$, then:
A *is a* Σ^0_α*-set in* X *if* $A = \bigcup_{i=0}^\infty A_i$*, where each* A_i *is a* $\Pi^0_{\beta_i}$*-set in* X *for some* $\beta_i < \alpha$;
A *is a* Π^0_α*-set in* X *if* $A = \bigcap_{i=0}^\infty A_i$*, where each* A_i *is a* $\Sigma^0_{\beta_i}$*-set in* X *for some* $\beta_i < \alpha$ *(i.e. if* $X \setminus A$ *is a* Σ^0_α*-set in* X*).*
For each $\xi \in [1,\omega_1)$, we put $\Delta^0_\xi = \Pi^0_\xi \cap \Sigma^0_\xi$.

Comparing this definition with Kuratowski's indexing of the Borel hierarchy, we see that unfortunately, there is a discrepancy at the finite levels: if $n < \omega$, then A is of additive (resp. multiplicative, resp. ambiguous) class n in X if and only if A is a Σ^0_{n+1}-set (resp. Π^0_{n+1}-set, resp. Δ^0_{n+1}-set) in X. If $\alpha \geq \omega$, then there is no such discrepancy; thus, for all $\xi < \omega_1$, we have:

A *is of additive (resp. multiplicative, resp. ambiguous) class* α *in* X *if and only if* A *is a* $\Sigma^0_{1+\alpha}$*-set (resp.* $\Pi^0_{1+\alpha}$*-set, resp.* $\Delta^0_{1+\alpha}$*-set) in* X.

If we say that A is a Σ^0_ξ-set (resp. Π^0_ξ-set, resp. Δ^0_ξ-set) without mentioning a space X, then we mean that A is a Σ^0_ξ-set, etc. *absolutely*, i.e. A is a Σ^0_ξ-set, etc., in X for every space X, whenever A is embedded in X; in Kuratowski's terminology, A is *absolutely* of additive (resp. multiplicative, resp. ambiguous) class α, where $\xi = 1+\alpha$. The notion of an absolute Borel set is now clear.

It is easy to see that

A *is* Π^0_1 *if and only if* A *is compact;*
A *is* Σ^0_2 *if and only if* A *is* σ*-compact;*

and it is well-known that

A is Π^0_2 *if and only if A is complete* ($\overset{def}{=}$ *Čech-complete* $\equiv$ *completely metrizable);*
if $\xi \geq 3$ *then A is* Π^0_ξ *(resp.* Σ^0_ξ*) if and only if A is* Π^0_ξ *(resp.* Σ^0_ξ*) in some completely metrizable space.*

The last two statements can be deduced from Lavrentieff's theorem ([29]; see also Engelking [15] 4.3.21). Thus, for all classes $\Sigma^0_\xi, \Pi^0_\xi, \Delta^0_\xi$, excluding Σ^0_1, Δ^0_1, a space is absolutely of that class if and only if it is of that class when embedded in some compact space; since zero-dimensional spaces can be embedded in the Cantor set, this is the reason that we mainly consider Borel subsets of the Cantor set in this monograph.

A space A is *exactly of additive class* α if it is absolutely of additive class α, but not absolutely of multiplicative class α, and A is *exactly of multiplicative class* α if it is absolutely of multiplicative class α but not absolutely of additive class α; A is *of exact class* α if it is exactly of additive class α, or exactly of multiplicative class α. A is *exactly of ambiguous class* α if it is absolutely of ambiguous class α, but not absolutely of additive or multiplicative class β for any $\beta < \alpha$.

1.11 A class $\Gamma \subset \mathcal{P}(X)$ (the power set of X) is said to satisfy the *reduction property* if for each $A_0, A_1 \in \Gamma$, there exist $B_0, B_1 \in \Gamma$ such that

(i) $B_0 \subset A_0$, $B_1 \subset A_1$;
(ii) $A_0 \cup A_1 = B_0 \cup B_1$;
(iii) $B_0 \cap B_1 = \emptyset$.

It is well-known (Kuratowski [28] §21.II and §27.VII) that the reduction property holds for the class of Σ^0_ξ-sets in X, if $\xi \geq 2$; if X is zero-dimensional, then the reduction property also holds for the class of Σ^0_1-sets in X. In fact, for these classes, we can also reduce *infinite* sequences: if $\langle A_n : n < \omega \rangle$ is a sequence of Σ^0_ξ-sets in X (where $\xi \geq 2$, and $\xi \geq 1$ if X is zero-dimensional), then there exists a sequence $\langle B_n : n < \omega \rangle$ of Σ^0_ξ-sets in X reducing $\langle A_n : n < \omega \rangle$, i.e. for all n, $B_n \subset A_n$; for all $n \neq m$, $B_n \cap B_m = \emptyset$; and $\bigcup_{n=0}^\infty A_n = \bigcup_{n=0}^\infty B_n$.

1.12 A space X is *first category* if $X = \bigcup_{i=0}^\infty X_i$, where each X_i is nowhere dense in X; X is *Baire* if the intersection of countably many dense open subsets of X is dense in X.

1.12.1 *A homogeneous space X is either Baire or first category.*

Proof: If X is not Baire, then there exists a family $\{U_n : n < \omega\}$ of dense open subsets of X, and a non-empty open subset V of X, such that $V \subset X\setminus(\bigcap_{n=0}^{\infty} U_n) = \bigcup_{n=0}^{\infty}(X\setminus U_n)$; note that each $X\setminus U_n$ is nowhere dense in X. Let $x \in V$; by homogeneity of X, we can find, for each $y \in X$, a homeomorphism $h_y : X \to X$ such that $h_y(x) = y$. Now if $\{h_{y_n}[V] : n < \omega\}$ is a countable subcovering of $\{h_y[V] : y \in X\}$, then $X = \bigcup_{n=0}^{\infty}\bigcup_{m=0}^{\infty} h_{y_n}[X\setminus U_m]$, so X is first category. □

The next statement is a special case of a result of Levi [31].

1.12.2 *Let* A *be an absolute Borel set. Then* A *is Baire if and only if* A *contains a dense complete subspace.*

Proof: Suppose A is Baire, and embed A densely in a compact space X. By Kuratowski [28] §35.II, there exist an open subset U of X, and nowhere dense subsets P_i, R_i of X such that

$$A = (U\setminus\bigcup_{i=0}^{\infty} P_i) \cup \bigcup_{i=0}^{\infty} R_i .$$

Then $U\setminus(\bigcup_{i=0}^{\infty} \overline{P}_i)$ is dense in A, and a G_δ in X, whence complete. The converse is trivial. □

1.12.3 *Let* A *be a dense and co-dense Borel set in a compact space* X. *Then* A *is Baire if and only if* $X\setminus A$ *is first category.*

Proof: If A is Baire, then A contains a dense complete subspace G by 1.12.2; then G is a G_δ in X, say $G = \bigcap_{n=0}^{\infty} U_n$, where each U_n is open in X. Since A is dense in X, $X\setminus U_n$ is nowhere dense in X, and since $X\setminus A$ is dense in X, $(X\setminus A) \cap (X\setminus U_n)$ is nowhere dense in $X\setminus A$; thus,

$$X\setminus A = \bigcup_{n=0}^{\infty}((X\setminus A) \cap (X\setminus U_n))$$

is first category.

Conversely, if $X\setminus A = \bigcup_{n=0}^{\infty} F_n$, where F_n is closed and nowhere dense in $X\setminus A$, then each $\overline{F}_n$ is nowhere dense in X; since X is compact, $\bigcup_{n=0}^{\infty} \overline{F}_n$ has empty interior in X, so $\bigcap_{n=0}^{\infty}(X\setminus\overline{F}_n)$ is a dense subset of A which is a G_δ in X, whence complete. □

CHAPTER 2: BOREL SETS OF CLASS 1

In this chapter, we consider the characterizations of zero-dimensional absolute Borel sets that were obtained between 1910 and 1930. In section 2.1, we prove Brouwer's characterization of the Cantor set. From this, Alexandroff and Urysohn's characterization of $C\setminus\{p\}$ follows rather easily (section 2.2). The space of irrationals P, also characterized by Alexandroff and Urysohn, is discussed in section 2.3, and in section 2.4, we obtain characterizations of the homogeneous σ-compact spaces Q (due to Sierpiński) and $Q\times C$ (again due to Alexandroff and Urysohn).
The proofs that we give are essentially the same as the original ones, although of course, terminology has somewhat changed.

2.1 <u>The multiplicative class 0</u>

The only space which really deserves to be called *the* Cantor set is the set defined by Cantor in 1883 ([5]), consisting of all points in the closed unit interval [0,1] that do not require the use of "1" in their ternary expansion; this space, which is denoted by C, is compact, zero-dimensional, and dense in itself. In this monograph, by the name Cantor set we usually denote *any* compact zero-dimensional space without isolated points; by the following theorem, this is topologically justified.

2.1.1 THEOREM (Brouwer [4]): *Up to homeomorphism, the Cantor set* C *is the only zero-dimensional compact space without isolated points.*

Proof: Let X and Y be such spaces. First note that if $\ell \in N$, and A is an arbitrary non-empty clopen subset of X or Y, then since A is dense in itself, we can write A as a disjoint union of ℓ non-empty clopen sets. To prove the theorem, we will inductively construct finite indexing sets E_j, and non-empty clopen subsets $X_{i_0\ldots i_n}$ of X, $Y_{i_0\ldots i_n}$ of Y, for each $(i_0,\ldots,i_n) \in \Pi_{j=0}^n E_j$, and each $n < \omega$, such that

(1) $X_0 = X$, $Y_0 = Y$, $E_0 = \{0\}$;

(2) $X_{i_0\ldots i_n} = \cup_{i\in E_{n+1}} X_{i_0\ldots i_n i}$, $Y_{i_0\ldots i_n} = \cup_{i\in E_{n+1}} Y_{i_0\ldots i_n i}$;

(3) $\mathrm{diam}(X_{i_0\ldots i_n}) \le 1/(n+1)$, $\mathrm{diam}(Y_{i_0\ldots i_n}) \le 1/(n+1)$;

(4) $X_{i_0\ldots i_n} \cap X_{j_0\ldots j_n} = \emptyset = Y_{i_0\ldots i_n} \cap Y_{j_0\ldots j_n}$ if $(i_0,\ldots,i_n) \neq (j_0,\ldots,j_n)$.

Define X_0, Y_0, and E_0 as in (1); then suppose that E_n, $X_{i_0\ldots i_n}$, $Y_{i_0\ldots i_n}$ have been defined for each $n \le k$, and each $(i_0,\ldots,i_n) \in \Pi_{j=0}^n E_j$. Since the non-empty clopen subsets of X of diameter $\le 1/(k+2)$ form a basis for X, we can write the compact set $X_{i_0\ldots i_k}$ as a finite disjoint union of such sets, say

$$X_{i_0\ldots i_k} = \cup\{A_{i_0\ldots i_k i}: i \le s(i_0\ldots i_k)\};$$

similarly, write

$$Y_{i_0\ldots i_k} = \cup\{B_{i_0\ldots i_k i}: i \le t(i_0\ldots i_k)\}.$$

Put

$$m = \max\{s(i_0\ldots i_k), t(i_0\ldots i_k): (i_0,\ldots,i_k) \in \Pi_{j=0}^k E_j\},$$

and $E_{k+1} = \{0,\ldots,m\}$.

By the above remark, we can write each $A_{i_0\ldots i_k s(i_0\ldots i_k)}$ as a disjoint union of $m+1-s(i_0\ldots i_k)$ non-empty disjoint clopen sets, say

$$A_{i_0\ldots i_k s(i_0\ldots i_k)} = \cup\{X_{i_0\ldots i_k i}: i = s(i_0\ldots i_k),\ldots,m\};$$

similarly,

$$B_{i_0\ldots i_k t(i_0\ldots i_k)} = \cup\{Y_{i_0\ldots i_k i}: i = t(i_0\ldots i_k),\ldots,m\}.$$

Finally, for $i < s(i_0\ldots i_k)$, put $X_{i_0\ldots i_k i} = A_{i_0\ldots i_k i}$, and for $i < t(i_0\ldots i_k)$, put $Y_{i_0\ldots i_k i} = B_{i_0\ldots i_k i}$. Then (2) -(4) are satisfied for $n = k+1$; this completes the induction. Now by compactness of X and Y, for each $\sigma \in \Pi_{j=0}^\infty E_j$, $\cap_{s<\sigma} X_s \neq \emptyset \neq \cap_{s<\sigma} Y_s$, hence by condition (3), both $\cap_{s<\sigma} X_s$ and $\cap_{s<\sigma} Y_s$ consist of exactly one point, say x_σ resp. y_σ. Define $f: X \to Y$ by $f(x_\sigma) = y_\sigma$, for each $\sigma \in \Pi_{j=0}^\infty E_j$; it is easily verified that f is a homeomorphism. □

2.1.2 COROLLARY: *C is strongly homogeneous, hence homogeneous.*

Proof: A non-empty clopen subset of $\mathcal{C}$ is compact, zero-dimensional, and dense in itself, hence homeomorphic to $\mathcal{C}$ by the theorem. □

Clearly, it also follows from the theorem that $\mathcal{C}$ and the finite spaces are the only zero-dimensional homogeneous compact spaces, i.e. the only zero-dimensional homogeneous absolute Borel sets of multiplicative class 0.

Since every zero-dimensional space can be embedded in $\mathcal{C}$, the *compact* zero-dimensional spaces are exactly the *closed* subsets of $\mathcal{C}$; this can also be regarded as a consequence of the following:

2.1.3 COROLLARY: *Let* X *be a non-empty compact zero-dimensional space. Then* $X \times \mathcal{C} \approx \mathcal{C}$.

Proof: Apply the theorem. □

Another corollary, which we will frequently use, is:

2.1.4 COROLLARY: *If* X *is a zero-dimensional space without isolated points, then* X *can be densely embedded in* $\mathcal{C}$*; if* X *is also nowhere compact, then any such embedding is co-dense.*

Proof: Embed X in $\mathcal{C}$, and let $K = \overline{X}$. Then K is compact zero-dimensional, and since X contains no isolated points, neither does K; so $K \approx \mathcal{C}$ by theorem 2.1.1. If X is nowhere compact, and U is a non-empty clopen subset of K, then $U \not\subset X$, so $U \cap K \backslash X \neq \emptyset$, i.e. X is co-dense in K. □

As yet another example of the strength of the above characterization, we give a short proof of the following special case of a theorem of Suslin:

2.1.5 COROLLARY: *Let* X *be an uncountable zero-dimensional* σ*-compact space; then* X *contains a copy of the Cantor set.*

Proof: Write $X = \bigcup_{i=0}^{\infty} X_i$, with each X_i compact. Then some X_j is uncountable. By the Cantor-Bendixson theorem (see e.g. Engelking [15] 1.7.11), X_j contains a closed subset K which is either empty or dense in itself, and a countable subset L, such that $X_j = K \cup L$. Since X_j is uncountable, K is non-empty, and hence $K \approx \mathcal{C}$ by theorem 2.1.1. □

2.2 The ambiguous class 1

Absolute Borel sets of ambiguous class 1 are those sets which are both σ-compact and complete. The following theorem gives an alternative description; it also appears in Kuratowski [28](§12.IX), but the proof there is rather indirect.

2.2.1 THEOREM: *A space* X *is both σ-compact and complete if and only if each non-empty closed subset of* X *contains a non-empty open locally compact subspace.*

Proof: If X is σ-compact and complete, and A is non-empty and closed in X, then A is also σ-compact, say $A = \bigcup_{i=0}^{\infty} A_i$ with A_i compact. A is also complete, hence Baire, so $\mathrm{Int}_A(A_j) \neq \emptyset$ for some j; then $\mathrm{Int}_A(A_j)$ is a non-empty locally compact open subset of A.
For the converse, first suppose that X is not σ-compact; put

$$V = \bigcup\{U: U \text{ is open in } X, \text{ and } \sigma\text{-compact}\}.$$

Then V is also σ-compact, so $A = X\setminus V \neq \emptyset$. Let W be a non-empty open subset of A, say $W = W' \cap A$, with W' open in X. If W were locally compact, then it would be σ-compact. Since $W' \cap V$ is also σ-compact, W' would be σ-compact, contradicting the fact that $W' \not\subset V$.
If X is not complete, then the proof is wordly the same once every occurrence of "σ-compact" is replaced by "complete"; the fact that V is complete is a consequence of Montgomery's theorem that a locally complete space is complete ([42]; see also Michael [37] and Stone [57]). □

2.2.2 COROLLARY: *A homogeneous space* X *is both σ-compact and complete if and only if it is locally compact.*

Proof: Sufficiency is clear. If X is σ-compact and complete, then by theorem 2.2.1, X contains some non-empty open locally compact subset; hence by homogeneity, X itself is locally compact. □

Remark: Without the homogeneity condition, the corollary is false: the space $[0,1]\setminus\{1/n: n \in \mathbb{N}\}$ is a counterexample.

From corollary 2.2.2, we see that if we want to characterize homogeneous zero-dimensional absolute Borel sets of ambiguous class 1, then we have to consider locally compact homogeneous subsets of $\mathcal{C}$. Since compact spaces were discussed in section 2.1, we will now look at the non-compact sets. The homogeneity requirement yields that the space either consists exclusively of isolated points, and thus is infinite and discrete, or is dense in itself. The following theorem shows that the latter condition determines precisely one topological type.

2.2.3 THEOREM (Alexandroff and Urysohn [1]): *Up to homeomorphism, the Cantor set minus one point* $C \setminus \{p\}$ *is the only zero-dimensional locally compact non-compact space without isolated points.*

Proof: Clearly, if $p \in C$, then $C \setminus \{p\}$ is a zero-dimensional locally compact non-compact space without isolated points. If X is any such space, then $\tilde{X} = X \cup \{\infty\}$, the Alexandroff one-point compactification of X, is a compact zero-dimensional space without isolated points, hence $X \approx C$ by theorem 2.1.1. Since C is homogeneous (corollary 2.1.2), there exists a homeomorphism $h: \tilde{X} \to C$ such that $h(\infty) = p$; then $h[X] = C \setminus \{p\}$. □

Remark: The above proof is the original one, as given by Alexandroff and Urysohn. We feel that the following proof is slightly more elementary, since it does not mention the one-point compactification: Cover X by disjoint non-empty compact open sets; then this covering is infinite, and the compact open sets are Cantor sets by theorem 2.1.1, so $X \approx \omega \times C$.

Since $C \setminus \{p\} \approx \omega \times C$, it is homogeneous. However, it is *not* strongly homogeneous, since it contains non-empty open compact subsets; of all the spaces discussed in this monograph, apart from the discrete spaces, $C \setminus \{p\}$ is the only one which is homogeneous, but not strongly homogeneous (see corollary 4.4.6).

2.2.4 COROLLARY: *Let* X *be a non-empty locally compact zero-dimensional space. Then* $X \times C \setminus \{p\} \approx C \setminus \{p\}$; *thus,* X *can be embedded as a closed subspace of* $C \setminus \{p\}$.

Proof: Apply theorem 2.2.3. □

2.3 <u>The multiplicative class 1</u>

By corollary 2.2.2, a homogeneous complete space is σ-compact if and only if it is locally compact. Thus, the homogeneous zero-dimensional absolute Borel sets that are exactly of multiplicative class 1 are those homogeneous subsets of C that are complete and nowhere compact.

2.3.1 THEOREM (Alexandroff and Urysohn [1]): *Up to homeomorphism, the space of irrationals* P *is the only zero-dimensional complete space which is nowhere compact.*

Proof: It is well-known, and easy to show, that P is complete and nowhere compact. So suppose that X and Y are two zero-dimensional, complete, nowhere compact spaces. Using an argument much like that in the proof of theorem 2.1.1, we will show that $X \approx Y$.
Fix complete metrics d and d' for X and Y, respectively. We will construct non-empty clopen subsets X_s of X, Y_s of Y, for each $s \in M$ (= $\omega^{<\omega}$, see 1.4), such that

(1) $X_\emptyset = X$, $Y_\emptyset = Y$;
(2) $X_s = \bigcup_{i=0}^{\infty} X_{s^\frown i}$, $Y_s = \bigcup_{i=0}^{\infty} Y_{s^\frown i}$;
(3) $d\text{-diam}(X_s) \leq 1/(|s|+1)$, $d'\text{-diam}(Y_s) \leq 1/(|s|+1)$;
(4) $X_s \cap X_t = \emptyset = Y_s \cap Y_t$ if $|s| = |t|$, $s \neq t$.

The construction is a triviality once it is noted that, since X and Y are nowhere compact, any non-empty clopen subset of X or Y can be written as a disjoint union of *infinitely* many non-empty clopen sets (of arbitrarily small diameter).
By completeness of d and d', and by condition (3), for each $\sigma \in \omega^\omega$, the sets $\bigcap_{s<\sigma} X_s$, $\bigcap_{s<\sigma} Y_s$ contain exactly one point, say x_σ resp. y_σ. If we put $f(x_\sigma) = y_\sigma$, then $f\colon X \to Y$ is easily seen to be a homeomorphism. □

Remark: If X is just a zero-dimensional nowhere compact space, then the above proof shows that X can be densely embedded in Y: indeed, if $x \in X$, then for some $\sigma \in \omega^\omega$ we have $x \in \bigcap_{s<\sigma} X_s$; now put $f(x) = y_\sigma$, then f is a dense embedding.

Remark: The above characterization of P was also obtained by Hausdorff [22], much later, in 1937.

2.3.2 COROLLARY: P *is strongly homogeneous, hence homogeneous.* □

Thus, P is the only zero-dimensional homogeneous absolute Borel set which is exactly of multiplicative class 1. Note that from theorem 2.3.1 it follows, among others, that P is homeomorphic to Baire space ω^ω.

Similar to 2.1.3 and 2.2.4, we have:

2.3.3 COROLLARY: *Let* X *be a non-empty zero-dimensional complete space. Then* $X \times \boldsymbol{P} \approx \boldsymbol{P}$*; thus,* X *can be embedded as a closed subspace of* $\boldsymbol{P}$. □

2.4 The additive class 1

By corollary 2.2.2, a homogeneous σ-compact space is complete if and only if it is locally compact. Thus, the homogeneous zero-dimensional absolute Borel sets that are exactly of additive class 1 are those homogeneous subsets of $\boldsymbol{C}$ that are σ-compact and nowhere compact. Such a space is either countable or nowhere countable; we will show that each possibility yields exactly one topological type.

2.4.1 THEOREM (Sierpiński [50]): *Up to homeomorphism, the space of rationals* $\boldsymbol{Q}$ *is the only countable space without isolated points.*

Proof: Let X and Y be countable spaces without isolated points. Then X and Y are zero-dimensional, so X and Y can be considered as dense subsets of $\{0,1\}^{\omega} \approx \boldsymbol{C}$ (corollary 2.1.4). Enumerate X,Y as $\{x_n : n < \omega\}$, $\{y_n : n < \omega\}$, respectively; we write $x_n \preccurlyeq x_m$, $y_n \preccurlyeq y_m$ if $n \le m$.
For each $n < \omega$, and each $(\alpha_0, \dots, \alpha_n) \in \{0,1\}^{n+1}$, define

$$\text{(i)}\ p_0 = \min\{x \in X : \pi_0(x) = 0\},\ p_1 = \min\{x \in X : \pi_0(x) = 1\};$$
$$\text{(ii)}\ p_{\alpha_0 \dots \alpha_n 0} = p_{\alpha_0 \dots \alpha_n}$$
$$\text{(iii)}\ p_{\alpha_0 \dots \alpha_n 1} = \min\{x \in X : \pi_{\le n}(x) = \pi_{\le n}(p_{\alpha_0 \dots \alpha_n}),\ \pi_{n+1}(x) = 1-\pi_{n+1}(p_{\alpha_0 \dots \alpha_n})\}.$$

Observe that

$$(1)\ p_{\alpha_0 \dots \alpha_n} = \min(X \cap \pi_{\le n}^{-1}\pi_{\le n}(p_{\alpha_0 \dots \alpha_n}));$$
$$(2)\ \text{if } p_{\alpha_0 \dots \alpha_n} = p_{\beta_0 \dots \beta_n}, \text{ then } \alpha_i = \beta_i \text{ for each } i \le n.$$

Also, note that

(3) each point of X is some $p_{\alpha_0 \dots \alpha_n}$.

Indeed, let $x \in X$. Put $\alpha_0 = \pi_0(x)$, and if $\alpha_0, \dots, \alpha_k$ have been defined, put $\alpha_{k+1} = 0$ if $\pi_{k+1}(x) = \pi_{k+1}(p_{\alpha_0 \dots \alpha_k})$, $\alpha_{k+1} = 1$ otherwise. It is easily verified that $\pi_{\le n}(x) = \pi_{\le n}(p_{\alpha_0 \dots \alpha_n})$ for each n. Hence by (1), $p_{\alpha_0 \dots \alpha_n} \preccurlyeq x$ for each n, and thus for some $y \preccurlyeq x$, $y = p_{\alpha_0 \dots \alpha_n}$ for infinitely many n. Since $\pi_{\le k}^{-1}\pi_{\le k}(p_{\alpha_0 \dots \alpha_k}) \supset \pi_{\le k+1}^{-1}\pi_{\le k+1}(p_{\alpha_0 \dots \alpha_{k+1}})$ for each k, we have

$$y \in \bigcap_{n=0}^{\infty} \pi_{\le n}^{-1}\pi_{\le n}(p_{\alpha_0 \dots \alpha_n}),$$

a set which consists of exactly one point. But also

$$x \in \bigcap_{n=0}^{\infty} \pi_{\le n}^{-1}\pi_{\le n}(p_{\alpha_0\ldots\alpha_n}),$$

so $y = x$, whence $x = p_{\alpha_0\ldots\alpha_n}$ for some n.
Define points $q_{\alpha_0\ldots\alpha_n}$ of Y in the same way, with similar properties. We claim that $f\colon X \to Y$ defined by $f(p_{\alpha_0\ldots\alpha_n}) = q_{\alpha_0\ldots\alpha_n}$ is a homeomorphism. By (2) and (3), f is a bijection. We will show that f is continuous; the continuity of f^{-1} can be proved in exactly the same way.
Let $q_{\alpha_0\ldots\alpha_n} \in U = \pi_{\le m}^{-1}\pi_{\le m}(q_{\alpha_0\ldots\alpha_n})$. Of course we may assume that $m \ge n$, and by (ii) we can in fact take $m = n$. We will show that $f[X \cap \pi_{\le n}^{-1}\pi_{\le n}(p_{\alpha_0\ldots\alpha_n})] \subset U$. Indeed, take $p_{\beta_0\ldots\beta_k} \in \pi_{\le n}^{-1}\pi_{\le n}(p_{\alpha_0\ldots\alpha_n})$. Again by (ii), we may assume that $k > n$. Then

$$\pi_{\le n}(p_{\alpha_0\ldots\alpha_n}) = \pi_{\le n}(p_{\beta_0\ldots\beta_k}) = \pi_{\le n}(p_{\beta_0\ldots\beta_n}),$$

so $p_{\alpha_0\ldots\alpha_n} = p_{\beta_0\ldots\beta_n}$ by (1), whence $\alpha_i = \beta_i$ for each $i \le n$ by (2). Thus,

$$f(p_{\beta_0\ldots\beta_k}) = q_{\beta_0\ldots\beta_k} = q_{\alpha_0\ldots\alpha_n\beta_{n+1}\ldots\beta_k},$$

and since $\pi_{\le n}(q_{\alpha_0\ldots\alpha_n\beta_{n+1}\ldots\beta_k}) = \pi_{\le n}(q_{\alpha_0\ldots\alpha_n})$, we find that $f(p_{\beta_0\ldots\beta_k}) \in U$. □

Remark: Sierpiński, in his original proof, considered subsets X,Y of some $\mathbb{R}^n$, but the technique described here is the same. The proof presented here is certainly not the most elegant one possible, e.g. consider the following argument: since X and Y are countable and dense in itself, they are zero-dimensional and nowhere compact; hence by the remark following theorem 2.3.1, each can be densely embedded in the irrationals considered as a subset of $\mathbb{R}$, thus X and Y are order-isomorphic, and hence they are homeomorphic. Yet another proof will be given in Chapter 3 (remark following theorem 3.2.4).

2.4.2 COROLLARY: *$\mathbb{Q}$ is strongly homogeneous, hence homogeneous.* □

2.4.3 COROLLARY: *Let X be a non-empty countable space. Then $X \times \mathbb{Q} \approx \mathbb{Q}$; thus, X can be embedded as a closed subspace of $\mathbb{Q}$.*

We now turn to the "nowhere countable" case; we start with a lemma (here, "unbounded" means: unbounded left *and* right).

2.4.4 LEMMA: (a) *Let* X_0, X_1 *be Cantor sets in* $\mathbb{R}$. *Then there exists an order-preserving homeomorphism* $h: X_0 \to X_1$.

(b) *Let* X_0, X_1 *be closed, nowhere dense, unbounded subsets of* $\mathbb{R}$ *without isolated points. Then there exists an order-preserving homeomorphism* $h: X_0 \to X_1$.

Proof: (a) For $i \in \{0,1\}$, let $\mathcal{U}_i = \{(a^i_n, b^i_n): n < \omega\}$ be the collection of components of $[\min X_i, \max X_i] \setminus X_i$. Since X_i is dense in itself, $b^i_n \neq a^i_m$ for each n,m; thus, we can linearly order $\mathcal{U}_i$ by: $(a^i_n, b^i_n) < (a^i_m, b^i_m)$ if and only if $b^i_n < a^i_m$. If $b^i_n < a^i_m$, then since X_i is nowhere dense in $\mathbb{R}$, we have $(a^i_k, b^i_k) \subset (b^i_n, a^i_m)$ for some k, i.e. we have $(a^i_n, b^i_n) < (a^i_k, b^i_k) < (a^i_m, b^i_m)$. Furthermore, for each n, we have $b^i_k < a^i_n$ and $b^i_n < a^i_\ell$ for some k, ℓ, i.e. we have $(a^i_k, b^i_k) < (a^i_n, b^i_n) < (a^i_\ell, b^i_\ell)$. Thus, $(\mathcal{U}_i, <)$ is order-isomorphic to $\mathbb{Q}$ (see Sierpiński [53] XI.8, theorem 1), and we can define an order-preserving homeomorphism $f: (\mathcal{U}_0, <) \to (\mathcal{U}_1, <)$; it is trivial to define an order-preserving homeomorphism $g: \cup\mathcal{U}_0 \to \cup\mathcal{U}_1$, induced by f. Now define $h: X_0 \to X_1$ by

$$h(\min X_0) = \min X_1;$$
$$h(x) = \sup\{g(y): y < x,\ y \in \cup\mathcal{U}_0\} \quad \text{if } x \neq \min X_0.$$

It is routine to verify that h is as required.
The proof of (b) is similar. □

Remark: In fact, the homeomorphisms h in the above lemma can be defined as restrictions of order-preserving autohomeomorphisms of $\mathbb{R}$.

2.4.5 THEOREM (Alexandroff and Urysohn [1]): *Up to homeomorphism, the product of the rationals with the Cantor set,* $\mathbb{Q} \times C$, *is the only zero-dimensional σ-compact space which is nowhere compact and nowhere countable.*

Proof: It is easily seen that $\mathbb{Q} \times C$ is σ-compact, nowhere compact, and nowhere countable.
In Alexandroff and Urysohn's paper, the space $\mathbb{Q} \times C$ is not mentioned at all; they use a "test space" Ψ which is defined as follows. If $U = (a,b) \subset \mathbb{R}$, denote by $P(U)$ the "real" Cantor set in $[a,b]$, obtained by subsequently deleting open middle third intervals. Put $P_0 = \cup_{n \in \mathbb{Z}} P((n,n+1))$; note that P_0 is closed in $\mathbb{R}$. If the closed subset P_n of $\mathbb{R}$ has been defined, then we put $P_{n+1} = P_n \cup \cup\{P(U): U \text{ is a component of } \mathbb{R} \setminus P_n\}$. Finally, define

$$\Psi = \cup_{n=0}^{\infty} P_n.$$

The sets P_n have the following properties:

(1) P_n is a closed, nowhere dense, unbounded subset of $\mathbb{R}$ without isolated points;

(2) $P_n \subset P_{n+1}$;

(3) if (a,b) is a component of $\mathbb{R}\backslash P_n$, then $a = \inf(P_{n+1} \cap (a,b))$, $b = \sup(P_{n+1} \cap (a,b))$.

Now let X be any zero-dimensional, σ-compact, nowhere countable, nowhere compact space. By the remark following the proof of theorem 2.3.1, we can consider X as a dense subset of $\mathbb{P} \subset \mathbb{R}$. We will show that we can write $X = \bigcup_{n=0}^{\infty} X_n$, where the sets X_n satisfy properties similar to those of the sets P_n.

Let $X = \bigcup_{n=0}^{\infty} K_n$, with K_n compact, and let $A \subset X$ be discrete, and closed unbounded in $\mathbb{R}$. Then $Y_n = K_n \cup A$ is closed, unbounded, and nowhere dense in $\mathbb{R}$. Fix $n < \omega$, and let $\{z_i : i < \omega\}$ be the set of isolated points of Y_n. Let $\{U_i : i < \omega\}$ be a disjoint collection of open intervals such that $U_i \cap Y_n = \{z_i\}$ for $i < \omega$. Since X is σ-compact, so is each open subset of X; hence, since X is nowhere countable, we can apply corollary 2.1.5 to obtain, for each $k \in \mathbb{N}$, a Cantor set C_k^i in $U_i \cap (z_i - \frac{1}{k}, z_i + \frac{1}{k}) \cap X$. Define

$$Z_n = Y_n \cup \bigcup\{C_k^i : k \in \mathbb{N},\ i < \omega\};$$

then $Z_n \subset X$ is a closed unbounded subset of $\mathbb{R}$ without isolated points. Finally, put $X_0 = Z_0$, and suppose that X_n has been defined for each $n \leq k$, satisfying the obvious analogues to properties (1), (2), and (3) above, and such that $Z_n \subset X_n$. Now if $\{(a_i,b_i) : i < \omega\}$ is the family of components of $\mathbb{R}\backslash X_n$, then as above, we can find Cantor sets $C_k^i(0)$ in $(a_i,b_i) \cap (a_i, a_i+\frac{1}{k}) \cap X$, $C_k^i(1)$ in $(a_i,b_i) \cap (b_i-\frac{1}{k}, b_i) \cap X$. Defining

$$X_{n+1} = X_n \cup Z_n \cup \bigcup\{C_k^i(j) : j \in \{0,1\},\ i < \omega,\ k \in \mathbb{N}\},$$

(1)-(3) are satisfied.

Applying lemma 2.4.4(b) to P_0 and X_0, we obtain an order-preserving homeomorphism $h_0 : P_0 \to X_0$, by (1). Let (a,b) be a component of $\mathbb{R}\backslash P_0$; by the construction of Ψ, $[a,b] \cap P_1 \approx \mathbb{C}$. Since h_0 is order-preserving, $(h_0(a), h_0(b))$ is a component of $\mathbb{R}\backslash X_0$, so by (1), (3), and theorem 2.1.1, we find that $[h_0(a), h_0(b)] \cap X_1 \approx \mathbb{C}$. By lemma 2.4.4(a), there exists an order-preserving homeomorphism $h_1(a,b) : [a,b] \cap P_1 \to [h_0(a), h_0(b)] \cap X_1$, and by (3),

$$h_1 = h_0 \cup \bigcup\{h_1(a,b) : (a,b) \text{ is a component of } \mathbb{R}\backslash P_0\}$$

extends h_0. It is now easy to construct subsequently order-preserving homeomorphisms $h_n: P_n \to X_n$. Then $\bigcup_{n=0}^{\infty} h_n = h: \Psi \to X$ is a homeomorphism. □

Remark: In [38], van Mill gave a new proof of this theorem. Again, our remark following theorem 3.2.4 will provide yet another proof (cf. remark following theorem 2.4.1).

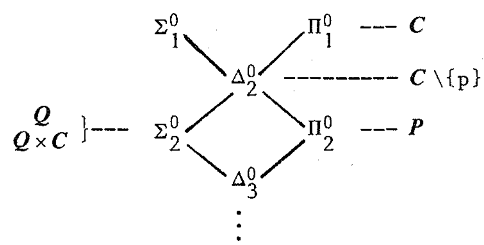
We have the usual corollaries:

2.4.6 COROLLARY: *$\mathbb{Q}\times C$ is strongly homogeneous, hence homogeneous.* □

2.4.7 COROLLARY: *Let X be a non-empty zero-dimensional σ-compact space. Then $X \times \mathbb{Q}\times C \approx \mathbb{Q}\times C$; thus, X can be embedded as a closed subspace of $\mathbb{Q}\times C$.* □

CHAPTER 3: BOREL SETS OF AMBIGUOUS CLASS 2

Consider the following picture of the lower levels of the Borel hierarchy in the Cantor set:



The results in the preceding chapter indicate, that each of the spaces in the diagram is almost determined by its level in the Borel hierarchy; the most conspicuous example is P, a space which is completely characterized by its being Π_2^0 and nowhere Σ_2^0. When looking for characterizations of spaces at level Δ_3^0, i.e. the ambiguous class 2, the first idea that comes to mind might be that perhaps we are so lucky that there is precisely one zero-dimensional space which is Δ_3^0, nowhere Σ_2^0, and nowhere Π_2^0. However, results of van Mill ([38] and [39]) show that there are at least two homogeneous spaces with these properties, and, as we shall see, there are in fact ω_1 of them.

Thus, if we want to characterize spaces by way of their level in the Borel hierarchy, then we will have to refine this hierarchy at level Δ_3^0, i.e. create a subhierarchy in Δ_3^0, in such a way that there will be only one space (or maybe finitely many) at a certain level which is nowhere at a lower level. Fortunately, such a hierarchy is already available, viz. the hierarchy of "small Borel classes", which has length ω_1; in section 3.1, we discuss this hierarchy, and change it slightly to make it better suit our needs. This will give the reader a rough idea of how the collection of homogeneous zero-dimensional absolute Borel sets of ambiguous class 2 is built up. Sections 3.2 and 3.3 contain the technical tools necessary to obtain the main results of this chapter, which are presented in sections 3.4 - 3.6. In section 3.4, we describe and characterize the spaces at the finite levels of our subhierarchy, and in section 3.5, the spaces at the infinite levels of this hierarchy; section 3.6 contains the main theorem, stating that the results of sections 3.4 and 3.5, together with those of Chapter 2, yield a complete picture of the homogeneous zero-dimensional absolute Borel sets of ambiguous class 2.

The main results of this chapter are taken from van Engelen [10].

3.1 Small Borel classes

3.1.1 DEFINITION: (a) *Let* X *be a space,* $A \subset X$, *let* $\eta < \omega_1$, *and let* $\langle A_\zeta : \zeta < \eta \rangle$ *be an increasing sequence of subsets of* X. *Then* $A = D_\eta(\langle A_\zeta : \zeta < \eta \rangle)$ *if and only if* $A = \cup\{A_\zeta \setminus (\cup_{\beta<\zeta} A_\beta) : \zeta \text{ odd} < \eta\}$ *when* η *is even,*
resp. $= \cup\{A_\zeta \setminus (\cup_{\beta<\zeta} A_\beta) : \zeta \text{ even} < \eta\}$ *when* η *is odd.*

(b) *Let* X *be a space,* $A \subset X$, *and let* $\eta, \xi \in [1, \omega_1)$. *Then* $A \in D_\eta^X(\Sigma_\xi^0)$ *if and only if* $A = D_\eta(\langle A_\zeta : \zeta < \eta \rangle)$ *for some increasing sequence* $\langle A_\zeta : \zeta < \eta \rangle$ *of* Σ_ξ^0*-sets in* X.

(c) *Let* $\eta, \xi \in [1, \omega_1)$. *Then* $A \in D_\eta(\Sigma_\xi^0)$ *if and only if* $A \in D_\eta^X(\Sigma_\xi^0)$ *for every space* X, *whenever* A *is embedded in* X.

Part (c) says that A is $D_\eta(\Sigma_\xi^0)$ *absolutely* (note that $D_1(\Sigma_\xi^0) = \Sigma_\xi^0$). It easily follows from Lavrentieff's theorem that if $\xi \geq 2$, then this is equivalent to $A \in D_\eta^X(\Sigma_\xi^0)$ for some compact space X; thus, if A is zero-dimensional, then $A \in D_\eta(\Sigma_\xi^0)$ if and only if $A \in D_\eta^{\mathcal{C}}(\Sigma_\xi^0)$ for some embedding of A in $\mathcal{C}$.

The above definition is basically due to Kuratowski ([28] §33.IV; for $\xi = 1$, Hausdorff [21]); he called the $D_\eta(\Sigma_\xi^0)$ the "small Borel classes". Here, we have more or less adopted the notation from Louveau [32] since this definition and other ones from that paper will be needed anyway in Chapters 4 and 5.

3.1.2 THEOREM (Kuratowski [28]; for $\xi = 1$, Hausdorff [21]): *Let* A *be a subset of a compact space* X, *and let* $\xi \in [1, \omega_1)$. *Then* $A \in \Delta_{\xi+1}^0$ *if and only if* $A \in D_\eta^X(\Sigma_\xi^0)$ *for some* $\eta \in [1, \omega_1)$.

Of course, in this chapter we will be interested in the case $\xi = 2$, $X = \mathcal{C}$ of the above theorem; the sets A_ζ of definition 3.1.1(a) will then be σ-compact subsets of $\mathcal{C}$. Equivalently, we will work with zero-dimensional elements of the absolute classes $D_\eta(\Sigma^0_2)$. Clearly, $D_\alpha(\Sigma^0_2) \subset D_\beta(\Sigma^0_2)$ if $\alpha \le \beta$ (and in fact, $D_\alpha(\Sigma^0_2) \subsetneq D_\beta(\Sigma^0_2)$ if $\alpha < \beta$, see Lavrentieff [30], and also Luzin [34], Sierpiński [52]). Thus, the classes $D_\eta(\Sigma^0_2)$ form a hierarchy, which refines Δ^0_3.

For even η, we are now going to give an alternative description of the classes $D_\eta(\Sigma^0_2)$, which is often easier to work with; it has the additional advantage over the original definition that it describes the elements of the classes *internally*.

First consider the "building blocks" of the elements of $D_\eta(\Sigma^0_2)$; they are sets of the form $F_1 \setminus F_0$, where F_0, F_1 are σ-compact, i.e. they are elements of $D_2(\Sigma^0_2)$.

3.1.3 LEMMA: $A \in D_2(\Sigma^0_2)$ *if and only if there exist closed, complete subsets* A_i *of* A *such that* $A = \bigcup_{i=0}^\infty A_i$.

Proof: Embed A in a compact space X, and first suppose that $A \in D_2(\Sigma^0_2)$, say $A = F_1 \setminus F_0$, with F_0, F_1 σ-compact; then $G = X \setminus F_0$ is complete, and $F_1 = \bigcup_{i=0}^\infty K_i$, with K_i compact. Since $K_i \cap G$ is closed in G, it is complete, and closed in A. But $A = \bigcup_{i=0}^\infty (K_i \cap G)$.

Conversely, let $A = \bigcup_{i=0}^\infty A_i$, where A_i is complete and closed in A; then both $F_0 = \bigcup_{i=0}^\infty (\overline{A}_i \setminus A_i)$ and $F_1 = \bigcup_{i=0}^\infty \overline{A}_i$ are σ-compact, and $A = F_1 \setminus F_0$. □

Because of the above lemma, elements of $D_2(\Sigma^0_2)$ will also be called *strongly σ-complete* spaces (see also definition 3.3.2(a)).

Note that the lemma internally describes the elements of $D_2(\Sigma^0_2)$; we will now do the same for $D_{2n}(\Sigma^0_2)$, where $n \in \mathbb{N}$.

3.1.4 LEMMA: *Let* $n \in \mathbb{N}$. *Then* $A \in D_{2n}(\Sigma^0_2)$ *if and only if* A *is the union of* n *strongly σ-complete subspaces.*

Proof: The "only if" part follows from lemma 3.1.3. For the converse, assume that A is embedded in a compact space X. We first establish the following CLAIM: *If* $B \subset X$, $m < \omega$, *and* $B = \bigcup_{i=0}^m (X_i \setminus Y_i)$, *where* X_i *and* Y_i *are σ-compact subsets of* X, *then there exist σ-compact* $\tilde{X}_i$ *and* $\tilde{Y}_i$ *in* X *such that* $B = \bigcup_{i=0}^m (\tilde{X}_i \setminus \tilde{Y}_i)$, *and* $\bigcup_{i<m} (\tilde{X}_i \setminus \tilde{Y}_i) \subset \tilde{Y}_m$.

Indeed, for $i < m$, put

$$\widetilde{X}_i = \bigcup_{j=0}^{i}(Y_j \cap X_{i+1}) \cup \bigcup_{j=0}^{i}(X_j \cap Y_{i+1}),$$
$$\widetilde{Y}_i = \bigcup_{j=0}^{i}(Y_j \cap Y_{i+1}),$$

and let

$$\widetilde{X}_m = \bigcup_{j=0}^{m} X_j,\ \widetilde{Y}_m = \bigcup_{j=0}^{m} Y_j.$$

Clearly, $\bigcup_{i<m}(\widetilde{X}_i\setminus\widetilde{Y}_i) \subset \widetilde{Y}_m$, and $\bigcup_{i=0}^{m}(\widetilde{X}_i\setminus\widetilde{Y}_i) \subset \bigcup_{i=0}^{m}(X_i\setminus Y_i)$. So let $n \le m$, and $x \in X_n\setminus Y_n$. If $x \notin Y_i$ for all i, then $x \in X_n\setminus(\bigcup_{j=0}^{m} Y_j) \subset \widetilde{X}_m\setminus\widetilde{Y}_m$. Otherwise, there is a minimal i with $x \in Y_i$. If $i > n$, then $i-1 \ge n \ge 0$, and we claim that $x \in \widetilde{X}_{i-1}\setminus\widetilde{Y}_{i-1}$; indeed, $x \in \bigcup_{j=0}^{i-1}(X_j \cap Y_{(i-1)+1})$, and $x \notin \bigcup_{j=0}^{i-1} Y_j$ by minimality of i. Since $x \notin Y_n$, the last possibility is $i < n$; but then $x \in \bigcup_{j=0}^{n-1}(Y_j \cap X_{(n-1)+1})$, and $x \notin Y_{(n-1)+1}$, whence $x \in \widetilde{X}_{n-1}\setminus\widetilde{Y}_{n-1}$. This proves the claim.

Now suppose that $A = \bigcup_{i=0}^{n-1} A_i$, where A_i is strongly σ-complete; by lemma 3.1.3, and by successive applications of the claim, we can write $A = \bigcup_{i=0}^{n-1}(X_i\setminus Y_i)$, where X_i and Y_i are σ-compact subsets of X, and $\bigcup_{i<m}(X_i\setminus Y_i) \subset Y_m$ for each $m < n$. Now if $i < n$, put

$$\widetilde{C}_{2i} = \bigcap_{j\ge i} Y_j,\ \widetilde{C}_{2i+1} = X_i \cap \bigcap_{j>i} Y_j;$$

then $\widetilde{C}_{2i+1}\setminus\widetilde{C}_{2i} = X_i\setminus Y_i$. Finally, for each $\ell < 2n$, put

$$C_\ell = \bigcup_{m\le\ell} \widetilde{C}_m;$$

then $B = D_{2n}(\langle C_\ell\colon \ell < 2n\rangle)$. Indeed, the C_ℓ are clearly increasing; and if $i < n$, then $C_{2i+1}\setminus C_{2i} \subset \widetilde{C}_{2i+1}\setminus\widetilde{C}_{2i}$. Furthermore, $\bigcup_{j<2i} \widetilde{C}_j \subset Y_i$, so $\widetilde{C}_{2i+1}\setminus\widetilde{C}_{2i} \subset C_{2i+1}\setminus(\widetilde{C}_{2i} \cup Y_i) \subset C_{2i+1}\setminus C_{2i}$. □

The following lemma covers the limit case.

3.1.5 LEMMA: *Let* $\alpha < \omega_1$ *be a limit. Then* $A \in D_\alpha(\Sigma_2^0)$ *if and only if there exist closed subsets* A_i *of* A, *and ordinals* $\alpha_i \in [1,\alpha)$, *such that* $A_i \in D_{\alpha_i}(\Sigma_2^0)$ *and* $A = \bigcup_{i=0}^{\infty} A_i$.

Proof: Embed A in a compact space X. If $A = D_\alpha(\langle C_\zeta\colon \zeta < \alpha\rangle)$, with each C_ζ a σ-compact subset of X, and if $\langle\alpha_i\colon i < \omega\rangle$ is a sequence of even ordinals in $[1,\alpha)$ such that $\sup_{i<\omega} \alpha_i = \alpha$, then $A = \bigcup_{i=0}^{\infty} D_{\alpha_i}(\langle C_\zeta\colon \zeta < \alpha_i\rangle)$. Since $\bigcup_{\zeta<\alpha_i} C_\zeta$ is σ-compact,

$$D_{\alpha_i}(\langle C_\zeta\colon \zeta < \alpha_i\rangle) = \bigcup_{j=0}^{\infty} D_{\alpha_i}(\langle C_\zeta \cap K_j^i\colon \zeta < \alpha_i\rangle)$$

for certain compact subsets K_j^i of $\bigcup_{\zeta<\alpha_i} C_\zeta$. Since $D_{\alpha_i}(\langle C_\zeta \cap K_j^i\colon \zeta < \alpha_i\rangle)$

$= K^i_j \cap A$ is closed in A, A has the required form.
Conversely, suppose that $A = \bigcup_{i=0}^{\infty} A_i$, with A_i closed in A, and $A_i \in D_{\alpha_i}(\Sigma^0_2)$ for certain $\alpha_i \in [1,\alpha)$; then also $A_i \in D_\alpha(\Sigma^0_2)$, say $A_i = D_\alpha(\langle A^i_\zeta : \zeta < \alpha \rangle)$. For each $i < \omega$, let K_i be a compact subset of X such that $A_i = K_i \cap A$, and put $L_i = K_i \setminus (\bigcup_{j<i} K_j)$; note that L_i is σ-compact, and $A = \bigcup_{i=0}^{\infty}(L_i \cap A_i)$.
Now for $\zeta < \alpha$, put

$$A_\zeta = \bigcup_{i=0}^{\infty}(L_i \cap A^i_\zeta).$$

Clearly, the sets A_ζ are σ-compact, and increasing. Furthermore,

$$\begin{aligned} D_\alpha(\langle A_\zeta : \zeta < \alpha \rangle) &= \bigcup\{A_{\zeta+1} \setminus A_\zeta : \zeta \text{ even} < \alpha\} \\ &= \bigcup\{\bigcup_{i=0}^{\infty}(L_i \cap A^i_{\zeta+1}) \setminus \bigcup_{i=0}^{\infty}(L_i \cap A^i_\zeta) : \zeta \text{ even} < \alpha\} \\ &= \bigcup\{\bigcup_{i=0}^{\infty}((L_i \cap A^i_{\zeta+1}) \setminus (L_i \cap A^i_\zeta)) : \zeta \text{ even} < \alpha\} \\ &= \bigcup_{i=0}^{\infty}(L_i \cap \bigcup\{A^i_{\zeta+1} \setminus A^i_\zeta : \zeta \text{ even} < \alpha\}) \\ &= \bigcup_{i=0}^{\infty}(L_i \cap A_i) = A. \end{aligned}$$

The third equality follows from the observation that $L_i \cap L_j = \emptyset$ if $i \neq j$.□

We are now ready for the alternative description of the $D_\alpha(\Sigma^0_2)$.

3.1.6 THEOREM: *Let* $\alpha \in [1,\omega_1)$ *be even, say* $\alpha = \beta+2n$, *where* $\beta = 0$ *or* $\lim(\beta)$, *and* $n < \omega$. *Then* $A \in D_\alpha(\Sigma^0_2)$ *if and only if* $A = \bigcup_{i=0}^{\infty} A_i \cup \bigcup_{i<n} B_i$, *where* A_i *is closed in* A, $A_i \in D_{\beta_i}(\Sigma^0_2)$ *for some* $\beta_i \in [1,\beta)$, *and* B_i *is strongly* σ*-complete.*

Proof: Embed A as a subset of a compact space X. If $A \in D_\alpha(\Sigma^0_2)$, then we can find the sets A_i as in the first part of the proof of lemma 3.1.5; the sets B_i are obtained from lemma 3.1.3. For the converse, let F be a σ-compact subset of X satisfying $F \cap A = \bigcup_{i=0}^{\infty} A_i$. By lemma 3.1.5, there exist σ-compact subsets $\{\widetilde{A}_\zeta : \zeta < \beta\}$ of X such that $\bigcup_{i=0}^{\infty} A_i = D_\beta(\langle \widetilde{A}_\zeta : \zeta < \beta \rangle)$, and by lemma 3.1.4, there exist σ-compact subsets $\{\widetilde{A}_{\beta+k} : k < 2n\}$ such that $\bigcup_{i<n} B_i = D_{2n}(\langle \widetilde{A}_{\beta+k} : k < 2n \rangle)$. Now for $\zeta < \beta$, put

$$A_\zeta = \widetilde{A}_\zeta \cap F,$$

and for $k < 2n$, put

$$A_{\beta+k} = F \cup \widetilde{A}_{\beta+k};$$

then $A = D_\alpha(\langle A_\zeta : \zeta < \alpha \rangle)$. □

Since $D_{2n}(\Sigma_2^0)$ is described internally by lemma 3.1.4, and since the β_i in the above theorem can be chosen to be even, we indeed have found (inductively) an *internal* description of the classes $D_{\alpha+2n}(\Sigma_2^0)$.

We will now describe the subhierarchy of Δ_3^0 that we will need. It is obtained from the classes $D_\alpha(\Sigma_2^0)$, for even α, by making a distinction between different types of "building blocks". These building blocks are just the strongly σ-complete spaces; however, a strongly σ-complete space might have the stronger property of being complete, or σ-compact, or even countable. Combining these topological properties, we find that the union of two building blocks can be of any of the following types (note that complete ∪ complete ≡ complete, countable ∪ countable ≡ countable, and σ-compact ∪ σ-compact ≡ σ-compact):

complete ∪ countable;
strongly σ-complete ∪ countable;
complete ∪ σ-compact;
strongly σ-complete ∪ σ-compact;
strongly σ-complete ∪ complete;
strongly σ-complete ∪ strongly σ-complete.

Adding another building block again yields six new topological properties; and proceeding in this way, we obtain topological properties describing elements of $\bigcup_{n=1}^{\infty} D_{2n}(\Sigma_2^0)$ as in the following definition; here, as in the remainder of this chapter,

> *if* $n < \omega$, *then a space* X *is said to have property* S_n *if it is the union of* n *strongly σ-complete subsets (i.e.* $X \in D_{2n}(\Sigma_2^0)$ *if* $n > 0$, $X = \emptyset$ *if* $n = 0$*).*

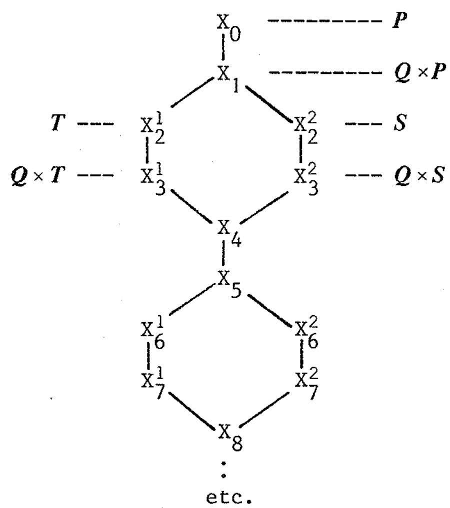
3.1.7 DEFINITION: *Let* X *be a topological space. Then for each* $k < \omega$,
X *has property* P_{4k} *if and only if* X *is* S_k ∪ *complete;*
X *has property* P_{4k+1} *if and only if* X *is* S_{k+1}*;*
X *has property* P^1_{4k+2} *if and only if* X *is* S_k ∪ *complete* ∪ *countable;*
X *has property* P^1_{4k+3} *if and only if* X *is* S_{k+1} ∪ *countable (also for* $k = -1$*);*
X *has property* P^2_{4k+2} *if and only if* X *is* S_k ∪ *complete* ∪ *σ-compact;*
X *has property* P^2_{4k+3} *if and only if* X *is* S_{k+1} ∪ *σ-compact (also for* $k = -1$*).*

For sake of simplicity, we will write "X is $P_n^{(i)}$" if X has one of the properties defined above, meaning that the index "i" may or may not be there.

Now suppose that the properties $P_n^{(i)}$ are ordered as in the definition, i.e.

$$P_{4k} < P_{4k+1} < P^1_{4k+2} < P^1_{4k+3} < P^2_{4k+2} < P^2_{4k+3} < P_{4(k+1)}.$$

In section 4 of this chapter, we will show that for each $n < \omega$, $i \in \{1,2\}$, there exists, up to homeomorphism, exactly one zero-dimensional space $X_n^{(i)}$ which is $P_n^{(i)}$ and nowhere $P_m^{(j)}$ for each $P_m^{(j)} < P_n^{(i)}$.
The spaces $X_n^{(i)}$ are roughly described by the following diagram:



Here, the spaces $\boldsymbol{S}$ and $\boldsymbol{T}$ are the homogeneous complements of $\boldsymbol{Q}\times\boldsymbol{P}$ in the Cantor set (see the introduction of this monograph), X_4 is obtained as the homogeneous complement of $\boldsymbol{Q}\times\boldsymbol{S}$ or $\boldsymbol{Q}\times\boldsymbol{T}$ in the Cantor set, $X_5 = \boldsymbol{Q}\times X_4$, and X_6^1 and X_6^2 are the only homogeneous complements of X_5 in the Cantor set. Then $X_7^1 = \boldsymbol{Q}\times X_6^1$, $X_7^2 = \boldsymbol{Q}\times X_6^2$, etc..

The spaces described above turn out to be the only zero-dimensional homogeneous (non σ-compact) elements of $\bigcup_{n=1}^{\infty} D_{2n}(\Sigma_2^0)$.
We now turn to the classes $D_{\alpha+2n}(\Sigma_2^0)$, where $n < \omega$, and $\alpha < \omega_1$ is a limit ordinal. Let us consider the class $D_{\omega+2}(\Sigma_2^0)$; each member X of this class is, by theorem 3.1.6, of the form $\bigcup_{i=0}^{\infty} A_i \cup B$, where A_i is closed in X, $A_i \in \bigcup_{n=1}^{\infty} D_{2n}(\Sigma_2^0)$, and B is strongly σ-complete. If B is σ-compact, then $X = \bigcup_{i=0}^{\infty} A_i \cup \bigcup_{j=0}^{\infty} B_j$, with B_j compact; hence X is already in $D_\omega(\Sigma_2^0)$:

the σ-compact part of X is "absorbed" by the part $\bigcup_{i=0}^{\infty} A_i$. Hence, if we want to define topological properties as in definition 3.1.7, then we need only distinguish between "B is complete" and "B is strongly σ-complete".

3.1.8 DEFINITION: *Let* X *be a topological space, and let* $\alpha < \omega_1$ *be a limit ordinal. Then for each* $n < \omega$,
X *has property* $P_{\alpha+2n}$ *if and only if* $X \in D_{\alpha+2n}(\Sigma_2^0)$;
X *has property* $P_{\alpha+2n+1}$ *if and only if* X *is* $P_{\alpha+2n} \cup$ *complete.*

In section 5, we will show that for each $m < \omega$, $i \in \{1,2\}$, and limit ordinal $\alpha < \omega_1$, there exists, up to homeomorphism, exactly one zero-dimensional space $X^i_{\alpha+m}$, such that

- $X^1_\alpha = X^2_\alpha = X_\alpha$ is P_α, and nowhere P_β for each $\beta < \alpha$;
- $X^1_{\alpha+n}$ is $P_{\alpha+n}$, nowhere $P_{\alpha+n-1}$, and contains no closed copies of $X^2_{\alpha+n-1}$ $(n > 0)$;
- $X^2_{\alpha+n}$ is $P_{\alpha+n}$, nowhere $P_{\alpha+n-1}$, and every non-empty clopen subset contains a closed copy of $X^2_{\alpha+n-1}$ $(n > 0)$.

The spaces can be described as follows:

$$X_\alpha \text{ --- } X^1_{\alpha+1} \text{ --- } X^1_{\alpha+2} = \mathbb{Q} \times X^1_{\alpha+1} \text{ --- } X^2_{\alpha+1} \text{ --- } X^2_{\alpha+2} = \mathbb{Q} \times X^2_{\alpha+1} \text{ --- }$$

$$\text{--- } X^1_{\alpha+3} \text{ --- } X^1_{\alpha+4} \text{ --- } X^2_{\alpha+3} \text{ --- } X^2_{\alpha+4} \text{ --- etc.}$$

Here, $X^1_{\alpha+1}$ is a complement of X_α in the Cantor set, $X^2_{\alpha+1}$ is a complement of $X^1_{\alpha+2}$, $X^1_{\alpha+3}$ is a complement of $X^2_{\alpha+2}$, and so on. Furthermore, $X^1_{\alpha+4} = \mathbb{Q} \times X^1_{\alpha+3}$, $X^2_{\alpha+4} = \mathbb{Q} \times X^2_{\alpha+3}$, etc..
X_α is obtained as a countable union of closed nowhere dense copies of previously defined spaces.

In this way, we obtain all homogeneous Borel sets of ambiguous class 2 in the Cantor set.

3.2 Knaster-Reichbach coverings and their applications

From section 3.1, we can see that, to obtain "new spaces from old ones", we frequently apply the following operations:

(1) taking the product with $\mathbb{Q}$;
(2) taking complements in the Cantor set.

In this section, we will establish two very general theorems. The first (theorem 3.2.4) is due to Ostrovskiǐ [44], and will enable us to handle (1); since Ostrovskiǐ's paper is in Russian, we feel that it might be useful to include a proof of this theorem. The second (theorem 3.2.6) will be used to show that, in all cases under consideration in this chapter, if we have two subsets X_0, X_1 of the Cantor set which have as their complements the copies Y_0, Y_1 of the previously characterized space $Q \times X$, then there exists an autohomeomorphism h of the Cantor set such that $h[Y_0] = Y_1$, and hence $X_0 \approx X_1$; this will take care of (2) above.

3.2.1 DEFINITION: *Let* X *and* Y *be zero-dimensional spaces, and let* A *be a closed nowhere dense subset of* X *and* B *a closed nowhere dense subset of* Y. *Suppose that* $h: A \to B$ *is a homeomorphism,* $\mathcal{U} = \{U_n : n < \omega\}$ *is a covering of* $X \setminus A$ *by disjoint non-empty clopen subsets of* X, *and* $\mathcal{V} = \{V_n : n < \omega\}$ *is a covering of* $Y \setminus B$ *by disjoint non-empty clopen subsets of* Y. *Then* $\{(U_n, V_n) : n < \omega\}$ *is a Knaster-Reichbach covering, or KR-covering, for* $(X \setminus A, Y \setminus B, h)$ *if, whenever* $h_n : U_n \to V_n$ *is a bijection for each* $n < \omega$, *the combination mapping* $\tilde{h} = h \cup \bigcup_{n=0}^{\infty} h_n : X \to Y$ *is continuous in points of* A, *and* $\tilde{h}^{-1}$ *is continuous in points of* B.

Note that we do not require the bijections h_n to exist. Objects similar to those defined above were used by Knaster and Reichbach in [25] to prove some theorems on extensions of homeomorphisms ; their technique has afterwards been used by various authors, e.g. Pollard [46], Ravdin [47], and Gutek [20].

The following lemma is a slight generalization of Knaster and Reichbach's theorem.

3.2.2 LEMMA: *Let* X *and* Y *be zero-dimensional spaces, and let* A *and* B *be non-empty closed nowhere dense subspaces of* X *and* Y, *respectively. If* $h: A \to B$ *is a homeomorphism, then there exists a KR-covering for* $(X \setminus A, Y \setminus B, h)$.

Proof: Embed X densely in a compact zero-dimensional space K. For each $x \in K \setminus \overline{A}$, let D_x be a clopen neighborhood of x such that $\operatorname{diam}(D_x) < \frac{1}{2} d(x, \overline{A})$; let $\{D_i : i < \omega\}$ be a countable subcovering of $\{D_x : x \in K \setminus \overline{A}\}$, and enumerate the non-empty elements of $\{D_i \setminus (\bigcup_{j<i} D_j) : i < \omega\}$ as $\{\tilde{U}_i : i < \omega\}$ (indeed, this set is infinite since $\overline{A}$ is nowhere dense in K). Note that $\operatorname{diam}(\tilde{U}_i) < d(\tilde{U}_i, \overline{A})$ for each $i < \omega$. Also, since $\{x \in K : d(x, \overline{A}) \geq \varepsilon\}$ is compact,

$d(\widetilde{U}_i,\overline{A}) \to 0$ $(i \to \infty)$. Put $U_i = \widetilde{U}_i \cap X$, then $\{U_i\colon i < \omega\}$ is a covering of $X\setminus A$ by non-empty pairwise disjoint clopen subsets of X satisfying $\mathrm{diam}(U_i) < d(U_i,A) \to 0$ $(i \to \infty)$. Similarly, let $\{V_i\colon i < \omega\}$ be a covering of $Y\setminus B$ by non-empty pairwise disjoint clopen subsets of Y satisfying $\mathrm{diam}(V_i) < d(V_i,B) \to 0$ $(i \to \infty)$.
Now define bijections $\rho,\sigma\colon \omega \to \omega$, and points $a_n \in A$, $b_n = h(a_n) \in B$, such that

$$\begin{aligned} &\text{if } n \text{ is even: } d(U_{\rho(n)},a_n) < 2d(U_{\rho(n)},A);\\ &\qquad V_{\sigma(n)} \subset B(b_n,d(U_{\rho(n)},A));\\ &\text{if } n \text{ is odd: } d(V_{\sigma(n)},b_n) < 2d(V_{\sigma(n)},B);\\ &\qquad U_{\rho(n)} \subset B(a_n,d(V_{\sigma(n)},B)),\end{aligned}$$

as follows:
Suppose a_k, b_k, $\rho(k)$, $\sigma(k)$ have been defined for $k < n$; if n is even, let $\rho(n) = \min \omega\setminus\{\rho(k)\colon k < n\}$; let $a_n \in A$ be such that $d(U_{\rho(n)},a_n) < 2d(U_{\rho(n)},A)$, and let $b_n = h(a_n)$. Put $\varepsilon = \frac{1}{2}d(U_{\rho(n)},A)$. Since $d(V_i,B) > 0$ for each $i < \omega$, and since B is nowhere dense in Y, $B(b_n,\varepsilon) \cap V_i \neq \emptyset$ for infinitely many $i < \omega$. Let $\sigma(n) \in \omega\setminus\{\sigma(k)\colon k < n\}$ be such that $V_{\sigma(n)} \cap B(b_n,\varepsilon) \neq \emptyset$ and $\mathrm{diam}(V_{\sigma(n)}) < \varepsilon$. Then $V_{\sigma(n)} \subset B(b_n,2\varepsilon)$.
If n is odd, let $\sigma(n) = \min \omega\setminus\{\sigma(k)\colon k < n\}$, and proceed as above.
We claim that $\{(U_{\rho(n)},V_{\sigma(n)})\colon n < \omega\}$ is a KR-covering for $(X\setminus A,Y\setminus B,h)$. Indeed, let $h_n\colon U_n \to V_n$ be a bijection, and let $\widetilde{h} = h \cup \bigcup_{n=0}^{\infty} h_n$. Suppose that $a \in A$, and $x_n \to a$. Since $\widetilde{h}|A\colon A \approx B$, we may assume that $x_n \in U_{\rho(i_n)}$ for some $i_n < \omega$, for all $n < \omega$. Since each U_i is clopen in X, we have $\rho(i_n) \to \infty$, hence $\mathrm{diam}(U_{\rho(i_n)})$, $d(U_{\rho(i_n)},A)$, $\mathrm{diam}(V_{\sigma(i_n)})$, $d(V_{\sigma(i_n)},B) \to 0$.
Now for each $n < \omega$,

$$\begin{aligned} d(a_{i_n},a) &\le d(a_{i_n},x_n) + d(x_n,a) \le d(a_{i_n},U_{\rho(i_n)}) + \mathrm{diam}(U_{\rho(i_n)}) + d(x_n,a)\\ &\le \max[d(V_{\sigma(i_n)},B),2d(U_{\rho(i_n)},A)] + \mathrm{diam}(U_{\rho(i_n)}) + d(x_n,a) \to 0.\end{aligned}$$

So $a_{i_n} \to a$, hence $\widetilde{h}(a_{i_n}) = b_{i_n} \to \widetilde{h}(a)$. Hence,

$$\begin{aligned} d(\widetilde{h}(x_n),\widetilde{h}(a)) &\le d(\widetilde{h}(x_n),b_{i_n}) + d(b_{i_n},\widetilde{h}(a))\\ &\le d(b_{i_n},V_{\sigma(i_n)}) + \mathrm{diam}(V_{\sigma(i_n)}) + d(b_{i_n},\widetilde{h}(a))\\ &\le \max[d(U_{\rho(i_n)},A),2d(V_{\sigma(i_n)},B)] + \mathrm{diam}(V_{\sigma(i_n)}) +\\ &\quad + d(b_{i_n},\widetilde{h}(a)) \to 0.\end{aligned}$$

So $\widetilde{h}(x_n) \to \widetilde{h}(a)$. Similarly, if $b \in B$, $y_n \to b$, then $\widetilde{h}^{-1}(y_n) \to \widetilde{h}^{-1}(b)$. □

For the sake of completeness, we mention the following corollary:

3.2.3 COROLLARY (van Mill [38]): *Let* X *be strongly homogeneous and zero-dimensional, and let* A *and* B *be closed nowhere dense subspaces of* X. *If* $h: A \to B$ *is a homeomorphism, then there exists a homeomorphism* $\tilde{h}: X \to X$ *extending* h.

Proof: If $A = \emptyset$, then there is nothing to prove. If $A \neq \emptyset$, then by the lemma, there is a KR-covering $\{(U_n,V_n): n < \omega\}$ for $(X\backslash A, X\backslash B, h)$, and by strong homogeneity, there is a homeomorphism $h_n: U_n \to V_n$ for each $n < \omega$. Now $\tilde{h} = h \cup \bigcup_{n=0}^{\infty} h_n$ is as required. □

Remark: Taking A to be a one-point set, we obtain the result of 1.9.1. In fact, in that proof, $\{(U'_n,V'_n): n < \omega\}$ is a KR-covering for $(X\backslash\{x\}, X\backslash\{y\}, g)$, where $g: \{x\} \to \{y\}$.

We now come to the first of the main results of this section.

3.2.4 THEOREM (Ostrovskiǐ [44]): *Let* A *be a strongly homogeneous, zero-dimensional space, and suppose that* $X = \bigcup_{i=0}^{\infty} X_i$, *where each* X_i *is closed and nowhere dense in* X, *and* $X_i \approx A$ *for each* i. *Then* $X \approx \mathbb{Q} \times A$.

Proof: If $A = \emptyset$, then there is nothing to prove, so suppose that $A \neq \emptyset$. Let $Y = \mathbb{Q} \times A$; put $\mathbb{Q} = \{q_i: i < \omega\}$, and let $Y_i = \{q_i\} \times A$. Note that for each i, Y_i is closed and nowhere dense in Y, and that $Y_i \approx A$.
Recall that $M = \omega^{<\omega}$, see 1.4. For each $s \in M$, we will define collections $\mathcal{U}(s) = \{U(s^\frown n): n < \omega\}$ of disjoint clopen subsets of X, $\mathcal{V}(s) = \{V(s^\frown n): n < \omega\}$ of disjoint clopen subsets of Y, closed nowhere dense subsets D(s) of U(s), E(s) of V(s), and homeomorphisms $h(s): D(s) \to E(s)$, such that

(1) $D(\emptyset) = X_0$, $E(\emptyset) = Y_0$, $U(\emptyset) = X$, $V(\emptyset) = Y$;

(2) $\{(U(s^\frown n),V(s^\frown n)): n < \omega\}$ is a KR-covering for $(U(s)\backslash D(s), V(s)\backslash E(s), h(s))$;

(3) $X_k \subset \bigcup_{|s|\le k} D(s)$, $Y_k \subset \bigcup_{|s|\le k} E(s)$.

Define $D(\emptyset)$, $E(\emptyset)$, $U(\emptyset)$, and $V(\emptyset)$ as in (1); then there exists by lemma 3.2.2 a KR-covering $\{(U(n),V(n)): n < \omega\}$ for $(U(\emptyset)\backslash D(\emptyset), V(\emptyset)\backslash E(\emptyset), h(\emptyset))$, where $h(\emptyset): D(\emptyset) \to E(\emptyset)$ is an arbitrary homeomorphism.
Now suppose that $\mathcal{U}(s)$, $\mathcal{V}(s)$, D(s), E(s), and h(s) have been defined for $|s| \le k$, and fix $s \in M$ with $|s| = k$. Let

$$k_n = \min\{j: U(s^\frown n) \cap X_j \neq \emptyset\},\quad \ell_n = \min\{j: V(s^\frown n) \cap Y_j \neq \emptyset\},$$

and put

$$D(s^\frown n) = U(s^\frown n) \cap X_{k_n},\ E(s^\frown n) = V(s^\frown n) \cap Y_{\ell_n}.$$

Since A is strongly homogeneous, there exists a homeomorphism $h(s^\frown n)$: $D(s^\frown n) \to E(s^\frown n)$. Since $D(s^\frown n)$ (resp. $E(s^\frown n)$) is closed and nowhere dense in $U(s^\frown n)$ (resp. $V(s^\frown n)$), and non-empty, there exist by lemma 3.2.2 collections $\mathcal{U}(s^\frown n) = \{U(s^\frown n^\frown i): i < \omega\}$ and $\mathcal{V}(s^\frown n) = \{V(s^\frown n^\frown i): i < \omega\}$ such that $\{(U(s^\frown n^\frown i), V(s^\frown n^\frown i)): i < \omega\}$ is a KR-covering for $(U(s^\frown n)\backslash D(s^\frown n), V(s^\frown n)\backslash E(s^\frown n), h(s^\frown n))$. It is easily seen that (3) is satisfied.
This completes the induction. We claim that $h = \bigcup_{s\in\mathcal{M}} h_s: X \approx Y$. Since $D(s_1) \cap D(s_2) = \emptyset$ if $s_1 \neq s_2$, h is well-defined, and by (3), the domain of h is all of X, and the range is all of Y. Since h is clearly bijective, it suffices to show that h and h^{-1} are continuous. So let $x \in X$, say $x \in D(s)$. By (2), it suffices to show that $h[U(s^\frown n)] = V(s^\frown n)$ for each $n < \omega$; but this follows immediately from the observation that $U(s^\frown n) = \bigcup_{t\in\mathcal{M}} D(s^\frown n^\frown t)$, $V(s^\frown n) = \bigcup_{t\in\mathcal{M}} E(s^\frown n^\frown t)$, and $h[D(s^\frown n^\frown t)] = E(s^\frown n^\frown t)$ for each $t \in \mathcal{M}$. The continuity of h^{-1} follows in exactly the same way. □

Remark: If the strongly homogeneous space A in the above theorem contains a closed nowhere dense copy of itself, then the result can also be deduced from a theorem of van Mill [38].

Remark: If $X = \{x_i: i < \omega\}$ has no isolated points, then taking $X_i = \{x_i\}$ in the above theorem yields $X \approx \mathbb{Q}$, substantiating the claim made in section 2.4. To extract the characterization of $\mathbb{Q}\times\mathbb{C}$ from the above theorem, let X be zero-dimensional, σ-compact, nowhere countable, and nowhere compact, say $X = \bigcup_{n=0}^{\infty} Y_n$, with Y_n compact and non-empty. Fix $n < \omega$. Since X is nowhere compact, Y_n is nowhere dense in X, so as in the proof of lemma 3.2.2, we can construct a covering $\{U_i: i < \omega\}$ of $X\backslash Y_n$ by non-empty clopen subsets of X satisfying $\mathrm{diam}(U_i) < d(U_i,Y_n) \to 0$ $(i \to \infty)$. Since X is nowhere countable, each U_i is uncountable, and hence it contains a Cantor set C_i (corollary 2.1.5). Using theorem 2.1.1, it is easily verified that

$$X_n = Y_n \cup \bigcup_{i=0}^{\infty} C_i \approx \mathbb{C},$$

and that X_n is nowhere dense in X. Thus, $X = \bigcup_{n=0}^{\infty} X_n \approx \mathbb{Q}\times\mathbb{C}$ by the theorem.

In the proof of our next result, we will use the following modification of a convergence criterion of Anderson [2].

3.2.5 LEMMA: *Let* X *be compact, and for each* $n \in N$, *let* $h_n: X \to X$ *be a homeomorphism such that* $d(h_{n+1},h_n) < \varepsilon_n = \min\{2^{-n}, 3^{-n}.\min\{\min\{d(h_i(x),h_i(y)): d(x,y) \geq 1/n\}: 1 \leq i \leq n\}\}$. *Then* $h = \lim_{n\to\infty} h_n$ *is an autohomeomorphism of* X.

Proof: Since $d(h_{n+1},h_n) < 2^{-n}$, for each $x \in X$ we have that $(h_n(x))_{n\in N}$ is a Cauchy sequence in X, and thus it converges to $h(x) \in X$. Since the convergence is uniform, h is continuous, and hence it is closed since X is compact. So it suffices to show that h is a bijection. Assume that h is not surjective, say $y \in X\setminus h[X]$. Since $h[X]$ is closed in X, $d(y,h[X]) = \varepsilon > 0$. Let $k \in N$ be such that $2^{-k} < \frac{1}{2}\varepsilon$, and put $x = h_k^{-1}(y)$; then

$$\begin{aligned} d(y,h(x)) &= \lim_{n\to\infty} d(h_k(x),h_n(x)) \\ &\leq \lim_{n\to\infty} (d(h_k(x),h_{k+1}(x)) + \ldots + d(h_{k+n}(x),h_{k+n+1}(x))) \\ &\leq \lim_{n\to\infty} \Sigma_{i=k}^{k+n} 2^{-i} \leq 2^{-k+1} < \varepsilon, \end{aligned}$$

a contradiction. So h is onto. Now suppose that $x,y \in X$, $x \neq y$, and let $n \in N$ be such that $d(x,y) \geq 1/n$. Let $\varepsilon = d(h_n(x),h_n(y)) > 0$. Then for each $k \geq n$, $d(h_{k+1},h_k) < 3^{-k}.\varepsilon$ since $1 \leq n \leq k$ and $d(x,y) \geq 1/n \geq 1/k$; thus, $d(h_n(x),h(x)) < \Sigma_{k=n}^{\infty} 3^{-k}.\varepsilon \leq \frac{1}{2}\varepsilon$, and similarly $d(h_n(y),h(y)) < \frac{1}{2}\varepsilon$. So

$$\begin{aligned} d(h(x),h(y)) &\geq d(h_n(x),h_n(y)) - d(h_n(x),h(x)) - d(h_n(y),h(y)) \\ &> 0, \end{aligned}$$

i.e. $h(x) \neq h(y)$. □

3.2.6 THEOREM: *Let* A *be a zero-dimensional, strongly homogeneous space, and suppose that the Cantor set is homogeneous with respect to dense copies of* A *if* $|A| > 1$. *Let* X *and* Y *be dense subsets of* C *such that* $X = \bigcup_{i=0}^{\infty} X_i$, $Y = \bigcup_{i=0}^{\infty} Y_i$, X_i *is closed and nowhere dense in* X, *and* Y_i *is closed and nowhere dense in* Y. *If*

(1) $X_i \approx A \approx Y_i$ *for each* $i < \omega$,

or (2) $\overline{X}_i\setminus X_i \approx A \approx \overline{Y}_i\setminus Y_i$ *and* $\overline{X}_i\setminus X_i$ *is dense in* $\overline{X}_i$, $\overline{Y}_i\setminus Y_i$ *is dense in* $\overline{Y}_i$ *for each* $i < \omega$,

then there exists a homeomorphism $h: C \to C$ *such that* $h[X] = Y$.

Proof: Let U be a clopen subset of C such that $U \cap X_i \neq \emptyset$. Then in case (1), $U \cap X_i \approx A$, and in case (2), $(\overline{U \cap X_i})\setminus(U \cap X_i) = U \cap (\overline{X}_i\setminus X_i) \approx A$, since A is strongly homogeneous. In case (2), also $((\overline{U \cap X_i})\setminus(U \cap X_i))^- = (U \cap (\overline{X}_i\setminus X_i))^- = \overline{U \cap X_i}$ since $\overline{X}_i\setminus X_i$ is dense in $\overline{X}_i$. Similarly, if V is a clopen subset of C such that $V \cap Y_j \neq \emptyset$, then in case (1), $V \cap Y_j \approx A$, and in case (2), $(\overline{V \cap Y_j})\setminus(V \cap Y_j) \approx A$ is dense in $\overline{V \cap Y_j}$. In both cases, it follows that there exists a homeomorphism $f\colon \overline{U \cap X_i} \to \overline{V \cap Y_j}$ such that $f[U \cap X_i] = V \cap Y_j$ (this is trivial if $|A| = 1$: then only case (1) can occur; if $|A| > 1$, it follows from the fact that $\overline{U \cap X_i} \approx \mathcal{C} \approx \overline{V \cap Y_j}$ is homogeneous with respect to dense copies of A).

For each $s \in M$, we will now construct collections $\mathcal{U}(s) = \{U(s^\frown n)\colon n < \omega\}$, $\mathcal{V}(s) = \{V(s^\frown n)\colon n < \omega\}$ of clopen subsets of $\mathcal{C}$, closed nowhere dense subsets $D(s), E(s)$ of $\mathcal{C}$, and for each $m \in \mathbb{N}$ a homeomorphism $h_m\colon \mathcal{C} \to \mathcal{C}$ such that if $n = |s|$, then

(1) $X_n \subset \bigcup_{|t|\le n} D(t)$, $Y_n \subset \bigcup_{|t|\le n} E(t)$;

(2) $D(s) \subset U(s)$, $E(s) \subset V(s)$, $U(\emptyset) = V(\emptyset) = \mathcal{C}$;

(3) $h_{n+1}[U(s^\frown i)] = V(s^\frown i)$ for each $i < \omega$;

(4) $h_{n+1}[D(s)] = E(s)$, $h_{n+1}[D(s) \cap X] = E(s) \cap Y$;

(5) if $n > 0$, then $h_{n+1}|\bigcup_{|t|<n} D(t) = h_n|\bigcup_{|t|<n} D(t)$;

(6) $\{(U(s^\frown i), V(s^\frown i))\colon i < \omega\}$ is a KR-covering for $(U(s)\setminus D(s), V(s)\setminus E(s), h_{n+1}|D(s))$;

(7) if $V \in \mathcal{V}(s)$, then $\operatorname{diam}(V) < \varepsilon_{n+1}$, where ε_{n+1} is as in lemma 3.2.5.

Suppose that this has been done; we claim that $d(h_n, h_{n+1}) < \varepsilon_n$ and that $\lim_{n\to\infty} h_n = h\colon \mathcal{C} \approx \mathcal{C}$ has the property that $h[X] = Y$.

If $n > 0$, and $x \in \bigcup_{|t|<n} D(t)$, then by (5), $h_{n+1}(x) = h_n(x)$, so $d(h_n(x), h_{n+1}(x)) < \varepsilon_n$. If $x \notin \bigcup_{|t|<n} D(t)$, then $x \notin D(\emptyset)$, hence $x \in U(m)$ for some $m < \omega$, and proceeding inductively, using (6), $x \in U(s^\frown i)$ for some $s \in M$ with $|s| = n-1$, and some $i < \omega$, so by (3), $h_n(x) \in V(s^\frown i)$; if $x \in D(s^\frown i)$, then by (4), $h_{n+1}(x) \in E(s^\frown i) \subset V(s^\frown i)$ by (2), and if $x \notin D(s^\frown i)$, then $x \in U(s^\frown i^\frown j)$ for some $j < \omega$, hence by (3), $h_{n+1}(x) \in V(s^\frown i^\frown j) \subset V(s^\frown i)$ by (6). Thus, $\{h_n(x), h_{n+1}(x)\} \subset V(s^\frown i) \in \mathcal{V}(s)$, $|s| = n-1$, hence by (7), $d(h_n(x), h_{n+1}(x)) < \varepsilon_n$. So, by lemma 3.2.5, $\lim_{n\to\infty} h_n$ is an autohomeomorphism of $\mathcal{C}$. Suppose that $x \in X_n$; by (1), $x \in D(s)$ for some $s \in M$ with $|s| \le n$, hence by (4), $h_{|s|+1}(x) \in Y$; by (5), $h_m(x) = h_{|s|+1}(x)$ for each $m > |s|+1$, so $h(x) = h_{|s|+1}(x)$, i.e. $h(x) \in Y$. Finally, if $y \in Y_n$, then $y \in E(s)$ for some $s \in M$ with $|s| \le n$, hence $y = h_{|s|+1}(x)$ for some $x \in D(s) \cap X$; as above $h(x) = h_{|s|+1}(x)$, so $y \in h[X]$.

We will now show how to carry out the inductive construction. First, put $U(\emptyset) = V(\emptyset) = \mathcal{C}$, $D(\emptyset) = \overline{X}_0$, $E(\emptyset) = \overline{Y}_0$, and let $\tilde{h}\colon D(\emptyset) \to E(\emptyset)$ be a homeomorphism such that $\tilde{h}[X_0] = Y_0$ (see the remarks at the begin of this proof). Let $\mathcal{R} = \{R_i\colon i < \omega\}$, $\mathcal{S} = \{S_i\colon i < \omega\}$ be such that $\{(R_i, S_i)\colon i < \omega\}$ is a

KR-covering for $(\mathbf{C}\setminus D(\emptyset), \mathbf{C}\setminus E(\emptyset), \tilde{h})$; this is possible by lemma 3.2.2. Since $R_i \approx \mathbf{C} \approx S_i$, there exists a homeomorphism $h^i\colon R_i \to S_i$. Put

$$h_1 = \tilde{h} \cup \bigcup_{i=0}^{\infty} h^i\colon \mathbf{C} \approx \mathbf{C}.$$

For each $i < \omega$, let $\{S(i,0),\dots,S(i,n_i)\}$ be a disjoint clopen covering of S_i by sets of diameter less than ε_1 (as in (7)), and for $i < \omega$, $k \le n_i$, put $R(i,k) = h_1^{-1}[S(i,k)]$; then $\mathcal{U}(\emptyset) = \{R(i,k)\colon i < \omega,\ k \le n_i\}$, $\mathcal{V}(\emptyset) = \{S(i,k)\colon i < \omega,\ k \le n_i\}$ can be enumerated as $\{U(n)\colon n < \omega\}$, $\{V(n)\colon n < \omega\}$ such that $\{(U(n),V(n))\colon n < \omega\}$ is a KR-covering for $(\mathbf{C}\setminus D(\emptyset), \mathbf{C}\setminus E(\emptyset), h_1|D(\emptyset))$ satisfying $h_1[U(n)] = V(n)$.
Now suppose that $D(s)$, $E(s)$, $\mathcal{U}(s)$, $\mathcal{V}(s)$, and h_m have been constructed for $|s| < n$ (≥ 1) and $m \le n$. Fix $s \in M$ with $|s| = n-1$. Let

$$k_i = \min\{j\colon U(s^\frown i) \cap X_j \ne \emptyset\},\ \ell_i = \min\{j\colon V(s^\frown i) \cap Y_j \ne \emptyset\},$$

and put

$$D(s^\frown i) = \overline{U(s^\frown i) \cap X_{k_i}},\ E(s^\frown i) = \overline{V(s^\frown i) \cap Y_{\ell_i}}.$$

Let $\tilde{h}(s^\frown i)\colon D(s^\frown i) \to E(s^\frown i)$ be a homeomorphism such that $\tilde{h}(s^\frown i)[D(s^\frown i) \cap X] = E(s^\frown i) \cap Y$. Choose $\mathcal{R}(s^\frown i) = \{R(s^\frown i,j)\colon j < \omega\}$, $\mathcal{S}(s^\frown i) = \{S(s^\frown i,j)\colon j < \omega\}$ such that $\{(R(s^\frown i,j),S(s^\frown i,j))\colon j < \omega\}$ is a KR-covering for $(U(s^\frown i)\setminus D(s^\frown i), V(s^\frown i)\setminus E(s^\frown i), \tilde{h}(s^\frown i))$, and let $h^j(s^\frown i)\colon R(s^\frown i,j) \to S(s^\frown i,j)$ be a homeomorphism. Put

$$h(s^\frown i) = \tilde{h}(s^\frown i) \cup \bigcup_{j=0}^{\infty} h^j(s^\frown i)\colon U(s^\frown i) \approx V(s^\frown i),$$

and

$$h_{n+1} = \bigcup\{h(s^\frown i)\colon |s| = n-1,\ i < \omega\} \cup h_n|\bigcup_{|s|<n} D(s).$$

As above, we can refine each $\mathcal{S}(s^\frown i)$ by disjoint clopen sets of diameter less than ε_{n+1}, and obtain $\mathcal{U}(s^\frown i) = \{U(s^\frown i^\frown j)\colon j < \omega\}$, $\mathcal{V}(s^\frown i) = \{V(s^\frown i^\frown j)\colon j < \omega\}$ such that $\{(U(s^\frown i^\frown j),V(s^\frown i^\frown j))\colon j < \omega\}$ is a KR-covering for $(U(s^\frown i)\setminus D(s^\frown i), V(s^\frown i)\setminus E(s^\frown i), h_{n+1}|D(s^\frown i))$ satisfying $h_{n+1}[U(s^\frown i^\frown j)] = V(s^\frown i^\frown j)$. As in the proof of theorem 3.2.4, it is easily seen that h_{n+1} is a homeomorphism, and that the inductive hypotheses are satisfied. □

Taking $|A| = 1$ in the above theorem, we obtain the following corollary (see e.g. Bennett [3]).

3.2.7 COROLLARY: *The Cantor set is homogeneous with respect to dense copies of* $\mathbf{Q}$ *(i.e.* $\mathbf{C}$ *is countable dense homogeneous).* □

Applying case (1) of theorem 3.2.6 for non-trivial A, we obtain:

3.2.8 THEOREM: *Let* A *be a zero-dimensional, strongly homogeneous space, with* $|A| > 1$. *If* $\mathbf{C}$ *is homogeneous with respect to dense copies of* A, *then* $\mathbf{C}$ *is also homogeneous with respect to dense copies of* $\mathbf{Q} \times A$. □

3.2.9 COROLLARY (van Mill [39]): *The Cantor set is homogeneous with respect to dense copies of* $\mathbf{Q} \times \mathbf{C}$.

3.3 Hurewicz-type theorems

In this chapter, we will often be concerned with the construction of closed copies of previously characterized spaces in the space which is yet to be characterized. The usefulness of this idea has already been demonstrated in the second remark following theorem 3.2.4, where the existence of Cantor sets in uncountable σ-compact spaces was used to obtain the characterization of $\mathbf{Q} \times \mathbf{C}$. In fact, Cantor sets exist in arbitrary uncountable analytic spaces; this is a theorem of Suslin, see Luzin [33]. Another theorem, which better exemplifies the type of result that we are looking for, was proved by Hurewicz [23]: he showed that, if a Borel subset A of a compact space X is not complete, then X contains a closed subset K such that $K \cap A \approx \mathbf{Q}$, and $K \backslash A \approx \mathbf{P}$; clearly, the condition that A is not complete is also necessary, since $\mathbf{Q}$ is not complete.
In [49], Saint-Raymond proved a theorem of a similar nature: he showed that a Borel subset A of a compact space X is not the union of a σ-compact subspace and a complete subspace if and only if A contains a closed copy of $\mathbf{Q} \times \mathbf{P}$; this is also equivalent to $X \backslash A$ not being the intersection of a complete subspace and a σ-compact subspace, i.e. not being strongly σ-complete (see the proof of lemma 3.1.3).
In van Engelen and van Mill [14], the above results were improved, and some more Hurewicz-type theorems were established (here, $\mathbf{S}$ and $\mathbf{T}$ are the homogeneous complements of $\mathbf{Q} \times \mathbf{P}$ in $\mathbf{C}$, see section 3.1 and the introduction to this monograph):

3.3.1 THEOREM (van Engelen and van Mill [14]): *Let* X *be a compact space, and let* A *be a Borel subset of* X.
(a) A *is not complete if and only if* X *contains a Cantor set* K *such that* $K \cap A \approx \mathbf{Q}$ and $K \backslash A \approx \mathbf{P}$;

(b) A *is not complete* $\cup$ *countable if and only if* X *contains a Cantor set* K *such that* $K \cap A \approx \boldsymbol{Q}\times\boldsymbol{C}$ *and* $K \setminus A \approx \boldsymbol{P}$*;*
(c) A *is not strongly* σ*-complete if and only if* X *contains a Cantor set* K *such that* $K \cap A \approx \boldsymbol{T}$ *and* $K \setminus A \approx \boldsymbol{Q}\times\boldsymbol{P}$*;*
(d) A *is not strongly* σ*-complete* $\cup$ *countable if and only if* X *contains a Cantor set* K *such that* $K \cap A \approx \boldsymbol{S}$ *and* $K \setminus A \approx \boldsymbol{Q}\times\boldsymbol{P}$.

Parts (a) and (b) will be proved in this section, (c) and (d) in section 3.4.

This theorem, and the other Hurewicz-type theorems that we will use in this chapter, will all be deduced from lemmas 4 and 5 of this section. First, however, we state the following definition and lemma:

3.3.2 DEFINITION: *Let* $\mathcal{P}$ *be a topological property.*
(a) *A space* X *is strongly* σ-$\mathcal{P}$ *if* $X = \bigcup_{i=0}^{\infty} X_i$, *where each* X_i *is a closed subspace of* X *which has property* $\mathcal{P}$.
(b) $\mathcal{P}$ *is strongly* σ*-additive if a space is* $\mathcal{P}$ *whenever it is strongly* σ-$\mathcal{P}$.

3.3.3 LEMMA: *Let* A *be a non-empty, compact, nowhere dense subset of a space* X, *and let* $(\varepsilon_n)_{n<\omega}$ *be a given sequence of positive real numbers. Then there exists a countable discrete subset* $D = \{d_n : n < \omega\}$ *of* $X \setminus A$ *such that* $\overline{D} = D \cup A$ *and* $d(d_n, A) < \varepsilon_n$ *for each* $n < \omega$.

Proof: For each $i < \omega$, let $\mathcal{D}_i = \{D(i,j) : j < n_i\}$ be a collection of open subsets of X of diameter less than $1/(i+1)$ such that $A \subset \bigcup\mathcal{D}_i$, and each $A \cap D(i,j) \neq \emptyset$, say $p(i,j) \in A \cap D(i,j)$. For each $n < \omega$, there exists a unique $i < \omega$ such that $n = (\Sigma_{k<i} n_k) + j$ for some $j < n_i$, and we choose $d_n \in (B(p(i,j),\varepsilon_n) \cap D(i,j)) \setminus A$. Then $D = \{d_n : n < \omega\}$ is as required. $\square$

We now come to our first Hurewicz-type result; the statement is rather awkward since we want to apply the lemma in a variety of situations. The technique of proof is inspired by that of Saint-Raymond [49]; see also van Douwen [9], and Topsøe and Hoffman-Jørgensen [58], where similar techniques are used.

3.3.4 LEMMA: *Let* $\mathcal{P}$ *be a topological property which is closed-hereditary and strongly* σ*-additive. Let* Z *be a non-empty strongly homogeneous space, and suppose that if* B *is a Borel subset of a compact space* Y *which is not* $\mathcal{P}$, *then* Y *contains a compact zero-dimensional subset* K *such that*

$K = \overline{K \cap B}$, $K \cap B \approx Z$, *and* $K \setminus B$ *is contained in a σ-compact subset* G *of* Y *with the property that* $G \cap B$ *is* $\mathcal{P}$. *Then if* A *is a Borel subset of a compact space* X *which is not* $\mathcal{P} \cup$ *complete, then* X *contains a Cantor set* K *such that* $\overline{K \cap A} = K$ *and* $K \cap A \approx \mathbf{Q} \times Z$.

Proof: Since A is a Borel subset of X there exists a continuous surjection $\phi \colon \mathbf{P} \to X \setminus A$. Put

$$W = \{x \in \mathbf{P}\colon \text{there exist a neighborhood } V_x \text{ of } x \text{ in } \mathbf{P}, \text{ and a } \sigma\text{-compact subset } E_x \text{ of } X, \text{ such that } \phi[V_x] \subset E_x, \text{ and } E_x \cap A \text{ is } \mathcal{P}\}.$$

Then W is open in $\mathbf{P}$, so there exist countably many open V_i in $\mathbf{P}$, and σ-compact E_i in X such that $W = \bigcup_{i=0}^{\infty} V_i$, $\phi[V_i] \subset E_i$, and $E_i \cap A$ is $\mathcal{P}$. Since E_i is σ-compact, we can write $E_i = \bigcup_{j=0}^{\infty} K_j^i$, where each K_j^i is compact; then $E = \bigcup_{i=0}^{\infty}\bigcup_{j=0}^{\infty} K_j^i$ is σ-compact, and since $K_j^i \cap A$ is $\mathcal{P}$, being a closed subspace of $E_i \cap A$, $E \cap A = \bigcup_{i=0}^{\infty}\bigcup_{j=0}^{\infty}(K_j^i \cap A)$ is strongly σ-$\mathcal{P}$, hence $\mathcal{P}$. Now suppose that $X \setminus A \subset E$; then $A = (E \cap A) \cup (X \setminus E)$ is $\mathcal{P} \cup$ complete, a contradiction. Hence,

$$F = \mathbf{P} \setminus \phi^{-1}[E]$$

is non-empty, and complete.

CLAIM 1: *If* U *is non-empty and open in* F, $G \subset X$ *is σ-compact, and* $\phi[U] \subset G$, *then* $G \cap A$ *is not* $\mathcal{P}$.

Indeed, suppose $G = \bigcup_{i=0}^{\infty} G_i$, where G_i is compact, and $G \cap A$ is $\mathcal{P}$; then also $G_i \cap A$ is $\mathcal{P}$, and hence $(G \cup E) \cap A = \bigcup_{i=0}^{\infty}(G_i \cap A) \cup \bigcup_{i=0}^{\infty}\bigcup_{j=0}^{\infty}(K_j^i \cap A)$ is strongly σ-$\mathcal{P}$, hence $\mathcal{P}$. Now let U' be open in $\mathbf{P}$ such that $U' \cap F = U$; then $\phi[U'] = \phi[U] \cup \phi[U' \setminus U] \subset G \cup E$, and since $U' \not\subset W$, $(G \cup E) \cap A$ is not $\mathcal{P}$, a contradiction.

From this claim, it follows that if U is any non-empty open subset of F, then $\overline{\phi[U]} \cap A$ is not $\mathcal{P}$, and hence $\overline{\phi[U]}$ contains a compact zero-dimensional subspace K such that $K = \overline{K \cap A}$, $K \cap A \approx Z$, and $K \setminus A$ is contained in a σ-compact subset G of $\overline{\phi[U]}$ (and hence of X) with the property that $G \cap A$ is $\mathcal{P}$.

We will now construct compact zero-dimensional subsets K_s of X, open subsets U_s of X, open subsets W_s of F, and finite collections $\mathcal{V}_i$ of open subsets of X, for each $s \in M$ and each $i < \omega$, such that the following hold:

(1) $K_s = \overline{K_s \cap A} \subset \overline{\phi[W_s]} \subset U_s$;

(2) for each $n < \omega$: $\overline{U}_{s^\frown n} \cap K_s = \emptyset$;

(3) for each $n,m < \omega$: $\overline{U}_{s^\frown n} \cap \overline{U}_{s^\frown m} = \emptyset$ if $n \neq m$;

(4) for each $n < \omega$: $Cl_F(\overline{W}_{s^\frown n}) \subset W_s$;

(5) for each $n < \omega$: $\overline{U}_{s^\frown n} \subset U_s$;

(6) $diam(W_s) \leq 2^{-|s|}$ (here, the diameter is taken with respect to a complete metric for F);

(7) $diam(U_s) \leq 2^{-\nu(s)}$;

(8) for each $n < \omega$: $d(K_s, K_{s^\frown n}) \leq 2^{1-\nu(s,n)}$;

(9) $K_s \cap A \approx Z$, and $K_s \backslash A \subset G_s$ for some σ-compact subset G_s of X such that $G_s \cap A$ is P;

(10) K_s is nowhere dense in $K_s \cup \bigcup_{n=0}^{\infty} K_{s^\frown n}$;

(11) $B_i = \bigcup\{K_s\colon |s| \leq i\}$ is compact zero-dimensional, and $B_i \subset \bigcup \mathcal{V}_i$;

(12) $\bigcup \mathcal{V}_{i+1} \subset \bigcup \mathcal{V}_i$;

(13) for each $V \in \mathcal{V}_i$: $diam(V) < 1/(i+1)$;

(14) $\{\overline{V}\colon V \in \mathcal{V}_i\}$ is pairwise disjoint.

We proceed by induction on $|s|$ and i. First, we put $W_\emptyset = F$, $U_\emptyset = X$. Then $\overline{\phi[W_\emptyset]}$ contains a compact zero-dimensional subspace $K_\emptyset = \overline{K_\emptyset \cap A}$ such that $K_\emptyset \cap A \approx Z$ and $K_\emptyset \backslash A \subset G_\emptyset$ for some σ-compact subset $G_\emptyset$ of X such that $G_\emptyset \cap A$ is P. Put $\mathcal{V}_0 = \{X\}$. Then (1), (9), (11), and (14) are satisfied, and so are (6), (7), and (13), since all metrics are assumed to be bounded by 1 (see 1.5).

Next, suppose that K_s, U_s, W_s, and $\mathcal{V}_i$ have been constructed for $|s| \leq k$ and $i \leq k$, in accordance with conditions (1) - (14). Fix $s \in M$ with $|s| = k$.

CLAIM 2: K_s *is nowhere dense in* $K_s \cup \phi[W_s]$.

We distinguish two cases.

<u>Case 1</u>: Let $y \in K_s \cap \phi[W_s]$, say $y = \phi(x)$, with $x \in W_s$, and let U be an open neighborhood of y in $K_s \cup \phi[W_s]$. By continuity of $\phi\colon W_s \to \phi[W_s]$, there exists an open neighborhood V of x in W_s such that $\phi[V] \subset U \cap \phi[W_s]$. Suppose that $U \cap \phi[W_s] \subset K_s$. Then $\phi[V] \subset K_s \backslash A \subset G_s$, and G_s is a σ-compact subset of X such that $G_s \cap A$ is P (apply (9)). Since V is a non-empty open subset of F, this contradicts claim 1, and hence $U \backslash K_s \neq \emptyset$.

<u>Case 2</u>: Let $y \in K_s \backslash \phi[W_s]$, and let U be an open neighborhood of y in $K_s \cup \phi[W_s]$. Since $K_s \subset \overline{\phi[W_s]}$ by (1), $U \cap \phi[W_s] \neq \emptyset$, say $z \in U \cap \phi[W_s]$. If $z \notin K_s$, then we are done, and if $z \in K_s$, then $z \in K_s \cap \phi[W_s]$, and case 1 applies. This proves the claim.

By this claim, we can apply lemma 3.3.3 to obtain a countable discrete subset $D_s = \{y_{s\hat{}n}: n < \omega\}$ of $\phi[W_s]\backslash K_s$ such that $\overline{D}_s = D_s \cup K_s$, and $d(y_{s\hat{}n}, K_s) \le 2^{-\nu(s\hat{}n)}$ for each $n < \omega$. Of course, we may assume that $D_s \subset \cup V_k$. Let $U_{s\hat{}n}$ be a neighborhood of $y_{s\hat{}n}$ in X such that $\overline{U}_{s\hat{}n} \cap K_s = \emptyset$, $\overline{U}_{s\hat{}n} \cap \overline{U}_{s\hat{}m} = \emptyset$ if $n \ne m$, $\mathrm{diam}(U_{s\hat{}n}) \le 2^{-\nu(s\hat{}n)}$, and $\overline{U}_{s\hat{}n} \subset U_s \cap \cup V_k$ for each $n,m < \omega$. Since $y_{s\hat{}n} \in \phi[W_s]$, $y_{s\hat{}n} = \phi(x_{s\hat{}n})$ for some $x_{s\hat{}n} \in W_s$; hence, there is an open neighborhood $W_{s\hat{}n}$ of $x_{s\hat{}n}$ in F such that $\mathrm{Cl}_F(W_{s\hat{}n}) \subset W_s$, $\mathrm{diam}(W_{s\hat{}n}) \le 2^{-|s|-1}$, and $\overline{\phi[W_{s\hat{}n}]} \subset U_{s\hat{}n}$. Then $\overline{\phi[W_{s\hat{}n}]}$ contains a compact zero-dimensional subset $K_{s\hat{}n}$ such that $K_{s\hat{}n} = \overline{K_{s\hat{}n} \cap A}$, $K_{s\hat{}n} \cap A \approx Z$, and $K_{s\hat{}n}\backslash A \subset G_{s\hat{}n}$ for some σ-compact subset $G_{s\hat{}n}$ of X such that $G_{s\hat{}n} \cap A$ is P. Then (1) - (7) and (9) are satisfied. For (8), note that

$$d(K_s,K_{s\hat{}n}) \le d(K_s,U_{s\hat{}n}) + \mathrm{diam}(U_{s\hat{}n}) \le d(K_s,y_{s\hat{}n}) + 2^{-\nu(s\hat{}n)} \le 2^{1-\nu(s\hat{}n)}.$$

To prove (10), take $x \in K_s$, and $\varepsilon > 0$. Choose $n < \omega$ so large that $2^{-\nu(s\hat{}n)} < \frac{1}{2}\varepsilon$. Since $\overline{D}_s = D_s \cup K_s$, $y_{s\hat{}m} \in B(x,\frac{1}{2}\varepsilon)$ for some $m > n$; and since $y_{s\hat{}m} \in U_{s\hat{}m}$ and $\mathrm{diam}(U_{s\hat{}m}) \le 2^{-\nu(s\hat{}m)} \le 2^{-\nu(s\hat{}n)} < \frac{1}{2}\varepsilon$, we have $U_{s\hat{}m} \subset B(x,\varepsilon)$, so $K_{s\hat{}m} \subset B(x,\varepsilon)$. By (1) and (2), $K_{s\hat{}m} \cap K_s = \emptyset$, so $B(x,\varepsilon) \cap (K_s \cup \bigcup_{n=0}^{\infty} K_{s\hat{}n})\backslash K_s \supset K_{s\hat{}m} \ne \emptyset$.

Since $B_{k+1} = \cup\{K_s: |s| \le k+1\}$ is a countable union of closed zero-dimensional subspaces, it is zero- dimensional (see e.g. Engelking [15] 7.2.1); so in order to establish (11), we must show that B_{k+1} is closed in X. For each $\varepsilon > 0$, put

$$B_k^\varepsilon = \{x \in X: d(x,B_k) \le \varepsilon\},$$

and

$$M_k^\varepsilon = \{s \in M: |s| = k+1, K_s \not\subset B_k^\varepsilon\};$$

then each B_k^ε is compact, and M_k^ε is finite by (1), (7), and (8). Since B_k is compact by the inductive hypothesis, we have

$$B_k = \bigcap_{\varepsilon>0} B_k^\varepsilon,$$

and

$$B_{k+1} = \bigcap_{\varepsilon>0}(B_k^\varepsilon \cup \cup\{K_s: s \in M_k^\varepsilon\})$$

is compact, being the intersection of compacta.

To complete the induction, let V be a covering of B_{k+1} by disjoint clopen subsets of B_{k+1} of diameter less than $1/(k+2)$. By normality, there exist open subsets V' of X (for $V \in V$), which can be taken to have diameter less than $1/(k+2)$, such that $V' \cap B_{k+1} = V$ and $V' = \{V': V \in V\}$

is pairwise disjoint; also, since $B_{k+1} \subset \bigcup\mathcal{V}_k$, we can assume that $\bigcup\mathcal{V}' \subset \bigcup\mathcal{V}_k$. Again by normality, we can shrink $\mathcal{V}'$ to obtain a covering $\mathcal{V}_{k+1}$ of B_{k+1} by open subsets of X, satisfying (11) - (14).
Now put

$$K = (\bigcup_{i=0}^{\infty} B_i)^-;$$

clearly, K is compact, and by (10), no point of B_i is isolated in B_{i+1}, so K contains no isolated points. Also, from (11) - (14) it follows that, for each $i < \omega$, $K \subset \bigcup\{\overline{V}: V \in \mathcal{V}_i\}$, which is a pairwise disjoint finite closed covering of K by sets of diameter less than $1/(i+1)$; hence, $\{\overline{V} \cap K: V \in \mathcal{V}_i, i < \omega\}$ is a clopen basis for K, so K is zero-dimensional. Thus, $K \approx \mathcal{C}$ by theorem 2.1.1.
We claim that $K \cap A = \bigcup_{i=0}^{\infty}(B_i \cap A)$. Suppose to the contrary that $x \in (K \cap A)\setminus(\bigcup_{i=0}^{\infty} B_i)$, and fix $i < \omega$. Since $x \notin B_i$, also $x \notin B_i^{\varepsilon}$ for some $\varepsilon > 0$. From (1) and (4) it follows that $\bigcup_{j=0}^{\infty} B_j \subset B_i \cup \bigcup_{|s|=i} \overline{\phi[W_s]}$, and from (1), (7), and (8) that $\overline{\phi[W_s]} \subset B_i^{\varepsilon}$ for all but finitely many $s \in \mathcal{M}$ with $|s| = i$. Hence for some finite $\mathcal{M}_0 \subset \{s \in \mathcal{M}: |s| = i\}$, we have

$$K \subset B_i^{\varepsilon} \cup \bigcup_{s\in\mathcal{M}_0} \overline{\phi[W_s]}.$$

Then $x \in \overline{\phi[W_s]}$ for some $s \in \mathcal{M}_0$, and this s is unique with $|s| = i$ by (1), (3), and (5) (or trivially, if $i = 0$). So by (4), there exists $\sigma \in \omega^{\omega}$ such that $x \in \bigcap_{s<\sigma} \overline{\phi[W_s]}$, which is a one-point set by (1) and (7). Also, $\bigcap_{s<\sigma} \overline{W}_s = \bigcap_{s<\sigma} W_s$ is a one-point set by (6), and by completeness of the metric on F. Hence, if $z \in \bigcap_{s<\sigma} W_s$, then $\phi(z) = x$, so $x \in \phi[\mathcal{P}] = X\setminus A$, a contradiction since $x \in K \cap A$.
Thus, $K \cap A = \bigcup_{i=0}^{\infty}(B_i \cap A) = \bigcup_{s\in\mathcal{M}}(K_s \cap A)$. Clearly, $K_s \cap A$ is closed in $K \cap A$, and by (1) and (10), $K_s \cap A$ is nowhere dense in $K \cap A$ for each $s \in \mathcal{M}$. So by (9) and theorem 3.2.4, $K \cap A \approx \mathcal{Q} \times Z$. Finally, using (1),

$$K \supset \overline{K \cap A} = (\bigcup_{s\in\mathcal{M}}\overline{(K_s \cap A)})^- = (\bigcup_{s\in\mathcal{M}} K_s)^- = K,$$

so $K = \overline{K \cap A}$. □

Before stating our second Hurewicz-type result, we show how to deduce theorem 3.3.1(a) and (b) from the above lemma.

Proof of theorem 3.3.1(a): Let a space have the property $\mathcal{P}$ if and only if it is empty, and let Z be a one-point space. Then clearly $\mathcal{P}$ is closed-hereditary and strongly σ-additive, and if Y is compact and B is a non-empty subset of Y, say $y \in B$, then $K = \{y\}$ and $G = \emptyset$ satisfy the requirements. So if A in X is not complete, then it is not $P \cup$ complete, so by the lemma, there is a closed copy Q of $\boldsymbol{Q}$ in A such that $K = \overline{Q} \approx \boldsymbol{C}$; then $K\backslash A = K\backslash Q \approx \boldsymbol{P}$, using theorem 2.3.1. □

Proof of theorem 3.3.1(b): Let a space have the property P if and only if it is countable, and let $Z = \boldsymbol{C}$. Again, P is as required, and if B is an uncountable Borel subset of the compact space Y, then B contains a Cantor set K by Suslin's theorem. So if A in X is not countable $\cup$ complete, then it is not $P \cup$ complete, so there is a closed copy H of $\boldsymbol{Q}\times\boldsymbol{C}$ in A such that $K = \overline{H} \approx \boldsymbol{C}$; again by theorem 2.3.1, $K\backslash A = K\backslash H \approx \boldsymbol{P}$. □

We only outline the proof of the following lemma, since it is entirely similar to that of lemma 3.3.4.

3.3.5 LEMMA: *Let* P *be as in lemma* 3.3.4, *and suppose that for each* $i < \omega$, Y_i *and* Z_i *are non-compact strongly homogeneous spaces. Furthermore, assume that if* B *is a Borel subset of a compact space* Y *which is not* P, *then* Y *contains, for each* $i < \omega$, *a Cantor set* K_i *such that* $K_i = \overline{K_i\backslash B}$, *and*

(I) $K_i\backslash B \approx Z_i$;
(II) $K_i \cap B \approx Y_i$ *is* P.

Then if A *is a Borel subset of a compact space* X *which is not* P, *then* X *contains Cantor sets* B_i, *for* $i < \omega$, *such that* $B_i = \overline{B_i\backslash A}$, $K = (\bigcup_{i=0}^{\infty} B_i)^{-} \approx \boldsymbol{C}$, *and for each* $i < \omega$,

(i) B_i *is nowhere dense in* K;
(ii) $K\backslash A = \bigcup_{i=0}^{\infty}(B_i\backslash A)$;
(iii) $B_0\backslash A \approx Z_0$, *and* $(B_{i+1}\backslash B_i)\backslash A \approx Z_{i+1}$;
(iv) $B_0 \cap A \approx Y_0$, *and* $(B_{i+1}\backslash B_i) \cap A \approx Y_{i+1}$.

Proof: Since A is a Borel subset of X, there exists a continuous surjection $\phi: \boldsymbol{P} \to A$. Put

$W = \{x \in \boldsymbol{P}$: there exist a neighborhood V_x of x in $\boldsymbol{P}$, and a σ-compact subset E_x of X, such that $\phi[V_x] \subset E_x$, and $E_x \cap A$ is $P\}$.

Then W is open in $\boldsymbol{P}$, so there exist countably many open V_i in $\boldsymbol{P}$, and σ-compact E_i in X, such that $W = \bigcup_{i=0}^{\infty} V_i$, $\phi[V_i] \subset E_i$, and $E_i \cap A$ is P. Put $E = \bigcup_{i=0}^{\infty} E_i$, then it follows as in the proof of lemma 3.3.4 that $E \cap A$

is P. Now suppose that $A \subset E$; then $A = E \cap A$ is P, a contradiction. Hence $F = \boldsymbol{P}\backslash\phi^{-1}[E]$ is non-empty, and complete.
As in lemma 3.3.4, we can show that if U is a non-empty open subset of F, then $\overline{\phi[U]} \cap A$ is not P, and hence that $\overline{\phi[U]}$ contains Cantor sets K_i satisfying (I) and (II). Construct K_s, U_s, W_s, and V_i as above, satisfying (1) - (8), (10) - (14), and

$$(9') \quad K_s\backslash A \approx Z_{|s|}, \text{ and } K_s \cap A \approx Y_{|s|} \text{ is } P.$$

Since each K_s is a Cantor set, B_i is also a Cantor set, and from (10) it easily follows that B_i is nowhere dense in K. Since this time, ϕ is a mapping onto A, we find that $K\backslash(\cup_{i=0}^{\infty} B_i) \subset \phi[\boldsymbol{P}] = A$, so that $K\backslash A = \cup_{i=0}^{\infty}(B_i\backslash A)$.
Since $B_{i+1}\backslash B_i = \cup\{K_s\colon |s| = i+1\}$, and since $K_s \subset U_s$ and $U_s \cap U_t = \emptyset$ if $s,t \in M$, $|s| = |t|$, $s \neq t$ (use (3) and (5)), we have $(B_{i+1}\backslash B_i)\backslash A = \cup\{K_s\backslash A\colon |s| = i+1\} \approx \omega \times Z_{i+1} \approx Z_{i+1}$ since Z_{i+1} is non-compact, strongly homogeneous and zero-dimensional (see 1.9.2). That $(B_{i+1}\backslash B_i) \cap A \approx Y_{i+1}$ follows in exactly the same way. □

3.4 The small Borel classes $D_n(\Sigma_2^0)$ for $n \in \boldsymbol{N}$

Recall that if $n < \omega$, then a space is S_n if it is the union of n strongly σ-complete subspaces. In this section, we describe and characterize certain homogeneous Borel sets in the Cantor set that are S_n for some $n < \omega$; in section 3.6 we will see that they are the only ones, apart from those of Chapter 2.
In section 1 of this chapter, we defined topological properties describing subclasses of Δ_3^0:

X is P_{4k} if and only if X is $S_k \cup$ complete;
X is P_{4k+1} if and only if X is S_{k+1};
X is P_{4k+2}^1 if and only if X is $S_k \cup$ complete $\cup$ countable;
X is P_{4k+3}^1 if and only if X is $S_{k+1} \cup$ countable;
X is P_{4k+2}^2 if and only if X is $S_k \cup$ complete $\cup$ σ-compact;
X is P_{4k+3}^2 if and only if X is $S_{k+1} \cup$ σ-compact.

Note the following equivalences:

$P_{4(k+1)} \equiv P_{4k+1} \cup$ complete;
$P_{4k+2}^1 \equiv P_{4k} \cup$ countable $\equiv P_{4(k-1)+3}^1 \cup$ complete;
$P_{4k+3}^1 \equiv P_{4k+1} \cup$ countable;

$P^2_{4k+2} \equiv P_{4k} \cup \sigma\text{-compact} \equiv P^2_{4(k-1)+3} \cup \text{complete};$
$P^2_{4k+3} \equiv P_{4k+1} \cup \sigma\text{-compact}.$

3.4.1 LEMMA: (a) *For each* $n < \omega$, *and each* $i \in \{1,2\}$, $P^{(i)}_n$ *is closed-hereditary.*
(b) *For each* $n < \omega$, $P^{(1)}_n$ *is hereditary with respect to* G_δ*-subsets.*

Proof: (a) It suffices to show that a closed subset of a strongly σ-complete space is strongly σ-complete. But if $X = \bigcup_{i=0}^{\infty} X_i$, with X_i closed in X and complete, then for a closed $A \subset X$, we have $A = \bigcup_{i=0}^{\infty}(X_i \cap A)$, $X_i \cap A$ is closed in A, and also in X_i, so $X_i \cap A$ is complete.
The proof of (b) is similar; P^2_{4k+2} and P^2_{4k+3} are not G_δ-hereditary since σ-compactness is not. Now □

3.4.2 LEMMA: *Let* X *be a topological space.*
(a) *For each* $k < \omega$, X *is* P_{4k+1} *if and only if* X *is strongly* σ-P_{4k}.
(b) *For each* $k < \omega$, *and each* $i \in \{1,2\}$, X *is* P^i_{4k+3} *if and only if* X *is strongly* σ-P^i_{4k+2}.

Proof: The proof is similar to that of lemma 3.1.5; consider X as a subset of the compact space Y.
(a) If X is P_{4k+1}, then by lemma 3.1.4, there exist σ-compact subsets $\{A_m: m < 2(k+1)\}$ of Y such that $X = D_{2(k+1)}(\langle A_m: m < 2(k+1)\rangle)$. Let $A_{2k+1} = \bigcup_{i=0}^{\infty} K_i$, where each K_i is compact; then $K_i \cap X$ is closed in X, and $X = X \cap A_{2k+1} = \bigcup_{i=0}^{\infty}(K_i \cap X)$. Now

$$K_i \cap X = D_{2k}(\langle A_m \cap K_i: m < 2k\rangle) \cup K_i \backslash A_{2k};$$

since $D_{2k}(\langle A_m \cap K_i: m < 2k\rangle)$ is S_k (lemma 3.1.4), and $K_i \backslash A_{2k}$ is complete, being a G_δ in K_i, we have that $K_i \cap X$ is P_{4k}, so X is strongly σ-P_{4k}.
For the converse, assume that X is strongly σ-P_{4k}; then X is certainly strongly σ-P_{4k+1}, so by lemma 3.1.4, we can write $X = \bigcup_{i=0}^{\infty} X_i$, where $X_i = D_{2(k+1)}(\langle A^i_m: m < 2(k+1)\rangle)$, each A^i_m is σ-compact, and X_i is closed in X, say $X_i = K_i \cap X$, with K_i compact in Y. Put $L_i = K_i \backslash (\bigcup_{j<i} K_j)$, and for $m < 2(k+1)$,

$$A_m = \bigcup_{i=0}^{\infty}(L_i \cap A^i_m).$$

As in the proof of theorem 3.1.5, we obtain that

$$X = \bigcup_{i=0}^{\infty}(L_i \cap X_i) = D_{2(k+1)}(\langle A_m: m < 2(k+1)\rangle),$$

so A is S_{k+1}, i.e. A is P_{4k+1}.

(b) Let X be P^i_{4k+3}, say $X = D_{2(k+1)}(\langle A_m : m < 2(k+1)\rangle) \cup B$, where B is countable or σ-compact, depending on whether $i = 1$ or $i = 2$. Let $A_{2k+1} \cup B = \bigcup_{j=0}^{\infty} K_j$, where K_j is compact. Then

$$K_j \cap X = D_{2k}(\langle A_m \cap K_j : m < 2k\rangle) \cup K_j \setminus A_{2k} \cup (K_j \cap B)$$

is $S_k \cup$ complete $\cup$ countable (resp. σ-compact), i.e. $K_j \cap X$ is P^i_{4k+2}; so $X = \bigcup_{j=0}^{\infty}(K_j \cap x)$ is strongly σ-P^i_{4k+2}.
Conversely, if X is strongly σ-P^i_{4k+2}, then X is strongly σ-P^i_{4k+3}, so we can write $X = \bigcup_{j=0}^{\infty} X_j$, where $X_j = D_{2(k+1)}(\langle A^j_m : m < 2(k+1)\rangle) \cup B_j$ is closed in X, B_j is countable or σ-compact. Let K_j be compact such that $K_j \cap X = X_j$, and put $L_j = K_j \setminus (\bigcup_{n<j} K_n)$. Then if

$$A_m = \bigcup_{j=0}^{\infty}(L_j \cap A^j_m)$$

for $m < 2(k+1)$, we have

$$X = D_{2(k+1)}(\langle A_m : m < 2(k+1)\rangle) \cup \bigcup_{j=0}^{\infty} B_j$$

is P^i_{4k+3}. □

3.4.3 COROLLARY: *For odd* $n < \omega$, *and each* $i \in \{1,2\}$, $P^{(i)}_n$ *is a strongly* σ-*additive property.* □

3.4.4 LEMMA: *Let* X *be compact, and let* A *be a subset of* X. *Then for each* $k < \omega$,

(a) A *is* P_{4k} *if and only if* $X \setminus A$ *is* $P^2_{4(k-1)+3}$.

(b) A *is* P_{4k+1} *if and only if* $X \setminus A$ *is* P^2_{4k+2}.

Proof:(a) This is trivial if $k = 0$, so suppose that $k > 0$. If A is P_{4k} is $S_k \cup$ complete, then $A = D_{2k}(\langle A_i : i < 2k\rangle) \cup B$, where each A_i is σ-compact and B is complete, i.e. $X \setminus B$ is σ-compact. Hence

$$X \setminus A = (A_0 \cap X \setminus B) \cup D_{2(k-1)}(\langle A_{i+1} \cap X \setminus B : i < 2(k-1)\rangle) \cup (X \setminus B) \setminus A_{2k-1}$$

is σ-compact $\cup S_{k-1} \cup S_1$ is $P^2_{4(k-1)+3}$.
Conversely, if $X \setminus A$ is $P^2_{4(k-1)+3}$ is $S_k \cup \sigma$-compact, then $X \setminus A$ can be written as A above, but with B being σ-compact; hence

$$A = (A_0 \setminus B) \cup D_{2(k-1)}(\langle A_{i+1} \cup B : i < 2(k-1)\rangle) \cup X \setminus (B \cup A_{2k-1})$$

is $S_1 \cup S_{k-1} \cup$ complete is P_{4k}.
(b) The case $k = 0$ follows from the observation that A is the difference of two σ-compacta if and only if it is the intersection of a σ-compact space and a complete space. So suppose that $k > 0$. If A is P_{4k+1} is S_{k+1},

then $A = D_{2(k+1)}(\langle A_i : i < 2(k+1)\rangle)$, so

$$X\backslash A = A_0 \cup D_{2k}(\langle A_{i+1} : i < 2k\rangle) \cup X\backslash A_{2k+1}$$

is σ-compact ∪ S_k ∪ complete is P^2_{4k+2}.
Conversely, if $X\backslash A$ is P^2_{4k+2} is S_k ∪ complete ∪ σ-compact, then $X\backslash A = D_{2k}(\langle A_i : i < 2k\rangle) \cup G \cup B$, where G is complete and B is σ-compact; hence

$$A = (A_0 \cap X\backslash G)\backslash B \cup D_{2(k-1)}(\langle (A_{i+1} \cup B) \cap X\backslash G : i < 2(k-1)\rangle) \cup (X\backslash G)\backslash(B \cup A_{2k-1})$$

is $S_1 \cup S_{k-1} \cup S_1$ is S_{k+1} is P_{4k+1}. □

3.4.5 LEMMA: *The following implications hold between the properties* $P^{(i)}_n$:

$$\begin{array}{ccccccc} P_{4k} & \to & P^1_{4k+2} & \to & P^2_{4k+2} & \to & P_{4(k+1)} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ P_{4k+1} & \to & P^1_{4k+3} & \to & P^2_{4k+3} & \to & P_{4(k+1)+1} \end{array}$$

□

Corresponding to each property $P^{(i)}_n$ we now define a class of spaces $X^{(i)}_n$, and show that, up to homeomorphism, each $X^{(i)}_n$ contains exactly one element.

3.4.6 DEFINITION: (a) $X^1_{-1} = \{\mathbf{Q}\}$; $X^2_{-1} = \{\mathbf{Q} \times \mathbf{C}\}$; $X_0 = \{\mathbf{P}\}$.
(b) *Let* X *be zero-dimensional; then for each* $k < \omega$,
$X \in X_{4(k+1)}$ *if and only if* X *is* $P_{4(k+1)}$, *nowhere* P^2_{4k+2}, *nowhere* P_{4k+1};
$X \in X_{4k+1}$ *if and only if* X *is* P_{4k+1}, *nowhere* P_{4k}, *nowhere* $P^2_{4(k-1)+3}$;
$X \in X^1_{4k+2}$ *if and only if* X *is* P^1_{4k+2}, *nowhere* P_{4k}, *nowhere* $P^2_{4(k-1)+3}$;
$X \in X^2_{4k+2}$ *if and only if* X *is* P^2_{4k+2}, *nowhere* P^1_{4k+2}, *nowhere* $P^2_{4(k-1)+3}$;
$X \in X^1_{4k+3}$ *if and only if* X *is* P^1_{4k+3}, *nowhere* P^1_{4k+2}, *nowhere* P_{4k+1};
$X \in X^2_{4k+3}$ *if and only if* X *is* P^2_{4k+3}, *nowhere* P^2_{4k+2}, *nowhere* P^1_{4k+3} .

Observe that if $X \in X^{(i)}_n$, then X is nowhere compact, so X can be densely embedded in $\mathbf{C}$, and any such embedding is also co-dense (corollary 2.1.4).

3.4.7 LEMMA: *Let* $n < \omega$, $i \in \{1,2\}$, *and* $X \in X^{(i)}_n$.
(a) *If* n *is odd, then* X *is first category.*
(b) *If* n *is even, then* X *contains a dense complete subset, and hence* X *is Baire.*

Proof: If n is odd, then X is strongly σ-$P^{(i)}_{n-1}$, and nowhere $P^{(i)}_{n-1}$, so X is first category. If $n = 0$, then X itself is complete. If $n = 4k > 0$, then X is P_{4k}, hence $X = A \cup B$, where A is S_k and B is complete; since X is nowhere $P_{4(k-1)+1}$, X is nowhere S_k, hence B is dense in X. The other cases are proved similarly. □

The following lemma gives alternative descriptions of the classes $X_n^{(i)}$ for even n, and of X^1_{4k+3}; in some cases, these are easier to use than those of definition 3.4.6.

3.4.8 LEMMA: *Let* X *be zero-dimensional. Then for each* $k < \omega$,

(a) $X \in X_{4k}$ *if and only if* X *is* P_{4k} *and nowhere* $P^2_{4(k-1)+3}$.

(b) $X \in X^1_{4k+2}$ *if and only if* X *is* P^1_{4k+2} *and nowhere* P_{4k+1}.

(c) $X \in X^2_{4k+2}$ *if and only if* X *is* P^2_{4k+2} *and nowhere* P^1_{4k+3}.

(d) $X \in X^1_{4k+3}$ *if and only if* X *is* P^1_{4k+3}, *nowhere* P^2_{4k+2}, *nowhere* P_{4k+1}.

Proof: We need only consider the "nowhere" properties. The "if" part follows immediately from lemma 3.4.5 ((a) from theorem 2.3.1 if $k = 0$).
The "only if" of (a) is trivial if $k = 0$, so suppose that $k > 0$ and $X \in X_{4k}$. Let U be a non-empty open subset of X, and suppose that U is $P^2_{4(k-1)+3}$; then by lemma 3.4.2, U is strongly σ-$P^2_{4(k-1)+2}$. But X is nowhere $P^2_{4(k-1)+2}$, hence U is not Baire, contradicting lemma 3.4.7.
For part (b), suppose that $X \in X^1_{4k+2}$, U is non-empty and open in X and P_{4k+1}. Again, since X is nowhere P_{4k}, U is not Baire, a contradiction. Part (c) is similar.
For (d), note that if U is a non-empty open subset of X which is P^2_{4k+2}, then $U \in X^2_{4k+2}$, so U is Baire, contradicting lemma 3.4.7. □

We now prove that each of the classes defined above contains at least one element.

3.4.9 LEMMA: *Let* $n < \omega$ *be even*, $i \in \{1,2\}$, *and* $X \in X_n^{(i)}$. *Then* $\mathbb{Q} \times X \in X_{n+1}^{(i)}$.

Proof: Since $\mathbb{Q} \times X = \bigcup_{q \in \mathbb{Q}}(\{q\} \times X)$, in all cases $\mathbb{Q} \times X$ is strongly σ-$P_n^{(i)}$, hence $P_{n+1}^{(i)}$ by lemma 3.4.2. Let $U \times V$ be a non-empty basic clopen subset of $\mathbb{Q} \times X$.
<u>Case 1</u>: $n = 4k$. Assume that $U \times V$ is $P^2_{4(k-1)+3}$, and non-empty. Then by lemma 3.4.1, V is $P^2_{4(k-1)+3}$. But $X \in X_{4k}$, and by lemma 3.4.8, X is nowhere $P^2_{4(k-1)+3}$, a contradiction. So $\mathbb{Q} \times X$ is nowhere $P^2_{4(k-1)+3}$, hence so is $U \times V$. Now suppose that $U \times V$ is P_{4k}; then by lemma 3.4.8, $U \times V \in X_{4k}$,

so $U \times V$ is Baire by lemma 3.4.7. But $U \times V = \bigcup_{q \in U}(\{q\} \times V)$ is first category, contradiction. So $\mathcal{Q} \times X$ is nowhere P_{4k}.

Case 2: $n = 4k+2$, $i=1$. If $U \times V$ is P_{4k+1}, and non-empty, then so is V; but $V \in X^1_{4k+2}$, a contradiction. If $U \times V$ is P^1_{4k+2}, then $U \times V$ is P^1_{4k+2} and nowhere P_{4k+1}, so $U \times V \in X^1_{4k+2}$ by lemma 3.4.8, so $U \times V$ is Baire by lemma 3.4.7, and as in case 1, we have a contradiction.

Case 3: $n = 4k+2$, $i=2$. As above. □

As was noted in section 3, to obtain elements of $X^{(i)}_n$ for even n, we have to consider complements in the Cantor set of spaces belonging to $X^{(j)}_m$ for some odd m.

3.4.10 LEMMA: *Let* X *be dense and co-dense in the Cantor set, and let* $k < \omega$. *Then*

(a) $X \in X_{4(k+1)}$ *if and only if* $\mathcal{C} \setminus X$ *is* P^2_{4k+3}, *nowhere* P^2_{4k+2}, *nowhere* P_{4k+1}.

(b) $X \in X_{4k+1}$ *if and only if* $\mathcal{C} \setminus X$ *is* P^2_{4k+2}, *nowhere* P_{4k}, *nowhere* $P^2_{4(k-1)+3}$.

Proof: (a) If $X \in X_{4(k+1)}$, then X is $P_{4(k+1)}$, so by lemma 3.4.4, $\mathcal{C} \setminus X$ is P^2_{4k+3}. If V is a non-empty clopen subset of $\mathcal{C}$, and $V \setminus X$ is P^2_{4k+2}, then $\emptyset \neq V \cap X$ is P_{4k+1} by lemma 3.4.4, contradicting the fact that X is nowhere P_{4k+1}. Similarly, $V \setminus X$ being P_{4k+1} would contradict X being nowhere P^2_{4k+2}. The converse of (a), and (b), are proved in the same way. □

3.4.11 COROLLARY: *Let* X *be a dense subset of the Cantor set. Then for each* $k < \omega$, *and* $i \in \{1,2\}$,

(a) *if* $X \in X^i_{4k+2}$, *then* $\mathcal{C} \setminus X \in X_{4k+1}$;

(b) *if* $X \in X^i_{4(k-1)+3}$, *then* $\mathcal{C} \setminus X \in X_{4k}$.

Proof: As noted before, X is co-dense in $\mathcal{C}$. If $X \in X^i_{4k+2}$, then X is P^2_{4k+2}, nowhere P_{4k}, and nowhere $P^2_{4(k-1)+3}$, using lemma 3.4.5. So by lemma 3.4.10, $\mathcal{C} \setminus X \in X_{4k+1}$. Part (b) is proved similarly. □

3.4.12 LEMMA: *For each* $n < \omega$ *and* $i \in \{1,2\}$, $X^{(i)}_n \neq \emptyset$.

Proof: This is clear for $n = 0$, so assume that $X^{(j)}_m \neq \emptyset$ for $m < n$ (≥ 1), $j \in \{1,2\}$. If n is odd, then $X^{(i)}_n \neq \emptyset$ by lemma 3.4.9. So suppose that n is even.

Case 1: $n = 4k$. By the inductive hypothesis, $X^1_{4(k-1)+3} \neq \emptyset$, say $X \in X^1_{4(k-1)+3}$. Embed X as a dense subset of $\mathcal{C}$; then by corollary 3.4.11, $\mathcal{C} \setminus X \in X_{4k}$.

Case 2: $n = 4k+2$, $i \in \{1,2\}$. Let $X_i \in \mathcal{X}^i_{4(k-1)+3}$, and embed X_i as a dense subset of the Cantor set; also embed $\mathbb{Q}$ densely in $\mathbb{C}$, and let

$$Y_i = (\mathbb{C} \times \mathbb{C}) \setminus (\mathbb{Q} \times (\mathbb{C} \setminus X_i)).$$

By corollary 3.4.11, $\mathbb{C} \setminus X_i \in \mathcal{X}_{4k}$; so by lemma 3.4.9, $\mathbb{Q} \times (\mathbb{C} \setminus X_i) \in \mathcal{X}_{4k+1}$. Hence by lemma 3.4.10, Y_i is $\mathcal{P}^2_{4k+2}$, nowhere $\mathcal{P}_{4k}$, and nowhere $\mathcal{P}^2_{4(k-1)+3}$.

Case 2a: $i = 1$. To show that $Y_1 \in \mathcal{X}^1_{4k+2}$, it suffices to show that Y_1 is $\mathcal{P}^1_{4k+2}$. But

$$Y_1 = (\mathbb{Q} \times X_1) \cup ((\mathbb{C} \setminus \mathbb{Q}) \times \mathbb{C});$$

and since X_1 is $\mathcal{P}^1_{4(k-1)+3}$, so is $\mathbb{Q} \times X_1$ by corollary 3.4.3. So Y_1 is $\mathcal{P}^1_{4(k-1)+3}$ $\cup$ complete is $\mathcal{P}^1_{4k+2}$.

Case 2b: $i = 2$. To show that $Y_2 \in \mathcal{X}^2_{4k+2}$, it suffices to show that Y_2 is nowhere $\mathcal{P}^1_{4k+2}$. So suppose that $U \times V$ is a non-empty basic clopen subset of $\mathbb{C} \times \mathbb{C}$, and suppose that $(U \times V) \cap Y_2$ is $\mathcal{P}^1_{4k+2}$. Since $\mathbb{Q}$ is dense in $\mathbb{C}$, $U \cap \mathbb{Q} \neq \emptyset$, say $x \in U \cap \mathbb{Q}$; then $(U \times V) \cap Y_2 \cap (\{x\} \times \mathbb{C}) = \{x\} \times (V \cap X_2)$, so $V \cap X_2$ is also $\mathcal{P}^1_{4k+2}$, i.e. $V \cap X_2 = A \cup B$, where A is $\mathcal{P}^1_{4(k-1)+3}$, and B is complete. Since X_2 is nowhere $\mathcal{P}^1_{4(k-1)+3}$, B is dense in $V \cap X_2$, so $V \cap X_2$ is Baire. This contradicts lemma 3.4.7. □

We now state and prove the main theorems of this section.

3.4.13 THEOREM: *Up to homeomorphism, each* $X^{(i)}_n$ *contains exactly one element, which is strongly homogeneous, whence homogeneous.*

3.4.14 THEOREM: *If* $X \in \mathcal{X}^i_{4k+2} \cup \mathcal{X}^i_{4k+3}$ *for some* $k < \omega$ *and* $i \in \{1,2\}$, *then* $\mathbb{C}$ *is homogeneous with respect to dense copies of* X.

3.4.15 THEOREM: *Let* A *be a Borel subset of a compact space* X. *Then for each* $k < \omega$,

(a) A *is not* $\mathcal{P}_{4k}$ *if and only if* X *contains a Cantor set* K *such that* $K \cap A \in \mathcal{X}^1_{4(k-1)+3}$ *and* $K \setminus A \in \mathcal{X}_{4k}$.

(b) A *is not* $\mathcal{P}^1_{4k+2}$ *if and only if* X *contains a Cantor set* K *such that* $K \cap A \in \mathcal{X}^2_{4(k-1)+3}$ *and* $K \setminus A \in \mathcal{X}_{4k}$.

(c) A *is not* $\mathcal{P}_{4k+1}$ *if and only if* X *contains a Cantor set* K *such that* $K \cap A \in \mathcal{X}^1_{4k+2}$ *and* $K \setminus A \in \mathcal{X}_{4k+1}$.

(d) A *is not* $\mathcal{P}^1_{4k+3}$ *if and only if* X *contains a Cantor set* K *such that* $K \cap A \in \mathcal{X}^2_{4k+2}$ *and* $K \setminus A \in \mathcal{X}_{4k+1}$.

Remark: Theorem 3.4.13 for X_1 is the characterization of $\boldsymbol{Q}\times\boldsymbol{P}$ due to van Mill [38]. The unique elements of X_2^1 and X_2^2 are the sets $\boldsymbol{T}$, resp. $\boldsymbol{S}$, mentioned in the introduction to this monograph; for these classes, theorem 3.4.13 is due to van Douwen [7] (see also van Engelen and van Mill [14], which includes theorem 3.4.14 for $k = 0$, $i = 1$), resp. van Mill [39] (which includes theorem 3.4.14 for $k = 0$, $i = 2$). Theorem 3.4.13 for $n = 3$, $i \in \{1,2\}$ is due to van Engelen and van Mill [14], and so is theorem 3.4.15 for $k = 0$ (see the introduction to section 3.3, and also theorem 3.3.1).

We will prove the above theorems by induction on n, using the following statements as inductive hypotheses:

(1) Up to homeomorphism, $X_n^{(i)}$ contains at most one element.

(2) If $n \in \{4k+2, 4k+3\}$ for some $k < \omega$, $i \in \{1,2\}$, and $X \in X_n^{(i)}$, then $\boldsymbol{C}$ is homogeneous with respect to dense copies of X.

(3) If $n = 4k$, then theorem 3.4.15(a) and (b) hold for this k;
If $n = 4k+2$, then theorem 3.4.15(c) and (d) hold for this k.

From (1), it follows that if $X \in X_n^{(i)}$, and V is a non-empty clopen subset of X, then $V \approx X$ since $V \in X_n^{(i)}$; thus X is strongly homogeneous. So if (1), (2), and (3) have been proved for all $n < \omega$, then theorem 3.4.13 follows from (1) and lemma 3.4.12, and theorems 3.4.14 and 3.4.15 are clear from (2) and (3).

Since (1) is clear for $n = 0$, and since (3) has been proved in section 3.3 for $n = 0$, we may assume that (1), (2), and (3) have been proved for $m < n$ (≥ 1). The proof of (1), (2), and (3) for $m = n$ will consist of a series of lemmas.

3.4.16 LEMMA: *If* n *is odd, then* (1) *holds.*

Proof: Let $X \in X_n^{(i)}$; we will show that $X \approx \boldsymbol{Q}\times Y$, where Y is the unique element of $X_{n-1}^{(i)}$.
Since X is $P_n^{(i)}$, we can write $X = \bigcup_{j=0}^{\infty} X_j$, where X_j is closed in X and $\mathcal{P}_{n-1}^{(i)}$ (lemma 3.4.2). Fix $j < \omega$, and let $\mathcal{D}$ be a disjoint covering of $X \backslash X_j$ by clopen subsets of X such that for each $D \in \mathcal{D}$, $\text{diam}(D) < d(D,X_j)$ (see the proof of lemma 3.2.2).
CLAIM: *If* U *is open in* X, *and* $U \cap X_j \neq \emptyset$, *then* $U \supset D$ *for some* $D \in \mathcal{D}$.
Indeed, let $x \in U \cap X_j$, and $B(x,\varepsilon) \subset U$. Since X is nowhere $\mathcal{P}_{n-1}^{(i)}$,

X_j is nowhere dense in X, so $B(x,\frac{1}{2}\varepsilon) \cap D \neq \emptyset$ for some $D \in \mathcal{D}$.
Then $\mathrm{diam}(D) < d(D,X_j) < \frac{1}{2}\varepsilon$ for this $D \in \mathcal{D}$, so $D \subset B(x,\varepsilon) \subset U$.

Case 1: $n = 4k+1$. Let $D \in \mathcal{D}$. Then D is not $\mathcal{P}^2_{4(k-1)+3}$ since X is nowhere $\mathcal{P}^2_{4(k-1)+3}$, so if we embed D in $\mathcal{C}$, then $\mathcal{C}\backslash D$ is not $\mathcal{P}_{4k}$ by lemma 3.4.4; since (3) holds for $m = 4k$, we can apply theorem 3.4.15(a) to obtain a closed subset E(D) of D such that $E(D) \approx Y \in \mathcal{X}_{4k}$. Then

$$A_j = X_j \cup \bigcup_{D\in\mathcal{D}} E(D)$$

is closed in X, and it is easily seen that A_j is $\mathcal{P}_{4k}$, and hence nowhere dense in X. If U is a non-empty open subset of A_j, say $U = U' \cap A_j$, with U' open in X, then either $U' \cap X_j = \emptyset$, hence $U \cap E(D) \neq \emptyset$ for some $D \in \mathcal{D}$, or $U' \cap X_j \neq \emptyset$, hence $U \supset E(D)$ for some $D \in \mathcal{D}$, by the claim. In both cases it follows that U is not $\mathcal{P}^2_{4(k-1)+3}$ since E(D) is nowhere $\mathcal{P}^2_{4(k-1)+3}$, and hence $A_j \approx Y$ (lemma 3.4.8). Thus $X = \bigcup_{j=0}^{\infty} A_j$ satisfies the hypotheses of theorem 3.2.4, and hence $X \approx \mathbf{Q}\times Y$.

Case 2: $n = 4k+3$. This is similar to case 1: since (3) holds for $m = 4k+2$, we here apply theorem 3.4.15(c) (if $i = 1$) or (d) (if $i = 2$) to obtain $E(D) \in \mathcal{X}^i_{4k+2}$. Then each $A_j \approx Y \in \mathcal{X}^i_{4k+2}$, and again $X \approx \mathbf{Q}\times Y$ by theorem 3.2.4. □

3.4.17 LEMMA: *If* $n = 4k+3$ *for some* $k < \omega$, *then* (2) *holds.*

Proof: Let $X \in \mathcal{X}^i_{4k+2}$ ($i \in \{1,2\}$). By lemmas 3.4.9 and 3.4.16, $\mathbf{Q}\times X$ is the unique element of $\mathcal{X}^i_{4k+3}$. Since (2) holds for $m = 4k+2$, the Cantor set is homogeneous with respect to dense copies of X. Now apply theorem 3.2.8. □

Lemmas 3.4.16 and 3.4.17 show that the inductive hypotheses are satisfied for $m = n$ if n is odd. The remaining lemmas of this section deal with the case where n is even.

3.4.18 LEMMA: *Let* $k \in \mathbb{N}$, *and let* $X \in \mathcal{X}_{4k}$. *Then* X *can be embedded in the Cantor set such that* $\mathcal{C}\backslash X \in \mathcal{X}^1_{4(k-1)+3}$.

Proof: Embed X in $\mathcal{C}$ as a dense subset. By lemma 3.4.10, $\mathcal{C}\backslash X$ is $\mathcal{P}^2_{4(k-1)+3}$, nowhere $\mathcal{P}^2_{4(k-1)+2}$, and nowhere $\mathcal{P}_{4(k-1)+1}$. Thus we can write $\mathcal{C}\backslash X = A \cup B$, where A is $\mathcal{P}_{4(k-1)+1}$, and B is σ-compact; since $A\backslash B$ is a G_δ-subset of A, it is $\mathcal{P}_{4(k-1)+1}$ by lemma 3.4.1, so we may assume that $A \cap B = \emptyset$. Note that $X \subset \mathcal{C}\backslash B$, and that $\mathcal{C}\backslash B$ is complete. Also, since $\mathcal{C}\backslash X$ is nowhere $\mathcal{P}_{4(k-1)+1}$, B is dense in $\mathcal{C}\backslash X$, and hence in $\mathcal{C}$, so $\mathcal{C}\backslash B$ is nowhere compact.

Hence by theorem 2.3.1, $\mathcal{C}\setminus B \approx \boldsymbol{P}$. Put $Y = \mathcal{C}\setminus B$, and embed Y in $K \approx \mathcal{C}$ such that $K\setminus Y \approx \boldsymbol{Q}$. We claim that $K\setminus X \in X^1_{4(k-1)+3}$.
Indeed, by lemma 3.4.10, $K\setminus X$ is nowhere $P^2_{4(k-1)+2}$, and nowhere $P_{4(k-1)+1}$; and $K\setminus X = (Y\setminus X) \cup \boldsymbol{Q} = A \cup \boldsymbol{Q}$ is $P_{4(k-1)+1} \cup$ countable is $P^1_{4(k-1)+3}$. □

3.4.19 LEMMA: *If* $n = 4k$ *for some* $k \in \boldsymbol{N}$, *then* (1) *holds.*

Proof: Let $X_0, X_1 \in X_{4k}$. By lemma 3.4.18, X_0 and X_1 can be embedded in $\mathcal{C}$ such that $\mathcal{C}\setminus X_0, \mathcal{C}\setminus X_1 \in X^1_{4(k-1)+3}$. Since (1) holds for $m = 4(k-1)+3$, $\mathcal{C}\setminus X_0 \approx \mathcal{C}\setminus X_1$, and hence by (2), there exists a homeomorphism $h: \mathcal{C} \to \mathcal{C}$ such that $h[\mathcal{C}\setminus X_0] = \mathcal{C}\setminus X_1$, and thus $h[X_0] = X_1$. □

3.4.20 LEMMA: *If* $n = 4k$ *for some* $k \in \boldsymbol{N}$, *then* (3) *holds.*

Proof: We have to prove theorem 3.4.15(a) and (b) for this k. For the "if" part of (a), observe that if A is P_{4k}, then $K\setminus A$ is $P^2_{4(k-1)+3}$ by lemmas 3.4.4 and 3.4.1, contradicting lemma 3.4.8. For "if" of (b), note that if A is P^1_{4k+2}, then $K \cap A$ is P^1_{4k+2} is $P^1_{4(k-1)+3} \cup$ complete, and since $K \cap A \in X^2_{4(k-1)+3}$, it is nowhere $P^1_{4(k-1)+3}$, hence Baire, contradicting lemma 3.4.7. We will now prove the "only if " parts.
(a) Let $P = P_{4(k-1)+1}$, and let Z be the unique element of $X^1_{4(k-1)+2}$; we will show that the hypotheses of lemma 3.3.4 are satisfied. By lemma 3.4.1, P is closed-hereditary, and P is strongly σ-additive by corollary 3.4.3. Let B be a Borel subset of a compact space Y which is not P; since (3) holds for $m = 4(k-1)+2$, Y contains a Cantor set K such that $K \cap B \in X^1_{4(k-1)+2}$ and $K\setminus B \in X_{4(k-1)+1}$. Then $K \cap B \approx Z$, and by lemma 3.4.2 we can write $K\setminus B = \bigcup_{i=0}^{\infty} X_i$, where X_i is closed in $K\setminus B$ and $P_{4(k-1)}$. Put $G = \bigcup_{i=0}^{\infty} \overline{X}_i$; then G is σ-compact, and since $\overline{X}_i \cap B = \overline{X}_i\setminus X_i$ is $P^2_{4(k-2)+3}$ by lemma 3.4.4, $G \cap B$ is strongly σ-$P^2_{4(k-2)+3}$, hence $P^2_{4(k-2)+3}$ by corollary 3.4.3, and thus $G \cap B$ is P by lemma 3.4.5. So by lemma 3.3.4, if A is a Borel subset of a compact space X which is not P_{4k}, or equivalently, not $P \cup$ complete, then X contains a Cantor set K such that $K = \overline{K \cap A}$ and $K \cap A \approx \boldsymbol{Q}\times Z$; then $K \cap A \in X^1_{4(k-1)+3}$ by lemma 3.4.9, and hence $K\setminus A \in X_{4k}$ by corollary 3.4.11.
(b) Since $P^1_{4k+2} \equiv P^1_{4(k-1)+3} \cup$ complete, here we put $P = P^1_{4(k-1)+3}$; then again P is closed-hereditary and strongly σ-additive. Let Z be the unique element of $X^2_{4(k-1)+2}$, and suppose that B is a Borel subset of a compact space Y which is not P; since (3) holds for $m = 4(k-1)+2$, Y contains a Cantor set K such that $K \cap B \in X^2_{4(k-1)+2}$, and $K\setminus B \in X_{4(k-1)+1}$. As above,

we put $K\backslash B = \bigcup_{i=0}^{\infty} X_i$, and $G = \bigcup_{i=0}^{\infty} \overline{X}_i$; then $G \cap B$ is $P^2_{4(k-2)+3}$, and hence $G \cap B$ is P by lemma 3.4.5. So by lemma 3.3.4, if A is a Borel subset of a compact space X which is not P^1_{4k+2}, then X contains a Cantor set K such that $K = \overline{K \cap A}$ and $K \cap A \approx \mathbb{Q} \times Z$; then $K \cap A \in X^2_{4(k-1)+3}$ by lemma 3.4.9, and hence $K\backslash A \in X_{4k}$ by corollary 3.4.11. □

Lemmas 3.4.18, 3.4.19, and 3.4.20 show that the inductive hypotheses are satisfied if $n = 4k$ for some $k \in \mathbb{N}$.

3.4.21 LEMMA: *Let* $k < \omega$, $i \in \{1,2\}$, *and let* $X \in X^i_{4k+2}$ *be a dense subset of the Cantor set. Then there exist closed nowhere dense subspaces* A_j *of* $C\backslash X$ *such that* $C\backslash X = \bigcup_{j=0}^{\infty} A_j$ *and* $\overline{A}_j\backslash A_j \in X^i_{4(k-1)+3}$ *is dense in* $\overline{A}_j$.

Proof: By corollary 3.4.11, $C\backslash X \in X_{4k+1}$, so by lemma 3.4.2 we can write $C\backslash X = \bigcup_{\ell=0}^{\infty} X_\ell$, with X_ℓ closed in $C\backslash X$ and P_{4k}. Since $\overline{X}_\ell\backslash X_\ell = \overline{X}_\ell \cap X$ is closed in X, it is P^i_{4k+2} by lemma 3.4.1; hence $\overline{X}_\ell\backslash X_\ell = G_\ell \cup H_\ell$, where G_ℓ is complete, and H_ℓ is $P^i_{4(k-1)+3}$. Then $\overline{X}_\ell\backslash G_\ell$ is σ-compact, say $\overline{X}_\ell\backslash G_\ell = \bigcup_{m=0}^{\infty} K(\ell,m)$, with $K(\ell,m)$ compact; note that $X_\ell \subset \overline{X}_\ell\backslash G_\ell$. Since $\{K(\ell,m)\backslash X\colon \ell,m < \omega\}$ is countable, we can enumerate the non-empty elements of this set as $\{M_j\colon j \in I\}$, where I is countable.
Fix $j \in I$. Since M_j is a closed subset of some X_ℓ, M_j is P_{4k}; and $\overline{M}_j \cap X = \overline{M}_j \cap H_\ell$ for this same ℓ, hence $\overline{M}_j \cap X$ is $P^i_{4(k-1)+3}$. Observe that since $C\backslash X$ is nowhere P_{4k}, each M_j is nowhere dense in $C\backslash X$, so we can take $I = \omega$. Now let $\mathcal{D}$ be a disjoint covering of $C\backslash\overline{M}_j$ by clopen subsets of C such that for each $D \in \mathcal{D}$, $\operatorname{diam}(D) < d(D,\overline{M}_j)$. Since X is nowhere P_{4k} (if $i = 1$), resp. nowhere P^1_{4k+2} (if $i = 2$), and since (3) holds for $m = 4k$, each $D \in \mathcal{D}$ contains a Cantor set $K(D)$ such that $K(D) \cap X \in X^i_{4(k-1)+3}$ and $K(D)\backslash X \in X_{4k}$. Put

$$K_j = \overline{M}_j \cup \bigcup_{D\in\mathcal{D}} K(D),$$

and

$$A_j = K_j\backslash X;$$

we claim that the sets A_j constructed in this way are as required.
First note that $C\backslash X = \bigcup_{j=0}^{\infty} A_j$ since $K_j\backslash X \supset M_j$. Furthermore, since M_j is nowhere dense in $C\backslash X$, also $\overline{M}_j$ is nowhere dense in C; it then follows from the fact that $\operatorname{diam}(D) < d(D,\overline{M}_j)$ for each $D \in \mathcal{D}$ that $(\bigcup_{D\in\mathcal{D}} K(D))^- = K_j$, and thus

$$K_j \supset \overline{A}_j \supset (\overline{A}_j\backslash A_j)^- \supset (\bigcup_{D\in\mathcal{D}}(K(D) \cap X))^- = (\bigcup_{D\in\mathcal{D}} K(D))^- = K_j,$$

so $\overline{A}_j\backslash A_j$ is dense in $\overline{A}_j$. It remains to show that $\overline{A}_j\backslash A_j \in X^i_{4(k-1)+3}$. Clearly, $\overline{A}_j\backslash A_j = (\overline{M}_j \cap X) \cup \bigcup_{D\in\mathcal{D}}(K(D) \cap X)$ is strongly σ-$\mathcal{P}^i_{4(k-1)+3}$ is $\mathcal{P}^i_{4(k-1)+3}$ by corollary 3.4.3. And as in the proof of lemma 3.4.16, it follows that if U is a non-empty open subset of $\overline{A}_j\backslash A_j$, then $U \cap K(D) \cap X \neq \emptyset$ for some $D \in \mathcal{D}$; hence, since $K(D) \cap X \in X^i_{4(k-1)+3}$, U is not $\mathcal{P}^2_{4(k-1)+2}$; and not $\mathcal{P}_{4(k-1)+1}$ (if $i = 1$), resp. not $\mathcal{P}^1_{4(k-1)+3}$ (if $i = 2$). Thus, $\overline{A}_j\backslash A_j \in X^i_{4(k-1)+3}$; by corollary 3.4.11, $A_j \in X_{4k}$, and thus A_j is nowhere dense in $C\backslash X$ since $C\backslash X$ is nowhere $\mathcal{P}_{4k}$. □

3.4.22 LEMMA: *If* $n = 4k+2$ *for some* $k < \omega$, $i \in \{1,2\}$, *and* $X_0, X_1 \in X^i_n$ *are dense subsets of* C, *then there exists a homeomorphism* $h: C \to C$ *such that* $h[X_0] = X_1$. *Thus,* (1) *and* (2) *are satisfied if* $n = 4k+2$.

Proof: Let A be the unique element of $X^i_{4(k-1)+3}$. By lemma 3.4.21, we can write $C\backslash X_0 = \bigcup_{j=0}^{\infty} A_j$, $C\backslash X_1 = \bigcup_{j=0}^{\infty} B_j$, such that $\overline{A}_j\backslash A_j \approx \overline{B}_j\backslash B_j \approx A$, $\overline{A}_j\backslash A_j$ is dense in $\overline{A}_j$, $\overline{B}_j\backslash B_j$ is dense in $\overline{B}_j$, A_j is closed and nowhere dense in $C\backslash X_0$, and B_j is closed and nowhere dense in $C\backslash X_1$. Hence the required homeomorphism can be obtained from lemma 3.2.6. □

3.4.23 LEMMA: *If* $n = 4k+2$ *for some* $k < \omega$, *then* (3) *holds.*

Proof: We have to prove theorem 3.4.15(c) and (d) for this k. The "if" part is easy: if A is $\mathcal{P}_{4k+1}$ (resp. $\mathcal{P}^1_{4k+3}$), then so is $K \cap A$, so $K \cap A \notin X^1_{4k+2}$ (resp. X^2_{4k+2}) by lemma 3.4.8. For the "only if" parts, we will apply lemma 3.3.5.

(c) Let $\mathcal{P} = \mathcal{P}_{4k+1}$, let Z' be the unique element of X_{4k}, and let Y' be the unique element of $X^1_{4(k-1)+3}$. By lemma 3.4.1, $\mathcal{P}$ is closed-hereditary, and $\mathcal{P}$ is strongly σ-additive by corollary 3.4.3. Let B be a Borel subset of a compact space Y which is not $\mathcal{P}$. Then B is not $\mathcal{P}_{4k}$ by lemma 3.4.5, hence since (3) holds for $m = 4k$, Y contains a Cantor set K such that $K \cap B \in X^1_{4(k-1)+3}$ and $K\backslash B \in X_{4k}$; thus, $K \cap B \approx Y'$, $K\backslash B \approx Z'$, and $K \cap B$ is $\mathcal{P}$ by lemma 3.4.5. Hence by lemma 3.3.5, if A is a Borel subset of a compact space X which is not $\mathcal{P}$, then X contains Cantor sets B_i, for $i < \omega$, such that $B_i = \overline{B_i\backslash A}$, $K = (\bigcup_{i=0}^{\infty} B_i)^-$, and for each $i < \omega$:

(i) B_i is nowhere dense in K;
(ii) $K\backslash A = \bigcup_{i=0}^{\infty}(B_i\backslash A)$;
(iii) $B_0\backslash A \approx Z' \approx (B_{i+1}\backslash B_i)\backslash A$;
(iv) $B_0 \cap A \approx Y' \approx (B_{i+1}\backslash B_i) \cap A$.

Since $B_{i+1}\backslash B_i$ is open in B_{i+1}, we can write $B_{i+1}\backslash B_i = \bigcup_{j=0}^{\infty} K_j^i$, where K_j^i is clopen in B_{i+1}. Since both Y' and Z' are strongly homogeneous, either $K_j^i \cap A \approx Y'$ or $K_j^i \cap A = \emptyset$, and $K_j^i\backslash A \approx Z'$ or $K_j^i\backslash A = \emptyset$. Hence,

$$K\backslash A = (B_0\backslash A) \cup \bigcup_{i=0}^{\infty}\bigcup_{j=0}^{\infty}(K_j^i\backslash A) \approx \mathcal{Q}\times Z'$$

by (i) and theorem 3.2.4, so $K\backslash A \in X_{4k+1}$ by lemma 3.4.9. Then $K \cap A$ is nowhere P_{4k}, and nowhere $P^2_{4(k-1)+3}$ by lemma 3.4.10. So we need only show that $K \cap A$ is P^1_{4k+2}. However, by (ii),

$$K \cap A = (B_0 \cap A) \cup \bigcup_{i=0}^{\infty}\bigcup_{j=0}^{\infty}(K_j^i \cap A) \cup K\backslash(\bigcup_{i=0}^{\infty} \overline{B}_i)$$

is strongly σ-$P^1_{4(k-1)+3}$ $\cup$ complete is $P^1_{4(k-1)+3}$ $\cup$ complete is P^1_{4k+2}.
(d) Let $P = P^1_{4k+3}$, let Z' be the unique element of X_{4k}, and let Y' be the unique element of $X^2_{4(k-1)+3}$. Then P is closed-hereditary and strongly σ-additive. Let B be a Borel subset of a compact space Y which is not P. Then B is not P^1_{4k+2}, so since (3) holds for $m = 4k$, Y contains a Cantor set K such that $K \cap B \approx Y'$, $K\backslash B \approx Z'$, whence $K \cap B$ is P by lemma 3.4.5. So again by lemma 3.3.5, we find Cantor sets B_i as under (a); then $K\backslash A \in X_{4k+1}$, hence by lemma 3.4.10, $K \cap A$ is P^2_{4k+2}, and nowhere $P^2_{4(k-1)+3}$, so it suffices to show that $K \cap A$ is nowhere P^1_{4k+2}. Put $B_{i+1}\backslash B_i = \bigcup_{j=0}^{\infty} K_j^i$ as in (a), and note that if U is a non-empty open subset of $K \cap A$, then $U \cap K_j^i \neq \emptyset$ for some $i,j < \omega$; thus, U contains a closed copy of Y'. Now suppose that U is P^1_{4k+2}; then so is Y' by the preceding remark, i.e. $Y' = A \cup B$ where A is $P^1_{4(k-1)+3}$, and B is complete. Since $Y' \in X^2_{4(k-1)+3}$, Y' is nowhere $P^1_{4(k-1)+3}$, hence B is dense in Y', so Y' is Baire, contradicting lemma 3.4.7. □

This completes the proof of theorems 3.4.13, 3.4.14, and 3.4.15. In section 6 of this chapter we will show that there are no other homogeneous non-σ-compact S_n-subsets of the Cantor set than those defined and characterized above.

We conclude this section with the following theorem, announced in section 3.1.

3.4.24 THEOREM: *If we order the properties* $P_n^{(i)}$ *by* $P^1_{-1} < P^2_{-1} < P_0$, *and* $P_{4k} < P_{4k+1} < P^1_{4k+2} < P^1_{4k+3} < P^2_{4k+2} < P^2_{4k+3} < P_{4(k+1)}$ *if* $k < \omega$, *then for each* $n < \omega$, *and each* $i \in \{1,2\}$, *we have* $X \in X_n^{(i)}$ *if and only if* X *is* $P_n^{(i)}$ *and nowhere* $P_m^{(j)}$ *for all* $P_m^{(j)} < P_n^{(i)}$, *where* $m = -1$ *or* $m < \omega$, *and* $j \in \{1,2\}$.

Proof: Apply lemmas 3.4.5 and 3.4.8. □

3.4.25 COROLLARY: *If* $P_m^{(j)} < P_n^{(i)}$, *then* $X_m^{(j)} \cap X_n^{(i)} = \emptyset$; *thus, all spaces characterized in this section are topologically distinct.* □

3.5 The small Borel classes $D_\alpha(\Sigma_2^0)$ for $\alpha \in [\omega,\omega_1)$

We recall the topological properties P_β determined by definition 3.1.8 and theorem 3.1.6; to avoid unnecessary complications, we will just write "Y is P_γ for some $\gamma < \omega$" for "Y is $P_m^{(j)}$ for some $m < \omega$, $j \in \{1,2\}$". Let $\alpha < \omega_1$ be a limit ordinal, and let $n < \omega$. Then

X is $P_{\alpha+2n}$ if and only if $X \in D_{\alpha+2n}(\Sigma_2^0)$ if and only if $X = \bigcup_{i=0}^\infty A_i \cup B$, where B is S_n, A_i is closed in X, and A_i is P_{α_i} for some $\alpha_i < \alpha$;

X is $P_{\alpha+2n+1}$ if and only if X is $P_{\alpha+2n} \cup$ complete.

3.5.1 LEMMA: *For each* $\beta \in [\omega,\omega_1)$, P_β *is hereditary with respect to* G_δ-*subsets.*

Proof: Embed X in a compact space Y. If β is even, then there are certain σ-compact subsets A_ζ of Y such that $X = D_\beta(\langle A_\zeta: \zeta < \beta\rangle)$; if G is a G_δ-subset of Y, then $X \cap G = D_\beta(\langle A_\zeta \cup Y\backslash G: \zeta < \beta\rangle) \in D_\beta(\Sigma_2^0)$. The case β is odd follows trivially. □

3.5.2 LEMMA: *Let* X *be a topological space, let* $\alpha < \omega_1$ *be a limit ordinal, and let* $n < \omega$. *Then*

(a) X *is* $P_{\alpha+2n}$ *if and only if* X *is* $P_\alpha \cup S_n$;

(b) X *is* $P_{\alpha+2n+1}$ *if and only if* $X = \bigcup_{i=0}^\infty A_i \cup B \cup G$, *where* A_i *is* P_{α_i} *for some* $\alpha_i < \alpha$, B *is* S_n, G *is complete, and* A_i *is closed in* X.

Proof: (a) If X is $P_{\alpha+2n}$, then clearly X is $P_\alpha \cup S_n$. We will prove the converse by induction. Suppose that $X = \bigcup_{i=0}^\infty A_i \cup B$, where A_i is closed in $\bigcup_{i=0}^\infty A_i$, A_i is P_{α_i} for some $\alpha_i < \alpha$, and B is S_n, and first consider the case $\alpha = \omega$. Then $\overline{A}_i = A_i \cup (\overline{A}_i \cap B)$ is $P_{\alpha_i} \cup S_n$, hence $\overline{A}_i$ is P_{β_i} for some $\beta_i < \omega$, so $X = \bigcup_{i=0}^\infty \overline{A}_i \cup B$ is $P_{\omega+2n}$. Now assume that we are done for limit ordinals $\delta < \alpha$ ($> \omega$); then we may assume that for each i, $\alpha_i \geq \omega$, and hence $\alpha_i = \gamma_i + n_i$ for some limit ordinal $\gamma_i < \alpha$, and some $n_i < \omega$. If n_i is even, then $\overline{A}_i$ is $P_{\gamma_i+n_i} \cup S_n$ is $P_{\gamma_i} \cup S_{\frac{1}{2}n_i} \cup S_n$, hence $P_{\gamma_i+n_i+2n}$ by the inductive hypothesis, and if n is odd, then $\overline{A}_i$ is $P_{\gamma_i+n_i-1} \cup$ complete $\cup S_n$ is $P_{\gamma_i} \cup S_{\frac{1}{2}(n_i-1)} \cup S_n \cup$ complete, hence certainly $P_{\gamma_i+n_i-1+2n+2}$, again by the inductive hypothesis. So each $\overline{A}_i$ is P_{β_i}

for some $\beta_i < \alpha$, and thus $X = \bigcup_{i=0}^{\infty} \overline{A}_i \cup B$ is $P_{\alpha+2n}$.

(b) The "if" part is clear, so suppose that $X = A \cup G$, where A is $P_{\alpha+2n}$, and G is complete. Put $A = \bigcup_{i=0}^{\infty} B_i \cup B$, where B_i is closed in A and P_{α_i} for some $\alpha_i < \alpha$, and B is S_n, and let $A_i = \overline{B}_i = B_i \cup (\overline{B}_i \cap G)$. Then A_i is $P_{\alpha_i} \cup$ complete, hence certainly A_i is $P_{\alpha_i} \cup S_1$, and thus, using (a), A_i is P_{β_i} for some $\beta_i < \alpha$. Since A_i is closed in X, $X = \bigcup_{i=0}^{\infty} A_i \cup B \cup G$ is as required. □

3.5.3 COROLLARY: *If* $n < \omega$, $i \in \{1,2\}$, $\beta,\gamma \in [\omega,\omega_1)$, *and* $\beta \leq \gamma$, *then* $P_n^{(i)} \to P_\beta \to P_\gamma$. □

3.5.4 LEMMA: *Let* X *be a topological space, let* $\alpha < \omega_1$ *be a limit ordinal, and let* $n \in \mathbb{N}$. *Then* X *is* $P_{\alpha+2n}$ *if and only if* X *is strongly* σ-$P_{\alpha+2n-1}$.

Proof: The proof is exactly the same as that of lemma 3.4.2. We find that X is $P_{\alpha+2n}$ if and only if X is strongly σ-$(P_{\alpha+2(n-1)} \cup$ complete), i.e. if and only if X is strongly σ-$P_{\alpha+2n-1}$. □

3.5.5 COROLLARY: *For even* $\beta \in [\omega,\omega_1)$, P_β *is a strongly* σ*-additive property.* □

3.5.6 LEMMA: *Let* X *be compact, and let* A *be a subset of* X. *If* A *is* P_β *for some* $\beta \in [\omega,\omega_1)$, *then* $X \backslash A$ *is* $P_{\beta+1}$.

Proof: If β is even, then there exist σ-compact subsets A_ζ of X such that $A = D_\beta(\langle A_\zeta \colon \zeta < \beta\rangle) = \bigcup\{A_\zeta \backslash (\bigcup_{\delta<\zeta} A_\delta) \colon \zeta \text{ odd} < \beta\}$. Then

$$X \backslash A = \bigcup\{A_\zeta \backslash (\bigcup_{\delta<\zeta} A_\delta) \colon \zeta \text{ even} < \beta\} \cup X \backslash (\bigcup_{\zeta<\beta} A_\zeta).$$

So if we put $B_\zeta = \bigcup_{\delta<\zeta} A_\delta$ for $\zeta < \beta$, then

$$\begin{aligned} X \backslash A &= \bigcup\{B_{\zeta+1} \backslash B_\zeta \colon \zeta \text{ even} < \beta\} \cup X \backslash (\bigcup_{\zeta<\beta} A_\zeta) \\ &= D_\beta(\langle B_\zeta \colon \zeta < \beta\rangle) \cup X \backslash (\bigcup_{\zeta<\beta} A_\zeta) \end{aligned}$$

is $P_\beta \cup$ complete is $P_{\beta+1}$.

If β is odd, say $\beta = \alpha+2n+1$, with $\lim(\alpha)$, $n < \omega$, then there exist σ-compact A_ζ in X such that $A = D_{\alpha+2n}(\langle A_\zeta \colon \zeta < \alpha+2n\rangle) \cup G$, where G is complete. Defining B_ζ as above for $\zeta < \alpha+2n = \gamma$, we find that

$$\begin{aligned} X \backslash A &= (D_\gamma(\langle B_\zeta \colon \zeta < \gamma\rangle) \cup (X \backslash (\bigcup_{\zeta<\gamma} A_\zeta)) \cap X \backslash G \\ &= D_\gamma(\langle B_\zeta \cap X \backslash G \colon \zeta < \gamma\rangle) \cup (X \backslash G) \backslash (\bigcup_{\zeta<\gamma} A_\zeta) \end{aligned}$$

is $P_\gamma \cup S_1$. By lemma 3.5.2, $X\backslash A$ is $P_\alpha \cup S_n \cup S_1$ is $P_{\alpha+2n+2}$ is $P_{\beta+1}$. □

For each pair (β,i), with $\beta \in [\omega,\omega_1)$, and $i = 2$ if β is a limit, resp. $i \in \{1,2\}$ if β is a successor, we now inductively define a class of spaces X^i_β, and show that, up to homeomorphism, each X^i_β contains exactly one element.

3.5.7 DEFINITION: *Let* X *be zero-dimensional, let* $\alpha < \omega_1$ *be a limit ordinal, and* $n \in N$*; then*

$X \in X^2_\alpha$ *if and only if* X *is* P_α*, nowhere* P_β *for* $\beta < \alpha$*;*

$X \in X^1_{\alpha+n}$ *if and only if* X *is* $P_{\alpha+n}$*, nowhere* $P_{\alpha+n-1}$*, and* X *does not contain any closed subsets belonging to* $X^2_{\alpha+n-1}$*;*

$X \in X^2_{\alpha+n}$ *if and only if* X *is* $P_{\alpha+n}$*, nowhere* $P_{\alpha+n-1}$*, and every non-empty clopen subset of* X *contains a closed subset belonging to* $X^2_{\alpha+n-1}$.

3.5.8 LEMMA: *Let* $\beta \in [\omega,\omega_1)$*,* $i \in \{1,2\}$*, and* $X \in X^i_\beta$.

(a) *If* β *is even, then* X *is first category.*

(b) *If* β *is odd, then* X *contains a dense complete subset, hence* X *is Baire.*

Proof:(a) By the definition of P_β (if β is a limit), resp. by lemma 3.5.4 (if β is a successor), we can write $X = \cup_{i=0}^{\infty} X_i$, where X_i is closed in X, and P_{β_i} for some $\beta_i < \beta$. Since X is nowhere P_{β_i}, each X_i is nowhere dense in X.

(b) Write $X = A \cup B$, where A is $P_{\beta-1}$, and B is complete; since X is nowhere $P_{\beta-1}$, B is dense in X. □

We now state and prove the main theorems of this section.

3.5.9 THEOREM: *If* $\beta \in [\omega,\omega_1)$*, and* $i = 2$ *if* β *is a limit,* $i \in \{1,2\}$ *if* β *is a successor, then up to homeomorphism, each* X^i_β *contains exactly one element, which is strongly homogeneous, whence homogeneous.*

3.5.10 THEOREM: *If* $X \in X^i_\beta$ *for some* $\beta \in [\omega,\omega_1)$ *and* $i \in \{1,2\}$*, then* C *is homogeneous with respect to dense copies of* X.

3.5.11 THEOREM: *Let* A *be a Borel subset of a compact space* X*, and let* $\beta \in [\omega,\omega_1)$*. Then* A *is not* P_β *if and only if* X *contains a Cantor set* K *such that* $K \cap A \in X^1_{\beta+1}$ *and* $K\backslash A \in X^2_\beta$.

We will prove the above theorems by induction on (β,i), ordered lexicographically, using the following statements as inductive hypotheses (here, $\alpha < \omega_1$ is a limit ordinal, and $n < \omega$):

If $\beta = \alpha+2n$, $i = 2$, then:

(1) $|\mathcal{X}^2_{\alpha+2n}| = 1$, and if $X \in \mathcal{X}^2_{\alpha+2n}$, then $\mathcal{C}$ is homogeneous with respect to dense copies of X.

(2) If A is a Borel subset of a compact space X which is not $\mathcal{P}_{\alpha+2n}$, then X contains a Cantor set K such that $K\setminus A \in \mathcal{X}^2_{\alpha+2n}$.

If $\beta = \alpha+2n+1$, $i = 1$, then:

(3) Let X be dense and co-dense in $\mathcal{C}$; then $X \in \mathcal{X}^1_{\alpha+2n+1}$ if and only if $\mathcal{C}\setminus X \in \mathcal{X}^2_{\alpha+2n}$.

(4) $|\mathcal{X}^1_{\alpha+2n+1}| = 1$, and if $X \in \mathcal{X}^1_{\alpha+2n+1}$, then $\mathcal{C}$ is homogeneous with respect to dense copies of X.

(5) If A is a Borel subset of a compact space X which is not $\mathcal{P}_{\alpha+2n}$, then X contains a Cantor set K such that $K\setminus A \in \mathcal{X}^2_{\alpha+2n}$ and $K \cap A \in \mathcal{X}^1_{\alpha+2n+1}$.

If $\beta = \alpha+2n+1$, $i = 2$, then:

(6) Let X be dense and co-dense in $\mathcal{C}$; then $X \in \mathcal{X}^2_{\alpha+2n+1}$ if and only if $\mathcal{C}\setminus X \approx \mathbb{Q}\times Y$ for some $Y \in \mathcal{X}^1_{\alpha+2n+1}$.

(7) $|\mathcal{X}^2_{\alpha+2n+1}| = 1$, and if $X \in \mathcal{X}^2_{\alpha+2n+1}$, then $\mathcal{C}$ is homogeneous with respect to dense copies of X.

(8) If A is a Borel subset of a compact space X which is not $\mathcal{P}_{\alpha+2n+1}$, then X contains a Cantor set K such that $K\setminus A \in \mathcal{X}^2_{\alpha+2n+1}$ and $K \cap A \approx \mathbb{Q}\times Y$ for some $Y \in \mathcal{X}^1_{\alpha+2n+1}$.

If $\beta = \alpha+2n+2$, $i = 1$, then:

(9) Let X be dense and co-dense in $\mathcal{C}$; then $X \in \mathcal{X}^1_{\alpha+2n+2}$ if and only if $\mathcal{C}\setminus X \in \mathcal{X}^2_{\alpha+2n+1}$.

(10) $|\mathcal{X}^1_{\alpha+2n+2}| = 1$, and if $X \in \mathcal{X}^1_{\alpha+2n+2}$, then $\mathcal{C}$ is homogeneous with respect to dense copies of X.

(11) If A is a Borel subset of a compact space X which is not $\mathcal{P}_{\alpha+2n+1}$, then X contains a Cantor set K such that $K\setminus A \in \mathcal{X}^2_{\alpha+2n+1}$ and $K \cap A \in \mathcal{X}^1_{\alpha+2n+2}$.

So let $\alpha < \omega_1$ be a limit ordinal, let $m < \omega$ and $i \in \{1,2\}$, and suppose that the inductive hypotheses are satisfied for all $(\beta,j) < (\alpha+m,i)$; note that from the fact that $|\mathcal{X}^j_\beta| = 1$ it follows as in section 3.4 that the

unique element of X^j_β is strongly homogeneous. We now show that the inductive hypotheses are satisfied for $(\alpha+m,i)$.
We start with the case $m = 0$ (hence $i = 2$); of course, we have to use the results of section 3.4 in case $\alpha = \omega$.

3.5.12 LEMMA: $X^2_\alpha \neq \emptyset$.

Proof: Let $\langle \alpha_i : i < \omega \rangle$ be an increasing sequence of ordinals less than α, such that $\sup_{i<\omega} \alpha_i = \alpha$; for convenience of notation, let $\alpha_i \equiv 3 \pmod 4$ if $\alpha_i < \omega$. Let $Y_i \in X^2_{\alpha_i}$ be densely embedded in C. Enumerate Q as $\{q_i : i < \omega\}$, and put

$$X = \bigcup_{i=0}^{\infty}(\{q_i\} \times Y_i) \subset Q \times C.$$

Then clearly, X is P_α. Also, if $W = X \cap (U \times V)$ is a non-empty basic clopen subset of X, then $W = \bigcup_{j \in E}(\{q_j\} \times (V \cap Y_j))$ for some infinite E. Since Y_j is nowhere P_γ for each $\gamma < \alpha_j$, $\{q_j\} \times (V \cap Y_j)$ is not P_γ for each $\gamma < \alpha_j$, and hence by lemma 3.4.1 or 3.5.1, W is not P_γ for each $\gamma < \alpha_j$ since $\{q_j\} \times (V \cap Y_j)$ is closed in W. Hence X is nowhere P_γ for each $\gamma < \alpha$ since $\sup_{j \in E} \alpha_j = \alpha$. So $X \in X^2_\alpha$. □

3.5.13 LEMMA: *If* $n = 0$*, then* (1) *holds.*

Proof: By lemma 3.5.12, $|X^2_\alpha| \geq 1$; so suppose that $X, Y \in X^2_\alpha$ are dense subsets of the Cantor set. In a way analogous to the proof of theorem 3.2.6, we construct a homeomorphism $h: C \to C$ such that $h[X] = Y$. We first consider the case where $\alpha > \omega$. Since X is P_α, we can write $X = \bigcup_{i=0}^{\infty} X_i$, where X_i is closed in X and P_{α_i} for some $\alpha_i < \alpha$; clearly, we may assume that $\alpha_i \geq \omega$. Note that, since X is nowhere P_{α_i} for each $i < \omega$, each X_i is nowhere dense in X. Similarly, $Y = \bigcup_{i=0}^{\infty} Y_i$, with Y_i closed and nowhere dense in Y, and Y_i is P_{β_i} for some $\beta_i \in [\omega,\alpha)$. As in section 3.2, we construct, for each $s \in M$, collections $U(s) = \{U(s^\frown n) : n < \omega\}$, $V(s) = \{V(s^\frown n) : n < \omega\}$ of clopen subsets of C, closed nowhere dense subsets $D(s)$, $E(s)$ of C, and for each $n \in N$ a homeomorphism $h_n : C \to C$, satisfying the hypotheses of the proof of theorem 3.2.6; then again, $\lim_{n\to\infty} h_n = h: C \approx C$, and $h[X] = Y$.
First, let $\gamma_0 < \alpha$ be an even successor ordinal such that $\alpha_0, \beta_0 \leq \gamma_0$; then both X_0 and Y_0 are P_{γ_0}. Let $\mathcal{D}$ be a disjoint covering of $C \setminus X_0$ by clopen subsets of C such that for each $D \in \mathcal{D}$, $\mathrm{diam}(D) < d(D,\overline{X}_0)$. Since X is a Borel subset of C which is nowhere P_{γ_0+1}, $C \setminus X$ is nowhere P_{γ_0} by

lemma 3.5.6. Hence, applying hypothesis (5) above, each $D \in \mathcal{D}$ contains a Cantor set $K(D)$ such that $K(D) \cap X = E(D) \in X^2_{\gamma 0}$. Then

$$A_0 = X_0 \cup \bigcup_{D \in \mathcal{D}} E(D)$$

is closed in X, and strongly σ-$P_{\gamma 0}$, hence $P_{\gamma 0}$ by corollary 3.5.5. As in the proof of lemma 3.4.16, it can be shown that, for each non-empty open subset U of A_0, $U \cap E(D) \neq \emptyset$ for some $D \in \mathcal{D}$, hence A_0 is nowhere $P_{\gamma 0-1}$ since $E(D)$ is nowhere $P_{\gamma 0-1}$. Also, $U \cap E(D)$ contains a subset F which is closed in $E(D)$, hence in A_0, such that $F \in X^2_{\gamma 0-1}$; thus $A_0 \in X^2_{\gamma 0}$. Similarly, we can find a closed subset B_0 of Y such that $Y_0 \subset B_0$, and $B_0 \in X^2_{\gamma 0}$; note that A_0 is nowhere dense in X, and B_0 is nowhere dense in Y. Now put $D(\emptyset) = \overline{A}_0$, $E(\emptyset) = \overline{B}_0$; by hypothesis (1), applied to γ_0, there exists a homeomorphism $\tilde{h}: D(\emptyset) \to E(\emptyset)$ such that $\tilde{h}[A_0] = B_0$. We construct $U(\emptyset)$, $V(\emptyset)$, and h_1 as in the proof of theorem 3.2.6.
If $D(s)$, $E(s)$, $U(s)$, $V(s)$, and h_m have been constructed for $|s| < n$ (≥ 1) and $m \leq n$, then fix $s \in M$ with $|s| = n-1$, and $i < \omega$. Let

$$k_i = \min\{j: U(s^\frown i) \cap X_j \neq \emptyset\},\ \ell_i = \min\{j: V(s^\frown i) \cap Y_j \neq \emptyset\},$$

and find an even successor $\gamma(s^\frown i) < \alpha$ such that $\alpha_{k_i}, \beta_{\ell_i} \leq \gamma(s^\frown i)$; then both $U(s^\frown i) \cap X_{k_i}$ and $V(s^\frown i) \cap Y_{\ell_i}$ are $P_{\gamma(s^\frown i)}$. As above, we obtain closed subsets $A(s^\frown i)$ of $U(s^\frown i) \cap X$, $B(s^\frown i)$ of $V(s^\frown i) \cap Y$, such that $A(s^\frown i) \supset U(s^\frown i) \cap X_{k_i}$, $B(s^\frown i) \supset V(s^\frown i) \cap Y_{\ell_i}$, and both $A(s^\frown i), B(s^\frown i) \in X^2_{\gamma(s^\frown i)}$. Put $D(s^\frown i) = \overline{A(s^\frown i)}$, $E(s^\frown i) = \overline{B(s^\frown i)}$, and let $\tilde{h}(s^\frown i): D(s^\frown i) \to E(s^\frown i)$ be a homeomorphism such that $\tilde{h}(s^\frown i)[A(s^\frown i)] = B(s^\frown i)$, applying hypothesis (1) to $\gamma(s^\frown i)$; now finish the proof as in section 3.2.
In case $\alpha = \omega$, the proof is similar; the sets A_0, B_0, $A(s^\frown i)$, and $B(s^\frown i)$ should be chosen in some X^i_{4k+2} or X^i_{4k+3}, because we want to apply theorem 3.4.14. □

3.5.14 LEMMA: *If* $n = 0$, *then* (2) *holds.*

Proof: As in 3.5.13, for sake of simplicity, we will only consider the case $\alpha > \omega$, the case $\alpha = \omega$ being entirely analogous. Let A be a Borel subset of a compact space X which is not P_α, let $\langle \alpha_i : i < \omega \rangle$ be an increasing sequence of even successor ordinals in $[\omega, \alpha)$ such that $\sup_{i<\omega} \alpha_i = \alpha$, let Z_i be the unique element of $X^2_{\alpha_i - 1}$, and let Y_i be the unique element of $X^1_{\alpha_i}$. Put $P = P_\alpha$; then P is closed-hereditary and strongly σ-additive. Let B be a Borel subset of a compact space Y which is not P. Then for each $i < \omega$, B is not $P_{\alpha_i - 1}$, hence by hypothesis (11), Y contains a Cantor set

K_i such that $K_i \backslash B \in X^2_{\alpha_i-1}$ and $K_i \cap B \in X^1_{\alpha_i}$. Thus $K_i \backslash B \approx Z_i$, and $K_i \cap B \approx Y_i$ is P_{α_i}, hence P. So by lemma 3.3.5, X contains Cantor sets B_i, for $i < \omega$, such that $B_i = \overline{B_i \backslash A}$, and $K = (\bigcup_{i=0}^{\infty} B_i)^{-} \approx C$, and such that for each $i < \omega$,

(i) B_i is nowhere dense in K;

(ii) $K \backslash A = \bigcup_{i=0}^{\infty}(B_i \backslash A)$;

(iii) $B_0 \backslash A \approx Z_0$, $(B_{i+1} \backslash B_i) \backslash A \approx Z_{i+1}$.

Since $(B_{i+1} \backslash B_i) \backslash A$ is open in $B_{i+1} \backslash A$, we can write $(B_{i+1} \backslash B_i) \backslash A = \bigcup_{j=0}^{\infty} K^i_j$, where K^i_j is clopen in $B_{i+1} \backslash A$; then $K^i_j = \emptyset$ or $K^i_j \approx Z_{i+1}$, so K^i_j is $P_{\alpha_{i+1}-1}$, and hence

$$K \backslash A = (B_0 \backslash A) \cup \bigcup_{i=0}^{\infty} \bigcup_{j=0}^{\infty} K^i_j$$

is P_α. By (i), each open subset of $K \backslash A$ intersects some $K^i_{j_i}$ for infinitely many i; hence since $K^i_{j_i}$ is nowhere P_β for each $\beta < \alpha_{i+1}-1$, $K \backslash A$ is nowhere P_β for each $\beta < \alpha$. Thus, $K \backslash A \in X^2_\alpha$. □

Lemmas 3.5.13 and 3.5.14 show that the inductive hypotheses are satisfied for $(\alpha+m,i)$ if $m = 0$. We now turn to the case where $m = 2n+1$ for some $n < \omega$, and $i = 1$.

3.5.15 LEMMA: (3) *holds*.

Proof: First, let $X \in X^1_{\alpha+2n+1}$; then no closed subset of X belongs to $X^2_{\alpha+2n}$, and hence $C \backslash X$ is $P_{\alpha+2n}$ by hypothesis (2). Let U be a non-empty clopen subset of C.

<u>Case 1</u>: $n = 0$. Suppose that $U \backslash X$ is P_β for some $\beta < \alpha$. By lemmas 3.4.4 and 3.5.6, $U \cap X$ is P_γ for some $\gamma < \alpha$, a contradiction, hence $C \backslash X \in X^2_\alpha$.

<u>Case 2</u>: $n > 0$. Suppose that $U \backslash X$ is $P_{\alpha+2n-1}$. By lemma 3.5.6, $U \cap X$ is $P_{\alpha+2n}$, a contradiction; hence $C \backslash X$ is nowhere $P_{\alpha+2n-1}$. Also, since $U \cap X$ is not $P_{\alpha+2n-1}$, we obtain from hypothesis (8) a Cantor set K in U such that $K \backslash X \in X^2_{\alpha+2n-1}$; so $C \backslash X \in X^2_{\alpha+2n}$.

Conversely, suppose that $C \backslash X \in X^2_{\alpha+2n}$; then by lemma 3.5.6, X is $P_{\alpha+2n+1}$. Let U be a non-empty clopen subset of C.

<u>Case 1</u>: $n = 0$. As in case 1 above, X is nowhere P_β for each $\beta < \alpha$. Suppose that $U \cap X$ is P_α; then $U \cap X \in X_\alpha$, so $U \cap X$ is first category by lemma 3.5.8. Hence $U \backslash X \in X^2_\alpha$ is Baire (see 1.12.3), contradicting lemma 3.5.8; so X is nowhere P_α. Finally, suppose that X contains a closed subset B belonging to X^2_α. By the same argument as above, $\overline{B} \backslash B$ is nowhere P_α; but $\overline{B} \backslash B$

is a closed subset of $\mathcal{C}\setminus X$, so we have a contradiction with lemma 3.5.1.
Case 2: $n > 0$. We must show that $U \cap X$ is not $P_{\alpha+2n}$. Since $U\setminus X$ is nowhere $P_{\alpha+2n-1}$, $U \cap X$ is nowhere $P_{\alpha+2n-2}$ by lemma 3.5.6. Assume that $U \cap X$ is $P_{\alpha+2n-1}$; since $U\setminus X$ is nowhere $P_{\alpha+2n-2}$ (corollary 3.5.3), it follows from hypothesis (5) that every clopen subset of $U \cap X$ contains a closed subset belonging to $X^2_{\alpha+2n-2}$, and hence that $U \cap X \in X^2_{\alpha+2n-1}$. But then by hypothesis (9), $U\setminus X \in X^1_{\alpha+2n}$, so $U\setminus X$ is a non-empty clopen subset of $\mathcal{C}\setminus X$ which contains no closed subset belonging to $X^2_{\alpha+2n-1}$, contradicting $\mathcal{C}\setminus X \in X^2_{\alpha+2n}$; so $U \cap X$ is not $P_{\alpha+2n-1}$. But then $U \cap X$ is not $P_{\alpha+2n}$ either, since by lemma 3.5.4, this would imply that $U \cap X$ is first category, and hence that $U\setminus X \in X^2_{\alpha+2n}$ is Baire, contradicting lemma 3.5.8. It remains to be shown that X contains no closed subsets belonging to $X^2_{\alpha+2n}$. So suppose to the contrary that X contains such a subset B. By the argument above, $\overline{B}\setminus B$ is nowhere $P_{\alpha+2n}$; but $\overline{B}\setminus B$ is a closed subset of $\mathcal{C}\setminus X \in X^2_{\alpha+2n}$, a contradiction with lemma 3.5.1. □

3.5.16 LEMMA: (4) *and* (5) *hold*.

Proof: By hypothesis (1), $X^2_{\alpha+2n} \neq \emptyset$, say $X \in X^2_{\alpha+2n}$. Embed X densely in $\mathcal{C}$, then by lemma 3.5.15, $\mathcal{C}\setminus X \in X^1_{\alpha+2n+1}$, so $|X^1_{\alpha+2n+1}| \geq 1$. And if $X,Y \in X^1_{\alpha+2n+1}$ are dense subsets of $\mathcal{C}$, then by lemma 3.5.15, $\mathcal{C}\setminus X, \mathcal{C}\setminus Y \in X^2_{\alpha+2n}$, and hence by hypothesis (1), there exists a homeomorphism $h: \mathcal{C} \to \mathcal{C}$ such that $h[\mathcal{C}\setminus X] = \mathcal{C}\setminus Y$, and thus also $h[X] = Y$. This proves hypothesis (4). To prove (5), let K' be the Cantor set obtained from hypothesis (2), and put $K = \overline{K'\setminus A}$; then $K \cap A \in X^1_{\alpha+2n+1}$ by lemma 3.5.15. □

The next case is $m = 2n+1$ for some $n < \omega$, and $i = 2$; we must show that hypotheses (6), (7), and (8) are valid.

3.5.17 LEMMA: (6) *holds*.

Proof: First, let $X \in X^2_{\alpha+2n+1}$. Then X is $P_{\alpha+2n+1}$, so we can write $X = A \cup B$, where A is $P_{\alpha+2n}$, and B is complete. Put $\mathcal{C}\setminus B = \bigcup_{i=0}^{\infty} K_i$, where K_i is compact, and let $L_i = K_i\setminus X$; then L_i is closed in $\mathcal{C}\setminus X$, and $\mathcal{C}\setminus X = \bigcup_{i=0}^{\infty} L_i$. Since $K_i \cap X$ is a closed subset of A, $K_i \cap X$ is $P_{\alpha+2n}$, and hence L_i is $P_{\alpha+2n+1}$, by lemma 3.5.6. We will show that L_i is nowhere dense in $\mathcal{C}\setminus X$, and that L_i does not contain any closed subsets belonging to $X^2_{\alpha+2n}$. Then we will construct closed nowhere dense subsets A_i of $\mathcal{C}\setminus X$ such that $L_i \subset A_i \in X^1_{\alpha+2n+1}$; since $|X^1_{\alpha+2n+1}| = 1$ by hypothesis (4), it

then follows from theorem 3.2.4 that $\mathcal{C}\setminus X \approx \mathcal{Q}\times Y$ for $Y \in \mathcal{X}^1_{\alpha+2n+1}$.
Now let U be a non-empty clopen subset of $\mathcal{C}$, and suppose that $U\setminus X$ is $\mathcal{P}_{\alpha+2n}$.

Case 1: $n = 0$. Since $U \cap X$ is nowhere $\mathcal{P}_\alpha$, $U\setminus X$ is nowhere $\mathcal{P}_\beta$ for each $\beta < \alpha$, so $U\setminus X \in \mathcal{X}^2_\alpha$.

Case 2: $n > 0$. Since $U \cap X$ is nowhere $\mathcal{P}_{\alpha+2n}$, $U\setminus X$ is nowhere $\mathcal{P}_{\alpha+2n-1}$ by lemma 3.5.6. Let V be a non-empty clopen subset of U. Since $V \cap X$ is not $\mathcal{P}_{\alpha+2n-1}$, we can apply hypothesis (8) to obtain a closed subset of $V\setminus X$ which belongs to $\mathcal{X}^2_{\alpha+2n-1}$; hence $U\setminus X \in \mathcal{X}^2_{\alpha+2n}$.

So in both cases, if $U\setminus X$ is $\mathcal{P}_{\alpha+2n}$, then $U\setminus X \in \mathcal{X}^2_{\alpha+2n}$. But then by hypothesis (3), $U \cap X \in \mathcal{X}^1_{\alpha+2n+1}$ is a non-empty clopen subset of X containing no closed subset belonging to $\mathcal{X}^2_{\alpha+2n}$, contradicting $X \in \mathcal{X}^2_{\alpha+2n+1}$; hence $\mathcal{C}\setminus X$ is nowhere $\mathcal{P}_{\alpha+2n}$. From this it follows that if $U\setminus X$ were $\mathcal{P}_{\alpha+2n+1}$, say $U\setminus X = G \cup H$, where G is $\mathcal{P}_{\alpha+2n}$, and H is complete, then H would be a dense complete subset of $U\setminus X$, so $U \cap X$ would be first category, contradicting lemma 3.5.8 since $U \cap X \in \mathcal{X}^2_{\alpha+2n+1}$. Thus, $\mathcal{C}\setminus X$ is nowhere $\mathcal{P}_{\alpha+2n+1}$, and, in particular, each L_i is nowhere dense in $\mathcal{C}\setminus X$.

If B is a closed subset of L_i belonging to $\mathcal{X}^2_{\alpha+2n}$, then $\overline{B}\setminus B \in \mathcal{X}^1_{\alpha+2n+1}$ by hypothesis (3), hence $\overline{B}\setminus B$ is not $\mathcal{P}_{\alpha+2n}$; but on the other hand, $\overline{B}\setminus B = \overline{B} \cap X \subset K_i \cap X \subset A$ is a closed subset of A, and A is $\mathcal{P}_{\alpha+2n}$, hence $\overline{B}\setminus B$ is $\mathcal{P}_{\alpha+2n}$ by lemma 3.5.1. Thus, no closed subset of L_i belongs to $\mathcal{X}^2_{\alpha+2n}$.

Fix $i < \omega$, and let $\mathcal{D}$ be a disjoint covering of $(\mathcal{C}\setminus X)\setminus L_i$ by clopen subsets of $\mathcal{C}\setminus X$ such that for each $D \in \mathcal{D}$, $\operatorname{diam}(D) < d(D,L_i)$. Since $\mathcal{C}\setminus X$ is nowhere $\mathcal{P}_{\alpha+2n}$, it follows from hypothesis (5) that each $D \in \mathcal{D}$ contains a closed subset $E(D) \in \mathcal{X}^1_{\alpha+2n+1}$. Then

$$A_i = L_i \cup \bigcup_{D\in\mathcal{D}} E(D)$$

is closed in $\mathcal{C}\setminus X$ and $\mathcal{P}_{\alpha+2n+1}$, hence nowhere dense in $\mathcal{C}\setminus X$. Clearly, A_i contains no closed subsets belonging to $\mathcal{X}^2_{\alpha+2n}$, and since each open subset of A_i intersects some $E(D)$, A_i is nowhere $\mathcal{P}_{\alpha+2n}$; hence $A_i \in \mathcal{X}^1_{\alpha+2n+1}$. This proves the first part of the lemma.

Conversely, assume that $\mathcal{C}\setminus X \approx \mathcal{Q}\times Y$ for $Y \in \mathcal{X}^1_{\alpha+2n+1}$, say $\mathcal{C}\setminus X = \bigcup_{i=0}^\infty A_i$, where each $A_i \in \mathcal{X}^1_{\alpha+2n+1}$ is a closed nowhere dense subset of $\mathcal{C}\setminus X$. Then

$$X = \bigcup_{i=0}^\infty (\overline{A}_i\setminus A_i) \cup \mathcal{C}\setminus(\bigcup_{i=0}^\infty \overline{A}_i).$$

By hypothesis (3), $\overline{A}_i\setminus A_i \in \mathcal{X}^2_{\alpha+2n}$, hence X is strongly σ-$\mathcal{P}_{\alpha+2n}$ $\cup$ complete is $\mathcal{P}_{\alpha+2n+1}$ by corollary 3.5.5; and since $\bigcup_{i=0}^\infty(\overline{A}_i\setminus A_i)$ is dense in X, every non-empty clopen subset of X contains a closed subset belonging to

$X^2_{\alpha+2n}$. It remains to be shown that X is nowhere $P_{\alpha+2n}$; so let U be a non-empty clopen subset of C.

Case 1: n = 0. Suppose that $U \cap X$ is P_β for some $\beta < \alpha$; then $U\backslash X$ is P_γ for some $\gamma < \alpha$, hence for each i, $A_i \cap (U\backslash X)$ is P_γ. Since $A_i \cap (U\backslash X) \neq \emptyset$ for some i, this contradicts the fact that A_i is nowhere P_γ for each $\gamma < \alpha$; hence $U \cap X$ is nowhere P_β for each $\beta < \alpha$. So if $U \cap X$ were P_α, then it would be first category, hence $U\backslash X$ would be Baire, a contradiction.

Case 2: n > 0. Suppose that $U \cap X$ is $P_{\alpha+2n-1}$; then $U\backslash X$ is $P_{\alpha+2n}$ by lemma 3.5.6, and hence $A_i \cap (U\backslash X)$ is $P_{\alpha+2n}$ for each i. Since some $A_i \cap (U\backslash X)$ is non-empty, this contradicts the fact that $A_i \in X^1_{\alpha+2n+1}$ is nowhere $P_{\alpha+2n}$; hence $U \cap X$ is nowhere $P_{\alpha+2n-1}$. Again, if $U \cap X$ were $P_{\alpha+2n}$, hence strongly σ-$P_{\alpha+2n-1}$ by lemma 3.5.4, then $U \cap X$ would be first category, and as in case 1, we obtain a contradiction. □

3.5.18 LEMMA: (7) *holds*.

Proof: By hypothesis (4), $X^1_{\alpha+2n+1} \neq \emptyset$, say $Y \in X^1_{\alpha+2n+1}$. Let X be a dense copy of $Q \times Y$ in the Cantor set; then by lemma 3.5.17, $C\backslash X \in X^2_{\alpha+2n+1}$, and hence $X^2_{\alpha+2n+1} \neq \emptyset$. Now assume that $X_0, X_1 \in X^2_{\alpha+2n+1}$ are dense subsets of the Cantor set; by lemma 3.5.17, and since $|X^1_{\alpha+2n+1}| = 1$, we have $C\backslash X_0 \approx C\backslash X_1 \approx Q \times Y$ for some $Y \in X^1_{\alpha+2n+1}$. By theorem 3.2.8, and hypothesis (4), there exists a homeomorphism $h: C \to C$ such that $h[C\backslash X_0] = C\backslash X_1$; hence $h[X_0] = X_1$. □

3.5.19: (8) *holds*.

Proof: Let $P = P_{\alpha+2n}$, and let Z be the unique element of $X^1_{\alpha+2n+1}$. We will show that the hypotheses of lemma 3.3.4 are satisfied. By lemma 3.5.1, P is closed-hereditary, and P is strongly σ-additive by corollary 3.5.5. Let B be a Borel subset of a compact space Y which is not P; then by hypothesis (5), Y contains a Cantor set K such that $K\backslash B \in X^2_{\alpha+2n}$ and $K \cap B \in X^1_{\alpha+2n+1}$. Then $K \cap B \approx Z$, and $K\backslash B = \bigcup_{i=0}^\infty X_i$, where X_i is closed in $K\backslash B$ and P_{β_i} for some $\beta_i < \alpha$ (if n = 0), resp. $P_{\alpha+2n-1}$ (if n > 0, applying lemma 3.5.4). Put $G = \bigcup_{i=0}^\infty \overline{X}_i$, then G is σ-compact, and $G \cap B = \bigcup_{i=0}^\infty (\overline{X}_i \backslash X_i)$ is P_α (if n = 0), resp. strongly σ-$P_{\alpha+2n}$ and hence $P_{\alpha+2n}$ (if n > 0, applying lemma 3.5.6 and corollary 3.5.5). So by lemma 3.3.4, if A is a Borel subset of a compact space X which is not $P_{\alpha+2n} \cup$ complete, then X contains a Cantor set $K = \overline{K \cap A}$ such that $K \cap A \approx Q \times Z$. Then $K\backslash A \in X^2_{\alpha+2n+1}$ by lemma 3.5.17. □

We now consider the case $m = 2n+2$ for some $n < \omega$, and $i = 1$ (recall that if $m = 0$, then we only have the case $(\alpha+m,2)$); we must show that hypotheses (9), (10), and (11) are true for this n.

3.5.20 LEMMA: *Let* Y *be the unique element of* $\mathcal{X}^1_{\alpha+2n+1}$*; then* $\mathbf{Q}\times Y$ *is the unique element of* $\mathcal{X}^1_{\alpha+2n+2}$*, up to homeomorphism.*

Proof: We first show that $\mathbf{Q}\times Y \in \mathcal{X}^1_{\alpha+2n+2}$. Clearly, $\mathbf{Q}\times Y = \bigcup_{q\in\mathbf{Q}}(\{q\}\times Y)$ is strongly σ-$\mathcal{P}_{\alpha+2n+1}$ is $\mathcal{P}_{\alpha+2n+2}$ by lemma 3.5.4. Note that, since every non-empty clopen subset of $\mathbf{Q}\times Y$ contains a closed copy of Y, and since Y is not $\mathcal{P}_{\alpha+2n}$, $\mathbf{Q}\times Y$ is nowhere $\mathcal{P}_{\alpha+2n}$ by lemma 3.5.1. Suppose that $U\times V$ is a non-empty basic clopen subset of $\mathbf{Q}\times Y$ which is $\mathcal{P}_{\alpha+2n+1}$; then we can write $U\times V = A\cup B$, where A is $\mathcal{P}_{\alpha+2n}$, and B is complete. Since $U\times V$ is nowhere $\mathcal{P}_{\alpha+2n}$ by the preceding remark, B is dense in $U\times V$, contradicting the fact that $U\times V$ is not Baire; so $\mathbf{Q}\times Y$ is nowhere $\mathcal{P}_{\alpha+2n+1}$. It remains to be shown that $\mathbf{Q}\times Y$ contains no closed subset belonging to $\mathcal{X}^2_{\alpha+2n+1}$; so embed $\mathbf{Q}\times Y$ as a dense subset of the Cantor set, and suppose to the contrary that $B\in\mathcal{X}^2_{\alpha+2n+1}$ is closed in $\mathbf{Q}\times Y$. Then by hypothesis (6), $\overline{B}\setminus B \approx \mathbf{Q}\times Y$, so $\overline{B}\setminus B$ is not $\mathcal{P}_{\alpha+2n+1}$ by the above argument. On the other hand, $\overline{B}\setminus B$ is a closed subset of $\mathbf{C}\setminus(\mathbf{Q}\times Y)$, and $\mathbf{C}\setminus(\mathbf{Q}\times Y)\in\mathcal{X}^2_{\alpha+2n+1}$ by another application hypothesis (6); so $\overline{B}\setminus B$ is $\mathcal{P}_{\alpha+2n+1}$ by lemma 3.5.1, and we have the required contradiction.
To show that $\mathbf{Q}\times Y$ is the only element of $\mathcal{X}^1_{\alpha+2n+2}$, it suffices to show that if $X\in\mathcal{X}^1_{\alpha+2n+2}$ is a dense subset of $\mathbf{C}$, then $\mathbf{C}\setminus X\in\mathcal{X}^2_{\alpha+2n+1}$; hypothesis (6) then yields that $X\approx\mathbf{Q}\times Y$. First note that, since X contains no closed subsets belonging to $\mathcal{X}^2_{\alpha+2n+1}$, it follows from hypothesis (8) that $\mathbf{C}\setminus X$ is $\mathcal{P}_{\alpha+2n+1}$. Also, since X is nowhere $\mathcal{P}_{\alpha+2n+1}$, $\mathbf{C}\setminus X$ is nowhere $\mathcal{P}_{\alpha+2n}$ by lemma 3.5.6. And if U is clopen in $\mathbf{C}$, then since $U\cap X$ is not $\mathcal{P}_{\alpha+2n}$, we can apply hypothesis (2) and obtain a closed subset of $U\setminus X$ belonging to $\mathcal{X}^2_{\alpha+2n}$. □

3.5.21 LEMMA: (9), (10), *and* (11) *hold.*

Proof: (9) follows from (6) and lemma 3.5.20. If $X_0, X_1 \in \mathcal{X}^1_{\alpha+2n+2}$ are dense subsets of the Cantor set, then $\mathbf{C}\setminus X_0, \mathbf{C}\setminus X_1 \in \mathcal{X}^2_{\alpha+2n+1}$, so by (7) there exists a homeomorphism $h\colon \mathbf{C}\to\mathbf{C}$ such that $h[\mathbf{C}\setminus X_0] = \mathbf{C}\setminus X_1$, and hence $h[X_0] = X_1$. (11) follows from (8) and lemma 3.5.20. □

Finally, we must show that the hypotheses are satisfied if $m = 2n+2$ for some $n < \omega$, and $i = 2$; i.e. we must show that (1) and (2) hold if $n > 0$.

3.5.22 LEMMA: *Let* $n \in \mathbb{N}$*, and let* Y *be the unique element of* $\mathcal{X}^2_{\alpha+2n-1}$*; then* $\mathbb{Q} \times Y$ *is the unique element of* $\mathcal{X}^2_{\alpha+2n}$*, up to homeomorphism.*

Proof: As in the proof of lemma 3.5.20, it can be shown that $\mathbb{Q} \times Y$ is $\mathcal{P}_{\alpha+2n}$, and nowhere $\mathcal{P}_{\alpha+2n-1}$; and since Y is strongly homogeneous, every non-empty clopen subset of $\mathbb{Q} \times Y$ contains a closed copy of Y. Hence, $\mathbb{Q} \times Y \in \mathcal{X}^2_{\alpha+2n}$. Now suppose that $X \in \mathcal{X}^2_{\alpha+2n}$. Then X is $\mathcal{P}_{\alpha+2n}$, so by lemma 3.5.4, we can write $X = \bigcup_{i=0}^{\infty} X_i$, where each X_i is closed in X, and $\mathcal{P}_{\alpha+2n-1}$. Since X is nowhere $\mathcal{P}_{\alpha+2n-1}$, each X_i is nowhere dense in X. Fix $i < \omega$, and let $\mathcal{D}$ be a disjoint covering of $X \backslash X_i$ by clopen subsets of X such that $\operatorname{diam}(D) < d(D, X_i)$ for each $D \in \mathcal{D}$. Then each $D \in \mathcal{D}$ contains a closed copy $E(D)$ of Y, and as before, it is easily shown that

$$A_i = X_i \cup \bigcup_{D \in \mathcal{D}} E(D)$$

is a closed nowhere dense subset of X such that $A_i \approx Y$. Hence, $X = \bigcup_{i=0}^{\infty} A_i \approx \mathbb{Q} \times Y$ by theorem 3.2.4. □

3.5.23 LEMMA: *If* $n > 0$*, then* (1) *holds.*

Proof: By lemma 3.5.22, $|\mathcal{X}^2_{\alpha+2n}| = 1$. So suppose that $X_0, X_1 \in \mathcal{X}^2_{\alpha+2n}$ are dense in $\mathcal{C}$. If Y is the unique element of $\mathcal{X}^2_{\alpha+2n-1}$ (hypothesis (7)), then $X_0 \approx X_1 \approx \mathbb{Q} \times Y$ by lemma 3.5.22, so by hypothesis (7), we can apply theorem 3.2.8 to obtain a homeomorphism $h: \mathcal{C} \to \mathcal{C}$ such that $h[X_0] = X_1$. □

3.5.24 LEMMA: *If* $n > 0$*, then* (2) *holds.*

Proof: Let $\mathcal{P} = \mathcal{P}_{\alpha+2n}$, let Z' be the unique element of $\mathcal{X}^2_{\alpha+2n-1}$, and let Y' be the unique element of $\mathcal{X}^1_{\alpha+2n}$. By lemma 3.5.1, $\mathcal{P}$ is closed-hereditary, and $\mathcal{P}$ is strongly σ-additive by corollary 3.5.5. Let B be a Borel subset of a compact space Y which is not $\mathcal{P}$. Then by lemma 3.5.6, B is not $\mathcal{P}_{\alpha+2n-1}$, and hence by hypothesis (11), Y contains a Cantor set K such that $K \backslash B \in \mathcal{X}^2_{\alpha+2n-1}$ and $K \cap B \in \mathcal{X}^1_{\alpha+2n}$; so $K \backslash B \approx Z'$, and $K \cap B \approx Y'$ is $\mathcal{P}$. Thus, by lemma 3.3.5, if A is a Borel subset of a compact space X which is not $\mathcal{P}_{\alpha+2n}$, then there exist Cantor sets B_i in X, for each $i < \omega$, such that $K = (\bigcup_{i=0}^{\infty} B_i)^{-} \approx \mathcal{C}$, and such that for each $i < \omega$,

(i) B_i is nowhere dense in K;
(ii) $K \backslash A = \bigcup_{i=0}^{\infty} (B_i \backslash A)$;
(iii) $B_0 \backslash A \approx Z \approx (B_{i+1} \backslash B_i) \backslash A$.

As in the proof of lemma 3.4.23, we deduce from theorem 3.2.4 that $K\backslash A \approx \mathbb{Q}\times Z$, and hence by lemma 3.5.22, $K\backslash A \in X^2_{\alpha+2n}$. □

Since the "if" part of theorem 3.5.11 is trivial, we have completed the proof of theorems 3.5.9, 3.5.10, and 3.5.11.

We finish this section with the analogue to theorem 3.4.25, that all spaces defined above are topologically different:

3.5.25 THEOREM: *Let* $\beta,\gamma \in [\omega,\omega_1)$, *and* $i,j \in \{1,2\}$ *be such that* $i = 2$ *if* $\lim(\beta)$, *and* $j = 2$ *if* $\lim(\gamma)$. *If* $(\beta,i) \neq (\gamma,j)$, then $X^i_\beta \cap X^j_\gamma = \emptyset$. *Furthermore,* $X^i_\beta \cap \bigcup_{n=1}^\infty D_{2n}(\Sigma^0_2) = \emptyset$, *so the spaces defined in this section are topologically distinct from those of section* 3.4 *and Chapter* 2.

Proof: If $\beta \neq \gamma$, say $\beta < \gamma$, then $X \in X^j_\gamma$ is nowhere P_β by corollary 3.5.3, so $X \notin X^i_\beta$. If $\beta = \gamma$, then $i \neq j$, say $i = 1$, $j = 2$; hence $X \in X^j_\gamma$ contains a closed subset belonging to $X^2_{\gamma-1}$, whereas $X \in X^i_\beta$ does not. The last statement is clear since each of the spaces defined in this section is nowhere P_γ for $\gamma < \omega$ (corollary 3.5.3). □

3.6 The main theorem, part 1

The principal purpose of this section is to show that the spaces, which were defined and characterized in sections 3.4 and 3.5, together with those of Chapter 2, are the only homogeneous zero-dimensional absolute Borel sets of ambiguous class 2.

3.6.1 LEMMA: *Let* X *be homogeneous and zero-dimensional.*
(a) *If* $\gamma = -1$ *or* $\gamma < \omega_1$, $i \in \{1,2\}$, *and* X *is not* $P^{(i)}_\gamma$, *then* X *is nowhere* $P^{(i)}_\gamma$.
(b) *If* Y *is a strongly homogeneous closed subset of* X, *then every non-empty clopen subset of* X *contains a closed copy of* Y.

Proof: (a) Suppose that U is a non-empty clopen subset of X which is $P^{(i)}_\gamma$. Let $x \in U$, and for each $y \in X$, let $h_y: X \to X$ be a homeomorphism such that $h_y(x) = y$. If $\{U_j: j < \omega\}$ is a countable subcovering of $\{h_y[U]: y \in X\}$, then for each $n < \omega$, the clopen set $V_n = U_n\backslash(\bigcup_{j<n} U_j)$ is $P^{(i)}_\gamma$ by lemma 3.4.1 or 3.5.1. Hence $X = \oplus_{n=0}^\infty V_n$ is also $P^{(i)}_\gamma$.
(b) Let U be a non-empty clopen subset of X, $x \in U$, and $y \in Y$. If $h: X \to X$ is a homeomorphism such that $h(y) = x$, then $h[Y] \cap U$ is a non-

empty clopen subset of $h[Y] \approx Y$, so $h[Y] \cap U \approx Y$ since Y is strongly homogeneous. Since $h[Y]$ is closed in X, $h[Y] \cap U$ is closed in X. □

We are now ready to prove the main theorem of this chapter. For each $\alpha < \omega_1$, $i \in \{1,2\}$, let $X_\alpha^{(i)}$ denote the unique element of $\mathcal{X}_\alpha^{(i)}$ (theorems 3.4.13 and 3.5.9).

3.6.2 THEOREM: *The topological types of homogeneous zero-dimensional absolute Borel sets of ambiguous class* 2 *are precisely those of* $\boldsymbol{C}$, $\boldsymbol{C}\backslash\{p\}$, $\boldsymbol{Q}$, $\boldsymbol{Q}\times\boldsymbol{C}$, $\boldsymbol{P}$, *the discrete spaces, and the spaces* $X_\alpha^{(i)}$, *for* $i \in \{1,2\}$ *and* $\alpha < \omega_1$.

Proof: From the results of Chapter 2, and sections 3.4 and 3.5, it follows that all these spaces are homogeneous. So let X be a homogeneous zero-dimensional absolute Borel set of ambiguous class 2. By theorem 3.1.2, $X \in D_\gamma(\Sigma_2^0)$ for some even $\gamma \in [1,\omega_1)$; let α be minimal in $[1,\omega_1)$ such that α is even and $X \in D_\alpha(\Sigma_2^0)$.
Case 1: $\alpha < \omega$. Let $<$ be the ordering on the properties $\mathcal{P}_m^{(j)}$, defined in theorem 3.4.24, and let $k \in \boldsymbol{N}$, $i \in \{1,2\}$ be such that $\mathcal{P}_k^{(i)}$ is minimal with respect to $<$ such that X is $\mathcal{P}_k^{(i)}$. If X is σ-compact or complete, then either X is discrete, or $X \in \{\boldsymbol{C}, \boldsymbol{C}\backslash\{p\}, \boldsymbol{Q}, \boldsymbol{Q}\times\boldsymbol{C}, \boldsymbol{P}\}$, see the introduction of this monograph; otherwise, X is nowhere $\mathcal{P}_m^{(j)}$ for each $\mathcal{P}_m^{(j)} < \mathcal{P}_k^{(i)}$ $(m \geq -1,\ j \in \{1,2\})$ by lemma 3.6.1, so $X \in \mathcal{X}_k^{(i)}$ by theorem 3.4.24.
Case 2: $\alpha \geq \omega$. Then X is $\mathcal{P}_\alpha$. If α is a limit, then X is $\mathcal{P}_\alpha$ and nowhere $\mathcal{P}_\beta$ for each $\beta < \alpha$ by lemma 3.6.1, so $X \in \mathcal{X}_\alpha^2$. If α is a successor, then $\alpha = \beta+2$ for some $\beta < \omega_1$, and X is $\mathcal{P}_\beta \cup \mathcal{S}_1$ by lemma 3.5.2. First suppose that X is not $\mathcal{P}_\beta \cup$ complete; then by lemma 3.6.1, X is nowhere $(\mathcal{P}_\beta \cup \text{complete})$. So if X contains no closed subsets belonging to $\mathcal{X}_{\beta+1}^2$, then $X \in \mathcal{X}_\alpha^1$; otherwise, applying lemma 3.6.1(b), we find that $X \in \mathcal{X}_\alpha^2$. If on the other hand X is $\mathcal{P}_\beta \cup$ complete, then since X is nowhere $\mathcal{P}_\beta$ by lemma 3.6.1 and by minimality of α, we have in the same way as above that $X \in \mathcal{X}_{\beta+1}^1$ or $X \in \mathcal{X}_{\beta+1}^2$. □

3.6.3 COROLLARY: *There are exactly* ω_1 *homogeneous zero-dimensional absolute Borel sets of ambiguous class* 2. □

Since there are spaces of arbitrarily high level in our subhierarchy of Δ_3^0, it is impossible to find a single element of Δ_3^0 that contains all zero-dimensional absolute Borel sets of ambiguous class 2 as a closed subspace.

Also, "product theorems" similar to 2.1.3, 2.2.4, 2.3.3, 2.4.3, and 2.4.7 do not hold, i.e. if A is $P_\gamma^{(i)}$ in C, then it is not necessarily true that $A \times X_\gamma^{(i)} \approx X_\gamma^{(i)}$. For example, since S is neither σ-compact nor complete, by theorem 3.3.1 it contains a closed copy of P and a closed copy of Q; so $S \times S$ contains a closed copy of $Q \times P$, and thus $S \times S \not\approx S$ since $Q \times P$ is not σ-compact ∪ complete.
However, we do have the following theorem.

3.6.4 THEOREM: *Let* A *be a zero-dimensional space.*
(a) *If* $n \in N$, $i \in \{1,2\}$, *and* A *is* $P_n^{(i)}$, *then* A *can be embedded as a closed subspace of* $X_n^{(i)}$.
(b) *If* $\alpha \in [\omega,\omega_1)$, *and* A *is* P_α, *then* A *can be embedded as a closed subspace of* X_α^2.
(c) *If* $\alpha \in [\omega,\omega_1)$ *is a successor,* A *is* P_α, *and* A *does not contain any closed copies of* $X_{\alpha-1}^2$, *then* A *can be embedded as a closed subspace of* X_α^1.
Proof:(a) For each $k \in N$, let Y_k be a dense copy of $X_n^{(i)}$ in $\{1/k\} \times C$, and consider A as a subset of $\{0\} \times C$. We claim that, as a subset of $[0,1] \times C$,

$$\tilde{A} = A \cup \bigcup_{k=1}^\infty Y_k \approx X_n^{(i)}.$$

Indeed, since A is nowhere dense in $\tilde{A}$, $\tilde{A}$ has the same "nowhere" properties as $X_n^{(i)}$, so it suffices to show that $\tilde{A}$ is $P_n^{(i)}$. Since $\tilde{A}$ is strongly σ-$P_n^{(i)}$, this is trivial if n is odd, by corollary 3.4.3. If n is even, then $P_n^{(i)} \equiv P_{n-3}^{(i)}$ ∪ complete; so write $A = A_0 \cup A_1$, $Y_k = Y_k^0 \cup Y_k^1$, where A_0, Y_k^0 are $P_{n-3}^{(i)}$, and A_1, Y_k^1 are complete. Then $A_0 \cup \bigcup_{k=1}^\infty Y_k^0$ is strongly σ-$P_{n-3}^{(i)}$ is $P_{n-3}^{(i)}$ by corollary 3.4.3, and $A_1 \cup \bigcup_{k=1}^\infty Y_k^1 = A_1 \cup \bigoplus_{k=1}^\infty Y_k^1$ is complete ∪ complete is complete; so $\tilde{A}$ is $P_{n-3}^{(i)}$ ∪ complete is $P_n^{(i)}$. It is clear that $A = \tilde{A} \cap (\{0\} \times C)$ is closed in $\tilde{A}$.
(b) Define $\tilde{A}$ as above, with Y_k a copy of X_α^2. Again, $\tilde{A}$ is nowhere P_β for $\beta < \alpha$, and if α is a successor, then each non-empty clopen subset of $\tilde{A}$ contains a closed copy of $X_{\alpha-1}^2$, since it intersects some Y_k. If α is even, then $\tilde{A}$ is P_α by corollary 3.5.5, and if α is odd, then $P_\alpha \equiv P_{\alpha-1}$ ∪ complete, so since $P_{\alpha-1}$ is strongly σ-additive, we can argue as under (a). Also as in (a), A is closed in $\tilde{A}$.
(c) As in (b), we can show that $\tilde{A}$ is P_α and nowhere $P_{\alpha-1}$. If B is a closed copy of $X_{\alpha-1}^2$ in $\tilde{A}$, then $B \not\subset A$, so $B \cap Y_k \neq \emptyset$ for some k. However, $B \cap Y_k = B \cap (\{1/k\} \times C)$ is clopen in B, so $B \cap Y_k$ is a closed copy of $X_{\alpha-1}^2$ in Y_k, a contradiction. Thus, $\tilde{A} \approx X_\alpha^1$, and A is closed in $\tilde{A}$. □

The aim of van Douwen in constructing $\boldsymbol{T}$ was to give an example of a homogeneous absolute Borel set which cannot be given the structure of a topological group. We conclude this chapter with the theorem that neither $\boldsymbol{S}$ nor $\boldsymbol{T}$ admits a group structure; the proof is taken from van Mill [40]. We do not have an answer to the general problem of which of the spaces characterized in this chapter can be made into a topological group.

3.6.5 THEOREM: *Suppose that* X *is the union of a* σ*-compact subset and a complete subset, and* X *is neither* σ*-compact nor complete. Then* X *does not admit the structure of a topological group.*

Proof: Write $X = A \cup B$, where A is σ-compact and B is complete. Since X is not complete, by theorem 3.3.1, X contains a closed copy Q of $\boldsymbol{Q}$. If there is a group structure on X, and we let $\tilde{A}$ be the subgroup generated by $A \cup Q$, then $\tilde{A}$ is σ-compact. Since X is not σ-compact, $X\backslash\tilde{A}$ is non-empty, say $x \in X\backslash\tilde{A}$. Then the coset $xQ \approx \boldsymbol{Q}$ is closed in X, and contained in $X\backslash\tilde{A} \subset B$. But B is complete, a contradiction. □

3.6.6 COROLLARY: $\boldsymbol{S}$ *and* $\boldsymbol{T}$ *are homogeneous absolute Borel sets that do not admit the structure of a topological group.* □

CHAPTER 4: BOREL SETS OF HIGHER CLASS

In Chapter 3, we used the hierarchy of small Borel classes in Δ_3^0 to characterize all homogeneous zero-dimensional absolute Borel sets of ambiguous class 2. However, when extended to classes Δ_α^0 for $\alpha < \omega_1$, this hierarchy does not suffice to describe *all* homogeneous Borel sets in C in a way similar to the results of Chapter 3; in other words, the hierarchy of small Borel classes is too coarse to distinguish between all homogeneous zero-dimensional absolute Borel sets. Thus, we have to make further refinements in our hierarchy. And again, we are so fortunate that a description of such a hierarchy is already available: the so-called *Wadge hierarchy of Borel sets*, developed by Wadge in [60] (see Wadge [59] or Moschovakis [43]), suits our needs so well that it seems to have been defined just for the purpose of characterizing homogeneous zero-dimensional Borel sets. Since the reader is not assumed to be familiar with the theory of Wadge classes, we discuss the necessary background in section 4.1; among others, we present the interpretation of the Wadge hierarchy in terms of games, which enables one to prove very strong results with very little work (modulo a highly non-trivial, but well-known result, viz. determinacy of Borel games, Martin [35]). Also, we state a theorem of Steel [56], which will serve in many cases to prove that there exists at most one zero-dimensional space of a given type.

In section 4.2, we discuss an inductive construction process of the Wadge classes, similar to (but much more complicated than) the usual definition of the Borel classes given in 1.10; this construction process is due to Louveau [32].

Then, in section 4.3, we give a definition of certain classes of topological spaces, based on Louveau's description of the Wadge hierarchy of Borel sets; we determine which of the classes are non-empty, and, using Steel's theorem, we show that the non-empty ones contain exactly one element, up to homeomorphism, and that this space is homogeneous. In section 4.4, we show that every homogeneous zero-dimensional absolute Borel set which is not in Δ_3^0 belongs to one of the classes of section 4.3 and hence is topologically characterized by the properties describing the class.

Thus, we have obtained topological characterizations of *all* homogeneous zero-dimensional absolute Borel sets; it should be noted that the results of Chapter 3 provided us with *internal* topological characterizations, but that in general, this is *not* the case for the descriptions in terms of Wadge classes obtained in the present chapter.

Since the results in this chapter are rather abstract, we take a closer look at some special Wadge classes in section 4.5; in that section, we also prove some more Hurewicz-type theorems, and we partially answer a question of Sikorski from [54].

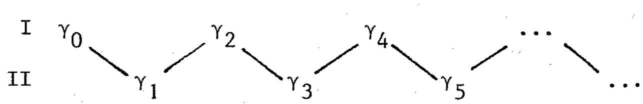
Finally, in section 4.6, we will exhibit some connections between the results of Chapters 2 and 3 and the Wadge hierarchy.

The main results of this chapter are taken from van Engelen [11].

4.1 The Wadge hierarchy of Borel sets

This section covers the basic notions from game theory; we list the main results about Wadge games and Wadge classes. A more detailed discussion can be found in Moschovakis [43] and Van Wesep [61], or in the unpublished dissertations of Wadge [60] (see also [59]), Steel [55] and Van Wesep [62]; a nice presentation of the subject is also given in Martin and Kechris [36].

Let X be a space, and let $A \subset X^\omega$. The *game* $G_X(A)$ is defined as follows: there are two players, I and II, who alternately choose elements of X, in this way:



In the end, we find a point $\gamma = (\gamma_n)_{n<\omega} \in X^\omega$; I wins if $\gamma \in A$, II wins if $\gamma \notin A$. Since I and II are assumed to have knowledge of all previous moves, these games were called *infinite games with perfect information* by Gale and Stewart [18].

A *strategy* for player I is a function σ assigning an element of X to each finite sequence of elements of X with even length; player I *follows* the strategy σ if for each $n > 0$, his n^{th} move (i.e. γ_{2n}) is $\sigma(\gamma_0,\ldots,\gamma_{2n-1})$. Thus, a strategy prescribes the moves, depending only on previous

moves. Similarly, a *strategy* for player II is a function τ assigning an element of X to each finite sequence of elements of X of odd length; player II follows τ if $\gamma_{2n+1} = \tau(\gamma_0,\ldots,\gamma_{2n})$ for all $n < \omega$.
If player I follows σ and player II follows τ in the above game, then we write $\gamma = \sigma*\tau$. We say that σ is a *winning strategy* for player I if, for each strategy τ of II, $\sigma*\tau \in A$; τ is a *winning strategy* for player II if, for each strategy σ of I, $\sigma*\tau \notin A$. The game $G_X(A)$ is *determined* if either player I or player II has a winning strategy.
We have the following famous theorem, due to Martin:

4.1.1 THEOREM (Martin [35]): *If* A *is a Borel subset of* X^ω, *then* $G_X(A)$ *is determined.*

Some special cases of this theorem were established earlier by Gale and Stewart [18], Wolfe [63], Davis [6], and Paris [45]. For a discussion of the *axiom of determinacy*, i.e. the statement (contradictory with the axiom of choice) that all games $G_\omega(A)$ are determined, see Moschovakis [43] and the references cited therein.

We will now describe a slightly different type of game, the *Lipschitz game* $G^X_\ell(A,B)$, where $A,B \subset X^\omega$. Again, we have two players, I and II, taking turns playing points of X, as follows:

I	II
α_0	β_0
α_1	β_1
α_2	β_2
$\vdots$	$\vdots$

In the end, I has played $\alpha = (\alpha_n)_{n<\omega}$, and II has played $\beta = (\beta_n)_{n<\omega}$; II wins $G^X_\ell(A,B)$ if and only if $(\alpha \in A \leftrightarrow \beta \in B)$, otherwise I wins. Note that if $h\colon X^\omega \times X^\omega \approx X^\omega$ is defined by $h(x,y)_{2n} = x_n$, $h(x,y)_{2n+1} = y_n$ for each $n < \omega$, then II wins $G^X_\ell(A,B)$ if and only if $(\alpha,\beta) \in (A\times B) \cup (X^\omega\backslash A \times X^\omega\backslash B)$ if and only if $h(\alpha,\beta) \in E = h[(A\times B) \cup (X^\omega\backslash A \times X^\omega\backslash B)]$ if and only if II wins $G_X(X^\omega\backslash E)$; in particular, if A and B are Borel subsets of X^ω, then $G^X_\ell(A,B)$ is determined by theorem 4.1.1.
If player I follows the strategy σ in $G^X_\ell(A,B)$, and ends up having played α against II's β, then we write $\sigma*[\beta] = \alpha$; similarly, if II follows τ, then we write $[\alpha]*\tau = \beta$. Define $f_\tau\colon X^\omega \to X^\omega$ by $f_\tau(\alpha) = [\alpha]*\tau$, and g_σ:

$X^\omega \to X^\omega$ by $g_\sigma(\beta) = \sigma*[\beta]$. If X is discrete, then since an initial segment of $f_\tau(\alpha)$, resp. $g_\sigma(\beta)$, only depends upon an initial segment of α, resp. β, both f_τ and g_σ are continuous. Furthermore, we have

τ is a winning strategy for player II if and only if $(\forall\alpha: \alpha \in A \leftrightarrow [\alpha]*\tau \in B)$ if and only if $(\forall\alpha: \alpha \in A \leftrightarrow f_\tau(\alpha) \in B)$ if and only if $A = f_\tau^{-1}[B]$;

and

σ is a winning strategy for player I if and only if $(\forall\beta: \sigma*[\beta] \notin A \leftrightarrow \beta \in B)$ if and only if $(\forall\beta: g_\sigma(\beta) \notin A \leftrightarrow \beta \in B)$ if and only if $B = g_\sigma^{-1}[X^\omega\setminus A]$.

Thus we have proved: if X is discrete, and A,B are Borel subsets of X^ω, then either there is a continuous $f: X^\omega \to X^\omega$ such that $A = f^{-1}[B]$ or there is a continuous $g: X^\omega \to X^\omega$ such that $B = g^{-1}[X^\omega\setminus A]$. The case $X = \omega$ is the so-called "Wadge lemma".

Remark: One might wonder, whether the existence of a continuous $f: X^\omega \to X^\omega$ satisfying $A = f^{-1}[B]$, where X is discrete, yields a winning strategy τ for II in the game $G_\ell^X(A,B)$. This is not the case. However, the mapping f_τ defined above has the special property that an initial segment of length n of $f_\tau(\alpha)$ is determined by an initial segment of length n of α. In other words, if X^ω is given the Baire space metric $\rho(\alpha,\tilde{\alpha}) = 1/(\min\{k: \alpha(k) \neq \tilde{\alpha}(k)\} + 1)$ if $\alpha \neq \tilde{\alpha}$, then f_τ satisfies the Lipschitz condition $\rho(f_\tau(\alpha),f_\tau(\tilde{\alpha})) \leq \rho(\alpha,\tilde{\alpha})$ (whence the name *Lipschitz game*). It is not hard to show that each $f: X^\omega \to X^\omega$ with $A = f^{-1}[B]$ and $\rho(f(\alpha),f(\tilde{\alpha})) \leq \rho(\alpha,\tilde{\alpha})$ for all $\alpha,\tilde{\alpha} \in X^\omega$ is some f_τ, where τ is a winning strategy for II in the game $G_\ell^X(A,B)$.

If f is just continuous, then in order to determine an initial segment of length n of $f(\alpha)$ we might need a much longer initial segment of α. In terms of games, player II might have to know more than n moves of player I before being able to play *his* n^{th} move such that he wins. And indeed, it can be shown that the existence of a continuous $f: X^\omega \to X^\omega$ with $A = f^{-1}[B]$ is equivalent to II having a winning strategy in the *Wadge game* $G_w^X(A,B)$, which is played just as the Lipschitz game, granting however player II the possibility of passing.

If $X = 2 = \{0,1\}$, then we write $A \leq_w B$ (A is *Wadge-reducible* to B) for "$A = f^{-1}[B]$ for some continuous $f: X^\omega \to X^\omega$". So we have:

4.1.2 THEOREM (Wadge [59]): *If* A, B *are Borel subsets of* 2^ω, *then either* $A \leq_w B$ *or* $B \leq_w 2^\omega \backslash A$.

The reader should realize that here it is important to consider subsets of *a particular copy* of the Cantor set, and *not* the topological type; the relation $A \leq_w B$ is not the same as "for certain embeddings $\tilde{A}, \tilde{B}$ of A, B in certain Cantor sets C_0, C_1, we have a continuous mapping $g: C_0 \to C_1$ such that $\tilde{A} = g^{-1}[\tilde{B}]$". For example, if $B = 2^\omega$, then 2^ω itself is the only copy A of the Cantor set for which $A \leq_w B$ (see also theorems 4.6.6 and 4.6.8). *This is the reason that, to avoid inaccuracies, IN THIS CHAPTER WE USE* 2^ω *AS THE CANTOR SET.*

We will now show how the relation $\leq_w$ gives rise to a refinement of the Borel hierarchy in 2^ω. It will be convenient to have available the following notions:

4.1.3 DEFINITION: *Let* X *be a space, and* $\Gamma \subset P(X)$.
(a) $\check{\Gamma} = \{X \backslash A : A \in \Gamma\}$ *is the dual class of* Γ; $\Delta(\Gamma) = \Gamma \cap \check{\Gamma}$ *is the ambiguous class associated with* Γ.
(b) Γ *is self-dual if* $\Gamma = \check{\Gamma}$ $(= \Delta(\Gamma))$.
(c) Γ *is continuously closed if for each* $A \in \Gamma$, *and each continuous* $f: X \to X$, *also* $f^{-1}[A] \in \Gamma$.

4.1.4 DEFINITION: (a) $A \equiv_w B$ *if and only if both* $A \leq_w B$ *and* $B \leq_w A$.
(b) $A <_w B$ *if and only if* $A \leq_w B$ *but* $B \not\leq_w A$.

4.1.5 LEMMA: *Let* A *and* B *be Borel subsets of* 2^ω.
(a) *If* $A <_w B$, *then* $A <_w 2^\omega \backslash B$.
(b) *Either* $A <_w B$, *or* $B <_w A$, *or* $A \equiv_w B$, *or* $2^\omega \backslash A \equiv_w B$.

Proof: (a) Since $B \not\leq_w A$, $A \leq_w 2^\omega \backslash B$ by theorem 4.1.2. If $2^\omega \backslash B \equiv_w A$, then $B \leq_w 2^\omega \backslash A \leq_w 2^\omega \backslash B \leq_w A$, a contradiction.
(b) By theorem 4.1.2, either $A \leq_w B$ or $B \leq_w 2^\omega \backslash A$.
Case 1: $A \leq_w B$. Then either $B \leq_w A$, whence $A \equiv_w B$; or $B \not\leq_w A$, whence $A <_w B$.
Case 2: $B \leq_w 2^\omega \backslash A$. Then either $2^\omega \backslash A \leq_w B$, whence $2^\omega \backslash A \equiv_w B$; or $2^\omega \backslash A \not\leq_w B$, whence $B <_w 2^\omega \backslash A$, and hence $B <_w A$ by (a). □

The $\equiv_w$-equivalence classes are called *Wadge degrees*. Since we want to construct a hierarchy, we prefer classes that are "closed downwards", i.e. continuously closed; these classes are called *Wadge classes*:

4.1.6 DEFINITION: (a) *If* A *is a subset of* 2^ω, *then the Wadge class of* A *is* $[A] = \{B \subset 2^\omega : B \leq_w A\}$.
(b) *If* A *is a Borel subset of* 2^ω, *then* $[A]$ *is called a Borel Wadge class.*

Note that a Borel Wadge class consists exclusively of Borel sets.
By the discussion following theorem 4.1.2, it is not always true that if $B \in [A]$ and $B \approx \tilde{B}$, then $\tilde{B} \in [A]$.

4.1.7 DEFINITION: *The Wadge ordering* $<$ *on dual pairs* $\{\Gamma, \check{\Gamma}\}$ *of Wadge classes is defined by* $\{\Gamma_0, \check{\Gamma}_0\} < \{\Gamma_1, \check{\Gamma}_1\}$ *if and only if* $\Gamma_0 \subsetneqq \Gamma_1$.

Since $[A] \subsetneqq [B]$ if and only if $A <_w B$, it follows from lemma 4.1.5(a) that for Borel Wadge classes, if $\{\Gamma_0, \check{\Gamma}_0\} < \{\Gamma_1, \check{\Gamma}_1\}$, then $\Gamma_0 \subsetneqq \Gamma_1$ and $\Gamma_0 \subsetneqq \check{\Gamma}_1$, hence $\check{\Gamma}_0 \subsetneqq \check{\Gamma}_1$ and $\check{\Gamma}_0 \subsetneqq \Gamma_1$. We will often abuse notation and write $\Gamma_0 < \Gamma_1$ for $\Gamma_0 \subsetneqq \Gamma_1$, etc..
A translation of lemma 4.1.5(b) into Wadge classes yields:

4.1.8 LEMMA: $<$ *is a linear ordering on the pairs of Borel Wadge classes.* □

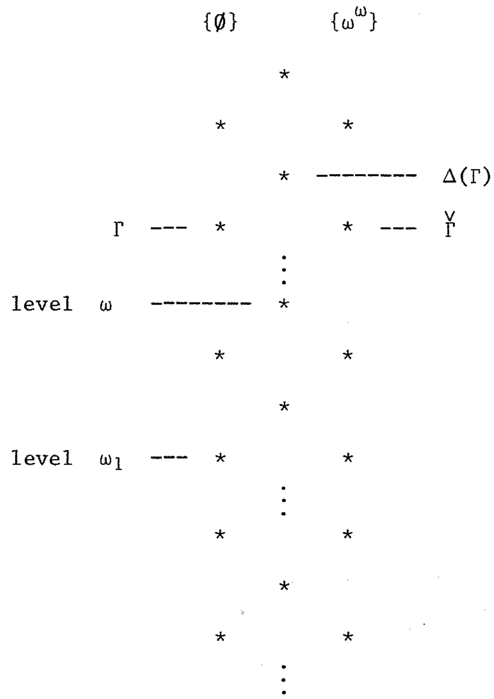
In fact, we have the following theorem, whose proof is beyond the scope of this section.

4.1.9 THEOREM (Wadge [59]): $<$ *well-orders the pairs of Borel Wadge classes.*

The analogues of the above results for ω^ω instead of 2^ω were all established by Wadge in his thesis [60]; theorem 4.1.9, for ω^ω, was announced by Wadge [59] in the 1972 Notices of the AMS, under the assumption of Borel determinacy (which was only proved in 1975). Martin has shown that the axiom of determinacy implies that $<$ well-orders all Wadge classes (for a proof, see Martin and Kechris [36], Moschovakis [43], or Van Wesep [61]).

Remark: The reason that we have presented the above discussion for 2^ω rather than ω^ω is, that for our characterization of homogeneous Borel sets, we heavily rely on the theorem of Steel, which will be stated at the end of this section; this theorem deals with classes of subsets of 2^ω, not of ω^ω.

For ω^ω, Van Wesep [61] has shown that the pattern of the Borel Wadge classes is as follows: the first element is $\{\{\emptyset\},\{\omega^\omega\}\}$; a successor is self-dual if and only if its predecessor is not; at limit stages of cofinality ω stands a self-dual class, and at limit stages of cofinality ω_1 a non-self-dual pair. Furthermore, the predecessor of the non-self-dual pair $\{\Gamma,\check{\Gamma}\}$ is $\Delta(\Gamma)$. Thus we have:



As we shall see in section 4.2, the only difference between the situation for ω^ω and that for 2^ω is that in the Borel Wadge ordering on 2^ω, the limit stages of cofinality ω are occupied by non-self-dual pairs.

Remark: Wadge [59] has proved that the order-type of the Wadge ordering on pairs of Borel Wadge classes in ω^ω is $\varepsilon_1^{\omega_1}$, where for each pair of ordinals γ,μ we define ε_γ^μ as follows:

(1) ε_γ^0 is the γ^{th} ε-number (δ is an ε-number if $\omega^\delta = \delta$, see Rubin [48] or Sierpiński [53])

(2) for $\mu > 0$, ε_γ^μ is the γ^{th} ordinal number δ such that $\varepsilon_\delta^\nu = \delta$ for all $\nu < \mu$.

The cardinality of $\varepsilon_1^{\omega_1}$ is ω_1. In section 4.2, we will see that the order-type of the Borel Wadge ordering on 2^ω is $\varepsilon_1^{\omega_1}$ as well.

Remark: The fact that the Wadge hierarchy really refines the Borel hierarchy can be easily deduced from the observation that if Γ is a continuously closed class of Borel sets, and $A \in \Gamma\backslash\check{\Gamma}$, then $\Gamma = [A]$. Indeed, clearly $[A] \subset \Gamma$; and if $B \in \Gamma$, then by theorem 4.1.2 we have either $B \leq_w A$ or $A \leq_w 2^\omega\backslash B$. However, since $2^\omega\backslash B \in \check{\Gamma}$, and since $\check{\Gamma}$ is continuously closed if Γ is, $A \leq_w 2^\omega\backslash B$ would imply that $A \in \check{\Gamma}$, a contradiction; so $B \in [A]$. Since Borel sets of exact class α exist for each $\alpha < \omega_1$ (see also section 4.5), this implies that each Σ_α^0 and each Π_α^0 is a Wadge class.
The same method can be used to prove that in fact the Wadge hierarchy of Borel sets refines the hierarchy of small Borel classes $D_\alpha(\Sigma_\xi^0)$.

The last part of this section is devoted to a theorem of Steel [56]. Let $Q_i = \{x \in 2^\omega: \exists m \forall n \geq m: x_n = i\}$, for $i \in \{0,1\}$; then $Q_0 \approx \mathbb{Q} \approx Q_1$. If $x \notin Q_0 \cup Q_1$, then x consists of blocks of zeros separated by blocks of ones; define $\phi: 2^\omega\backslash(Q_0 \cup Q_1) \to 2^\omega$ by $\phi(x)_n = 0$ if the n^{th} block of zeros in x has even length, $\phi(x)_n = 1$ otherwise (we start counting with the 0^{th} block). Note that ϕ is continuous.

4.1.10 DEFINITION: (a) *A class* $\Gamma \subset P(2^\omega)$ *is reasonably closed if* Γ *is continuously closed, and* $\phi^{-1}[A] \cup Q_0 \in \Gamma$ *for each* $A \in \Gamma$.
(b) *A subset* A *of* 2^ω *is everywhere properly* Γ *if for each non-empty open subset* U *of* 2^ω, *we have* $U \cap A \in \Gamma\backslash\check{\Gamma}$.

Note that in (b), if $\emptyset \in \check{\Gamma}$, then A is dense in 2^ω.

We are now ready to state Steel's theorem (in fact, a special case of it).

4.1.11 THEOREM (Steel [56]): *If* Γ *is a reasonably closed class of Borel subsets of* 2^ω, *and* $A, B \subset 2^\omega$ *are everywhere properly* Γ *and either both first category or both Baire, then* $h[A] = B$ *for some autohomeomorphism* h *of* 2^ω.

In sections 4.4, 4.5, and 4.6, it will be made clear to what kind of classes Steel's theorem can be applied.

4.2 An inductive description of the Borel Wadge classes

This section is devoted to a discussion of part of Louveau's paper [32]; in that part of his paper, Louveau provides an inductive definition of the Borel Wadge classes in ω^ω. Since we want to work inside the Cantor set 2^ω, we briefly indicate what changes are to be made in order to obtain analogous results for 2^ω. Finally, using this description of the Borel Wadge classes, we prove some closure properties of these classes, which will be used in sections 4.3 and 4.4.

Unfortunately, the inductive defintion is rather complicated, and we need many new notions; all are taken from Louveau [32], except that we define them for 2^ω instead of ω^ω.
One of the operations we use is the difference operation, which was defined in definition 3.1.1. However, since we will need results about the Wadge hierarchy, and since Wadge classes are not defined for arbitrary spaces, it is not convenient to consider arbitrary classes $D^X_\eta(\Sigma^0_\xi)$, or the absolute class $D_\eta(\Sigma^0_\xi)$: we have to work with $D^{2^\omega}_\eta(\Sigma^0_\xi)$. To simplify notation, therefore, *IN THIS CHAPTER,* $D_\eta(\Sigma^0_\xi)$ *DENOTES* $D^{2^\omega}_\eta(\Sigma^0_\xi)$
Similarly, all the other operations will only be explicitly defined for 2^ω.

4.2.1 DEFINITION: *Let* $\Gamma,\Gamma' \subset P(2^\omega)$, *and let* $A \subset 2^\omega$.
(a) $A \in \mathrm{Sep}(D_\eta(\Sigma^0_\xi),\Gamma)$ *if and only if*

$$A = (A_0 \cap C) \cup (A_1\backslash C)$$

for some $C \in D_\eta(\Sigma^0_\xi)$, $A_0 \in \check{\Gamma}$, *and* $A_1 \in \Gamma$.
(b) $A \in \mathrm{Bisep}(D_\eta(\Sigma^0_\xi),\Gamma,\Gamma')$ *if and only if*

$$A = (A_0 \cap C_0) \cup (A_1 \cap C_1) \cup B\backslash(C_0 \cup C_1)$$

for some disjoint $C_0, C_1 \in D_\eta(\Sigma^0_\xi)$, *and some* $A_0 \in \check{\Gamma}$, $A_1 \in \Gamma$, *and* $B \in \Gamma'$.
(c) $A \in \mathrm{SU}(\Sigma^0_\xi,\Gamma)$ *if and only if*

$$A = \cup_{n=0}^\infty (A_n \cap C_n)$$

for some collection $\{C_n : n < \omega\}$ *of pairwise disjoint* Σ^0_ξ*-sets in* 2^ω, *and some family* $\{A_n : n < \omega\}$ *of elements of* Γ.
The set $\cup_{n=0}^\infty C_n$ *is called the envelop of* A. *If the envelop of* A *is all of* 2^ω *(i.e.* $\{C_n : n < \omega\}$ *partitions* 2^ω*), then we write* $A \in \mathrm{PU}(\Sigma^0_\xi,\Gamma)$.
(d) $A \in \mathrm{SD}_\eta(\langle \Sigma^0_\xi, \mathrm{SU}(\Sigma^0_\xi,\Gamma)\rangle,\Gamma')$ *if and only if*

$$A = \cup_{\zeta<\eta}(A_\zeta\backslash(\cup_{\beta<\zeta} C_\beta)) \cup B\backslash(\cup_{\zeta<\eta} C_\zeta)$$

for some increasing sequences $\langle A_\zeta : \zeta < \eta \rangle$ *of elements of* $SU(\Sigma^0_\xi,\Gamma)$, *and* $\langle C_\zeta : \zeta < \eta \rangle$ *of* Σ^0_ξ*-sets in* 2^ω *such that* $A_\zeta \subset C_\zeta \subset A_{\zeta+1}$ *and* C_ζ *is the envelop of* A_ζ, *and some* $B \in \Gamma'$.

In (b) and (d), we always omit Γ' if $\Gamma' = \{\emptyset\}$.

To simplify the exposition, if we write e.g. "$A \in Sep(D_\eta(\Sigma^0_\xi),\Gamma)$, say $A = (A_0 \cap C) \cup (A_1 \backslash C)$", then it is tacitly understood that the sets A_0, A_1, and C are chosen as required by the above definition.

Louveau now selects a certain subset D of $(\omega_1)^\omega$, its elements being called *descriptions*, and for each $u \in D$, he defines a *non-self-dual Borel Wadge class* Γ_u, according to the following inductive definition (where sometimes $v \in (\omega_1)^\omega$ is considered as a pair $\langle v_0,v_1 \rangle$ or a sequence $\langle v_n : n < \omega \rangle$ of elements of $(\omega_1)^\omega$; $\underline{0} \in (\omega_1)^\omega$ has all coordinates 0):

4.2.2 DEFINITION: (a) $\underline{0} \in D$, $\Gamma_0 = \{\emptyset\}$.
(b) *If* $u = \xi^\frown 1^\frown \eta^\frown \underline{0}$, *where* $\xi \geq 1$, $\eta \geq 1$, *then* $u \in D$, *and* $\Gamma_u = D_\eta(\Sigma^0_\xi)$.
(c) *If* $u = \xi^\frown 2^\frown \eta^\frown u^*$, *where* $\xi \geq 1$, $\eta \geq 1$, $u^* \in D$, *and* $u^*(0) > \xi$, *then* $u \in D$, *and* $\Gamma_u = Sep(D_\eta(\Sigma^0_\xi),\Gamma_{u^*})$.
(d) *If* $u = \xi^\frown 3^\frown \eta^\frown \langle u_0,u_1 \rangle$, *where* $\xi \geq 1$, $\eta \geq 1$, $u_0,u_1 \in D$, $u_0(0) > \xi$, $u_1(0) \geq \xi$ *or* $u_1 = \underline{0}$, *and* $\Gamma_{u_1} < \Gamma_{u_0}$, *then* $u \in D$, *and* $\Gamma_u = Bisep(D_\eta(\Sigma^0_\xi),\Gamma_{u_0},\Gamma_{u_1})$.
(e) *If* $u = \xi^\frown 4^\frown \langle u_n : n < \omega \rangle$, *where* $\xi \geq 1$, *each* $u_n \in D$, $\Gamma_{u_n} < \Gamma_{u_{n+1}}$ *for each* n, $\langle u_n(0) : n < \omega \rangle$ *is non-decreasing, and* $\sup_{n<\omega} u_n(0) > \xi$, *then* $u \in D$, *and* $\Gamma_u = SU(\Sigma^0_\xi,\bigcup^\infty_{n=0} \Gamma_{u_n})$.
(f) *If* $u = \xi^\frown 5^\frown \eta^\frown \langle u_0,u_1 \rangle$, *where* $\xi \geq 1$, $\eta \geq 2$, $u_0,u_1 \in D$, $u_0(0) = \xi$, $u_0(1) = 4$, $u_1(0) \geq \xi$ *or* $u_1 = \underline{0}$, *and* $\Gamma_{u_1} < \Gamma_{u_0}$, *then* $u \in D$, *and* $\Gamma_u = SD_\eta(\langle \Sigma^0_\xi, \Gamma_{u_0} \rangle,\Gamma_{u_1})$.

Note that in part (f), since $u_0(0) = \xi$ and $u_0(1) = 4$, we have that $\Gamma_{u_0} = SU(\Sigma^0_\xi,\Gamma)$ for some Γ, so that Γ_u is well-defined (definition 4.2.1(d)).

Remark: The restrictions that we put on $\langle u_n : n < \omega \rangle$ in case $u = \xi^\frown 4^\frown \langle u_n : n < \omega \rangle \in D$ are slightly different from those in Louveau [32], definition 1.2(e); it can be seen that the definition in Louveau does not yield the desired results. For example, if $\Gamma_{u_0} = Sep(\Sigma^0_1,\Sigma^0_\omega)$, $\Gamma_{u_n} = \Sigma^0_{n+1}$ for $n \in \mathbb{N}$, and $u = 1^\frown 4^\frown \langle u_n : n < \omega \rangle$, then u would be a description for Louveau; however, $\Gamma_u = SU(\Sigma^0_1,\bigcup^\infty_{n=0} \Gamma_{u_n}) = SU(\Sigma^0_1,\Gamma_{u_0}) = \Gamma_{u_0}$ by Louveau [32], lemma 1.4 (see lemma 11(a) of this section). This contradicts Louveau's own lemma 1.23.

Remark: If we put $D_0 = \{\underline{0}\} \cup \{\xi\hat{\ }1\hat{\ }\eta\hat{\ }\underline{0}\colon \xi,\eta \in [1,\omega_1)\} \subset (\omega_1)^\omega$, $D_{\alpha+1} = D_\alpha \cup \{u \in (\omega_1)^\omega\colon u$ is defined as in 4.2.2(c)-(f), with u^*, u_0, u_1, and $\langle u_n\colon n < \omega\rangle$ elements of $D_\alpha\}$, and $D_\beta = \bigcup_{\alpha<\beta} D_\alpha$ if $\lim(\beta)$, then $D = D_{\omega_1}$. Furthermore, since we have increasing sequences $\langle \Gamma_{u_n}\colon n < \omega\rangle$ in 4.2.2(e), each set $\{\Gamma_u\colon u \in D_\alpha\}$ has cardinality ω_1, so $|\{\Gamma_u\colon u \in D\}| = \omega_1$.

For each description u, the *type* $t(u) \in \{0,1,2,3\}$ of u is defined (inductively) as follows:

4.2.3 DEFINITION: *Let* u *be a description.*

(a) *If* $u = \underline{0}$, *then* $t(u) = 0$.

(b) *If* $u = \xi\hat{\ }1\hat{\ }\eta\hat{\ }\underline{0}$, *then* $t(u) = 1$ *if* η *is a successor,* *and* $t(u) = 2$ *if* η *is a limit.*

(c) *If* $u(1) = 2$, *then* $t(u) = 3$.

(d) *If* $u = \xi\hat{\ }3\hat{\ }\eta\hat{\ }\langle u_0,u_1\rangle$, *then* $t(u) = 1$ *if* $u_1 = \underline{0}$ *and* η *is a successor,*
$t(u) = 2$ *if* $u_1 = \underline{0}$ *and* η *is a limit,*
$t(u) = t(u_1)$ *if* $u_1(0) = \xi$,
and $t(u) = 3$ *if* $u_1(0) > \xi$.

(e) *If* $u(1) = 4$, *then* $t(u) = 2$.

(f) *If* $u = \xi\hat{\ }5\hat{\ }\eta\hat{\ }\langle u_0,u_1\rangle$, *then* $t(u) = 2$ *if* $u_1 = \underline{0}$,
$t(u) = t(u_1)$ *if* $u_1(0) = \xi$,
and $t(u) = \xi$ *if* $u_1(0) > \xi$.

For the meaning of the type of a description u in relation with the ordinal of $\{\Gamma_u,\check{\Gamma}_u\}$ in the Wadge well-ordering, see Louveau [32].

4.2.4 DEFINITION: *If* u *is a description of type* 1, *then we associate a description* $\bar{u}$ *with* u, *as follows:*

(a) *If* $u = \xi\hat{\ }1\hat{\ }\eta\hat{\ }\underline{0}$, *say* $\eta = \eta_0+1$, *then* $\bar{u} = \underline{0}$ *if* $\eta_0 = 0$,
and $\bar{u} = \xi\hat{\ }1\hat{\ }\eta_0\hat{\ }\underline{0}$ *otherwise.*

(b) *If* $u = \xi\hat{\ }3\hat{\ }\eta\hat{\ }\langle u_0,\underline{0}\rangle$, *say* $\eta = \eta_0+1$, *then* $\bar{u} = u_0$ *if* $\eta_0 = 0$,
and $\bar{u} = \xi\hat{\ }2\hat{\ }\eta_0\hat{\ }u_0$ *otherwise.*

(c) *If* $u = \xi\hat{\ }3\hat{\ }\eta\hat{\ }\langle u_0,u_1\rangle$, *and* $t(u_1) = 1$, *then* $\bar{u} = \xi\hat{\ }3\hat{\ }\eta\hat{\ }\langle u_0,\bar{u}_1\rangle$.

(d) *If* $u = \xi\hat{\ }5\hat{\ }\eta\hat{\ }\langle u_0,u_1\rangle$, *then* $\bar{u} = \xi\hat{\ }5\hat{\ }\eta\hat{\ }\langle u_0,\bar{u}_1\rangle$.

Parts (c) and (d) are in fact induction steps. Note that in all cases, either $\bar{u} = \underline{0}$ or $\bar{u}(0) \geq \xi$, and $\Gamma_{\bar{u}} \subset \Gamma_u$, so that in (c) and (d), we have $\Gamma_{\bar{u}_1} < \Gamma_{u_0}$, and $\bar{u}_1 = \underline{0}$ or $\bar{u}_1(0) \geq \xi$; hence, $\bar{u}$ is well-defined in (c) and (d).

The main result in the first part of Louveau's paper is, that the Borel Wadge classes in ω^ω are exactly

$$\{\Gamma_u^\dagger: u \in D\} \cup \{\check{\Gamma}_u^\dagger: u \in D\} \cup \{\Delta(\Gamma_u^\dagger): u \in D,\ u(0) = 1, \text{ and } t(u) \in \{1,2\}\},$$

where we let $\Gamma_u^\dagger$ denote the classes Γ_u defined in ω^ω instead of 2^ω. Using the fact that the Wadge-ordering well-orders the Wadge classes, it is not hard to deduce the above from the following statements:

(A) *If* $u(0) = 1$ *and* $t(u) = 1$, *then* $\Delta(\Gamma_u^\dagger)$ *is the unique Borel Wadge class* Γ *in* ω^ω *such that* $\Gamma_{u^-}^\dagger < \Gamma < \Gamma_u^\dagger$.
(B) *If* $u(0) = 1$ *and* $t(u) = 2$, *then there exists a strictly increasing sequence* $\langle \Gamma_{u_n}^\dagger: n < \omega \rangle$ *of described classes such that* $\Delta(\Gamma_u^\dagger)$ *is the unique Borel Wadge class* Γ *in* ω^ω *such that* $\Gamma_{u_n}^\dagger < \Gamma < \Gamma_u^\dagger$ *for all* $n < \omega$.
(C) *If* $u(0) > 1$ *or* $t(u) = 3$, *then there exists a strictly increasing sequence* $\langle \Gamma_{u_\alpha}^\dagger: \alpha < \omega_1 \rangle$ *of described classes such that* $\Delta(\Gamma_u^\dagger) = \bigcup\{\Gamma_{u_\alpha}^\dagger: \alpha < \omega_1\}$.

The statement (C) is proved in Louveau's lemmas 1.14, 1.19, 1.24, 1.25, and 1.28; analyzing the proofs of these lemmas, it can be seen that they go through for the Cantor set, and thus we find that statement (C) holds for the Cantor set as well.
Statements (A) and (B) are deduced by Louveau from his lemmas 1.11(b) and 1.23(b), respectively (note: the last one, as well as 1.23(a), is stated incorrectly: for $\bigcup\{\Gamma_{u'}: u' \in Qu\}$, read $\bigcup\{\Gamma_{s_u(n)}: n < \omega\}$); he appeals to "the game theoretical characterization of the Wadge ordering", but does not give any details. Since it is for (B) that the situation in the Cantor set is going to be different, we will give the correct statement of the lemma 1.23(b) of Louveau, show how to prove (B) from it, and then indicate what has to be changed in order to describe the situation for the Cantor set; this will also give the reader an impression of how to work with the original definition of the Wadge classes. The deduction of (A) from Louveau's lemma 1.11(b) is similar, and both this lemma and this deduction go through for the Cantor set.

4.2.5 LEMMA ([32], lemma 1.23(b)): *Let* u *be a description with* $u(0) = 1$ *and* $t(u) = 2$. *Then* $\Delta(\Gamma_u^\dagger) = PU(\Sigma_1^0, \bigcup_{n=0}^\infty \Gamma_{u_n}^\dagger)$ *for some strictly increasing sequence* $\langle \Gamma_{u_n}^\dagger: n < \omega \rangle$ *of described classes*.

Remark: This lemma also holds for the Cantor set (so without the †), with the same proof.

Proof of (B): Let $\langle \Gamma^\dagger_{u_n}: n < \omega\rangle$ be as in the lemma. Since this sequence is strictly increasing, we may assume that $\{\{\emptyset\},\{\omega^\omega\}\} < \Gamma^\dagger_{u_0}$, and hence if $\Gamma^\dagger_{u_n} = [E_n]$, then $\emptyset \neq E_n \neq \omega^\omega$, for each $n < \omega$. Fix $n < \omega$, and put $B_n = \{n^\frown x: x \in E_n\}$. Define $\psi_0: \omega^\omega \to \omega^\omega$ by $\psi_0(x) = n^\frown x$; then $E_n = \psi_0^{-1}[B_n]$, so $E_n \leq_w B_n$. Also, define $\psi_1: \omega^\omega \to \omega^\omega$ by $\psi_1(n^\frown x) = x$, $\psi_1(m^\frown x) = p \notin E_n$ if $m \neq n$; then $B_n = \psi_1^{-1}[E_n]$, so $B_n \leq_w E_n$, whence $B_n \equiv_w E_n$ and $\Gamma^\dagger_{u_n} = [B_n]$. Put $B = \bigcup_{n=0}^\infty B_n$, and $\Gamma' = \bigcup_{n=0}^\infty \Gamma^\dagger_{u_n}$.

CLAIM: $[B] = PU(\Sigma^0_1,\Gamma')$.

Indeed, if $f: \omega^\omega \to \omega^\omega$ is continuous, then $f^{-1}[B_n] \in \Gamma^\dagger_{u_n}$ for each $n < \omega$, so if we put $U_n = \{n^\frown x: x \in \omega^\omega\}$, then $f^{-1}[B] = \bigcup_{n=0}^\infty (f^{-1}[U_n] \cap f^{-1}[B_n]) \in PU(\Sigma^0_1,\Gamma')$, i.e. $[B] \subset PU(\Sigma^0_1,\Gamma')$.

Conversely, let $A \in PU(\Sigma^0_1,\Gamma')$, say $A = \bigcup_{n=0}^\infty (A_n \cap C_n)$, where $\{C_n: n < \omega\}$ partitions ω^ω. For each $n < \omega$, let $k(n) < \omega$ be such that $A_n \in \Gamma^\dagger_{u_{k(n)}}$, and let $f_n: \omega^\omega \to \omega^\omega$ be continuous such that $f_n^{-1}[B_{k(n)}] = A_n$; since $B_{k(n)} \subsetneq U_n$, we may assume that $f_n(x)_0 = k(n)$ for each $x \in \omega^\omega$. Put $f = \bigcup_{n=0}^\infty (f_n | C_n)$; then $A = f^{-1}[B]$. For if $x \in A$, say $x \in A_n \cap C_n$, then $f(x) = f_n(x) \in B_{k(n)} \subset B$; and if $f(x) \in B$, say $f(x) \in B_k$, then $x \in C_n$ for some $n < \omega$ with $k(n) = k$, and thus $x \in f^{-1}[B_{k(n)}] \cap C_n = f_n^{-1}[B_{k(n)}] \cap C_n = A_n \cap C_n \subset A$.

Thus, we have shown that $\Delta(\Gamma^\dagger_u)$ is a Wadge class, and clearly $\Gamma^\dagger_{u_n} < \Delta(\Gamma^\dagger_u) < \Gamma^\dagger_u$, since $\Gamma^\dagger_u$ is non-self-dual.

To complete the proof of (B), note that if Γ is a Borel Wadge class such that $\Gamma^\dagger_{u_n} < \Gamma < \Gamma^\dagger_u$ for all n, then $\Gamma \subset \Delta(\Gamma^\dagger_u)$, and also $\Delta(\Gamma^\dagger_u) = PU(\Sigma^0_1,\Gamma) \subset PU(\Sigma^0_1,\Gamma)$. Now if $\Gamma = [E]$, and $A = \bigcup_{n=0}^\infty (A_n \cap C_n) \in PU(\Sigma^0_1,\Gamma)$, where $\{C_n: n < \omega\}$ partitions ω^ω, and each $A_n \in \Gamma$, then taking $g_n: \omega^\omega \to \omega^\omega$ such that $A_n = g_n^{-1}[E]$, and defining $g: \omega^\omega \to \omega^\omega$ by $g = \bigcup_{n=0}^\infty (g_n | C_n)$, we see that $A = g^{-1}[E] \in \Gamma$, so $\Delta(\Gamma^\dagger_u) \subset \Gamma$; Thus, $\Delta(\Gamma^\dagger_u) = \Gamma$. □

It is immediately clear that the above proof cannot be copied for the Cantor set; indeed, being compact, 2^ω cannot be partitioned into infinitely many non-empty open sets.

So let $\langle \Gamma_{u_n}: n < \omega\rangle$ be the strictly increasing sequence of described classes obtained from the analogue to lemma 4.2.5 for 2^ω. Then each $A \in \Delta(\Gamma_u) = PU(\Sigma^0_1, \bigcup_{n=0}^\infty \Gamma_{u_n})$ can be written as $\bigcup_{n<k} (A_n \cap C_n)$ for some $k < \omega$, and thus $A \in PU(\Sigma^0_1,\Gamma_{u_m})$ for some $m < \omega$. So we find that

$$\Delta(\Gamma_u) = PU(\Sigma^0_1, \bigcup^\infty_{n=0} \Gamma_{u_n}) = \bigcup^\infty_{n=0} PU(\Sigma^0_1, \Gamma_{u_n}) = \bigcup^\infty_{n=0} \Gamma_{u_n},$$

the last equality by an argument as in the proof of (B). Clearly, $\bigcup^\infty_{n=0} \Gamma_{u_n}$ is not a Wadge class, for if $[A] = \bigcup^\infty_{n=0} \Gamma_{u_n}$, then $A \in \Gamma_{u_n}$ for some n, and hence $[A] \subset \Gamma_{u_n} \subsetneq \Gamma_{u_{n+1}} \subset [A]$.

So for the Cantor set, we obtain the following theorem as the equivalent of statements (A), (B), and (C).

4.2.6 THEOREM: (a) *If* $u(0) = 1$ *and* $t(u) = 1$, *then* $\Delta(\Gamma_u)$ *is the unique Borel Wadge class* Γ *such that* $\Gamma_{\bar{u}} < \Gamma < \Gamma_u$.
(b) *If* $u(0) = 1$ *and* $t(u) = 2$, *then there exists a strictly increasing sequence* $\langle \Gamma_{u_n} : n < \omega \rangle$ *of described classes such that* $\Delta(\Gamma_u) = \bigcup^\infty_{n=0} \Gamma_{u_n}$.
(c) *If* $u(0) > 1$ *or* $t(u) = 3$, *then there exists a strictly increasing sequence* $\langle \Gamma_{u_\alpha} : \alpha < \omega_1 \rangle$ *of described classes such that* $\Delta(\Gamma_u) = \bigcup\{\Gamma_{u_\alpha} : \alpha < \omega_1\}$.

From this, we easily obtain:

4.2.7 THEOREM: *The collection of Borel Wadge classes in* 2^ω *is* $\{\Gamma_u : u \in D\} \cup \{\check{\Gamma}_u : u \in D\} \cup \{\Delta(\Gamma_u) : u \in D, u(0) = 1, \text{and } t(u) = 1\}$.

Thus, if we compare the pattern of the Borel Wadge classes in 2^ω with that of ω^ω as described in the first remark following theorem 4.1.9, then we see that the only difference is that in the Borel Wadge ordering on 2^ω, the limit stages of cofinality ω are occupied by non-self-dual pairs. Indeed, statement (A) and theorem 4.2.6(a) are similar, and so are statement (C) and theorem 4.2.6(c); since they cover the successor case and the cofinality ω_1 case, respectively no difference occurs at those levels. Theorem 4.2.6(b) says that the would-be self-dual class $\Delta(\Gamma_u)$ at a limit stage of cofinality ω is not a Wadge class, and hence its place is occupied by the non-self-dual pair $\{\Gamma_u, \check{\Gamma}_u\}$.

Remark: Since removing the ordinals of cofinality ω from a limit ordinal does not change the order-type, the order-type of the Borel Wadge oredering on 2^ω is still $\varepsilon_1^{\omega_1}$ (see the second remark following theorem 4.1.9).

We now derive some properties of the classes defined in definition 4.2.2. We start with a theorem concerning the classes $D_\alpha(\Sigma^0_\xi)$. It is used extensively in Louveau's paper, but no proof seems to be available in the literature. Since we need this result for lemma 4.2.15, we include a proof here; it was suggested to us by A. Louveau.

Recall the definition of the reduction property from 1.11.

4.2.8 THEOREM: *For each* $\alpha, \xi \in [1, \omega_1)$, *the reduction property holds for* $D_\alpha(\Sigma^0_\xi)$.

Before proving the theorem, we state two lemmas:

4.2.9 LEMMA: *If* $A_0 \subset A_1$, $B_0 \subset B_1$ *are* Σ^0_ξ*-sets in* 2^ω, *then there exist* Σ^0_ξ*-sets* A_1', B_1' *such that* $A_0 \subset A_1' \subset A_1$, $B_0 \subset B_1' \subset B_1$, *and* $A_1' \backslash A_0, B_1' \backslash B_0$ *reduce* $A_1 \backslash A_0, B_1 \backslash B_0$.

Proof: Let $\tilde{A}_1, \tilde{B}_1$ reduce A_1, B_1 (see 1.11), and put $A_1' = A_0 \cup (A_1 \cap B_0) \cup \tilde{A}_1$, and $B_1' = B_0 \cup (B_1 \cap A_0) \cup \tilde{B}_1$. □

4.2.10 LEMMA: *Let* $A_0 \subset A_1$ *be* Σ^0_ξ*-sets in* 2^ω, *and let* $B \in D_n(\Sigma^0_\xi)$ *for some* $n \in \mathbb{N}$. *Then* $(A_1 \backslash A_0) \backslash B = D_m(\langle C_k : k < m \rangle)$ *for some even* $m \in \mathbb{N}$, *and some increasing sequence* $\langle C_k : k < m \rangle$ *of* Σ^0_ξ*-sets in* 2^ω *such that* $C_0 = A_0$ *and* $C_{k-1} = A_1$.

Proof: Clearly, it suffices to prove the lemma for $n = 2$. So let $B_0 \subset B_1$ be Σ^0_ξ-sets in 2^ω; then $(A_1 \backslash A_0) \backslash (B_1 \backslash B_0) = (A_1 \cap B_0) \backslash A_0 \cup A_1 \backslash (A_0 \cup B_1) = (A_0 \cup (A_1 \cap B_0)) \backslash A_0 \cup A_1 \backslash (A_1 \cap (A_0 \cup B_1))$. So take $C_0 = A_0$, $C_1 = A_0 \cup (A_1 \cap B_0)$, $C_2 = A_1 \cap (A_0 \cup B_1)$, and $C_3 = A_1$; then $(A_1 \backslash A_0) \backslash (B_1 \backslash B_0) = D_4(\langle C_k : k < 4 \rangle)$. □

Proof of theorem 4.2.8: Let $A, B \in D_\alpha(\Sigma^0_\xi)$, say $A = D_\alpha(\langle A_\zeta : \zeta < \alpha \rangle)$, $B = D_\alpha(\langle B_\zeta : \zeta < \alpha \rangle)$ for certain increasing sequences $\langle A_\zeta : \zeta < \alpha \rangle$, $\langle B_\zeta : \zeta < \alpha \rangle$ of Σ^0_ξ-sets in 2^ω.

<u>Case 1</u>: $\alpha = n \in \mathbb{N}$ is even. This is easily proved by successive applications of lemma 4.2.9.

<u>Case 2</u>: $\alpha = n \in \mathbb{N}$ is odd. By case 1, we can reduce $D_{n-1}(\langle A_{k+1} : k < n-1 \rangle)$, $D_{n-1}(\langle B_{k+1} : k < n-1 \rangle)$ by $\tilde{A} = D_{n-1}(\langle \tilde{A}_{k+1} : k < n-1 \rangle)$, $\tilde{B} = D_{n-1}(\langle \tilde{B}_{k+1} : k < n-1 \rangle)$. By the reduction property for Σ^0_ξ-sets, let A_0', B_0' reduce A_0, B_0; for $0 < m < n$, put $A_m' = \tilde{A}_m \cup A_0' \cup B_0'$, $B_m' = \tilde{B}_m \cup A_0' \cup B_0'$. Then $\langle A_m' : m < n \rangle$, $\langle B_m' : m < n \rangle$ are increasing, and

$$A' = D_n(\langle A_m' : m < n \rangle) = A_0' \cup \tilde{A} \backslash (A_0' \cup B_0'),$$
$$B' = D_n(\langle B_m' : m < n \rangle) = B_0' \cup \tilde{B} \backslash (A_0' \cup B_0')$$

reduce A, B.

Case 3: $\alpha = \beta+n$, where $\beta < \omega_1$ is a limit, and $n < \omega$. We will prove the case where α is odd, the other case is similar.
For even $\zeta < \beta$, put $C_\zeta = A_\zeta \setminus (\bigcup_{\gamma<\zeta} A_\gamma)$, $D_\zeta = B_\zeta \setminus (\bigcup_{\gamma<\zeta} B_\gamma)$, and let $\psi: \beta \to \omega$ be a bijection. Also, put

$$\widetilde{A} = \bigcup\{A_{\beta+m} \setminus (\bigcup_{\gamma<\beta+m} A_\gamma) : m \text{ even} < n\} = A \setminus (\bigcup_{\gamma<\beta} A_\gamma),$$
$$\widetilde{B} \quad \bigcup\{B_{\beta+m} \setminus (\bigcup_{\gamma<\beta+m} B_\gamma) : m \text{ even} < n\} = B \setminus \bigcup_{\gamma<\beta} B_\gamma).$$

For each even $\zeta < \beta$, define

$$C'_\zeta = C_\zeta \setminus (\widetilde{B} \cup \bigcup\{D_\gamma : \gamma \text{ even} < \beta,\ \psi(\gamma) < \psi(\zeta)\}),$$
$$D'_\zeta = D_\zeta \setminus (\widetilde{A} \cup \bigcup\{C_\gamma : \gamma \text{ even} < \beta,\ \psi(\gamma) \le \psi(\zeta)\}).$$

By lemma 4.2.10, there exist even $k(\zeta), \ell(\zeta) \in \mathbb{N}$, and increasing sequences $\langle E^\zeta_m : m < k(\zeta) \rangle$, $\langle F^\zeta_m : m < \ell(\zeta) \rangle$ of Σ^0_ξ-sets in 2^ω, such that

$$C'_\zeta = D_{k(\zeta)}(\langle E^\zeta_m : m < k(\zeta) \rangle),$$
$$D'_\zeta = D_{\ell(\zeta)}(\langle F^\zeta_m : m < \ell(\zeta) \rangle),$$

where furthermore $E^\zeta_0 = \bigcup_{\gamma<\zeta} A_\gamma$, $F^\zeta_0 = \bigcup_{\gamma<\zeta} B_\gamma$, and $E^\zeta_{k(\zeta)-1} = A_\zeta$, $F^\zeta_{\ell(\zeta)-1} = B_\zeta$.
Fix $\gamma < \beta$ satisfying $[\gamma = 0$ or $\lim(\gamma)]$, and $t < \omega$. Then there are unique $n < \omega$ and $i < k(\gamma+2n)$ such that

$$t = (\Sigma_{m<n}\, k(\gamma+2m)) + i - 1$$

(note that $i \ge 1$ if $n = 0$!). Now put

$$A'_{\gamma+t} = E^{\gamma+2n}_i,$$

i.e.	A'_γ	$A'_{\gamma+1}$	$\dots$ $A'_{\gamma+k(\gamma)-2}$	$A'_{\gamma+k(\gamma)-1}$	$\dots$	$A'_{\gamma+k(\gamma)+k(\gamma+2)-2}$	$\dots$
$\equiv$	E^γ_1	E^γ_2	$\dots$ $E^\gamma_{k(\gamma)-1}$	$E^{\gamma+2}_0$	$\dots$	$E^{\gamma+2}_{k(\gamma+2)-1}$	$\dots$
$\equiv$	E^γ_1	E^γ_2	$\dots$ A_γ	$A_{\gamma+1}$	$\dots$	$A_{\gamma+2}$	$\dots$.

Then $\langle A'_\zeta : \zeta < \beta \rangle$ is an increasing sequence of Σ^0_ξ-sets, and

$$A' = \bigcup\{C'_\zeta : \zeta \text{ even} < \beta\} = \bigcup\{A'_\zeta \setminus (\bigcup_{\gamma<\zeta} A'_\gamma) : \zeta \text{ even} < \beta\}.$$

Similarly, define an increasing sequence $\langle B'_\zeta : \zeta < \beta \rangle$ such that

$$B' = \bigcup\{D'_\zeta : \zeta \text{ even} < \beta\} = \bigcup\{B'_\zeta \setminus (\bigcup_{\gamma<\zeta} B'_\gamma) : \zeta \text{ even} < \beta\}.$$

Note that $\bigcup_{\zeta<\beta} A'_\zeta = \bigcup_{\zeta<\beta} A_\zeta$, and $\bigcup_{\zeta<\beta} B'_\zeta = \bigcup_{\zeta<\beta} B_\zeta$. Clearly, $A' \cap (B' \cup \widetilde{B}) = \emptyset = B' \cap (A' \cup \widetilde{A})$, and $A' \cup \widetilde{A} \cup B' \cup \widetilde{B} = A \cup B$.
By lemma 4.2.9, we can reduce $\widetilde{A}, \widetilde{B}$ by $D_{n+1}(\langle E_k : k < n+1 \rangle), D_{n+1}(\langle F_k : k < n+1 \rangle)$ such that $E_0 = \bigcup_{\zeta<\beta} A_\zeta$, $F_0 = \bigcup_{\zeta<\beta} B_\zeta$. Hence, if we put

$$A'_{\beta+m} = E_{m+1},\ B'_{\beta+m} = F_{m+1}$$

for all $m < n$, then $D_\alpha(\langle A'_\zeta : \zeta < \alpha\rangle), D_\alpha(\langle B'_\zeta : \zeta < \alpha\rangle)$ reduce A,B. □

The other lemmas that we derive in this section are concerned with closure properties of the classes Γ_u. We first list some results from Louveau [32] that also hold for 2^ω, with the same proof.

4.2.11 LEMMA: *Let* $u \in D$, *with* $u(0) = \xi$.
(a)([32], lemma 1.4 and its proof) $SU(\Sigma^0_\xi,\Gamma_u) = \Gamma_u$, *and if* $\eta < \xi$, *then* $SU(\Sigma^0_\eta,\check{\Gamma}_u) = \check{\Gamma}_u$.
(b)([32], lemma 1.4) Γ_u *and* $\check{\Gamma}_u$ *are closed under union and intersection with a* Δ^0_ξ*-set.*
(c)([32], proof of lemma 1.4) *If* $u(1) = 4$, *then* Γ_u *is closed under union with a* Σ^0_ξ*-set.*
(d)([32], lemma 1.26, and proof of lemma 1.4) *If* $t(u) = 3$, *then* Γ_u *and* $\check{\Gamma}_u$ *are closed under union with a* Σ^0_ξ*-set and under intersection with a* Π^0_ξ*-set.*
(e)([32], corollary 1.6) *If* $u = \xi^\frown 3^\frown \eta^\frown\langle u_0,u_1\rangle$, *and* $A \in \Gamma_u$, *then there exist* $C \in \Sigma^0_\xi$ *and* $B \in \Gamma_{u_1}$ *such that* $A = (A \cap C) \cup (B\setminus C)$, *and both* $A \cap C$ *and* $C\setminus A$ *are in* $\text{Bisep}(D_\eta(\Sigma^0_\xi),\Gamma_{u_0})$.
(f)([32], lemma 1.11) *If* $t(u) = 1$, *then* $\Gamma_u = \text{Bisep}(\Sigma^0_\xi,\Gamma_{\bar{u}})$, *with* $\bar{u}$ *as in definition* 4.2.4.
(g)([32], lemma 1.23) *If* $t(u) = 2$, *then* $\Gamma_u = SU(\Sigma^0_\xi,\bigcup_{n=0}^\infty \Gamma_{u_n})$ *for some strictly increasing sequence* $\langle \Gamma_{u_n} : n < \omega\rangle$ *of described classes with* $u_n(0) \geq \xi$ *for all* $n < \omega$.

The following lemmas are of a similar nature. We note here that, since the homogeneous zero-dimensional absolute Borel sets of ambiguous class 2 were characterized in the preceding chapters, it would suffice now to consider Wadge classes Γ_u with $\Delta^0_3 \subset \Gamma_u$ in order to obtain the complete picture. However, for the discussion in section 4.6, it will be convenient to have the results of this section for classes Γ_u containing $\Delta(D_\omega(\Sigma^0_2))$.

4.2.12 LEMMA: *If* $\tilde{\Delta} = \Delta(D_\omega(\Sigma^0_2)) \subset \Gamma_u$, *and* $u(0) \geq 2$, *then* Γ_u *is closed under intersection with a* Π^0_2*-set and under union with a* Σ^0_2*-set; hence so is* $\check{\Gamma}_u$.

Proof: First note that, by lemma 3.5.1, $D_\omega(\Sigma^0_2)$ is closed under intersection with a Π^0_2-set; and if $A = D_\omega(\langle A_\zeta : \zeta < \omega\rangle) \in D_\omega(\Sigma^0_2)$, and F is σ-compact,

then $A \cup F = D_\omega(\langle A'_\zeta : \zeta < \omega\rangle)$, where $A'_0 = A_0$, and $A'_\zeta = A_\zeta \cup F$ if $0 < \zeta < \omega$. So the lemma holds if Γ_u or $\check{\Gamma}_u$ is $D_\omega(\Sigma^0_2)$.
If the lemma is false, then there is a minimal class Γ_u for which it fails. By lemma 4.2.11(d), the lemma holds if $t(u) = 3$, and by lemma 4.2.11(b) also if $u(0) \geq 3$; so we have $u(0) = 2$ and $t(u) \in \{1,2\}$.
<u>Case 1</u>: $t(u) = 1$. By lemma 4.2.11(f), $\Gamma_u = \mathrm{Bisep}(\Sigma^0_2, \Gamma_{\bar{u}})$. Suppose that $\tilde{\Delta} \not\subset \Gamma_{\bar{u}}$; then $\Gamma_{\bar{u}} \subset \tilde{\Delta}$, so $\tilde{\Delta} \subset \Gamma_u = \mathrm{Bisep}(\Sigma^0_2, \Gamma_{\bar{u}}) \subset \mathrm{Bisep}(\Sigma^0_2, \tilde{\Delta}) \subset \mathrm{SU}(\Sigma^0_2, D_\omega(\Sigma^0_2)) = D_\omega(\Sigma^0_2)$ by lemma 4.2.11(a), and hence, since Γ_u is non-self-dual, $D_\omega(\Sigma^0_2) \in \{\Gamma_u, \check{\Gamma}_u\}$, contradicting the above remark. Thus, $\Gamma_{\bar{u}} \subset \tilde{\Delta}$, and from definition 4.2.4 it is easily seen that $\bar{u}(0) \geq 2$, since $\bar{u} = \underline{0}$ would imply $\Gamma_u = D_1(\Sigma^0_2) \not\supset \tilde{\Delta}$. Since $\Gamma_u = \mathrm{Bisep}(\Sigma^0_2, \Gamma_{\bar{u}}) \supset \Gamma_{\bar{u}} \cup \check{\Gamma}_{\bar{u}}$, we have $\Gamma_{\bar{u}} < \Gamma_u$, so by minimality of Γ_u, $\Gamma_{\bar{u}}$ and $\check{\Gamma}_{\bar{u}}$ have the desribed closure properties. Now let $A \in \Gamma_u$, say $A = (A_0 \cap C_0) \cup (A_1 \cap C_1)$, where C_0, C_1 are disjoint Σ^0_2-sets, $A_0 \in \check{\Gamma}_{\bar{u}}$, and $A_1 \in \Gamma_{\bar{u}}$. If $F \in \Pi^0_2$, then $A_0 \cap F \in \check{\Gamma}_{\bar{u}}$, $A_1 \cap F \in \Gamma_{\bar{u}}$, so $A \cap F = (A_0 \cap F \cap C_0) \cup (A_1 \cap F \cap C_1) \in \Gamma_u$. If $G \in \Sigma^0_2$, let C^*_0, C^*_1 reduce $C_0 \cup G$, $C_1 \cup G$. Then $A \cup G = ((A_0 \cup G) \cap C^*_0) \cup ((A_1 \cup G) \cap C^*_1) \in \Gamma_u$ since $A_0 \cup G \in \check{\Gamma}_{\bar{u}}$, $A_1 \cup G \in \Gamma_{\bar{u}}$.
<u>Case 2</u>: $t(u) = 2$. By lemma 4.2.11(g), $\Gamma_u = \mathrm{SU}(\Sigma^0_2, \bigcup_{n=0}^\infty \Gamma_{u_n})$ for some strictly increasing sequence $\langle \Gamma_{u_n} : n < \omega\rangle$ of described classes with $u_n(0) \geq 2$ for all $n < \omega$. If each $\Gamma_{u_n} \subset \tilde{\Delta}$, then as in case 1 we obtain a contradiction; thus we conclude that $\tilde{\Delta} \subset \Gamma_{u_k}$ for some k, and hence $\tilde{\Delta} \subset \Gamma_{u_m}$ for all $m \geq k$. Since $u_m(0) \geq 2$ and $\Gamma_{u_m} < \Gamma_u$, each Γ_{u_m} with $m \geq k$ has the described closure properties. Now let $A \in \Gamma_u$, say $A = \bigcup_{n=0}^\infty (A_n \cap C_n)$, where the C_n are pairwise disjoint Σ^0_2-sets, and $A_n \in \Gamma_{u_{m(n)}}$ for some $m(n) < \omega$; of course, we may assume that $m(n) \geq k$. As in case 1, if $F \in \Pi^0_2$, then $A_n \cap F \in \Gamma_{u_{m(n)}}$, so $A \cap F = \bigcup_{n=0}^\infty (A_n \cap F \cap C_n) \in \Gamma_u$; and if $G \in \Sigma^0_2$, and $\langle C^*_n : n < \omega\rangle$ reduces $\langle C_n \cup G : n < \omega\rangle$, then $A \cup G = \bigcup_{n=0}^\infty ((A_n \cup G) \cap C^*_n) \in \Gamma_u$. □

4.2.13 LEMMA: *If* $u(0) \geq 3$, *or* $u(0) = 2$ *and* $t(u) = 3$, *then* Γ_u *is closed under union with a* Π^0_2*-set.*

Proof: If the lemma fails, then there is a minimal Γ_u for which it does. Since the lemma is true if $u(0) \geq 3$ by lemma 4.2.11(b), we have $u(0) = 2$ and $t(u) = 3$. Let $F \in \Pi^0_2$.
<u>Case 1</u>: $u(1) = 2$, so $\Gamma_u = \mathrm{Sep}(D_\eta(\Sigma^0_2), \Gamma_{u*})$ for some $u* \in D$ with $u*(0) > 2$. If $A \in \Gamma_u$, say $A = (A_0 \cap C) \cup (A_1 \setminus C)$, then $A_0 \cup F \in \check{\Gamma}_{u*}$, $A_1 \cup F \in \Gamma_{u*}$ by lemma 4.2.11(b), so $A \cup F = ((A_0 \cup F) \cap C) \cup (A_1 \cup F) \setminus C \in \Gamma_u$.
<u>Case 2</u>: $u(1) = 3$, so $\Gamma_u = \mathrm{Bisep}(D_\eta(\Sigma^0_2), \Gamma_{u_0}, \Gamma_{u_1})$ for some $u_0, u_1 \in D$ with

$\Gamma_{u_1} < \Gamma_{u_0}$, $u_0(0) > 2$, and $u_1(0) > 2$ or $(u_1(0) = 2$ and $t(u_1) = 3)$. If $A \in \Gamma_u$, say $A = (A_0 \cap C_0) \cup (A_1 \cap C_1) \cup B\backslash(C_0 \cup C_1)$, then $A_0 \cup F \in \check{\Gamma}_{u_0}$, $A_1 \cup F \in \Gamma_{u_0}$ by lemma 4.2.11(b); also, $B \cup F \in \Gamma_{u_1}$ by lemma 4.2.11(b) if $u_1(0) > 2$, and by minimality of Γ_u if $u_1(0) = 2$ (note that $\Gamma_{u_1} < \Gamma_{u_0} \subset \Gamma_u$). So $A \cup F = ((A_0 \cup F) \cap C_0) \cup ((A_1 \cup F) \cap C_1) \cup (B \cup F)\backslash(C_0 \cup C_1) \in \Gamma_u$.

Case 3: $u(1) = 5$, so $\Gamma_u = SD_\eta(\langle\Sigma^0_2,\Gamma_{u_0}\rangle,\Gamma_{u_1})$ for some $u_0,u_1 \in D$ with $\Gamma_{u_1} < \Gamma_{u_0}$, $u_0(0) = 2$, $u_0(1) = 4$, and $u_1(0) > 2$ or $(u_1(0) = 2$ and $t(u_1) = 3)$. Let $A \in \Gamma_u$, say $A = \bigcup_{\zeta<\eta}(A_\zeta\backslash(\bigcup_{\beta<\zeta} C_\beta)) \cup B\backslash(\bigcup_{\zeta<\eta} C_\zeta)$. Since $2^\omega\backslash F \in \Sigma^0_2$, and $u_0(0) = 2$, by lemma 4.2.11(a) we have $A_\zeta \cap (2^\omega\backslash F) \in \Gamma_{u_0}$, and it is easily verified that we can choose $C_\zeta \cap (2^\omega\backslash F)$ as envelop of $A_\zeta \cap (2^\omega\backslash F)$; also $B \cup F \in \Gamma_{u_1}$ as in case 2. So

$$A \cup F = \bigcup_{\zeta<\eta}((A_\zeta \cap (2^\omega\backslash F))\backslash\bigcup_{\beta<\zeta}(C_\beta \cap (2^\omega\backslash F))) \cup (B \cup F)\backslash\bigcup_{\zeta<\eta}(C_\zeta \cap (2^\omega\backslash F))$$

$\in \Gamma_u$. □

4.2.14 COROLLARY: *If* $u(0) \geq 3$, *or* $u(0) = 2$ *and* $t(u) = 3$, *then* $SU(\Sigma^0_2,\check{\Gamma}_u) = \check{\Gamma}_u$.

Proof: If $A \in SU(\Sigma^0_2,\check{\Gamma}_u)$, say $A = \bigcup^\infty_{n=0}(A_n \cap C_n)$, then $2^\omega\backslash A = \bigcup^\infty_{n=0}(C_n \cap (2^\omega\backslash A_n)) \cup 2^\omega\backslash(\bigcup^\infty_{n=0} C_n)$. Now $2^\omega\backslash A_n \in \Gamma_u$, so $\bigcup^\infty_{n=0}(C_n \cap (2^\omega\backslash A_n)) \in SU(\Sigma^0_2,\Gamma_u) = \Gamma_u$ by lemma 4.2.11(a), and $2^\omega\backslash(\bigcup^\infty_{n=0} C_n) \in \Pi^0_2$; thus, $2^\omega\backslash A \in \Gamma_u$ by lemma 4.2.13, whence $A \in \check{\Gamma}_u$. □

For the final lemmas of this section, it will be convenient to introduce the following notation: if $X \subset 2^\omega$, then $\Gamma_u(X)$ is the class of all subsets of X that are formed by applying the corresponding operations to the corresponding subsets of X. For example, $\Sigma^0_\xi(X)$ consists of all Σ^0_ξ-sets in X, $D_\eta(\Sigma^0_\xi)(X)$ is the class $D^X_\eta(\Sigma^0_\xi)$ of definition 3.1.1(b), $Sep(D_\eta(\Sigma^0_\xi),\Gamma_{u*})(X) = Sep(D_\eta(\Sigma^0_\xi)(X),\Gamma_{u*}(X))$, etc..

4.2.15 LEMMA: *If* $X \subset 2^\omega$, *and* $A \subset X$, *then* $A \in \Gamma_u(X)$ *if and only if* $A = \tilde{A} \cap X$ *for some* $\tilde{A} \in \Gamma_u$; *similarly for* $\check{\Gamma}_u$.

Proof: The "if" part is clear (use induction), and it is also elementary to show that if the lemma holds for Γ_u, then it also holds for $\check{\Gamma}_u$. The "only if" is trivial for $\Gamma_u = D_1(\Sigma^0_\xi) = \Sigma^0_\xi(2^\omega)$; and if $A = D_\eta(\langle A_\zeta: \zeta < \eta\rangle)$ for some increasing sequence $\langle A_\zeta: \zeta < \eta\rangle$ of Σ^0_ξ-sets in X, say $A_\zeta = B_\zeta \cap X$, where B_ζ is Σ^0_ξ in 2^ω, then $A = D_\eta(\langle B'_\zeta: \zeta < \eta\rangle) \cap X$, where $B'_\zeta = \bigcup_{\gamma\leq\zeta} B_\gamma$. If the lemma holds for Γ_{u*}, and $A \in \Gamma_u(X)$, where $\Gamma_u = Sep(D_\eta(\Sigma^0_\xi),\Gamma_{u*})$,

say $A = (A_0 \cap C) \cup (A_1 \setminus C)$, then there exist $\tilde{C} \in D_\eta(\Sigma^0_\xi)$, $\tilde{A}_0 \in \check{\Gamma}_{u*}$, and $\tilde{A}_1 \in \Gamma_{u*}$ such that $\tilde{C} \cap X = C$, $\tilde{A}_0 \cap X = A_0$, and $\tilde{A}_1 \cap X = A_1$; hence, $\tilde{A} = (\tilde{A}_0 \cap \tilde{C}) \cup (\tilde{A}_1 \setminus \tilde{C}) \in \Gamma_u$, and $\tilde{A} \cap X = A$.
If the lemma holds for Γ_{u0} and Γ_{u1}, and $A \in \Gamma_u(X)$, where $\Gamma_u = \mathrm{Bisep}(D_\eta(\Sigma^0_\xi), \Gamma_{u0}, \Gamma_{u1})$, say $A = (A_0 \cap C_0) \cup (A_1 \cap C_1) \cup B \setminus (C_0 \cup C_1)$, then there exist $\tilde{C}_0, \tilde{C}_1 \in D_\eta(\Sigma^0_\xi)$, $\tilde{A}_0 \in \check{\Gamma}_{u0}$, $\tilde{A}_1 \in \Gamma_{u0}$, and $\tilde{B} \in \Gamma_{u1}$ such that $\tilde{C}_0 \cap X = C_0$, $\tilde{C}_1 \cap X = C_1$, $\tilde{A}_0 \cap X = A_0$, $\tilde{A}_1 \cap X = A_1$, and $\tilde{B} \cap X = B$. By theorem 4.2.8, there are $C'_0, C'_1 \in D_\eta(\Sigma^0_\xi)$ reducing $\tilde{C}_0, \tilde{C}_1$. Then $\tilde{A} = (\tilde{A}_0 \cap C'_0) \cup (\tilde{A}_1 \cap C'_1) \cup \tilde{B} \setminus (C'_0 \cup C'_1) \in \Gamma_u$, and $\tilde{A} \cap X = A$. The case $\Gamma_u = SU(\Sigma^0_\xi, \bigcup_{n=0}^\infty \Gamma_{u_n})$ is proved similarly, using the reduction property for infinite sequences of Σ^0_ξ-sets (see 1.11).
Finally, let $A \in \Gamma_u(X)$, where $\Gamma_u = SD_\eta(\langle \Sigma^0_\xi, \Gamma_{u0} \rangle, \Gamma_{u1})$, say $A = \bigcup_{\zeta<\eta}(A_\zeta \setminus (\bigcup_{\beta<\zeta} C_\beta)) \cup B \setminus (\bigcup_{\zeta<\eta} C_\zeta)$. Let $\tilde{A}_\zeta \in \Gamma_{u0}$, $\tilde{B} \in \Gamma_{u1}$ be such that $\tilde{A}_\zeta \cap X = A_\zeta$, $\tilde{B} \cap X = B$, and let $\tilde{C}_\zeta$ be the envelop of $\tilde{A}_\zeta$. Put $A'_\zeta = \tilde{A}_\zeta \cup \bigcup_{\beta<\zeta} \tilde{C}_\beta$, and $C'_\zeta = \bigcup_{\beta\leq\zeta} \tilde{C}_\beta$. By lemma 4.2.11(c), we have $A'_\zeta \in \Gamma_{u0}$, and in fact, arguing as in the proof of lemma 4.2.12, case 2, it can be shown that we can choose C'_ζ as its envelop. Thus, $\tilde{A} = \bigcup_{\zeta<\eta}(A'_\zeta \setminus (\bigcup_{\beta<\zeta} C'_\beta)) \cup \tilde{B} \setminus (\bigcup_{\zeta<\eta} C'_\zeta) \in \Gamma_u$, and $\tilde{A} \cap X = A$. □

4.2.16 LEMMA: *If* $\Delta(D_\omega(\Sigma^0_2)) \subset \Gamma_u$, *and* $u(0) \geq 2$, *and if* $B \subset 2^\omega$, $A \in \Gamma_u$, $B \approx A$, *then* $B \in \Gamma_u$; *similarly for* $\check{\Gamma}_u$, *and hence for* $\Delta(\Gamma_u)$.

Proof: Let $f: A \approx B$. By Lavrentieff's theorem, there exist Π^0_2-sets G,H in 2^ω with $A \subset G$, $B \subset H$, and a homeomorphism $\tilde{f}: G \to H$ extending f. Since $A \in \Gamma_u$, also $A \in \Gamma_u(G)$ by lemma 4.2.15, and hence $B \in \Gamma_u(H)$. Again by lemma 4.2.15, there exists $\tilde{B} \in \Gamma_u$ with $\tilde{B} \cap H = B$. Since $H \in \Pi^0_2$, by lemma 4.2.12 we have $\tilde{B} \cap H \in \Gamma_u$, i.e. $B \in \Gamma_u$. The proof for $\check{\Gamma}_u$ is analogous. □

4.2.17 LEMMA: *If* $\Delta(D_\omega(\Sigma^0_2)) \subset \Gamma_u$, *and* $u(0) \geq 2$, *then* Γ_u *is reasonably closed; similarly for* $\check{\Gamma}_u$.

Proof: Being a Wadge class, Γ_u is continuously closed. Let $\phi: 2^\omega \setminus (Q_0 \cup Q_1) \to 2^\omega$ be as in definition 4.1.10, and put $P = 2^\omega \setminus (Q_0 \cup Q_1)$. If $A \in \Gamma_u$, then clearly $\phi^{-1}[A] \in \Gamma_u(P)$. Hence for some $\tilde{A} \in \Gamma_u$, we have $\phi^{-1}[A] = \tilde{A} \cap P$. Since $P \in \Pi^0_2$, $\phi^{-1}[A] \in \Gamma_u$ by lemma 4.2.12, and hence $\phi^{-1}[A] \cup Q_0 \in \Gamma_u$ by lemma 4.2.12 since $Q_0 \in \Sigma^0_2$. The proof for $\check{\Gamma}_u$ is the same. □

4.3 Existence of homogeneous Borel sets

Since we want to describe topological types rather than subsets of 2^ω, we use the following topological properties instead of membership of Γ_u or $\check{\Gamma}_u$:

4.3.1 DEFINITION: *Let* $u \in D$, *and let* X *be a zero-dimensional space.*
(a) X *has property* $\mathcal{P}_u$ *if and only if each copy of* X *in* 2^ω *is in* Γ_u.
(b) X *has property* $\check{\mathcal{P}}_u$ *if and only if each copy of* X *in* 2^ω *is in* $\check{\Gamma}_u$.

Remark: By lemma 4.2.16, if $\Delta(D_\omega(\Sigma_2^0)) \subset \Gamma_u$, and $u(0) \geq 2$, then X is $\mathcal{P}_u$ (resp. $\check{\mathcal{P}}_u$) if and only if *some* copy of X in 2^ω is in Γ_u (resp. $\check{\Gamma}_u$).

4.3.2 DEFINITION: *Let* $u \in D$, *and let* X *be a zero-dimensional space.*
(a) $X \in \mathcal{Y}_u^0$ *if and only if* X *is* $\mathcal{P}_u$, *nowhere* $\check{\mathcal{P}}_u$, *and first category.*
(b) $X \in \mathcal{Y}_u^1$ *if and only if* X *is* $\mathcal{P}_u$, *nowhere* $\check{\mathcal{P}}_u$, *and Baire.*
(c) $X \in \mathcal{Z}_u^0$ *if and only if* X *is* $\check{\mathcal{P}}_u$, *nowhere* $\mathcal{P}_u$, *and first category.*
(d) $X \in \mathcal{Z}_u^1$ *if and only if* X *is* $\check{\mathcal{P}}_u$, *nowhere* $\mathcal{P}_u$, *and Baire.*

4.3.3 LEMMA: *Let* $u \in D$, $\Delta(D_\omega(\Sigma_2^0)) \subset \Gamma_u$, $u(0) \geq 2$, *and let* X *be a zero-dimensional space.*
(a) X *is* $\mathcal{P}_u$, *nowhere* $\check{\mathcal{P}}_u$ *if and only if some copy of* X *in* 2^ω *is everywhere properly* Γ_u.
(b) X *is* $\check{\mathcal{P}}_u$, *nowhere* $\mathcal{P}_u$ *if and only if some copy of* X *in* 2^ω *is everywhere properly* $\check{\Gamma}_u$.

Proof: We prove (a); (b) is similar. Let X be $\mathcal{P}_u$ and nowhere $\check{\mathcal{P}}_u$, and let Y be a dense copy of X in 2^ω (such a copy exists, since any isolated point of X would be in $\check{\Gamma}_u$ when "embedded" in 2^ω). Then $Y \in \Gamma_u$ by definition, and hence $U \cap Y \in \Gamma_u$ for each open U in 2^ω, by lemma 4.2.11(b). Now let U be any non-empty open subset of 2^ω, and suppose that $U \cap Y \in \check{\Gamma}_u$; then by the remark following definition 4.3.1, X contains a non-empty open subset which is $\check{\mathcal{P}}_u$, a contradiction.
Conversely, suppose that Y is a copy of X in 2^ω which is everywhere properly Γ_u. Then some copy of X in 2^ω is Γ_u, so X is $\mathcal{P}_u$; and if U is a non-empty open subset of X which is $\check{\mathcal{P}}_u$, then U corresponds to $V \cap Y$ for some non-empty open subset V of 2^ω, whence $V \cap Y \in \check{\Gamma}_u$, a contradiction. □

We will now show exactly which of the classes $\mathcal{Y}_u^0, \mathcal{Y}_u^1, \mathcal{Z}_u^0, \mathcal{Z}_u^1$, with $u(0) \geq 2$ and $\Delta(D_\omega(\Sigma_2^0)) \subset \Gamma_u$, are non-empty, and we prove that the non-empty ones contain exactly one topological type, which is homogeneous.

4.3.4 LEMMA: *Let* $u \in D$, $\Delta(D_\omega(\Sigma_2^0)) \subset \Gamma_u$, *and* $u(0) \geq 2$. *Then* $\mathcal{Y}_u^0 \neq \emptyset$ *if and only if* $\mathcal{Z}_u^1 \neq \emptyset$, *and* $\mathcal{Y}_u^1 \neq \emptyset$ *if and only if* $\mathcal{Z}_u^0 \neq \emptyset$.

Proof: Since $\Delta(D_\omega(\Sigma_2^0)) \subset \Gamma_u$, both Γ_u and $\check{\Gamma}_u$ contain all compact subsets of 2^ω; hence if a subset X of 2^ω is everywhere properly Γ_u or everywhere properly $\check{\Gamma}_u$, then X is dense and co-dense in 2^ω. Thus, by 1.12.3, X is first category if and only if $2^\omega \backslash X$ is Baire. So by lemma 4.3.3, it suffices to show that $X \subset 2^\omega$ is everywhere properly Γ_u if and only if $2^\omega \backslash X$ is everywhere properly $\check{\Gamma}_u$; for then, if $Y \in \mathcal{Y}_u^0$, and $\tilde{Y}$ is a copy of Y in 2^ω which is everywhere properly Γ_u, then $2^\omega \backslash \tilde{Y} \in \mathcal{Z}_u^1$, etc..
So let X be everywhere properly Γ_u. Since $X \in \Gamma_u$, $2^\omega \backslash X \in \check{\Gamma}_u$, and hence for each open subset U of 2^ω, we have $U \cap (2^\omega \backslash X) \in \check{\Gamma}_u$ by lemma 4.2.11(b), since $u(0) \geq 2$. Now let U be any non-empty open subset of 2^ω, and suppose that $V = U \cap (2^\omega \backslash X) \in \Gamma_u$. Then $2^\omega \backslash V \in \check{\Gamma}_u$, and again by lemma 4.2.11(b), also $U \cap (2^\omega \backslash V) = U \cap X \in \check{\Gamma}_u$, a contradiction. The converse is established in a similar way. □

4.3.5 LEMMA: *If* $\Delta(D_\omega(\Sigma_2^0)) \subset \Gamma_u$, *and* $u(0) \geq 2$, *then* $\mathcal{Y}_u^0$ *and* $\mathcal{Z}_u^1$ *are non-empty*.

Proof: By lemma 4.3.4, it suffices to show that $\mathcal{Y}_u^0$ is non-empty. Since Γ_u is non-self-dual, there is a subset Z of 2^ω such that $Z \in \check{\Gamma}_u \backslash \Gamma_u$. Let

$$O = \cup\{U\colon U \text{ is open in } Z, \text{ and } U \in \Gamma_u\}.$$

Then for some $U_n \in \Gamma_u$ with U_n open in Z, we have $O = \cup_{n=0}^\infty U_n$. Let $\tilde{U}_n$ be open in 2^ω with $\tilde{U}_n \cap Z = U_n$, and let $\langle V_n\colon n < \omega\rangle$ be open subsets of 2^ω reducing $\langle \tilde{U}_n\colon n < \omega\rangle$. Then $V_n \cap Z = V_n \cap \tilde{U}_n \cap Z = V_n \cap U_n$, and since $U_n \in \Gamma_u$, $u(0) \geq 2$, and $V_n \in \Sigma_1^0$, we have $V_n \cap Z \in \Gamma_u$ by lemma 4.2.11(b). So $O = \cup_{n=0}^\infty (V_n \cap Z) \in SU(\Sigma_1^0, \Gamma_u) = \Gamma_u$ by lemma 4.2.11(a). Put $\tilde{Z} = Z \backslash O$; then $\tilde{Z}$ is non-empty since $Z \notin \Gamma_u$. Since $\tilde{Z} = Z \backslash (\cup_{n=0}^\infty V_n)$, and $2^\omega \backslash (\cup_{n=0}^\infty V_n) \in \Pi_1^0 \subset \Delta_2^0$, we have $\tilde{Z} \in \check{\Gamma}_u$ by lemma 4.2.11(b). We claim that no non-empty open subset U of $\tilde{Z}$ is in Γ_u. Indeed, if $U \in \Gamma_u$, then choose $\tilde{U}$ open in 2^ω with $\tilde{U} \cap \tilde{Z} = U$; then

$$\tilde{U} \cap Z = ((2^\omega \backslash (\cup_{n=0}^\infty V_n)) \cap (\tilde{U} \cap \tilde{Z})) \cup \cup_{n=0}^\infty ((\tilde{U} \cap V_n) \cap (V_n \cap Z))$$

$\in SU(\Sigma_2^0, \Gamma_u) = \Gamma_u$ by lemma 4.2.11(a), so $\tilde{U} \cap Z \subset O$, contradicting $\emptyset \neq U \subset$

$(\tilde{U} \cap Z)\backslash O$.

Now let Z' be a densely embedded copy of $\tilde{Z}$ in 2^ω (which exists since $\tilde{Z}$ contains no isolated points), and put $Y = 2^\omega \backslash Z'$. Also, let Q be a countable dense subset of 2^ω, let $h: 2^\omega \times 2^\omega \to 2^\omega$ be a homeomorphism, and put

$$Y_u^0 = h[Q \times Y].$$

We claim that $Y_u^0 \in \mathcal{Y}_u^0$.

First note that $Z' \in \check{\Gamma}_u$ by lemma 4.2.16, whence $Y \in \Gamma_u$. Hence $Q \times Y = \bigcup_{q \in Q}(\{q\} \times Y \cap \{q\} \times 2^\omega) \in SU(\Sigma_2^0, \Gamma_u) = \Gamma_u$, and thus $U \cap Y_u^0 \in \Gamma_u$ for each open subset U of 2^ω, by lemmas 4.2.11(a) and 4.2.16. Now let U be any non-empty open subset of 2^ω, and suppose that $U \cap Y_u^0 \in \check{\Gamma}_u$. If U_0, U_1 are non-empty open subsets of 2^ω such that $h[U_0 \times U_1] \subset U$, then also $h[U_0 \times U_1] \cap Y_u^0 \in \check{\Gamma}_u$ by lemma 4.2.11(b), and hence $(U_0 \times U_1) \cap (Q \times Y) = V \in \check{\Gamma}_u$ by lemma 4.2.16. Let $q \in U_0 \cap Q$, then $(\{q\} \times 2^\omega) \cap V = \{q\} \times (Y \cap U_1) \in \check{\Gamma}_u$ by lemma 4.2.11(b), and hence $Y \cap U_1 \in \check{\Gamma}_u$ by lemma 4.2.16. But now $U_1 \cap 2^\omega \backslash (Y \cap U_1) = U_1 \cap Z'$ is a non-empty open subset of Z' which is in Γ_u. Hence $\tilde{Z}$ also contains such a subspace, and we have a contradiction. So Y_u^0 is everywhere properly Γ_u, and first category, so $Y_u^0 \in \mathcal{Y}_u^0$ using lemma 4.3.3. □

If we try to prove that $\mathcal{Z}_u^0$ and $\mathcal{Y}_u^1$ are non-empty by replacing Γ_u in the above argument by $\check{\Gamma}_u$, then we see that we need that $SU(\Sigma_2^0, \check{\Gamma}_u) = \check{\Gamma}_u$ holds; as we shall see in lemma 4.3.7, this is not always the case. However, from corollary 4.2.14, we see that the following holds:

4.3.6 LEMMA: *If* $\Delta(D_\omega(\Sigma_2^0)) \subset \Gamma_u$, *and* $u(0) \geq 3$ *or* $(u(0) = 2$ *and* $t(u) = 3)$, *then* $\mathcal{Z}_u^0$ *and* $\mathcal{Y}_u^1$ *are non-empty.* □

In fact, if $Z_u^1 \in \mathcal{Z}_u^1$, then $\mathbb{Q} \times Z_u^1 \in \mathcal{Z}_u^0$.

4.3.7 LEMMA: *If* $\Delta(D_\omega(\Sigma_2^0)) \subset \Gamma_u$, $u(0) = 2$, *and* $t(u) \in \{1,2\}$, *then* $\mathcal{Y}_u^1 = \emptyset$, *and* $\mathcal{Z}_u^0 = \emptyset$.

Proof: By lemma 4.3.4, it suffices to show that $\mathcal{Y}_u^1 = \emptyset$. We will show that if $X \subset 2^\omega$ is everywhere properly Γ_u, then X is first category. First take $t(u) = 2$. By lemma 4.2.11(g), $\Gamma_u = SU(\Sigma_2^0, \bigcup_{n=0}^\infty \Gamma_{u_n})$ for some increasing sequence $\langle \Gamma_{u_n} : n < \omega \rangle$ of described classes with $u_n(0) \geq 2$ for all $n < \omega$. Write $X = \bigcup_{n=0}^\infty (A_n \cap C_n)$, and let $C_n = \bigcup_{m=0}^\infty C_m^n$, with $C_m^n \in \Pi_1^0$; then if $A_n \in \Gamma_{u_k}$, also $C_m^n \cap X = C_m^n \cap A_n \in \Gamma_{u_k}$ since $u_k(0) \geq 2$ and $C_m^n \in \Delta_2^0$, using lemma 4.2.11(b). If U is non-empty and open in X, say $U = \tilde{U} \cap X$, with $\tilde{U}$

open in 2^ω, and if $U \subset C_m^n \cap A_n$, then $U = \tilde{U} \cap C_m^n \cap A_n \in \Gamma_{u_k}$; thus $U \in \check{\Gamma}_u$ since $\Gamma_{u_k} < \Gamma_u$, a contradiction. So C_m^n is closed and nowhere dense in X, whence $X = \bigcup_{n=0}^\infty \bigcup_{m=0}^\infty (C_m^n \cap X)$ is first category. If $t(u) = 1$, then note that since $\Delta(D_\omega(\Sigma_2^0)) \subset \Gamma_u$, we have $\bar{u}(0) \geq 2$. Since $\Gamma_u = \mathrm{Bisep}(\Sigma_2^0, \Gamma_{\bar{u}})$ by lemma 4.2.11(f), we can argue as above. □

4.3.8 THEOREM: *Let* $\Delta(D_\omega(\Sigma_2^0)) \subset \Gamma_u$, *and* $u(0) \geq 2$. *Then up to homeomorphism, each of* $\mathcal{Y}_u^0, \mathcal{Y}_u^1, \mathcal{Z}_u^0, \mathcal{Z}_u^1$ *contains at most one space, and if it exists, then this space is strongly homogeneous, whence homogeneous.*

Proof: If $X, Y \in \mathcal{Y}_u^0$, then there are copies $\tilde{X}, \tilde{Y}$ of X,Y, respectively, in 2^ω such that $\tilde{X}, \tilde{Y}$ are everywhere properly Γ_u. By lemma 4.2.17, Γ_u is reasonably closed, whence by theorem 4.1.11, there is a homeomorphism $h: 2^\omega \to 2^\omega$ such that $h[\tilde{X}] = \tilde{Y}$, i.e. $\tilde{X} \approx \tilde{Y}$. For the other classes, the proof is similar. For the second part of the lemma, it now suffices to show that if $X \in \mathcal{Y}_u^0$ (resp. $\mathcal{Y}_u^1, \mathcal{Z}_u^0, \mathcal{Z}_u^1$), and U is a non-empty clopen subset of X, then $U \in \mathcal{Y}_u^0$ (resp. $\mathcal{Y}_u^1, \mathcal{Z}_u^0, \mathcal{Z}_u^1$). Again, we only give the argument for $\mathcal{Y}_u^0$. So let X be a subset of 2^ω which is everywhere properly Γ_u, and first category, and let U be non-empty and open in X. Embed U as a dense subset A of 2^ω; this is possible since U contains no isolated points. Let V be a non-empty open subset of 2^ω; then $V \cap A \approx W$ for some open subset W of U. Since U is open in X, so is W, and hence $W = \tilde{W} \cap X$ for some non-empty open subset $\tilde{W}$ of 2^ω. Since X is everywhere properly Γ_u, we have $W \in \Gamma_u \setminus \check{\Gamma}_u$, and thus $V \cap A \in \Gamma_u \setminus \check{\Gamma}_u$ by lemma 4.2.16. So A is everywhere properly Γ_u, hence by theorem 4.3.3, $U \in \mathcal{Y}_u^0$ since U is clearly first category. □

Remark: From the proof of lemma 4.3.7 it follows that if $\Delta(D_\omega(\Sigma_2^0)) \subset \Gamma_u$, $u(0) = 2$, and $t(u) \in \{1,2\}$, then the unique element of $\mathcal{Y}_u^0$ (resp. $\mathcal{Z}_u^1$) is in fact characterized by being zero-dimensional, $\mathcal{P}_u$, and nowhere $\check{\mathcal{P}}_u$ (resp. $\check{\mathcal{P}}_u$, and nowhere $\mathcal{P}_u$).

Finally, from the proof of theorem 4.3.8, we see that

4.3.9 THEOREM: *If* $\Delta(D_\omega(\Sigma_2^0)) \subset \Gamma_u$, $u(0) \geq 2$, *and* $X \in \mathcal{Y}_u^0 \cup \mathcal{Y}_u^1 \cup \mathcal{Z}_u^0 \cup \mathcal{Z}_u^1$, *then the Cantor set is homogeneous with respect to dense copies of* X. □

4.4 The main theorem, part 2

This section will provide the justification for the use of the hypothesis "$u(0) \geq 2$" in many of the lemmas of the preceding sections: we will show that, if X is a homogeneous Borel set in 2^ω with $X \notin \Delta(D_\omega(\Sigma_2^0))$, then $[X] \in \{\Gamma_u, \check{\Gamma}_u\}$ for some $u \in D$ with $u(0) \geq 2$. Together with the results of section 4.3, this will easily yield the main theorem of this chapter, completing the classification of all homogeneous zero-dimensional absolute Borel sets according to topological type.

4.4.1 LEMMA: *Let* $u \in D$, $\Delta(D_\omega(\Sigma_2^0)) \subset \Gamma_u$, *and* $u(0) = 1$. *If* $X \in \Gamma_u \cup \check{\Gamma}_u$ *is non-empty, then* X *contains a non-empty clopen subset* U *such that* $U \in \Gamma_v \cup \check{\Gamma}_v$ *for some described class* Γ_v *with* $\Gamma_v < \Gamma_u$.

Proof: Case 1: $t(u) = 1$. By lemma 4.2.11(f), $\Gamma_u = \mathrm{Bisep}(\Sigma_1^0, \Gamma_{\bar{u}})$, and since $\Delta(D_\omega(\Sigma_2^0)) \subset \Gamma_u$, we have $\bar{u}(0) \geq 1$. If $X \in \Gamma_u$, say $X = (A_0 \cap C_0) \cup (A_1 \cap C_1)$, then there are clopen subsets C_m^n of 2^ω, for each $n \in \{0,1\}$ and $m < \omega$, such that $C_n = \bigcup_{m=0}^\infty C_m^n$. Since $X = \bigcup_{m=0}^\infty (A_0 \cap C_m^0) \cup \bigcup_{m=0}^\infty (A_1 \cap C_m^1) \neq \emptyset$, also $A_n \cap C_m^n \neq \emptyset$ for some $n \in \{0,1\}$, $m < \omega$. But $A_n \cap C_m^n = X \cap C_m^n$ since $C_0 \cap C_1 = \emptyset$, so $A_n \cap C_m^n$ is clopen in X; and by lemma 4.2.11(b), $A_n \cap C_m^n \in \Gamma_{\bar{u}} \cup \check{\Gamma}_{\bar{u}}$. Since $\Gamma_u = \mathrm{Bisep}(\Sigma_1^0, \Gamma_{\bar{u}}) \supset \Gamma_{\bar{u}} \cup \check{\Gamma}_{\bar{u}}$, we have $\Gamma_{\bar{u}} < \Gamma_u$.
If $X \in \check{\Gamma}_u$, say $2^\omega \backslash X = (A_0 \cap C_0) \cup (A_1 \cap C_1)$, then $X = (C_0 \cap 2^\omega \backslash A_0) \cup (C_1 \cap 2^\omega \backslash A_1) \cup 2^\omega \backslash (C_0 \cup C_1)$. If some $C_n \cap 2^\omega \backslash A_n \neq \emptyset$, then proceed as above; otherwise, $X = 2^\omega \backslash (C_0 \cup C_1) \in \Pi_1^0 = \check{\Gamma}_{1\frown 1\frown 1\frown 0} < \Gamma_u$.
Case 2: $t(u) = 2$. By lemma 4.2.11(g), $\Gamma_u = \mathrm{SU}(\Sigma_1^0, \bigcup_{n=0}^\infty \Gamma_{u_n})$ for some strictly increasing sequence $\langle \Gamma_{u_n} : n < \omega \rangle$ of described classes with $u_n(0) \geq 1$ for each $n < \omega$. If $X \in \Gamma_u$, say $X = \bigcup_{n=0}^\infty (A_n \cap C_n)$, then we can write $C_n = \bigcup_{m=0}^\infty C_m^n$ for certain clopen subsets C_m^n of 2^ω, and proceed as in case 1. Similarly if $X \in \check{\Gamma}_u$.
Case 3: $u = 1\frown 2\frown \eta\frown u*$ for some $u* \in D$ with $u*(0) > 1$; then $\Gamma_u = \mathrm{Sep}(D_\eta(\Sigma_1^0), \Gamma_{u*})$. If $X \in \Gamma_u$, then $X = (A_0 \cap C) \cup (A_1 \backslash C)$ for some $C \in D_\eta(\Sigma_1^0)$, $A_0 \in \check{\Gamma}_{u*}$, $A_1 \in \Gamma_{u*}$. Let $C = D_\eta(\langle C_\zeta : \zeta < \eta \rangle)$.
(i) If $C_\zeta \cap X = \emptyset$ for all $\zeta < \eta$, then $X = A_1 \backslash (\bigcup_{\zeta<\eta} C_\zeta)$. Since $A_1 \in \Gamma_{u*}$, and $2^\omega \backslash (\bigcup_{\zeta<\eta} C_\zeta) \in \Pi_1^0 \subset \Delta_2^0$, we have $X \in \Gamma_{u*}$ by lemma 4.2.11(b), and $\Gamma_{u*} < \Gamma_u$ since $\Gamma_{u*} \cup \check{\Gamma}_{u*} \subset \Gamma_u$.
(ii) Let $\alpha < \eta$ be minimal with $C_\alpha \cap X \neq \emptyset$. If α and η are both even or both odd, then $C_\alpha \backslash (\bigcup_{\beta<\alpha} C_\beta) \subset 2^\omega \backslash C$, so $C_\alpha \cap X = C_\alpha \cap A_1 \backslash C$. Since $C_\alpha \cap 2^\omega \backslash C \in \Delta_2^0$, we have $C_\alpha \cap X \in \Gamma_{u*}$ as above, and $C_\alpha \cap X$ is a non-empty open subset of X; if U is clopen in 2^ω with $\emptyset \neq U \cap X \subset C_\alpha \cap X$, then $U \cap X$

$\in \Gamma_{u*}$ by lemma 4.2.11(b), so $U \cap X$ is the required clopen subset of X. If α is even and η is odd or conversely, then $C_\alpha \setminus (\bigcup_{\beta<\alpha} C_\beta) \subset C$, so $C_\alpha \cap X = C_\alpha \cap C \cap A_0 \in \check{\Gamma}_{u*}$; we find the required clopen subset of X as above.

If $X \in \check{\Gamma}_u$, then $2^\omega \setminus X = (A_0 \cap C) \cup (A_1 \setminus C)$, so $X = ((2^\omega \setminus A_0) \cap C) \cup (2^\omega \setminus A_1) \setminus C$. Put $\tilde{A}_0 = 2^\omega \setminus A_0 \in \Gamma_{u*}$, $\tilde{A}_1 = 2^\omega \setminus A_1 \in \check{\Gamma}_{u*}$, and argue as above.

<u>Case 4</u>: $u = 1^\frown 3^\frown \eta^\frown \langle u_0, u_1 \rangle$, where $\Gamma_{u_1} < \Gamma_{u_0}$, $u_0(0) > 1$, and $u_1(0) \geq 1$; then $\Gamma_u = \mathrm{Bisep}(D_\eta(\Sigma^0_1), \Gamma_{u_0}, \Gamma_{u_1})$. If $X \in \Gamma_u$, then by lemma 4.2.11(e), we can write $X = (X \cap C) \cup B \setminus C$ for some $C \in \Sigma^0_1$, $B \in \Gamma_{u_1}$ such that $X \cap C \in \mathrm{Bisep}(D_\eta(\Sigma^0_1), \Gamma_{u_0}) = \Gamma_v$, where $v = 1^\frown 3^\frown \eta^\frown \langle u_0, \underline{0} \rangle$. Clearly, $\Gamma_v \subset \Gamma_u$, and if $Y \in \check{\Gamma}_v$, say $2^\omega \setminus Y = (A_0 \cap C_0) \cup (A_1 \cap C_1)$, then

$$Y = ((2^\omega \setminus A_1) \cap C_1) \cup ((2^\omega \setminus A_0) \cap C_0) \cup 2^\omega \setminus (C_0 \cup C_1)$$

$\in \Gamma_u$, because $2^\omega \in \Gamma_{u_1}$ since $u_1(0) \geq 1$; so $\check{\Gamma}_v \subset \Gamma_u$, whence $\Gamma_v < \Gamma_u$. Now if $X \cap C \neq \emptyset$, and U is a clopen subset of 2^ω such that $\emptyset \neq U \cap X \subset C \cap X$, then $U \cap X \in \Gamma_v$ and we are done. Otherwise, $X = B \setminus C$; but $B \in \Gamma_{u_0}$, $2^\omega \setminus C \in \Delta^0_2$, and $u_0(0) \geq 2$, so $B \setminus C \in \Gamma_{u_0}$ by lemma 4.2.11(b). Hence $X \in \Gamma_{u_0} < \Gamma_u$.

If $X \in \check{\Gamma}_u$, then again by lemma 4.2.11(e), we have $2^\omega \setminus X = ((2^\omega \setminus X) \cap C) \cup B \setminus C$ for some $C \in \Sigma^0_1$, $B \in \Gamma_{u_1}$ with $C \setminus (2^\omega \setminus X) = C \cap X \in \mathrm{Bisep}(D_\eta(\Sigma^0_1), \Gamma_{u_0})$. Thus, $X = (X \cap C) \cup (2^\omega \setminus B) \setminus C$. Since $2^\omega \setminus B \in \check{\Gamma}_{u_1} < \Gamma_{u_0}$, we can argue as above.

<u>Case 5</u>: $u = 1^\frown 5^\frown \eta^\frown \langle u_0, u_1 \rangle$, where $\Gamma_{u_1} < \Gamma_{u_0}$, $u_1(0) \geq 1$, and $\Gamma_{u_0} = \mathrm{SU}(\Sigma^0_1, \bigcup_{n=0}^\infty \Gamma_{v_n})$ for some strictly increasing sequence $\langle \Gamma_{v_n} : n < \omega \rangle$, where $\langle v_n(0) : n < \omega \rangle$ is non-decreasing, and $\sup_{n<\omega} v_n(0) > 1$. Then $\Gamma_u = \mathrm{SD}_\eta(\langle \Sigma^0_1, \Gamma_{u_0} \rangle, \Gamma_{u_1})$. If $X \in \Gamma_u$, say $X = \bigcup_{\zeta<\eta}(A_\zeta \setminus (\bigcup_{\beta<\zeta} C_\beta)) \cup B \setminus (\bigcup_{\zeta<\eta} C_\zeta)$, and if we put $C = \bigcup_{\zeta<\eta} C_\zeta$, then $X = (X \cap C) \cup B \setminus C$, with $C \in \Sigma^0_1$, $B \in \Gamma_{u_1}$, and $X \cap C \in \mathrm{SD}_\eta(\langle \Sigma^0_1, \Gamma_{u_0} \rangle) = \Gamma_v$, where $v = 1^\frown 5^\frown \eta^\frown \langle u_0, \underline{0} \rangle$. As in case 4, we will be done if we can show that $\Gamma_v < \Gamma_u$. Clearly, $\Gamma_v \subset \Gamma_u$, so it suffices to prove:

CLAIM: $\check{\Gamma}_v \subset \Gamma_u$.

Indeed, let $Y \in \check{\Gamma}_v$, say $2^\omega \setminus Y = \bigcup_{\zeta<\eta}(B_\zeta \setminus (\bigcup_{\beta<\zeta} D_\beta))$. Then

$$Y = \bigcup_{\zeta<\eta}(\tilde{B}_\zeta \setminus (\bigcup_{\beta<\zeta} D_\beta)) \cup 2^\omega \setminus (\bigcup_{\zeta<\eta} D_\zeta),$$

where $\tilde{B}_\zeta = (D_\zeta \setminus B_\zeta) \cup \bigcup_{\beta<\zeta} D_\beta$. Now since $B_\zeta \in \Gamma_{u_0}$ with envelop D_ζ, say $B_\zeta = \bigcup_{n=0}^\infty (B^\zeta_n \cap E^\zeta_n)$, with $D_\zeta = \bigcup_{n=0}^\infty E^\zeta_n$, we can write

$$\tilde{B}_\zeta = \bigcup_{n=0}^\infty ((2^\omega \setminus B^\zeta_n) \cap E^\zeta_n) \cup \bigcup_{\beta<\zeta} D_\beta.$$

Since each B^ζ_n is in some $\Gamma_{v_m} < \Gamma_{v_{m+1}}$, we have $2^\omega \setminus B^\zeta_n \in \Gamma_{v_{m+1}}$, and of course we can assume that $v_{m+1}(0) \geq 2$. By lemma 4.2.11(b), also

$(2^\omega \setminus B_n^\zeta) \cup \bigcup_{\beta<\zeta} D_\zeta \in \Gamma_{v_{m+1}}$, and if we let $\langle F_n^\zeta : n < \omega \rangle$ reduce $\langle E_n^\zeta \cup \bigcup_{\beta<\zeta} D_\beta : n < \omega \rangle$, then

$$\widetilde{B}_\zeta = \bigcup_{n=0}^\infty (((2^\omega \setminus B_n^\zeta) \cup \bigcup_{\beta<\zeta} D_\beta) \cap F_n^\zeta)$$

$\in \Gamma_{u0}$, with envelop $\bigcup_{n=0}^\infty E_n^\zeta \cup \bigcup_{\beta<\zeta} D_\beta = \bigcup_{\beta\le\zeta} D_\beta = D_\zeta$. Since clearly $\widetilde{B}_\zeta \subset D_\zeta \subset \widetilde{B}_{\zeta+1}$, and since $2^\omega \in \Gamma_{u1}$ because $u_1(0) \ge 1$, we have $Y \in \Gamma_{u0}$. This proves the claim.

Finally, let $X \in \check{\Gamma}_u$, say $2^\omega \setminus X = ((2^\omega \setminus X) \cap C) \cup B \setminus C$, with $C \in \Sigma_1^0$, $B \in \Gamma_{u1}$, and $Y = (2^\omega \setminus X) \cap C \in SD_\eta(\langle \Sigma_1^0, \Gamma_{u0} \rangle) = \Gamma_v$, as above. Then $X = (X \cap C) \cup (2^\omega \setminus B) \setminus C$. Now if $X \cap C \neq \emptyset$, let U be a clopen subset of 2^ω such that $U \cap X \neq \emptyset$ and $U \subset C$; then $U \cap X = U \cap C \cap X = U \cap C \cap (2^\omega \setminus Y) = U \cap (2^\omega \setminus Y) \in \check{\Gamma}_v$ since $v(0) = 1$, by lemma 4.2.11(b). Otherwise, $X = (2^\omega \setminus B) \setminus C \in \Gamma_{u0} < \Gamma_u$. □

4.4.2 LEMMA: *Let* $u \in D$, $\Delta(D_\omega(\Sigma_2^0)) \subset \Gamma_u$, *and* $u(0) \ge 2$. *If* X *is a homogeneous subset of* 2^ω, *and* A *is a non-empty open subset of* X *such that* $A \in \Gamma_u$, *then* $X \in \Gamma_u$; *similarly for* $\check{\Gamma}_u$, *and hence for* $\Delta(\Gamma_u)$.

Proof: let $x \in A$, and for each $y \in X$, let $h_y : X \approx X$ be such that $h_y(x) = y$. Then $\{h_y[A] : y \in X\}$ is an open covering of X, which has a countable subcovering $\{U_n : n < \omega\}$. For each $n < \omega$, let V_n be open in 2^ω such that $V_n \cap X = U_n$, and let $\langle W_n : n < \omega \rangle$ reduce $\langle V_n : n < \omega \rangle$. Since $W_n \cap X = W_n \cap V_n \cap X = W_n \cap U_n$, and since $U_n \in \Gamma_u$ by lemma 4.2.16, we have $X = \bigcup_{n=0}^\infty (V_n \cap U_n) \in SU(\Sigma_1^0, \Gamma_u) = \Gamma_u$ by lemma 4.2.11(a). The proof for $\check{\Gamma}_u$ is identical. □

4.4.3 LEMMA: *Let* X *be a homogeneous Borel subset of* 2^ω *such that* $X \notin \Delta(D_\omega(\Sigma_2^0))$. *Then* $[X] \in \{\Gamma_u, \check{\Gamma}_u\}$ *for some* $u \in D$ *with* $\Delta(D_\omega(\Sigma_2^0)) \subset \Gamma_u$ *and* $u(0) \ge 2$.

Proof: Let $\{\Gamma_u, \check{\Gamma}_u\}$ be a minimal non-self-dual pair in the Borel Wadge hierarchy such that $A \in \Gamma_u \cup \check{\Gamma}_u$ for some non-empty open subset A of X. By lemma 4.4.2, if $A \in \Delta(D_\omega(\Sigma_2^0))$, then $X \in \Delta(D_\omega(\Sigma_2^0))$ (for $D_\omega(\Sigma_2^0) = \Gamma_v$, where $v = 2^\frown 1^\frown \omega^\frown \underline{0})$, a contradiction; thus, $\Delta(D_\omega(\Sigma_2^0)) \subset \Gamma_u$. So if $u(0) = 1$, then we can apply lemma 4.4.1 to obtain a non-empty open subset B of A such that $B \in \Gamma_v \cup \check{\Gamma}_v$ for some described class $\Gamma_v < \Gamma_u$, contradicting minimality of $\{\Gamma_u, \check{\Gamma}_u\}$. So we must have $u(0) \ge 2$, and thus we can apply lemma 4.4.2 again to obtain that $X \in \Gamma_u$ (if $A \in \Gamma_u$) or $X \in \check{\Gamma}_u$ (if $A \in \check{\Gamma}_u$), and hence clearly $[X] \in \{\Gamma_u, \check{\Gamma}_u\}$, again by minimality of $\{\Gamma_u, \check{\Gamma}_u\}$. □

We can now state the main theorem of this chapter.
Let $\mathcal{D}_0 = \{u \in \mathcal{D}: \Delta(D_\omega(\Sigma^0_2)) \subset \Gamma_u,\ u(0) \geq 2\}$, and $\mathcal{D}_1 = \{u \in \mathcal{D}_0: u(0) \geq 3$ or $t(u) = 3\}$. By lemmas 4.3.5 and 4.3.6, there are unique, homogeneous spaces $Y^0_u \in \mathcal{Y}^0_u$, $Z^1_u \in \mathcal{Z}^1_u$ for each $u \in \mathcal{D}_0$, and $Y^1_u \in \mathcal{Y}^1_u$, $Z^0_u \in \mathcal{Z}^0_u$ for each $u \in \mathcal{D}_1$. Let Δ denote the class of all spaces that are in $\Delta(D_\omega(\Sigma^0_2))$ in the sense of definition 3.1.1(c), i.e. in the absolute sense.

4.4.4 THEOREM: *The topological types of homogeneous zero-dimensional absolute Borel sets that are not in* Δ *are precisely those of the spaces* Y^0_u, Z^1_u *for* $u \in \mathcal{D}_0$, *and the spaces* Y^1_u, Z^0_u *for* $u \in \mathcal{D}_1$.

Proof: From theorem 4.3.8, we see that these spaces are homogeneous. So let X be a homogeneous zero-dimensional absolute Borel set which is not in Δ, and let Y be a copy of X in 2^ω; then $Y \notin \Delta(D_\omega(\Sigma^0_2))$, so by lemma 4.4.3, $[Y] \in \{\Gamma_u, \check{\Gamma}_u\}$ for some $u \in \mathcal{D}$ with $\Delta(D_\omega(\Sigma^0_2)) \subset \Gamma_u$ and $u(0) \geq 2$. Consider the case $[Y] = \Gamma_u$ (the other case is similar). By the remark following definition 4.3.1, X is $\mathcal{P}_u$; and if A is a non-empty open subset of Y which is in $\check{\Gamma}_u$, then $Y \in \check{\Gamma}_u$ by lemma 4.4.2, a contradiction, so X is nowhere $\check{\mathcal{P}}_u$. By 1.12.1, X is either first category or Baire, so $X \in \mathcal{Y}^0_u \cup \mathcal{Y}^1_u \cup \mathcal{Z}^0_u \cup \mathcal{Z}^1_u$. Since $\mathcal{Y}^1_u = \emptyset = \mathcal{Z}^0_u$ if $u \in \mathcal{D}_0 \setminus \mathcal{D}_1$ by lemma 4.3.7, we are done. □

With theorems 3.4.13, 3.5.9, 3.6.2, 4.3.8, and 4.4.4, we have now obtained topological characterizations of *all homogeneous zero-dimensional absolute Borel sets.*

Remark: The homogeneous zero-dimensional elements of $\Delta^0_3 \setminus \Delta$ have been described twice, by theorems 3.6.2 and 4.4.4. In section 4.6, we will relate the two classifications.

Remark: Since Steel's theorem always yields autohomeomorphisms of 2^ω, the methods used here can only produce spaces with respect to which the Cantor set is dense homogeneous (see theorem 4.3.9); thus it is clear that we can not extend theorem 4.4.4 to include the elements of Δ as well, since the Cantor set is not homogeneous with respect to dense copies of the spaces X_{4k} and X_{4k+1} of Chapter 3, where $k < \omega$ (see theorem 4.4.8).

Since there are only ω_1 Borel Wadge classes (see e.g. the second remark following definition 4.2.2), the above theorem, together with corollary 3.6.3, yields:

4.4.5 COROLLARY: *There are pecisely* ω_1 *homogeneous zero-dimensional absolute Borel sets.* □

In fact, by analyzing Louveau's description of the Wadge classes, it is not hard to see that there are precisely ω_1 homogeneous zero-dimensional absolute Borel sets of exactly ambiguous class α, for each $\alpha \geq 2$. In section 4.5, we will examine the homogeneous zero-dimensional absolute Borel sets that are exactly of additive or multiplicative class α, for $\alpha \geq 2$.

Another corollary to theorems 3.6.2 and 4.4.4 is the following strange characterization of $C\setminus\{p\}$.

4.4.6 COROLLARY: *Up to homeomorphism,* $C\setminus\{p\}$ *is the only non-discrete, zero-dimensional absolute Borel set which is homogeneous, but not strongly homogeneous.* □

As an analogue to theorem 3.6.4, we have:

4.4.7 THEOREM: *Let* A *be a zero-dimensional space.*
(a) *If* $u \in D_0$, *and* A *is* P_u, *then* A *can be embedded as a closed subspace of* Y^0_u.
(b) *If* $u \in D_0$, *and* A *is* $\not P_u$, *then* A *can be embedded as a closed subspace of* Z^1_u.
(c) *If* $u \in D_1$, *and* A *is* P_u, *then* A *can be embedded as a closed subspace of* Y^1_u.
(d) *If* $u \in D_1$, *and* A *is* $\not P_u$, *then* A *can be embedded as a closed subspace of* Z^0_u.

Proof: (a) For each $k \in N$, let X_k be a dense copy of Y^0_u in $\{1/k\} \times 2^\omega$, and consider A as a subset of $\{0\} \times 2^\omega$. We claim that, as a subset of $[0,1] \times 2^\omega$, $\tilde{A} = A \cup \bigcup_{k=1}^\infty X_k \approx Y^0_u$. Indeed, since A is nowhere dense in $\tilde{A}$, $\tilde{A}$ is nowhere $\not P_u$, and clearly $\tilde{A}$ is first category. Also, if we identify $(\{0\} \cup \{1/k\colon k \in N\}) \times 2^\omega$ with 2^ω, then by lemma 4.2.16, $\tilde{A} \in SU(\Pi^0_1,\Gamma_u) \subset SU(\Sigma^0_2,\Gamma_u) = \Gamma_u$ by lemma 4.2.11(a), so $\tilde{A}$ is P_u. Clearly, A is closed in $\tilde{A}$.
(b) Let X_k be as in (a), and put $B = (\{0\} \times 2^\omega)\setminus A \cup \bigcup_{k=1}^\infty X_k$; then $B \approx Y^0_u$ as in (a). Hence, since B is dense in $X = (\{0\} \cup \{1/k\colon k \in N\}) \times 2^\omega$, we have $\tilde{A} = X\setminus B \approx Z^1_u$ as in the proof of lemma 4.3.4; again, A is closed in $\tilde{A}$.
(d) is proved as (a), using corollary 4.2.14 instead of lemma 4.2.11(a), and (c) follows from (d) exactly as we proved (b) from (a). □

Finally, from corollaries 3.2.7 and 3.2.9, and theorems 3.4.14, 3.5.10, and 4.3.9, we can deduce:

4.4.8 THEOREM: *Let* X *be a homogeneous zero-dimensional absolute Borel set. Then the Cantor set is homogeneous with respect to dense copies of* X *if and only if* $\mathcal{C}\setminus X_0 \approx \mathcal{C}\setminus X_1$ *for all dense copies* X_0, X_1 *of* X *in* $\mathcal{C}$. □

4.5 Some applications

The results of the preceding sections are rather abstract; in particular, the construction of the elements of $\mathcal{Y}^0_u$, $\mathcal{Y}^1_u$, $\mathcal{Z}^0_u$, and $\mathcal{Z}^1_u$ in lemmas 4.3.5 and 4.3.6 does not make it very clear as to what kind of space we end up with. Therefore, we will begin this section by showing that if $u = \alpha^\frown 1^\frown 1^\frown \underline{0}$ for some $\alpha \geq 3$, then there is a very elegant way of describing elements of $\mathcal{Y}^0_u$, $\mathcal{Y}^1_u$, $\mathcal{Z}^0_u$, and $\mathcal{Z}^1_u$ (each of these classes is non-empty since $u(0) \geq 3$). The spaces that we want are obtained from the construction process for Borel sets of exact class, due to Sikorski [54] (see also Engelking, Holsztyński, and Sikorski [16], and Freiwald, McDowell, and McHugh [17]), which for 2^ω is the following:

Let $p \in 2^\omega$, and put $M_1 = \{p\}$, $A_1 = 2^\omega \setminus M_1$.
If $\alpha \in [2,\omega_1)$, and A_β, M_β have been defined for $\beta < \alpha$, then

$$M_\alpha = \Pi^\infty_{i=0} A_\gamma \subset \Pi^\infty_{i=0} 2^\omega \approx 2^\omega \quad \text{if} \quad \alpha = \gamma+1,$$

$$M_\alpha = \Pi_{\beta<\alpha} A_\beta \subset \Pi_{\beta<\alpha} 2^\omega \approx 2^\omega \quad \text{if} \quad \alpha \text{ is a limit.}$$

In both cases, put $A_\alpha = 2^\omega \setminus M_\alpha$.

Sikorski showed that A_α is Σ^0_α but not Π^0_α in 2^ω, and M_α is Π^0_α but not Σ^0_α in 2^ω.

For $\alpha \in [2,\omega_1)$, put $u(\alpha) = \alpha^\frown 1^\frown 1^\frown \underline{0}$; note that $\mathcal{Y}^0_{u(\alpha)}$ consists of all zero-dimensional spaces that are Σ^0_α, nowhere Π^0_α, and first category, $\mathcal{Y}^1_{u(\alpha)}$ consists of all zero-dimensional spaces that are Σ^0_α, nowhere Π^0_α, and Baire, and similarly for $\mathcal{Z}^0_u, \mathcal{Z}^1_u$.

4.5.1 THEOREM: *Let* $\alpha \in [2,\omega_1)$.
(a) *If* $1+\alpha$ *is even, then* $M_\alpha \in \mathcal{Z}^0_{u(\alpha)}$, $A_\alpha \in \mathcal{Y}^1_{u(\alpha)}$, $\tilde{A}_\alpha = \mathbb{Q} \times A_\alpha \in \mathcal{Y}^0_{u(\alpha)}$, *and* $\tilde{M}_\alpha = 2^\omega \setminus \tilde{A}_\alpha \in \mathcal{Z}^1_{u(\alpha)}$, *where* $\tilde{A}_\alpha$ *is densely embedded in* 2^ω.
(b) *If* $1+\alpha$ *is odd, then* $M_\alpha \in \mathcal{Z}^1_{u(\alpha)}$, $A_\alpha \in \mathcal{Y}^0_{u(\alpha)}$, *and if* $\alpha \geq 3$, *then* $\tilde{M}_\alpha = \mathbb{Q} \times M_\alpha \in \mathcal{Z}^0_{u(\alpha)}$, *and* $\tilde{A}_\alpha = 2^\omega \setminus \tilde{M}_\alpha \in \mathcal{Y}^1_{u(\alpha)}$, *where* $\tilde{M}_\alpha$ *is densely embedded in* 2^ω.

Proof: The proof is by induction on α. If $\alpha = 2$, then the theorem is clear, since it is not hard to see that $M_2 \approx \boldsymbol{P}$ and $A_2 \approx \boldsymbol{Q}\times\boldsymbol{C}$. So suppose that the theorem holds for all $\beta < \alpha$. Assume for example that α is a limit (the other cases are entirely similar); then $1+\alpha = \alpha$ is even, and $M_\alpha = \Pi_{\beta<\alpha} A_\beta$. Now $A_1 \times A_2 \approx \boldsymbol{C}\setminus\{p\}\times\boldsymbol{Q}\times\boldsymbol{C} \approx \boldsymbol{Q}\times\boldsymbol{C}$ is strongly homogeneous, and if $\beta \geq 3$, then $A_\beta \in \mathcal{Y}^0_{u(\beta)} \cup \mathcal{Y}^1_{u(\beta)}$ is strongly homogeneous by theorem 4.3.8. So if U is a non-empty basic clopen subset of M_α, then $U \approx M_\alpha \notin \Sigma^0_\alpha$; hence M_α is nowhere Σ^0_α. Since A_β is first category if $1+\beta$ is odd, M_α is first category, so $M_\alpha \in \mathcal{Z}^0_{u(\alpha)}$. It is now easily deduced that $A_\alpha \in \mathcal{Y}^1_{u(\alpha)}$, $\boldsymbol{Q}\times A_\alpha \in \mathcal{Y}^0_{u(\alpha)}$, and $2^\omega\setminus(\boldsymbol{Q}\times A_\alpha) \in \mathcal{Z}^1_{u(\alpha)}$. □

Thus, the homogeneous Borel sets of exact class 2 are precisely $\boldsymbol{Q}^\omega$, $2^\omega\setminus\boldsymbol{Q}^\omega$, $\boldsymbol{Q}\times(2^\omega\setminus\boldsymbol{Q}^\omega)$, and $2^\omega\setminus(\boldsymbol{Q}\times(2^\omega\setminus\boldsymbol{Q}^\omega))$; those of exact class 3 are $(2^\omega\setminus\boldsymbol{Q}^\omega)^\omega = Z$, $2^\omega\setminus Z$, $\boldsymbol{Q}\times Z$, and $2^\omega\setminus(\boldsymbol{Q}\times Z)$, etc..

Note that if $u = u(3) = 3^\frown 1^\frown 1^\frown \underline{0}$, then theorem 4.4.7(d) says that any zero-dimensional absolute $F_{\sigma\delta}$ can be embedded as a closed subspace of $\boldsymbol{Q}^\omega$; this was proved independently also by van Douwen [9] and Junnila [24]. In fact, if we let $Y_\alpha = \tilde{A}_\alpha$ (resp. A_α), and $Z_\alpha = M_\alpha$ (resp. $\tilde{M}_\alpha$) if $1+\alpha$ is even (resp. odd), for $\alpha \geq 3$ (so actually $Y_\alpha \in \mathcal{Y}^0_{u(\alpha)}$, $Z_\alpha \in \mathcal{Z}^0_{u(\alpha)}$, both first category), then we have the following analogue to corollaries 2.1.3, 2.2.4, 2.3.3, 2.4.3, and 2.4.7:

4.5.2 THEOREM: *Let* X *be a non-empty zero-dimensional space, and let* $\alpha \geq 3$.
(a) *If* X *is* Σ^0_α, *then* $X\times Y_\alpha \approx Y_\alpha$.
(b) *If* X *is* Π^0_α, *then* $X\times Z_\alpha \approx Z_\alpha$.

Proof: Immediate from the characterizations. □

Thus, if X is a non-empty zero-dimensional absolute $F_{\sigma\delta}$, then $X\times\boldsymbol{Q}^\omega \approx \boldsymbol{Q}^\omega$; this was first proved in van Engelen [12].

Remark: We could not have chosen $Y_\alpha \in \mathcal{Y}^1_{u(\alpha)}$, $Z_\alpha \in \mathcal{Z}^1_{u(\alpha)}$ in the above theorem since $X\times Y_\alpha, X\times Z_\alpha$ are not necessarily Baire.

For the spaces A_α, $\tilde{A}_\alpha$, M_α, and $\tilde{M}_\alpha$, we now derive Hurewicz-type theorems. We need three lemmas; the first is due to Harrington (see the proof of lemma 3 in Steel [56]).

4.5.3 LEMMA: *Let* Γ *be a reasonably closed class of Borel sets in* 2^ω*; let* $A, B \subset 2^\omega$ *be such that* $A \in \Gamma$, $B \notin \check{\Gamma}$. *Then there is a one-to-one continuous* $f: 2^\omega \to 2^\omega$ *such that* $A = f^{-1}[B]$.

4.5.4 LEMMA (Kunen and Miller [26], theorem 4): *Suppose* A *is a Borel subset of a complete space* X. *Then for any* $\alpha < \omega_1$, A *is* Π^0_α *in* X *if and only if for all compact zero-dimensional* $P \subset X$, $P \cap X$ *is* Π^0_α *in* P.

The final lemma is a Hurewicz-type result itself, but only for zero-dimensional spaces.

4.5.5 LEMMA: *Let* B *be a zero-dimensional absolute Borel set, and let* $u \in D$ *be such that* $\Delta(D_\omega(\Sigma^0_2)) \subset \Gamma_u$*, and* $u(0) \geq 2$. *Let* Y *be* P_u*, nowhere* $\check{P}_u$*, and let* Z *be* $\check{P}_u$*, nowhere* P_u.

(a) *If* B *is not* $\check{P}_u$*, then* B *contains a closed copy of* Y.

(b) *If* B *is not* P_u*, then* B *contains a closed copy of* Z.

Proof: (a) Embed B and Y in 2^ω; then $B \notin \check{\Gamma}_u$, $Y \in \Gamma_u$, so by lemma 4.2.17 we can apply lemma 4.5.3 to obtain a continuous one-to-one mapping $f: 2^\omega \to 2^\omega$ such that $Y = f^{-1}[B]$. Then $f: 2^\omega \to f[2^\omega]$ is a homeomorphism, so $Y \approx f[Y]$, and $f[Y] = f[2^\omega] \cap B$ is closed in B. The proof of (b) is analogous. □

Remark: The above proof, and the proof of the next theorem were motivated by a suggestion of A.W. Miller.

4.5.6 THEOREM: *Let* A *be a Borel subset of a compact space* X*, and* $\alpha \geq 3$.

(a) *If* A *is not* Π^0_α*, then* X *contains Cantor sets* K_0 *and* K_1 *such that* $K_0 \cap A \approx A_\alpha$, $K_0 \backslash A \approx M_\alpha$*, and* $K_1 \cap A \approx \tilde{A}_\alpha$, $K_1 \backslash A \approx \tilde{M}_\alpha$.

(b) *If* A *is not* Σ^0_α*, then* X *contains Cantor sets* K_0 *and* K_1 *such that* $K_0 \cap A \approx M_\alpha$, $K_0 \backslash A \approx A_\alpha$*, and* $K_1 \cap A \approx \tilde{M}_\alpha$, $K_1 \backslash A \approx \tilde{A}_\alpha$.

Proof: Clearly, (b) follows from (a). So suppose that A is not Π^0_α. By lemma 4.5.4, X contains a compact zero-dimensional space K such that $B = K \cap A$ is not Π^0_α. Now apply lemma 4.5.5 to B and $u = u(\alpha) = \alpha^\frown 1^\frown 1^\frown \underline{0}$ to obtain closed subsets $A_0 \approx A_\alpha$, $A_1 \approx \tilde{A}_\alpha$ of B; then $K_0 = \overline{A}_0$, $K_1 = \overline{A}_1$ are as required. □

The construction of the sets M_α and A_α naturally led Sikorski, in [54], to the following question (Coll. Math. problem P.215):

QUESTION: *Let* B_n *be a* Σ^0_α*-set in a space* X_n*, but not a* Π^0_α*-set in* X_n *(*$n = 0,1,2,\ldots$*). Prove or disprove that the set* $B = \Pi^\infty_{n=0} B_n$ *(which is, of course, a* $\Pi^0_{\alpha+1}$*-set) is not a* $\Sigma^0_{\alpha+1}$*-set in* $X = \Pi^\infty_{n=0} X_n$.

We give a partial answer to this question:

4.5.7 THEOREM: *Let* $\alpha \in [1,\omega_1)$*. In the above question,* $\Pi^\infty_{n=0} B_n$ *is not a* $\Sigma^0_{\alpha+1}$*-set absolutely.*

Proof: If $\alpha = 1$, then B_n is not compact, so it contains a closed copy of ω; hence B contains a closed copy of $\omega^\omega \approx \boldsymbol{P}$, which is not σ-compact, so B is not Σ^0_2. If $\alpha = 2$, then by theorem 3.3.1, B_n contains a closed copy of $\boldsymbol{Q}$, so B contains a closed copy of $\boldsymbol{Q}^\omega \approx M_3$; and if $\alpha \geq 3$, then by theorem 4.5.6, B_n contains a closed copy of A_α, so B contains a closed copy of $\Pi^\infty_{n=0} A_\alpha = M_{\alpha+1}$. In both cases, since $M_{\alpha+1}$ is not a $\Sigma^0_{\alpha+1}$-set, neither is B. □

Remark: Sikorski took X_n to be a metric space, not mentioning separability; thus, the question is also open for the non-separable case.

4.6 The ambiguous class 2 revisited

Recall the definitions of the classes X^i_β, $i \in \{1,2\}$, $\beta \in [\omega,\omega_1)$ from definition 3.5.7: if X is zero-dimensional, $\alpha < \omega_1$ is a limit, and $n \in \mathbb{N}$, then:

$X \in X^2_\alpha$ if and only if X is P_α, and nowhere P_β for $\beta < \alpha$;

$X \in X^1_{\alpha+n}$ if and only if X is $P_{\alpha+n}$, nowhere $P_{\alpha+n-1}$, and X does not contain any closed subsets belonging to $X^2_{\alpha+n-1}$;

$X \in X^2_{\alpha+n}$ if and only if X is $P_{\alpha+n}$, nowhere $P_{\alpha+n-1}$, and every non-empty clopen subset of X contains a closed subset belonging to $X^2_{\alpha+n-1}$.

In theorem 3.6.2 it was proved that the unique elements of these classes are precisely the homogeneous zero-dimensional absolute Borel sets in $\Delta^0_3\backslash\Delta$, where Δ consists of all spaces that are $\Delta(D_\omega(\Sigma^0_2))$ absolutely. From theorem 4.4.4 it follows that each of the above classes is some Y^0_u, Y^1_u, Z^0_u, or Z^1_u; in this section, we establish the exact correspondence.

We first determine which $u \in D$ determine a relevant Wadge class.

4.6.1 LEMMA: *Let* $u \in D$ *be such that* $\Delta(D_\omega(\Sigma_2^0)) \subset \Gamma_u \subset \Delta_3^0$, *and* $u(0) \geq 2$. *Then* $u = 2^\frown 1^\frown \eta^\frown \underline{0}$ *for some* $\eta \in [\omega,\omega_1)$, *i.e.* $\Gamma_u = D_\eta(\Sigma_2^0)$.

Proof: It is easily seen from definition 4.2.2 that if $u(0) \geq 3$, then $\Gamma_u \supset \Sigma_3^0$; so $u(0) = 2$. If $u(1) = 2$ (resp. 3, resp. 4), then Γ_u contains Γ_{u*} (resp. Γ_{u_0}, resp. some Γ_{u_n}), where $u*(0) \geq 3$ (resp. $u_0(0) \geq 3$, resp. $u_n(0) \geq 3$), and thus $\Gamma_u \supset \Sigma_3^0$; if $u(1) = 5$, then Γ_u contains $\Gamma_{u_0} = SU(\Sigma_2^0, \bigcup_{n=0}^\infty \Gamma_{u_n})$, which contains Σ_3^0 by the previous case $u(1) = 4$. So $u(1) = 1$, and clearly $u(2) \geq \omega$. □

Since for each $u = 2^\frown 1^\frown \eta^\frown \underline{0}$ we have $t(u) \neq 3$ and $u(0) \not\geq 3$, we see from theorem 4.4.4 that the homogeneous zero-dimensional absolute Borel sets in $\Delta_3^0 \backslash \Delta$ are precisely

$$\{Y_u^0, Z_u^1 : u = 2^\frown 1^\frown \eta^\frown \underline{0} \text{ for some } \eta \in [\omega,\omega_1)\};$$

and by the remark following theorem 4.3.8, these spaces are characterized by being $\mathcal{P}_u$ and nowhere $\check{\mathcal{P}}_u$, respectively $\check{\mathcal{P}}_u$ and nowhere $\mathcal{P}_u$.

Let $u(\eta) = 2^\frown 1^\frown \eta^\frown \underline{0}$.

4.6.2 THEOREM: *Let* $\alpha < \omega_1$ *be a limit ordinal.*

(a) *For each* $n < \omega$, $\mathcal{X}_{\alpha+2n}^2 = \mathcal{Y}_{u(\alpha+2n)}^0$ *and* $\mathcal{X}_{\alpha+2n+1}^1 = \mathcal{Z}_{u(\alpha+2n)}^1$.

(b) *For each* $n \in \mathbb{N}$, $\mathcal{X}_{\alpha+2n}^1 = \mathcal{Y}_{u(\alpha+2n-1)}^0$ *and* $\mathcal{X}_{\alpha+2n-1}^2 = \mathcal{Z}_{u(\alpha+2n-1)}^1$.

Proof: If we consider all classes as consisting of dense subsets of 2^ω, then $\mathcal{X}_{\alpha+2n}^2 = \check{\mathcal{X}}_{\alpha+2n+1}^1$ and $\mathcal{X}_{\alpha+2n}^1 = \check{\mathcal{X}}_{\alpha+2n-1}^2$ (inductive hypotheses (3) and (9) for the proof of theorems 3.5.9 - 3.5.11), and $\mathcal{Y}_{u(\beta)}^0 = \check{\mathcal{Z}}_{u(\beta)}^1$ (proof of lemma 4.3.4). So it suffices to prove only the first statement in (a) and (b). Also, we need only prove one inclusion since each class contains exactly one topological type.

(a) Let $X \in \mathcal{X}_{\alpha+2n}^2$; then X is $\mathcal{P}_{\alpha+2n}$ is $\mathcal{P}_{u(\alpha+2n)}$. And if X is densely embedded in 2^ω, then $2^\omega \backslash X \in \mathcal{X}_{\alpha+2n+1}^1$ is nowhere $\mathcal{P}_{\alpha+2n}$, so $2^\omega \backslash X$ is nowhere $\mathcal{P}_{u(\alpha+2n)}$, so X is nowhere $\check{\mathcal{P}}_{u(\alpha+2n)}$, whence $X \in \mathcal{Y}_{u(\alpha+2n)}^0$.

(b) Embed $X \in \mathcal{X}_{\alpha+2n}^1$ densely in 2^ω; then $2^\omega \backslash X \in \mathcal{X}_{\alpha+2n-1}^2$ is $\mathcal{P}_{\alpha+2(n-1)} \cup$ complete, say $2^\omega \backslash X = D_\beta(\langle A_\zeta : \zeta < \beta \rangle) \cup G$, where G is complete, and $\beta = \alpha+2(n-1)$. Then

$$X = (\bigcup\{A_\zeta \backslash (\bigcup_{\gamma<\zeta} A_\gamma) : \zeta \text{ even} < \beta\} \cup 2^\omega \backslash (\bigcup_{\zeta<\beta} A_\zeta)) \cap 2^\omega \backslash G;$$

thus, if we put $A_\beta = 2^\omega$, then $X = D_{\beta+1}(\langle A_\zeta \cap 2^\omega \backslash G : \zeta < \beta+1 \rangle) \in D_{\alpha+2n-1}(\Sigma_2^0)$, i.e. X is $\mathcal{P}_{u(\alpha+2n-1)}$. If U is a non-empty clopen subset of 2^ω, and

$U \cap X$ is $\check{P}_{u(\alpha+2n-1)}$, then $U\backslash X$ is $P_{u(\alpha+2n-1)}$, say $U\backslash X = D_{\alpha+2n-1}(\langle A_\zeta: \zeta < \alpha+2n-1\rangle)$: hence,

$$U \cap X = \{A_\zeta\backslash(\cup_{\gamma<\zeta} A_\gamma): \zeta \text{ odd} < \alpha+2(n-1)\} \cup 2^\omega\backslash A_{\alpha+2(n-1)}$$
$$= D_\beta(\langle A_\zeta: \zeta < \beta\rangle) \cup 2^\omega\backslash A_\beta,$$

where $\beta = \alpha+2(n-1)$. So $U \cap X$ is $P_\beta \cup$ complete is $P_{\alpha+2n-1}$, a contradiction. Thus, X is nowhere $\check{P}_{u(\alpha+2n-1)}$, i.e. $X \in Y^0_{u(\alpha+2n-1)}$. □

We finish this chapter by computing the Wadge classes of the other homogeneous absolute Borel sets, i.e. the spaces which were described and characterized in section 3.4 and Chapter 2. For this, recall the last remark following theorem 4.1.9, that if $\Gamma \subset P(2^\omega)$ is a continuously closed class of Borel sets, and $A \in \Gamma\backslash\check{\Gamma}$, then $\Gamma = [A]$.

4.6.3 LEMMA: *Let* $A \subset 2^\omega$, *and* $n \in \mathbb{N}$.

(a) $A \in \check{D}_{2n}(\Sigma^0_2)$ *if and only if* A *is* $P^2_{4(n-1)+2}$.

(b) $A \in D_{2n+1}(\Sigma^0_2)$ *if and only if* A *is* $P^2_{4(n-1)+3}$.

(c) $A \in \check{D}_{2n+1}(\Sigma^0_2)$ *if and only if* A *is* P_{4n}.

Proof: Since $A \in D_{2n}(\Sigma^0_2)$ if and only if A is $P_{4(n-1)+1}$, (a) is merely a reformulation of lemma 3.4.4(b). Similarly, (c) follows from (b) and lemma 3.4.4(a). To prove (b), let $A \in D_{2n+1}(\Sigma^0_2)$, say $A = D_{2n+1}(\langle A_m: m < 2n+1\rangle)$; then $A = A_0 \cup D_{2n}(\langle A_{m+1}: m < 2n\rangle)$ which is σ-compact $\cup\ S_n$ by lemma 3.1.4, so A is $P^2_{4(n-1)+3}$. Conversely, if A is $P^2_{4(n-1)+3}$ is $S_n \cup \sigma$-compact, then by lemma 3.1.4 we can write $A = A_0 \cup D_{2n}(\langle A'_m: m < 2n\rangle)$, where A_0 and each A'_m are σ-compact; if we put $A_{m+1} = A'_m \cup A_0$ for each $m < 2n$, then $\langle A_m: m < 2n+1\rangle$ is increasing, and $A = D_{2n+1}(\langle A_m: m < 2n+1\rangle)$. □

4.6.4 THEOREM: *Let* $i \in \{1,2\}$, $k < \omega$, *and* $X \subset 2^\omega$.

(a) *If* $X \in X_{4k+1}$, *then* $[X] = D_{2(k+1)}(\Sigma^0_2)$.

(b) *If* $X \in X^i_{4k+2}$, *then* $[X] = \check{D}_{2(k+1)}(\Sigma^0_2)$.

(c) *If* $X \in X^i_{4k+3}$, *then* $[X] = D_{2(k+1)+1}(\Sigma^0_2)$.

(d) *If* $X \in X_{4(k+1)}$, *then* $[X] = \check{D}_{2(k+1)+1}(\Sigma^0_2)$.

Proof: (a) Clearly, $X \in D_{2(k+1)}(\Sigma^0_2)$. If $X \in \check{D}_{2(k+1)}(\Sigma^0_2)$, then by lemma 4.6.3, X is P^2_{4k+2} is $P^2_{4(k-1)+3} \cup$ complete; since X is nowhere $\check{P}^2_{4(k-1)+3}$, X is Baire, contradicting lemma 3.4.7. Thus, $X \in D_{2(k+1)}(\Sigma^0_2)\backslash\check{D}_{2(k+1)}(\Sigma^0_2)$, whence $[X] = D_{2(k+1)}(\Sigma^0_2)$.

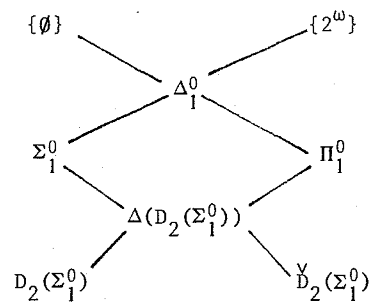
(b) By lemma 4.6.3, $X \in \check{D}_{2(k+1)}(\Sigma^0_2)$. If $X \in D_{2(k+1)}(\Sigma^0_2)$, then X is strong-

ly σ-P_{4k}, whence first category since X is nowhere P_{4k}. This contradicts lemma 3.4.7, so $X \notin D_{2(k+1)}(\Sigma_2^0)$, and $[X] = \check{D}_{2(k+1)}(\Sigma_2^0)$.
The proof of (c) and (d) is similar. □

The only thing left is to determine the Wadge classes of $\boldsymbol{C}$, $\boldsymbol{C}\setminus\{p\}$, $\boldsymbol{Q}$, $\boldsymbol{Q}\times\boldsymbol{C}$, and $\boldsymbol{P}$.
The fact that $\boldsymbol{P}$ is Π_2^0 but not Σ_2^0, and $\boldsymbol{Q}$ and $\boldsymbol{Q}\times\boldsymbol{C}$ are Σ_2^0 but not Π_2^0, yields the following theorem:

4.6.5 THEOREM: *Let* $X \subset 2^\omega$.
(a) *If* $X \approx \boldsymbol{P}$, *then* $[X] = \Pi_2^0$.
(b) *If* $X \approx \boldsymbol{Q}$ *or* $X \approx \boldsymbol{Q}\times\boldsymbol{C}$, *then* $[X] = \Sigma_2^0$. □

For $\boldsymbol{C}$ and $\boldsymbol{C}\setminus\{p\}$, the Wadge class turns out to depend upon the embedding in 2^ω. Note that from theorem 4.2.7 it is easily deduced that the Wadge hierarchy starts with



4.6.6 THEOREM: *Let* $\boldsymbol{C} \approx X \subset 2^\omega$.
(a) *If* $X = 2^\omega$, *then* $[X] = \{2^\omega\}$.
(b) *If* $2^\omega \neq X$ *is clopen in* 2^ω, *then* $[X] = \Delta_1^0$.
(c) *If* X *is not open in* 2^ω, *then* $[X] = \Pi_1^0$.

Proof: (a) and (b) are trivial. If X is not open, then $X \in \Pi_1^0\setminus\Sigma_1^0$, so $[X] = \Pi_1^0$. □

4.6.7 LEMMA: *Let* $X \subset 2^\omega$. *Then* $X \in \check{D}_2(\Sigma_1^0)$ *if and only if* $X\setminus\mathrm{Int}(X)$ *is compact.*

Proof: Since $D_2(\Sigma_1^0)$ consists of all subsets of 2^ω that are the intersection of an open subset and a closed subset of 2^ω, $\check{D}_2(\Sigma_1^0)$ consists of all subsets of 2^ω that are the union of an open set and a closed set. Now if

$X = O \cup F$, with O open and F compact, then $O \subset \mathrm{Int}(X)$, so $X\setminus\mathrm{Int}(X)$ is closed in F, whence compact. Conversely, if $X\setminus\mathrm{Int}(X)$ is compact, then clearly $X = \mathrm{Int}(X) \cup (X\setminus\mathrm{Int}(X)) \in \check{D}_2(\Sigma^0_1)$. □

4.6.8 THEOREM: *Let* $\mathcal{C}\setminus\{p\} \approx X \subset 2^\omega$.

(a) *If* X *is open, then* $[X] = \Sigma^0_1$.

(b) *If* $X\setminus\mathrm{Int}(X)$ *is non-empty and compact, then* $[X] = \Delta(D_2(\Sigma^0_1))$.

(c) *If* $X\setminus\mathrm{Int}(X)$ *is non-compact, then* $[X] = D_2(\Sigma^0_1)$.

Proof: (a) Since $X \in \Sigma^0_1\setminus\Pi^0_1$, $[X] = \Sigma^0_1$.

(b) $X \in \check{D}_2(\Sigma^0_1)$ by lemma 4.6.7; also, $X \in D_2(\Sigma^0_1)$ since X is locally compact. Thus, $[X] \subset \Delta(D_2(\Sigma^0_1))$, and hence $[X] = \Delta(D_2(\Sigma^0_1))$ since $X \notin \Sigma^0_1 \cup \Pi^0_1$.

(c) $X \notin \check{D}_2(\Sigma^0_1)$ by lemma 4.6.7, so $\Delta(D_2(\Sigma^0_1)) \subsetneq [X] \subset D_2(\Sigma^0_1)$ since X is locally compact; hence $[X] = D_2(\Sigma^0_1)$. □

CHAPTER 5: RIGID BOREL SETS

The "propeller space" defined by de Groot and Wille in [19] is an example of a non-trivial compact rigid subset of $\mathbb{R}^2$. A space with these properties cannot be constructed in $\mathbb{R}$: indeed, note that such a space would have to be zero-dimensional (since one can move points around in an interval), that the space could contain at most one isolated point (since interchanging two distinct isolated points and leaving the other points in place yields a non-trivial autohomeomorphism), and that removing this isolated point if necessary, we would still have a compact rigid space (since a clopen subset of a rigid space is rigid). Thus, we would have a rigid compact zero-dimensional space without isolated points, i.e. a rigid Cantor set (theorem 2.1.1); this is impossible.

On the other hand, a method is known for constructing non-trivial zero-dimensional rigid spaces (see e.g. Kuratowski [27]), but this method only yields spaces which are not even analytic. Thus, the question arises whether it is possible at all to construct "nice" zero-dimensional rigid spaces, say rigid zero-dimensional absolute Borel sets; this question was asked by van Douwen in [8].

In this chapter, we answer van Douwen's question in the negative; also, we briefly comment on the question of whether rigid zero-dimensional analytic spaces exist.

5.1 THEOREM: *There are no non-trivial rigid zero-dimensional absolute Borel sets.*

Proof: If there is a non-trivial, rigid, zero-dimensional absolute Borel set, then by an argument as above, there is also one without isolated points.

Put

$$\mathcal{A}_0 = \{A \subset 2^\omega : A \in \Delta(D_\omega(\Sigma_2^0)),\ A \text{ rigid, dense in itself}\},$$

and

$$\mathcal{A}_1 = \{A \subset 2^\omega : A \notin \Delta(D_\omega(\Sigma_2^0)),\ A \text{ Borel, rigid, dense in itself}\}.$$

We first show that $\mathcal{A}_0 = \emptyset$; recall the definitions of $\mathcal{P}_n^{(i)}$, $\mathcal{X}_n^{(i)}$ from section 3.4, and of $\mathcal{P}_\omega$, $\mathcal{X}_\omega^2$ from section 3.5.
Assume that for each $j \in \{1,2\}$, $m < \omega$, no element of $\mathcal{A}_0$ is $\mathcal{P}_m^{(j)}$. If $A \in \mathcal{A}_0$, then each non-empty clopen subset of A is in $\mathcal{A}_0$, so A is nowhere $\mathcal{P}_m^{(j)}$ for all $j \in \{1,2\}$, $m < \omega$; on the other hand, $A \in \Delta(D_\omega(\Sigma_2^0))$, so A is $\mathcal{P}_\omega$. Thus $A \in \mathcal{X}_\omega$, whence A is homogeneous, a contradiction.
So now let $i \in \{1,2\}$, $n \geq -1$ be such that some $X \in \mathcal{A}_0$ is $\mathcal{P}_n^{(i)}$, but no $A \in \mathcal{A}_0$ is $\mathcal{P}_m^{(j)} < \mathcal{P}_n^{(i)}$, in the ordering of theorem 3.4.24; then X is nowhere $\mathcal{P}_m^{(j)}$ for each $\mathcal{P}_m^{(j)} < \mathcal{P}_n^{(i)}$. If $n \geq 0$, then by theorem 3.4.24, $X \in \mathcal{X}_n^{(i)}$, so X is homogeneous by theorem 3.4.13, a contradiction, so $n = -1$, i.e. X is σ-compact. By the argument in the introduction to this chapter, X is nowhere compact; if X is nowhere countable, then $X \approx \mathbb{Q} \times \mathbb{C}$ by theorem 2.4.5, so again X is homogeneous, a contradiction. The last possibility is that X contains a non-empty clopen subset which is countable, but then by theorem 2.4.1 this subset is a copy of $\mathbb{Q}$, which is not rigid either, and we are done.
We now show that $\mathcal{A}_1 = \emptyset$. If not, then there is a minimal pair $\{\Gamma_u, \check{\Gamma}_u\}$ of described Wadge classes such that $X \in \Gamma_u \cup \check{\Gamma}_u$ for some $X \in \mathcal{A}_1$. Since $\mathcal{A}_0 = \emptyset$, $V \in \mathcal{A}_1$ for each non-empty clopen V in X, and thus $V \notin \Gamma_v \cup \check{\Gamma}_v$ for each $\Gamma_v < \Gamma_u$; because $X \notin \Delta(D_\omega(\Sigma_2^0))$, we have $\Delta(D_\omega(\Sigma_2^0)) \subset \Gamma_u$, so we can apply lemma 4.4.1 and obtain that $u(0) \geq 2$.
Suppose that $X \in \Gamma_u$ (the case $X \in \check{\Gamma}_u$ is similar). If V is a non-empty clopen subset of X, then $V \notin \check{\Gamma}_u$ since by theorem 4.2.6, $\Delta(\Gamma_u)$ is the union of described Wadge classes smaller than Γ_u; thus, X is $\mathcal{P}_u$, and nowhere $\check{\mathcal{P}}_u$ (definition 4.3.1 and the remark following it). Now if X is Baire, put $Y = X$; then $Y \in \mathcal{Y}_u^1$. Otherwise, there are dense open subsets U_n of X, for $n < \omega$, such that $\bigcap_{n=0}^\infty U_n$ is not dense, i.e. $Y \subset \bigcup_{n=0}^\infty (X \setminus U_n)$ for some non-empty clopen subset Y of X; then $Y = \bigcup_{n=0}^\infty (Y \setminus U_n)$ is first category, and also Y is $\mathcal{P}_u$, nowhere $\check{\mathcal{P}}_u$, so $Y \in \mathcal{Y}_u^0$. In both cases, Y is homogeneous by theorem 4.3.8, a contradiction. □

Remark: In van Engelen, Miller, and Steel [13], using a different method, a generalization of the above theorem will be proved. This generalization has as a consequence that if all analytic games (i.e. games $G_X(A)$ with A analytic) are determined, then there are no rigid zero-dimensional analytic spaces; determinacy of analytic sets can be proved granting MC (the hypothesis that there exists at least one measurable cardinal). On the other hand, under the axiom of constructibility $V = L$, a rigid zero-dimensional analytic space will be constructed in that paper.

APPENDIX: THE COUNTABLE INFINITE PRODUCT OF RATIONALS

As we noted in section 4.5, the case Z^0_u, $u = 3\hat{}1\hat{}1\hat{}\underline{0}$, of theorem 4.3.8 yields that the countable infinite product of rationals $\mathbf{Q}^\omega$ is the only zero-dimensional absolute $F_{\sigma\delta}$ which is first category and nowhere an absolute $G_{\delta\sigma}$. In this appendix, we will give an elementary proof of this characterization. Our first step is to provide a characterization of $\mathbf{Q}^\omega$ which is inspired by Sierpiński's internal topological characterization of the absolute $F_{\sigma\delta}$ spaces ([51]). Then we use techniques similar to those of section 3.3 to get the intended result. We will also show how to deduce the characterizations of the three other zero-dimensional homogeneous Borel sets of exact class 2 from that of $\mathbf{Q}^\omega$ in an elementary way.

For obvious reasons, we will use the term "σ-complete space" for "absolute $G_{\delta\sigma}$ space".

A.1 The first characterization of $\mathbf{Q}^\omega$

We start with an "estimated homeomorphism extension theorem" for the Cantor set, which is essentially due to van Mill [39]; we include a proof.

A.1.1 THEOREM: *Let* $g: \mathbf{C} \to \mathbf{C}$ *be a homeomorphism, let* $\varepsilon > 0$, *and let* A *be closed and nowhere dense in* $\mathbf{C}$. If $h_0: A \to g[A]$ *is a homeomorphism such that* $d(g|A, h_0) < \varepsilon$, *then there exists an autohomeomorphism* h *of* $\mathbf{C}$ *such that* $h|A = h_0$, *and* $d(g,h) < \varepsilon$.

Proof: Put $\delta = \varepsilon - d(g|A,h_0)$, and let $\eta > 0$ be such that

(1) for all $x,y \in \mathbf{C}$, if $d(x,y) < \eta$, then $d(g(x),g(y)) < \frac{1}{2}\delta$, and

(2) for all $x,y \in A$, if $d(x,y) < \eta$, then $d(h_0(x),h_0(y)) < \frac{1}{2}\delta$.

Let $\{U_i: i = 0,\dots,n\}$ be a disjoint family of clopen subsets of $\mathbf{C}$ of diameter less than η, with $A \subset \bigcup_{i=0}^{n} U_i$, and $U_i \cap A \neq \emptyset$ for each i. By (2), we can find a family $\{V_i: i = 0,\dots,n\}$ of pairwise disjoint clopen subsets of $\mathcal{C}$ of diameter less than $\frac{1}{2}\delta$ such that $V_i \cap g[A] = h_0[U_i \cap A]$. Since $g[A] \subset \bigcup_{i=0}^{n} V_i$, we can find a clopen subset U of $\mathcal{C}$ such that $A \subset U \subset \bigcup_{i=0}^{n} U_i$, and $g[U] \subset \bigcup_{i=0}^{n} V_i$. Now put

$$W_i = U_i \cap U,\ Z_i = V_i \cap g[U].$$

Then $\bigcup_{i=0}^{n} W_i = U$, $\bigcup_{i=0}^{n} Z_i = g[U]$, $W_i \cap A = U_i \cap A$, and $Z_i \cap g[A] = V_i \cap g[A]$. For each i, define $h_i: W_i \to Z_i$ to be a homeomorphism extending $h_0|(W_i \cap A)$ (use corollary 3.2.3), and put

$$h = \bigcup_{i=0}^{n} h_i \cup g|(\mathcal{C}\setminus U).$$

Then $d(g(x),h(x)) = 0 < \varepsilon$ if $x \notin U$. If $x \in W_i$, then we can find a point $a \in W_i \cap A$. Since $\mathrm{diam}(W_i) < \eta$, we have $d(g(x),g(a)) < \frac{1}{2}\delta$ by (1), and since $\{h(x),h(a)\} \subset Z_i$, also $d(h(x),h(a)) < \frac{1}{2}\delta$. Hence, $d(g(x),h(x)) \leq d(g(x),g(a)) + d(g(a),h(a)) + d(h(a),h(x)) < \frac{1}{2}\delta + d(g|A,h_0) + \frac{1}{2}\delta = \varepsilon$. □

Recall the definition of M from 1.4.

A.1.2 DEFINITION: $\mathcal{X}$ *is the class of all non-empty zero-dimensional spaces* X *for which there exist closed subspaces* X_s, *for each* $s \in M$, *satisfying*

(i) $X = X_\emptyset$, *and* $X_s = \bigcup_{i=0}^{\infty} X_{s\frown i}$ *for each* $s \in M$;
(ii) *for each* $i < \omega$, *and each* $s \in M$, $X_{s\frown i}$ *is nowhere dense in* X_s;
(iii) *if* $\sigma \in \omega^\omega$, *and* $p_k \in X_{\sigma|k}$ *for each* $k < \omega$, *then the sequence* $(p_k)_{k<\omega}$ *converges in* X.

A.1.3 LEMMA: $\mathbb{Q}^\omega \in \mathcal{X}$.

Proof: Enumerate $\mathbb{Q}$ as $\{q_n: n < \omega\}$, and put $X_\emptyset = \mathbb{Q}^\omega$, $X_{i_0\dots i_k} = (q_{i_0},\dots,q_{i_k}) \times \mathbb{Q} \times \mathbb{Q} \times \cdots$. □

Our aim is to prove that, up to homeomorphism, $\mathbb{Q}^\omega$ is the only element of $\mathcal{X}$. We first show that the sets X_s of definition A.1.2 can be chosen to be disjoint. In fact, we have:

A.1.4 LEMMA: *Let* $X \in \mathcal{X}$ *be embedded in* $\mathcal{C}$. *Then the sets* X_s *from definition* A.1.2 *can be chosen to satisfy the additional property*

(iv) $\overline{X}_s \cap \overline{X}_t = \emptyset$ *if* $s,t \in M$, $|s| = |t|$, $s \neq t$.

Proof: For each $s \in M\backslash\{\emptyset\}$, let $\mathcal{U}_s$ be a disjoint clopen covering of $\overline{X}_s \backslash \bigcup_{i<f(s)} \overline{X}_{\hat{s}^\frown i}$; enumerate $\mathcal{U}_s$ as $\{U_j: j \in E_s\}$, using pairwise disjoint indexing sets. If $j_{n-1} \in E_{s|n}$ for each $1 \le n \le |s| = k$, then define a closed set $X(s,j_0\ldots j_{k-1}) = X_s \cap \bigcap_{n=0}^{k-1} U_{j_n}$. The reader can easily verify that if $|s| = k$, then

(1) $\overline{X}(s,j_0\ldots j_{k-1}) \cap \overline{X}(t,\ell_0\ldots\ell_{k-1}) = \emptyset$ if $(s,j_0\ldots j_{k-1}) \ne (t,\ell_0\ldots\ell_{k-1})$ and $|t| = |s|$;

(2) $X = \bigcup\{X(s,j): |s| = 1, j \in E_s\}$, and $X(s,j)$ is nowhere dense in X;

(3) $X(s,j_0\ldots j_{k-1}) = \bigcup\{X(t,j_0\ldots j_{k-1}j): \hat{t} = s, j \in E_t\}$, and $X(t, j_0\ldots j_{k-1}j)$ is nowhere dense in $X(s,j_0\ldots j_{k-1})$.

Furthermore, if $\sigma \in \omega^\omega$, $j_{k-1} \in E_{\sigma|k}$ for each $k \in N$, $p_k \in X(\sigma|k,j_0\ldots j_{k-1})$ for each $k \in N$, and $p_0 \in X$, then $p_k \in X_{\sigma|k}$ for each $k < \omega$, and hence the sequence $(p_k)_{k<\omega}$ converges to a point of X. It is now obvious that we can reindex the sets $X(s,j_0\ldots j_{k-1})$ to obtain the required representation of X.□

A.1.5 LEMMA: *Let* $X,Y \in \mathcal{X}$ *be densely embedded in* C. *Then there exists a homeomorphism* $h: C \to C$ *such that* $h[X] = Y$.

Proof: Since $X,Y \in \mathcal{X}$, there exist closed non-empty subsets X_s of X, Y_s of Y, for each $s \in M$, staisfying properties (i), (ii), and (iii) of definition A.1.2, and (iv) of lemma A.1.4. Throughout this proof, (i), (ii), (iii), and (iv) will always refer to those properties. We will construct, for each $n \in N$, a homeomorphism $h_n: C \to C$, such that

(*) $d(h_n,h_{n+1}) < \varepsilon_n = \min\{2^{-n},3^{-n}.\min\{\min\{d(h_i(x),h_i(y)): d(x,y) \ge 1/n\}: 1 \le i \le n\}\}$;

(**) $\forall s \in M: \forall x \in X_s: \exists t \in M: |t| = |s|$, and $\forall n \ge \nu(s): h_n(x) \in \overline{Y}_t$;

(***) $\forall s \in M: \forall y \in Y_s: \exists t \in M: |t| = |s|$, and $\forall n \ge \nu(s): h_n^{-1}(y) \in \overline{X}_t$.

Suppose this has been done; then $\lim_{n\to\infty} h_n$ is an autohomeomorphism of C by (*) and lemma 3.2.5. We claim that $h[X] = Y$. Indeed, let $x \in X$, say $x \in \bigcap_{s<\sigma} X_s$. By (**), for each $k < \omega$ we can find $t(k) \in M$ such that $|t(k)| = k$, and such that $h_n(x) \in \overline{Y}_{t(k)}$ for each $n \ge \nu(\sigma|k)$; thus, if $k < \ell$, then $h_{\nu(\sigma|\ell)}(x) \in \overline{Y}_{t(k)} \cap \overline{Y}_{t(\ell)} \subset \overline{Y}_{t(k)} \cap \overline{Y}_{t(\ell)|k}$, and hence $t(\ell)|k = t(k)$ by (iv). So in fact, we can find $\tau \in \omega^\omega$ such that $\tau|k = t(k)$. Then $h_{\nu(\sigma|k)}(x) \in \overline{Y}_{\tau|k}$; and if $p_k \in Y_{\tau|k}$ satisfies $d(h_{\nu(\sigma|k)}(x),p_k) < 1/(k+1)$, then $(p_k)_{k<\omega}$ converges in Y by (iii), and hence $h(x) = \lim_{k\to\infty} h_{\nu(\sigma|k)}(x) = \lim_{k\to\infty} p_k \in Y$. A similar argument shows that (***) implies $h^{-1}[Y] \subset X$.

Thus, roughly speaking, if $x \in X_s$, and $\nu(s) = n$, then h_n determines the Y_t with $|t| = |s|$ which, at the end of the process, will contain $h(x)$; and if $y \in Y_s$, and $\nu(s) = n$, then h_n^{-1} determines the X_t with $|t| = |s|$ which, at the end of the process, will contain $h^{-1}(x)$.

We will construct the homeomorphisms h_n inductively, together with finite collections $\mathcal{A}_s = \{A_\alpha : \alpha \in E_s\}$, $\mathcal{B}_s = \{B_\alpha : \alpha \in E_s\}$ (using pairwise disjoint indexing sets), each consisting of pairwise disjoint Cantor sets in $\mathbf{C}$, such that the following hold for each $n < \omega$, and each $s \in M$ with $\nu(s) = n$:

(1) if $n \geq 2$, then $d(h_n, h_{n-1}) < \varepsilon_{n-1}$;

(2) if $\nu(t) \leq n$, and $\alpha \in E_t$, then $h_n[A_\alpha] = B_\alpha$;

(3) if $|t| = |s|$, $t \neq s$, and $\nu(t) \leq n$, then $\cup\mathcal{A}_s \cap \cup\mathcal{A}_t = \emptyset = \cup\mathcal{B}_s \cap \cup\mathcal{B}_t$;

(4) $\overline{X}_s \subset \cup\{\cup\mathcal{A}_t : |t| = |s|, \nu(t) < n\} \cup \cup\mathcal{A}_s$;
$\overline{Y}_s \subset \cup\{\cup\mathcal{B}_t : |t| = |s|, \nu(t) < n\} \cup \cup\mathcal{B}_s$;

(5) if $\alpha \in E_s$, then there exist $t_0, t_1 \in M$ with $|t_0| = |s| = |t_1|$, such that A_α is a clopen subset of $\overline{X}_{t_0}$, and B_α is a clopen subset of $\overline{Y}_{t_1}$;

(6) if $\alpha \in E_{\hat{s}}$, $|t| = |\hat{s}|$, and $A_\alpha \subset \overline{X}_t$, then $A_\alpha \cap \overline{X}_{t^\frown f(s)} \subset \cup\{\cup\mathcal{A}_{\hat{s}^\frown j} : j \leq f(s)\}$;
if $\alpha \in E_{\hat{s}}$, $|t| = |\hat{s}|$, and $B_\alpha \subset \overline{Y}_t$, then $B_\alpha \cap \overline{Y}_{t^\frown f(s)} \subset \cup\{\cup\mathcal{B}_{\hat{s}^\frown j} : j \leq f(s)\}$

(7) if $\alpha \in E_s$, then for some $\beta \in E_{\hat{s}}$, A_α is a nowhere dense subset of A_β, and B_α is a nowhere dense subset of B_β.

First note that from (ii) it follows that no X_s or Y_s can contain isolated points (in the relative topology), so that $\overline{A} \approx \mathbf{C}$ for any non-empty clopen subset A of X_s or Y_s.

Put $\mathcal{A}_\emptyset = \{\mathbf{C}\} = \mathcal{B}_\emptyset$. Since $\overline{X}_0, \overline{Y}_0$ are nowhere dense in $\mathbf{C}$, we can define a homeomorphism $h_1 : \mathbf{C} \to \mathbf{C}$ such that $h_1[\overline{X}_0] = \overline{Y}_0$; if we put $\mathcal{A}_0 = \{\overline{X}_0\}$, $\mathcal{B}_0 = \{\overline{Y}_0\}$, then (1) - (7) are satisfied ((5) is satisfied for $s = \emptyset$ since X, Y are dense in $\mathbf{C}$). So suppose that h_m, $\mathcal{A}_s$, and $\mathcal{B}_s$, satisfying (1) - (7), have been constructed for $m \leq n$, $\nu(s) \leq n$ (≥ 1).

Fix $s \in M$ with $\nu(s) = n+1$, and fix $\alpha \in E_{\hat{s}}$. By (5), there exist $t_0, t_1 \in M$ with $|t_0| = |\hat{s}| = |t_1|$, such that $A_\alpha \subset \overline{X}_{t_0}$, $B_\alpha \subset \overline{Y}_{t_1}$. Put $s_0 = t_0{}^\frown f(s)$, $s_1 = t_1{}^\frown f(s)$. By (5), $\overline{Y}_{s_1} \setminus \cup\{\cup\mathcal{B}_{\hat{s}^\frown i} : i < f(s)\}$ is closed in $\overline{Y}_{s_1}$, so we can find a clopen V' in B_α, satisfying

$$B_\alpha \cap \cup\{\cup\mathcal{B}_{\hat{s}^\frown i} : i < f(s)\} \subset V' \subset B_\alpha \setminus (\overline{Y}_{s_1} \setminus \cup\{\cup\mathcal{B}_{\hat{s}^\frown i} : i < f(s)\}).$$

Since $\nu(\hat{s}) \le n$, $h_n[A_\alpha] = B_\alpha$ by (2), and also by (2), $h_n[\cup\{\cup\mathcal{A}_{\hat{s}^\frown i}: i < f(s)\}] = \cup\{\cup\mathcal{B}_{\hat{s}^\frown i}: i < f(s)\}$ since $\nu(\hat{s}^\frown i) \le n$ for each $i < f(s)$. Thus,

$$h_n[A_\alpha \cap \cup\{\cup\mathcal{A}_{\hat{s}^\frown i}: i < f(s)\}] = B_\alpha \cap \cup\{\cup\mathcal{B}_{\hat{s}^\frown i}: i < f(s)\},$$

and since $\overline{X}_{s_0}\backslash\cup\{\cup\mathcal{A}_{\hat{s}^\frown i}: i < f(s)\}$ is closed in $\overline{X}_{s_0}$ by (5), we can find a clopen $U_{s,\alpha}$ in A_α such that

$$A_\alpha \cap \cup\{\cup\mathcal{A}_{\hat{s}^\frown i}: i < f(s)\} \subset U_{s,\alpha} \subset A_\alpha\backslash(\overline{X}_{s_0}\backslash\cup\{\cup\mathcal{A}_{\hat{s}^\frown i}: i < f(s)\}),$$

while moreover

$$h_n[U_{s,\alpha}] = V_{s,\alpha} \subset V'.$$

Since $\mathcal{A}_{\hat{s}}$ is disjoint, $A_\alpha \cap \cup\{\cup\mathcal{A}_{\hat{s}^\frown i}: i < f(s)\}$ is nowhere dense in A_α by (7), so we may assume that $A_\alpha\backslash U_{s,\alpha} \neq \emptyset$, and hence $B_\alpha\backslash V_{s,\alpha} \neq \emptyset$. Let $\mathcal{V}_{s,\alpha}$ be a clopen disjoint covering of $B_\alpha\backslash V_{s,\alpha}$ by non-empty sets of diameter less than ε_n. For each $W \in \mathcal{V}_{s,\alpha}$, put

$$p_W = \min\{p: h_n^{-1}[W] \cap \overline{X}_{t_0^\frown p} \neq \emptyset\},\ q_W = \min\{q: W \cap \overline{Y}_{t_1^\frown q} \neq \emptyset\},$$

and

$$A(W,s,\alpha) = h_n^{-1}[W] \cap \overline{X}_{t_0^\frown p_W},\ B(W,s,\alpha) = W \cap \overline{Y}_{t_1^\frown q_W}.$$

Note that p_W (resp. q_W) is well-defined since A_α (resp. B_α) is clopen in $\overline{X}_{t_0}$ (resp. $\overline{Y}_{t_1}$), and $\cup_{p<\omega} \overline{X}_{t_0^\frown p}$ (resp. $\cup_{q<\omega} \overline{Y}_{t_1^\frown q}$) is dense in $\overline{X}_{t_0}$ (resp. $\overline{Y}_{t_1}$). Now define

$$\mathcal{A}_s = \{A(W,s,\alpha): W \in \mathcal{V}_{s,\alpha},\ \alpha \in E_{\hat{s}}\},\ \mathcal{B}_s = \{B(W,s,\alpha): W \in \mathcal{V}_{s,\alpha},\ \alpha \in E_{\hat{s}}\},$$

and put $\mathcal{A}_s = \{A_\beta: \beta \in E_s\}$, $\mathcal{B}_s = \{B_\beta: \beta \in E_s\}$, such that if $A_\beta = A(W,s,\alpha)$, then $B_\beta = B(W,s,\alpha)$. Before we define h_{n+1}, we will show that (3) – (7) are satisfied for each $s \in M$ with $\nu(s) = n+1$. Fix $s \in M$ with $\nu(s) = n+1$.
To prove (3), let $|t| = |s|$, $t \neq s$, and $\nu(t) \le n+1$. If $\hat{s} \neq \hat{t}$, then since $\nu(\hat{s}),\nu(\hat{t}) \le n$, we have $\cup\mathcal{A}_{\hat{s}} \cap \cup\mathcal{A}_{\hat{t}} = \emptyset$, and hence by (7), $\cup\mathcal{A}_s \cap \cup\mathcal{A}_t = \emptyset$; if $\hat{s} = \hat{t}$, then $\cup\mathcal{A}_t \subset \cup\{\cup\mathcal{A}_{\hat{s}^\frown i}: i < f(s)\} \subset \mathcal{C}\backslash\cup\mathcal{A}_s$ by the construction of $\mathcal{A}_s$. Similarly, $\cup\mathcal{B}_s \cap \cup\mathcal{B}_t = \emptyset$.
For (4), fix $x \in \overline{X}_s$; note that $x \in \overline{X}_{\hat{s}}$. We must show that $x \in \cup\{\cup\mathcal{A}_r: |r| = |s|,\ \nu(r) < n+1\} \cup \cup\mathcal{A}_s$. First note that if $x \notin \cup\{\cup\mathcal{A}_r: |r| = |s|,\ \nu(r) < n+1\}$, then $x \notin \cup\{\cup\mathcal{A}_t: |t| = |\hat{s}|,\ \nu(t) < \nu(\hat{s})\}$. Indeed, if $x \in A$ for some $A \in \mathcal{A}_t$ for some $t \in M$ with $|t| = |\hat{s}|$, $\nu(t) < \nu(\hat{s})$, then by (5), $A \subset \overline{X}_{t_0}$ for some $t_0 \in M$ with $|t_0| = |t|$, and since $|\hat{s}| = |t|$ and $A \cap \overline{X}_{\hat{s}} \neq \emptyset$, we must have $t_0 = \hat{s}$ by (iv). Put $q = t^\frown f(s)$. Since $\nu(q) = \nu(t)+f(s)+1 < \nu(\hat{s})+f(s)+1 = \nu(s) = n+1$, we can apply (6), and obtain that $x \in A \cap \overline{X}_s = A \cap \overline{X}_{\hat{s}^\frown f(q)} \subset \cup\{\cup\mathcal{A}_{\hat{q}^\frown j}: j \le f(q)\} \subset \cup\{\cup\mathcal{A}_r: |r| = |s|,\ \nu(r) < n+1\}$, a contra-

diction. So if $x \notin \cup\{\cup\mathcal{A}_r: |r| = |s|, \nu(r) < n+1\}$, then by (4) applied to $\hat{s}$, we find that $x \in A_\alpha$ for some $\alpha \in E_{\hat{s}}$. Using notation as in the construction of $\mathcal{A}_s$, we find that $t_0 = \hat{s}$, and hence $t_0\hat{\ }f(s) = s_0 = s$. Since $\cup\{\cup\mathcal{A}_r: |r| = |s|, \nu(r) < n+1\} \supset \cup\{\cup\mathcal{A}_{\hat{s}\hat{\ }i}: i < f(s)\}$, we have $x \in A_\alpha \cap (\overline{X}_{s_0}\setminus\cup\{\cup\mathcal{A}_{\hat{s}\hat{\ }i}: i < f(s)\})$, whence $x \in h_n^{-1}[W]$ for some $W \in \mathcal{V}_{s,\alpha}$. We claim that $x \in A(W, s,\alpha)$; since $x \in h_n^{-1}[W] \cap \overline{X}_{t_0\hat{\ }f(s)}$, it suffices to show that $h_n^{-1}[W] \cap \overline{X}_{t_0\hat{\ }p} = \emptyset$ if $p < f(s)$. So take $p < f(s)$; then $\nu(t_0\hat{\ }p) \leq n$, so by (4) and (7), $\overline{X}_{t_0\hat{\ }p} \subset \cup\{\cup\mathcal{A}_t: |t| = |t_0\hat{\ }p|, \nu(t) < n\} \cup \cup\mathcal{A}_{t_0\hat{\ }p} \subset \cup\{\cup\mathcal{A}_t: |t| = |\hat{s}|, t \neq \hat{s}\} \cup \cup\{\cup\mathcal{A}_{\hat{s}\hat{\ }i}: i < f(s)\}$. Now $\cup\mathcal{A}_t \cap A_\alpha = \emptyset$ if $|t| = |\hat{s}|, t \neq \hat{s}$, by (3); and $h_n^{-1}[W] \subset A_\alpha\setminus\cup\{\cup\mathcal{A}_{\hat{s}\hat{\ }i}: i < f(s)\}$. This proves the claim. The proof that $\overline{Y}_s \subset \cup\{\cup\mathcal{B}_r: |r| = |s|, \nu(r) < n+1\} \cup \cup\mathcal{B}_s$ is similar, so (4) holds. (5) is trivial, and so is (7). It remains so check (6); we will only do so for the first part.

Let $\alpha \in E_{\hat{s}}$, and suppose that $t \in M$ is such that $|t| = |\hat{s}|$, and $A_\alpha \subset \overline{X}_t$. Then $A_\alpha \cap \cup\{\cup\mathcal{A}_{\hat{s}\hat{\ }j}: j < f(s)\} \subset U_{s,\alpha} = A_\alpha\setminus\cup\{h_n^{-1}[W]: W \in \mathcal{V}_{s,\alpha}\} \subset A_\alpha\setminus(\overline{X}_{t\hat{\ }f(s)}\setminus \cup\{\cup\mathcal{A}_{\hat{s}\hat{\ }j}: j < f(s)\}$, where $U_{s,\alpha}$ and $\mathcal{V}_{s,\alpha}$ are as in the construction of $\mathcal{A}_s$. If $p < f(s)$, then $\nu(\hat{s}\hat{\ }p) \leq n$, so by (6), $A_\alpha \cap \overline{X}_{t\hat{\ }p} \subset \cup\{\cup\mathcal{A}_{\hat{s}\hat{\ }j}: j \leq p\} \subset \cup\{\cup\mathcal{A}_{\hat{s}\hat{\ }j}: j < f(s)\}$, whence $h_n^{-1}[W] \cap \overline{X}_{t\hat{\ }p} = h_n^{-1}[W] \cap A_\alpha \cap \overline{X}_{t\hat{\ }p} \subset h_n^{-1}[W] \cap A_\alpha \cap \cup\{\cup\mathcal{A}_{\hat{s}\hat{\ }j}: j < f(s)\} = \emptyset$. So if $h_n^{-1}[W] \cap \overline{X}_{t\hat{\ }f(s)} \neq \emptyset$, then $A(W,s,\alpha) = h_n^{-1}[W] \cap \overline{X}_{t\hat{\ }f(s)}$. Now let $x \in A_\alpha \cap (\overline{X}_{t\hat{\ }f(s)}\setminus\cup\{\cup\mathcal{A}_{\hat{s}\hat{\ }j}: j < f(s)\})$; then $x \in h_n^{-1}[W] \cap \overline{X}_{t\hat{\ }f(s)}$ for some $W \in \mathcal{V}_{s,\alpha}$, and hence $x \in A(W,s,\alpha) \in \mathcal{A}_s$. This completes the proof of (6).

We will now define h_{n+1} satisfying (1) and (2).

Since $A(W,s,\alpha) \approx B(W,s,\alpha) \approx h_n^{-1}[W] \approx W \approx C$ for each $W \in \mathcal{V}_{s,\alpha}$, each $s \in M$ with $\nu(s) = n+1$, and each $\alpha \in E_{\hat{s}}$, and since $A(W,s,\alpha)$ (resp. $B(W,s,\alpha)$) is closed and nowhere dense in $h_n^{-1}[W]$ (resp. W), there exist homeomorphisms $g(W,s,\alpha): h_n^{-1}[W] \to W$ such that $g(W,s,\alpha)[A(W,s,\alpha)] = B(W,s,\alpha)$. Since $\mathcal{V}_{s,\alpha}$ is a disjoint clopen covering of $B_\alpha\setminus V_{s,\alpha}$, we can define a homeomorphism $g_{s,\alpha}: A_\alpha\setminus U_{s,\alpha} \to B_\alpha\setminus V_{s,\alpha}$ by

$$g_{s,\alpha} = \cup\{g(W,s,\alpha): W \in \mathcal{V}_{s,\alpha}\}.$$

Note that $d(g_{s,\alpha}, h_n|(A_\alpha\setminus U_{s,\alpha})) < \varepsilon_n$ since $\mathrm{diam}(W) < \varepsilon_n$ for each $W \in \mathcal{V}_{s,\alpha}$. Now put $I_j = \{s \in M: \nu(s) = n+1, f(s) = j\}$, for each $j \leq n$. Using induction on j, we will define for each $s \in I_j$, and each $\alpha \in E_{\hat{s}}$, a homeomorphism $h_{s,\alpha}: A_\alpha \to B_\alpha$, such that

(I) $h_{s,\alpha}|(A_\alpha\setminus U_{s,\alpha}) = g_{s,\alpha}$;

(II) if $\ell \leq j$, $t \in I_\ell$, $\beta \in E_{\hat{t}}$, and $A_\beta \subset A_\alpha$, then $h_{s,\alpha}|A_\beta = h_{t,\beta}$;

(III) $d(h_{s,\alpha}, h_n|A_\alpha) < \varepsilon_n$.

Suppose that the $h_{s,\alpha}$ can be constructed. Let $s_0 \in M$ be the sequence $(n+1)$, and let α_0 be the unique element of $E_{\hat{s}} = E_{\emptyset}$; then $A_{\alpha_0} = \mathcal{C} = B_{\alpha_0}$, so h_{s_0,α_0} is an autohomeomorphism of $\mathcal{C}$. We claim that $h_{n+1} = h_{s_0,\alpha_0}$ is as required. Indeed, by (III), h_{n+1} clearly satisfies (1). To prove (2), let $t \in M$ with $\nu(t) \leq n+1$, and let $\gamma \in E_t$. If $\nu(t) = n+1$, then $A_\gamma = A(W,t,\beta)$ for some $\beta \in E_{\hat{t}}$, and some $W \in \mathcal{V}_{t,\beta}$. Hence $A_\gamma \subset A_\beta \setminus U_{t,\beta} \subset A_\beta$, so applying (II) (for $\ell = f(t)$, $j = n$, $\alpha = \alpha_0$, and $s = s_0$), we find that $h_{n+1}[A_\gamma] = (h_{s_0,\alpha_0}|A_\beta)[A_\gamma] = h_{t,\beta}[A_\gamma]$, and by (I), $h_{t,\beta}[A_\gamma] = g_{t,\beta}[A_\gamma] = B(W,t,\beta) = B_\gamma$. If $\nu(t) \leq n$, then $t = \hat{s}$ for some $s \in M$ with $\nu(s) = n+1$; hence by (II) (for $\ell = f(s)$, $j = n$, $t = s$, $\beta = \gamma$, $\alpha = \alpha_0$, $s = s_0$), we find that $h_{n+1}[A_\gamma] = h_{s,\gamma}[A_\gamma] = B_\gamma$.

The homeomorphisms $h_{s,\alpha}$ are constructed as follows.

For $s \in I_0$, $\alpha \in E_{\hat{s}}$, define $h_{s,\alpha}$ by

$$h_{s,\alpha}|U_{s,\alpha} = h_n|U_{s,\alpha};$$
$$h_{s,\alpha}|(A_\alpha \setminus U_{s,\alpha}) = g_{s,\alpha}.$$

Since $g_{s,\alpha}[A_\alpha \setminus U_{s,\alpha}] = h_n[A_\alpha \setminus U_{s,\alpha}]$, and $h_n[A_\alpha] = B_\alpha$, $h_{s,\alpha}$ maps A_α onto B_α, and $h_{s,\alpha}$ is a homeomorphism since $U_{s,\alpha}$ is clopen in A_α. Clearly, (I) and (III) are satisfied. For (II), note that from (3) and (7) it follows that $A_\beta \subset A_\alpha$ for some $\beta \in E_{\hat{t}}$, $t \in I_0$, can only occur if $\hat{s} \leq \hat{t}$; since $\nu(s) = \nu(t)$ and $f(s) = f(t)$, we have $\nu(\hat{s}) = \nu(\hat{t})$, and hence $\hat{s} = \hat{t}$, so $s = t$. Then $\alpha = \beta$, and we are done.

Now suppose that $h_{t,\beta}$ has been defined for $t \in \bigcup_{\ell=0}^{j} I_\ell$, $\beta \in E_{\hat{t}}$, such that (I), (II), and (III) are satisfied, and fix $s \in I_{j+1}$, $\alpha \in E_{\hat{s}}$. For $\ell < f(s) = j+1$, put $s_\ell = \hat{s}^\frown\ell^\frown j-\ell$. Then $s_\ell \in I_{j-\ell}$, and $j-\ell \leq j$, so $h_{s_\ell,\gamma}\colon A_\gamma \to B_\gamma$ has been defined for each $\gamma \in E_{\hat{s}_\ell}$. Let $\tilde{E}_{\hat{s}_\ell} = \{\gamma \in E_{\hat{s}_\ell}\colon A_\gamma \subset A_\alpha\} = \{\gamma \in E_{\hat{s}_\ell}\colon B_\gamma \subset B_\alpha\}$. Then

$$g_\ell = \bigcup\{h_{s_\ell,\gamma}\colon \gamma \in \tilde{E}_{\hat{s}_\ell}\}\colon \bigcup\{A_\gamma\colon \gamma \in \tilde{E}_{\hat{s}_\ell}\} \to \bigcup\{B_\gamma\colon \gamma \in \tilde{E}_{\hat{s}_\ell}\}$$

is a well-defined homeomorphism since $\mathcal{A}_{\hat{s}_\ell}$ and $\mathcal{B}_{\hat{s}_\ell}$ consist of pairwise disjoint sets; and

$$g = \bigcup_{\ell=0}^{j} g_\ell\colon \bigcup\{A_\gamma\colon \gamma \in \tilde{E}_{\hat{s}_\ell},\ \ell \leq j\} \to \bigcup\{B_\gamma\colon \gamma \in \tilde{E}_{\hat{s}_\ell},\ \ell \leq j\}$$

is a well-defined homeomorphism since by (3), $\bigcup\mathcal{A}_{\hat{s}_\ell} \cap \bigcup\mathcal{A}_{\hat{s}_{\ell'}} = \emptyset = \bigcup\mathcal{B}_{\hat{s}_\ell} \cap \bigcup\mathcal{B}_{\hat{s}_{\ell'}}$, if $\ell \neq \ell'$. Let D_0 denote the domain, and D_1 the range of g. Then $D_0 \subset A_\alpha \cap \bigcup\{\bigcup\mathcal{A}_{\hat{s}^\frown i}\colon i < f(s)\} \subset U_{s,\alpha}$, $D_1 \subset B_\alpha \cap \bigcup\{\bigcup\mathcal{B}_{\hat{s}^\frown i}\colon i < f(s)\} \subset V_{s,\alpha}$, $D_0 \approx \mathcal{C}$ (resp. $D_1 \approx \mathcal{C}$) is nowhere dense in $U_{s,\alpha} \approx \mathcal{C}$ (resp. $V_{s,\alpha} \approx \mathcal{C}$) by (7), and $d(g,h_n|D_0) < \varepsilon_n$ by (III). So by theorem A.1.1, there exists a homeomorphism $\tilde{g}\colon U_{s,\alpha} \to V_{s,\alpha}$ such that $\tilde{g}|D_0 = g$, and $d(\tilde{g},h_n|U_{s,\alpha}) < \varepsilon_n$. Define $h_{s,\alpha}$:

$A_\alpha \to B_\alpha$ by

$$h_{s,\alpha}|U_{s,\alpha} = \tilde{g};$$
$$h_{s,\alpha}|(A_\alpha \setminus U_{s,\alpha}) = g_{s,\alpha}.$$

Then $h_{s,\alpha}$ satisfies (I) and (III). If $\ell \leq j+1$, $t \in I_\ell$, $\beta \in E_{\hat{t}}$, and $A_\beta \subset A_\alpha$, then by (3) and (7), $\hat{s} \leq \hat{t}$. If $\hat{s} = \hat{t}$, then $s = t$, $\alpha = \beta$, and we are done. If $\hat{s} < \hat{t}$, then for some $k \leq j$, we have $\hat{s} < \hat{s}_k \leq \hat{t}$. By (7), there exist $\gamma \in E_{\hat{s}_k}$, $\delta \in E_{\hat{s}}$, such that $A_\beta \subset A_\gamma \subset A_\delta$. Since $A_{\hat{s}}$ consists of pairwise disjoint sets, we have $\delta = \alpha$. Hence, $h_{s,\alpha}|A_\beta = (h_{s,\alpha}|A_\gamma)|A_\beta = (\tilde{g}|A_\gamma)|A_\beta = (g_k|A_\gamma)|A_\beta = h_{s_k,\gamma}|A_\beta$. Since (II) holds for $j = f(s_k)$, $h_{s_k,\gamma}|A_\beta = h_{t,\beta}$, and we are done.

This completes the inductive construction of the homeomorphisms $h_{s,\alpha}$, and hence of the autohomeomorphisms h_n of $\mathcal{C}$. To complete the proof of the lemma, we must show that the conditions (*), (**), and (***) at the begin of this proof, follow from (1) - (7). Now (*) is clear from (I), and since (***) is similar to (**), we will only prove (**). So let $s \in M$, and $x \in \overline{X}_s$. By (4), $x \in A_\alpha$ for some $\alpha \in E_{t'}$, for some $t' \in M$ with $|t'| = |s|$, $\nu(t') \leq \nu(s)$. Hence by (2), $h_n(x) \in B_\alpha$ for each $n \geq \nu(t')$, in particular for each $n \geq \nu(s)$. By (5), $B_\alpha \subset \overline{Y}_t$ for some $t \in M$ with $|t| = |t'|$, so $|t| = |s|$, and $h_n(x) \in \overline{Y}_t$ for each $n \geq \nu(s)$. □

A.1.6 THEOREM: *Up to homeomorphism, $\mathcal{Q}^\omega$ is the unique element of X.*

Proof: $\mathcal{Q}^\omega \in X$ by lemma A.1.3; and if $X \in X$, then X contains no isolated points, so X can be densely embedded in $\mathcal{C}$. Now apply lemma A.1.5. □

A.1.7 COROLLARY: *The Cantor set is homogeneous with respect to dense copies of $\mathcal{Q}^\omega$.* □

In [34], Luzin "effectively" described an absolute $F_{\sigma\delta}$ which is not an absolute $G_{\delta\sigma}$, viz. the subspace of ω^ω consisting of all sequences of elements of ω which converge to infinity. As a corollary to our first characterization, we show that in fact this space is homeomorphic to $\mathcal{Q}^\omega$.

A.1.8 THEOREM: *Let* $X = \{(x_i)_{i<\omega} \in \omega^\omega : \lim_{i\to\infty} x_i = \infty\}$. *Then* $X \approx \mathcal{Q}^\omega$.

Proof: Note that X consists of those sequences which, for each $n < \omega$, take the value n at only finitely many coordinates. Let $\{E_i : i < \omega\}$ be an enumeration of the collection of finite subsets of ω.

For $s,t \in M$, if $|s| = |t| \geq 1$, $s = (i_0,\dots,i_k)$, put

$$X(s,t) = \{\sigma = (x_m)_m \in X\colon t < \sigma, \text{ and for each } n \leq k,\ x_m = n \text{ if and only if } m \in E_{i_n}\}.$$

Then $X(s,t)$ is closed in X. If we also put $X(\emptyset,\emptyset) = X$, then it is easily seen that, for each $s_0,t_0 \in M$ with $|s_0| = |t_0|$,

$$X(s_0,t_0) = \cup\{X(s,t)\colon s,t \in M,\ \hat{s} = s_0,\ \hat{t} = t_0\},$$

and that $X(s,t)$ is nowhere dense in $X(s_0,t_0)$ if $\hat{s} = s_0$, $\hat{t} = t_0$. Finally, if $\sigma,\tau \in \omega^\omega$, and $p_k = (p_i^k)_{i<\omega} \in X(\sigma|k,\tau|k)$ for each $k < \omega$, then $p_i^k = p_i^{k+1}$ if $i < k$, so $(p_k)_k$ converges to a point of X. □

A.2 The second characterization of $\mathcal{Q}^\omega$

Throughout this section, X_0 denotes the class of all zero-dimensional absolute $F_{\sigma\delta}$ spaces which are nowhere σ-complete, and of the first category. Using theorem A.1.6, we will show that, up to homeomorphism, $\mathcal{Q}^\omega$ is the unique element of X_0, yielding the result we desire.

A.2.1 LEMMA: *If* X *is an analytic space which is not σ-complete, then* X *contains a closed nowhere σ-complete subspace* Y *which is nowhere dense in* X.

Proof: First note that any non-σ-complete space A contains a nowhere σ-complete closed subspace B, viz. $B = A\backslash\cup\{U\colon U$ is an open σ-complete subset of $A\}$. So we may assume that X is nowhere σ-complete. If X is Baire, then by 1.12.2, X contains a dense complete subset G. Since G is an absolute G_δ, we can write $X\backslash G = \cup_{i=0}^\infty F_i$, with F_i closed in X. Then for some j, F_j is not σ-complete. By the above remark, F_j contains a closed nowhere σ-complete subspace Y; then Y is as required. If X is not Baire, then there exist a non-empty open set U, and closed nowhere dense sets A_i in X, such that $U \subset \cup_{i=0}^\infty A_i$. Since U is an F_σ in X, and since U is not σ-complete, U contains a subset F which is not σ-complete, and closed in X. Then $F = \cup_{i=0}^\infty (A_i \cap F)$, and hence some $A_j \cap F$ is not σ-complete; again, if Y is nowhere σ-complete and closed in $A_j \cap F$, then Y is as required. □

The following lemma is the key to our second characterization; the proof is similar to that of lemmas 3.3.4 and 3.3.5.

A.2.2 LEMMA: *Let* A *be a Borel set in* $\mathcal{C}$ *which is not σ-complete, and let* F *be a σ-compact space such that* $A \subset F \subset \mathcal{C}$. *Then* A *contains a closed no-*

where dense subset Y *which is nowhere σ-complete and first category, such that* $\mathrm{Cl}_{\mathbf{C}}(Y) \subset F$.

Proof: We let $\overline{\ \ }$ denote closure in $\mathbf{C}$. Since $F\backslash A$ is Borel in $\mathbf{C}$, there exists a continuous surjection $\phi: \mathbf{P} \to F\backslash A$. Let

$$W = \{x \in \mathbf{P}: \text{there exists a neighborhood } V_x \text{ of } x \text{ in } \mathbf{P}, \text{ and a } \sigma\text{-compact subset } E_x \text{ of } F, \text{ such that } \phi[V_x] \subset E_x, \text{ and } E_x \cap A \text{ is } \sigma\text{-complete}\}.$$

Then W is open in $\mathbf{P}$, so there exist countably many open V_i in $\mathbf{P}$, and σ-compact E_i in F, such that $W = \bigcup_{i=0}^{\infty} V_i$, $\phi[V_i] \subset E_i$, and $E_i \cap A$ is σ-complete. Suppose that $F\backslash A \subset E = \bigcup_{i=0}^{\infty} E_i$; then $A = (E \cap A) \cup (F\backslash E)$ is σ-complete, a contradiction. So $G = \mathbf{P}\backslash\phi^{-1}[E\backslash A]$ is non-empty, and a G_δ in $\mathbf{P}$, whence complete. If U is a non-empty open subset of G, say $U = U' \cap G$, with U' open in $\mathbf{P}$, then $\phi[U'] = \phi[U] \cup \phi[U'\backslash U] \subset (\overline{\phi[U]} \cap F) \cup E$, which is a σ-compact subset of F. Since $U' \not\subset W$, $((\overline{\phi[U]} \cap F) \cup E) \cap A$ is not σ-complete; but $E \cap A$ is σ-complete, so $\overline{\phi[U]} \cap A$ is not σ-complete.
Now write $F = \bigcup_{i=0}^{\infty} F_i$, with F_i compact, and let $\{B_i: i < \omega\}$ be a basis for the topology of A. We will construct compact sets K_s, open subsets U_s of $\mathbf{C}$, open subsets W_s of G, and points $x_i \in B_i$, for each $s \in M$ and each $i < \omega$, such that the following hold:

(1) $K_s = \overline{K_s \cap A} \subset \overline{\phi[W_s]} \subset U_s$;
(2) for each $n < \omega$: $\overline{U}_{s^\frown n} \cap K_s = \emptyset$;
(3) for each $n,m < \omega$: $\overline{U}_{s^\frown n} \cap \overline{U}_{s^\frown m} = \emptyset$ if $n \neq m$;
(4) for each $n < \omega$: $\mathrm{Cl}_G(W_{s^\frown n}) \subset W_s$;
(5) for each $n < \omega$: $\overline{U}_{s^\frown n} \subset U_s$;
(6) $\mathrm{diam}(W_s) \leq 2^{-|s|}$ (here, the diameter is taken with respect to a complete metric for G);
(7) $\mathrm{diam}(U_s) \leq 2^{-\nu(s)}$;
(8) for each $n < \omega$: $d(K_s, K_{s^\frown n}) \leq 2^{1-\nu(s^\frown n)}$;
(9) $K_s \cap A$ is nowhere σ-complete, and nowhere dense in $\overline{\phi[W_s]} \cap A$;
(10) K_s is nowhere dense in $K_s \cup \bigcup_{n=0}^{\infty} K_{s^\frown n}$;
(11) K_s is contained in some F_j;
(12) $Z_i = \bigcup\{K_s: |s| \leq i\}$ is compact, and $Z_i \cap A$ is nowhere dense in A;
(13) for each $j \leq i$: $x_j \notin Z_i$.

We use induction on $|s|$ and i. First, put $W_\emptyset = G$, $U_\emptyset = \mathbf{C}$. Then $\overline{\phi[W_\emptyset]} \cap A = \bigcup_{i=0}^{\infty}(\overline{\phi[W_\emptyset]} \cap A \cap F_i)$ is not σ-complete, so some $\overline{\phi[W_\emptyset]} \cap A \cap F_j$ is not

σ-complete. By lemma A.2.1, $\overline{\phi[W_\emptyset]} \cap A \cap F_j$ contains a nowhere σ-complete, closed nowhere dense subset $H_\emptyset$; put $K_\emptyset = \overline{H}_\emptyset$. Since $H_\emptyset$ is nowhere dense in A, $B_0 \not\subset H_\emptyset$, say $x_0 \in B_0 \setminus H_\emptyset$. Then (1), (9), and (11) - (13) are satisfied, and so are (6) and (7) since all metrics are assumed to be bounded by 1. Next, suppose that K_s, U_s, W_s, and x_i have been defined for $|s| \le k$, $i \le k$, in accordance with conditions (1) - (13). Fix $s \in M$ with $|s| = k$. From (1) and (9), it is not hard to see that K_s is nowhere dense in $K_s \cup \phi[W_s]$, so by lemma 3.3.3 there exists a countable discrete subset $D_s = \{y_{s^\frown n}: n < \omega\}$ of $\phi[W_s] \setminus K_s$, such that $\overline{D}_s = D_s \cup K_s$, and $d(y_{s^\frown n}, K_s) \le 2^{-\nu(s^\frown n)}$ for each $n < \omega$. Now let $U_{s^\frown n}$ be an open neighborhood of $y_{s^\frown n}$ such that $\overline{U}_{s^\frown n} \subset U_s$, $\overline{U}_{s^\frown n} \cap K_s = \emptyset$, $\overline{U}_{s^\frown n} \cap \overline{U}_{s^\frown m} = \emptyset$ if $n \ne m$, and diam $(U_{s^\frown n}) \le 2^{-\nu(s^\frown n)}$, for each $n,m < \omega$. Since $y_{s^\frown n} \in \phi[W_s]$, $y_{s^\frown n} = \phi(x_{s^\frown n})$ for some $x_{s^\frown n} \in W_s$; hence there is an open neighborhood $W_{s^\frown n}$ of $x_{s^\frown n}$ in G such that $Cl_G(W_{s^\frown n}) \subset W_s$, $\mathrm{diam}(W_{s^\frown n}) \le 2^{-|s|-1}$, and $\overline{\phi[W_{s^\frown n}]} \subset U_{s^\frown n}$. Then $\overline{\phi[W_{s^\frown n}]} \cap A$ is not σ-complete, so as above, $\overline{\phi[W_{s^\frown n}]} \cap A$ contains a nowhere σ-complete, closed nowhere dense subset $H'_{s^\frown n}$ which is contained in some F_j; let $H_{s^\frown n}$ be a non-empty clopen subset of $H'_{s^\frown n}$ which is disjoint from $\{x_j: j \le k\}$, and put $K_{s^\frown n} = \overline{H}_{s^\frown n}$. Then (1) - (7), (9), and (11) are satisfied; furthermore, (8), (10), and the first part of (12) are proved as in lemma 3.3.4. To prove the second part of (12), suppose that V is a non-empty open subset of A which is contained in Z_{k+1}. Since $Z_k \cap A$ is closed and nowhere dense in A, $V \setminus Z_k$ is a non-empty open subset of A, contained in $\bigcup_{|s|=k+1} K_s$. So for some $s \in M$ with $|s| = k+1$, $(V \setminus Z_k) \cap K_s \ne \emptyset$; however, by (1), (3), and (5), $(V \setminus Z_k) \cap K_s = (V \setminus Z_k) \cap U_s$, contradicting the fact that $K_s \cap A = H_s$ is nowhere dense in A. Hence (12) holds. In particular, $B_{k+1} \not\subset Z_{k+1} \cap A$, so we can find a point $x_{k+1} \in B_{k+1} \setminus Z_{k+1}$; then (13) is also satisfied. This completes the induction.

Now put $Y = \bigcup_{i=0}^{\infty} (Z_i \cap A)$; we claim that Y is as required. First, that $\overline{Y} \setminus (\bigcup_{i=0}^{\infty} Z_i) \subset \phi[P] = F \setminus A$ is shown exactly as in the proof of lemma 3.3.4; so $\overline{Y} \subset \bigcup_{i=0}^{\infty} Z_i \cup (F \setminus A) = \bigcup_{s \in M} K_s \cup (F \setminus A) \subset F$ by (11), and $\overline{Y} \cap A = Y$. By (13), $B_j \not\subset Y$ for each $j < \omega$, so Y is closed and nowhere dense in A. Now Y is clearly nowhere σ-complete, and each $K_s \cap A$ is nowhere dense in Y by (1) and (10), so Y is first category. □

A.2.3 LEMMA: *Let* $X \in \mathcal{X}_0$, *let* F *be a σ-compact space such that* $X \subset F \subset \boldsymbol{C}$, *and let* $\varepsilon > 0$. *Then there exist closed nowhere dense subsets* X_i *of* X *such that*

(i) $X = \bigcup_{i=0}^{\infty} X_i$;

(ii) $X_i \in X_0$ *for each* $i < \omega$;

(iii) $Cl_{C}(X_i) \subset F$;

(iv) $diam(X_i) < \varepsilon$.

Proof: Again, let $\overline{}$ denote closure in C. If $F = \bigcup_{i=0}^{\infty} F_i$, with F_i compact, and $X = \bigcup_{i=0}^{\infty} Y_i$, with Y_i closed and nowhere dense in X, then $X = \bigcup_{i=0}^{\infty} \bigcup_{j=0}^{\infty}(Y_i \cap F_j)$, i.e. we can write $X = \bigcup_{i=0}^{\infty} A_i$, where A_i is closed and nowhere dense in X, and $\overline{A}_i \subset F$; of course, we may assume that each A_i is non-empty. Fix $i < \omega$, and let $\mathcal{D}$ be a covering of $\overline{X}\setminus\overline{A}_i$ by non-empty disjoint clopen subsets of $\overline{X}$, such that $diam(D) < d(D,\overline{A}_i)$ for each $D \in \mathcal{D}$. Since $D \cap X$ is non-empty, it is not σ-complete, and since $D \cap X \subset F \subset C$, we can apply lemma A.2.2 to obtain, for each $D \in \mathcal{D}$, a closed nowhere dense subset $E(D)$ of $D \cap X$ which is nowhere σ-complete and first category, such that $\overline{E(D)} \subset F$. Put $B_i = A_i \cup \bigcup_{D\in\mathcal{D}} E(D)$. Since $\overline{X}\setminus(\overline{A}_i \cup \bigcup_{D\in\mathcal{D}}\overline{E(D)}) = \bigcup_{D\in\mathcal{D}}(D\setminus\overline{E(D)})$ is open in $\overline{X}$, we have $\overline{B}_i = \overline{A}_i \cup \bigcup_{D\in\mathcal{D}}\overline{E(D)} \subset F$, and B_i is closed in X. From the diameter condition on the elements of $\mathcal{D}$ it follows that A_i is nowhere dense in B_i; thus, since each $E(D)$ is first category, B_i is first category. Also, if U is a non-empty open subset of B_i, then $U \cap E(D) \neq \emptyset$ for some $D \in \mathcal{D}$, so U is not σ-complete, i.e. B_i is nowhere σ-complete, whence $B_i \in X_0$. Finally, B_i is nowhere dense in X: if V is non-empty and open in X, and $V \subset B_i$, then $V \cap D = V \cap E(D) \neq \emptyset$ for some $D \in \mathcal{D}$, contradicting the fact that $E(D)$ is nowhere dense in X. Now let $\mathcal{U}_i$ be a clopen covering of B_i by non-empty sets of diameter less than ε, and enumerate $\bigcup_{i=0}^{\infty} \mathcal{U}_i$ as $\{X_i: i < \omega\}$; then the sets X_i are as required.□

To keep everything elementary, we here also give a new proof of the fact that $\mathbb{Q}^{\omega}$ is not σ-complete.

A.2.4 LEMMA: $\mathbb{Q}^{\omega} \in X_0$.

Proof: Being a countable product of σ-compacta, $\mathbb{Q}^{\omega}$ is an absolute $F_{\sigma\delta}$, and clearly it is first category. Since every non-empty open subset of $\mathbb{Q}^{\omega}$ contains a closed copy of $\mathbb{Q}^{\omega}$, it suffices to prove that $\mathbb{Q}^{\omega}$ is not σ-complete. Suppose that $\{A_i: i < \omega\}$ is a family of complete subsets of $\mathbb{Q}^{\omega}$. Since $\mathbb{Q}^{\omega}$ is not Baire, A_0 is not dense in $\mathbb{Q}^{\omega}$, so there exists a non-empty basic open subset U of $\mathbb{Q}^{\omega}$ such that $U \cap A_0 = \emptyset$. Let $n_0 \in \mathbb{N}$, $(q_0,\dots,q_{n_0-1}) \in \mathbb{Q}^{n_0}$ be such that $X_0 = (q_0,\dots,q_{n_0-1}) \times \mathbb{Q} \times \mathbb{Q} \times \cdots \subset U$. Since $A_1 \cap X_0$ is closed in A_1, it is complete, and since $X_0 \approx \mathbb{Q}^{\omega}$ is not Baire, as above we can find $n_0 < n_1 < \omega$, $(q_{n_0},\dots,q_{n_1-1}) \in \mathcal{Q}^{n_1-n_0}$, such that $X_1 = (q_0,\dots,q_{n_1-1}) \times \mathcal{Q} \times \mathcal{Q} \times \cdots \subset \mathcal{Q}^{\omega} \setminus (A_0 \cup A_1)$. Proceeding in this way, we find a point $(q_i)_{i<\omega} \in \mathcal{Q}^{\omega} \setminus (\bigcup_{i=0}^{\infty} A_i)$; so $\mathcal{Q}^{\omega}$ is not σ-complete. □

We are now ready to prove the main theorem of this appendix.

A.2.5 THEOREM: *Up to homeomorphism, $\mathcal{Q}^{\omega}$ is the unique element of X_0.*

Proof: By lemma A.2.4, $\mathcal{Q}^{\omega} \in X_0$. So suppose $X \in X_0$; embed X in C, and let $\{F_k : k < \omega\}$ be a family of σ-compact subsets of C such that $X = \bigcap_{k=0}^{\infty} F_k$, and $F_0 = C$. We will construct closed subspaces X_s of X, for each $s \in M$, satisfying conditions (i) and (ii) of definition A.1.2, as well as

(*) for each $s \in M$, $X_s \in X_0$;

(**) for each $s \in M$, $\operatorname{diam}(X_s) < (|s|+1)^{-1}$;

(***) for each $s \in M$, $\operatorname{Cl}_C(X_s) \subset F_{|s|}$.

The construction is a triviality: put $X_{\emptyset} = X$, and if X_s has been defined for all $s \in M$ with $|s| \le k$, then we obtain the sets $X_{s^\frown i}$ by applying lemma A.2.3 to $X_s \subset F_{|s|+1} \subset C$, $\varepsilon = (|s|+2)^{-1}$. We claim that the sets X_s also satisfy condition (iii) of definition A.1.2. Indeed, let $\sigma \in \omega^{\omega}$. Since $\overline{X}_{\sigma|0} \supset \overline{X}_{\sigma|1} \supset \overline{X}_{\sigma|2} \supset \cdots$ is a decreasing sequence of compacta, $\bigcap_{k=0}^{\infty} \overline{X}_{\sigma|k} \ne \emptyset$, say $x \in \bigcap_{k=0}^{\infty} \overline{X}_{\sigma|k}$. By (***), $x \in \bigcap_{k=0}^{\infty} F_k = X$. Thus, $x \in \bigcap_{k=0}^{\infty} X_{\sigma|k}$, and if U is any open neighborhood of x in X, then by (**), $X_{\sigma|n} \subset U$ for some $n < \omega$. Hence, if $p_k \in X_{\sigma|k}$ for each $k < \omega$, then $p_k \in U$ for each $k \ge n$, so $(p_k)_k$ converges to x. □

From this characterization of $\mathcal{Q}^{\omega}$ we can, by elementary methods, obtain characterizations of the other homogeneous zero-dimensional absolute Borel sets of exact class two. Let X_1 be the class of all zero-dimensional nowhere σ-complete absolute $F_{\sigma\delta}$ spaces that are Baire, and let W_0 (resp. W_1) denote the class of all zero-dimensional σ-complete spaces that are first category (resp. Baire), and nowhere an absolute $F_{\sigma\delta}$. By 1.12.3, it is clear that if X is dense and co-dense in C, then $X \in X_0$ (resp. X_1) if and only if $C \setminus X \in W_1$ (resp. W_0).

A.2.6 THEOREM: *Let $\mathcal{Q}^{\omega}$ be densely embedded in C. Then up to homeomorphism, $C \setminus \mathcal{Q}^{\omega}$ is the only element of W_1; furthermore, C is homogeneous with respect to dense copies of $C \setminus \mathcal{Q}^{\omega}$.*

Proof: By the above remark, $\mathcal{C}\setminus\mathcal{Q}^{\omega} \in \mathcal{W}_1$; and if $A,B \in \mathcal{W}_1$ are densely embedded in $\mathcal{C}$, then by the same remark and theorem A.2.5, $\mathcal{C}\setminus A \approx \mathcal{Q}^{\omega} \approx \mathcal{C}\setminus B$, so by corollary A.1.7, there is an autohomeomorphism h of $\mathcal{C}$ such that $h[\mathcal{C}\setminus A] = \mathcal{C}\setminus B$, whence $h[A] = B$. □

We will just write $\mathcal{C}\setminus\mathcal{Q}^{\omega}$ for the unique element of $\mathcal{W}_1$.

A.2.7 THEOREM: (a) *Up to homeomorphism,* $\mathcal{Q}\times(\mathcal{C}\setminus\mathcal{Q}^{\omega})$ *is the unique element of* $\mathcal{W}_0$.
(b) *Let* $\mathcal{Q}\times(\mathcal{C}\setminus\mathcal{Q}^{\omega})$ *be densely embedded in* $\mathcal{C}$. *Then up to homeomorphism,* $\mathcal{C}\setminus(\mathcal{Q}\times(\mathcal{C}\setminus\mathcal{Q}^{\omega}))$ *is the unique element of* $\mathcal{X}_1$.

Proof: (a) It is clear that $\mathcal{Q}\times(\mathcal{C}\setminus\mathcal{Q}^{\omega}) \in \mathcal{W}_0$. So suppose that $X \in \mathcal{W}_0$, say $X = \bigcup_{i=0}^{\infty} X_i$, with X_i closed and nowhere dense in X. Fix $i < \omega$, and let $\mathcal{D}$ be a covering of $X\setminus X_i$ by non-empty clopen disjoint subsets of X such that $\mathrm{diam}(D) < d(D,X_i)$ for each $D \in \mathcal{D}$. If we embed D densely in $\mathcal{C}$, then since D is σ-complete and not an absolute $F_{\sigma\delta}$, $\mathcal{C}\setminus D$ is an absolute $F_{\sigma\delta}$ which is not σ-complete. By lemma A.2.2, $\mathcal{C}\setminus D$ contains a closed nowhere dense subset Y such that $Y \in \mathcal{X}_0$, i.e. $Y \approx \mathcal{Q}^{\omega}$; then $E(D) = \overline{Y}\setminus Y \approx \mathcal{C}\setminus\mathcal{Q}^{\omega}$. Note that E(D) is closed and nowhere dense in D. Put $A_i = X_i \cup \bigcup_{D\in\mathcal{D}} E(D)$; then A_i is closed and nowhere dense in X, and it is easily seen that $A_i \in \mathcal{W}_1$, i.e. $A_i \approx \mathcal{C}\setminus\mathcal{Q}^{\omega}$. So by theorem 3.2.4, $\bigcup_{i=0}^{\infty} A_i = X \approx \mathcal{Q}\times(\mathcal{C}\setminus\mathcal{Q}^{\omega})$.
(b) By theorems A.2.6 and 3.2.8, $\mathcal{C}$ is homogeneous with respect to dense copies of $\mathcal{Q}\times(\mathcal{C}\setminus\mathcal{Q}^{\omega})$; now proceed as in the proof of theorem A.2.6. □

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SPECIAL SYMBOLS

AUTHOR INDEX

SUBJECT INDEX

SAMENVATTING

Alle ruimten zijn separabel en metrizeerbaar.

In het begin van deze eeuw werden topologische karakteriseringen gegeven van bekende homogene, nul-dimensionale ruimten als de Cantor verzameling $\mathbf{C}$, de verzameling der irrationale getallen $\mathbf{P}$, en de verzameling der rationale getallen $\mathbf{Q}$, bijvoorbeeld:

> als X een (niet-lege) nul-dimensionale compacte ruimte is zonder geïsoleerde punten, dan is X homeomorf met $\mathbf{C}$.

Veel later, rond 1980, werden door van Mill en van Douwen nog enkele homogene nul-dimensionale absolute Borel verzamelingen gekarakteriseerd. In dit proefschrift worden soortgelijke karakteriseringen gegeven van *alle homogene nul-dimensionale absolute Borel verzamelingen*; er blijken ω_1 veel topologische typen van zulke ruimten te bestaan.
Zoals de karakterisering van de Cantor verzameling voornamelijk berust op het precies aangeven van haar plaats in de Borel hiërarchie (compactheid), zo bestaan ook de karakteriseringen in dit proefschrift voor een groot deel uit de beschrijving van het niveau in de Borel hiërarchie van de betreffende ruimten. De gewone Borel hiërarchie (F_σ, G_δ, $F_{\sigma\delta}$, $G_{\delta\sigma}$, etc.) is echter niet fijn genoeg om onderscheid te kunnen maken tussen de verschillende homogene verzamelingen, en we zullen dan ook gebruik maken van (bestaande) verfijningen van de Borel hiërarchie, te weten de hiërarchie van de kleine Borel klassen (Kuratowski), en de Wadge hiërarchie (Wadge); de laatste is weer een verfijning van de hiërarchie van kleine Borel klassen. Gebruik makend van technieken voor het uitbreiden van homeomorfismen op nergens dichte deelverzamelingen komen we tot *interne* topologische karakteriseringen van alle homogene

Borel verzamelingen die zowel een $F_{\sigma\delta}$ als een $G_{\delta\sigma}$ in $\mathcal{C}$ zijn, in termen van de kleine Borel klassen. De Wadge hiërarchie gebruiken we voor de Borel verzamelingen van hogere klasse; deze hiërarchie berust op reductie door middel van continue functies: een verzameling A is "strikt eenvoudiger" dan een verzameling B in $\mathcal{C}$ als er een continue $f: \mathcal{C} \to \mathcal{C}$ is met $A = f^{-1}[B]$, maar geen continue $g: \mathcal{C} \to \mathcal{C}$ met $B = g^{-1}[A]$. Resultaten uit de speltheorie, onder andere van Martin, Steel, en Louveau spelen een belangrijke rol. De karakteriseringen die we hier verkrijgen zijn wel topologisch, maar in het algemeen *niet intern.*
De karakteriseringen van *homogene* ruimten, die in dit proefschrift gegeven worden, blijken verrassend genoeg vrij eenvoudig te leiden tot de stelling dat niet-triviale *rigide* nul-dimensionale absolute Borel verzamelingen niet bestaan; dit beantwoordt een vraag van van Douwen.
In een appendix bij dit proefschrift wordt tenslotte nog een elementair bewijs gegeven van de eerder verkregen karakterisering van $\mathbb{Q}^{\omega}$, het aftelbaar produkt van de rationale getallen.

STELLINGEN

1. Er bestaat een homeomorfisme $h: \mathbb{R}\backslash\mathbb{Q} \to \mathbb{R}\backslash\mathbb{Q}$ dat niet voortzetbaar is over enige $q \in \mathbb{Q}$.

2. Er bestaat een decompositie van $\mathbb{R}$ in twee homeomorfe rigide delen.

3. (a) Er bestaat een rigide deelverzameling van $\mathbb{R}$ die opgesplitst kan worden in twee homeomorfe rigide delen.
(b) Er bestaat een rigide deelverzameling van $\mathbb{R}$ die opgesplitst kan worden in twee homeomorfe homogene delen.

4. Er bestaat een overaftelbare, sterk homogene deelverzameling A van $\mathbb{R}$, en een aftelbare dichte deelverzameling D van A, zodat $A\backslash D$ rigide is.

5. Onder aanname van de continuum hypothese bestaat er, voor elke $n \in \mathbb{N}$, een samenhangende, lokaal samenhangende, separabel metrische topologische groep G_n, zodat voor elke $x,y \in G_n$ er precies $\phi(n)$ autohomeomorfismen van G_n zijn die x op y afbeelden; hierin is ϕ de Euler-functie: $\phi(n) = |\{m \in \{1,\ldots,n\}: (m,n) = 1\}|$.

6. Zij X een niet-pseudocompacte Tychonoff-ruimte, en zij $X^* = \beta X\backslash X$. Definieer

$A_0 = \{x \in X^*: x \in \overline{D}$ voor zekere aftelbare, nergens dichte, discrete $D \subset \beta X\backslash\{x\}\}$;

$A_1 = \{x \in X^*: x \in \overline{D}$ voor zekere aftelbare, nergens dichte, π-homogene $D \subset \beta X\backslash\{x\}$ met π-gewicht $\omega\}$;

$A_2 = \{x \in X^*: x \in \overline{D}$ voor zekere aftelbare, nergens dichte, π-homogene $D \subset \beta X\backslash\{x\}$ met π-gewicht $\omega_1\}$.

Onder aanname van Martin's axioma bestaat er, voor elke $F \subset \{0,1,2\}$, een punt $x_F \in X^*$ met ($x_F \in A_i$ dan en slechts dan als $i \in F$).

7. Zij X de verzameling van alle functies $f: \mathbb{N} \to \mathbb{N}$ met $\lim_{n\to\infty} f(n) = \infty$, voorzien van de topologie van puntsgewijze convergentie. Dan is X homeomorf met $\mathbb{Q}^\omega$, het aftelbaar oneindige product van de rationale getallen (zie pag.120, dit proefschrift).

8. Zij $\alpha < \omega_1$, en zij $\{A_n : n < \omega\}$ een collectie (separabele) Borel verzamelingen die absoluut van additieve klasse α zijn, maar niet absoluut van multiplicatieve klasse α. Dan is $\Pi_{n=0}^{\infty} A_n$ absoluut van multiplicatieve klasse $\alpha+1$, maar niet absoluut van additieve klasse $\alpha+1$ (zie pag. 105, dit proefschrift).

9. Een bespreking van een congresverslag dient de lezer een indruk te geven van de inhoud der opgenomen artikelen. Dit zou bewerkstelligd kunnen worden door enkele belangrijke artikelen volledig te recenseren; het is niet voldoende om de titels van alle artikelen op te sommen, zoals in de Mededelingen van het Wiskundig Genootschap bij dergelijke besprekingen vaak gebeurt.

10. De laatste stelling bij het proefschrift van I. Moerdijk is onjuist.