ABEL-JACOBI ISOGENIES FOR CERTAIN TYPES OF FANO THREEFOLDS

G.E. WELTERS

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PROEFSCHRIFT

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A mi familia

CONTENTS

PART TWO : THE INTERSECTION OF THREE QUADRICS

INTRODUCTION

In contrast with the advances in the theory of Fano threefolds, their intermediate Jacobians have been studied far less than expected, since the beautiful paper [8] of Clemens and Griffiths on the cubic threefold was published, now nine years ago. The basic tool in the study of the intermediate Jacobian $J(X)$ of a threefold X being the Abel-Jacobi map for families of curves on X, the problem lies in our bad understanding of the latter ones. Already for $X = \mathbb{P}^3$, the most simple threefold, very little is known in general about its Chow varieties of curves - for instance about their dimension, irregularity, Kodaira dimension, etc.

From a more optimistic viewpoint however, to establish their results about the cubic threefold, Clemens and Griffiths needed only one such family of curves, in fact the most simple ones: the two-dimensional family of lines of X, parametrized by the Fano surface F of X. Their main results - Torelli and non rationality Theorems - rely heavily on their analogue of the Riemann parametrization Theorem, giving a description in geometrical terms (by means of an Abel-Jacobi map) of the theta divisor of $J(X)$. A basic ingredient of the latter result is their so-called Gherardelli-Todd isomorphism Theorem stating that, with the above notations, the Abel-Jacobi map

$$Alb(F) \longrightarrow J(X)$$

is an isomorphism of abelian varieties.

In studying the intermediate Jacobian of other Fano varieties, it is therefore natural to start looking at the Abel-Jacobi map for the most simple curves - lines, conics - on these varieties and ask whether similar results hold. This has motivated our study.

Among the results we have gotten, there is an extension of the last mentioned theorem to two other Fano varieties. More exactly:

I) Let X be a smooth double cover of $\mathbb{P}^3$ with quartic discriminant locus $S \subset \mathbb{P}^3$. We shall call this henceforth the quartic double solid - or, shortly, the double solid - after [7]. This is a Fano variety of index 2 and its lines (curves of degree 1 with respect to the positive generator of Pic(X)) are parametrized by a surface F which we shall call also the Fano surface of X . This surface is smooth exactly when S doesn't contain lines of $\mathbb{P}^3$. The result is now: if X has smooth Fano surface F , then the Abel-Jacobi map Alb(F) $\longrightarrow$ J(X) is an isomorphism of abelian varieties.

II) Take now X to be the smooth complete intersection of three quadrics in $\mathbb{P}^6$. This Fano variety is of index 1 and its lines are parametrized by a curve, so they dont cover the variety X . The conics of X (i.e. curves of degree 2 with respect to the positive generator of Pic(X)) are parametrized by a surface which again is called the Fano surface of X , being smooth if X is sufficiently general. We shall prove: if X has smooth Fano surface F , then Alb(F) $\longrightarrow$ J(X) is an isogeny of abelian varieties. We dont know yet if it is, or not, an isomorphism.

Let us add inmediately that the surjectivity of the above maps is not a surprising fact: if non zero, an Abel-Jacobi map

$$\mathrm{Alb}(B) \longrightarrow J(X)$$

(for a smooth projective variety B parametrizing a family of curves on X) can be seen to be surjective, by standard monodromy arguments, if, for example,

i) X moves in a Lefschetz pencil of hypersurfaces of a fourfold W with vanishing 3^d Betti number, and

ii) the family of curves parametrized by B survives as X moves in that pencil.

However, the property of being isomorphic (resp. at least isogenous) still escapes to a common interpretation. Accordingly, the proofs which we present are very different from each other.

The work is divided into two independent parts, each one dealing with one of the two cases, and being quite asymmetric in their construction and further purposes.

Part One is devoted to the double solid. We carry out an exhaustive study of the corresponding Fano surface in the smooth case, by using cohomological methods. On the other side we specialize the yoga of the tangent bundle sequence to this situation, and both things together yield the Isogeny Theorem. As a consequence we get that the threefold can be recovered from its Fano surface. A topological argument largely inspirated from [5] finally gives the main result of this part.

As a first application, we prove that the intermediate Jacobian of our threefold is a generalized Prym variety in the sense of Tyurin (cf [28] , [5]) . This implies, as in [5] , that the underlying abstract group of the intermediate Jacobian can be identified with the Chow group of rational equivalence classes of algebraic 1-cycles algebraically equivalent to zero.

Section 6 is devoted to the second application of the Isomorphism Theorem. Still far from being complete, it aims to make more transparent the relevance of Abel-Jacobi isomorphisms in the study of the theta divisor of intermediate Jacobians, towards analogues in this case of the Riemann parametrization Theorem for curves. Basically, we study the traces of the theta divisor Θ_X on the Fano surface, getting in this way some insight in the (seemingly difficult) question of finding families of curves on X parametrizing Θ_X .

In Part Two we prove our second main result, stated in II) above, and which was conjectured by Tyurin in [27], p.103. Inspirating on his suggestions, the question is translated into a problem dealing only with irreducible étale (2:1) coverings of smooth plane septic curves and the linear system of linear sections of these. It is solved within the framework of curves, in absence of any further reference to threefolds. More explicitly, the theory of intermediate Jacobians and the theory of Prym varieties are known to overlap above the intermediate Jacobians of conic bundles (we are dealing only with threefolds) . Underlying this relationship there is a similar one at the level of cycles: curves on conic bundles yield - generally speaking - divisors on the covering curve of the associated Prym situation. The isogeny question we are looking at translates in this way into a problem concerning a certain variety of divisors on that curve. This new problem extends naturally beyond the range of those varieties of divisors which come from Chow varieties of curves on the threefold, and seems to have an affirmative answer in many cases, at least in sufficiently many as to settle our original question by an inductive argument.

The groundfield will be taken always to be the field $\mathbb{C}$ of complex numbers.

We notice that several of the results of Part One until the Isogeny Theorem and its consequences have been got independently by A.S. Tikhomirov (The geometry of the Fano surface of the double cover of $\mathbb{P}^3$ branched in a quartic, Izv. Akad Nauk SSSR 44 (1980) (russian), Maths. of the USSR Izv. 14(1980)) Also, a quite different method to get the results on the Fano surface of the double solid consists in using the degeneration methods of Clemens ([7]). In Section 6 we shall use these in an essential way.

Finally, the contents of Part One appeared previously, with exception (mainly) of Section 6 , in the Preprint Series of the Math. Inst. of the University of Utrecht (The Fano surface of lines on a double $\mathbb{P}^3$ with 4^{th} order discriminant locus, Part I, prep. n° 123, August 1979 ; idem, Part II, prep. n° 164, July 1980) . Similarly, Part Two is: Divisor varieties, Prym varieties and a conjecture of Tyurin, prep. n° 139, January 1980 , of the same series.

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PART ONE : THE QUARTIC DOUBLE SOLID

0. PRELIMINARIES

Generalities on the quartic double solid

With the aim of fixing notations, we start recalling briefly some elementary facts we will be concerned with. We denote by X the double solid and by S its discriminant locus, writing f for the projection map onto $\mathbb{P}^3$. Thus X is a double cover of $\mathbb{P}^3$ by means of f, and $S \subset \mathbb{P}^3$ is the discriminant locus of f, a smooth quartic surface by hypothesis. As usual for double covers, we identify branch and discriminant loci, hence $S \subset X$ will make sense too. We denote by $\phi_4 = 0$ a fixed equation for S in $\mathbb{P}^3$, and the covering involution of X will be written i. One has an embedding

$$\begin{array}{ccc} X & \hookrightarrow & E \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f} & \\ & \mathbb{P}^3 & \end{array}$$

where $E = \mathrm{Spec}(S_{\mathbb{P}^3}(\mathcal{O}_{\mathbb{P}^3}(-2))) \longrightarrow \mathbb{P}^3$ is the line bundle over $\mathbb{P}^3$ with fibre at a point $P \in \mathbb{P}^3$ the vector space of 2-forms at P. Denoting by

$$T \in H^0(E, f^*\mathcal{O}_{\mathbb{P}^3}(2))$$

the tautological section of $f^*\mathcal{O}_{\mathbb{P}^3}(2)$, X is identified with the zero scheme of $T^2 - \phi_4 \in H^0(E, f^*\mathcal{O}_{\mathbb{P}^3}(4))$ in E. This presentation of X as an embedded variety will be needed later on.

The sheaves $f^*\mathcal{O}_{\mathbb{P}^3}(n)$ on X and E will be denoted respectively by $\mathcal{O}_X(n)$ and $\mathcal{O}_E(n)$. In particular, we have an isomorphism $\mathcal{O}_E(X) \simeq \simeq \mathcal{O}_E(4)$ and S is given inside X by the equation $T = 0$, that is, $\mathcal{O}_X(S) \simeq \mathcal{O}_X(2)$. We shall use also the standard sequences:

(0.1) $$0 \longrightarrow f^*\Omega^1_{\mathbb{P}^3} \longrightarrow \Omega^1_E \longrightarrow \Omega^1_{E/\mathbb{P}^3} \longrightarrow 0 \qquad \text{on } E$$

$$(0.2)\qquad 0 \longrightarrow f^*\Omega^1_{\mathbb{P}^3} \longrightarrow \Omega^1_X \longrightarrow \Omega^1_{X/\mathbb{P}^3} \longrightarrow 0 \qquad \text{on } X$$

together with the identifications

$$(0.3)\qquad \Omega^1_{E/\mathbb{P}^3} \simeq \mathcal{O}_E(-2) \quad \text{and} \quad \Omega^1_{X/\mathbb{P}^3} \simeq \mathcal{O}_S(-2) \quad .$$

There are several ways to get the known facts about X , and we just choose one of them. Below we shall recall the proof of the unirationality of X ; in particular, this gives $H^i(X,\mathcal{O}_X) = 0$, $i > 0$. From (0.2),(0.3) we get an exact sequence

$$H^2(X,f^*\Omega^1_{\mathbb{P}^3}) \longrightarrow H^2(X,\Omega^1_X) \longrightarrow H^2(S,\mathcal{O}_S(-2)) \longrightarrow H^3(X,f^*\Omega^1_{\mathbb{P}^3}) \quad .$$

By the formulae of Bott, both ends of this sequence are zero, hence $H^2(X,\Omega^1_X) \simeq H^2(S,\mathcal{O}_S(-2))$. By Kodaira-Serre duality and the fact that $K_S = 0$, we have on the other side that

$$H^2(S,\mathcal{O}_S(-2)) \simeq H^0(S,\mathcal{O}_S(2))^\vee \simeq H^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(2))^\vee \quad .$$

Thus $H^2(X,\Omega^1_X) \simeq \mathbb{C}^{10}$. In the same way one gets $H^1(X,\Omega^1_X) \simeq \mathbb{C}$, hence the Hodge numbers of X are $h^{i,i} = 1$, $0 \le i \le 3$, and $h^{i,j} = 0$ if $i \neq j$, except $h^{1,2} = h^{2,1} = 10$.

Next, $\mathrm{Pic}(X) \simeq \mathbb{Z}$ holds. This goes as follows ([14], Section 1) : from the exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}^h_X \longrightarrow \mathcal{O}^{h*}_X \longrightarrow 0$$

together with GAGA and $H^1(X,\mathcal{O}_X) = H^2(X,\mathcal{O}_X) = 0$ we derive $\mathrm{Pic}(X) \simeq H^2(X,\mathbb{Z})$; on the other hand, if $\mathbb{P}^2 \subset \mathbb{P}^3$ denotes for a moment a generally chosen 2-plane in $\mathbb{P}^3$, Lefschetz' weak theorem gives $H^2(X,\mathbb{Z}) \subset H^2(f^{-1}(\mathbb{P}^2),\mathbb{Z})$. The surface $f^{-1}(\mathbb{P}^2)$ is a Del Pezzo one, isomorphic with the blowing up of a 2-plane at a finite number of points (cf e.g. [12], p.549) ; Therefore $H^2(f^{-1}(\mathbb{P}^2),\mathbb{Z})$ is a direct sum of copies of $\mathbb{Z}$, hence has no torsion, and the same holds now for $H^2(X,\mathbb{Z})$. Since $h^{1,1} = 1$, $h^{2,0} = h^{0,2} = 0$, we have $H^2(X,\mathbb{C}) \simeq \mathbb{C}$, hence $H^2(X,\mathbb{Z}) \simeq \mathbb{Z}$, as claimed.

We write

$h \in \mathrm{Pic}(X)$

the class of the divisor $f^{-1}(\mathbb{P}^2)$ of X. Being clearly $(h)^3 = 2$, this is a generator of the Picard group. Remark that, by our choice of notations, one has $\mathcal{O}_X(nh) = \mathcal{O}_X(n)$.

Taking first Chern classes in (0.2) and using (0.3) we get $K_X = -4h + c_1(\Omega^1_{X/\mathbb{P}^3}) = -2h$. This formula shows X to be a Fano variety of index 2 (cf [15], for a general study of Fano varieties).

We define the "lines" of X to be the curves of degree 1 with respect to the generator h of $\mathrm{Pic}(X)$, i.e. such that their intersection product with h yields 1. Equivalently, lines of X are the curves mapping isomorphically onto lines of $\mathbb{P}^3$ by f. To get some insight, we look for the possible shapes of the inverse images in X of the lines $\overline{L}$ of $\mathbb{P}^3$. Assuming first $\overline{L} \not\subset S$, the intersection cycle $S\cdot\overline{L}$ on $\overline{L}$ is defined and $f^{-1}(\overline{L})$ is completely described as the double covering of $\overline{L} \simeq \mathbb{P}^1$ with discriminant divisor $\delta = S\cdot\overline{L}$. The shape of this divisor therefore determines that of $f^{-1}(\overline{L})$, according to the following description:

If $\delta = \overline{P}_1 + \overline{P}_2 + \overline{P}_3 + \overline{P}_4$, $f^{-1}(\overline{L})$ is a smooth elliptic curve; if $\delta = \overline{P}_1 + \overline{P}_2 + 2\overline{Q}$, it is a rational curve with a node, and, if $\delta = \overline{P} + 3\overline{Q}$, a rational curve with a cusp. For $\delta = 2\overline{P} + 2\overline{Q}$ we get two copies of $\mathbb{P}^1$ intersecting transversally at two points and, if $\delta = 4\overline{Q}$, two copies of $\mathbb{P}^1$ meeting tangentially at a single point, like a conic and a tangent line in the plane.

If $\overline{L} \subset S$ then $f^{-1}(\overline{L}) \simeq \mathrm{Spec}(\mathcal{O}_{\overline{L}}[T]/T^2)$ is a line counted twice. We see therefore that the lines of X are exactly the components of the inverse images of the bitangent lines to $S \subset \mathbb{P}^3$; here we define a line $\overline{L} \subset \mathbb{P}^3$ to be bitangent to S if and only if the equation ϕ_4 of S restricts to a perfect square in $H^0(\overline{L}, \mathcal{O}_{\overline{L}}(4))$. There are thus three possible types of bitangents: proper bitangents (meeting S twice with multiplicity 2), hyperflexes (meeting S once with multiplicity 4) and lines contained in S.

(0.4) In connection with the above, we notice already that, being $A^2(\mathbb{P}^3) = 0$, full inverse images are irrelevant as far as the Chow group of X is concerned. Therefore our interest concentrates on "halves" of inverse images, and we may ask therefore: given an arbitrary curve $\overline{C} \subset \mathbb{P}^3$, when does it happen that $f^{-1}(\overline{C}) = C' + C''$ with $C'' = i(C')$? Since $f^{-1}(\overline{C})$ is given inside the line bundle $\mathcal{O}_{\overline{C}}(2)$ (see (0.6.b) below) by the equation $T^2 = \overline{\phi_4}$ where $\overline{\phi_4}$ is the restriction of ϕ_4 , the splitting is obviously equivalent with the existence of a rational section of $\mathcal{O}_{\overline{C}}(2)$ such that $\tau^2 = \overline{\phi_4}$ holds (clearly, this section will be defined at every smooth point of $\overline{C}$ in that case). Writing $\overline{N}$ for the normalization of $\overline{C}$, this means that there exists a (regular) section τ of $\mathcal{O}_{\overline{N}}(2)$ with $\tau^2 = \overline{\phi_4}$. Thus the splitting is equivalent with $S \cdot \overline{N} = 2D$, with D belonging to $|\mathcal{O}_{\overline{N}}(2)|$. Notice that, if $\overline{N}$ is rational, the latter condition is superfluous.

A slightly different question, which is nevertheless the same if $\overline{C}$ is smooth, is: when does there exist a curve C in X mapping isomorphically onto $\overline{C}$? The answer is clearly that there has to be a globally defined section τ of $\mathcal{O}_{\overline{C}}(2)$ such that $\tau^2 = \overline{\phi_4}$. A relevant example for our later purposes is the following one:

(0.5) Let $\overline{C} \subset \mathbb{P}^3$ be the plane curve consisting of four coplanar bitangents to S and assume for simplicity that these yield eight distinct points of contact. Then a necessary and sufficient condition for the existence of a configuration of four lines in X mapping isomorphically onto $\overline{C}$ is that these eight points ly on a conic. Furthermore, all such configurations are complete intersection curves in X , hence are rationally constant.

In fact, if $\mathbb{P}^2 \subset \mathbb{P}^3$ is the supporting plane, one has clearly $H^0\mathcal{O}_{\mathbb{P}^2}(2) \simeq H^0\mathcal{O}_{\overline{C}}(2)$. Therefore, if a τ as above exists, it is $\tau = \psi_2|\overline{C}$ for a well-determined 2-form ψ_2 on $\mathbb{P}^2$, and $\psi_2 = 0$ passes through the 8-ple of contact. Conversely, if a conic $\psi_2 = 0$ contains these eight points, the 4-forms $(\psi_2)^2$ and ϕ_4 of $\mathbb{P}^2$ induce the same divisor on $\overline{C}$, hence differ there by a constant factor, say $\overline{\phi_4} = c(\overline{\psi_2})^2$. Then $\tau = c^{\frac{1}{2}}\overline{\psi_2}$ satisfies $\tau^2 = \overline{\phi_4}$. The two configurations produced in this way are the complete intersections of $f^{-1}(\mathbb{P}^2)$ with each one of the surfaces $T \pm c^{\frac{1}{2}}\psi_2 = 0$ of X (the 2-form ψ_2 having been previously extended to the whole of $\mathbb{P}^3$) , q.e.d.

For completeness sake, we recall a proof of the unirationality of X . Fix a line $\overline{L} \subset \mathbb{P}^3$ bitangent to S , and a line L' in X above $\overline{L}$ (Figure 1) . The variety

$$Y = \{ (\overline{P},\overline{l}) \mid \overline{l} \text{ is a line in } \mathbb{P}^3 \text{ meeting } \overline{L} \text{ at } \overline{P} \}$$

is a $\mathbb{P}^2$ - bundle over $\overline{L}$. We will be done if we exhibit a dominant rational map $\Phi : Y \dashrightarrow X$. Take a general $(\overline{P},\overline{l}) \in Y$; let $\{P'\} = L' \cap f^{-1}(\overline{l})$ and $\{P''\} = i\{P'\} = L'' \cap f^{-1}(\overline{l})$, where $L'' = iL'$. By Riemann-Roch, there is a unique $Q \in f^{-1}(\overline{l})$ such that $P'' + Q \equiv 2P'$ on $f^{-1}(\overline{l})$. We put $\Phi(\overline{P},\overline{l}) = Q$, and notice that the definition still makes sense when $\overline{l}$ is simply tangent to S at a point other than $\overline{l} \cap \overline{L}$.

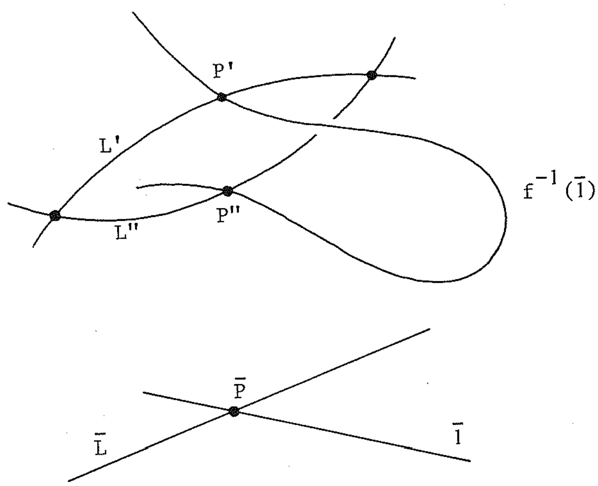


Figure 1.

It remains to show that Φ is dominant . Take $Q \in X$ sufficiently general as to have, if $\overline{Q} = f(Q) \in \mathbb{P}^3$: $\overline{Q} \notin S$, $\overline{Q} \notin \overline{L}$ and, putting $\mathbb{P}^2 = \overline{L} \vee \overline{Q} \subset \mathbb{P}^3$, the tangent lines from $\overline{Q}$ to the quartic $\mathbb{P}^2 \cap S$ are simple tangents, at points outside $\overline{L} \cap S$. Consider the $\mathbb{P}^1$ of lines in $\mathbb{P}^2$ through $\overline{Q}$ and define a morphism $\Phi' : \mathbb{P}^1 \longrightarrow f^{-1}(\mathbb{P}^2)$ as follows: $\Phi'(\overline{l}) = x \in f^{-1}(\overline{l})$ such that $x + Q \equiv 2P'$ on $f^{-1}(\overline{l})$, where again $\{P'\} = L' \cap f^{-1}(\overline{l})$. The image of Φ' is either a curve in $f^{-1}(\mathbb{P}^2)$ or a point. In the latter case, it ought to be $\Phi'(\mathbb{P}^1) = Q$ or $\Phi'(\mathbb{P}^1) = iQ$. If $\Phi'(\mathbb{P}^1) = iQ$, we would have on each $f^{-1}(\overline{l})$: $2P' \equiv Q + iQ \equiv P' + P''$, hence $P' \equiv P''$ always, which is impossible. If $\Phi'(\mathbb{P}^1) = Q$ we get that $2P' \equiv 2Q$ on each $f^{-1}(\overline{l})$. Taking a line joining $\overline{Q}$ with one of the two points of $\overline{L} \cap S$ we have, on $f^{-1}(\overline{l})$: $2P' \equiv Q + iQ$. Hence $Q \equiv iQ$ there, i.e. : $Q = iQ$. But this is impossible, since $Q \notin S$.

So $\Phi'(\mathbb{P}^1)$ is a curve in $f^{-1}(\mathbb{P}^2)$. If this curve meets L'' we are done. But $\Phi'(\mathbb{P}^1)$ meets certainly $f^{-1}(\overline{L}) = L' + L''$, and has empty intersection with L' : if $x + Q \equiv 2P'$ on $f^{-1}(\overline{l})$ and $x \in L'$, we would have $x = P'$, hence $P' \equiv Q$, i.e. : $P' = Q$; but $\overline{Q} \notin \overline{L}$ by hypothesis. This finishes the proof.

We recall that, in [2], Théorème 5.6 , Beauville has shown that a general quartic double solid is non rational. As far as we know, it is still an open question as whether this is true for all (smooth) quartic double solids.

Remarks and conventions

To end this preliminary section, we make some conventions and recall a few basic facts for easier reference.

(0.6) Conventions :

a) If $\mathbb{P}E \longrightarrow Y$ denotes a projective bundle, $\mathcal{O}_{\mathbb{P}E}(1)$ will stand for the fundamental sheaf of $\mathbb{P}E$. We follow Grothendieck's notation for projective bundles, hence the points of $\mathbb{P}E$ above $y \in Y$ are the codimension one subspaces of $E(y)$.

b) If $\mathcal{F}$ is a locally free sheaf on Y , we shall identify $\mathcal{F}$, for geometrical purposes, with its associated vector bundle on Y , i.e. with $V(\mathcal{F}^{\vee}) = \mathrm{Spec}(S_Y(\mathcal{F}^{\vee}))$.

c) Distinct mappings will be sometimes denoted by the same symbol, if they are deduced from one of them by base extension or by restriction. The context will decide about the precise meaning in each case.

d) Idem as in c) , with cycle classes, cohomology classes, etc.

e) The pullback of vector bundles is denoted by subscripts, i.e. the inverse image of $E \longrightarrow Y$ to $Y' \longrightarrow Y$ is written $E_{Y'} \longrightarrow Y'$.

f) Unless otherwise specified, the terms 'general' , 'generally' and 'sufficiently general' , referred to objects parametrized by a certain variety V , mean: "for all such objects parametrized by a certain Zariski open and dense subset V^0 of V ".

(0.7) If $g : Z \longrightarrow Y$ is a (2:1) covering, we say that it is given by $\mu \in \mathrm{Pic}(Y)$ if it is isomorphic with $\mathrm{Spec}(\mathcal{O}_Y \oplus \mathcal{O}_Y(-\mu))$, the algebra structure coming from a choice of a non vanishing section of $\mathcal{O}_Y(2\mu)$.

The scheme of zeros of the latter is then the discriminant locus of g .

If $\mathcal{F}$ is a coherent sheaf on Y , a degenerating Leray spectral sequence yields

$$(0.8) \qquad H^i(Z, g^*\mathcal{F}) \simeq H^i(Y,\mathcal{F}) \oplus H^i(Y,\mathcal{F}(-\mu))$$

for all i , and this is also the invariant-antiinvariant part decomposition of $H^i(Z,g^*\mathcal{F})$ under the action of the covering involution.

(0.9) Consider a commutative diagram

$$\begin{array}{ccc} Z & \hookrightarrow & \mathbb{P}E \\ & {\scriptstyle g}\searrow \quad \swarrow{\scriptstyle g} & \\ & Y & \end{array}$$

where Z is a divisor of $\mathbb{P}E$, Y being smooth and complete, $g : Z \longrightarrow Y$ finite and surjective, and $\mathbb{P}E$ a $\mathbb{P}^1$- bundle over Y . We interprete this situation as a moving set of points $Z(y)$ on a moving line $(\mathbb{P}E)(y)$, $y \in Y$. If m is the cardinality of that set, i.e. the degree of the map $g : Z \longrightarrow Y$, the equations of these sets are the (non zero) elements of a well determined sub-bundle $\mathcal{L} \subset S^m(E)$, which we shall call "bundle of equations of the fibres of $Z \longrightarrow Y$ in the fibres of $\mathbb{P}E \longrightarrow Y$ ". The ideal defining the subscheme $Z \subset \mathbb{P}E$ is isomorphic with $g^*\mathcal{L} \otimes \mathcal{O}_{\mathbb{P}E}(-m)$.

To justify the last statement we remark that, by the structure of $\mathrm{Pic}(\mathbb{P}E)$, we have $\mathcal{O}_{\mathbb{P}E}(-Z) \simeq (g^*\mathcal{L}')(r)$ for some $\mathcal{L}' \in \mathrm{Pic}(Y)$ and some $r \in \mathbb{Z}$. Restriction to a fibre yields $r = -m$. Then, taking direct images by g in the exact sequence $0 \to g^*\mathcal{L}' \to \mathcal{O}_{\mathbb{P}E}(m) \to \mathcal{O}_Z(m) \to 0$, we get $\mathcal{L}' \simeq \mathcal{L}$.

The following is also easily seen:

(0.10) If $m = 2$ in the above, the class $\mu \in \mathrm{Pic}(Y)$ giving the covering $g : Z \longrightarrow Y$ is $\mu = c_1(E) - c_1(\mathcal{L})$.

Our last remarks concern bitangents of smooth plane quartics; such a curve has exactly 28 bitangents (the odd theta characteristics) . The following can be found in several classical books, for example in [26], p.p. 321-322 , or reproved in terms of theta characteristics. (Compare also with (0.5).)

(0.11) Given a pair (a_1,b_1) of distinct bitangents there are exactly 6 pairs of bitangents (a_i,b_i) , $i=1,\dots,6$, yielding 12 distinct lines, such that for each $i\neq j$ the 8 points of contact of the bitangents a_i , b_i , a_j , b_j ly on a conic (of course, if a pair of contact degenerates, we ask the conic for tangency to the corresponding line) . Moreover, the above 10 lines a_i , b_i , $i=2,\dots,6$ are the only bitangents c , distinct from a_1 , b_1 , such that a_1 , b_1 , c have their 6 points of contact on a conic.

The following is an easy exercise:

(0.12) Given a pair (a_1,b_1) as in (0.11) , the other 5 pairs of the 6-tuple intersect on each line a_1 and b_1 5 pairs of points which belong to a unique linear pencil of degree 2 on the line a_1 (resp. b_1) . Moreover, this pencil includes the pair of contact of a_1 (resp. b_1) . Thus, if we take a bitangent a_1 , the choice of a second one b_1 induces a pencil g^1_2 on a_1 which contains the pair of contact of that line.

1. THE SURFACES F , F_0 : INFINITESIMAL STUDIES

The surfaces F , F_0

We start with the following remarks:

a) There are ∞^2 bitangent lines to the smooth quartic surface $S \subset \mathbb{P}^3$.

b) If S is sufficiently general, it contains no lines of $\mathbb{P}^3$.

The first one follows from the fact that any 2-plane of $\mathbb{P}^3$ contains only a finite number of bitangents (28 , if the corresponding plane section of S is smooth) . As for b) , there are ∞^{29} quartic surfaces through any given line of $\mathbb{P}^3$, hence at most ∞^{33} quartics containing lines. Since the quartics are parametrized by a $\mathbb{P}^{34}$, this proves the claim.

We define

F = Hilbert scheme parametrizing the lines on X .

F_0 = Hilbert scheme parametrizing the bitangents of $S \subset \mathbb{P}^3$.

There is an obvious morphism

$$\pi : F \longrightarrow F_0$$

which is (2:1) except for points of F_0 corresponding to lines contained in S , where the fibre consists of a single point. Let $L \in F$ and $\bar{L} = \pi(L)$; one has:

(1.1) LEMMA. *If* $L \not\subset S$ *then* F *is smooth at* L , F_0 *is smooth at* $\bar{L}$ *and the mapping* π *is étale there.*

PROOF. We choose a 2-plane $\mathbb{P}^2 \subset \mathbb{P}^3$ containing $\bar{L}$ and cutting S along a smooth quartic; the inverse image $f^{-1}(\mathbb{P}^2)$ is then a smooth surface which contains L . Consider the sequence of normal bundles

$$0 \longrightarrow N_{L/f^{-1}(\mathbb{P}^2)} \longrightarrow N_{L/X} \longrightarrow N_{f^{-1}(\mathbb{P}^2)/X} \otimes \mathcal{O}_L \longrightarrow 0 \ .$$

Being $N_{f^{-1}(\mathbb{P}^2)/X} = \mathcal{O}_{f^{-1}(\mathbb{P}^2)}(1)$, the last term equals $\mathcal{O}_L(1)$; on the other side, having on $f^{-1}(\mathbb{P}^2)$: $1 = L\cdot(L + iL) = L^2 + 2$, it is $L^2 = -1$, hence $N_{L/f^{-1}(\mathbb{P}^2)} = \mathcal{O}_L(-1)$. By the associated cohomology sequence we get therefore $h^0(L,N_{L/X}) = 2$, $h^1(L,N_{L/X}) = 0$. Hence F is smooth at L .

Next, identifying L with $\bar{L}$ by f , the exact diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_L & \longrightarrow & T_X \otimes \mathcal{O}_L & \longrightarrow & N_{L/X} & \longrightarrow & 0 \\ & & \downarrow \simeq & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_{\bar{L}} & \longrightarrow & T_{\mathbb{P}^3} \otimes \mathcal{O}_{\bar{L}} & \longrightarrow & N_{\bar{L}/\mathbb{P}^3} & \longrightarrow & 0 \end{array}$$

yields an injection $N_{L/X} \hookrightarrow N_{\bar{L}/\mathbb{P}^3}$. Hence, by taking global sections of this morphism we get $T_F(L) \hookrightarrow T_G(\bar{L})$, where we have put

$$G = \text{Grassmann variety of lines of } \mathbb{P}^3 \ .$$

So, the composite map $F \longrightarrow F_0 \hookrightarrow G$ is an inmersion in the differential-geometric sense, both at L , $iL \in F$. Since in a neighbourhood of $\bar{L}$ it remains exactly (2:1) onto its image F_0 , the latter has to be smooth at $\bar{L}$ and π is étale there, as claimed.

(1.2) If instead $L \subset S$, then F is singular at L .

To see this, we consider the following exact commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & N_{L/S} & \longrightarrow & N_{L/X} & \longrightarrow & N_{S/X} \otimes \mathcal{O}_L & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & T_S \otimes \mathcal{O}_L & \longrightarrow & T_X \otimes \mathcal{O}_L & \longrightarrow & N_{S/X} \otimes \mathcal{O}_L & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & T_L & = & T_L & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & . & & &
\end{array}$$

Being $N_{S/X} \simeq \mathcal{O}_X(2)$ (cf Section 0), it is $N_{S/X} \otimes \mathcal{O}_L \simeq \mathcal{O}_L(2)$. To compute $N_{L/S} \simeq \mathcal{O}_L(L)$, take a 2-plane in $\mathbb{P}^3$ through L; this cuts S along $L + C$ with C a plane cubic, hence we get, on S : $L^2 + 3 = L\cdot(L + C) = 1$. So $L^2 = -2$ and $N_{L/S} \simeq \mathcal{O}_L(-2)$. By the first row above we get therefore $N_{L/X} \simeq \mathcal{O}_L(-d) \oplus \mathcal{O}_L(d)$ with $d = 0,1$ or 2. We claim that $d = 2$ holds in this case. In fact, since $T_L \simeq \mathcal{O}_L(2)$, we have $T_S \otimes \mathcal{O}_L \simeq \mathcal{O}_L(2) \oplus \mathcal{O}_L(-2)$ by the first column above. Next, since the middle row splits by taking $df : T_X \otimes \mathcal{O}_L \longrightarrow T_S \otimes \mathcal{O}_L$, we derive an isomorphism $T_X \otimes \mathcal{O}_L \simeq \mathcal{O}_L(2) \oplus \mathcal{O}_L(2) \oplus \mathcal{O}_L(2)$. From this it follows that $h^0(L,N_{L/X}) = 3$, by using the second column; hence $N_{L/X} \simeq \mathcal{O}_L(-2) \oplus \mathcal{O}_L(2)$ as claimed. In particular we see that $\dim T_F(L) = 3$, and F is singular at L, q.e.d.

From now on, and unless otherwise specified, *we shall assume that* S *contains no lines*. Therefore:

(1.3) COROLLARY. F *and* F_0 *are smooth surfaces and the projection map* $\pi : F \longrightarrow F_0$ *is* (2:1) , *étale*.

The curve Δ of hyperflexes

Among the most relevant curves on F_0 and F we have, respectively, the curve of hyperflexes and the curves of incidence; we put $(L \in F)$:

(1.4) $\Delta \subset F_0$, Δ = curve of hyperflexes of $S \subset \mathbb{P}^3$

(1.5) $D_L \subset F$, D_L = closure, in F, of $\{l \in F \mid l \cap L \neq \emptyset , l \neq L\}$.

We want to study these curves infinitesimally. We start with Δ ; fix a hyperflex $\overline{L}$ of $S \subset \mathbb{P}^3$ and call $\overline{P}$ its (unique) point of contact with S . Let Π be the tangent plane of S at $\overline{P}$. We may choose coordinates in $\mathbb{P}^3$ in such a way that $\Pi = \{X_0 = 0\}$, $\overline{L} = \{X_0 = X_1 = 0\}$ and $\overline{P} = \{X_0 = X_1 = X_3 = 0\}$. The intersection of S with the plane Π is a quartic curve with a singular point at $\overline{P}$ and cutting out on $\overline{L}$ the divisor $4\overline{P}$; therefore it will be given in Π by an equation

$$X_3^4 + A_2(X_1,X_2,X_3)X_1^2 + B_2(X_2,X_3)X_1X_3 = 0 .$$

Hence the equation $\phi_4 = 0$ of S in $\mathbb{P}^3$ reads

$$X_3^4 + A_2(X_1,X_2,X_3)X_1^2 + B_2(X_2,X_3)X_1X_3 + C_3(X_0,X_1,X_2,X_3)X_0 = 0$$

and we put, more explicitly:

$$B = b_0X_2^2 + b_1X_2X_3 + b_2X_3^2$$
$$C = c_0X_2^3 + c_1X_2^2X_3 + c_2X_2X_3^2 + c_3X_3^3 + (\dots)X_0 + (\dots)X_1 .$$

A general (linear) infinitesimal deformation $\overline{L}_\varepsilon$ of $\overline{L}$ inside $\mathbb{P}^3$ is given by equations

$$X_0 = (\alpha_2X_2 + \alpha_3X_3)\varepsilon , \quad X_1 = (\beta_2X_2 + \beta_3X_3)\varepsilon ,$$

with $\varepsilon^2 = 0$ and where $(\alpha_2,\alpha_3,\beta_2,\beta_3) \in \mathbb{C}^4$ can be regarded as coordinates in the vector space $T_G(\overline{L}) \simeq H^0(L,N_{\overline{L}/\mathbb{P}^3})$. The intersection of $\overline{L}_\varepsilon$ with S is given by the 4-form on $\overline{L}_\varepsilon$:

$$\phi_4^\varepsilon = X_3^4 + X_3(\beta_2X_2 + \beta_3X_3)(b_0X_2^2 + b_1X_2X_3 + b_2X_3^2)\varepsilon + \\ + (\alpha_2X_2 + \alpha_3X_3)(c_0X_2^3 + c_1X_2^2X_3 + c_2X_2X_3^2 + c_3X_3^3)\varepsilon .$$

In order that $\overline{L}_\varepsilon$ be a hyperflex of S , one has to have:

$$\phi_4^\varepsilon = (1 + \gamma\varepsilon)(X_3 + \sigma X_2\varepsilon)^4$$

for suitable $\gamma,\sigma \in \mathbb{C}^4$. Equating both expressions we derive:

$$\begin{aligned}
c_3\alpha_3 \qquad\qquad + b_2\beta_3 &= \gamma \\
c_3\alpha_2 + c_2\alpha_3 + b_2\beta_2 + b_1\beta_3 &= 3\sigma \\
c_2\alpha_2 + c_1\alpha_3 + b_1\beta_2 + b_0\beta_3 &= 0 \\
c_1\alpha_2 + c_0\alpha_3 + b_0\beta_2 \qquad &= 0 \\
c_0\alpha_2 \qquad\qquad\qquad &= 0 \quad .
\end{aligned}$$

The last three ones are therefore the equations of the tangent space $T_\Delta(\overline{L}) \subset T_G(\overline{L})$. Remark that $c_0 \neq 0$ by smoothness of S at $\overline{P}$. Hence, in order that the last three equations become dependent, i.e. that Δ be singular at $\overline{L}$, it is necessary and sufficient that $b_0 = b_1 = 0$.

We next count constants. Consider the Schubert variety M of triples $(\overline{P},\overline{L},\Pi)$ where $\overline{P} \in \overline{L} \subset \Pi \subset \mathbb{P}^3$ are a point on a line, in a plane of $\mathbb{P}^3$. Clearly M is smooth of dimension 6 . Let $N \subset M \times \mathbb{P}^{34}$ be the subvariety of pairs $((\overline{P},\overline{L},\Pi),S)$ such that S is a smooth quartic surface of $\mathbb{P}^3$ and $\overline{L}$ is a hyperflex of S with contact point $\overline{P}$, Π is the tangent plane of S at $\overline{P}$ and furthermore the curve of hyperflexes of S is singular at $\overline{L}$. We claim that N has dimension 33 . In fact, if we take $(\overline{P},\overline{L},\Pi) \in M$, the surfaces S occurring in the fibre of the projection map $N \longrightarrow M$ above that triple are given by equations

$$\phi_4 = X_3^4 + A_2(X_1,X_2,X_3)X_1^2 + b_2X_3^3X_1 + C_3(X_0,X_1,X_2,X_3)X_0$$

in a coordinate system as above; hence the fibre has dimension $6+1+20 = 27$, whence $\dim N = 33$. The projection mapping $N \longrightarrow \mathbb{P}^{34}$ fails to be surjective, and we get finally:

(1.6) *If* X *is sufficiently general, the curve of hyperflexes* $\Delta \subset F_0$ *is smooth.*

The incidence curves D_L : infinitesimal study

The infinitesimal study of the curves D_L (cf (1.5)) can be carried out like that of F , by using Grothendieck's deformation theory, since D_L is a Chow component of curves of the threefold $\tilde{X}$ gotten from X by blowing up along $L \subset X$. Alternatively, one may use methods like the above ones applied to the projection of D_L in F_0 , namely

$$D_{\overline{L}} = \text{closure, in } F_0 \text{, of } \{\overline{l} \in F_0 \mid \overline{l} \cap \overline{L} \neq \emptyset , \overline{l} \neq \overline{L}\} .$$

We shall follow the second method but omitting the details, these being similar to the former ones.

Fix $\overline{l} \in D_{\overline{L}}$, $\overline{l} \neq \overline{L}$. We may choose a coordinate system in $\mathbb{P}^3$ such that the line $\overline{l}$ is given by the equations $X_0 = 0$, $X_1 = 0$. The tangent space of G at $\overline{l}$, $T_G(\overline{l}) \simeq H^0(\overline{l}, N_{\overline{l}/\mathbb{P}^3})$, can be identified then (by differentiation of Plücker coordinates) with the vector space of differential operators spanned by $X_i \cdot (\partial/\partial X_j)$, $i = 2,3$, $j = 0,1$, i.e. :

$$T_G(\overline{l}) \simeq \langle X_2 \cdot (\partial/\partial X_0) , X_3 \cdot (\partial/\partial X_0) , X_2 \cdot (\partial/\partial X_1) , X_3 \cdot (\partial/\partial X_1) \rangle .$$

At $\overline{l} \in D_{\overline{L}}$, the curve $D_{\overline{L}}$ is the intersection, in the Grassmann variety G , of F_0 and $V_{\overline{L}}$, the latter being the Schubert variety of lines of $\mathbb{P}^3$ which meet $\overline{L}$. Hence

$$T_{D_{\overline{L}}}(\overline{l}) = T_{F_0}(\overline{l}) \cap T_{V_{\overline{L}}}(\overline{l}) .$$

To describe $T_{F_0}(\overline{l}) \subset T_G(\overline{l})$, introduce the scheme of contact $\gamma \subset \overline{l}$ of $\overline{l}$, given by $T = 0$ on this line. With the above identifications one gets

$$T_{F_0}(\overline{l}) = \{v \in T_G(\overline{l}) \mid v \cdot \phi_4 \text{ yields zero in } H^0(\gamma, \mathcal{O}_\gamma(4))\} .$$

Similarly, if $\psi = 0$ is the 2-plane $\overline{l} \vee \overline{L}$ of $\mathbb{P}^3$:

$$T_{V_{\overline{L}}}(\overline{l}) = \{v \in T_G(\overline{l}) \mid v \cdot \psi \text{ vanishes at } \overline{l} \cap \overline{L}\} .$$

The curve $D_{\overline{L}}$ is singular at $\overline{l}$ if and only if $T_{F_0}(\overline{l})$ is contained in $T_{V_{\overline{L}}}(\overline{l})$. With the descriptions of these spaces this can be worked out in coordinates, getting :

(1.7) If $\overline{l} \in F_0$ is a singular point of the curve $D_{\overline{L}}$, $\overline{l} \neq \overline{L}$, then (at least) one of the following things takes place:

i) $\overline{l}$ and $\overline{L}$ have a point of contact in common.

ii) The plane spanned by $\overline{l}$ and $\overline{L}$ in $\mathbb{P}^3$ is tangent to S at both points of contact of $\overline{l}$ if $\overline{l}$ is a pure bitangent; if $\overline{l}$ is a hyperflex, this plane is tangent to S at the point of contact $\overline{P}$ of $\overline{l}$ and cuts on S a curve having $2\bar{l}$ as tangent cone at $\bar{P}$ or a triple point there.

(1.8) A general choice of $\bar{L}$ allows us to assume that case ii) above doesn't occur; it suffices to avoid the lines of the ∞^1 2-planes which are either tangent to S at the point of contact of a hyperflex or tangent to S at two distinct points at least.

(1.9) Assume for a moment the surface F_0 allowed to be singular. Taking $\bar{L} \in F_0$ smooth, the whole of (1.7) goes through , by adding:

iii) $\bar{l}$ is contained in S .

Completing (1.8) , a general choice of $\bar{L}$ allows us to avoid cases ii) and iii) above, since a smooth quartic S contains at most a finite number of lines.

The infinitesimal study of the incidence curves will be pursued by geometrical means in Section 5 below, where a fairly complete picture of the general case will be given.

2. INFINITESIMAL ABEL-JACOBI MAPPINGS

Ordinary and infinitesimal Abel-Jacobi mappings

We start recalling some well known facts on Abel-Jacobi mappings ([8], [9],[10],[11],[17],[28], etc.) , restricting ourselves to the case of curves on threefolds. Let Y denote a smooth projective threefold; the Griffiths intermediate Jacobian of Y is defined as the complex torus

$$J(Y) = (H^{3,0}(Y) \oplus H^{2,1}(Y))^{\vee} \; / \; H_3(Y)$$

- where $H_3(Y) = H_3(Y,\mathbb{Z})/(\text{torsion})$ is embedded in the above vector space by integration of forms along cycles - , together with the "principal polarization" stemming from the Poincaré pairing on Y (cf [8] , for instance). If Y is a Fano threefold, i.e. ([15]) a threefold with ample anticanonical class, one has in particular $H^{3,0}(Y) = 0$ and $J(Y)$ becomes a principally polarized abelian variety.

The Abel-Jacobi map is a group homomorphism from $\theta(Y)$, the group of algebraic 1-cycles on Y which are homologous to zero, into $J(Y)$. Given $\zeta \in \theta(Y)$, one writes $\zeta = \partial\Gamma$ for a 3-chain Γ of Y and considers the linear form

$$\omega \longmapsto \int_\Gamma \omega$$

on $H^{3,0}(Y) \oplus H^{2,1}(Y)$; the image of the latter in $J(Y)$ is independent of the particular choice of Γ and is, by definition, the image of ζ by the Abel-Jacobi map.

If B is a connected variety parametrizing a family $\{Z_b\}_{b\in B}$ of algebraic 1-cycles on Y, there is an evident map from B to $\theta(Y)$ gotten after the choice of a base point $\beta \in B$. By composition one gets a set--theoretical map from B into $J(Y)$ which is also called Abel-Jacobi map. If B is smooth, this is known to be a morphism of analytic varieties (cf [9], II, p.826, [22], p.9, [17]).

In the latter case it is shown in [9] (II, Theorem 2.25) that, if the family is effective and $Z = Z_0$ is a smooth curve, the differential of the Abel-Jacobi map $B \longrightarrow J(Y)$ at $\beta \in B$ is given by the composition of the characteristic map of Kodaira

$$T_B(\beta) \longrightarrow H^0(Z,N_{Z/Y}) \tag{2.1}$$

with the following one, which we shall call the infinitesimal Abel-Jacobi map at Z :

$$\psi_Z : H^0(Z,N_{Z/Y}) \longrightarrow H^0(Y,\Omega_Y^3)^\vee \oplus H^1(Y,\Omega_Y^2)^\vee = T_{J(Y)}(0) \quad , \tag{2.2}$$

whose transpose

$$\psi_Z^\vee : \Omega^1_{J(Y)}(0) = H^0(Y,\Omega_Y^3) \oplus H^1(Y,\Omega_Y^2) \longrightarrow H^0(Z,N_{Z/Y})^\vee \tag{2.3}$$

is described as follows. On the first summand it is the zero map; on the second one it is the composition of three morphisms:

$$\begin{array}{l} H^1(Y,\Omega_Y^2) \longrightarrow H^1(Z,\Omega_Y^2 \otimes \mathcal{O}_Z) \\ H^1(Z,\Omega_Y^2 \otimes \mathcal{O}_Z) \longrightarrow H^1(Z,\Omega_Z^1 \otimes N_{Z/Y}^\vee) \end{array} \tag{2.4}$$

$$H^1(Z,\Omega^1_Z \otimes N^\vee_{Z/Y}) \longrightarrow H^0(Z,N_{Z/Y})^\vee$$

where the first one is the obvious restriction map and the third one is the Kodaira-Serre duality isomorphism. The second one is derived from the exact sequence

$$0 \longrightarrow \Lambda^2 N^\vee_{Z/Y} \longrightarrow \Omega^2_Y \otimes \mathcal{O}_Z \longrightarrow \Omega^1_Z \otimes N^\vee_{Z/Y} \longrightarrow 0$$

gotten by exterior squaring from the sequence

$$0 \longrightarrow N^\vee_{Z/Y} \longrightarrow \Omega^1_Y \otimes \mathcal{O}_Z \longrightarrow \Omega^1_Z \longrightarrow 0 \ .$$

The vanishing of $\psi^\vee_Z$ on $H^0(Y,\Omega^3_Y)$ reflects the fact that the image of B in $J(Y)$ is contained in the "abelian part" of this torus, i.e. inside

$$H^{2,1}(Y)^\vee \ / \ H_3(Y) \cap (H^{2,1}(Y)^\vee) \quad \subset J(Y) \quad .$$

With the same hypotheses on B (or, more generally, if B is irreducible - by taking first a non singular model of B) the above Abel--Jacobi map $B \longrightarrow J(Y)$ induces a morphism of complex tori

$$\mathrm{Alb}(B) \longrightarrow J(Y)$$

which is again referred to as Abel-Jacobi morphism.

The infinitesimal Abel-Jacobi map, which by the above may be considered as a morphism

$$H^0(Z,N_{Z/Y}) \longrightarrow H^2(Y,\Omega^1_Y) \subset H^3(Y,\mathcal{O}_Y) \oplus H^2(Y,\Omega^1_Y) = T_{J(Y)}(0) \quad ,$$

is far from being well understood. One should compare this with the codimension one case; if W is a n-dimensional variety, the preceding considerations are the analoga of those concerning the Picard variety of W,

$$\underline{\mathrm{Pic}}^o_W \simeq H^{n,n-1}(W)^\vee \ / \ H_{2n-1}(W) \simeq H^{0,1}(W) \ / \ H^1(W) \quad .$$

Here the infinitesimal Abel-Jacobi map for an effective (but otherwise arbitrary) divisor D of W

$$\psi_D : H^0(D,N_{D/W}) \longrightarrow H^1(W,\mathcal{O}_W)$$

is the differential of a well defined morphism from the corresponding component of the Hilbert scheme of W to the Picard variety $\underline{\text{Pic}}^{\text{o}}{}_W$, whose infinitesimal structure at D is given exactly by the first part of the cohomology sequence of the standard sequence

$$0 \longrightarrow \mathcal{O}_W \longrightarrow \mathcal{O}_W(D) \longrightarrow \mathcal{O}_D(D) \longrightarrow 0 \quad ;$$

in particular, the first connecting homomorphism $H^0(D,\mathcal{O}_D(D)) \longrightarrow H^1(W,\mathcal{O}_W)$ coincides with ψ_D .

Back to the case of curves on threefolds, we shall use a method of analyzing (2.3) which is less intrinsic than the above but sufficiently explicit in the cases we are interested in. Namely, given an embedding of Y in a smooth - but non necessarily complete - fourfold, one constructs natural exact sequences involving respectively $H^1(Y,\Omega^2_Y)$ and $H^0(Z,N_{Z/Y})^\vee$ and a mapping between them which extends (the essential part of) $\psi^\vee_Z$. The second one appears to be, in various cases, the pointwise sequence associated with an analogous exact sequence of vector bundles which in the folklore is unanimously called (co)tangent bundle sequence, by generalization of the tangent bundle isomorphism of [8], p.338 .

The method of the pointwise TBS

Assume given an embedding $Y \hookrightarrow W$ in a smooth (non necessarily complete) fourfold, and $Z \subset Y$ a smooth curve. The announced exact sequence involving $H^0(Z,N_{Z/Y})^\vee$ is constructed as follows : First, take the exact sequence of normal bundles

$$0 \longrightarrow N_{Z/Y} \longrightarrow N_{Z/W} \longrightarrow N_{Y/W} \otimes \mathcal{O}_Z \longrightarrow 0 \quad ,$$

and tensor it with Ω^3_Y ; then consider the associated cohomology sequence, and use the canonical isomorphism

$$H^i(Z,N_{Z/Y}\otimes\Omega^3_Y)\simeq H^{1-i}(Z,N_{Z/Y})^\vee \quad ,$$

gotten from the chain of isomorphisms

$$N_{Z/Y}\otimes\Omega^3_Y\simeq N^\vee_{Z/Y}\otimes\Lambda^2 N_{Z/Y}\otimes\Omega^3_Y\simeq N^\vee_{Z/Y}\otimes\Omega^1_Z$$

(the first one coming from $\mathrm{rk}(N_{Z/Y})=2$, and the second one from the adjunction formula) , and using Kodaira-Serre duality. The final result is the exact sequence

$$\begin{aligned}(2.5)\qquad & 0\longrightarrow H^1(Z,N_{Z/Y})^\vee\longrightarrow H^0(Z,N_{Z/W}\otimes\Omega^3_Y)\xrightarrow{\alpha_Z}\\ &\longrightarrow H^0(Z,N_{Y/W}\otimes\Omega^3_Y\otimes\mathcal{O}_Z)\xrightarrow{\beta_Z}H^0(Z,N_{Z/Y})^\vee\longrightarrow\\ &\longrightarrow H^1(Z,N_{Z/W}\otimes\Omega^3_Y)\longrightarrow H^1(Z,N_{Y/W}\otimes\Omega^3_Y\otimes\mathcal{O}_Z)\to 0 \quad .\end{aligned}$$

(2.6) Under the same hypotheses as above, the exact sequence involving the space $H^1(Y,\Omega^2_Y)$ is gotten from the standard one

$$0\longrightarrow N^\vee_{Y/W}\longrightarrow\Omega^1_W\otimes\mathcal{O}_Y\longrightarrow\Omega^1_Y\longrightarrow 0$$

by taking exterior cubes first,

$$0\longrightarrow\Omega^2_Y\otimes N^\vee_{Y/W}\longrightarrow\Omega^3_W\otimes\mathcal{O}_Y\longrightarrow\Omega^3_Y\longrightarrow 0 \quad ,$$

twisting then with $N_{Y/W}$ and taking the cohomology sequence. We are mainly interested in the first connecting homomorphism

$$(2.7)\qquad R : H^0(Y,N_{Y/W}\otimes\Omega^3_Y)\longrightarrow H^1(Y,\Omega^2_Y) \quad .$$

(2.8) <u>LEMMA</u>. *We use the above notations and hypotheses. The following diagram is commutative:*

$$\begin{array}{ccc} H^0(Y,N_{Y/W}\otimes\Omega^3_Y) & \xrightarrow{R} & H^1(Y,\Omega^2_Y)\\ r_Z\downarrow & & \downarrow\psi^\vee_Z\\ H^0(Z,N_{Y/W}\otimes\Omega^3_Y\otimes\mathcal{O}_Z) & \xrightarrow{\beta_Z} & H^0(Z,N_{Z/Y})^\vee \end{array} \quad ,$$

where r_Z *is ordinary restriction.*

PROOF. This follows at once from the definitions and the existence of a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \Omega_Y^2 \otimes \mathcal{O}_Z & \longrightarrow & \Omega_W^3 \otimes N_{Y/W} \otimes \mathcal{O}_Z & \longrightarrow & \Omega_Y^3 \otimes N_{Y/W} \otimes \mathcal{O}_Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \Omega_Y^3 \otimes N_{Z/Y} & \longrightarrow & \Omega_Y^3 \otimes N_{Z/W} & \longrightarrow & \Omega_Y^3 \otimes N_{Y/W} \otimes \mathcal{O}_Z & \longrightarrow & 0
\end{array} ,
$$

with as first vertical arrow the composition of the map

$$\Omega_Y^2 \otimes \mathcal{O}_Z \longrightarrow \Omega_Z^1 \otimes N_{Z/Y}^{\vee} ,$$

used to define (2.4) , with the canonical isomorphism

$$\Omega_Z^1 \otimes N_{Z/Y}^{\vee} \simeq N_{Z/Y} \otimes \Omega_Y^3 .$$

The claimed diagram comes from

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N_{Y/W}^{\vee} \otimes \mathcal{O}_Z & \longrightarrow & N_{Z/W}^{\vee} & \longrightarrow & N_{Z/Y}^{\vee} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N_{Y/W}^{\vee} \otimes \mathcal{O}_Z & \longrightarrow & \Omega_W^1 \otimes \mathcal{O}_Z & \longrightarrow & \Omega_Y^1 \otimes \mathcal{O}_Z & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \Omega_Z^1 & = & \Omega_Z^1 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

by taking exterior cubes; in the so obtained diagram

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & N_{Y/W}^{\vee} \otimes \Lambda^2 N_{Z/Y}^{\vee} & \xrightarrow{\simeq} & \Lambda^3 N_{Z/W}^{\vee} & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & N_{Y/W}^{\vee} \otimes \mathcal{O}_Z \otimes \Omega_Y^2 & \longrightarrow & \Omega_W^3 \otimes \mathcal{O}_Z & \longrightarrow & \Omega_Y^3 \otimes \mathcal{O}_Z & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \simeq & & \\
0 & \longrightarrow & N_{Y/W}^{\vee} \otimes N_{Z/Y}^{\vee} \otimes \Omega_Z^1 & \longrightarrow & \Lambda^2 N_{Z/W}^{\vee} \otimes \Omega_Z^1 & \longrightarrow & \Lambda^2 N_{Z/Y}^{\vee} \otimes \Omega_Z^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} ,
$$

the two lower rows yield the desired result after twisting with $N_{Y/W}$, and using obvious identifications, q.e.d.

(2.9) REMARK. As the proof shows, the above square extends in fact to a morphism between the sequence of (2.6) and the sequence (2.5) .

EXAMPLES.

a) Take $Y = Y_3^3$, $W = \mathbb{P}^4$ (cf [8]) . Here $N_{Y/W} \simeq \mathcal{O}_Y(3)$ and $\Omega_Y^3 \simeq \simeq \mathcal{O}_Y(-2)$; one sees easily that R is an isomorphism. We put (2.5) and (2.7) together, getting for $Z \subset Y$ a smooth curve on the cubic threefold:

(2.10)
$$\begin{array}{ccccccccccc} & & & & H^0\mathcal{O}_{\mathbb{P}^4}(1) & \xrightarrow{\simeq} & \Omega^1_{J(Y)}(0) & & & & \\ & & & & r_Z \downarrow & & \downarrow \psi_Z^{\vee} & & & & \\ 0 \to (H^1N_{Z/Y})^{\vee} \to & H^0N_{Z/\mathbb{P}^4}(-2) & \to & & H^0\mathcal{O}_Z(1) & \xrightarrow{\beta_Z} & (H^0N_{Z/Y})^{\vee} & \to H^1N_{Z/\mathbb{P}^4}(-2) & \to H^1\mathcal{O}_Z(1) \to 0 \,. \end{array}$$

b) Consider our case $Y = X$, $W = E$ (cf Section 0) . Here (ibid) one has $N_{X/E} \simeq \mathcal{O}_X(4)$, $\Omega_X^3 \simeq \mathcal{O}_X(-2)$; we claim that (2.5) and (2.7) yield here, for a smooth curve $Z \subset X$:

(2.11)
$$\begin{array}{ccccccccccc} & & & & H^0\mathcal{O}_{\mathbb{P}^3}(2) & \xrightarrow{\simeq} & \Omega^1_{J(X)}(0) & & & & \\ & & & & r_Z \downarrow & & \downarrow \psi_Z^{\vee} & & & & \\ 0 \to (H^1N_{Z/X})^{\vee} \to & H^0N_{Z/E}(-2) & \to & & H^0\mathcal{O}_Z(2) & \xrightarrow{\beta_Z} & (H^0N_{Z/X})^{\vee} & \to H^1N_{Z/E}(-2) & \to H^1\mathcal{O}_Z(2) \to 0 \,. \end{array}$$

Only the square needs to be explained; it is gotten from (2.8) by composition with $H^0\mathcal{O}_{\mathbb{P}^3}(2) \hookrightarrow H^0\mathcal{O}_X(2)$. Furthermore, the sequence of (2.6) yields in this case

$$0 \longrightarrow H^0(X,\Omega_E^3 \otimes \mathcal{O}_X(4)) \longrightarrow H^0(X,\mathcal{O}_X(2)) \xrightarrow{R} H^1(X,\Omega_X^2) \longrightarrow 0 \ ,$$

for, by using (0.1) and (0.3) , one has $H^1(X,\Omega_E^3 \otimes \mathcal{O}_X(4)) = 0$. The injection of the above sequence can be identified with the natural map

$$H^0(X,(f^*\Omega^3_{\mathbb{P}^3})(4)) \longrightarrow H^0(X,\Omega_X^3(4)) \ ,$$

from which one easily deduces that the image of the former one is the subspace $\langle T\rangle \subset H^0(X,\mathcal{O}_X(2))$. The result now follows, since

$H^0(X,\mathcal{O}_X(2)) = \langle T\rangle \oplus H^0\mathcal{O}_{\mathbb{P}^3}(2)$.

(2.12) The following proposition is an application of (2.11) above. For expository reasons, at this point - and only in connection with this proposition - we make the assumption that F is connected. This will be proved to be true in (3.57) (cf also (3.58)) . The reader will notice that (2.13) is not used in the sequel, before the proof of the connectedness of F , in (3.57) .

(2.13) PROPOSITION.

i) Let $L \in F$ *be a line in* X *and consider the Abel-Jacobi map* $F \longrightarrow J(X)$ *(with arbitrary base point* $L_0 \in F$*). The codifferential of this map followed by translation to the origin of* $J(X)$ *is described by the following commutative diagram:*

$$(2.14)\qquad \begin{array}{ccc} H^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(2)) & \xrightarrow{\simeq} & \Omega^1_{J(X)}(0) \\ \downarrow & & \downarrow \\ H^0(L,\mathcal{O}_L(2))/\langle T|L\rangle & \xrightarrow{\simeq} & \Omega^1_F(L) \end{array} ,$$

the left hand side arrow being the obvious map. In particular, $F \longrightarrow J(X)$ *is an inmersion in the differential-geometric sense.*

ii) The Abel-Jacobi map $F^5 \longrightarrow J(X)$ *is onto, hence generically finite to one. In particular, the Abel-Jacobi map* $\mathrm{Alb}(F) \longrightarrow J(X)$ *is surjective.*

PROOF. i) Consider (2.11) with $Z = L$. As shown in the proof of (1.1), $H^1N_{L/X} = 0$. On the other side $H^1N_{L/E}(-2) = 0$, which is seen as follows: by means of the standard sequence $0 \to \mathcal{O}_E(2) \to T_E \to f^*T_{\mathbb{P}^3} \to 0$, we construct the diagram (identifying L with its image $\overline{L}$ in $\mathbb{P}^3$)

$$\begin{array}{ccccccccc} & & & & 0 & & 0 & & \\ & & & & \downarrow & & \downarrow & & \\ & & & & \mathcal{O}_L(2) & = & \mathcal{O}_L(2) & & \\ & & & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_L & \longrightarrow & T_E \otimes \mathcal{O}_L & \longrightarrow & N_{L/E} & \longrightarrow & 0 \\ & & \simeq\downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & T_{\overline{L}} & \longrightarrow & T_{\mathbb{P}^3} \otimes \mathcal{O}_{\overline{L}} & \longrightarrow & N_{\overline{L}/\mathbb{P}^3} & \longrightarrow & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array} .$$

The third column yields, after tensoring with $\mathcal{O}_L(-2)$,

$$0 \longrightarrow \mathcal{O}_L \longrightarrow N_{L/E}(-2) \longrightarrow \mathcal{O}_L(-1) \oplus \mathcal{O}_L(-1) \longrightarrow 0 \quad ,$$

from whose cohomology sequence our claim follows at once.

By (2.9) and the proof of (2.11) we get a morphism of exact sequences

$$(2.15) \qquad \begin{array}{ccccccccc} 0 & \longrightarrow & \langle T\rangle & \longrightarrow & H^0(X,\mathcal{O}_X(2)) & \longrightarrow & H^1(X,\Omega_X^2) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \langle T|L\rangle & \longrightarrow & H^0(L,\mathcal{O}_L(2)) & \longrightarrow & H^0(L,N_{L/X})^{\vee} & \longrightarrow & 0 \end{array}$$

giving the desired diagram, since the characteristic map (2.1) is the identity in this case. The remainder of i) is clear, by the surjectivity of the vertical arrows in (2.14) .

ii) By i) , the kernel of the codifferential of $F^5 \longrightarrow J(X)$ at $\underline{L} = (L_1,\ldots,L_5) \in F^5$ consists of the 2-forms on $\mathbb{P}^3$ vanishing at the 10-ple of contact of the bitangent lines on which project these L_i's . It suffices to show that this kernel vanishes for at least one 5-tuple of bitangents; taking for example three of these in a general plane and two more in general position we are done.

The surjectivity of $\mathrm{Alb}(F) \longrightarrow J(X)$ is now clear. It could have been gotten alternatively from the commutative diagram

$$(2.16) \qquad \begin{array}{ccc} H^1(X,\Omega_X^2) & \xrightarrow{\simeq} & \Omega^1_{J(X)}(0) \\ \downarrow & & \downarrow \\ H^0(F,\Omega_F^1) & \longrightarrow & \Omega_F^1(L) \end{array} \quad ,$$

the left hand side being the corresponding cotangent map at the origin. Its injectivity then comes from the fact that there are obviously no quadrics in $\mathbb{P}^3$ containing all the points of contact of all bitangent lines to $S \subset \mathbb{P}^3$, q.e.d.

3. THE SURFACES F , F_0 : GLOBAL STUDY

Preliminaries

Write $M \subset \mathbb{P}^3 \times G$ the natural correspondence between $\mathbb{P}^3$ and the Grassmann variety of lines on $\mathbb{P}^3$, consisting on the pairs $(\overline{P},\overline{L})$ with $\overline{P} \in \overline{L}$. The variety M is a $\mathbb{P}^2$-bundle over $\mathbb{P}^3$ and a $\mathbb{P}^1$-bundle over G. We denote the corresponding projections by p and q respectively. Writing

$$(3.1) \qquad 0 \longrightarrow K \longrightarrow \mathcal{O}^4_{\mathbb{P}^3} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow 0$$

the standard sequence of $\mathbb{P}^3$, we may identify $(M,p) = \mathbb{P}K$. Also, if we let

$$(3.2) \qquad R \longrightarrow G$$

be the rank two bundle on G with fibre at $\overline{L} \in G$ the linear forms on $\overline{L}$, $H^0(\overline{L},\mathcal{O}_{\overline{L}}(1))$, we identify $(M,q) = \mathbb{P}R$. The fundamental sheaf of $\mathbb{P}R$ can be written as

$$(3.3) \qquad \mathcal{O}_{\mathbb{P}R}(1) \simeq p^*\mathcal{O}_{\mathbb{P}^3}(1) \quad ,$$

and the fundamental sequence of $\mathbb{P}R$ yields

$$(3.4) \qquad 0 \longrightarrow \mathcal{O}_{\mathbb{P}K}(1) \longrightarrow R_M \longrightarrow p^*\mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow 0$$

(cf e.g. [1] for more details).

Take next the standard sequence of differentials

$$0 \longrightarrow \mathcal{O}_S(-4) \longrightarrow \Omega^1_{\mathbb{P}^3} \otimes \mathcal{O}_S \longrightarrow \Omega^1_S \longrightarrow 0$$

and twist it by $\mathcal{O}_S(1)$; we get, putting

$$(3.5) \qquad H = \Omega^1_S \otimes \mathcal{O}_S(1) \quad ,$$

the sequence

$$(3.6) \qquad 0 \longrightarrow \mathcal{O}_S(-3) \longrightarrow K_S \longrightarrow H \longrightarrow 0 \quad .$$

The latter provides us with an embedding $\mathbb{P}H \hookrightarrow \mathbb{P}K_S$ of projective bundles over S ; one checks easily that these subvarieties of M can be described respectively as follows:

$$\mathbb{P}K_S = \{(\overline{P},\overline{L}) \in M \mid \overline{P} \in S\} \quad ,$$
$$\mathbb{P}H = \{(\overline{P},\overline{L}) \in M \mid \overline{L} \text{ is tangent to } S \text{ at } \overline{P}\} \quad .$$

The following diagram gives a survey of the varieties we shall be mainly concerned with in this section:

$$(3.7) \qquad \begin{array}{ccccc}
 & & & & F \\
 & & & & \downarrow \pi \\
B & \hookrightarrow & \mathbb{P}R_{F_0} & \xrightarrow{q} & F_0 \\
\downarrow & & \downarrow & & \downarrow \\
 & & \mathbb{P}R & \xrightarrow{q} & G \\
\downarrow & & \| & & \\
\mathbb{P}H & \hookrightarrow & \mathbb{P}K & & \\
\downarrow p & & \downarrow p & & \\
S & \hookrightarrow & \mathbb{P}^3 & &
\end{array} \qquad (\mathbb{P}K = M = \mathbb{P}R)$$

where we have put

$$B = \{(\overline{P},\overline{L}) \in M \mid \overline{L} \text{ is a bitangent of } S \text{ and } \overline{P} \text{ is a point of contact}\}.$$

This is a (2:1) covering of F_0 , branched above the curve of hyperflexes $\Delta \subset F_0$. By (1.6) , the surface B is smooth if X is sufficiently general. Our way of carrying out, later on in this section, cohomological computations on F consists in dropping things to F_0 , then lifting them to B and using finally the embeddings $B \hookrightarrow \mathbb{P}H \hookrightarrow \mathbb{P}K$. The first two steps use systematically (0.8) ; we need therefore the classes in $\mathrm{Pic}(F_0)$ giving these coverings (cf (0.7)) .

Let $Q_{F_0} \subset S^2R_{F_0}$ be the bundle of equations of the pairs of contact of the bitangents inside these (cf (0.9)) . Equivalently, by using the embedding $B \hookrightarrow \mathbb{P}R_{F_0}$, where B intersects each fibre of $\mathbb{P}R_{F_0} \xrightarrow{q} F_0$ along its pair of contact,

$$(3.8) \qquad Q_{F_0} = R^0 q_* \mathcal{O}_{\mathbb{P}R_{F_0}}(2-B) = \mathrm{Ker}(S^2 R_{F_0} \longrightarrow R^0 q_* \mathcal{O}_B(2)) \ .$$

We define also

$$(3.9) \qquad \kappa \in \mathrm{Pic}(F_0) \ , \quad \kappa = c_1 Q_{F_0} \ ,$$
$$(3.10) \qquad \rho \in \mathrm{Pic}(F_0) \ , \quad \rho = c_1 R_{F_0} \ .$$

(3.11) PROPOSITION.

a) The étale covering $\pi : F \longrightarrow F_0$ *is given by* $\kappa \in {}_2\mathrm{Pic}(F_0)$.

b) The branched covering $q : B \longrightarrow F_0$ *is given by* $\rho + \kappa \in \mathrm{Pic}(F_0)$.

PROOF. a) The bundle $Q_{F_0} \otimes Q_{F_0} \subset S^4 R_{F_0}$ has an everywhere non vanishing section (because S doesn't contain lines, cf p.16) , given by the 4-form ϕ_4 . Thus the subvariety $Y \subset Q_{F_0}$:

$$Y = \{ \omega_{\bar{L}} \in Q_{F_0} \mid \omega_{\bar{L}}^2 = \phi_4 | \bar{L} \ \text{ in } \ Q_{F_0} \otimes Q_{F_0} \subset S^4 R_{F_0} \}$$

yields an étale (2:1) covering of F_0 with class κ . The following map is now easily seen to be an isomorphism between F and Y over F_0 , settling thereby part a) ; if $L \in F$ and $\bar{L} \in F_0$ is its projection in $\mathbb{P}^3$, we map L into the image of $T|L$ in $H^0(\bar{L}, \mathcal{O}_{\bar{L}}(2))$.

b) This follows at once from the definition of Q_{F_0} above together with (0.10) , and the fact that $2\kappa = 0$, q.e.d.

Notice that it is not clear, a priori, that $\kappa \neq 0$; this will follow later on, in (3.56) .

Another preliminary result will be needed, too. We define first:

$$(3.12) \qquad h \in \mathrm{Pic}(\mathbb{P}H) \ , \quad h = c_1 p^* \mathcal{O}_S(1) \ ,$$
$$(3.13) \qquad \sigma \in \mathrm{Pic}(\mathbb{P}H) \ , \quad \sigma = c_1 \mathcal{O}_{\mathbb{P}H}(1) \ .$$

(3.14) PROPOSITION. *In* $\mathrm{Pic}(\mathbb{P}H)$, $[B] = 6\sigma + 2h$ *holds.*

We give two different proofs of this fact.

1[st] PROOF. We get B as scheme of zeros of a suitable line bundle on $\mathbb{P}H$:

a line of $\mathbb{P}^3$ parametrized by $\mathbb{P}H$ meets S at 4 points, twice the point of contact plus a residual pair. The latter one is a quadric of dimension 0 and we ask for the locus where it degenerates.

Let $U \subset S^2R_{\mathbb{P}H}$ be the bundle of equations of the residual pairs on their respective lines (cf (0.9)). Let $V \subset R_{\mathbb{P}H}$ be the bundle of equations on the lines of their point of contact with S. Clearly:

$$U \otimes V \otimes V \simeq \mathcal{O}_{\mathbb{P}H} \cdot \phi_4 \simeq \mathcal{O}_{\mathbb{P}H} .$$

To compute V, we notice that it is the restriction to $\mathbb{P}H$ of the bundle of equations of the tautological section of $\mathbb{P}R_M$. By (3.4) the latter bundle equals $\mathcal{O}_{\mathbb{P}K}(1)$, hence $V = \mathcal{O}_{\mathbb{P}H}(1) = \mathcal{O}_{\mathbb{P}H}(\sigma)$. Therefore $U = \mathcal{O}_{\mathbb{P}H}(-2\sigma)$.

Thinking now of $U \subset S^2R_{\mathbb{P}H}$ as the equation of a 0-dimensional quadric on a moving line, we have a natural morphism $U \longrightarrow R_{\mathbb{P}H} \otimes R_{\mathbb{P}H}$ giving the matrix form of these equations ; equivalently, a morphism

$$\Phi : U \otimes R^{\vee}_{\mathbb{P}H} \longrightarrow R_{\mathbb{P}H} .$$

The locus $B \subset \mathbb{P}H$ where the quadrics degenerate is given by

$$\Lambda^2\Phi = 0 ,$$

hence it is the scheme of zeros of a section of the bundle

$$\underline{\mathrm{Hom}}(\Lambda^2(U \otimes R^{\vee}_{\mathbb{P}H}), \Lambda^2R_{\mathbb{P}H}) \simeq U^{\vee} \otimes U^{\vee} \otimes \Lambda^2R_{\mathbb{P}H} \otimes \Lambda^2R_{\mathbb{P}H} .$$

Hence $[B] = -2c_1(U) + 2c_1(R_{\mathbb{P}H}) = 6\sigma + 2h$, by using the relation $c_1(R_{\mathbb{P}H}) = \sigma + h$ (cf (3.4)).

<u>2^d PROOF</u>. For a general quartic surface S in $\mathbb{P}^3$, $\mathrm{Pic}(S) \simeq \mathbb{Z}$ holds (by Noether's theorem, [16], p.108 ; here the term 'general' means: outside a countable union of proper subvarieties of the $\mathbb{P}^{34}$ parametrizing all quartic surfaces in $\mathbb{P}^3$). It will suffice to prove the formula in this case. Since $\mathrm{Pic}(\mathbb{P}H) = \mathrm{Pic}(S) \oplus \mathbb{Z}\sigma$, we have $\mathrm{Pic}(\mathbb{P}H) = \mathbb{Z}h \oplus \mathbb{Z}\sigma$. So, $[B] = nh + m\sigma$ with $m,n \in \mathbb{Z}$.

We use the elementary facts that through a general point of S there pass 6 bitangents to S and that a general plane of $\mathbb{P}^3$ contains 28 bitangents . Intersection of B with a fibre of $\mathbb{P}H$ gives:

$$6 = [B]\cdot p^*(x) = n(h\cdot p^*(x)) + m(\sigma\cdot p^*(x)) = m \quad .$$

Secondly, if $C \subset S$ is a general plane section, the variety of pairs $(\overline{P},\overline{L})$ with $\overline{P} \in C$ and $\overline{L}$ the tangent line to C at $\overline{P}$ yields a section of $\mathbb{P}H$ above C, according to the diagram

$$\begin{array}{ccc} \mathbb{P}(\Omega^1_C \otimes \mathcal{O}_C(1)) & \hookrightarrow & \mathbb{P}H \\ \simeq\downarrow & & \downarrow \\ C & \hookrightarrow & S \end{array} \quad .$$

We look for the intersection number $[\mathbb{P}(\Omega^1_C \otimes \mathcal{O}_C(1))]\cdot[B]$. On one side, this is the number of pairs $(\overline{P},\overline{L})$ with $\overline{L}$ bitangent to C, hence yields $2\cdot 28 = 56$. Secondly, $[\mathbb{P}(\Omega^1_C \otimes \mathcal{O}_C(1))]\cdot h$ is the number of tangent lines to C at all of the 4 points of intersection of C with a general plane, thus it equals 4 . Furthermore, since the section corresponds to a projective subbundle of $\mathbb{P}H_C$, we have

$$\mathcal{O}_{\mathbb{P}(\Omega^1_C \otimes \mathcal{O}_C(1))}(1) \simeq \mathcal{O}_{\mathbb{P}H}(1) \otimes \mathcal{O}_{\mathbb{P}(\Omega^1_C \otimes \mathcal{O}_C(1))} \quad ,$$

and the intersection number $[\mathbb{P}(\Omega^1_C \otimes \mathcal{O}_C(1))]\cdot\sigma$ is the degree of this sheaf. The section being a projective bundle of projective dimension 0 , its fundamental sequence degenerates into

$$(\Omega^1_C \otimes \mathcal{O}_C(1))_{\mathbb{P}(\Omega^1_C \otimes \mathcal{O}_C(1))} \simeq \mathcal{O}_{\mathbb{P}(\Omega^1_C \otimes \mathcal{O}_C(1))}(1) \quad ;$$

therefore $\deg(\Omega^1_C \otimes \mathcal{O}_C(1)) = 2p_a(C) - 2 + 4 = 8$. Putting it all together, we have gotten

$$56 = 4n + 8m \quad ,$$

hence $n = 2$, q.e.d.

<u>The tangent bundle sequences (TBS)</u>

Our next step is to globalize (2.5) to the Fano surface F . While

(2.5) is thought to analyze the infinitesimal Abel-Jacobi map, its global version (the tangent bundle sequence) is used to get information about the global geometry of the Chow variety of curves itself.

(3.15) PROPOSITION. (TBS) *There are natural exact sequences*

$$\text{i) on } F : \quad 0 \longrightarrow \mathcal{O}_F \xrightarrow{\alpha} S^2R_F \longrightarrow \Omega^1_F \longrightarrow 0 \quad ,$$

$$\text{ii) on } F_0 : \quad 0 \longrightarrow Q_{F_0} \xrightarrow{\alpha_0} S^2R_{F_0} \longrightarrow \Omega^1_{F_0} \otimes Q_{F_0} \longrightarrow 0 \quad ,$$

α *being multiplication by* $T \in H^0(F, S^2R_F)$ *and* α_0 *being the natural inclusion map (cf* (3.8)*).*

PROOF. i) This is a globalized copy of the proof of (2.5). We start with the inclusion $X \hookrightarrow E$ (cf Section 0) and consider the following situation, fibered above F :

$$\begin{array}{ccccc} \mathbb{P}R_F & \hookrightarrow & X_F & \hookrightarrow & E_F \\ & \searrow & \downarrow & \swarrow q & \\ & & F & & \end{array}$$

where $E_F = E \times F$, $X_F = X \times F$ and (recall) $\mathbb{P}R_F$ is the universal line on X . Next, we take the exact sequence of normal bundles, twist it by the relative dualizing sheaf $\omega = \omega_{X_F/F}$ and take the exact sequence of higher direct images; this gives, on F :

$$\begin{aligned} 0 &\longrightarrow R^0q_*(N_{\mathbb{P}R_F/X_F} \otimes \omega) \longrightarrow R^0q_*(N_{\mathbb{P}R_F/E_F} \otimes \omega) \longrightarrow \\ &\longrightarrow R^0q_*(N_{X_F/E_F} \otimes \mathcal{O}_{\mathbb{P}R_F} \otimes \omega) \longrightarrow R^1q_*(N_{\mathbb{P}R_F/X_F} \otimes \omega) \longrightarrow \\ &\longrightarrow R^1q_*(N_{\mathbb{P}R_F/E_F} \otimes \omega) \longrightarrow R^1q_*(N_{X_F/E_F} \otimes \mathcal{O}_{\mathbb{P}R_F} \otimes \omega) \longrightarrow 0 \quad . \end{aligned} \tag{3.16}$$

The bundle $N_{\mathbb{P}R_F/X_F}$ being of rank 2 , one has

$$N_{\mathbb{P}R_F/X_F} \simeq N^\vee_{\mathbb{P}R_F/X_F} \otimes \Lambda^2 N_{\mathbb{P}R_F/X_F}$$

and hence

$$N_{\mathbb{P}R_F/X_F} \otimes \omega_{X_F/F} \simeq N^\vee_{\mathbb{P}R_F/X_F} \otimes \Lambda^2 N_{\mathbb{P}R_F/X_F} \otimes \omega_{X_F/F} \simeq N^\vee_{\mathbb{P}R_F/X_F} \otimes \omega_{\mathbb{P}R_F/F} \quad .$$

By duality one has therefore, for $i = 1,2$:

$$R^i q_*(N_{\mathbb{P}R_F/X_F} \otimes \omega) \simeq [R^{1-i} q_*(N_{\mathbb{P}R_F/X_F})]^v ,$$

hence finally, by the fact that $H^1(L, N_{L/X}) = 0$ for all $L \in F$ (cf the proof of (1.1)) and Grothendieck's deformation theory ([14]) :

$$(3.17) \qquad R^0 q_*(N_{\mathbb{P}R_F/X_F} \otimes \omega) = 0 \quad , \quad R^1 q_*(N_{\mathbb{P}R_F/X_F} \otimes \omega) = \Omega^1_F \quad .$$

Secondly, since $H^1(L, N_{L/E}(-2)) = 0$ for all $L \in F$ (cf the proof of (2.13.i)) we have

$$(3.18) \qquad R^1 q_*(N_{\mathbb{P}R_F/X_F} \otimes \omega) = 0 \quad ,$$

and, using $N_{X_F/E_F} = \mathcal{O}_{X_F}(4h)$, $\omega_{X_F/F} = \mathcal{O}_{X_F}(-2h)$ together with (3.12) and (3.3) ,

$$(3.19) \qquad R^0 q_*(N_{X_F/E_F} \otimes \mathcal{O}_{\mathbb{P}R_F} \otimes \omega) = S^2 R_F \quad .$$

Putting (3.16) - (3.19) together we obtain an exact sequence

$$0 \longrightarrow R^0 q_*(N_{\mathbb{P}R_F/E_F} \otimes \omega) \longrightarrow S^2 R_F \longrightarrow \Omega^1_F \longrightarrow 0 \quad ,$$

whose pointwise fibre sequence at $L \in F$ is given by the bottom row of (2.15) . The first term being an invertible sheaf with an everywhere non vanishing section T (ibid) , it has to be

$$R^0 q_*(N_{\mathbb{P}R_F/E_F} \otimes \omega) \simeq \mathcal{O}_F \quad ,$$

and this finishes the proof of i) .

ii) We have the identifications

$$\begin{array}{ccccccc} 0 \longrightarrow & \mathcal{O}_F \cdot T & \longrightarrow & S^2 R_F & \longrightarrow & \Omega^1_F & \longrightarrow 0 \\ & \downarrow \simeq & & \simeq \downarrow & & \simeq \downarrow & \\ & \pi^* Q_{F_0} & \underset{\pi^*\alpha_0}{\hookrightarrow} & \pi^* S^2 R_{F_0} & & \pi^* \Omega^1_{F_0} & . \end{array}$$

Moreover, $\mathrm{Hom}(\pi^* S^2 R_{F_0} , \pi^* \Omega^1_{F_0}) = \mathrm{Hom}(S^2 R_{F_0} , \Omega^1_{F_0}) \oplus \mathrm{Hom}(S^2 R_{F_0} , \Omega^1_{F_0} \otimes Q_{F_0})$, by (0.8) , (3.11) , this being the decomposition in invariant and antiinvariant morphisms. Call

$$\overline{\beta} : \pi^* S^2 R_{F_0} \longrightarrow \pi^* \Omega^1_{F_0}$$

the composite arrow in the diagram above. We shall show below (cf (3.21)) that $\overline{\beta}$ is antiinvariant, hence $\overline{\beta}$ is the inverse image of a well defined morphism $\beta_0 : S^2 R_{F_0} \longrightarrow \Omega^1_{F_0} \otimes Q_{F_0}$. The sequence

$$0 \longrightarrow Q_{F_0} \xrightarrow{\alpha_0} S^2 R_{F_0} \xrightarrow{\beta_0} \Omega^1_{F_0} \otimes Q_{F_0} \longrightarrow 0$$

is exact, since so is its inverse image by π , q.e.d.

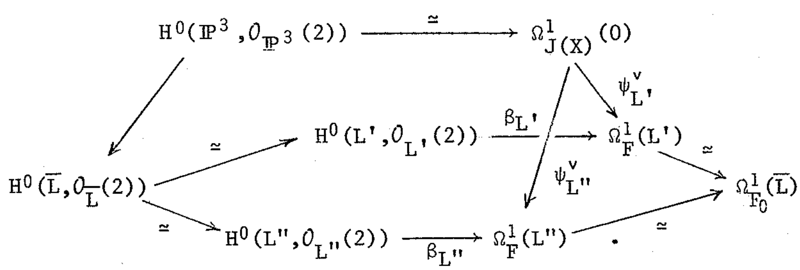
(3.20) REMARK. Part ii) above can be restated by saying that the direct image of the sequence i) by π ,

$$0 \longrightarrow \mathcal{O}_{F_0} \oplus Q_{F_0} \longrightarrow S^2 R_{F_0} \oplus (S^2 R_{F_0} \otimes Q_{F_0}) \longrightarrow \Omega^1_{F_0} \oplus (\Omega^1_{F_0} \otimes Q_{F_0}) \longrightarrow 0 ,$$

decomposes into two sequences, namely ii) and its twist by Q_{F_0} .

(3.21) LEMMA. *The morphism* $\overline{\beta}$ *above is antiinvariant.*

PROOF. Let $\overline{L} \in F_0$ and call L' , $L'' \in F$ the two lines in X above $\overline{L}$. Consider the diagram



The claim $i\overline{\beta} = -\overline{\beta}$ amounts to say that tha bottom hexagon is anticommutative. By (2.14) , the face $\Omega^1_{J(X)}(0)$, $\Omega^1_F(L'')$, $H^0(L'',\mathcal{O}_{L''}(2))$, $H^0(\overline{L},\mathcal{O}_{\overline{L}}(2))$, $H^0(\mathbb{P}^3,\mathcal{O}_{\mathbb{P}^3}(2))$ is commutative, and similarly for the opposite one. Hence all is reduced to show the anticommutativity of the face involving the four vertices $\Omega^1_{J(X)}(0)$, $\Omega^1_F(L')$, $\Omega^1_F(L'')$, $\Omega^1_{F_0}(\overline{L})$. Otherwise said, the sum of both composite arrows of that face has to be zero; this sum is the cotangent map at $\overline{L}$ of the mapping $F_0 \longrightarrow J(X)$ attaching to each $\overline{L} \in F_0$ the sum of the images of L' and L'' under the Abel-Jacobi map for F. Since the 1-cycles $L' + L''$ are all rationally equivalent in X as $\overline{L}$ describes F_0, the above map is constant and so its codifferential is zero, q.e.d.

Numerical data

We shall get next some numerical results about F and F_0. Recall the already defined classes (cf (3.10), (3.12), (3.13), (0.6.d)) :

$$\rho = c_1(R) \in \mathrm{Pic}(G) \ ,$$
(3.22) $$h = c_1(p^*\mathcal{O}_{\mathbb{P}^3}(1)) \in \mathrm{Pic}(M) \ ,$$
$$\sigma = c_1(\mathcal{O}_{\mathbb{P}K}(1)) \in \mathrm{Pic}(M) \ .$$

We put moreover

(3.23) $$\Pi = c_2(R) \in CH^2(G)$$

and notice that, on G,

(3.24) ρ = class of { lines in $\mathbb{P}^3$ meeting a given line } ,
(3.25) Π = class of { lines in $\mathbb{P}^3$ lying on a given 2-plane } .

Putting also

(3.26) $p \in CH^2(G)$, p = class of { lines in $\mathbb{P}^3$ through a given point } ,

there is a well known relation in G :

(3.27) $$\rho^2 = \Pi + p \ .$$

Furthermore, from (3.4) we get, on M,

(3.28) $$\rho = \sigma + h \ ,$$
(3.29) $$\Pi = \sigma \cdot h \ .$$

We need a set of basic formulae on F_0 :

(3.30) <u>LEMMA</u>. *The following holds in* F_0 *:*

$$\deg(\rho^2) = 40 \quad , \quad \deg(\Pi) = 28 \quad , \quad \deg(p) = 12 \quad .$$

<u>PROOF</u>. By (3.27) it suffices to compute $\deg(\rho^2)$ and $\deg(\Pi)$. For all $\xi \in CH^2(G)$ one has (cf (3.7)) :

$$\deg(\xi | F_0) = \tfrac{1}{2}([B]\cdot\xi)_{\mathbb{P}H} \quad .$$

To work in $\mathbb{P}H$ we use

$$CH(\mathbb{P}H) = CH(S)[\sigma] \;/\; (\sigma^2 - c_1(H)\sigma + c_2(H)) \quad ,$$

together with $c(H) = 1 + 2h + 7h^2$ (cf (3.6)) . Using the relations $\deg(h^3) = 0$, $\deg(\sigma h^2) = 4$, we get in this way two other ones: $\deg(\sigma^2 h) =$ $= 8$, $\deg(\sigma^3) = -12$.

The result now follows from these data together with (3.14) , (3.28) and (3.29) , q.e.d.

(Of course, there are several more elementary ways to get these formulae.)

In order to introduce the following important algebraic equivalence class on F , we assume for a moment that F is connected . This assumption ranges until (3.33) below, including also the statement in brackets in Proposition (3.34) and the conclusion concerning the curves D_L in (3.36) . We do so (again) by expository reasons, observing that neither of these facts is used before the connectedness proof of F in (3.57). We put

(3.31) $\quad \upsilon \in NS^1(F) \quad , \quad \upsilon = [D_L] \quad .$

Clearly, in $NS^1(F)$ the relation $\rho = 2\upsilon$ holds. From this one derives

(3.32) $\quad \deg(\upsilon^2) = 20 \quad .$

(3.33) Therefore, through a general point of X there pass 12 lines; two lines of X in general position meet 20 other lines. Notice that, if $x \in S$ is general, the 12 lines through x are pairwise mapped onto the

6 bitangents of S through x (Figure 2) .

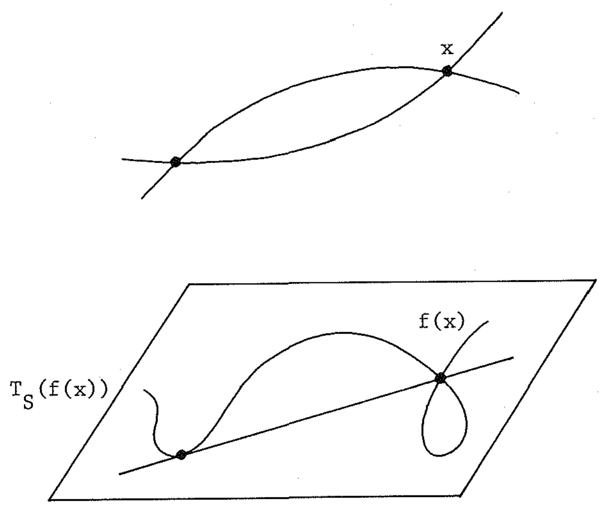


Figure 2.

(3.34) PROPOSITION. *In* F *we have the formulae :*

$K_F = 3\rho$ in $\mathrm{Pic}(F)$ $[= 6\upsilon$ in $NS^1(F)]$,
$E(F) = 384$,
$\chi(\mathcal{O}_F) = 92$,
$K_F^2 = 720$.

PROOF. From the TBS (3.15.i) we get

$$c(\Omega_F^1) = c(S^2 R_F) = 1 + 3\rho + (2\rho^2 + 4\Pi)$$

(cf (3.22) , (3.23)) . From this K_F , $E(F)$ and hence K_F^2 follow, by using (3.30) (and $\deg(\pi) = 2$) . The Noether formula then yields $\chi(\mathcal{O}_F)$, q.e.d.

(3.35) PROPOSITION. *In* F_0 *we have the formulae :*

$K_{F_0} = 3\rho + \kappa$ in $\mathrm{Pic}(F_0)$,
$E(F_0) = 192$,
$\chi(\mathcal{O}_{F_0}) = 46$,
$K_{F_0}^2 = 360$.

PROOF. From (3.15.ii) one has

$$c_1(\Omega^1_{F_0}) = c_1(S^2R_{F_0}) + 3c_1(Q_{F_0}) = 3\rho + \kappa \quad ,$$

since $3\kappa = \kappa$. The remainder is now clear, e.g. from (3.34) , q.e.d.

(3.36) Furthermore, the virtual genus of D_L yields, by the adjunction formula, $p_a(D_L) = 71$. In a similar way, the curve of hyperflexes $\Delta \subset F_0$ yielding $[\Delta] = 2\rho$ in $\mathrm{Pic}(F_0)$ by (3.11) (cf (0.7)) , one has $p_a(\Delta) =$ $= 201$.

Cohomological study

We turn to the cohomological computations on F and F_0. As said before, we use (0.8) and (3.11) to reduce things to questions on B ; by (3.14) we reconduct them to $\mathbb{P}H$ and, by the exact sequences (cf (3.6))

$$(3.37) \qquad 0 \longrightarrow \mathcal{O}_{\mathbb{P}K_S}(-3h-\sigma) \longrightarrow \mathcal{O}_{\mathbb{P}K_S} \longrightarrow \mathcal{O}_{\mathbb{P}H} \longrightarrow 0 \quad ,$$

$$(3.38) \qquad 0 \longrightarrow \mathcal{O}_{\mathbb{P}K}(-4h) \longrightarrow \mathcal{O}_{\mathbb{P}K} \longrightarrow \mathcal{O}_{\mathbb{P}K_S} \longrightarrow 0 \quad ,$$

we are finally concerned with cohomological computations in $\mathbb{P}K$. The results we shall need hereabout are contained in Proposition (3.40) below. Recall that, for a projective bundle $\mathbb{P}E \longrightarrow Y$ with $\mathrm{rk}(E) = k$, the relative and absolute dualizing sheaves are : $\omega_{\mathbb{P}E/Y} = \Lambda^k E_{\mathbb{P}E} \otimes \mathcal{O}_{\mathbb{P}E}(-k) =$ $= \mathcal{O}_{\mathbb{P}E}(-k + c_1(E))$ and $\omega_{\mathbb{P}E} = \omega_{\mathbb{P}E/Y} \otimes \omega_Y$ respectively, hence

$$(3.39) \qquad \begin{aligned} \omega_{\mathbb{P}K} &= \mathcal{O}_{\mathbb{P}K}(-5h - 3\sigma) \quad , \\ \omega_{\mathbb{P}K/\mathbb{P}^3} &= \mathcal{O}_{\mathbb{P}K}(-h - 3\sigma) \quad , \\ \omega_{\mathbb{P}K/G} &= \mathcal{O}_{\mathbb{P}K}(\rho - 2h) \quad . \end{aligned}$$

(3.40) PROPOSITION. *The cohomology of the sheaves* $\mathcal{O}_{\mathbb{P}K}(mh + n\sigma)$ *is given by the following table :*

i) If $n \geq 0$ $[h^i\mathcal{O}_{\mathbb{P}K}(mh + n\sigma) = h^i(S^nK)(m)]$ *:*

if $m > n-2$ *then* $h^0 = \frac{1}{3}\binom{m+3}{2}\binom{n+2}{2}(m-n+1)$ *and* $h^i = 0$, $i \neq 0$;

if $n-2 \ge m \ge -3$ *then* $h^1 = \frac{1}{3}\binom{m+3}{2}\binom{n+2}{2}(n-m-1)$, $h^i = 0$, $i \ne 1$ *;*

if $-3 > m$ *then* $h^3 = \frac{1}{3}\binom{m+3}{2}\binom{n+2}{2}(n-m-1)$, $h^i = 0$, $i \ne 3$ *;*

ii) if $n = -1,-2$ *then* $h^i = 0$ *for all* i .

(If $n < -2$ *we apply duality, using* (3.39)*).*

PROOF. The class $\rho = c_1(R)$ is ample in G and $h = c_1\mathcal{O}_{\mathbb{P}^3}(1)$ is ample in $\mathbb{P}^3$; therefore $mh + n\rho$ is ample in $\mathbb{P}^3 \times G$, hence also in the subvariety $\mathbb{P}K \subset \mathbb{P}^3 \times G$, if $m,n > 0$. So, by (3.28) , $mh + n\sigma$ is ample in $\mathbb{P}K$ if $m > n > 0$. By Kodaira's vanishing theorem and (3.39) we have therefore $H^i\mathcal{O}_{\mathbb{P}K}(mh + n\sigma) = 0$ if $i > 0$ and $m > n-2$, $n > -3$. This is ii) and the first part of i) , except for the statements about h^0 . To get the latter ones, if $n = -1,-2$ we use $R^ip_*\mathcal{O}_{\mathbb{P}K}(mh + n\sigma) = 0$ $\forall i$ (because on the fibres of p the induced sheaves have no cohomology at all) . This gives ii) . If $n \ge 0$ we have $R^ip_*\mathcal{O}_{\mathbb{P}K}(mh + n\sigma) = 0$ if $i > 0$ and $R^0p_*\mathcal{O}_{\mathbb{P}K}(mh + n\sigma) = (S^nK)(m)$; therefore $h^0\mathcal{O}_{\mathbb{P}K}(mh + n\sigma) = h^0(S^nK)(m)$. The standard sequence

$$0 \longrightarrow K \longrightarrow \mathcal{O}^4_{\mathbb{P}^3} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow 0$$

yields

$$(3.41) \qquad 0 \longrightarrow (S^nK)(m) \longrightarrow (S^n\mathcal{O}^4_{\mathbb{P}^3})(m) \longrightarrow (S^{n-1}\mathcal{O}^4_{\mathbb{P}^3})(m+1) \longrightarrow 0 ,$$

from whose cohomology sequence we obtain (knowing already $h^1(S^nK)(m) = 0$) the formula for h^0 .

Assume now that $n \ge 0$ and $n-2 \ge m \ge -3$. We keep the above sequence (3.41) . The last two sheaves being acyclic for $i > 0$, it suffices to see that $h^0(S^nK)(m) = 0$ in this case. But $R^iq_*\mathcal{O}_{\mathbb{P}K}(mh + n\sigma) = R^iq_*\mathcal{O}_{\mathbb{P}K}((m-n)h + n\rho) = [R^{1-i}q_*\mathcal{O}_{\mathbb{P}K}((n-m-2)h + (1-n)\rho)]^\vee$, which gives zero if $i = 0$ and $(S^{n-m-2}R)^\vee(n-1)$ if $i = 1$. Hence $H^0\mathcal{O}_{\mathbb{P}K}(mh + n\sigma) = H^{-1}(\dots) = 0$, as we wanted to see.

Finally, if $m < -3$, the cohomology sequence of (3.41) is explicit enough to give the expected result, q.e.d.

Using the above, we get the following list of cohomologies $[h^0, h^1, h^2, h^3]$ on $\mathbb{P}H$:

$\mathcal{O}_{\mathbb{P}H}$ $\qquad$ [1,0,1,0]

$$(3.42)\qquad \begin{array}{ll}
\mathcal{O}_{\mathbb{P}H}(-2h - 6\sigma) & [0,0,0,170] \\
\mathcal{O}_{\mathbb{P}H}(-h - 5\sigma) & [0,0,10,66] \\
\mathcal{O}_{\mathbb{P}H}(h + \sigma) & [6,10,0,0] \\
\mathcal{O}_{\mathbb{P}H}(2h + 2\sigma) & [21,15,0,0] \\
\mathcal{O}_{\mathbb{P}H}(-4\sigma) & [0,0,15,21] \\
\mathcal{O}_{\mathbb{P}H}(-2\sigma) & [0,0,0,10] \\
\mathcal{O}_{\mathbb{P}H}(-h - \sigma) & [0,0,0,0] \\
\mathcal{O}_{\mathbb{P}H}(-2h - 4\sigma) & [0,0,0,126]\ .
\end{array}$$

By (3.14), this implies inmediately:

$$(3.43)\qquad h^0\mathcal{O}_B = 1\ ,\quad h^1\mathcal{O}_B = 0\ ,\quad h^2\mathcal{O}_B = 171\ .$$

(3.44) COROLLARY. $h^0\mathcal{O}_{F_0} = 1$, $h^1\mathcal{O}_{F_0} = 0$, $h^2\mathcal{O}_{F_0} = 45$. *In particular, the surface* F_0 *is connected and regular.*

PROOF. The first two assertions follow from (3.43). The third one then comes from (3.35), q.e.d.

We look next for the cohomology of $\mathcal{O}_B(\kappa)$. To this end we rewrite the class κ in terms of divisor classes which extend to bigger spaces (with more transparent structure than B). Introduce therefore

$$P = \mathbb{P}R_{\mathbb{P}H} \xrightarrow{\ \gamma\ } \mathbb{P}H$$

whose points can be thought of as triples $(\overline{P},\overline{L},\overline{x})$ where $\overline{L}$ is a tangent line to $S \subset \mathbb{P}^3$ at $\overline{P}$ and $\overline{x} \in \overline{L}$. A fibre of the above projection map being identified with such a tangent line, the residual pair of the intersection of that line with S (after dropping the point of contact counted twice) describes a well defined subvariety Z of P as the fibre moves. This subvariety is mapped (2:1) onto $\mathbb{P}H$ and branched exactly above B . We have a commutative diagram

$$(3.45)\qquad \begin{array}{ccccc}
B_1 & \hookrightarrow & Z & \hookrightarrow & P \\
\downarrow{\scriptstyle\simeq} & & & \searrow & \downarrow{\scriptstyle\gamma} \\
B & & \hookrightarrow & & \mathbb{P}H
\end{array}\ .$$

Introduce furthermore the class

(3.46) $\xi \in \mathrm{Pic}(P)$, $\xi = c_1\mathcal{O}_{\mathbb{P}R_{\mathbb{P}H}}(1)$.

By the first proof of (3.14) , the bundle of equations of the fibres of $Z \longrightarrow \mathbb{P}H$ in those of γ is $\mathcal{O}_{\mathbb{P}H}(-2\sigma)$. Hence ((0.9)) , in Pic(P) :

(3.47) $[Z] = 2\sigma + 2\xi$,

and the covering $Z \longrightarrow \mathbb{P}H$ is given by $\rho + 2\sigma = h + 3\sigma$ (ibid) , hence, in Pic(Z) :

(3.48) $[B_1] = h + 3\sigma$.

(3.49) <u>LEMMA</u>. *In* B_1 *we have* $\kappa = h + 2\sigma - \xi$.

<u>PROOF</u>. Since B parametrizes bitangent lines with a distinguished point of contact, we have $Q_B \simeq Q'_B \otimes Q''_B$, where Q'_B is the bundle of equations of the distinguished contact points and Q''_B that of the second contact points. The bundle Q'_B is the restriction to B of the bundle V of (3.14) , hence $c_1(Q'_B) = \sigma$.

To compute Q''_B , take the fundamental sequence of P :

$$0 \longrightarrow L \longrightarrow R_P \longrightarrow \mathcal{O}_P(1) \longrightarrow 0 ,$$

and notice that the punctual fibre of L at a point $(\overline{P},\overline{L},\overline{x})$ of P is the vector space of equations of $\overline{x}$ in $\overline{L}$. Therefore the restriction $L \otimes \mathcal{O}_{B_1}$ is the lifting, by the isomorphism $B_1 \xrightarrow{\simeq} B$, of the bundle Q''_B . Since $c_1(L) = c_1(R_P) - \xi = \rho - \xi = h + \sigma - \xi$, we finally get, in B_1 :

$$\kappa = c_1(Q_{B_1}) = c_1(Q'_{B_1}) + c_1(Q''_{B_1}) = h + 2\sigma - \xi ,$$

q.e.d.

To get $h^i\mathcal{O}_B(\kappa) = h^i\mathcal{O}_{B_1}(h+2\sigma-\xi)$, we use (3.47) and (3.48) to write the sequences

(3.50) $$0 \longrightarrow \mathcal{O}_Z(-\sigma-\xi) \longrightarrow \mathcal{O}_Z(2\sigma+h-\xi) \longrightarrow \mathcal{O}_{B_1}(2\sigma+h-\xi) \longrightarrow 0 ,$$

(3.51) $$0 \longrightarrow \mathcal{O}_P(-3\sigma-3\xi) \longrightarrow \mathcal{O}_P(-\sigma-\xi) \longrightarrow \mathcal{O}_Z(-\sigma-\xi) \longrightarrow 0 ,$$

(3.52) $$0 \longrightarrow \mathcal{O}_P(h-3\xi) \longrightarrow \mathcal{O}_P(2\sigma+h-\xi) \longrightarrow \mathcal{O}_Z(2\sigma+h-\xi) \longrightarrow 0 .$$

We work them out backwards, starting with the last one. The presence of $-\xi$ in the middle term gives already $R^i\gamma_*\mathcal{O}_P(2\sigma+h-\xi) = 0$ for all i. Secondly, since $\omega_{P/\mathbb{P}H} = \mathcal{O}_P(\sigma+h-2\xi)$, we derive by duality that

$$R^1\gamma_*\mathcal{O}_P(h-3\xi) \simeq [R^0\gamma_*\mathcal{O}_P(\sigma+\xi)]^\vee \simeq R^\vee_{\mathbb{P}H} \otimes \mathcal{O}_{\mathbb{P}H}(-\sigma) \quad .$$

Therefore the direct image sequence of (3.52) gives $R^0\gamma_*\mathcal{O}_Z(2\sigma+h-\xi) \simeq$ $\simeq R^\vee_{\mathbb{P}H} \otimes \mathcal{O}_{\mathbb{P}H}(-\sigma)$ and $R^1\gamma_*\mathcal{O}_Z(2\sigma+h-\xi) = 0$. Hence, for all i, one has an isomorphism $H^i(Z,\mathcal{O}_Z(2\sigma+h-\xi)) \simeq H^i(\mathbb{P}H, R^\vee_{\mathbb{P}H} \otimes \mathcal{O}_{\mathbb{P}H}(-\sigma))$. To get further, we use sequence (3.4), which reads, on $\mathbb{P}H$:

(3.53) $$0 \longrightarrow \mathcal{O}_{\mathbb{P}H}(\sigma) \longrightarrow R_{\mathbb{P}H} \longrightarrow \mathcal{O}_{\mathbb{P}H}(h) \longrightarrow 0 \quad .$$

Dualizing and twisting by $\mathcal{O}_{\mathbb{P}H}(-\sigma)$ we deduce

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}H}(-h-\sigma) \longrightarrow R^\vee_{\mathbb{P}H} \otimes \mathcal{O}_{\mathbb{P}H}(-\sigma) \longrightarrow \mathcal{O}_{\mathbb{P}H}(-2\sigma) \longrightarrow 0 \quad .$$

From the corresponding cohomology sequence, together with (3.42), one obtains finally :

(3.54) The cohomology of $\mathcal{O}_Z(2\sigma+h-\xi)$ is [0,0,0,10] .

In a completely similar way, (3.51) leads to the less precise statement:

(3.55) The cohomology of $\mathcal{O}_Z(-\sigma-\xi)$ is [0,0,a,b] with a and b inserted in an exact sequence $0 \to a \to 10 \to 126 \to b \to 66 \to 0$.

Combination of (3.50), (3.54) and (3.55) gives finally the essential

(3.56) LEMMA. *The cohomology of* $\mathcal{O}_B(\kappa)$ *is* [0,a,172+a] *with* $a \leq 10$.

The cohomology of the surface F can be written down now :

(3.57) THEOREM. *The following formulae hold on* F *:*

i) $h^0\mathcal{O}_F = 1$, $h^1\mathcal{O}_F = 10$, $h^2\mathcal{O}_F = 101$;

ii) $h^1\Omega^1_F = 220$, *and the distribution in invariant and antiinvariant part of the (essential) Hodge numbers is:*

$(h^{0,1})^+ = 0$, $(h^{0,2})^+ = 45$, $(h^{1,1})^+ = 100$,
$(h^{0,1})^- = 10$, $(h^{0,2})^- = 56$, $(h^{1,1})^- = 120$.

(3.58) <u>REMARKS</u>.

a) $h^0\mathcal{O}_F = 1$ tells us that the surface F is connected ; this justifies the assumptions of (2.12) and p.38 .

b) $(h^{i,j})^+ = (h^{j,i})^+$, $(h^{i,j})^- = (h^{j,i})^-$, as follows from $h^{i,j} = h^{j,i}$ both for F and F_0 .

<u>PROOF OF (3.57)</u>. By (3.11) we have, for all i :

$$h^i\mathcal{O}_F = h^i\mathcal{O}_{F_0} + h^i\mathcal{O}_{F_0}(\kappa) \quad ,$$

$$h^i\mathcal{O}_B(\kappa) = h^i\mathcal{O}_{F_0}(\kappa) + h^i\mathcal{O}_{F_0}(-\rho) \quad .$$

Therefore, by (3.44) and (3.56) , we derive $h^0\mathcal{O}_F = 1$.

Next, by Kodaira's vanishing theorem , since ρ is ample on F_0 : $h^i\mathcal{O}_{F_0}(-\rho) = 0$ if $i < 2$. By the Riemann-Roch Theorem , $\chi\mathcal{O}_{F_0}(-\rho) = \frac{1}{2}(\rho^2 + K_{F_0}\cdot\rho) + \chi\mathcal{O}_{F_0} = \frac{1}{2}(40 + 120) + 46 = 126$ (cf (3.30) , (3.35)) , hence $\mathcal{O}_{F_0}(-\rho)$ has cohomology $[0,0,126]$.

Together with (3.56) , this tells us that $\mathcal{O}_{F_0}(\kappa)$ has cohomology $[0,a,46+a]$ with $a \leq 10$. The above therefore yields, by (3.44) , that $\mathcal{O}_F$ has cohomology $[1,a,91+a]$, with $a \leq 10$. On the other side,

$$a = q(F) \geq 10$$

by (2.13.ii) (sic) , hence $a = 10$ and i) is proved.

The value $h^{1,1}(F) = 220$ then follows from this and (3.34) ; similarly $h^{1,1}(F_0) = 100$ follows from (3.44) and (3.35) . This finishes the proof.

(3.59) <u>COROLLARY</u>. *The Abel-Jacobi map* $\mathrm{Alb}(F) \longrightarrow J(X)$ *is an isogeny.*

<u>PROOF</u>. Follows from (2.13.ii) and i) above, q.e.d.

We end this section with a further result on the cohomology of F , which will play a major rôle in Section 6 below (cf Proposition (6.1)). We recall that in [8] it is shown that the Fano surface F' of the cubic threefold satisfies $\Lambda^2H^1(F',\mathbb{C}) \simeq H^2(F',\mathbb{C})$ (loc.cit., p.326) . In the present case the image of $\Lambda^2H^1(F,\mathbb{C})$ in $H^2(F,\mathbb{C})$ consists of invariant cohomology classes, since the whole of $H^1(F,\mathbb{C})$ is antiinvariant. More precisely, one has:

(3.60) PROPOSITION. *The natural map yields an isomorphism* $\Lambda^2 H^1(F,\mathbb{C}) \simeq$ $\simeq H^2(F,\mathbb{C})^+$.

Equivalently, this says that the natural mappings

$$(3.61)\qquad \Lambda^2 H^0\Omega^1_F \longrightarrow (H^0\Omega^2_F)^+ \quad ,$$

$$(3.62)\qquad H^0\Omega^1_F \otimes H^1\mathcal{O}_F \longrightarrow (H^1\Omega^1_F)^+$$

are isomorphisms. Remark that the dimensions on the left and on the right hand sides coincide, by (3.57) . The proof of (3.60) will occupy the remainder of the present section. We shall use freely the identifications deduced from (3.15) with aid of (3.57).

(3.63) LEMMA. *The map* (3.61) *is an isomorphism* .

PROOF. As remarked above, only injectivity needs to be proved. Identifying $H^0\Omega^1_F$ with $H^0(S^2R_{F_0})$, and $(H^0\Omega^2_F)^+$ with $H^0\Omega^2_{F_0}$, this map is the composition

$$\Lambda^2 H^0(S^2R_{F_0}) \longrightarrow H^0(\Lambda^2 S^2 R_{F_0}) \xrightarrow{\eta} H^0\Omega^2_{F_0} \quad ,$$

where η comes from the sequence

$$0 \longrightarrow \Omega^1_{F_0} \longrightarrow \Lambda^2 S^2 R_{F_0} \longrightarrow \Omega^2_{F_0} \longrightarrow 0 \quad ,$$

gotten by exterior squaring from (3.15.ii) . Since $H^0\Omega^1_{F_0} = 0$, the map η is injective; it remains to prove the injectivity of the map

$$\Lambda^2 H^0\mathcal{O}_{\mathbb{P}^3}(2) = \Lambda^2 H^0(S^2R_{F_0}) \longrightarrow H^0(\Lambda^2 S^2 R_{F_0}) \quad ,$$

and this is an easy consequence of the following observation: an element $\Phi \in \Lambda^2 H^0\mathcal{O}_{\mathbb{P}^3}(2)$ vanishes, as soon as it vanishes at five 2-planes in general position; and Φ vanishes at a given 2-plane , as soon as it vanishes at five lines in general position in that plane, q.e.d.

(3.64) LEMMA. *The kernel of the morphism* (3.62) *has dimension* 1 .

PROOF. Our argument is quite long; we divide it into several parts, for the sake of clearness. To begin with, we introduce some notations to be used only during this proof: we put $W = H^0\mathcal{O}_{\mathbb{P}^3}(2)$ for shortness; J will denote the rank 9 bundle on $\mathbb{P}^3$ defined as the kernel of the natural map $W \otimes \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_{\mathbb{P}^3}(2)$. Also, I stands for the rank 7 bundle on G defined as the kernel of the natural map $W \otimes \mathcal{O}_G \longrightarrow S^2R$.

i) The map (3.62) can be identified with the cup-product morphism

$$H^0(\Omega^1_{F_0} \otimes Q_{F_0}) \otimes H^1Q_{F_0} \longrightarrow H^1\Omega^1_{F_0} ,$$

which in turn can be included in the following commutative diagram:

$$\begin{array}{ccc} H^1(W \otimes Q_B) & \longrightarrow & H^1(S^2R_B \otimes Q_B) \\ \uparrow \simeq & & \uparrow \\ H^1(W \otimes Q_{F_0}) & \longrightarrow & H^1(S^2R_{F_0} \otimes Q_{F_0}) \\ \downarrow \simeq & & \downarrow \\ H^0(\Omega^1_{F_0} \otimes Q_{F_0}) \otimes H^1Q_{F_0} & \longrightarrow & H^1\Omega^1_{F_0} \end{array} ,$$

the remaining morphisms being the obvious ones. The statements of injectivity, etc. which it contains are easily checked. The assertion of the lemma is therefore equivalent with the same assertion for the upper horizontal arrow.

The latter one comes from the sequence

$$0 \longrightarrow I_B \otimes Q_B \longrightarrow W \otimes Q_B \longrightarrow S^2R_B \otimes Q_B \longrightarrow 0 ,$$

hence the lemma is equivalent with being the rank of the morphism

$$H^1(I_B \otimes Q_B) \longrightarrow H^1(W \otimes Q_B)$$

at most 1 . Since $I_B \subset J_B \subset W \otimes \mathcal{O}_B$, we have $I_B \otimes Q_B \subset J_B \otimes Q_B \subset W \otimes Q_B$, and it is sufficient to prove that

(3.65) $\dim H^1(J_B \otimes Q_B) \leq 1$

holds. This will be done in the remainder of this proof.

ii) We proceed along the lines of the proof of (3.56) ; accordingly, we omit the most evident details. It will be more convenient to replace $J_B \otimes Q_B$ by $J_B^\vee \otimes Q_B^\vee \otimes \omega_B$, both sheaves having mutually dual first cohomology spaces.

By (3.35) and (3.11) , $\omega_B = \mathcal{O}_B(4\rho)$. Using also (3.49) , the above sheaf yields on B_1 (cf diagram (3.45)) the bundle

(3.66) $$D = J_{B_1}^\vee \otimes \mathcal{O}_{B_1}(3h+2\sigma+\xi) \quad .$$

By using (3.48) we get a resolution of D :

(3.67) $$0 \longrightarrow J_Z^\vee \otimes \mathcal{O}_Z(2h-\sigma+\xi) \longrightarrow J_Z^\vee \otimes \mathcal{O}_Z(3h+2\sigma+\xi) \longrightarrow D \longrightarrow 0 \quad ,$$

whose direct image sequence by γ (cf loc. cit.) gives

(3.68) $$0 \to J_{\mathbb{P}H}^\vee \otimes R_{\mathbb{P}H} \otimes \mathcal{O}_{\mathbb{P}H}(2h-\sigma) \to J_{\mathbb{P}H}^\vee \otimes R_{\mathbb{P}H} \otimes \mathcal{O}_{\mathbb{P}H}(3h+2\sigma) \to R^0\gamma_*(D) \to 0 \quad ,$$

the $R^1\gamma_*$ of all terms of (3.67) being zero. This is seen by means of resolutions of the first two $\mathcal{O}_P$-modules (using (3.47)) and writing down the corresponding sequences of direct images. In particular, the cohomology of D is the same as that of $R^0\gamma_*(D)$.

iii) Call the first two terms of (3.68) respectively $(3.68)_I$ and $(3.68)_{II}$. We look for the cohomology of these sheaves.

Term $(3.68)_I$ offers no difficulty. By using (3.53) we get

$$0 \longrightarrow J_{\mathbb{P}H}^\vee \otimes \mathcal{O}_{\mathbb{P}H}(2h) \longrightarrow (3.68)_I \longrightarrow J_{\mathbb{P}H}^\vee \otimes \mathcal{O}_{\mathbb{P}H}(3h-\sigma) \longrightarrow 0 \quad .$$

The third term of this sequence is clearly acyclic; we see that the first one has cohomology [99,1,0,0] , by using the defining sequence of $J^\vee$ in $\mathbb{P}^3$. Therefore:

(3.69) The sheaf $(3.68)_I$ has cohomology [99,1,0,0] .

As for $(3.68)_{II}$, we are less fortunate, since the above method yields no sufficiently explicit sequences. Instead we use (3.37) and (3.38) to work things back to $\mathbb{P}K$ (cf (3.7)) .

iv) Consider the sequence

$$(3.70) \quad 0 \to J^{\vee}_{\mathbb{P}K_S} \otimes R_{\mathbb{P}K_S} \otimes \mathcal{O}_{\mathbb{P}K_S}(\sigma) \to J^{\vee}_{\mathbb{P}K_S} \otimes R_{\mathbb{P}K_S} \otimes \mathcal{O}_{\mathbb{P}K_S}(3h+2\sigma) \to (3.68)_{II} \to 0$$

dedeuced from (3.37) . The cohomology of the second sheaf can be computed by using the $\mathcal{O}_{\mathbb{P}K}$-resolution deduced from (3.38) and applying to each of the two $\mathcal{O}_{\mathbb{P}K}$-modules gotten in this way analogous arguments as with $(3.68)_I$ above (i.e. using (3.4) and the defining sequence of $J^{\vee}$ in $\mathbb{P}^3$). We get:

(3.71) The sheaf $(3.70)_{II}$ has cohomology [1920,0,0,0,0] .

Term $(3.70)_I$ cannot be settled in this way; however, an $\mathcal{O}_{\mathbb{P}K}$-resolution being, by (3.38) ,

$$(3.72) \quad 0 \to J^{\vee}_{\mathbb{P}K} \otimes R_{\mathbb{P}K} \otimes \mathcal{O}_{\mathbb{P}K}(-4h+\sigma) \to J^{\vee}_{\mathbb{P}K} \otimes R_{\mathbb{P}K} \otimes \mathcal{O}_{\mathbb{P}K}(\sigma) \to (3.70)_I \to 0 \ ,$$

the following observation will be useful:

v) Sublemma. The following holds: $R^0 p_*(R_{\mathbb{P}K} \otimes \mathcal{O}_{\mathbb{P}K}(\sigma)) \simeq J$ and, if $i > 0$, $R^i p_*(R_{\mathbb{P}K} \otimes \mathcal{O}_{\mathbb{P}K}(\sigma)) = 0$.

Proof. Since $\mathcal{O}_{\mathbb{P}K}(\sigma) \otimes R_{\mathbb{P}K}$ is the vector bundle on $M = \mathbb{P}K$ with fibre at $(\overline{P},\overline{L})$ the vector space of 2-forms on $\overline{L}$ vanishing at $\overline{P}$, we have an exact sequence

$$0 \longrightarrow I_{\mathbb{P}K} \longrightarrow J_{\mathbb{P}K} \longrightarrow \mathcal{O}_{\mathbb{P}K}(\sigma) \otimes R_{\mathbb{P}K} \longrightarrow 0$$

and the result follows from $R^i p_* I_{\mathbb{P}K} = 0$ for all i (as easily checked fibrewise) , q.e.d.

vi) By v) , the direct image sequence of (3.72) by p yields, in $\mathbb{P}^3$:

$$0 \longrightarrow J^{\vee} \otimes J(-4) \longrightarrow J^{\vee} \otimes J \longrightarrow R^0 p_*(3.70)_I \longrightarrow 0 \ ,$$

and $R^i p_*(3.70)_I = 0$ if $i > 0$. Since the cohomology of $J^{\vee} \otimes J$ equals [1,0,0,0] and (hence) that of $J^{\vee} \otimes J(-4)$ is [0,0,0,1] , we get:

(3.73) Term $(3.70)_I$ has cohomology [1,0,1] .

Combination of (3.71) and (3.73) now gives:

(3.74) The cohomology of $(3.68)_{II}$ is [1919,1,0] .

By (3.68) , (3.69) and (3.74) we obtain finally $\dim H^1(R^0\gamma_* D) \leq 1$, hence (3.65) holds and Lemma (3.64) is proved.

(3.75) <u>END OF THE PROOF OF (3.60)</u>. By (3.63) and (3.64) we know the restriction map

$$(3.76) \qquad H^2(J(X),\mathbb{C}) \longrightarrow H^2(F,\mathbb{C})$$

to have a kernel of dimension ≤ 1 , and our purpose is to see that Ker = 0 holds. Assume this is not so; we shall derive a contradiction.

The kernel is gotten by $\otimes_{\mathbb{Z}} \mathbb{C}$ from the kernel of the restriction map

$$H^2(J(X),\mathbb{Z}) \longrightarrow H^2(F,\mathbb{Z}) \quad ,$$

hence the latter would be $\mathbb{Z}\omega \simeq \mathbb{Z}$ for a certain $\omega \in H^2(J(X),\mathbb{Z})$. The module $\mathbb{Z}\omega$ being invariant under the action of the monodromy (as X varies, its Fano surface staying smooth) , we get $T\omega = \pm\omega$ for each monodromy transformation T . We want to show that ω is a multiple of $[\Theta] \in H^2(J(X),\mathbb{Z})$, and this will be contradictory, for then $[\Theta]\cdot[F] = 0$ in J(X) , contradicting ampleness of Θ or effectiveness of F . Identifying $H^2(J(X),\mathbb{C})$ with the vector space of skew-symmetric bilinear forms on the vector space $H_1(J(X),\mathbb{C}) \simeq H_3(X,\mathbb{C})$, there exists certainly a constant $c \in \mathbb{C}$ such that

$$\omega^+ = \omega + c[\Theta]$$

is degenerate. If $\omega^+ = 0$, we are done. Otherwise, observe that each monodromy transformation T satisfies either $T\omega^+ = \omega^+$ or $T\omega^+ = \omega^-$, with

$$\omega^- = -\omega + c[\Theta] \quad .$$

Hence, denoting for a moment $W = H_3(X,\mathbb{C})$, $W^+ = \mathrm{Ker}(\omega^+)$ and $W^- = \mathrm{Ker}(\omega^-)$, we have $0 \neq W^+ \neq W$, and T satisfies $TW^+ = W^+$, $TW^- = W^-$ or $TW^+ = W^-$, $TW^- = W^+$. By (4.3) below , W is spanned by the vanishing cycles (we apologize for the double use of the symbol W ; the reader will notice that the proof of (4.3) is independent of our previous arguments). Therefore, by the irreducibility of the monodromy action on W, we obtain $W = W^+ + W^-$; by the same reason, $W^+ \cap W^-$ either is zero or equals W, but the latter case is impossible. So $W = W^+ \oplus W^-$.

Next we take a Lefschetz pencil of double solids (cf Section 4) and restrict ourselves to that situation. The monodromy transformation T_i associated with a vanishing cycle δ_i is given by the Picard-Lefschetz formula

$$T_i\alpha = \alpha \pm (\alpha,\delta_i)\delta_i \quad .$$

Decomposing δ_i along W^+ and W^- : $\delta_i = \delta_i^+ + \delta_i^-$, we get, for each $\alpha \in W^+$:

$$T_i\alpha = (\alpha \pm (\alpha,\delta_i)\delta_i^+) \pm (\alpha,\delta_i)\delta_i^- \quad .$$

If it were $T_iW^+ = W^-$, we would get $W^+ = \langle\delta_i^+\rangle$, $W^- = \langle\delta_i^-\rangle$, which is impossible. Therefore both spaces W^+ and W^- are invariant under monodromy, hence $W^+ = 0$ or $W^+ = W$. But both cases have been ruled out by assumption, so this is a contradiction.

Therefore the kernel of the restriction map (3.76) is zero, and this is (3.60), q.e.d.

APPENDIX TO SECTION 3

This appendix contains some geometrical aspects related with the preceding theory which, being perhaps interesting for themselves, will not be used in the sequel.

The Gauss map for F

Consider the Albanese map $F \longrightarrow Alb(F)$, for some fixed base point. By (3.59), the associated Gauss map can be identified with that of the Abel-Jacobi map $F \longrightarrow J(X)$ and, by (2.13), we have an identification, for each $L \in F$, writing $\overline{P}$, $\overline{Q}$ for the points of contact of the projection $\overline{L} \subset \mathbb{P}^3$ of L :

$$\begin{array}{ccc} T_{\mathrm{Alb}(F)}(0) & \xrightarrow{\ \simeq\ } & H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))^{\vee} \\ \big\downarrow & & \big\downarrow \\ T_F(L) & \xrightarrow{\ \simeq\ } & \left\{ \begin{array}{l} \text{2- forms of } \mathbb{P}^3 \text{ vani-} \\ \text{shing at } \overline{P} \text{ and } \overline{Q} \end{array} \right\}^{\perp} \end{array} .$$

If $\mathbb{P}^9$ stands for the projectivization of the vector space $H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(2))^{\vee}$, $\mathbb{G}$ for the Grassmann variety of lines in that $\mathbb{P}^9$, and $V : \mathbb{P}^3 \longrightarrow \mathbb{P}^9$ denotes the Veronese embedding, the above says that the Gauss map

$$G : F \longrightarrow \mathbb{G}$$

is given by $G(L) = V(\overline{P}) \vee V(\overline{Q})$. (If $\overline{P} = \overline{Q}$, $G(L)$ is the tangent line to the conic $V(\overline{L})$ at $V(\overline{P})$). In particular, G factors as

$$\text{(A.1)} \qquad \begin{array}{ccc} F & \xrightarrow{\ G\ } & \mathbb{G} \\ & {\scriptstyle\pi}\searrow \quad \nearrow{\scriptstyle G_0} & \\ & F_0 & \end{array}$$

where G_0 is a well defined morphism.

Let $L_1, L_2 \in F$ and $\overline{L}_1$, $\overline{L}_2$ their respective projections in $\mathbb{P}^3$. Then: $G(L_1) = G(L_2)$ if and only if $\overline{L}_1 = \overline{L}_2$; secondly, the lines $G(L_1)$ and $G(L_2)$ of $\mathbb{P}^9$ intersect at exactly one point if and only if $\overline{L}_1$ and $\overline{L}_2$ do so. In the latter case, the intersection of $G(L_1)$ and $G(L_2)$ is the image of that point by the Veronese embedding.

In fact, if $(\overline{P}_1, \overline{Q}_1)$, $(\overline{P}_2, \overline{Q}_2)$ are the pairs of contact of $\overline{L}_1$ and $\overline{L}_2$ respectively, $G(L_1) = G(L_2)$ means that the linear system of quadrics of $\mathbb{P}^3$ meeting L_1 at $\overline{P}_1 + \overline{Q}_1$ coincides with the system of quadrics meeting L_2 at $\overline{P}_2 + \overline{Q}_2$. This can happen only if $\{\overline{P}_1, \overline{Q}_1\} = \{\overline{P}_2, \overline{Q}_2\}$, i.e. if $\overline{L}_1 = \overline{L}_2$.

On the other side, $G(L_1) \cap G(L_2)$ to be a single point means that the system of quadrics of $\mathbb{P}^3$ meeting $\overline{L}_i$ at $\overline{P}_i + \overline{Q}_i$ for each $i = 1,2$ has dimension 6. This is equivalent with $\{\overline{P}_1, \overline{Q}_1\}$ and $\{\overline{P}_2, \overline{Q}_2\}$ sharing exactly one point. The rest is now clear.

Furthermore, as $\overline{L}$ varies over F_0, the line $G_0(\overline{L}) \subset \mathbb{P}^9$ describes a threefold in $\mathbb{P}^9$. Any point of this threefold meets exactly one of these lines, except the points of the Veronese image of $S \subset \mathbb{P}^3$, which meet exactly six lines in general (cf (3.33)). Summarizing, we have gotten thus:

(A.2) PROPOSITION. *The Gauss map* $\mathcal{G}$ *associated with the Albanese mapping of* F *factors as in* (A.1) , *with* $\mathcal{G}_0$ *injective. Moreover, if* $Y \subset \mathbb{P}^9$ *denotes the image of the composite map*

$$\mathbb{P}\Omega^1_F \longrightarrow \mathbb{P}\Omega^1_{Alb(F)} \simeq Alb(F) \times \mathbb{P}^9 \xrightarrow{pr} \mathbb{P}^9$$

then $\mathbb{P}\Omega^1_F$ *maps everywhere* (2:1) *onto* Y , *except above a certain surface in* Y , *where it is generically* (12:1) ; *this surface can be identified with the Veronese embedding of the discriminant locus* $S \subset \mathbb{P}^3$ *of the double solid* X . *In particular,* X *is determined by* F .

(A.3) COROLLARY. *The Abel-Jacobi map* $F \longrightarrow J(X)$ *is generically injective.*

PROOF. Assume it is (k:1) onto its image. Then $\mathcal{G}$ has to be at least (k:1) too. Hence k = 1 or 2 . If k = 2 , a general fibre ought to consist of a conjugate pair under the involution i . Calling the above map ψ for a moment, we would have, for each $x \in F : \psi(x) = \psi(ix)$. But $\psi(x) + \psi(ix) = \text{const.}$ for all $x \in F$, hence $2\psi(x) = \text{const.}$, which is impossible. Therefore k = 1 , q.e.d.

(A.4) As first remarked to us by Collino , the results of Clemens imply that $F \longrightarrow J(X)$ is injective if X is sufficiently general. We dont know whether it is injective for all X .

To end our remarks about the Gauss map of F we describe geometrically the canonical map for F_0 .

(A.5) PROPOSITION. *The map* $\mathcal{G}_0 : F_0 \longrightarrow \mathbb{G}$ *followed by the Plücker embedding* $\mathbb{G} \hookrightarrow \mathbb{P}^{44}$ *is the canonical map for* F_0 .

PROOF. The composition of these maps is given by

$$F_0 \longrightarrow Grass(2,(H^0\Omega^1_F)^\vee) \longrightarrow Grass(1,(\Lambda^2 H^0\Omega^1_F)^\vee) \quad ,$$

sending $\overline{L} \in F_0$ to

$$\{\omega \in \Lambda^2(H^0\Omega^1_F) \mid \omega \text{ vanishes at (one of) the } L \in F \text{ above } \bar{L}\}^\perp .$$

With the identification (3.63) this is the same as

$$\{\omega_0 \in H^0\Omega^2_{F_0} \mid \omega_0 \text{ vanishes at } \bar{L}\}^\perp ,$$

q.e.d.

In particular, if we take a pencil of quadrics in $\mathbb{P}^3$, the curve of bitangents of $S \subset \mathbb{P}^3$ such that their pair of contact lies on some quadric of that pencil yields a canonical divisor of F_0.

The geometry of the projective tangent bundle isomorphism

A geometric version of the TBS (3.15.i) is gotten by projectivization: the projective tangent bundle $\mathbb{P}\Omega^1_F$ of F can be identified with the subbundle of

$$\mathbb{P}(S^2R_F) = \{ g^1_2 \text{'s on the lines } l \in F \}$$

whose fibre at $l \in F$ consists on those g^1_2's which contain the divisor $S\cdot l$ of l. A natural question raises: given a direction of F at $l \in F$, describe the associated g^1_2 of l. We give an answer to this question:

(A.6) PROPOSITION. *Let* $l \in F$ *be given, and let* $L \in F$ *be a sufficiently general line of* X *meeting* l. *The curve* D_L *then has* l *as a smooth point* ((1.7), (1.8)), *hence provides us with a tangent direction of* F *at* l. *The corresponding* g^1_2 *on* l *is the one induced on the projection* $\bar{l} \subset \mathbb{P}^3$ *of* l *by the projection* $\bar{L}$ *of* L, *as in* (0.12).

PROOF. The main point is that the curve D_L is the Chow component, in the blown-up $\tilde{X}$ of X along the line L, of the proper transform $\tilde{l}$ of l. Hence

$$T_{D_L}(l) \simeq H^0(\tilde{l}, N_{\tilde{l}/\tilde{X}}) .$$

Call $\tilde{E}$ the blowing-up of E (cf Section 0) along L. We recall the formulae (ibid) : $\omega_X \simeq \mathcal{O}_X(-2)$, $\omega_{\tilde{X}} \simeq \mathcal{O}_{\tilde{X}}(-2+D)$, $N_{X/E} \simeq \mathcal{O}_X(4)$, $N_{\tilde{X}/\tilde{E}} \simeq$

$\simeq \mathcal{O}_{\tilde X}(4-D)$, where D denotes here the exceptional divisor of the blowing--up. We have an exact commutative diagram

$$\text{(A.7)}\qquad \begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & k & = & k & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & N_{\tilde 1/\tilde X}\otimes\omega_{\tilde X} & \longrightarrow & N_{\tilde 1/\tilde E}\otimes\omega_{\tilde X} & \longrightarrow & N_{\tilde X/\tilde E}\otimes\omega_{\tilde X}\otimes\mathcal{O}_{\tilde 1} & \longrightarrow & 0\\
 & & \uparrow & & \uparrow & & \Big|\simeq & & \\
0 & \longrightarrow & N_{1/X}\otimes\omega_X & \longrightarrow & N_{1/E}\otimes\omega_X & \longrightarrow & N_{X/E}\otimes\omega_X\otimes\mathcal{O}_1 & \longrightarrow & 0\\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

where the top "k" 's mean skyscraper sheaves with stalk $k = \mathbb{C}$ at the point $\tilde P \in \tilde 1$ where $\tilde 1$ meets the exceptional divisor D . The diagram is completely described by telling that its restriction to $\tilde 1 \smallsetminus \tilde P$ is the natural identification between the middle and the bottom rows and furthermore that its existence (i.e. the extension of the identification maps near $\tilde P \in \tilde 1$) follows from a straightforward local computation.

The injection morphism of the left hand side column yields, taking first cohomology:

$$\begin{array}{ccc}
H^1 N_{\tilde 1/\tilde X}\otimes\omega_{\tilde X} & \simeq & T_{D_L}(1)^{\vee}\\
\uparrow & & \uparrow r\\
H^1 N_{1/X}\otimes\omega_X & \simeq & T_F(1)^{\vee}
\end{array} ,$$

where r is the ordinary restriction map of cotangent vectors. Taking the cohomology sequences of the two lower rows in (A.7) we get an exact commutative diagram

$$\text{(A.8)}\qquad \begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \uparrow & & & & & & \\
 & & \mathbb{C} & & & & & & \\
 & & \uparrow & & & & & & \\
0 & \longrightarrow & H^0 N_{\tilde 1/\tilde E}\otimes\omega_{\tilde X} & \longrightarrow & H^0\mathcal{O}_{\tilde 1}(2) & \longrightarrow & T_{D_L}(1)^{\vee} & \longrightarrow & 0\\
 & & \uparrow & & \| & & \uparrow r & & \\
0 & \longrightarrow & \langle T|1\rangle & \longrightarrow & H^0\mathcal{O}_1(2) & \longrightarrow & T_F(1)^{\vee} & \longrightarrow & 0\\
 & & \uparrow & & & & & & \\
 & & 0 & & & & & &
\end{array} ,$$

where the following has been used:

i) The bottom row is the bottom row of (2.15) .

ii) The middle column of (A.7) yields a short exact sequence in cohomology, since $H^1 N_{l/E} \otimes \omega_X = 0$ (cf the proof of (2.13.i)) .

iii) l is a smooth point of D_L (by assumption) , hence $\dim T_{D_L}(l)^\vee = 1$, i.e. $h^1(N_{\tilde{l}/\tilde{X}} \otimes \omega_{\tilde{X}}) = 1$ and so, by the first column of (A.7) , we get $h^0(N_{\tilde{l}/\tilde{X}} \otimes \omega_{\tilde{X}}) = 0$. Thus the second row begins and ends with a zero term.

From (A.8) we learn that the g^1_2 we are looking for is made up by the loci on l of the 2-forms of the subspace $H^0 N_{\tilde{l}/\tilde{E}} \otimes \omega_{\tilde{X}}$ of $H^0 \mathcal{O}_l(2)$. Observe that $T|l = 0$ yields the pair of contact of l as a member of that pencil. It suffices now to exhibit an element of $H^0 N_{\tilde{l}/\tilde{E}} \otimes \omega_{\tilde{X}}$ having as zeros one of the pairs of points as mentioned in (0.12) .

To produce that section, take l_1 , l_2 to be two lines in X such that (L, l, l_1, l_2) are a configuration as described in (0.5) (cf (0.11)) . By (loc.cit.) there is a 2-form ψ_2 on $\mathbb{P}^3$ such that $L + l + l_1 + l_2$ is the complete intersection in X of the surfaces $T - \psi_2 = 0$ and $\phi_1 = 0$, the latter one being the pullback to X of the 2-plane $\Pi \subset \mathbb{P}^3$ spanned by the projections $\bar{L}$ and $\bar{l}$.

The threefold $V \subset E$ given by $T - \psi_2 = 0$ maps isomorphically onto $\mathbb{P}^3$; the inverse image of Π in E is a threefold $W \subset E$ defined by the equation $\phi_1 = 0$; hence the intersection surface $M = V \cap W \subset E$ maps isomorphically onto the plane Π and, by construction, $M \cap X = L \cup l \cup l_1 \cup l_2$. Let $\tilde{M}$ be the proper transform of M in $\tilde{E}$, $\tilde{M} \simeq M$; we have an obvious injection

$$H^0(N_{\tilde{l}/\tilde{M}} \otimes \omega_{\tilde{X}}) \hookrightarrow H^0(N_{\tilde{l}/\tilde{E}} \otimes \omega_{\tilde{X}}) \quad .$$

On the other side, $N_{\tilde{l}/\tilde{M}} \simeq \mathcal{O}_{\tilde{l}}(1)$, the situation being that of a line in the plane. Hence

$$N_{\tilde{l}/\tilde{M}} \otimes \omega_{\tilde{X}} \simeq \mathcal{O}_{\tilde{l}}(1) \otimes \mathcal{O}_{\tilde{X}}(-2+D) \simeq \mathcal{O}_{\tilde{l}} \quad ,$$

and this bundle has a non zero section, providing us with a section of $N_{\tilde{l}/\tilde{E}} \otimes \omega_{\tilde{X}}$. We consider the image of the latter in the bundle $N_{\tilde{X}/\tilde{E}} \otimes \omega_{\tilde{X}} \otimes \mathcal{O}_{\tilde{l}} \simeq \mathcal{O}_{\tilde{l}}(2)$ and ask for its locus of zeros. If $\tilde{Q}_1$, $\tilde{Q}_2$ denote the intersections of $\tilde{l}_1$, $\tilde{l}_2$ respectively with $\tilde{l}$, it is clear that $\tilde{M}$ is tangent to $\tilde{X}$ at $\tilde{Q}_1$ and $\tilde{Q}_2$, hence the above image section vanishes at these points and, since it doesn't vanish identically, this is its locus of zeros, q.e.d.

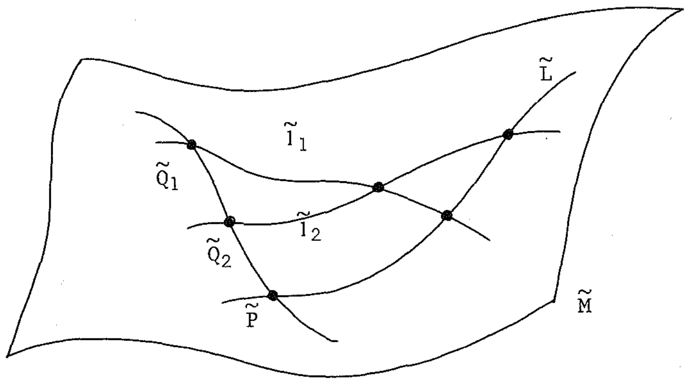


Figure 3.

4. THE ISOMORPHISM THEOREM

(4.1) THEOREM. *The Abel-Jacobi map* $\text{Alb}(F) \longrightarrow J(X)$ *is an isomorphism of abelian varieties.*

The proof of this statement will occupy the whole of this section. We use mainly ideas from [28] (which seem to go back in some aspects to ideas of Clemens) and [5] .

The sheaves defined by singular homology (as X varies, together with F) being locally constant, it suffices to prove the statement for a single X ; hence we may choose it in appropiate way. By [5], Proposition 1.1 , there exists a quartic threefold $T \subset \mathbb{P}^4$ such that the lines on T are parametrized by a smooth (connected) curve. We fix such a T and call $f : W \longrightarrow \mathbb{P}^4$ the double hypersolid with discriminant locus T . Finally, X is chosen as inverse image of a sufficiently general hyperplane of $\mathbb{P}^4$ under the map f .

By (3.59) , the proof of (4.1) will be complete if we show the surjectivity of the induced morphism in homology:

$$H_1(\mathrm{Alb}(F),\mathbb{Z}) \longrightarrow H_3(J(X),\mathbb{Z}) \quad .$$

Hence it suffices to see that the natural map

$$(4.2) \qquad H_1(F,\mathbb{Z}) \longrightarrow H_3(X,\mathbb{Z})$$

induced by the correspondence between F and X (the cylinder map, in Tyurin's terminology) is surjective. This will be done in the remainder of this section.

(4.3) <u>LEMMA</u>. $H_3(W,\mathbb{Z}) = 0$, $H_5(W,\mathbb{Z}) = 0$.

<u>PROOF</u>. We mimic the proof of Lemma 1.23 of [7] . (We drop the reference to the coefficient group $\mathbb{Z}$ in all our notations, during this proof). One certainly may assume that W is the hypersolid with discriminant surface $X_0{}^4+X_1{}^4+X_2{}^4+X_3{}^4+X_4{}^4 = 0$ in $\mathbb{P}^4$. We take $X \subset W$ as the inverse image of the hyperplane $X_0 = 0$ of $\mathbb{P}^4$. Take a tubular neighbourhood U of X in W , and consider the following piece of the Gysin sequence of the S^1--bundle $p : \partial U \longrightarrow X$:

$$H_3(\partial U) \xrightarrow{p_*^!} H_3(X) \xrightarrow{\psi'} H_1(X) \longrightarrow H_2(\partial U) \xrightarrow{p_*^{!!}} H_2(X) \xrightarrow{\psi''} H_0(X) \quad .$$

Here ψ' , ψ'' is intersection product with the first Chern class of the normal bundle $\mathcal{O}_X(1)$ of X in W ; thus with the class of a surface in $|\mathcal{O}_X(1)|$. By the mentioned lemma of [7] , $H_2(X) \simeq \mathbb{Z}$ with generator the class of a line L of X . Since $L\cdot c_1\mathcal{O}_X(1) = 1$, ψ'' is an isomorphism, and so $p_*^{!!} = 0$. On the other hand, $H_1(X) = 0$ (e.g. [7]) hence $H_2(\partial U) = 0$ follows, and also that $p_*^!$ is surjective. We shall need these two facts in a moment.

Consider next the Mayer-Vietoris sequence for the decomposition $W = (W \setminus U) \cup \overline{U}$; the piece we shall need is

$$H_3(\partial U) \longrightarrow H_3(W\setminus X) \oplus H_3(X) \longrightarrow H_3(W) \longrightarrow H_2(\partial U) \quad .$$

We know that $H_2(\partial U) = 0$. On the other hand, $W\setminus X$ is isomorphic with the hypersurface of $\mathbb{C}^5$ given by

$$y^2 = 1 + x_1^4 + x_2^4 + x_3^4 + x_4^4 \quad .$$

By [23], p.19 , this has the homotopy type of a bouquet of 4-spheres, hence $H_3(W \setminus X) = 0$. So, the first map of the above sequence can be identified with $p_*^!$, hence is surjective. Therefore $H_3(W) = 0$, as claimed.

As for $H_5(W)$, we use the foregoing. By Poincaré duality and universal coefficients for cohomology, this group has the same rank as $H_3(W)$ and the same torsion as $H_2(W)$. By Lefschetz' theorem , $H_2(W) \simeq H_2(X)$, and this has already been noticed to be isomorphic with $\mathbb{Z}$. Therefore $H_5(W) = 0$, q.e.d.

Define, as usual, the "lines" of W to be the curves mapping isomorphically onto lines of $\mathbb{P}^4$ by f . One has:

(4.4) <u>LEMMA</u>. *The lines of* W *are parametrized by a smooth connected fourfold* $F(W)$.

<u>PROOF</u>. Connectedness follows from the connectedness of our Fano surface F . As for smoothness, consider first a line $L \subset W$ such that $L \not\subset T$. Take a smooth hyperplane section $X' \subset W$ (i.e. a hypersurface of the linear system $|\mathcal{O}_W(1)|$) containing L , and write the standard sequence

$$0 \longrightarrow N_{L/X'} \longrightarrow N_{L/W} \longrightarrow N_{X'/W} \otimes \mathcal{O}_L \longrightarrow 0 \quad .$$

By using (1.1) and $N_{X'/W} \otimes \mathcal{O}_L \simeq \mathcal{O}_L(1)$ we get inmediately $h^0 N_{L/W} = 4$.

Assume now that $L \subset T$; we consider the sequence

$$0 \longrightarrow N_{L/T} \longrightarrow N_{L/W} \longrightarrow N_{T/W} \otimes \mathcal{O}_L \longrightarrow 0 \quad .$$

By our choice of T we have $h^0 N_{L/T} = 1$ and, since $N_{T/W} \otimes \mathcal{O}_L \simeq \mathcal{O}_L(2)$, it follows that $h^0 N_{L/W} = 4$. From this the lemma follows, q.e.d.

The graph of the correspondence between $F(W)$ and W will be denoted by $P(W)$; this is a $\mathbb{P}^1$-bundle over $F(W)$. Consider the diagram

(4.5)
$$\begin{array}{ccccc} \tilde{X} & \searrow & & \xrightarrow{\ \tilde{q}\ } & \\ {\scriptstyle \tilde{p}}\downarrow & & P(W) & \xrightarrow{\ q\ } & F(W) \\ & & \downarrow{\scriptstyle p} & & \\ X & \hookrightarrow & W & & \end{array}$$

where p and q are the natural projections and $\tilde{X}$ is defined as the inverse image of X by p. This is a 4-dimensional variety and the map $\tilde{q}$ exhibits it as the blowing-up of $F(W)$ along F.

This yields a natural decomposition

$$H_3(\tilde{X},\mathbb{Z}) \simeq H_3(F(W),\mathbb{Z}) \oplus H_1(F,\mathbb{Z})$$

where $H_3(F(W),\mathbb{Z})$ is included in $H_3(\tilde{X},\mathbb{Z})$ by means of the transfer morphism $\tilde{q}^*$, and the inclusion of $H_1(F,\mathbb{Z})$ in $H_3(\tilde{X},\mathbb{Z})$ is given by composition of the transfer morphism $H_1(F,\mathbb{Z}) \longrightarrow H_3(\tilde{q}^{-1}F,\mathbb{Z})$ with the natural map $H_3(\tilde{q}^{-1}F,\mathbb{Z}) \longrightarrow H_3(\tilde{X},\mathbb{Z})$. The cylinder map (4.2) is the restriction of

$$\tilde{p}_* : H_3(\tilde{X},\mathbb{Z}) \longrightarrow H_3(X,\mathbb{Z})$$

to $H_1(F,\mathbb{Z})$. But on the other side $\tilde{p}_* H_3(F(W),\mathbb{Z}) = 0$, since this is the restriction to X of $p_* q^* H_3(F(W),\mathbb{Z}) \subset H_5(W,\mathbb{Z}) = 0$ (cf (4.3)). It suffices therefore to show the surjectivity of $\tilde{p}_*$ itself.

This will follow, as in [5], from Lemma (7.15) of that paper, which we quote without proof, with a slight change in notations:

(4.6) <u>LEMMA</u> ([5], Lemma (7.15)). *Let* W *be a smooth projective variety defined over* $\mathbb{C}$, $\dim W = d+1$, $p : P \longrightarrow W$ *a proper map, generically finite (* P *irreducible)*, $X \hookrightarrow W$ *a smooth hyperplane section,* $\tilde{X} = p^{-1}(X)$. *Then the image of* $\tilde{p}_* : H_d(\tilde{X},\mathbb{Z}) \longrightarrow H_d(X,\mathbb{Z})$ *contains the vanishing cycles.*

We apply this to our situation, with $P = P(W)$. Notice that Lemma (4.6) goes through for a proper surjective p without the finiteness assumption, for it suffices to take an irreducible $P' \subset P$ such that the restriction of p to P' is proper and generically finite onto W.

Since by (4.3) and Lefschetz theory $H_3(X,\mathbb{Z})$ is spanned by vanishing cycles, the above yields the surjectivity of $\tilde{p}_*$, and Theorem (4.1) is proved.

5. THE INTERMEDIATE JACOBIAN AS A GENERALIZED PRYM VARIETY

The incidence curves D_L

Consider again the incidence curves $D_L \subset F$ (cf (1.5)). Some results of local infinitesimal type have been quoted already in (1.7) - (1.9) and we want to give now a more complete picture of these curves in the general case.

The D_L are ample curves of F and $[D_L] = \upsilon$ by definition. From Section 3 we get $h^0\mathcal{O}_{D_L} = 1$, $h^1\mathcal{O}_{D_L} = 71$. However, these curves are not smooth, as we shall see now.

Fix a sufficiently general line $L \in F$ and consider the curve D_L. Calling $\overline{L}$ the projection of L in $\mathbb{P}^3$, it is clear that the curve D_L projects onto the curve $D_{\overline{L}} \subset F_0$. In fact, we have

(5.1) $\qquad iD_L = D_{iL}$ and $\pi^{-1}(D_{\overline{L}}) = D_L + D_{iL}$.

We call furthermore $\overline{P}$, $\overline{Q}$ the contact points of $\overline{L}$; $\overline{L}_1, \dots, \overline{L}_5$ are the remaining bitangents to S with contact point $\overline{P}$, and $\overline{M}_1, \dots, \overline{M}_5$ those with contact point $\overline{Q}$. The lines in X above these bitangents are written respectively L_i', L_i'' and M_i', M_i'', $i = 1, \dots, 5$ (cf (3.33)). With these notations one has:

(5.2) <u>PROPOSITION</u>. *The curve* $D_{\overline{L}}$ *has exactly* 11 *ordinary double points corresponding to* $\overline{L}$, $\overline{L}_i$ *(i = 1,...,5) and* $\overline{M}_i$ *(i = 1,...,5). The curve* D_L *is the normalization of* $D_{\overline{L}}$ *at the latter* 10 *double points, remaining singular at* $iL \in D_L$.

<u>PROOF</u>. Clearly the projection $\pi : D_L \longrightarrow D_{\overline{L}}$ is (2:1) above $\overline{L}_i$, $\overline{M}_i$, $i = 1, \dots, 5$ and (1:1) above the remaining points of $D_{\overline{L}}$ except possibily above $\overline{L} \in D_{\overline{L}}$. But there it is in fact (1:1) too, since otherwise we would have $L \in D_L$; however, since $D_L \cdot D_{iL} = 20$ by (3.32), the intersections are already exhausted by L_i', L_i'', M_i', M_i'', $i = 1, \dots, 5$, and therefore, being $L \in D_{iL}$, it cannot happen that $L \in D_L$.

This shows also that D_L is smooth at these 20 points, and that $\overline{L}_i$, $\overline{M}_i$, $i = 1, \dots, 5$ are ordinary double points of $D_{\overline{L}}$. It remains to study D_L at iL. The fact that $\overline{L} \in D_{\overline{L}}$ is a point of multiplicity 2 can be seen as follows: if we take two bitangents $\bar{L}$, $\bar{M}$ which meet but otherwise are in general position, there can be counted $28 - 2 = 26$ other bitangents meeting $\bar{L}$ and $\bar{M}$ and lying in the 2-plane $\bar{L} \vee \bar{M}$, and $12 - 2 = 10$ bitangents passing through the intersection point of $\bar{L}$ and $\bar{M}$. We know on the other side that $D_{\bar{L}} \cdot D_{\bar{M}} = \rho^2 = 40$, hence 4 multiplicities have to be distributed among the intersection points $\bar{L}$ and $\bar{M}$ themselves. By symmetry, there correspond 2 to each one, as we wanted to show.

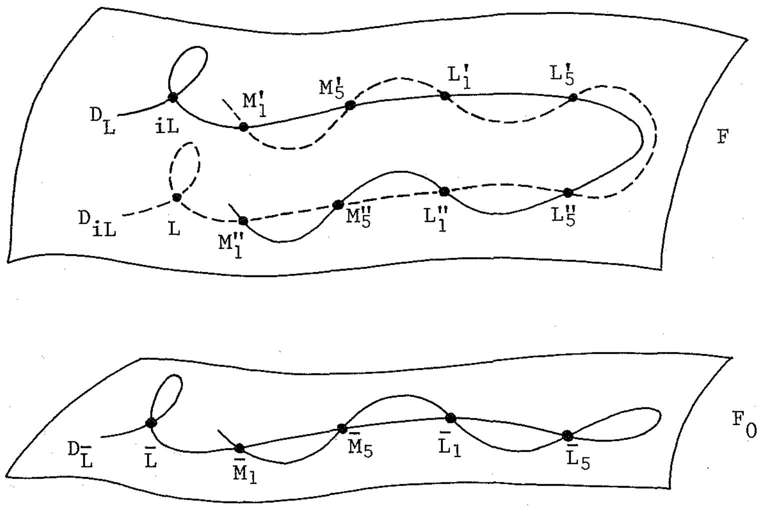


Figure 4.

To see that the double point $iL \in D_L$ is an ordinary one, take a sufficiently general 2-plane $\Pi \subset \mathbb{P}^3$ through $\bar{L}$ and consider the diagram (analogue of (4.5))

$$\begin{array}{ccccc} \widetilde{f^{-1}\Pi} & & \xrightarrow{\quad \tilde{q} \quad} & & F \\ & \searrow & & \nearrow & \\ \downarrow \tilde{p} & & \mathbb{P}R_F & \xrightarrow{q} & F \\ & & \downarrow p & & \\ f^{-1}\Pi & \hookrightarrow & X & & \end{array}$$

with $\widetilde{f^{-1}\Pi} = p^{-1}(f^{-1}\Pi)$. The map $\tilde{q}$ exhibits $\widetilde{f^{-1}\Pi}$ as the blown-up of F at the 56 lines of $f^{-1}\Pi$ lying above the 28 bitangents on Π . Set--theoretically, $\widetilde{f^{-1}\Pi}$ is described as

$$\widetilde{f^{-1}\Pi} = \{ (x,l) \mid l \in F \, , \, x \in l \cap f^{-1}\Pi \}$$

and the exceptional locus above iL is identified with the line iL itself. The proper transform of $D_L \subset F$ in $\widetilde{f^{-1}\Pi}$ has to meet iL with multiplicity two, as we already know. Since the points P and Q belong to this intersection and are distinct, iL is an ordinary double point of D_L, and this finishes the proof of Proposition (5.2) .

(5.3) REMARK. The above identification of the projective tangent space of F at iL $\in$ F with the line iL itself suggests the question as whether it would be possible to get an identification (like for the cubic threefold , cf [8]) between the projective tangent bundle of F and the universal line $\mathbb{P}R_F$. The answer is negative, as shown by the insolvability of the equation $\Omega^1_F \simeq L \otimes R_F$ for $L \in \mathrm{Pic}(F)$, which is an easy exercise concerning Chern classes (cf Section 3) .

To complete the above description we show the irreducibility of the curve D_L for general $L \in F$ and make a remark about the degeneration of D_L as L tends to a line L_0 above a hyperflex $\overline{L}_0$.

Suppose that $\overline{L}$ tends to a (sufficiently general) hyperflex $\overline{L}_0$. Then, in (5.2) , the two 5-tuples $\overline{L}_i$ and $\overline{M}_i$, i = 1,...,5 , collapse into a single 5-tuple $\overline{N}_1 ,\ldots, \overline{N}_5$, while the contact points $\overline{P}$ and $\overline{Q}$ tend to the unique contact point $\overline{R}$ of $\overline{L}_0$. The curve $D_{\overline{L}_0}$ has six double ponts, at $\overline{N}_1 ,\ldots, \overline{N}_5$ and $\overline{L}_0$. The first five points are tacnodal ones, each tacnode being the limit of two nodes of $D_{\overline{L}}$. The singularity at $\overline{L}_0$ will be shown in a moment to be a cusp, and the curve D_{L_0} has the cusp at iL_0 as only singularity. Furthermore, the curves D_{L_0} and D_{iL_0} meet tangentially at the 10 points N'_i , N''_i , i = 1,...,5 lying above $\overline{N}_1 ,\ldots, \overline{N}_5$.

To see that iL_0 is a cusp of D_{L_0} , observe first that, as L tends to L_0 , the pair of distinct tangent lines at $iL \in D_L$ tends to a double tangent line at $iL_0 \in D_{L_0}$, by the proof of (5.2) . So, D_{L_0} has either a cusp or a tacnode at iL_0 . To see that the first case holds, it suffices (e.g.) to exhibit a curve meeting D_{L_0} with multiplicity exactly 3 at iL_0 . For example, take the curve C of lines of X meeting $f^{-1}(\overline{l})$, where $\overline{l} \subset \mathbb{P}^3$ is a sufficiently general line through $\overline{R}$. Each of the remaining 27 bitangents of the 2-plane $\overline{L}_0 \vee \overline{l} \subset \mathbb{P}^3$ yields a single line of X meeting both L_0 and $f^{-1}(\overline{l})$. On the other side there are 5 other bitangents, besides $\overline{L}_0$, through $\overline{R}$, and each of them yields two lines of X meeting L_0 and $f^{-1}(\overline{1})$. We get in this way 37 intersection points of D_{L_0} with C. Being $D_{L_0}\cdot C = \upsilon\cdot\rho = 2\upsilon^2 = 40$, one easily concludes that the three remaining multiplicities have to correspond to L_0, q.e.d.

The curve D_{L_0} being connected and having a cusp as only singularity, it must be irreducible; hence D_L is irreducible, too, for general $L \in F$. We summarize our conclusions in the following

(5.4) COROLLARY. *For general choice of* $L \in F$, *the incidence curve* D_L *is irreducible and has an ordinary double point at* iL *as only singularity. This double point degenerates into a cusp, as* L *tends to a sufficiently general line* $L_0 \in F$ *above a hyperflex of* $S \subset \mathbb{P}^3$. *The effective genus of* D_L *is* 70.

J(X) as a generalized Prym variety

Our next purpose is to prove that J(X) is isomorphic with a generalized Prym variety in the sense of Tyurin ([28], Lecture 5). This will be the case for any smooth X, without the usual restriction about S containing no lines. Its interest lies in the fact that it yields, by work of Bloch and Murre [5], an isomorphism $A^2(X) \simeq J(X)$ between the Chow group of rational equivalence classes of algebraic 1-cycles of X algebraically equivalent with 0, with the underlying abstract group of J(X).

We start with the following description, which is a consequence of (0.5), (0.11), (3.32) and (3.33) (Figure 5):

(5.5) Let L and l be meeting lines of X, but otherwise in general position. The 20 lines meeting both L and l are divided into 10 lines $m^{(k)}$, $k = 1,\dots,10$ through the intersection $L \cap l$ and 10 other lines. The latter ones are divided into 5 groups of two meeting lines $n^{(k)}$, $n^{(k+5)}$, $k = 1,\dots,5$, with no other incidences. In the Chow group $CH^2(X)$ we have the equalities ($h = c_1\mathcal{O}_X(1)$, cf Section 0):

$$\begin{aligned} &L + iL = l + il = h^2 \ , \\ (5.6)\qquad &iL + il = 2h^2 - (L + l) = n^{(1)} + n^{(6)} = \dots = n^{(5)} + n^{(10)} \ , \\ &6h^2 = L + l + m^{(1)} + \dots + m^{(10)} \ . \end{aligned}$$

In particular, to any pair of meeting lines (L,l) one can associate naturally 5 other meeting couples $(n^{(1)}, n^{(6)})$,..., $(n^{(5)}, n^{(10)})$. The 5--tuple associated with anyone of the pairs $(n^{(k)}, n^{(k+5)})$ can be described in terms of these data: it consists of (L,l) plus the conjugates $(in^{(r)}, in^{(r+5)})$ of the $(n^{(r)}, n^{(r+5)})$, $r \neq k$.

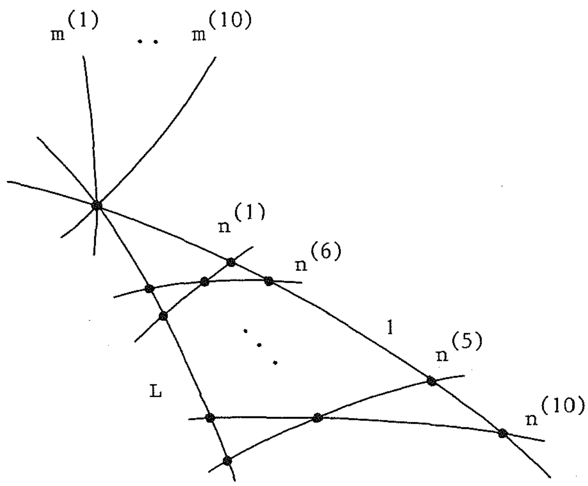
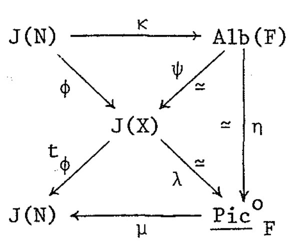


Figure 5.

We fix now a sufficiently general $L \in F$, and let N be the normalization of the curve D_L , a smooth curve of genus 70 , by (5.4) . The curve N parametrizes a family of lines of X , hence there is an Abel--Jacobi morphism

$$\phi : J(N) \longrightarrow J(X) \quad .$$

Together with its transpose ${}^t\phi$, these maps fit into a commutative diagram



where κ and μ are repectively the Albanese and Picard morphisms induced by the map $N \longrightarrow F$, the isomorphisms of the right hand triangle coming from (4.1) .

For all l_1 , $l_2 \in F$ we have: $\eta(l_1 - l_2) = D_{l_1} - D_{l_2}$ in $\underline{\mathrm{Pic}}^{\circ}{}_F$ (cf [8], p. 301 , or also (6.8) below) . We put:

$$\rho : J(N) \longrightarrow J(N) \quad , \quad \rho = {}^t\phi\phi \quad .$$

By the foregoing we get, for all l_1 , $l_2 \in N$:

$$\rho(l_1 - l_2) = D_{l_1}|N - D_{l_2}|N \quad ,$$

where $D_l|N$ means the divisor class of N gotten by pullback of D_l by the map $N \longrightarrow F$.

(5.7) LEMMA. *The following relation holds in* $J(N)$: $\rho^2 + 6\rho = 0$.

PROOF. It suffices to check this for $l_1 - l_2 \in J(N)$ with l_1 , l_2 sufficiently general in N . Write $m_i^{(\alpha)}$, $n_i^{(\beta)}$, $1 \le \alpha,\beta \le 10$ the lines which meet both L and l_i , for each $i = 1,2$ (here we follow the notations introduced in (5.5)) . Then

$$(5.8) \qquad \rho(l_1 - l_2) = (\textstyle\sum m_1{}^{(\alpha)} - \sum m_2{}^{(\alpha)}) + (\sum n_1{}^{(\beta)} - \sum n_2{}^{(\beta)}) \quad ,$$

the sums ranging from 1 to 10 . As recalled in (5.6) , the cycles $\sum m_1{}^{(\alpha)} + L + l_1$ and $\sum m_2{}^{(\alpha)} + L + l_2$ of X are rationally equivalent, hence $\phi(\sum m_1{}^{(\alpha)} - \sum m_2{}^{(\alpha)}) + \phi(l_1 - l_2) = 0$ and therefore

$$(5.9) \qquad \rho(\textstyle\sum m_1{}^{(\alpha)} - \sum m_2{}^{(\alpha)}) + \rho(l_1 - l_2) = 0 \quad .$$

Similarly, for $\beta = 1,\ldots,5$, $i = 1,2$, the cycles $n_i{}^{(\beta)} + n_i{}^{(\beta+5)} + L + l_i$ are rationally equivalent, hence $\phi(n_1{}^{(\beta)} + n_1{}^{(\beta+5)} - n_2{}^{(\beta)} - n_2{}^{(\beta+5)}) + \phi(l_1 - l_2) = 0$ and so

$$(5.10) \qquad \rho(\textstyle\sum n_1{}^{(\beta)} - \sum n_2{}^{(\beta)}) + 5\rho(l_1 - l_2) = 0 \quad .$$

Relations (5.8) - (5.10) imply finally:

$$\rho^2(l_1 - l_2) + 6\rho(l_1 - l_2) = 0 \quad , \quad \text{q.e.d.}$$

The second fact we shall need is:

(5.11) LEMMA. μ *is a closed inmersion.*

PROOF. This will follow if we show that the natural map

$$h : H_1(N,\mathbb{Z}) \longrightarrow H_1(F,\mathbb{Z})$$

is surjective, for then the transfer map $H_3(F,\mathbb{Z})/(\text{torsion}) \longrightarrow H_1(N,\mathbb{Z})$ will be an embedding as a direct summand, from which the injectivity of μ follows at once.

To rpove the surjectivity of h we remark that, D_L being an ample curve of F with an ordinary double point as only singularity, the condition of [6] at the bottom of p. 215, lines -8/-6, is fulfilled; hence, by the results of that paper, the morphism $\pi_1(D_L) \longrightarrow \pi_1(F)$ is surjective, and the same holds for $H_1(D_L,\mathbb{Z}) \longrightarrow H_1(F,\mathbb{Z})$.

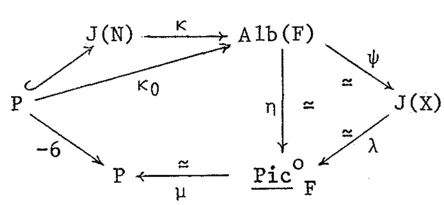
On the other hand, we can write a direct sum decomposition $H_1(D_L,\mathbb{Z}) = H_1(N,\mathbb{Z}) \oplus \mathbb{Z}\delta$, where δ is a singular 1-cycle on D_L which vanishes as L tends to a line over a hyperflex (cf (5.4)). So, the image of δ in $H_1(F,\mathbb{Z})$ is zero, and h is surjective, q.e.d.

Here we could stop and refer to [5], Section 7. Namely, we are in the situation of that paper by putting

$$\sigma = \rho - 1 \quad , \quad q = 6 \quad .$$

This is Assumption I of (loc.cit.) ; Assumption II follows from $\rho = {}^t\rho$, and Assumption III is essentially given by (5.11) together with (4.1). For completeness sake however, we work it out until the isomorphism statement.

Putting $P = \text{Im}(\mu) = \text{Im}(\rho)$ (the map κ is surjective), this is an abelian subvariety of $J(N)$, isomorphic with $J(X)$. We have a commutative diagram



and the following formulae hold:

i) $(\psi\kappa_0)^*\Theta_X \sim 6(\Theta_N|P)$.

In fact, if $\alpha,\beta \in H_1(P,\mathbb{Z})$ we have (cf e.g. [8]) :

$$
\begin{aligned}
((\psi\kappa_0)^*\Theta_X\cdot(\alpha\times\beta))_P &= (\Theta_X\cdot((\psi\kappa_0)_*\alpha \times (\psi\kappa_0)_*\beta))_{J(X)} = \\
&= -(\psi_*(\kappa_{0*}\alpha)\cdot\psi_*(\kappa_{0*}\beta))_X = -(\kappa_{0*}\alpha\cdot\eta_*(\kappa_{0*}\beta))_F = \\
&= -(\alpha\cdot\mu_*\eta_*\kappa_{0*}\beta)_N = -(\alpha\cdot(-6\beta))_N = 6(\alpha\cdot\beta)_N = \\
&= 6(\Theta_N\cdot(\alpha\times\beta))_{J(N)} = (6(\Theta_N|P)\cdot(\alpha\times\beta))_P \quad ,
\end{aligned}
$$

where the standard identifications between the various homology groups have been used, as well as the definition of the polarizations of $J(X)$ and $J(N)$ in terms of the Poincaré pairings of X and N respectively.

ii) If Ξ is an effective divisor of P with the property that $(\mu\lambda)^*\Xi = \Theta_X$, then: $36\Xi \sim (\mu\lambda\psi\kappa_0)^*\Xi \sim (\psi\kappa_0)^*\Theta_X$.

Combining i) and ii) we get $\Theta_N|P \sim 6\Xi$, i.e.: the abelian subvariety $P \subset J(N)$ carries a natural principal polarization Ξ , being $\frac{1}{6}$ of the restriction of the polarization of $J(N)$, and such that (P , Ξ) is isomorphic with $(J(X) , \Theta_X)$.

Since by (1.9) the description (5.4) goes through without change in the case where S is allowed to contain lines of $\mathbb{P}^3$, an obvious rigidity argument allows us to state finally (compare [5], Theorem 7.16) :

(5.12) <u>THEOREM</u>. *Let* X *be a double solid with smooth quartic discriminant* $S \subset \mathbb{P}^3$ *(but otherwise no conditions on* S*) and let* $L \subset X$ *be a sufficiently general line. Calling* N *the normalization of the incidence curve* D_L *, the transpose of the Abel-Jacobi map*

$$
{}^t\phi : J(X) \longrightarrow J(N)
$$

induces an isomorphism of $(J(X) , \Theta_X)$ *with the principally polarized generalized Prym variety* (P , Ξ) *associated to the incidence correspondence on* N .

As already mentioned, this yields, by using the same methods as in [5], Section 8 :

(5.13) <u>COROLLARY</u>. *The natural map* $A^2(X) \longrightarrow J(X)$ *is an isomorphism of* $A^2(X)$ *with the underlying abstract group of* $J(X)$.

6. ON THE THETA DIVISOR OF $J(X)$

Selected traces of Θ on F

Looking at the theory of curves, the Riemann parametrization theorem for the theta divisor of the polarized Jacobian can be viewed as a corollary of the study of the intersection behaviour of the translates of Θ_C with the curve C itself, embedded in its Jacobian. In the present case we have a principal polarization of the Picard variety of a surface whose geometry is fairly well understood. One may expect therefore to get information about Θ_X by inspecting the traces of its translates on this surface. We devote the present section to a study in this direction. The Abel-Jacobi map $F \longrightarrow J(X)$ will be denoted ψ .

The first step in the case of curves was to compute the degree (= the genus of the curve) of the traces of Θ_C on C . Similarly, we start looking for the cohomology class of the traces of Θ_X on F . This is done following [8] . To begin with, the classes of $\psi^*\Theta_X$, $\upsilon \in NS^1(F)$ in $H^2(F,\mathbb{C})$ are proportional. This is the standard invariant-theoretic argument, used in (loc.cit.) , p.p. 335-336 ; we sketch it briefly, keeping the above symbols to denote the corresponding cohomology classes. These belong both to $H^2(F,\mathbb{C})^+$; for υ it is obvious (cf (5.1)) , and for $\psi^*\Theta_X$ it follows from the fact that $H^2(J(X),\mathbb{C}) = \Lambda^2 H^1(J(X),\mathbb{C}) = \Lambda^2 H^1(F,\mathbb{C})$ and the elements of $H^1(F,\mathbb{C})$ being antiinvariant. Therefore, by using the non--degeneration of the intersection product on F , together with (3.60) , it is equivalent to prove the proportionality of the bilinear forms B_1 and B_2 on the vector space $H^1(F,\mathbb{C})$ defined by sending (α_1,α_2) respectively to

$$\int_F \alpha_1 \wedge \alpha_2 \wedge \psi^*\Theta_X \qquad \text{and} \qquad \int_F \alpha_1 \wedge \alpha_2 \wedge \upsilon \quad .$$

The variety X belongs to a Lefschetz pencil of hypersurfaces of a fourfold W with $H^3(W,\mathbb{C}) = 0$ (cf (4.3)) . The monodromy therefore acts irreducibily on $H^3(X,\mathbb{C})$ and hence on $H^1(F,\mathbb{C})$. The above bilinear forms clearly are invariant under the induced action. Taking a suitable constant $c \in \mathbb{C}$, the form $B_1 + cB_2$ is degenerate; its kernel is a non-zero invariant subspace of $H^1(F,\mathbb{C})$, hence equals the whole space, i.e. $B_1 + cB_2 = 0$, q.e.d.

To get the proportionality factor, we use the formula

$$\int_X \omega \wedge \omega' = -\frac{1}{12}\int_F \psi^*\omega \wedge \psi^*\omega' \wedge \rho \qquad \forall\ \omega,\omega' \in H^3(X,\mathbb{C}) = H^1(J(X),\mathbb{C}) ,$$

which is established exactly in the same way as the analogous formula of (loc.cit.) , (11.4), by using the correspondence $\mathbb{P}R_F$ between F and X (cf the diagram in the proof of (5.2)) . Next, if we let $\omega_1,\ldots,\omega_{10}$ (resp. $\omega_1',\ldots,\omega_{10}'$) be a basis of $H^{2,1}(X) = H^{1,0}(J(X))$ (resp. $H^{1,2}(X) = H^{0,1}(J(X))$) such that

$$\int_X \omega_i \wedge \omega_j' = -\delta_{ij} \qquad \forall\ i,j ,$$

then one has, in $H^2(J(X),\mathbb{C})$:

$$\Theta_X = \sum_1^{10} \omega_i \wedge \omega_i' .$$

Combining these expressions we get

$$\int_F \psi^*\Theta_X \wedge \rho = \sum_1^{10}\int_F \psi^*\omega_i \wedge \psi^*\omega_i' \wedge \rho = -12\sum_1^{10}\int_X \omega_i \wedge \omega_i' = 120 .$$

Since on the other side it is

$$\int_F \upsilon \wedge \rho = 40 ,$$

we conclude that $\psi^*\Theta_X = 3\upsilon$ in $H^2(F,\mathbb{C})$.

To finish this computation, as well as in a moment, in the proof of Proposition (6.3) , we shall use Clemens' degeneration of the surface F ([7]) . Letting degenerate the discriminant surface $S \subset \mathbb{P}^3$ of X inside a linear pencil into $2Q$, where $Q \subset \mathbb{P}^3$ is a (sufficiently generally chosen) quadric, the Fano surfaces of the smooth double solids above a small punctured disk around the critical value $t = 0$ fit into a variety which can be smoothly compactified above the origin with singular fibre F_{lim} described as follows:

If $C \subset \mathbb{P}^3$ denotes the smooth complete intersection curve $Q \cap S$, one takes two copies of $C^{(2)}$; each of them contains two disjoint smooth curves Γ_1, Γ_2 defined as the curves of pairs (x,y) such that the line $\overline{xy}$ belongs to one of the two rulings of the quadric Q. Since such a line meets C at 4 points, the curves Γ_k carry natural fixed point free involutions. The surface F_{lim} is gotten by glueing the two copies of $C^{(2)}$ along Γ_1 and Γ_2 by means of their respective involutions.

As mentioned in the introduction, this yields an alternative way of getting the results of sections 3 and 4. We use it here to prove that the surface F has no torsion. (If we have well understood, the proposition below is a particular case of a more general result of Clemens, to appear.)

PROPOSITION. *The group* $H^2(F,\mathbb{Z})$ *is torsion free, hence* $NS^1(F)$ *embeds into* $H^2(F,\mathbb{C})$.

PROOF. We show, equivalently, that $H_1(F,\mathbb{Z})$ has no torsion. We use freely some standard facts from the theory of degenerations of algebraic surfaces (cf e.g. [24]) as well as the above facts from [7]. For simplicity we skip the coefficient group $\mathbb{Z}$ in our notations in this proof.

Notice that the curves Γ_1 and Γ_2 are algebraically equivalent in $C^{(2)}$: if g^1_4, h^1_4 are the special series induced on C by the rulings of the quadric Q, we may write $\Gamma_1 = \{(x,y) \in C^{(2)} \mid x+y \leq \text{some member of } g^1_4\}$ and similarly for Γ_2. Hence, by choosing fixed divisors $D_1 \in |K_C - g^1_4|$ and $D_2 \in |K_C - h^1_4|$, we obtain the algebraic equivalences in $C^{(2)}$:

$$\begin{aligned} &\Gamma_1 + 12C \sim \\ &\sim \{(x,y) \in C^{(2)} \mid x+y \leq \text{some member of the pencil } D_1 + g^1_4 \subset |K_C|\} \sim \\ &\sim \{(x,y) \in C^{(2)} \mid x+y \leq \text{some member of the pencil } D_2 + h^1_4 \subset |K_C|\} \sim \\ &\sim \Gamma_2 + 12C \end{aligned}$$

(Here C stands for the curve of $C^{(2)}$ defined as $\{(x,P) \mid x \in C\}$ for some fixed $P \in C$). Hence $\Gamma_1 \sim \Gamma_2$ as claimed.

Therefore, since $\Gamma_1 \cdot \Gamma_2 = 0$, it is $\Gamma_1^2 = \Gamma_2^2 = 0$, and the normal bundles of Γ_i, $i = 1,2$ in $C^{(2)}$ are topologically trivial. This allows one to describe F topologically with aid of F_{lim} in a similar way as done in [8], p.p. 319-320.

By using the mapping cylinder of a degeneration map $F \longrightarrow F_{lim}$ together with the Thom isomorphism Theorem, one gets a long exact sequence

$$\longrightarrow H_{q-1}(\Gamma_1) \oplus H_{q-1}(\Gamma_2) \longrightarrow H_q(F) \longrightarrow H_q(F_{lim}) \longrightarrow H_{q-2}(\Gamma_1) \oplus H_{q-2}(\Gamma_2) \longrightarrow$$

and, in particular

$$H_2(F_{lim}) \xrightarrow{f} H_0(\Gamma_1) \oplus H_0(\Gamma_2) \longrightarrow H_1(F) \longrightarrow H_1(F_{lim}) \longrightarrow 0 .$$

Identifying the second term with $\mathbb{Z}^2$, one finds that, for classes $\alpha \in H_2(F_{lim})$ coming (by one of the two inclusion mappings) from classes $\alpha \in H_2(C^{(2)})$ it is $f(\alpha) = (\alpha\cdot\Gamma_1, \alpha\cdot\Gamma_2)$, the intersection products being taken in $C^{(2)}$.

We show in the first place that there exists $\alpha \in H_2(C^{(2)})$ such that $\alpha\cdot\Gamma_1 = 1$. By the unimodularity of the intersection product in $C^{(2)}$, it suffices to see that $H_2(C^{(2)})/\mathbb{Z}\gamma$ is torsion free, where γ = class of the curve Γ_1 . Embedding C in $C^{(2)}$ as $C \simeq \{(x_0,x) | x \in C\}$ we have $C\cdot\Gamma_1 = 3$. If it were $n\gamma = m\delta$ with $n,m \in \mathbb{Z}$, $(n,m) = 1$, and $\delta \in H_2(C^{(2)})$, one would get $3n = m(\delta\cdot C)$, hence m divides 3 . On the other hand, taking direct images by the Albanese mapping $\psi : C^{(2)} \longrightarrow J(C)$, one has in $J(C)$: $n\psi_*(\gamma) = m\psi_*(\delta)$. By the theorem of Recillas ([25]) , $J(C)$ is the Prym variety of the curve Γ_1 with its involution, hence the theory of Prym varieties tells us that $\psi_*(\gamma)$ is twice the class of $\Theta_C^8 / 8!$. If $\eta_1,\ldots,\eta_{18}$ is a basis of $H_1(C)$ such that $\eta_i\cdot\eta_{i+9} = -\eta_{i+9}\cdot\eta_i = 1$, $1 \le i \le 9$ and $\eta_i\cdot\eta_j = 0$ otherwise, the above class is written

$$\eta_1 \times \eta_{10} + \ldots + \eta_9 \times \eta_{18}$$

in $H_2(J(C))$. Since the $\eta_i \times \eta_j$, $i<j$ form a basis of $H_2(J(C))$ we finally deduce that $m = \pm 1$, as was to be shown.

Therefore, the image of $H_0(\Gamma_1) \oplus H_0(\Gamma_2)$ in $H_1(F)$ is a cyclic subgroup. In a moment we shall see that $H_1(F_{lim}) \simeq \mathbb{Z}^{19}$ holds. This will force the above cyclic group to be isomorphic with $\mathbb{Z}$, and $H_1(F) \simeq \mathbb{Z}^{20}$ will follow (notice that we have not used our former computations of Section 3 - but, of course, the theory of [7]) .

To compute $H_1(F_{lim})$, consider the Mayer-Vietoris sequence

$$(\dots) \longrightarrow H_1(\Gamma_1) \oplus H_1(\Gamma_2) \xrightarrow{g} H_1(C^{(2)}) \oplus H_1(C^{(2)}) \longrightarrow H_1(F_{lim}) \longrightarrow$$

$$\longrightarrow H_0(\Gamma_1) \oplus H_0(\Gamma_2) \longrightarrow H_0(C^{(2)}) \oplus H_0(C^{(2)}) \longrightarrow H_0(F_{lim}) \longrightarrow 0 \ .$$

The bottom part of this sequence shows that the upper one can be completed with $\longrightarrow \mathbb{Z} \longrightarrow 0$ at the right hand side. It suffices then to show that the cokernel of g yields $\mathbb{Z}^{18}$. Writing $\upsilon_k : \Gamma_k \hookrightarrow C^{(2)}$, $k = 1,2$ the natural inclusions and $i_k : \Gamma_k \longrightarrow \Gamma_k$, $k = 1,2$ the involutions, the map g has matrix

$$\begin{pmatrix} \upsilon_{1*} & \upsilon_{2*} \\ -\upsilon_{1*}i_{1*} & -\upsilon_{2*}i_{2*} \end{pmatrix} .$$

The maps υ_{k*} are also gotten by taking H_1 from the Albanese morphisms

$$J(\Gamma_k) \longrightarrow \mathrm{Alb}(C^{(2)}) \ .$$

But, as noticed earlier in this proof, these can be identified with the natural projection maps

$$1 - i_k : J(\Gamma_k) \longrightarrow \mathrm{Pr}(\Gamma_k, i_k) \ .$$

Therefore, by the theory of Prym varieties, the maps υ_{k*} are onto; moreover, $-\upsilon_{k*}i_{k*} = \upsilon_{k*}$. Thus the image of g is the diagonal of $H_1(C^{(2)}) \oplus H_1(C^{(2)})$ and the cokernel of g is isomorphic with $H_1(C^{(2)})$. Since $p_a(C) = 9$, the desired result follows, q.e.d.

The injectivity of the natural map $NS^1(F) \longrightarrow H^2(F,\mathbb{C})$ allows finally to conclude:

(6.1) <u>PROPOSITION</u>. *The relation* $\psi^*\Theta_X = 3\upsilon$ *holds in* $NS^1(F)$.

(6.2) <u>REMARK</u>. *In the case of the cubic threefold* $Y_3^3 \subset \mathbb{P}^4$ *and its Fano surface of lines* F' *one has* ([8]) *, calling* $\upsilon' \in NS^1(F')$ *the class of the curve of lines incident with a given one:* $\psi^*\Theta_Y = 2\upsilon'$ *in* $NS^1(F')$. This will be used throughout in the sequel, for comparison.

We denote by $\{3\upsilon\}$ the variety of effective divisors on F with algebraic equivalence class 3υ (cf [20], [13]). Fixing a copy Θ of the theta divisor of $J(X)$, the dual statement of (4.1) implies that the map from $J(X)$ to $\underline{\mathrm{Pic}}^{3\upsilon}_F$ assigning to $a \in J(X)$ the linear equivalence class $\psi^*(\Theta + a)$ is an isomorphism between these varieties. As a consequence, the natural map

$$\{3\upsilon\} \longrightarrow \underline{\mathrm{Pic}}^{3\upsilon}_F$$

is surjective; its fibres are the complete linear systems inside $\{3\upsilon\}$. Clearly, there is exactly one irreducible component of $\{3\upsilon\}$ dominating $\underline{\mathrm{Pic}}^{3\upsilon}_F$. We claim that this component maps birationally onto the latter variety; this will follow from the next

(6.3) PROPOSITION. *If* X *is sufficiently general and* l_i, $i = 1,2,3$ *are sufficiently general lines in* X, *then* $h^0\mathcal{O}_F(\Sigma D_{l_i}) = 1$ *holds.*

PROOF. If a line $l \in F$ specializes to $l' \in F_{lim}$, the curve D_l degenerates into the curve $D_{l'}$ of "lines" (cf [7]) in the limit threefold, which meet l'. If we show that there exist l'_1, l'_2, $l'_3 \in F_{lim}$ such that $h^0\mathcal{O}_{F_{lim}}(\Sigma D_{l'_i}) = 1$ then the proposition follows by flatness and semicontinuity.

The curves $D_{l'}$ are easily seen to consist of two halves, say α and β, each one lying on one of the two copies of $C^{(2)}$, and being described as follows. Call δ the pair of points of C where the bisecant which underlies l' meets this curve; then $\beta = (\delta, C)$, where we define, if $p \in C$: $(p,C) = \{(p,x) \mid x \in C\} \subset C^{(2)}$, and, if Σx_i is an effective divisor of C, $(\Sigma x_i, C) = \Sigma(x_i, C)$. Part α is described as $\alpha = \{(x,y) \mid x+y+\delta \le$ some member of the linear system $|\mathcal{O}_C(1)|\}$. Part α meets each curve Γ_k, $k = 1,2$ at six points, which are transformed by the involution of Γ_k into the six points where β meets Γ_k.

We take l'_1, l'_2. l'_3 sufficiently general and in such a way that, calling α_j, β_j the respective parts of the curves $D_{l'_j}$, the distribution of these parts among the two copies of $C^{(2)}$ yields $\alpha_1 + \alpha_2 + \beta_3$ on one of them and $\beta_1 + \beta_2 + \alpha_3$ on the other one. This means exactly that the "non exceptional" parts of the "lines" l'_j are dsitributed (2,1) among the two copies of the blowing-up of $\mathbb{P}^3$ in the limit threefold. Consider the standard exact sequence

$$0 \longrightarrow \mathcal{O}_{F_{\lim}}(\Sigma D_{l'_j}) \xrightarrow{\binom{1}{1}} \mathcal{O}_{C^{(2)}}(\alpha_1+\alpha_2+\beta_3) \oplus \mathcal{O}_{C^{(2)}}(\beta_1+\beta_2+\alpha_3) \xrightarrow{(1,-1)} \mathcal{O}_{\Gamma}(\xi) \longrightarrow 0$$

where ξ is the trace of $\Sigma D_{l'_j}$ on $\Gamma = \Gamma_1 + \Gamma_2$. If $x \in C$ is general, the curve $\beta_1 + \beta_2 + \alpha_3$ of $C^{(2)}$ cuts out on (x,C) a divisor which is linearly equivalent with $\delta_1 + \delta_2 + c_1\mathcal{O}_C(1) - \delta_3 - x$, where δ_j are the pairs of contact of the bisecants associated with the l'_j, $j = 1,2,3$. By Riemann-Roch and the generality assumption on the l'_j we have $\ell(\delta_1 + \delta_2 + c_1\mathcal{O}_C(1) - \delta_3) = 2$, hence $\ell(\delta_1 + \delta_2 + c_1\mathcal{O}_C(1) - \delta_3 - x) = 1$ for general $x \in C$. So, the curve $\beta_1 + \beta_2 + \alpha_3$ is linearly isolated in $C^{(2)}$, since so is its intersection with a general (x,C), on that curve. I.e. : $h^0\mathcal{O}_{C^{(2)}}(\beta_1 + \beta_2 + \alpha_3) = 1$. By the exact sequence

$$0 \longrightarrow H^0\mathcal{O}_{F_{\lim}}(\Sigma D_{l'_j}) \longrightarrow H^0\mathcal{O}_{C^{(2)}}(\alpha_1+\alpha_2+\beta_3) \oplus H^0\mathcal{O}_{C^{(2)}}(\beta_1+\beta_2+\alpha_3) \longrightarrow H^0\mathcal{O}_{\Gamma}(\xi),$$

the proof will be complete if we show that $H^0\mathcal{O}_{C^{(2)}}(\alpha_1 + \alpha_2 + \beta_3 - \Gamma) = 0$.

Taking again a general $x \in C$, the divisor $\alpha_1 + \alpha_2 + \beta_3 - \Gamma$ cuts out on (x,C) a divisor equivalent with $2c_1\mathcal{O}_C(1) - \delta_1 - \delta_2 + \delta_3 - g - h$, where g and h are members of the two 4^{th} degree series of C respectively, hence (being $g + h = c_1\mathcal{O}_C(1)$) this is the same as $c_1\mathcal{O}_C(1) - \delta_1 - \delta_2 + \delta_3$. By the previous computations in this proof, $\ell(c_1\mathcal{O}_C(1) - \delta_1 - \delta_2 + \delta_3) = 0$, hence, a fortiori, $h^0\mathcal{O}_{C^{(2)}}(\alpha_1 + \alpha_2 + \beta_3 - \Gamma) = 0$, q.e.d.

(6.4) <u>REMARK</u>. One proves in a similar way, by using the degeneration exhibited in [8], p.319 : *If* l_1, l_2 *are sufficiently general lines of a sufficiently general cubic threefold* $Y = Y_3^3 \subset \mathbb{P}^4$, *then* $h^0\mathcal{O}_{F'}(D_{l_1} + D_{l_2}) = 1$.

(6.5) Therefore, the general fibre of the map $\{3\upsilon\} \longrightarrow \underline{\mathrm{Pic}}^{3\upsilon}_F$ consists of a single point, and the traces on F of the translates of Θ describe a dense subset of a component of $\{3\upsilon\}$ which is birational with $J(X)$. However, we can make a stronger use of (6.3) by means of the following

(6.6) <u>PROPOSITION</u>. *If* $T \in \{3\upsilon\}$ *satisfies* $h^0\mathcal{O}_F(T) = 1$ *then* T *is the trace of a translate of* Θ. (*)

<u>PROOF</u>. Writing $[T] = \psi^*(\Theta + \varepsilon_0)$ in $\underline{\mathrm{Pic}}^{3\upsilon}_F$, it suffices to show that $\psi(F)$ is not contained in $\Theta + \varepsilon_0$. Assume on the contrary that $\psi(F) \subset \Theta + \varepsilon_0$. Consider the divisor $\tilde{\Theta} \subset J(X) \times J(X)$,

---

(*) It was C.H. Clemens' suggestion, to exploit the jumping phenomena of the linear systems $|\psi^{-1}(\Theta + a)|$, as the theta divisor contains $\psi(F)$.

$$\tilde{\Theta} = \cup_{\varepsilon\in J(X)} \{\varepsilon\} \times (\Theta + \varepsilon) \quad .$$

Its inverse image by the map $1 \times \psi : J(X) \times F \longrightarrow J(X) \times J(X)$ is the divisor

$$\tilde{\Theta}_F = \cup_{\varepsilon\in J(X)} \{\varepsilon\} \times \psi^{-1}(\Theta + \varepsilon) \quad .$$

Call $W = \{ \varepsilon \in J(X) \mid \psi F \subset \Theta + \varepsilon \} \subset J(X)$; by our assumption, $\varepsilon_0 \in W$. As $\varepsilon \in J(X) \setminus W$ tends to ε_0 , the curve $\psi^{-1}(\Theta + \varepsilon)$ tends to a representative of $\psi^*(\Theta + \varepsilon_0)$ hence, by our hypothesis, to the curve T . Thus $\{\varepsilon_0\} \times F$ is not contained in the closure of $\tilde{\Theta}_F \setminus (W \times F)$. This implies that $\tilde{\Theta}_F$ is reducible and that $W \times F$ contains a component of this divisor, hence that W contains an irreducible subset of dimension 9 . Being $\psi(1)-W \subset \Theta$, $\forall 1 \in F$, we conclude that Θ has an irreducible component M such that $\psi(1) + M = M$, $\forall 1 \in F$. By the theory of principally polarized abelian varieties we may write $J(X) = A \times B$ with (A,Θ_A) , (B,Θ_B) in this category and $M = \Theta_A \times B$, Θ_A being irreducible and B eventually zero. Writing $\psi(1) = (\psi_A(1), \psi_B(1))$ it is $\psi_A(1) + \Theta_A = \Theta_A$, hence $\psi_A(1) = 0$ for each $1 \in F$. Thus $\psi(F) \subset 0 \times B$, contradicting the fact that F generates $J(X)$, thereby finishing this proof, q.e.d.

Combination of (6.3) and (6.6) gives:

(6.7) <u>COROLLARY</u>. *Assume that* X *is sufficiently general. Then there exixts a translate* Θ *of the theta divisor such that, in* F^3 ,

$$\psi^{-1}(\Theta - \textstyle\sum_1^3 \psi(1_i)) = \textstyle\sum_1^3 D_{1_i}$$

for general $\underline{1} = (1_1, 1_2, 1_3) \in F^3$.

<u>PROOF</u>. Fix a sufficiently general X and choose a Θ in $J(X)$. By (6.3) and (6.5) , there is a well defined morphism $h : F^3 \longrightarrow J(X)$ such that, for general $\underline{1} \in F^3$:

$$\textstyle\sum_1^3 D_{1_i} = \psi^{-1}(\Theta + h(\underline{1})) \quad .$$

In $\underline{\mathrm{Pic}}^o{}_F$ we get therefore, for all $\underline{1}, \underline{1}^o \in F^3$:

$$\psi^*(\Theta + h(\underline{1})) - \psi^*(\Theta + h(\underline{1}^o)) = [D_{\Sigma 1_i} - D_{\Sigma 1_i^o}] =$$

$$= \psi^*(\Theta + \Sigma\psi(1_i^o)) - \psi^*(\Theta + \Sigma\psi(1_i))$$

(cf (6.8) for the last equality) . Hence $h(\underline{1}) - h(\underline{1}^o) = \Sigma\psi(1_i^o) - \Sigma\psi(1_i)$ and the corollary follows by replacing Θ by $\Theta + h(\underline{1}^o) + \Sigma\psi(1_i^o)$, q.e.d.

(6.8) REMARK. In the proof of Corollary (6.7) we used that the class $[D_{\Sigma 1_i}] + \psi^*(\Theta + \Sigma\psi(1_i))$ of Pic(F) is independent of $\underline{1} = (1_1, 1_2, 1_3) \in F^3$. This is a particular case of the property, for the polarization of J(X) , of being an incidence polarization (compare [2], (3.4.2)) . More generally, if X is any smooth projective threefold defined over $\mathbb{C}$ such that $h^{3,0}(X) = 0$ – hence its intermediate Jacobian yielding a principally polarized abelian variety – , the following holds:

Let S , T *be smooth projective varieties and* Z_S (*resp.* Z_T) *an algebraic cycle of codimension* 2 *on* $S \times X$ (*resp.* $T \times X$) . *Define the incidence divisor class* I(S,T) *in* $\mathrm{Pic}(S \times T)$ *as the projection in* $S \times T$ *of the cycle class* $(Z_S \times T) \cdot (S \times {}^tZ_T)$ *on* $S \times X \times T$. *Let furthermore*

$$\psi_S : S \longrightarrow J(X) \quad , \quad \psi_T : T \longrightarrow J(X)$$

be the Abel-Jacobi maps induced by Z_S *and* Z_T , *and* $\sigma : S \times T \longrightarrow J(X)$ *their sum. Then, if* $\Theta \subset J(X)$ *is a copy of the theta divisor, one has, in* $\mathrm{Pic}(S \times T)$:

$$\sigma^*[\Theta] \equiv I(S,T) \pmod{\mathrm{Pic}(S) \oplus \mathrm{Pic}(T)} .$$

For convenience of the reader, we give a proof of this fact, although this is essentially contained in Part One of [8] . By the Seesaw principle and the Theorem of the square ([18], p.p. 54, 59) this can be stated equivalently as follows.

For any $s \in S$, *we call* $D_s \in \mathrm{Pic}(T)$ *the projection, in* T , *of the cycle class* $I(S,T) \cdot (s \times T)$ *of* $S \times T$; *this is the same as the projection in* T *of the class* $(Z_s \times T) \cdot {}^tZ_T$ *of* $X \times T$. *Then*:

$$D_s + \psi_T^*(\Theta + \psi_S(s)) \in \mathrm{Pic}(T)$$

is independent of $s \in S$.

We prove it in this second form. Put $n = \dim S$, $m = \dim T$. By

[17], p. 1192 , there is a commutative diagram

$$\begin{array}{ccccc} A^n(S) & \xrightarrow[\phi_{o,S}]{} & A^2(X) & \xrightarrow[\lambda_{o,T}]{} & A^1(T) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Alb}(S) & \xrightarrow[\phi_S]{} & J(X) & \xrightarrow[\lambda_T]{} & \underline{\mathrm{Pic}}^o{}_T \end{array}$$

where $A^p(..)$ denotes the Chow group of algebraic cycles of codimension p, algebraically equivalent to zero, modulo rational equivalence; the vertical arrows are the Abel-Jacobi maps. The morphisms $\phi_{o,S}$, $\lambda_{o,T}$ are defined respectively as

$$\phi_{o,S}(z) = (pr_X)_*((z \times X)\cdot Z_S) \quad , \quad \text{for} \quad z \in A^n(S) \; ,$$

$$\lambda_{o,T}(z) = (pr_T)_*((z \times T)\cdot {}^tZ_T) \quad , \quad \text{for} \quad z \in A^2(X) \; .$$

The morphisms ϕ_S , λ_T are characterized by the maps which they induce on the first homology groups; these maps are respectively

$$(\phi_S)_* : H_1(S) \longrightarrow H_3(X) \quad : \quad (\phi_S)_*(\alpha) = (pr_X)_*((\alpha \times X)\cdot[Z_S]) \quad ,$$

$$(\lambda_T)_* : H_3(X) \longrightarrow H_{2m-1}(T) \quad : \quad (\lambda_T)_*(\beta) = (pr_T)_*((\beta \times T)\cdot[{}^tZ_T]) \quad ,$$

where $H_1(S)$, $H_3(X)$, $H_{2m-1}(T)$ are the homology groups with coefficients in $\mathbb{Z}$, taken modulo torsion.

Fix any $s_0 \in S$; our aim is to show that, for all $s \in S$,

$$D_s - D_{s_0} = \psi_T^*(\Theta + \psi_S(s_0)) - \psi_T^*(\Theta + \psi_S(s))$$

holds, in $\underline{\mathrm{Pic}}^o{}_T$. Clearly, identifying $A^1(T)$ with $\underline{\mathrm{Pic}}^o{}_T$,

$$D_s - D_{s_0} = \lambda_{o,T}\, \phi_{o,S}\,(s - s_0) = \lambda_T(\psi_S(s) - \psi_S(s_0)) \quad .$$

Since $\psi_T^*(\Theta + \psi_S(s_0)) - \psi_T^*(\Theta + \psi_S(s)) = \psi_T^*([\Theta - (\psi_S(s) - \psi_S(s_0))] - [\Theta])$, we will be done by showing that, for all $a \in J(X)$, $\lambda_T(a) = \psi_T^*([\Theta - a] - [\Theta])$ holds. This expresses that λ_T coincides with the composition

$$J(X) \xrightarrow[\rho]{\simeq} \underline{\mathrm{Pic}}^o{}_{J(X)} \xrightarrow[\psi_T^*]{} \underline{\mathrm{Pic}}^o{}_T \quad ,$$

ρ being derived from the principal polarization of $J(X)$. Notice that, defining

$$\phi_T : \mathrm{Alb}(T) \longrightarrow J(X)$$

in a similar way as ϕ_S above, the morphism ψ_T^* is identified with the transpose ${}^t\phi_T$, identifying $\underline{\mathrm{Pic}}^o{}_{\mathrm{Alb}(T)}$ with $\underline{\mathrm{Pic}}^o{}_T$. So, the equality $\lambda_T = \psi_T^*\rho$ roughly says that ϕ_T and λ_T are transposes of each other.

To prove this equality, we merely have to check that both terms yield the same maps on the first homology groups. We have ([8], (1.1)), for all $\alpha \in H_1(T)$, $\beta \in H_3(X)$:

$$(\alpha\cdot(\lambda_T)_*\beta)_T = ((\phi_T)_*\alpha\cdot\beta)_X \quad .$$

It suffices to see that the same relation holds, replacing $(\lambda_T)_*$ by $(\psi_T^*)_*\rho_*$. The induced morphism $(\psi_T^*)_*$ is the transfer map deduced from ψ_T :

$$(\psi_T^*)_* : H_{2g-1}(J(X)) \longrightarrow H_{2m-1}(T) \quad ,$$

where we have put $g = \dim J(X)$. So, by the projection formula :

$$(\alpha\cdot(\psi_T^*)_*\rho_*\beta)_T = ((\psi_T)_*\alpha\cdot\rho_*\beta)_{J(X)} = ((\phi_T)_*\alpha\cdot\rho_*\beta)_{J(X)} \quad .$$

The result then follows from the fact that, for all $\gamma,\beta \in H_3(X) = H_1(J(X))$:

$$(\gamma\cdot\beta)_X = (\gamma\cdot\rho_*\beta)_{J(X)} \quad .$$

The latter is a restatement of the definition of the polarization of $J(X)$ in terms of Poincaré duality on X. More explicitly, there are identifications

$$\begin{array}{ccc}
H_1(J(X)) & \xrightarrow{\rho_*} & H_{2g-1}(J(X)) \\
\| & & \simeq \downarrow P_{J(X)} \\
\| & & H^1(J(X)) \\
\| & & \| \\
\| & & \mathrm{Hom}(H_1(J(X),\mathbb{Z}) \\
\| & & \| \\
H_3(X) & \xrightarrow[P_X]{\simeq} & H^3(X)
\end{array}$$

where the symbol P stands for Poincaré duality (all homology and cohomology groups being taken with integral coefficients, modulo torsion) . Using this one derives, for all $\gamma,\beta \in H_3(X)$:

$$(\gamma\cdot\rho_*\beta)_{J(X)} = -(\rho_*\beta\cdot\gamma)_{J(X)} = -\int_\gamma P_{J(X)}\rho_*\beta =$$
$$= -\int_\gamma P_X\beta = -(\beta\cdot\gamma)_X = (\gamma\cdot\beta)_X \quad ,$$

q.e.d.

In our applications T will be the Fano surface of lines and S a family of curves on the threefold, and the classes D_s will be representable by the incidence curves D_{Z_s} on the Fano surface, the latter ones being equal, for general $s \in S$, to the sum of the codimension one components of the suvariety of lines meeting Z_s .

(6.9) REMARK. It is quite clear that (6.6) *holds similarly for the cubic threefold* $Y = Y_3^3$, *replacing* $\{3\upsilon\}$ *by* $\{2\upsilon'\}$ (cf (6.2)) . By using (6.4) one derives in this way a similar statement as that of (6.7) , namely: *If* Y *is sufficiently general, then* $\exists\ \Theta_Y$ *such that, if* $(l_1,l_2) \in (F')^2$ *is sufficiently general, then* $\psi^{-1}(\Theta_Y - \psi(l_1) - \psi(l_2)) = D_{l_1} + D_{l_2}$. We claim that *this yields the parametrization of* Θ_Y *as given in* [8], p. 348 .

In fact, it implies that

$$\psi(l_1) + \psi(l_2) + \psi(D_{l_1+l_2}) \subset \Theta_Y$$

for all $(l_1,l_2) \in (F')^2$. Recall next that the sum of three (distinct) coplanar lines of Y is a rationally constant 1-cycle of Y . Therefore, taking any $(l',l'') \in (F')^2$, if we choose two other lines l and m yielding together with l'' a plane section of Y , thus $\psi(l'') + \psi(l) + \psi(m) =$ = const. , we have

$$\psi(l') - \psi(l'') = \psi(l') + \psi(l) + \psi(m) - \text{const.} \in \Theta_Y - \text{const.} \quad .$$

This shows that the image of the map $(F')^2 \longrightarrow J(Y)$ sending (l',l'') to $\psi(l') - \psi(l'')$ is contained in a translate of the theta divisor. By the infinitesimal theory (cf (2.10)) we see that this map is generically finite onto its image, hence the latter is a divisor of J(Y) . By invariance under monodromy, this divisor has to be homologous with a multiple of Θ_Y

(cf e.g. the proofs of (3.75), (6.1)) hence, being contained in Θ_Y, they coincide. By continuity, the conclusion holds for any smooth cubic threefold, q.e.d.

For the double solid however, the above doesn't yield a parametrization of Θ, but only describes a 7-dimensional subvariety of the theta divisor. For its extension to this case, it is perhaps more interesting to consider the following somewhat sharper statement than (6.9). Call $I_{ij} \subset (F')^3$, $1 \leq i < j \leq 3$, the divisor of triples such that the i^{th} and the j^{th} members are incident. Then, if $\psi_3 : (F')^3 \longrightarrow J(Y)$ denotes the Abel-Jacobi map, one can show that there exists a copy of Θ such that the scheme theoretic inverse image yields $(\psi_3)^{-1}\Theta = \sum_{1\leq i<j\leq 3} I_{ij}$. We shall prove the similar statement for the double solid:

(6.10) PROPOSITION. *If* X *is sufficiently general and* ψ_4 *denotes the Abel-Jacobi map (for a given choice of a base point) of* F^4, *there is a unique translate of the theta divisor yielding (scheme theoretically) the inverse image* $\sum_{1\leq i<j\leq 4} I_{ij}$ *by* ψ_4.

PROOF. Uniqueness is clear. As for the existence, we compute first the cohomology class of the pullback of Θ to F^k, $k \geq 1$. Call this class $\xi_k \in H^2(F^k,\mathbb{C})$, and put $I_{ij} \subset F^k$ as above. Put also $D_i \subset F^k$, divisor of the k-ples of lines with the i^{th} member incident with a given (fixed) line. Denoting the cohomology classes of these divisors by the same symbols, we claim:

$$(6.11) \qquad \xi_k = \sum_{1\leq i<j\leq k} I_{ij} + (4-k)\sum_1^k D_i \quad .$$

This is clear for $k = 1$; we prove it by induction on k. We know that $\xi_{k+1} \in H^2(F^k \times F,\mathbb{C})$ is congruent, modulo inverse images from F^k and F, with the incidence divisor $I \subset F^k \times F$ consisting of the codimension one components of the subvariety of pairs $(\underline{1},1_{k+1})$, $\underline{1} = (1_1,\ldots,1_k)$ such that 1_{k+1} meets at least one of the lines 1_i, $i = 1,\ldots,k$ (cf (6.8)). With the preceding notations one may write thus

$$\xi_{k+1} = \sum_{1\leq i\leq k} I_{i,k+1} + (\alpha \times F) + (F^k \times \beta) \quad .$$

Restriction to F^k shows $\xi_k = \sum_1^k D_i + \alpha$, hence $\alpha \times F = \sum_{1\leq i<j\leq k} I_{ij} + (3-k)\sum_1^k D_i$. Similarly, restriction to F yields $\xi_1 = kD_{k+1} + \beta$, and so

$F^k \times \beta = (3-k)\, D_{k+1}$. This yields the expected expression for ξ_{k+1} , thereby proving (6.11) .

Following with the proof of (6.10) , we get from (6.7) that

$$\sum_{1\le i<j\le k} I_{ij} \subset (\psi_4)^{-1}\Theta \subsetneq F^4$$

for suitable choice of Θ . Hence there is an equality of divisors in F^4 : $(\psi_4)^{-1}\Theta = \Sigma\, I_{ij} + D$, with $D \ge 0$. By (6.11) the cohomology class of D is zero, hence $D = 0$, q.e.d.

(6.12) REMARK. The above theta divisor depends on the choice of the base point for ψ_4 . If we take (l_1, l_2, il_1, il_2) or a permutation of it (notice that ψ_4 , hence Θ , doesn't depend then on the particular choice of l_1 and l_2) , the uniqueness statement of (6.10) implies that the corresponding theta divisor is a symmetric one. Thus *(as happens for the cubic threefold,* cf (6.9)*) the variety* X *distinguishes canonically a symmetric theta divisor of its intermediate Jacobian.*

A general geometric approach

The preceding considerations of this section are based on the philosophy that a point of departure in the search for a parametrization of the theta divisor is the geometrical description of sufficiently many traces of it on the Fano surface. In the above, a common description pattern for the cubic threefold and the quartic double solid yeld enough such traces in the first case but not in the second one - notice that, even in the case of the cubic threefold, these are not all of them. One is naturally suggested to ask whether it is possible to describe geometrically broader families in some other way. We devote the remainder of the present section (and hence of Part One) to this question.

There exists a seemingly quite standard method of description which, roughly spoken, produces all the traces for both varieties and leads consequently to a very primitive sort of parametrization of Θ . However, this seems to have no practical use, unless one is able to translate it into more natural terms. As an example, we shall carry out this translation in the case of the cubic threefold, getting the parametrization of Θ by means of the rational twisted cubics. (Below we give also a short alternative proof of this fact. The existence of the latter parametrization is already known; we learned it from Beauville and from Clemens.) There are indications that a better knowledge of certain curves of higher degree on the double solid could lead to a similar argument in this case, but we shall not deal with this question here.

The idea comes from (6.8) : Suppose that Γ is a curve on X , moving in a family parametrized by - say - a smooth, complete, connected variety W . Then, in $\mathrm{Pic}(F)$:

$$(6.13) \qquad [D_\Gamma] + \psi^*(\Theta + \psi_W(\Gamma)) = \text{const.} \quad ,$$

where D_Γ is the curve of incidence with Γ , lying on the Fano surface, the symbol Θ stands for any fixed copy of the theta divisor and ψ_W is the Abel-Jacobi map for W with arbitrarily chosen base point. Assume furthermore that W *dominates* the intermediate Jacobian $J(X)$. Fix any rational equivalence class $\gamma \in \mathrm{Pic}(F)$ yielding $(d+3)\upsilon$ in $NS^1(F)$, where d is the degree of the curves parametrized by W . The curves D_Γ are homologous with dD_l $(l \in F)$, hence they have class $d\upsilon$ in $NS^1(F)$. Therefore $\gamma - [D_\Gamma]$ belongs to $\mathrm{Pic}^{3\upsilon}(F)$ and, by our hypothesis on W together with (6.13) , it describes this set completely.

Using (6.5) , we obtain therefore that the linear eqivalence equation on F

$$(6.14) \qquad D_\Gamma + T_\Gamma \equiv \gamma$$

has always an effective solution in T_Γ , and that this solution is unique if $\Gamma \in W$ is general enough, being then a trace of the theta divisor on F . Our aim is to choose W and γ in such a way that the general solution T_Γ can be described in geometrical terms. A choice in this sense will be suggested by the following considerations.

Let $M \subset H^0\mathcal{O}_X(k)$ be a vector space of dimension $k+1$ of forms of degree k on X . The variety of lines of X which ly on at least one of the surfaces $\Psi = 0$ with $\Psi \in M$ is either F itself or underlies a divisor of class $c_1(\Lambda^{k+1}(S^kR_F)) = c_1(S^kR_F) = \frac{1}{2}k(k+1)\rho$ in $\mathrm{Pic}(F)$. In the latter case , assuming furthermore that $d+3 = k(k+1)$, if the forms of M vanish on a certain curve $\Gamma \in W$, then, (at least) if D_Γ is reduced, D_Γ will be a part of the former divisor and the residual part is a solution T_Γ of (6.14) with $\gamma = \frac{1}{2}k(k+1)\rho$.

Proposition (6.17) below contains a precise formulation of our conclusions. We shall need two lemmas before.

(6.15) LEMMA. *Let* $\Theta \subset J(X)$ *be a copy of the theta divisor,* $S \subset J(X)$ *the closed subset defined as* $S = \{a \in J(X) \mid \psi(F) \subset \Theta + a\}$, *and* $U = J(X) \setminus S$ *its (open) complement. For any* $a \in U$, *we write* T_a *for the trace divisor* $\psi^{-1}(\Theta + a)$ *on* F. *The following holds: for any* $1 \in F$, *the divisor* $U_1 = \{a \in U \mid 1 \in T_a\}$ *of* U *yields a dense subset of* $\psi(1) - \Theta$.

PROOF. Clearly $U_1 = U \cap (\psi(1) - \Theta)$. Assume now that M is an irreducible component of $\psi(1) - \Theta$ which doesn't meet U. Then $M \subset S$, i.e. for all $a \in M$: $\psi(F) \subset \Theta + a$. Writing $M = \psi(1) - N$ with N an irreducible component of Θ, this means that, for all $b \in N$, $b + \psi(F) \subset \Theta + \psi(1)$; hence $N + \psi(F) \subset \Theta + \psi(1)$. Thus $N + \psi(F)$ is an irreducible component of $\Theta + \psi(1)$ and, since $N + \psi(1) \subset N + \psi(F)$, a fortiori $N + \psi(F) = N + \psi(1)$. As at the end of the proof of (6.6), this yields a contradiction, q.e.d.

(6.16) LEMMA. *We keep all the notations of the preceding lemma. Let* $V \subset U$, $V \neq \emptyset$ *be an open subset; one has: for all* $1 \in F$, *the divisor* $V_1 = \{a \in V \mid 1 \in T_a\}$ *of* V *is contained in* $\psi(1) - \Theta$ *and, if* $1 \in F$ *is sufficiently general, then* V_1 *is dense in* $\psi(1) - \Theta$.

PROOF. This amounts to see that V_1 is dense in U_1 if $1 \in F$ is sufficiently general. Write the closed complement of V in U as union of irreducible components: $U \setminus V = Y_1 \cup \ldots \cup Y_r$. We show that, for general $1 \in F$, U_1 has no component contained in Y_i, $i = 1,\ldots,r$. Fix any i; if $\dim Y_i < 9$ this is clear. Otherwise Y_i is a divisor of U. To say that U_1 has a component inside Y_i is the same as to say that Y_i is a component of U_1. The subset $X_i = \{1 \in F \mid U_1 \text{ has } Y_i \text{ as a component}\}$ is closed in F. Suppose that X_i equals F. Taking any $a \in Y_i$, this implies $a \in U_1$ for all $1 \in F$, i.e. $1 \in T_a$ for all $1 \in F$, a contradiction. Therefore $X_i \neq F$, and this proves the lemma.

(6.17) PROPOSITION. *Let* W *be a smooth connected (non necessarily complete) variety parametrizing a family* $\{\Gamma_\omega \mid \omega \in W\}$ *of reduced curves of degree* $d =$

$= k(k+1) - 3$ *on* X *with the following properties:*

i) The Abel-Jacobi map $\psi_W : W \longrightarrow J(X)$ *is dominant;*

ii) for all $\omega \in W$ *, the linear system* $|\mathcal{O}_X(k) - \Gamma_\omega|$ *of surfaces of* $|\mathcal{O}_X(k)|$ *containing* Γ_ω *has dimension* k *and the surfaces of that system dont contain all the lines of* X *;*

iii) the incidence divisor between the family of lines F *and the family* W *is reduced.* (This condition is put for simplicity; it is not inmediately clear to us, as whether it is superfluous or not.)

Then the following holds:

I) Let $\mathcal{D}$ *be the subvariety of* $F \times W$ *of those pairs* $(1,\omega)$ *such that* $1 + \Gamma_\omega$ *lies on a surface of degree* k *in* X *. There exists an effective divisor* $\mathcal{D}'$ *on* $F \times W$ *whose underlying variety is* $\mathcal{D}$ *and such that* $\mathcal{D}' = I + T$ *with* I *the incidence divisor and* T *an effective reduced divisor satisfying the properties below.*

Put, for any $\omega \in W$ *,* $T_\omega = (pr_F)_*(T\cdot(F \times \omega))$ *, effective divisor on* F *. The set* $W^1 \subset W$ *of those* $\omega \in W$ *such that* $h^0\mathcal{O}_F(T_\omega) = 1$ *is open, dense. For all* $\omega \in W^1$ *,* T_ω *is a trace of the theta divisor on* F *, and this describes a dense subset of these.*

II) For any $1 \in F$ *the subvariety of* W *defined as* $W_1 = pr_W(T\cap(1\times W))$ *is mapped into a copy of the theta divisor and, if* $1 \in F$ *is sufficiently general,* W_1 *dominates the latter.*

In particular, for any $1 \in F$ *, the curves of the family* W *which dont meet* 1 *and ly in a surface of degree* k *through* 1 *are mapped into a copy of the theta divisor.*

<u>PROOF</u>. Consider the obvious rank $(k+1)$ vector bundles E_1 and E_2 on $F \times W$ whose fibres above $(1,\omega)$ are given respectively by the space of k-forms on X vanishing on Γ_ω , and the space of k-forms on the line 1 . Restriction of forms yields a morphism $E_1 \longrightarrow E_2$, hence a morphism $\Lambda^{k+1}E_1 \longrightarrow \Lambda^{k+1}E_2$, i.e. a section of the line bundle $(\Lambda^{k+1}E_1)^\vee \otimes (\Lambda^{k+1}E_2)$. The scheme of zeros of this section yields a divisor $\mathcal{D}'$ of $F \times W$ whose underlying variety obviously coincides with $\mathcal{D}$. Clearly, $I \le \mathcal{D}'$ hence $\mathcal{D}' = I + T$ with $T \ge 0$. By the definition of $\mathcal{D}'$ together with our hypotheses and the discussion preceding (6.15) , the classes of the divisors T_ω describe a dense subset of $\underline{\mathrm{Pic}}^{3\upsilon}_F$. Therefore, by (6.5) , the points $\omega \in W$ with $h^0\mathcal{O}_F(T_\omega) = 1$ yield a non-empty open subset $W^1 \subset W$. From (6.6) we get that, for all $\omega \in W^1$, T_ω is a trace of the theta divisor on F . Since a general trace is reduced, the divisor T has to be reduced, too.

It remains to prove the first part of II) , the second one being an obvious consequence. To begin with, there is a copy Θ of the theta divisor such that, for each $\omega \in W^1$, $T_\omega = \psi^{-1}(\Theta + \psi_W(\omega))$. In fact, as ω moves in W^1 , the class of $T_\omega + D_{\Gamma_\omega}$ is constant in Pic(F) and, by (6.13) , the same holds for $\psi^*(\Theta + \psi_W(\omega)) + D_{\Gamma_\omega}$. Therefore $T_\omega - \psi^*(\Theta + \psi_W(\omega))$ is constant in $\underline{\mathrm{Pic}}^o{}_F$. Replacing Θ by a translate, we may assume this constant to be zero, hence $T_\omega \equiv \psi^{-1}(\Theta + \psi_W(\omega))$, i.e. $T_\omega = \psi^{-1}(\Theta + \psi_W(\omega))$.

With this choice of Θ , the map

$$F \times W \longrightarrow J(X) \quad , \quad (1,\omega) \longrightarrow \psi(1) - \psi_W(\omega)$$

sends $T \cap (F \times W^1)$ into Θ . By continuity, the same holds for T itself and, for all $1 \in F$, the Abel-Jacobi map $\psi_W : W \longrightarrow J(X)$ sends $W_1 = \mathrm{pr}_W(T \cap (1 \times W))$ into $\psi(1) - \Theta$. It remains to see that the image is dense in $\psi(1) - \Theta$ if $1 \in F$ is sufficiently general. By condition i) , the restriction of ψ_W to W^1 is dominant; hence we find a non-empty open subset $V \subset J(X)$ contained in the image of W^1 . But then the image of $\mathrm{pr}_W(T \cap (1 \times W^1)) = \{\omega \in W^1 \mid 1 \in T_\omega\}$ contains a subset V_1 like in Lemma (6.16) , and the result follows from that lemma, q.e.d.

As mentioned before, we shall not deal here with applications of (6.17) . Instead, we shall see how the slightly modified version of (6.17) works in the case of the cubic threefold. However, before doing so, we owe a proof that the hypotheses of (6.17) are non empty. This will be done in the following (boring) note, which is purely technical and adds nothing relevant to the foregoing.

(6.18) <u>NOTE</u>. Conditions i) , ii) , iii) of (6.17) are non empty.

<u>PROOF</u>. We consider the cheapest case; in order that the family W dominates $J(X)$ one needs $d \geq 5$, hence $k \geq 3$. Take thus $k = 3$ and $d = 9$. For a smooth connected curve $\Gamma \subset X$ of degree 9 and genus 8 one gets, by Riemann-Roch, that $h^0\mathcal{O}_\Gamma(3) = 20$ hence, being $h^0\mathcal{O}_X(3) = 24$, Γ lies on at least ∞^3 cubic surfaces of X . So these are natural candidates for our purposes. However, for convenience, we choose here a certain degenerate type of such curves and, moreover, the existence claim will be proved merely for general X .

To begin with, consider the smooth irreducible curves of degree 7 and genus 4 in $\mathbb{P}^3$. The system of hyperplane sections being non special, these curves yield an irreducible variety V of dimension 28. For a certain open subset $U \subset V$, any $\overline{\Gamma} \in U$ is gotten by considering two skew lines $\overline{L}'$ and $\overline{L}''$ and two smooth cubic surfaces through them, and taking the residual intersection curve of these surfaces. The lines $\overline{L}'$ and $\overline{L}''$ are then the only 4-secants of $\overline{\Gamma}$. If X is a (smooth) quartic double solid and $\Gamma \subset X$ maps isomorphically onto $\overline{\Gamma} \in U$ - a situation whose existence is quickly verified - , the curves $\Gamma + f^{-1}(\overline{L}')$ and $\Gamma + f^{-1}(\overline{L}'')$ are both of degree 9 and virtual genus 8 ; observe that they are rationally equivalent, since $f^{-1}(\text{line})$ is rationally constant in X. We claim that, at least for general X, an irreducible component of the family of these curves yields an open subset satisfying i), ii), iii) of (6.17).

Introduce the family U_E of curves on $E = \mathcal{O}_{\mathbb{P}^3}(2)$ (cf Section 0) which map isomorphically onto curves of U. Being $h^0\mathcal{O}_{\overline{\Gamma}}(2) = 11$ for all $\overline{\Gamma} \in V$, the variety U_E is irreducible (of dimension 39). In a similar way, since each $\overline{\Gamma} \in U$ lies on exactly ∞^9 quartic surfaces of $\mathbb{P}^3$, the variety of pairs (Γ, X) with $\Gamma \in U_E$, $X \subset E$ a smooth subvariety given by an equation $T^2 = \phi_4$ $(\phi_4 \in H^0\mathcal{O}_{\mathbb{P}^3}(4))$, and $\Gamma \subset X$, is irreducible (of dimension 49).

Back to our claim, condition i) is equivalent with the fact of $J(X)$ being dominated by the corresponding irreducible component of the variety $U_X = \{\Gamma \in U_E \mid \Gamma \subset X\}$. Furthermore, the condition on the dimension in ii) is easily verified for a general choice of (Γ, X) as above (compare with the discussion below). As for condition iii), this follows at once from the reducedness of $D_{\Gamma+f^{-1}(\overline{L}')}$, $D_{\Gamma+f^{-1}(\overline{L}'')}$ for a general choice of (Γ, X).

Therefore our task can be reduced to show that:

a) there exists (Γ, X) as above such that Γ is a smooth point of U_X and the Abel-Jacobi map $U_X \longrightarrow J(X)$ is submersive at Γ, and

b) there exists (Γ, X) with $\Gamma \in U_X$ and a line L in X such that no cubic surface of X contains $\Gamma + f^{-1}(\overline{L}') + L$ nor $\Gamma + f^{-1}(\overline{L}'') + L$, where $\overline{L}'$ and $\overline{L}''$ are the 4-secants of the projection curve $\overline{\Gamma} \subset \mathbb{P}^3$.

Consider b) in the first place. Take an arbitrary $\Gamma \in U_E$ and call $\overline{\Gamma} \subset \mathbb{P}^3$ its projection in $\mathbb{P}^3$ and $\overline{L}'$, $\overline{L}''$ the 4-secants of $\overline{\Gamma}$. The latter curve is contained in exactly ∞^1 cubic surfaces of $\mathbb{P}^3$, since $H^0\mathcal{O}_{\mathbb{P}^3}(3)$ maps onto $H^0\mathcal{O}_{\overline{\Gamma}}(3)$; call $\langle\psi', \psi''\rangle$ the vector space of 3-forms yielding this pencil. The vector space $H^0\mathcal{O}_{\mathbb{P}^3}(3) \oplus T \cdot H^0\mathcal{O}_{\mathbb{P}^3}(1) \subset H^0\mathcal{O}_E(3)$ maps onto $H^0\mathcal{O}_{\Gamma}(3)$; therefore the subspace of those forms which vanish on Γ

has dimension 6 , and can be written as $< X_0T - \psi_0, \ldots, X_3T - \psi_3, \psi', \psi'' >$ where $X_0, \ldots, X_3$ is any coordinate system of $\mathbb{P}^3$ and ψ_i , $i = 0, \ldots, 3$ are suitable cubic forms on $\mathbb{P}^3$. We may assume that the line $\overline{L}'$ (resp. $\overline{L}''$) is given by $X_0 = X_1 = 0$ (resp. $X_2 = X_3 = 0$) . If $X \subset E$ is any smooth subvariety $T^2 = \phi_4$, $\phi_4 \in H^0\mathcal{O}_{\mathbb{P}^3}(4)$, through Γ such that the discriminant surface $\phi_4 = 0$ doesn't contain $\overline{L}'$ (resp. $\overline{L}''$) then the subspace of $H^0\mathcal{O}_X(3)$ of those 3-forms which vanish on $\Gamma + f^{-1}(\overline{L}')$ (resp. $\Gamma + f^{-1}(\overline{L}'')$) is easily seen to be given by $H' = < X_0T - \psi_0, X_1T - \psi_1, \psi', \psi'' >$ (resp. $H'' = < X_2T - \psi_2, X_3T - \psi_3, \psi', \psi'' >$) . The above condition on the discriminant surface is satisfied if, for instance, the points of Γ projecting onto the points of $\overline{\Gamma} \cap (\overline{L}' \cup \overline{L}'')$ dont meet the section $T = 0$ of E . We assume that Γ satisfies this property, and look at the 7-dimensional irreducible variety of all lines on E , i.e. curves on E which map isomorphically onto lines of $\mathbb{P}^3$. There is an open set parametrizing lines which live in a smooth X through Γ , as above. But also an open set parametrizing lines such that no cubic form of H' or H'' vanishes there. Hence a choice of X and L can be made such as to satisfy, together with our Γ , condition b) above.

As for condition a) , we use the infinitesimal theory of Section 2 . Take an arbitrary pair (Γ, X) with $\Gamma \in U_X$; we have to show that the composite map $\beta_Z r_Z$ of (2.11) with $Z = \Gamma$ is injective if (Γ, X) is sufficiently general. To analyze this map we use the following exact diagram, assuming furthermore that $\Gamma \subset X$ doesn't ly on the branch surface $T = 0$:

(6.19)
$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \mathcal{O}_\Gamma & = & \mathcal{O}_\Gamma & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N_{\Gamma/X}(-2) & \longrightarrow & N_{\Gamma/E}(-2) & \longrightarrow & \mathcal{O}_\Gamma(2) & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N_{\Gamma/X}(-2) & \longrightarrow & N_{\overline{\Gamma}/\mathbb{P}^3}(-2) & \longrightarrow & F & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

It is fully described as the twist by $\mathcal{O}_\Gamma(-2)$ of the exact diagram gotten in the obvious way from the middle row (which is the standard sequence for the triple $\Gamma \subset X \subset E$) and the left hand side square, of evident definition.

The upper right hand side morphism is given by the (by assumption) non vanishing section $T \in H^0\mathcal{O}_\Gamma(2)$. From the cohomological display of (6.19) we consider

$$\begin{array}{ccccc} 0 & & 0 & & \\ \downarrow & & \downarrow & & \\ \langle T\rangle & = & \langle T\rangle & & \\ \downarrow & & \downarrow & & \\ H^0N_{\Gamma/E}(-2) & \longrightarrow & H^0\mathcal{O}_\Gamma(2) & \xrightarrow[\beta_\Gamma]{} & H^1N_{\Gamma/X}(-2) \\ \downarrow & & & & \\ H^0N_{\overline{\Gamma}/\mathbb{P}^3}(-2) & . & & & \end{array}$$

We know already that $h^0\mathcal{O}_\Gamma(2) = 11$ and, clearly, $H^0\mathcal{O}_{\mathbb{P}^3}(2)$ is injected into $H^0\mathcal{O}_{\overline{\Gamma}}(2)$ if $\overline{\Gamma} \in U$. Identifying $H^0\mathcal{O}_\Gamma(2)$ with $H\ \mathcal{O}_{\overline{\Gamma}}(2)$, the subspace $\langle T\rangle$ of the latter can take almost any preassigned position, by moving X with Γ in E . In particular, a general choice yields $\langle T\rangle$ and $H^0\mathcal{O}_{\mathbb{P}^3}(2)$ as complementary subspaces there. In this case the injectivity we want to prove is equivalent with $\operatorname{Ker}\beta_\Gamma = \langle T\rangle$. By the diagram above, this equality will be a consequence of the following

(6.20) LEMMA. *If* $\overline{\Gamma}$ *is a sufficiently general smooth irreducible curve of* $\mathbb{P}^3$ *of degree* 7 *and genus* 4 *, then* $H^0N_{\overline{\Gamma}/\mathbb{P}^3}(-2) = 0$.

PROOF. It suffices to exhibit one such curve, the family being irreducible. If $\overline{L}'$ and $\overline{L}''$ are two skew lines and $\overline{\Gamma}$ is the residual intersection of two general cubics V_1 , V_2 through these lines, we write $\overline{\Gamma}\cap\overline{L}' = \{P_1',..,P_4'\}$ and $\overline{\Gamma}\cap\overline{L}'' = \{P_1'',..,P_4''\}$. Consider the following exact sequence, of evident definition,

$$0 \longrightarrow N_{\overline{\Gamma}/\mathbb{P}^3} \longrightarrow (N_{V_1/\mathbb{P}^3}\otimes\mathcal{O}_{\overline{\Gamma}}) \oplus (N_{V_2/\mathbb{P}^3}\otimes\mathcal{O}_{\overline{\Gamma}}) \longrightarrow \mathcal{O}_{\overline{\Gamma}\cap\overline{L}'} \oplus \mathcal{O}_{\overline{\Gamma}\cap\overline{L}''} \longrightarrow 0 \ .$$

After twisting by $\mathcal{O}_{\overline{\Gamma}}(-2)$ one gets an exact sequence

$$0 \longrightarrow N_{\overline{\Gamma}/\mathbb{P}^3}(-2) \longrightarrow \mathcal{O}_{\overline{\Gamma}}(1) \oplus \mathcal{O}_{\overline{\Gamma}}(1) \longrightarrow \Sigma\mathcal{O}_{P_i'} \oplus \Sigma\mathcal{O}_{P_i''} \longrightarrow 0 \ .$$

By using the Euler sequence one checks that the morphism which the third map induces at the H^0-level,

$$\gamma : H^0\mathcal{O}_{\mathbb{P}^3}(1) \oplus H^0\mathcal{O}_{\mathbb{P}^3}(1) \longrightarrow \mathbb{C}^4 \oplus \mathbb{C}^4 \ ,$$

can be described as follows: fix an equation ϕ_3 (resp. ψ_3) of the cubic V_1 (resp. V_2). At each P_i', P_i'' both cubics have the same tangent plane, hence there exist constants c_i', c_i'' such that $d\phi_3(P_i') = c_i' \cdot d\psi_3(P_i')$, $d\phi_3(P_i'') = c_i'' \cdot d\psi_3(P_i'')$ for each $i = 1,..,4$. Then $\gamma(A,B) = (((A - c_i'B)(P_i')), ((A - c_i''B)(P_i'')))$, the latter being an element of $\Sigma \mathcal{O}_{P_i'}(1) \oplus \Sigma \mathcal{O}_{P_i''}(1) \simeq \mathbb{C}^4 \oplus \mathbb{C}^4$.

It is now easy to verify that there are choices of ϕ_3 and ψ_3 such that γ yields an isomorphism between these vector spaces. If $\bar{L}'$ is given by $X_2 = X_3 = 0$ and $\bar{L}''$ is given by $X_0 = X_1 = 0$, and furthermore $P_i' = (a_i' : b_i' : 0 : 0)$, $P_i'' = (0 : 0 : a_i'' : b_i'')$, $i = 1,...,4$, the determinant of γ can be taken as the product

$$\det \begin{pmatrix} a_1' & b_1' & c_1'a_1' & c_1'b_1' \\ a_2' & b_2' & c_2'a_2' & c_2'b_2' \\ a_3' & b_3' & c_3'a_3' & c_3'b_3' \\ a_4' & b_4' & c_4'a_4' & c_4'b_4' \end{pmatrix} \cdot \det \begin{pmatrix} a_1'' & b_1'' & c_1''a_1'' & c_1''b_1'' \\ a_2'' & b_2'' & c_2''a_2'' & c_2''b_2'' \\ a_3'' & b_3'' & c_3''a_3'' & c_3''b_3'' \\ a_4'' & b_4'' & c_4''a_4'' & c_4''b_4'' \end{pmatrix} .$$

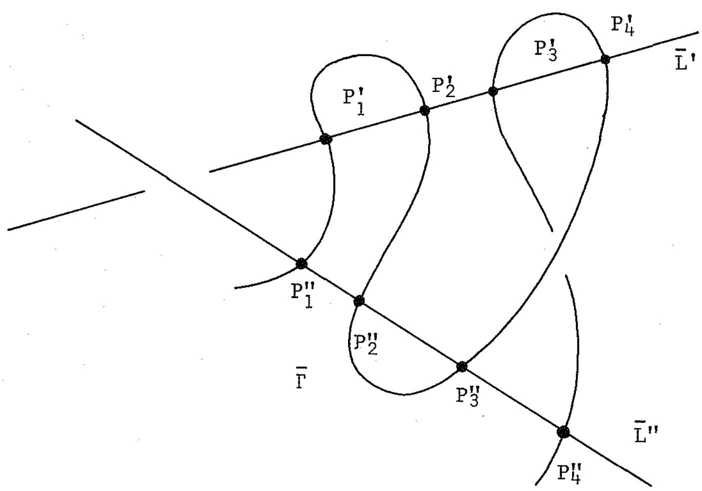


Figure 6.

A cubic form ϕ yielding a surface through $\bar{L}'$ and $\bar{L}''$ is a combination of X_0X_2, X_0X_3, X_1X_2, X_1X_3 with 1-forms as coefficients. It is equivalently determined by the following 2-forms, which can be preassigned arbitrarily:

$$A = \left(\frac{\partial \phi}{\partial X_2}\right)(X_0,X_1,0,0) \quad , \quad B = \left(\frac{\partial \phi}{\partial X_3}\right)(X_0,X_1,0,0) \quad ,$$

$$C = \left(\frac{\partial\phi}{\partial X_0}\right)(0,0,X_2,X_3) \quad , \quad D = \left(\frac{\partial\phi}{\partial X_1}\right)(0,0,X_2,X_3) \quad .$$

In these terms, if the above ϕ_3 and ψ_3 are given respectively by the 2-forms A_1 , B_1, C_1, D_1 and A_2, B_2, C_2, D_2 , the points P_i' , $i=1,\dots,4$ and P_i'' , $i=1,\dots,4$ are gotten respectively as

$$\det\begin{pmatrix} A_1 & A_2 \\ B_1 & B_2 \end{pmatrix} = 0 \text{ on } \overline{L}' \quad , \quad \det\begin{pmatrix} C_1 & C_2 \\ D_1 & D_2 \end{pmatrix} = 0 \text{ on } \overline{L}'' \quad .$$

Specific choices of ϕ_3 and ψ_3 can be made now, yielding $\det(\gamma) \neq 0$, q.e.d.

The example of the cubic threefold

(6.21) We turn finally to the cubic threefold, as an example to illustrate the preceding discussion. We repeat that the result which will be deduced in this way - namely, that the variety $\mathcal{R}_Y$ of smooth twisted cubics in the cubic threefold $Y = Y_3^3 \subset \mathbb{P}^4$ dominates Θ_Y - is a known one, and can be gotten alternatively from (6.9) . Let us see this in the first place.

Any curve $R \in \mathcal{R}_Y$ belongs to a unique hyperplane section $Y \cap \mathbb{P}^3$ of Y . There it moves in the 2-dimensional linear system of residual intersections of Y with the net of quadrics in $\mathbb{P}^3$ through a certain fixed twisted cubic R' on $Y \cap \mathbb{P}^3$. If L is a line in $Y \cap \mathbb{P}^3$ meeting R' , there exists certainly a reducible curve of the form $L + \gamma$ (with γ a conic) in that system. Hence R is rationally equivalent with $L + \gamma$ and, if $\gamma + L' \equiv$ (const.) is a plane section of Y , we get, on Y : $R \equiv L - L' +$ (const.) . One concludes now easily, by using (6.9) .

We notice also that the variety $\mathcal{R}_Y$ is irreducible and smooth. To see this, observe that for each $R \in \mathcal{R}_Y$ one has $H^0 N_{R/\mathbb{P}^4}(-2) = 0$, as follows from $N_{R/\langle R\rangle} \simeq \mathcal{O}_{\mathbb{P}^1}(5) \oplus \mathcal{O}_{\mathbb{P}^1}(5)$ with the identification $R \simeq \mathbb{P}^1$, and $\langle R\rangle$ denoting the span of R in $\mathbb{P}^4$. Therefore, by (2.10) , $\mathcal{R}_Y$ is a smooth variety of dimension 6 and the Abel-Jacobi map $\mathcal{R}_Y \longrightarrow J(Y)$ has everywhere rank 4 , hence its fibres are (smooth) of pure dimension 2 . On the other side, it is not difficult to see that the only hyperplane sections of Y which carry an infinity of lines are cones over smooth plane cubics. Therefore, by the former arguments, the contribution in $J(Y)$ of the smooth twisted cubics in a fixed hyperplane section of Y consists of a finite number of points. This implies, by the above, that the dimension of the subvariety $\{R \in \mathcal{R}_Y \mid \langle R\rangle$ is tangent to $Y\}$ is 5 . Thus it suffices to prove the irreducibility of the open subvariety $\mathcal{R}_Y^o = \{R \in \mathcal{R}_Y \mid \langle R\rangle \cap Y$ is a smooth cubic surface $\}$ of $\mathcal{R}_Y$. To this end, we consider the variety described as

$$\left\{(R,L,L') \in \mathcal{R}_Y^o \times F' \times F' \;\middle|\; \begin{array}{l} L, L' \subset \langle R\rangle \cap Y \text{ and } R \equiv L - L' + (\mathrm{ct.}) \\ \text{in the smooth surface } Y \cap \langle R\rangle . \end{array}\right\} .$$

(Here (ct.) denotes the class of a hyperplane section of the cubic surface $Y \cap \langle R\rangle$) . This variety maps onto $\mathcal{R}_Y^o$; on the other side, it maps onto the irreducible variety $(F')^2 \setminus \{\text{meeting couples}\}$, with open subsets of $\mathbb{P}^2$ as fibres. From this its irreducibility follows at once, and therefore that of $\mathcal{R}_Y^o$ too, q.e.d.

Our aim is now to proceed exactly in the same way as has been done for the quartic double solid in the preceding subsection. We refer to (6.2) , (6.4) and (6.9) (or, of course, to [8]) for details about the cubic threefold Y and its Fano surface F' .

The traces of Θ_Y on F' have class $2\upsilon'$ in $NS^1(F')$, where υ' denotes the class of an incidence curve D_L , L a line on Y . As before, we have: if the lines on Y which ly on some surface of degree k belonging to a fixed k-dimensional linear system on Y dont fill the whole F' , they underly a divisor of class $\frac{1}{2}k(k+1)\rho'$ in $\mathrm{Pic}(F')$, where ρ' is here the class of D_Δ with Δ a plane section of Y . Thus $\rho' = 3\upsilon'$ in $NS^1(F')$ and, if Γ is a curve of degree d lying in the basis locus of that system, the former divisor splits into D_Γ plus a part yielding the class $(\frac{3}{2}k(k+1) - d)\upsilon'$ in $NS^1(F')$ – here again we assume, for simplicity, that D_Γ is reduced – . If we want this to be equal to $2\upsilon'$, the relation $3k(k+1) = 2d + 4$ must hold.

It is clear now, that

(6.22) *The statements of Proposition* (6.17) *continue to hold if we replace* X , F *by* Y , F' *respectively and the relation* $d = k(k+1) - 3$ *by* $2d = 3k(k+1) - 4$.

We look agin for the cheapest case where this applies. Since W has to dominate $J(Y)$, we need $d \geq 3$, hence $k \geq 2$. Choose $k = 2$ and hence $d = 7$, looking therefore, in the first place, for smooth curves of degree 7 in Y lying in the basis locus of a net of quadrics of $\mathbb{P}^4$, and dominating $J(Y)$.

By Riemann Roch, natural candidates are the smooth curves of degree 7 and genus 3 in Y. Consider the twisted curves of this kind in $\mathbb{P}^4$. The system of hyperplane sections being non special, they yield an irreducible variety, of dimension 33. A general such curve Γ is gotten by taking sufficiently general nets of quadrics through lines of $\mathbb{P}^4$. The basis locus of such a net is a curve of degree 8 and virtual genus 5 which decomposes as $\Gamma + L$, with Γ smooth of degree 7 and genus 3, and L a line. The latter is then the only 3-secant of Γ and the quadrics of the net are the only quadrics of $\mathbb{P}^4$ through Γ. We shall need the following

(6.23) LEMMA. *With the above notations, a general* Γ *satisfies* $H^0N_{\Gamma/\mathbb{P}^4}(-2) = 0$.

PROOF. This is similar to the proof of (6.20). We assume that the line $L = \{X_0 = X_1 = X_2 = 0\}$ meets Γ at three distinct points P_1, P_2, P_3 and that $L + \Gamma$ is the complete intersection of three quadrics in $\mathbb{P}^4$. As in (loc. cit.), we write down the standard sequence

$$0 \longrightarrow N_{\Gamma/\mathbb{P}^4} \longrightarrow (\mathcal{O}_\Gamma(2))^3 \longrightarrow \mathcal{O}_{\Gamma\cap L} \longrightarrow 0 ,$$

which yields, after twisting by $\mathcal{O}_\Gamma(-2)$:

$$0 \longrightarrow N_{\Gamma/\mathbb{P}^4}(-2) \longrightarrow \mathcal{O}_\Gamma{}^3 \longrightarrow \textstyle\sum_1^3 \mathcal{O}_{P_i} \longrightarrow 0 .$$

The associated cohomology sequence starts with

$$0 \longrightarrow H^0N_{\Gamma/\mathbb{P}^4}(-2) \longrightarrow \mathbb{C}^3 \xrightarrow{\gamma} \mathbb{C}^3 ,$$

where, by using the Euler sequence, the morphism γ can be seen to be identifiable with

$$\gamma = \begin{pmatrix} c_1' & c_1'' & c_1''' \\ c_2' & c_2'' & c_2''' \\ c_3' & c_3'' & c_3''' \end{pmatrix}$$

with the entries chosen as follows: if ψ', ψ'', ψ''' are equations of three independent quadrics through Γ, the relation

$$c_i' d\psi' + c_i'' d\psi'' + c_i''' d\psi''' = 0$$

holds at P_i , $i = 1,2,3$.

A quadric through L can be written uniquely as

$$\psi = X_0\lambda_0(X_0,X_1,..,X_4) + X_1\lambda_1(X_1,..,X_4) + X_2\lambda_2(X_2,..,X_4) \quad ,$$

where the 1-forms $\lambda_0,\lambda_1,\lambda_2$, and hence their restrictions to L : $\overline{\lambda}_0 = \lambda_0(0,0,0,X_3,X_4)$, $\overline{\lambda}_1 = \lambda_1(0,0,X_3,X_4)$, $\overline{\lambda}_2 = \lambda_2(0,X_3,X_4)$, can be prescribed arbitrarily. If the above quadrics are given respectively by $\lambda_0',\lambda_1',\lambda_2'$, $\lambda_0'',\lambda_1'',\lambda_2''$, $\lambda_0''',\lambda_1''',\lambda_2'''$, then the points P_1,P_2,P_3 are the locus

$$\det \begin{pmatrix} \overline{\lambda}_0' & \overline{\lambda}_0'' & \overline{\lambda}_0''' \\ \overline{\lambda}_1' & \overline{\lambda}_1'' & \overline{\lambda}_1''' \\ \overline{\lambda}_2' & \overline{\lambda}_2'' & \overline{\lambda}_2''' \end{pmatrix} = 0$$

on L . The existence of choices yielding $\det(\gamma) \neq 0$ can be checked easily now, q.e.d.

Denote by U the irreducible variety of smooth twisted curves Γ of degree 7 and genus 3 in $\mathbb{P}^4$ such that $H^0 N_{\Gamma/\mathbb{P}^4}(-2) = 0$ and such that Γ is gotten by dropping a line from the basis locus of some net of quadrics in $\mathbb{P}^4$. (This last condition is put for simplicity). Call furthermore T the irreducible variety parametrizing the smooth cubic threefolds of $\mathbb{P}^4$, and consider the subvariety $V \subset U \times T$ defined as $V = \{(\Gamma,Y) \mid \Gamma \subset Y\}$. The latter is certainly non empty and, since $H^0\mathcal{O}_{\mathbb{P}^4}(3)$ maps onto $H^0\mathcal{O}_{\Gamma}(3)$ for any $\Gamma \in U$, V is irreducible (of dimension 48) . Moreover, it contains an open dense subvariety V^o such that, for any $(\Gamma,Y) \in V^o$,

a) there exist lines $l \subset Y$ with no quadric of $\mathbb{P}^4$ containing $l + \Gamma$, and

b) the divisor D_Γ of F' is reduced (This may be somewhat weak, but is sufficient for our purposes; one merely has to exhibit a single such pair (Γ,Y) , which is easily done) .

If Y is a smooth cubic threefold, we put $U_Y = \{\Gamma \in U \mid (\Gamma,Y) \in V^o\}$. By using (2.10) together with (6.23) we get that, if non empty, U_Y is smooth of dimension 14 and the Abel-Jacobi map $U_Y \longrightarrow J(Y)$ is everywhere submersive. This will be the case for all Y belonging to a certain open dense subset T^o of T . We conclude therefore:

(6.24) *If* $Y \in T^o$ *and* W *is an irreducible component of* U_Y *, the conditions of* (6.17) *(modified by* (6.22)*) are satisfied.*

(6.25) So far, the development has been similar to that in the case of the double solid. We shall sketch now how to use (6.24) to derive again the result that, for any smooth cubic threefold Y , the variety R_Y dominates the theta divisor of $J(Y)$. For the sake of clearness, we give first an outline of the idea.

Fix an $Y \in T^o$ and a line $l \subset Y$. Then, by (6.24) , a suitable copy of Θ_Y will contain the images of the curves $\Gamma \in U_Y$ which ly on some surface of $|\mathcal{O}_Y(2)|$ together with the line l , and dont meet this line. If Γ is such a curve and $V_1 , V_2 \in |\mathcal{O}_Y(2)|$ are two sufficiently general surfaces through Γ with V_1 containing also l , one "could expect" (cf (6.26)) the intersection of V_1 with V_2 to consist, besides Γ , of a curve Δ of degree 5 and virtual genus 1 meeting the line l at two points. And conversely, given such a curve Δ and two surfaces V_1 and V_2 of $|\mathcal{O}_Y(2)|$ through Δ , and V_1 containing also l , their residual intersection with respect to Δ "should yield" generally (ibid) a curve Γ as earlier described.

Given such curves Γ and Δ , with $\Gamma + \Delta = V_1 \cdot V_2$ in Y , the sum of their images in $J(Y)$ (with some fixed choice of the base points) is constant. Therefore, if the above guesses are right, the curves Δ will map into some copy of the theta divisor. Now, given a general twisted cubic $R \subset Y$, we may use it to construct in a natural way such a curve Δ as follows (Figure 7) . The linear projection of $\mathbb{P}^4$ from the line l maps R birationally onto a singular plane cubic; hence there is exactly one 2-plane through l meeting R twice. This plane intersects Y along the curve $l + \gamma$, where γ is a conic meeting both R and l twice. Hence the curve $\Delta = R + \gamma$ is a quintic of virtual genus 1 which intersects l twice, as claimed. The curves $R + \gamma$ have their image in $J(Y)$ inside some copy of the theta divisor, and we observe that, since l has been fixed, the curves γ are rationally constant in Y (plane sections minus l) and have therefore constant image in $J(Y)$. So, always assuming that the above hypotheses are right, this would imply again the twisted cubics mapping into a translate of Θ_Y .

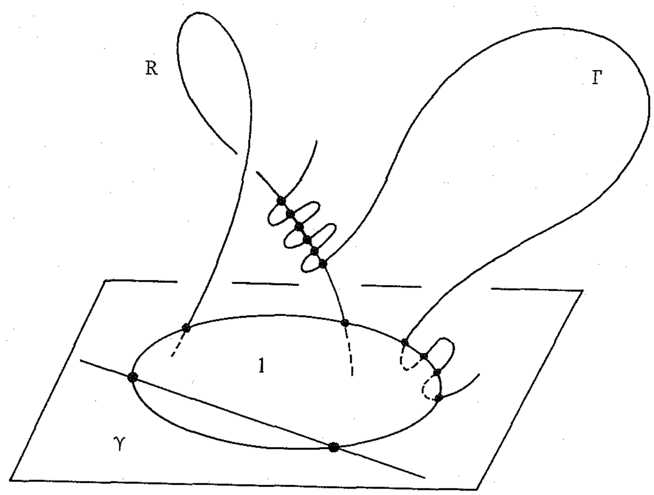


Figure 7.

(6.26) REMARK. The guessses in the preceding discussion are suggested by the following fact, which can be easily verified. Let Z be a smooth complete threefold and $S_1, S_2 \subset Z$ two surfaces such that their complete intersection yields $S_1 \cdot S_2 = C_1 + \ldots + C_r$, the C_i being smooth curves and the intersections of the different components yielding only ordinary double points. Denote by υ_i the number of intersection points of C_i with the remaining components, by g_i the genus of C_i and by K_Z the canonical class of Z ; one has the following relation:

$$\upsilon_i = C_i \cdot (S_1 + S_2 + K_Z) - 2g_i + 2 \quad .$$

Back to rigorous arguing, we recall first (cf (6.21)) that $\mathcal{R}_Y$ is irreducible and dominates a divisor of $J(Y)$ for any smooth cubic threefold Y. This divisor will move continuously with Y, a priori for general Y at least. To get this statement for all Y we ought to start for instance with a compactification of the $\mathcal{R}_Y$'s yielding a proper family, and so on. However, according to our purposes we shall not care about these aspects knowing that, a posteriori, the statement is true. If wanted, one can use the already proved fact that the divisor dominated by $\mathcal{R}_Y$ is the image of $(F')^2$ in $J(Y)$ by the difference mapping (cf (6.21)).

From its continuous variation we deduce, by the usual monodromy argument, that the divisor of $J(Y)$ dominated by $\mathcal{R}_Y$ is homologous with a multiple of Θ_Y. It suffices therefore to show that $\mathcal{R}_Y$ maps into a copy of the theta divisor for some Y.

Take for a moment a fixed smooth cubic threefold Y and a line $l \subset Y$, and consider the open subvariety $\mathcal{R}_Y^l \subset \mathcal{R}_Y$ consisting of those $R \in \mathcal{R}_Y$ which dont meet l and yield a cubic with a node by linear projection from l. As described in (6.25), there is exactly one conic $\gamma \subset Y$ which meets both R and l twice (Figure 7). The subsystem $\Lambda_1 \subset |\mathcal{O}_Y(2)|$ of surfaces through $\gamma + R$ has dimension 4, and the subsystem $\Lambda_2 \subset \Lambda_1$ of surfaces through $l + \gamma + R$ (i.e., through R and the supporting plane of γ) has dimension 3. If $V_1 \in \Lambda_1$ and $V_2 \in \Lambda_2$ have no common component, their residual intersection with respect to $\gamma + R$ is a curve $\Gamma \subset Y$ of degree 7, lying on a quadric of $\mathbb{P}^4$ together with l. The image of Γ in $J(Y)$ (with respect to a fixed base curve Γ_0) is independent of the specific choice of V_1 and V_2.

Suppose that for some V_1 and V_2 the curve Γ belongs to the family U_Y above (cf after (6.23)) and doesn't meet l. Then, since these are open conditions and the variety $\{(R,V_1,V_2) \in \mathcal{R}_Y^l \times |\mathcal{O}_Y(2)| \times |\mathcal{O}_Y(2)| \mid V_1 \in \Lambda_1, V_2 \in \Lambda_2, \text{ and } V_1, V_2 \text{ without common component}\}$ is irreducible, such a choice will be possible for a general $R \in \mathcal{R}_Y$. Therefore, by (6.25), the image of $\mathcal{R}_Y$ in $J(Y)$ will ly in some translate of Θ_Y, as we want to deduce.

In this way, thigs are finally reduced to checking the following fact, whose proof we shall omit (Figure 7):

There exists a smooth cubic threefold Y and two quadrics Q', Q'' in $\mathbb{P}^4$ such that $Y \cdot Q' \cdot Q'' = \gamma + R + \Gamma$ with γ a conic, R a twisted cubic and Γ an irreducible septic such that: a) the curves γ, R, Γ are smooth and $\Gamma \in U_Y$, b) the supporting plane of γ meets R transversally at exactly two points, and c) there is a line $l \subset Q'' \cap Y$ in the plane of γ, which doesn't meet $\Gamma + R$. (As a matter of fact, Q'' has to contain then the supporting plane of γ and l is the residual intersection of Y with this plane.)

PART TWO : THE INTERSECTION OF THREE QUADRICS

7. INTRODUCTION

In his paper [27] , Tyurin pointed out a relationship between the surface which parametrizes the conics lying on the smooth complete intersection X of three quadrics in $\mathbb{P}^6$ and a certain surface of divisors of degree 7 on the curve $\hat{C}$ which parametrizes the Chow components of the 3-planes living inside the degenerate quadrics of the net with basis locus X . In view of this relation and further remarks (ibid, p. 103) , he conjectured that the Albanese variety of the former surface is isogenous to the Prym variety of the covering of the discriminant curve C of the net by $\hat{C}$ and hence to the intermediate Jacobian of X .

In this part we give a proof of Tyurin's conjecture (cf (10.2)) . In the context of double covers and nets of divisors the isogeny considered here seems to hold more generally. It has been found to be actually an isomorhism in various cases, among which the well-known isomorphism between the intermediate Jacobian of the cubic threefold in $\mathbb{P}^4$ and the Albanese variety of its (Fano) surface of lines (cf [8] , and (8.22)) .

Being mainly interested in (10.2) , we have ignored several more general questions which arise in this context, using instead ad-hoc methods to settle our particular cases.

Throughout, the term 'double cover' means étale irreducible (2:1) covering.

8. THE VARIETY OF DIVISORS ATTACHED TO A LINEAR SYSTEM ON A SMOOTH CURVE WITH AN ETALE DOUBLE COVER

Connectedness

(8.1) We start recalling some well-known facts and easily derived consequences ([21] , [22] , [29]) . Consider a double cover

$$\pi : \hat{C} \longrightarrow C$$

of a smooth curve C. The kernel of the induced morphism

$$\pi_* : J(\hat{C}) \longrightarrow J(C)$$

has two connected components. The neutral component $\ker(\pi_*)^o$ is a (naturally principally polarized) abelian variety called the Prym variety of the covering, usually denoted $Pr(\hat{C}/C)$. We shall write it also Pr, the other component being written Pr'. The covering involution of $\hat{C}$ is written as usual

$$i : \hat{C} \longrightarrow \hat{C} \quad .$$

Fix an integer $d \geq 0$. It follows from the above that the fibres of the map $\pi_* : J(\hat{C})_d \longrightarrow J(C)_d$ consist of two connected components, the fibre of $\hat{\xi} \in J(\hat{C})_d$ being the disjoint union $(\hat{\xi} + Pr) \cup (\hat{\xi} + Pr')$.

For later purposes, we reformulate Lemma 1 of [22], p. 187, as follows:

(8.2) LEMMA. $\mathrm{Ker}(\pi_*)$ *is the set of elements of* $J(\hat{C})$ *of the form* $\xi = \sum_{k=1}^{N} [P_k - i(P_k)]$ *and* $\xi \in Pr$ *or* $\xi \in Pr'$ *depending on whether* $N \equiv 0$ *or* $1 \pmod 2$. *A set of generators of* Pr *is given by the elements* $2\,[P - i(P)]$, $P \in \hat{C}$.

The last statement follows from the fact that $Pr = 2\,Pr$. Take next a linear system Λ of dimension ≥ 1 of divisors of degree d on the curve C.

(8.3) DEFINITION. *The variety* X *of divisors on* $\hat{C}$ *lying above* Λ *is defined as the pullback*

$$\begin{array}{ccc} X = (\pi^{(d)})^{-1}\Lambda & \hookrightarrow & \hat{C}^{(d)} \\ \Pi \downarrow & & \downarrow \pi^{(d)} \\ \Lambda & \hookrightarrow & C^{(d)} \end{array} \quad .$$

The map Π is a ramified covering, étale above smooth divisors of the system Λ. It follows that the irreducible components of X have dimension equal to $\dim(\Lambda)$. A look at the diagram

$$\begin{array}{ccccc} X & \hookrightarrow & \hat{C}^{(d)} & \longrightarrow & J(\hat{C})_d \\ \downarrow \Pi & & \downarrow \pi^{(d)} & & \downarrow \pi_* \\ \Lambda & \hookrightarrow & C^{(d)} & \longrightarrow & J(C)_d \end{array}$$

shows – the horizontal mappings being the obvious ones and (therefore) the image of Λ in $J(C)_d$ being a single point – that X breaks up naturally into two disjoint subvarieties. We shall call them the *halves* of the variety of divisors X .

(8.4) Put $X = Y' \cup Y''$, Y' and Y'' being the halves of X . We want to know how the elements of a fibre of Π are distributed with respect to Y' and Y'' (cf also [29], p. 955) . To this end, let $D \in X$, $D = Q_1 + \ldots + Q_d$ and write $D' \in X$ for the divisor gotten by replacing one of the points Q_k by its image $i(Q_k)$ under the covering involution, $D' = D + (i(Q_k) - Q_k)$. Using (8.1) , (8.2) and (8.3) we get inmediately

(8.5) PROPOSITION. *With the above notations,* D *and* D' *belong to different halves of* X . *As a consequence, the fibre of* Π *outside the discriminant locus has* 2^{d-1} *elements lying on* Y' *and* 2^{d-1} *on* Y'' . *The elements of a fibre of* $Y' \longrightarrow \Lambda$ *(resp.* $Y'' \longrightarrow \Lambda$ *) are gotten from one of them by switching in the above sense an even number of times.*

It follows, by using the covering transformation i ,

(8.6) COROLLARY. *If the degree* d *of the system* Λ *is odd, both halves* Y' *and* Y'' *are naturally isomorphic. If the degree is even, each half carries a natural involution.*

(8.7) REMARK. *The examples in* (8.20) *will show that the halves may be non isomorphic in the even case.*

(8.8) PROPOSITION. *Let* Λ *be an arbitrary linear system on* C *, of dimension* ≥ 1 *. Then both halves of the variety of divisors are connected, hence are the connected components of* X .

PROOF. It obviously suffices to consider the case $\dim(\Lambda) = 1$, both halves of X being covered by linear systems of analogous halves for the one-dimensional case. Thus suppose $\dim(\Lambda) = 1$ and let Y be a fixed half of X . We assume that Λ has no base points, the general case being an easy consequence of this one. We consider the maps

$$\hat{C} \xrightarrow{\pi} C \xrightarrow{f} \mathbb{P}^1$$

where f is the morphism giving rise to the system Λ ; identifying Λ with $\mathbb{P}^1$, we have also the map

$$\Pi : Y \longrightarrow \mathbb{P}^1 .$$

Fix a point $t_0 \in \mathbb{P}^1$ corresponding to a smooth divisor $f^{-1}(t_0)$ on C . It suffices to exhibit a path in Y between two arbitrary elements of the fibre $Y(t_0)$ of Y above t_0 . We may even suppose that they differ by exactly two switches, e.g. $\hat{D} = P_1 + P_2 + \ldots + P_d$ and $\hat{D}' = iP_1 + iP_2 + \ldots + P_d$.

Remark that a continuous path in Y is equivalent to a family of paths $\gamma_1, \ldots, \gamma_d : [0,1] \longrightarrow \hat{C}$ and a path $\gamma : [0,1] \longrightarrow \mathbb{P}^1$ satisfying:

a) For each $i = 1, \ldots, d$ and each $\tau \in [0,1]$ one has $f\pi\gamma_i(\tau) = \gamma(\tau)$.

b) If $\gamma(\tau)$ is not a discriminant point of f , then the points $\pi\gamma_1(\tau), \ldots, \pi\gamma_d(\tau)$ are all distinct points of C .

The main point in our argument to prove (8.8) is the following

(8.9) LEMMA. *Let* $t_0 \in \mathbb{P}^1$ *be, as above, a non-discriminant point of the map* f *and* A , B *two points in* $\hat{C}$ *above* t_0 *, with* $\pi(A) \neq \pi(B)$. *Then there exists a path* Γ *in* $\mathbb{P}^1$ *from* t_0 *to a discriminant point of* f *and paths* Γ_A , Γ_B *in* $\hat{C}$ *projecting onto* Γ *such that* $\Gamma_A(0) = A$ *,* $\Gamma_B(0) = B$ *,* $\Gamma_A(1) = \Gamma_B(1)$ *and furthermore* $\pi\Gamma_A(\tau) \neq \pi\Gamma_B(\tau)$ *for all* τ *such that* $\Gamma(\tau)$ *is not a discriminant point of* f .

PROOF. $\hat{C}$ being connected, there is a path in $\hat{C}$ between A and B . We may assume that this path projects onto a composition of paths going from t_0 to a discriminant point δ_i and backwards to t_0 , these paths not meeting other discriminant points. The path from A to B in $\hat{C}$ thus breaks up in paths joining points of the fibre $(f\pi)^{-1}(t_0)$, which we shall call "intermediate points". We may suppose that the points of $(f\pi)^{-1}(t_0)$ appear at most once as intermediate points (otherwise we drop the superfluous part) . We distinguish cases:

a) Suppose that $i(B)$ doesn't appear as intermediate point. The path Γ_A will be a part of our path from A to B , to be precised in a moment. The path Γ_B goes as follows: we put $\Gamma_B(0) = B$; next, while Γ_A goes from A to Q_1 (Figure 8) , let Γ_B go its prescribed way - by êtaleness - above the projection of Γ_A in $\mathbb{P}^1$ until certain point in $\hat{C}$ above δ_1 . If that point was Q_1 we stop things here. Else Γ_A follows his travel from Q_1 to the first intermediate point and Γ_B goes back to B . We iterate this procedure, Γ_A now going to the second branch point Q_2 , and so on. Obviously, the first point where Γ_A and Γ_B can meet is one of the branch points Q_i , and we are shure that they meet, latest at Q_n . We stop at the first encounter.

The only thing to be checked for Γ_A and Γ_B is that $\Gamma_A(\tau)$ and $\Gamma_B(\tau)$ are not conjugate under the covering involution i , if $f\pi\Gamma_A(\tau)$ is not a discriminant point. In fact, if that wasn't the case, by êtaleness B would be conjugate to one of the intermediate points, but this has been ruled out by assumption.

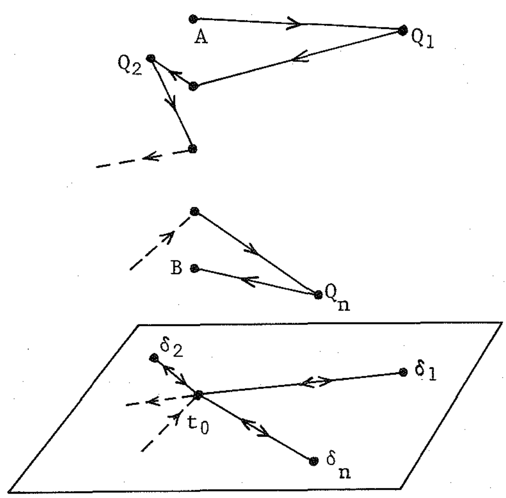


Figure 8.

b) Suppose that $i(B)$ appears as intermediate point but that $i(A)$ doesn't. We reverse the procedure, interchanging the roles of A and B .

c) Finally, if $i(A)$ and $i(B)$ both appear as intermediate points, the procedure can be applied to these points instead of A and B , fol -

lowing case a) . Once these paths are gotten, we apply the covering involution to the whole situation, getting Γ_A and Γ_B as wanted. This proves Lemma (8.9) .

To finish the proof of (8.8) , apply the above lemma to $A = P_1$ and $B = iP_2$. Say $Q := \Gamma_A(1) = \Gamma_B(1)$. We put $\gamma = \Gamma$, $\gamma_1 = \Gamma_A$, $\gamma_2 = i\Gamma_B$. Then $\gamma_1(0) = P_1$, $\gamma_1(1) = Q$, and $\gamma_2(0) = P_2$, $\gamma_2(1) = iQ$. It is easily seen that we can add paths $\gamma_3 , \dots , \gamma_d$ above γ such that $\gamma_i(0) = P_i$ and $\gamma_1, \gamma_2 , \dots , \gamma_d , \gamma$ satisfy conditions a) and b) above. This gives a path in Y between $\hat{D}$ and a certain divisor $\hat{D}'' = Q + iQ + \dots$. Next take the system of paths $(i\gamma_2)^{-1}$, $(i\gamma_1)^{-1}$, γ_3^{-1} , ... , γ_d^{-1} , γ^{-1} . We have $(i\gamma_2)^{-1}(0) = Q$, $(i\gamma_2)^{-1}(1) = iP_2$, $(i\gamma_1)^{-1}(0) = iQ$, $(i\gamma_1)^{-1}(1) = = iP_1$. Hence this gives a path between $\hat{D}''$ and $\hat{D}'$. Composition of the above two paths yields the required one between $\hat{D}$ and $\hat{D}'$, q.e.d.

(8.10) REMARK. *Even above a base point free pencil, the halves of the variety of divisors may be reducible.* For example, take a general pencil of canonical divisors on a hyperelliptic curve. This works, because one half of the divisor variety above the canonical system is mapped onto a copy of the theta divisor of the principally polarized Prym variety (cf [21] , p. 342) ; and, as discussed later on in Example (8.20) - and, of course, well--known - for suitable coverings of an hyperelliptic curve the polarized Prym decomposes into a product of Jacobians, hence its theta divisor is reducible.

Infinitesimal study

Let D be an effective divisor of degree d on a smooth curve C . The standard exact sequences on C :

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_C(D) \longrightarrow \mathcal{O}_D(D) \longrightarrow 0$$

$$0 \longrightarrow \Omega^1_C(-D) \longrightarrow \Omega^1_C \longrightarrow \Omega^1_C \otimes \mathcal{O}_D \longrightarrow 0$$

give rise to (mutually dual) cohomology sequences, from which we recall the geometrical meaning with aid of the diagrams below. We denote by $|D|$ the complete linear system of D and use the natural mappings

$$|D| \xhookrightarrow{j} C^{(d)} \xrightarrow{\gamma} J(C)_d$$

(8.11) One has identifications

$$\begin{array}{ccccccccccc}
0 \to H^0\mathcal{O}_C \to & H^0\mathcal{O}_C(D) & \longrightarrow & H^0\mathcal{O}_D(D) & \longrightarrow & H^1\mathcal{O}_C & \longrightarrow & H^1\mathcal{O}_C(D) & \to 0 \\
& \searrow \quad \nearrow & & \uparrow\simeq & & \uparrow\simeq & & \| & \\
& 0 \to T_{|D|}(D) & \xrightarrow[dj]{} & T_{C(d)}(D) & \xrightarrow[d\gamma]{} & T_{J(C)_d}(D) & \longrightarrow & H^1\mathcal{O}_C(D) & \to 0
\end{array}$$

and

$$\begin{array}{ccccccccccc}
0 \to & H^0\Omega^1_C(-D) & \longrightarrow & H^0\Omega^1_C & \longrightarrow & H^0(\Omega^1_C \otimes \mathcal{O}_D) & \longrightarrow & H^1\Omega^1_C(-D) & \to & H^1\Omega^1_C & \to 0 \\
& \| & & \downarrow\simeq & & \downarrow\simeq & \searrow & \nearrow & & & \\
0 \to & H^0\Omega^1_C(-D) & \to & T^\vee_{J(C)_d}(D) & \xrightarrow[d\gamma^*]{} & T^\vee_{C(d)}(D) & \xrightarrow[dj^*]{} & T^\vee_{|D|}(D) & \longrightarrow & 0 & .
\end{array}$$

Here $T_{C(d)}(D) \simeq H^0\mathcal{O}_D(D)$ follows from Grothendieck's deformation theory ([14]) , $\mathcal{O}_D(D)$ being the normal bundle of the embedding of D in C as a subscheme; $H^0(\Omega^1_C \otimes \mathcal{O}_D) \simeq T^\vee_{C(d)}(D)$ follows e.g. by duality.

(8.12) Let Λ be again an arbitrary linear system on C , of degree d and dimension ≥ 1 . Let X be the variety of divisors on $\hat{C}$ above Λ and $\hat{D} \in X$ a fixed element, D its image in Λ . The Zariski tangent spaces yield a pullback diagram

$$\begin{array}{ccc}
T_X(\hat{D}) & \hookrightarrow & T_{\hat{C}(d)}(\hat{D}) \\
{\scriptstyle d\Pi}\downarrow & & \downarrow{\scriptstyle d\pi^{(d)}} \\
T_\Lambda(D) & \xhookrightarrow[dj]{} & T_{C(d)}(D)
\end{array}$$

where $j : \Lambda \hookrightarrow C^{(d)}$ denotes the inclusion map. The variety X will be smooth at $\hat{D}$ if and only if $T_X(\hat{D})$ has the same dimension as X . But $\dim X = \dim \Lambda = \dim T_\Lambda(D)$; and $\dim T_{\hat{C}(d)}(\hat{D}) = \dim T_{C(d)}(D) = d$, hence

$$\begin{aligned}
\dim T_X(\hat{D}) &= \dim \operatorname{Ker} d\pi^{(d)} + \dim (\operatorname{Im} dj) \cap (\operatorname{Im} d\pi^{(d)}) = \\
&= d - \dim \operatorname{Im} d\pi^{(d)} + \dim (\operatorname{Im} dj) \cap (\operatorname{Im} d\pi^{(d)}) .
\end{aligned}$$

So X is smooth at $\hat{D}$ if and only if

$$T_{C(d)}(D) = dj\, T_\Lambda(D) + d\pi^{(d)}\, T_{\hat{C}(d)}(\hat{D}) ,$$

the standard transversality condition. Dually, this reads

$$\mathrm{Ker}\ (dj^*) \cap \mathrm{Ker}\ (d\pi^{(d)*}) = 0 \quad \text{in} \quad T^{\vee}_{C(d)}(D)\ .$$

If the system Λ is complete, this can be put in a more geometrical fashion (not needed in the sequel) :

PROPOSITION. *With the assumptions and notations of* (8.12) *, if furthermore the system* Λ *is complete, then* X *is smooth at* $\hat{D}$ *if and only if the following condition holds: For any effective canonical divisor* K_C *, if* $\hat{D} \leq \pi^* K_C$ *then* $D \leq K_C$.

PROOF. The second sequence of (8.11) can be used twice, giving a diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & H^0\Omega^1_{\hat{C}}(-\hat{D}) & \longrightarrow & T^{\vee}_{J(\hat{C})_d}(\hat{D}) & \longrightarrow & T^{\vee}_{\hat{C}(d)}(\hat{D}) & \longrightarrow & T^{\vee}_{|\hat{D}|}(\hat{D}) & \longrightarrow 0 \\
 & \uparrow & & \uparrow (d\pi_*)^* & & \uparrow d\pi^{(d)*} & & \uparrow & \\
0 \longrightarrow & H^0\Omega^1_C(-D) & \longrightarrow & T^{\vee}_{J(C)_d}(D) & \longrightarrow & T^{\vee}_{C(d)}(D) & \longrightarrow & T^{\vee}_{|D|}(D) & \longrightarrow 0
\end{array}$$

The condition $\mathrm{Ker}(dj^*) \cap \mathrm{Ker}(d\pi^{(d)*}) = 0$ is equivalent with the square of the left hand side being a pullback diagram. But this square is

$$\begin{array}{ccc}
H^0\mathcal{O}_{\hat{C}}(K_{\hat{C}} - \hat{D}) & \hookrightarrow & H^0\mathcal{O}_{\hat{C}}(K_{\hat{C}}) \\
\uparrow & & \uparrow \\
H^0\mathcal{O}_C(K_C - D) & \hookrightarrow & H^0\mathcal{O}_C(K_C)
\end{array}$$

from which the result follows at once, q.e.d.

We go back to the general case, the system Λ containing D being given by a subspace $E \subset H^0\mathcal{O}_C(D)$ which contains the image of $H^0\mathcal{O}_C$ in $H^0\mathcal{O}_C(D)$. The sequence of maps

$$T_\Lambda(D) \hookrightarrow T_{|D|}(D) \hookrightarrow T_{C(d)}(D)$$

can be identified with the following one:

$$E\ /\ H^0\mathcal{O}_C \hookrightarrow H^0\mathcal{O}_C(D)\ /\ H^0\mathcal{O}_C \hookrightarrow H^0\mathcal{O}_D(D)\ .$$

Recall also that the sheaf $\mathcal{O}_C(D)$ is gotten as asubsheaf of $\mathcal{R}_C$, the sheaf of rational functions on C, by putting:

$$\mathcal{O}_C(D)_P := \underline{m}_{C,P}^{-v_P(D)}\mathcal{O}_{C,P} \qquad \text{for each} \quad P \in C \ .$$

Write P' and P'' for the points of $\hat{C}$ lying above P. Then, if $D = \sum n_i P_i$, the points P_i being all distinct, and the divisor

$$\hat{D} = \sum n'_j P'_j + \sum n''_j P''_j \qquad (n'_i + n''_i = n_i \ , \ n'_i \geq n''_i \quad \forall i)$$

lies above D in X, we can identify the two diagrams below:

$$\begin{array}{ccc} & & T_{\hat{C}}(d)(\hat{D}) \\ & & \Big\downarrow {\scriptstyle d\pi(d)} \\ T_\Lambda(D) & \xhookrightarrow[dj]{} & T_C(d)(D) \end{array} \qquad \text{and}$$

$$\begin{array}{ccc} & & \sum(\underline{m}_{\hat{C},P'_i}^{-n'_i}\mathcal{O}_{\hat{C},P'_i} / \mathcal{O}_{\hat{C},P'_i}) \oplus \sum(\underline{m}_{\hat{C},P''_i}^{-n''_i}\mathcal{O}_{\hat{C},P''_i} / \mathcal{O}_{\hat{C},P''_i}) \\ & & \Big\downarrow {\scriptstyle \beta} \\ E / H^0\mathcal{O}_C & \xhookrightarrow[\alpha]{} & \sum(\underline{m}_{C,P_i}^{-n_i}\mathcal{O}_{C,P_i} / \mathcal{O}_{C,P_i}) \ . \end{array}$$

Here the mapping β is the transpose (by the residue pairing) of the natural map

$$\sum(\Omega^1_{C,P_i} / \underline{m}_{C,P_i}^{n_i}\Omega^1_{C,P_i}) \longrightarrow \sum(\Omega^1_{\hat{C},P'_i} / \underline{m}_{\hat{C},P'_i}^{n'_i}\Omega^1_{\hat{C},P'_i}) \oplus \sum(\Omega^1_{\hat{C},P''_i} / \underline{m}_{\hat{C},P''_i}^{n''_i}\Omega^1_{\hat{C},P''_i})$$

induced by the projection map π. Thus, if t_i is a local parameter of C at P_i and t'_i, t''_i are the local parameters induced on $\hat{C}$ at P'_i and P''_i respectively, we have

$$\beta({t'_i}^k) = \beta({t''_i}^k) = {t_i}^k \ .$$

The mapping α is the obvious one: if $\psi \in E \subset H^0\mathcal{O}_C(D) \subset k(C)$ then the i-th component of $\alpha(\overline{\psi})$ is the image of the element $\psi \in \underline{m}_{C,P_i}^{-n_i}\mathcal{O}_{C,P_i}$ in $\underline{m}_{C,P_i}^{-n_i}\mathcal{O}_{C,P_i} / \mathcal{O}_{C,P_i}$.

PROPOSITION. *With the assumptions of* (8.12) *and the above notations, the divisor variety* X *above* Λ *is smooth at* $\hat{D}$ *if and only if the following condition holds: For each* i *such that* $n_i > n_i'$ *and each* q, $0 \le q < n_i - n_i'$, *there exists* $\overline{D}$ *in* Λ *such that* $v_{P_i}(\overline{D}) = q$ *and* $\forall j \neq i$: $v_{P_j}(\overline{D}) \ge n_j - n_j'$.

PROOF. By (8.12), X is smooth at $\hat{D}$ if and only if the composition of α with the cokernel projection of β is surjective, i.e.:

$$E \,/\, H^0\mathcal{O}_C \longrightarrow\!\!\!\!\rightarrow \sum \left(\underline{m}_{C,P_i}^{-n_i}\mathcal{O}_{C,P_i} \,/\, \underline{m}_{C,P_i}^{-n_i'}\mathcal{O}_{C,P_i}\right)$$

where the right hand sum now ranges over those i for which $n_i > n_i'$ holds. Say, as before, that t_i is a local parameter of C at P_i. Then the vector space $\underline{m}_{C,P_i}^{-n_i}\mathcal{O}_{C,P_i} \,/\, \underline{m}_{C,P_i}^{-n_i'}\mathcal{O}_{C,P_i}$ has basis

$$t_i^{-n_i}\ ,\ t_i^{-(n_i-1)}\ ,\ \ldots\ ,\ t_i^{-(n_i'+1)}$$

over $\mathbb{C}$. An element $\psi \in E$ gives a divisor $\overline{D}$ of Λ by taking $\overline{D} = \operatorname{div}(\psi) + D$ and, if the image of $\overline{\psi}$ in the above vector space is written

$$c_k t_i^{-k} + c_{k-1} t_i^{-(k-1)} + \ldots$$

with $c_k \neq 0$, then $v_{P_i}(\overline{D}) = v_{P_i}(\psi) + v_{P_i}(D) = n_i - k$. From this the result follows at once, q.e.d.

(8.13) Let us describe, for example, necessary and sufficient conditions for a net or a pencil yielding smooth divisor varieties (the proofs are trivial).

a) If $\dim \Lambda = 1$, smoothness of the curve of divisors above Λ is equivalent with the following two conditions: i) The system Λ has no fixed part ; ii) its singular members have at most one singularity of type 2P or one of type 3P .

b) If $\dim \Lambda = 2$, smoothness is equivalent with the following three conditions: i) The system Λ is induced by a generically injective map $\sigma : C \longrightarrow \mathbb{P}^2$; ii) if present, the singular parts of the members of Λ have one of the following shapes: 2P , 3P , 4P , 5P or $2P_1 + 2P_2$, $2P_1 + 3P_2$, $3P_1 + 3P_2$; iii) in the latter three cases, $\sigma(P_1) \neq \sigma(P_2)$, and in the case of a singularity $4P$ or $5P$, σ is a local inmersion at P .

For later use in Section 10 , we distinguish a class of plane curves by the following

(8.14) CONVENTION. *A plane curve* $C \subset \mathbb{P}^2$ *will be called* admissible *if and only if*

a) C *is irreducible and has at most ordinary double points.*

b) If L *is any line of* $\mathbb{P}^2$ *then* C *cuts out on* L *either a smooth divisor or a divisor with exactly one of the following singularities:* $2P$, $3P$ *or* $2P + 2Q$.

Using (8.13) it is clear that if C is an admissible curve and N its normalization, then the system of linear sections of C induces on N a net Λ such that

i) For any double cover of N , the surface of divisors above Λ is smooth ;

ii) for any double cover $\hat{N}$ of N , and any singular point of C , if P_1 and P_2 are the points of N lying above the latter, the moving part of the pencil of Λ determined by the base points P_1 and P_2 yields a smooth curve of divisors on $\hat{N}$.

The Albanese variety

Notations and assumptions being as in (8.3) , let Y be one of the halves of X . Suppose that Y is irreducible - then it makes sense to speak of the Albanese variety $\mathrm{Alb}(Y)$ of Y . The diagram

$$\begin{array}{ccccc} \mathrm{Alb}(Y) & \longrightarrow & \mathrm{Alb}(\hat{C}^{(d)}) & \xrightarrow{\simeq} & J(\hat{C}) \\ \downarrow & & \downarrow & & \downarrow \\ 0 = \mathrm{Alb}(\Lambda) & \longrightarrow & \mathrm{Alb}(C^{(d)}) & \xrightarrow{\simeq} & J(C) \end{array}$$

(the morphisms being the natural ones) shows that the map $\mathrm{Alb}(Y) \longrightarrow J(\hat{C})$ factors through a well-determined morphism

$$\psi : \mathrm{Alb}(Y) \longrightarrow \mathrm{Pr}(\hat{C}/C) .$$

Exercising with known examples (cf the next subsection) , one is lead to the following question:

(8.15) *Suppose that the linear system* Λ *has dimension* ≥ 2 *and that* Y *is a smooth half of the variety of divisors* X *above* Λ . *Is then* ψ *an isogeny ? Is it an isomorphism ?*

The reason for imposing $\dim \Lambda \geq 2$ is that - except a few cases - the answer is negative if $\dim \Lambda = 1$ (cf (8.20)) . Smoothness of Y is required for technical reasons. We shall see in a moment that, if the degree d of Λ is ≥ 3 , ψ is surjective, by obvious reasons. In this case, the weak form of the above question is equivalent to ask whether, with the above assumptions, $q(Y) = p_a(C) - 1$ holds, with $q(Y) = h^1\mathcal{O}_Y$, $p_a(C) = h^1\mathcal{O}_C$. In fact, only the inequality $q(Y) \leq p_a(C) - 1$ is essential here.

(8.16) <u>PROPOSITION</u>. *Suppose that* Y *is an irreducible half of* X . *If the degree* d *of* Λ *is* ≥ 3 *then* ψ *is surjective.*

<u>PROOF</u>. In view of Lemma (8.2) it suffices to show that every element of the form $2[P - i(P)]$, $P \in \hat{C}$, lies in the image of ψ . Let $P \in \hat{C}$ be given. Take an arbitrary $D \in Y$ containing P in its support: $D = P + P_2 + P_3 + \ldots$. Then Y contains the point $D' = i(P) + i(P_2) + P_3 + \ldots$ by (8.5) , and also $D'' = i(P) + P_2 + i(P_3) + \ldots$ and $D''' = P + i(P_2) + i(P_3) + \ldots$. Now $\psi([D - D']) = [P - i(P)] + [P_2 - i(P_2)]$ and also $\psi([D''' - D'']) = [P - i(P)] + [i(P_2) - P_2]$. Hence $2[P - i(P)] = \psi([D - D'] + [D''' - D''])$, q.e.d.

(8.17) <u>REMARK</u>. *As* (8.20) *shows, the conclusion of* (8.16) *fails to be true in general, if* $d < 3$.

For systems of relative high degree, one gets quickly an answer to (8.15) :

(8.18) <u>PROPOSITION</u>. *Let* $\pi : \hat{C} \longrightarrow C$ *be a double cover of a smooth curve* C , *and* Λ *a linear system on* C , *of dimension* ≥ 2 *and degree* d . *Suppose furthermore that*

a) Y *is a smooth half of the variety of divisors* X *above* Λ *and*

b) $d \geq 4p_a(C) - 3$.

Then the mapping $\psi : \mathrm{Alb}(Y) \longrightarrow \mathrm{Pr}(\hat{C}/C)$ *is an isomorphism.*

PROOF. Suppose first that Λ is complete . In this case the diagram

$$\begin{array}{ccccc} X & \hookrightarrow & \hat{C}^{(d)} & \xrightarrow{\hat{h}} & J(\hat{C})_d \\ \downarrow & & \downarrow & & \downarrow \\ \Lambda & \hookrightarrow & C^{(d)} & \xrightarrow{h} & J(C)_d \end{array}$$

gives, if $\hat{\xi} \in \hat{h}(Y)$ and $\xi = h(\Lambda)$, that $\Lambda = h^{-1}(\xi)$ and hence that $Y = \hat{h}^{-1}(\hat{\xi} + \mathrm{Pr})$. Since $p_a(\hat{C}) = 2p_a(C) - 1$, the divisors of degree d in $\hat{C}$ are non special, so the fibres of $\hat{h}$ are all isomorphic with $\mathbb{P}^k$, $k = d + 1 - 2p_a(C)$. Hence Y is a projective bundle over $\hat{\xi} + \mathrm{Pr}$, and $\mathrm{Alb}(Y) \simeq \mathrm{Alb}(\hat{\xi} + \mathrm{Pr}) \simeq \mathrm{Pr}(\hat{C}/C)$, as claimed.

If Λ is arbitrary, consider the complete linear system determined by Λ, say $\tilde{\Lambda}$. The variety Y lies in a half $\tilde{Y}$ of the (smooth) divisor variety above $\tilde{\Lambda}$. Since Λ is a complete intersection of dimension ≥ 2 in $\tilde{\Lambda}$, Y is a smooth complete intersection of ample divisors in $\tilde{Y}$, and $\dim Y \geq 2$. Lefschetz theory then gives the required result: $\mathrm{Alb}(Y) \simeq$ $\simeq \mathrm{Alb}(\tilde{Y}) \simeq \mathrm{Pr}(\hat{C}/C)$, q.e.d.

REMARK. Of course, in the first part of the proof of (8.18) , the fact that Y is a projective bundle over $\hat{\xi} + \mathrm{Pr}$ is not fully needed (in the second part this guarantees smoothness of $\tilde{Y}$) . It suffices to have a morphism $Y \longrightarrow \hat{\xi} + \mathrm{Pr}$ with projective spaces as fibres. Therefore (8.18) holds also for complete systems Λ of degree $d \geq 2p_a(C) - 1$, without smoothness assumptions on Y .

We can now improve Proposition (8.18) as follows:

(8.19) THEOREM. *Let* $\pi : \hat{C} \longrightarrow C$ *be a double cover of a smooth curve* C . *If* Λ *is a non special linear system of dimension* ≥ 2 *on* C *, and* X *is a smooth half of the variety of divisors on* $\hat{C}$ *above* Λ *, then the map* $\psi : \mathrm{Alb}(X) \longrightarrow \mathrm{Pr}(\hat{C}/C)$ *is an isomorphism.*

PROOF. Put d = degree of Λ , $g = p_a(C)$. The fact of being Λ non special gives, for $D \in \Lambda : 3 \leq l(D) = d + 1 - g$, hence $d \geq g + 2$. Remark that the assertion is true if $d \geq 4g - 3$ by Proposition (8.18). We shall prove it by descending induction on d .

We first prove the following

LEMMA. *Suppose we can find a non special system* Λ *of degree* d *,* $\dim \Lambda =$ $= 2$ *, the divisor variety above* Λ *being smooth, and* ψ *being an isomor-*

phism for each one of the two halves . Then the assertion of (8.19) *is true for systems of degree* d .

PROOF. This is entirely standard. Consider the open subvariety $V_d \subset J(C)_d$ consisting of non special classes. The inverse image of V_d in $C^{(d)}$ yields an open subvariety $U_d \subset C^{(d)}$ and the projection map $U_d \longrightarrow V_d$ is a projective bundle with fibre $\mathbb{P}^{d-g}$. The associated Grassmann bundle of 2-planes inside the fibres of $U_d \longrightarrow V_d$ yields an irreducible variety $Gr_2(U_d)$, above which there lives a scheme Σ , the fibres of the projection

$$f : \Sigma \longrightarrow Gr_2(U_d)$$

being the surfaces of divisors above the 2-dimensional non-special linear systems of C . If necessary, we perform a base change to the Stein factorization of f , to insure finally the existence of a (smooth irreducible) parameter variety T and two T-schemes $f_i : \Sigma_i \longrightarrow T$, $i = 1 , 2$, the fibres of f_i being the halves of the surfaces of divisors. For suitable open subvarieties $T_i \subset T$, $i = 1 , 2$ we get two irreducible families

$$\Sigma_i^0 \longrightarrow T_i \quad , \quad i = 1 , 2$$

parametrizing all smooth halves of surfaces of divisors above non special 2-dimensional linear systems of degree d . Consider the natural morphisms

$$\underline{\mathrm{Pic}}^0{}_{\Sigma_i^0} / T_i \xleftarrow{t_\psi} \mathrm{Pr}(\hat{C}/C) \times T_i \quad , \quad i = 1 , 2 .$$

By standard arguments, t_ψ will yield isomorphisms

$$\underline{\mathrm{Pic}}^0{}_{\Sigma_i^0(\lambda)} / \mathbb{C} \xleftarrow{t_\psi} \mathrm{Pr}(\hat{C}/C) \qquad , \quad i = 1 , 2$$

for all $\lambda \in T_i(\mathbb{C})$ if and only if it does so for one of them. Transposing, we get an analogous statement for the morphisms

$$\mathrm{Alb}(\Sigma_i^0(\lambda)) \xrightarrow{\psi} \mathrm{Pr}(\hat{C}/C) \quad , \quad i = 1 , 2 .$$

Remark finally that for systems of dimension > 2 , the statement of the lemma follows from the 2-dimensional case by Bertini and Lefschetz. This proves the lemma, q.e.d.

Proceeding with the proof of (8.19), we keep all the above notations. The inverse image $\hat{U}_d$ of U_d in $\hat{C}(d)$ is an open subvariety of the latter. The fibres of the composite map $\hat{U}_d \longrightarrow V_d$ are the varieties of divisors above complete non special linear systems. Since $\hat{U}_d$, V_d are smooth, there exists a non empty open subset $V_d^0 \subset V_d$ such that the divisor varieties above the corresponding complete systems are smooth.

Fix a point $P \in C$ once for all in the remainder of this proof. Translation by P gives an isomorphism $J(C)_d \xrightarrow{\simeq} J(C)_{d+1}$. The inverse image of V_{d+1}^0 is a non empty open subset of $J(C)_d$, which therefore meets V_d^0. In this way we reach the following situation:

There exists a complete non special system $\Lambda = |D|$ of degree d yielding a smooth divisor variety in $\hat{C}$ and such that the system $\overline{\Lambda} = |D + P|$, of degree $d + 1$, has the same properties. Take one of the components, say $\overline{X}$, of the divisor variety above $\overline{\Lambda}$. For any point $x \in \hat{C}$ we denote by Y_x the subvariety of $\overline{X}$ consisting on those divisors which contain x in their support. Then $\{Y_x\}_{x \in \hat{C}}$ is an algebraic system of divisors of $\overline{X}$ and, if $P_1, P_2 \in \hat{C}$ are the points in $\hat{C}$ lying above P, we have:

$$\pi^*(\Lambda + P) = Y_{P_1} + Y_{P_2} .$$

Since $\Lambda + P \subset \overline{\Lambda}$ is an ample divisor of $\overline{\Lambda}$, so is $Y_{P_1} + Y_{P_2}$ in $\overline{X}$. Hence $2Y_x$, $x \in \hat{C}$, and therefore also Y_x, $x \in \hat{C}$, are ample divisors of $\overline{X}$. Consider the two halves of the divisor variety of $\hat{C}$ above Λ, say X_1, X_2. For a suitable choice of the indexes we have obvious isomorphisms $X_i \simeq Y_{P_i}$. In particular, the Y_{P_i}, $i = 1, 2$ are smooth. Since $\dim Y_{P_i} \geq 2$, we may therefore apply Lefschetz and get

$$\mathrm{Alb}(X_i) \simeq \mathrm{Alb}(Y_{P_i}) \simeq \mathrm{Alb}(\overline{X}) \simeq \mathrm{Pr}(\hat{C}/C) \quad ,$$

using the induction hypothesis. A final application of Bertini and Lefschetz gives a 2-dimensional system inside Λ satisfying the assumptions of the lemma, thereby proving the claimed assertion, q.e.d.

Examples

(8.20) In (8.15) we request $\dim \Lambda \geq 2$. This is done because $\dim \Lambda = 1$ gives a negative answer, except in two well-known cases: the presentation of the Prym of a double covering of an hyperelliptic curve as a product of Jacobians, and the result of Recillas ([25]) concerning trigonal curves. Let us describe the case $\dim \Lambda = 1$ in more detail.

a) If the degree d of Λ is 1 , C is a rational curve, hence it has no double covers (in our terminology). If $d = 2$ then C is either elliptic or hyperelliptic. If C is elliptic, there are ∞^1 such g_2^1 's . The halves of the divisor variety are smooth (2:1) coverings of $\mathbb{P}^1$ branched at two points, hence these curves are rational; we call them C_i . Thus $\mathrm{Alb}(C_i) = 0 = \mathrm{Pr}(\hat{C}/C)$ and (8.15) holds trivially. Suppose that C is hyperelliptic. Then the g_2^1 is unique. The halves C_i are smooth, by (8.13.a) , and they are mapped (2:1) onto $\Lambda = \mathbb{P}^1$. Hence they are either rational, elliptic or hyperelliptic curves. The morphisms

$$\psi_i : J(C_i) \longrightarrow \mathrm{Pr}(\hat{C}/C)$$

lead to an isomorphism (actually as principally polarized abelian varieties)

$$J(C_1) \times J(C_2) \xrightarrow{\simeq} \mathrm{Pr}(\hat{C}/C) \quad ,$$

as explained in [21], p.346 . One should compare this result with (8.7) and (8.17) : if neither of the curves C_i is rational, which happens for all but one cover of C (ibid), the mappings ψ_i are not surjective; moreover C_1 and C_2 are generally non isomorphic.

b) Suppose still $\dim \Lambda = 1$, but $d \geq 3$,the divisor variety being supposed smooth. Both halves of X , C_1 and C_2 , are $(2^{d-1} : 1)$ coverings of $\mathbb{P}^1$, branched above the discriminant points of the $(d : 1)$ mappings $C \longrightarrow \mathbb{P}^1$ associated to Λ . An easy reasonning shows that above a discriminant point in $\mathbb{P}^1$, corresponding to a point of multiplicity 2 in some divisor of Λ , each of the maps $g_i : C_i \longrightarrow \mathbb{P}^1$ exhibits 2^{d-3} clusters of two points; above a point belonging to a singularity of multiplicity 3 each map g_i produces 2^{d-3} clusters of 3 points. Say that there are δ discriminant points of the first type and τ of the second one. Then Zeuthen-Hurwitz, applied twice, yields:

$$2p_a(C) - 2 = -2d + \delta + 2\tau$$
$$2p_a(C_i) - 2 = -2\cdot 2^{d-1} + \delta\cdot 2^{d-3} + 2\tau\cdot 2^{d-3} \quad .$$

Hence $\quad 2p_a(C_i) - 2 - 2^{d-3}(2p_a(C) - 2) = -2^d + d\cdot 2^{d-2}$,

i.e. $\quad p_a(C_i) = 2^{d-3}\cdot p_a(C) + (d\cdot 2^{d-3} - 2^{d-1} - 2^{d-3} + 1)$.

This yields finally

$$\dim \mathrm{Alb}(C_i) - \dim \mathrm{Pr}(\hat{C}/C) = p_a(C_i) - (p_a(C) - 1) =$$
$$= (2^{d-3} - 1)p_a(C) + d\cdot 2^{d-3} - 2^{d-1} - 2^{d-3} + 2 \ .$$

This difference is strictly positive unless $d = 3$ (or $d = 4$ and $p_a(C) = 0$, this case being dropped because no cover of C can exist) . This is just the case of the theorem of Recillas ([25]) . It turns out that

$$\psi_i : J(C_i) \longrightarrow \mathrm{Pr}(\hat{C}/C) \quad , \quad i = 1 , 2$$

is an isomorphism of principally polarized abelian varieties.

(8.21) The following example, taken from [27], p.103, has been the point of departure for Part Two of this work.

Consider a smooth complete intersection X of three quadrics in $\mathbb{P}^6$ and look at the conics living inside X . The supporting plane of such a conic is characterized, among the 2-planes of $\mathbb{P}^6$, by the property of being contained in the quadrics of a pencil of the net of quadrics through X . Identify this net with $\mathbb{P}^2$. The discriminant curve C of the net, above which the quadrics are singular, has degree 7 . We suppose X general enough so that this curve will be smooth. Equivalently (cf [27], p.69) , the degenerate quadrics of the net are ordinary cones, i.e. quadrics of rank 6 .

The Chow variety of 3-planes living in such a cone has exactly two components. The variety parametrizing these components, as the cone varies above C , is a smooth curve $\hat{C}$, (2:1) étale covering of C . Furthermore, if a 2-plane lies in such a cone, it is contained in precisely one 3-plane of the latter (resp. in exactly two 3-planes, belonging to different families) if it doesn't meet the vertex of the cone (resp. if it meets the vertex) . If a conic in X is given, its plane determines a line in $\mathbb{P}^2$ (the pencil of quadrics containing this plane) inducing a divisor of degree 7 on C . Considering for each of the cones of the pencil the component (resp. the components) of 3-planes determined by the fact of containing, one of its members, this 2-plane , determines a 7th degree divisor of $\hat{C}$ lying above the former 7th degree divisor of C .

One obtains in this way (ibid) an isomorphism between the Fano surface F of conics in X with one half S of the surface of divisors of $\hat{C}$ above the system of linear sections of C .

We have the following data. First, the Abel-Jacobi map

$$\phi : \mathrm{Alb}(F) \longrightarrow J(X) \quad ;$$

secondly, the map

$$\psi : \mathrm{Alb}(S) \longrightarrow \mathrm{Pr}(\hat{C}/C) \quad ,$$

and finally, either from [27], p.98 or from [2], p.33 , an isomorphism $J(X) \simeq \mathrm{Pr}(\hat{C}/C)$ (actually as principally polarized abelian varieties) . Recall that the latter can be gotten as follows: Fix an arbitrary line L in X (there is a 1-dimensional family of lines in X) . In each cone of the net there are exactly two 3-planes (one in each family) containing L . Each such 3-plane meets X in a quartic curve, intersection of two quadrics , which therefore has to consist of the line L plus a (eventually reducible) twisted cubic meeting L at two points; The curve C parametrizes in this way a family of twisted cubics inside X ; we have the corresponding Abel-Jacobi map $J(\hat{C}) \longrightarrow J(X)$, and its transpose map induces an isomorphism $J(X) \xrightarrow{\simeq} \mathrm{Pr}(\hat{C}/C) \subset J(\hat{C})$. (Remark: Usually one considers the isomorphism $\mathrm{Pr}(\hat{C}/C) \xrightarrow{\simeq} J(X)$ transpose of the latter, which equals the opposite of its inverse; one is suggested to keep this in mind when arriving, in a moment, at an anticommutative diagram).

Remark that the above isomorphism is independent of the choice of L , because so is the Abel-Jacobi map $J(\hat{C}) \longrightarrow J(X)$. In fact, if L' is another line, and Z , Z' are the residual curves cut out on X by 3-planes belonging to the same family of one cone of the net, one has the rational equivalence in X : $L + Z \equiv L' + Z'$; from this our claim follows at once.

We compose, with these data, the following diagram:

$$\begin{array}{ccc} Alb(F) & \xrightarrow{\phi} & J(X) \\ \simeq\downarrow & & \downarrow\simeq \\ Alb(S) & \xrightarrow[\psi]{} & Pr(\hat{C}/C) \end{array} .$$

We claim that this diagram is *anti*commutative. Before giving a proof, we remark that this is not strictly necessary for our purposes; in fact, using the surjectivity of $J(\hat{C}) \longrightarrow J(X)$, one easily finds the surjectivity of ϕ. Hence the question of ϕ being an isogeny is equivalent with the dimension of $Alb(F)$ being 14, i.e. with ψ being an isogeny. We include nevertheless a proof of this fact, since we consider it as the "raison d'être" of the identification between F and S.

Consider thus, for a fixed line $L \subset X$, the curve $\Sigma_L \subset F$ of conics meeting L. We shall see that, for two general conics $\gamma_1, \gamma_2 \in \Sigma_L$, the element $\gamma_1 - \gamma_2 \in Alb(F)$ has opposite images under the two possible composite maps of the above diagram. Since the divisor Σ_L of F is ample ($2\Sigma_L$ is algebraically equivalent with the divisor of conics meeting a given conic, which is clearly ample by Nakai-Moishezon), this will settle the question. For, if some Σ_L is smooth, this follows from $J(\Sigma_L)$ mapping onto $Alb(F)$; in any case it follows from the fact that these curves are connected and always meeting.

Take thus general conics $\gamma_1, \gamma_2 \in \Sigma_L$. The image of $\gamma_1 - \gamma_2$ by the composite map $Alb(F) \xrightarrow{\phi} J(X) \xrightarrow{\simeq} Pr(\hat{C}/C)$ is the class of the divisor $D_1 - D_2$ of $\hat{C}$, where D_i is the divisor of points $\lambda \in \hat{C}$ such that the twisted cubic Z_λ meets γ_i. The divisors D_i can be written as $D_i = D_i' + D_i''$, where D_i' is the divisor of points $\lambda \in \hat{C}$ such that Z_λ meets γ_i at the point $P_i = L \cap \gamma_i$ (Figure 9). Since D_1' and D_2' are linearly equivalent (the point P_i moves on $L \simeq \mathbb{P}^1$), it follows that the class of $D_1 - D_2$ in $Pr(\hat{C}/C) \subset J(\hat{C})$ equals the class of $D_1'' - D_2''$.

On the other side, the image of $\gamma_1 - \gamma_2$ by the composite arrow $Alb(F) \xrightarrow{\simeq} Alb(S) \xrightarrow{\psi} Pr(\hat{C}/C)$ is the class of the divisor $\overline{D}_1 - \overline{D}_2$, where $\overline{D}_i$ is the divisor of points $\lambda \in \hat{C}$ such that the corresponding family of 3-planes of a cone of the net has a member containing the supporting plane $\Pi\gamma_i$ of the conic γ_i.

Denoting by $i : \hat{C} \longrightarrow \hat{C}$ the covering involution (sending one family of 3-planes into the other one, for any cone), we have $i(\overline{D}_k) = D_k''$

for $k=1,2$. To see this, suppose that, for $\lambda \in \hat{C}$, Z_λ and a (general) conic $\lambda \in \Sigma_L$ meet outside $P = L \cap \gamma$. Call Q the cone of the net herewith involved. The 3-plane V spanned by the twisted cubic Z_λ meets γ at two different points, hence the line joining these points is common to V and Π_γ . Then $V \subset Q$ says that Q meets Π_γ at least along γ and this line. Therefore $\Pi_\gamma \subset Q$. Let V' be a 3-plane of Q containing Π_γ . The intersection $V \cap V'$ consists at least of the above mentioned line on Π_γ plus the vertex of Q . By the generality assumption on γ , this vertex lies not on Π_γ and, a fortiori, not on the line in question. So V and V' meet along a 2-plane, and this tells us that they belong to different families of Q .

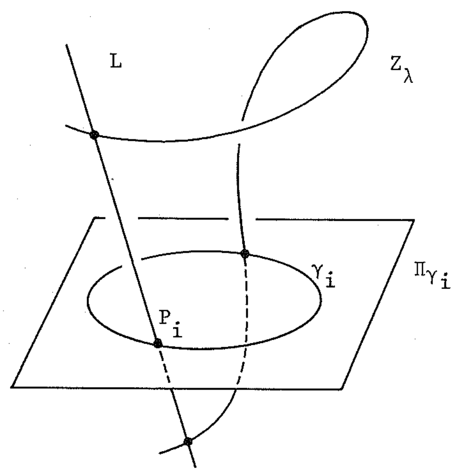


Figure 9.

Conversely, suppose that Π_γ is contained in a 3-plane V' of a cone Q of the net . Let Z_λ be the twisted cubic corresponding to the other family of 3-planes in Q . We want to see that Z_λ and γ meet at a point different from $P = L \cap \gamma$. The 3-plane spanned by Z_λ meets V' already at P and at the vertex of Q . Being of two different families, V and V' have to meet either at the vertex only, or along a whole 2-plane. The latter has to be the case here. This 2-plane meets the conic γ at P plus a second point which, belonging to X , has to belong also to the twisted cubic Z_λ , as was to be shown.

Therefore the second image of $\gamma_1 - \gamma_2$ in $Pr(\hat{C}/C)$ is $[\overline{D}_1 - \overline{D}_2] =$

$= [i(D_1'' - D_2'')] = i[D_1'' - D_2''] = -[D_1'' - D_2'']$, since the involution acts as minus the identity on $Pr(\hat{C}/C)$, q.e.d.

(8.22) In (8.21) it has been question of divisor varieties attached to the system of linear sections of a plane septic. Another important case is that of the plane quintics. These arise as discriminant curves of nets of quadrics in $\mathbb{P}^4$, in a similar way as before. More explicitly (cf [2], (6.23), (6.27)) , take a smooth plane curve C of degree 5 . The double covers $\hat{C}$ of C fall into two classes, depending on the parity of the theta characteristic ξ giving this covering. The even coverings such that $H^0(C,\mathcal{O}_C(\xi)) = 0$ correspond to smooth complete intersections X of three quadrics in $\mathbb{P}^4$, and the odd ones with $\dim H^0(C,\mathcal{O}_C(\xi)) = 1$ correspond to smooth cubic threefolds X in $\mathbb{P}^4$.

We recall how one gets the curve C and its covering $\hat{C}$ in each case. In the first one, C is the discriminant curve of the net $\mathbb{P}^2$ of quadrics in $\mathbb{P}^4$ containing X , and $\hat{C}$ is, as in (8.21) , the variety of Chow components of 2-planes lying in the cones of the net (quadrics of rank 4 parametrized by C) . In the second case one fixes a sufficiently general line L in the cubic threefold X . Here $\mathbb{P}^2$ is the set of 2-planes of $\mathbb{P}^4$ containing L . The curve $C \subset \mathbb{P}^2$ consists of those planes which meet X in a degenerate conic plus the line L . The variety of components of these degenerate conics (= lines in X which meet L) together with the natural map onto C , is the (2:1) covering $\hat{C} \longrightarrow C$.

We describe next how, in each case, a certain variety of cycles on X is naturally related with one half of the surface of divisors in $\hat{C}$ above the system of line sections of $C \subset \mathbb{P}^2$. In the even case one proceeds as in (8.21) . The variety X is a canonically embedded curve of genus 5 . The variety F of "conics" on X is simply the surface $X^{(2)}$. But, as before, a conic determines uniquely (and is determined by) a line of $\mathbb{P}^4$ which lies on the quadrics of a pencil of the net determined by X . Such a line therefore determines a divisor of degree 5 on $\hat{C}$ (the families of 2-planes on the cones, which contain a member containing this line) lying above a linear section of $C \subset \mathbb{P}^2$ (the pencil of quadrics containing the line) . This identifies the variety of "conics" $F = X^{(2)}$ with one half S of the surface of divisors of $\hat{C}$ above the linear sections of C .

In the odd case one considers the Fano surface F of lines on X . A line in X not meeting L is met, together with L , by exactly 5 lines of X . This yields a birational map from F to a half S of the surface of divisors of $\hat{C}$ above the linear sections of C and, more exactly, it can be shown that S is the blowing up of F at the point $L \in F$.

In both cases we have natural identifications of $J(X)$ with $Pr(\hat{C}/C)$ (loc. cit.) . It turns out that these allow us to identify the Abel-Jacobi map $\phi : Alb(F) \longrightarrow J(X)$ with the map $\psi : Alb(S) \longrightarrow Pr(\hat{C}/C)$. But in the even case ϕ is the isomorphism $Alb(X^{(2)}) \simeq J(X)$ and in the odd one ϕ is an isomorphism by [8], p. 329 . Accordingly, ψ is an isomorphism in both cases.

9. A DEGENERATION OF SURFACES OF DIVISORS

The limit case of a curve with a node

(9.1) Suppose that C is an irreducible curve with exactly one ordinary double point, and $\hat{C}$ a double cover of C . Normalizing both curves, we get a double cover

$$\hat{N} \longrightarrow N$$

involving smooth curves. Call P the double point of C and P_1 , P_2 the points in N above P . We suppose given a net Λ of degree d on N such that

a) Λ is induced by a morphism $\sigma : C \longrightarrow \mathbb{P}^2$;

b) the surface of divisors in $\hat{N}$ above Λ is smooth;

c) the curve of divisors in $\hat{N}$ above the pencil $\Lambda - P_1 - P_2$ is smooth;

d) no member of Λ has its support contained in $\{ P_1 , P_2 \}$.

(9.2) EXAMPLE. The above conditions are fulfilled if we take C to be the normalization at all but one singular point of an admissible curve $\overline{C} \subset \mathbb{P}^2$ (cf (8.14)) , the tangent lines at the branches of the remaining singular point meeting $\overline{C}$ outside this singular point again. In particular, this will be the case if $\overline{C}$ is an admissible curve of degree ≥ 4 .

In the situation of (9.1) , fix one half of the surface of divisors in $\hat{N}$ above Λ , and call it S . We want to study the image surface S_0

of S in $\hat{C}^{(d)}$ and the relationship between S and S_0.

Call P', P'' the points of $\hat{C}$ above P and P_1', P_1'' (resp. P_2', P_2'') the points of $\hat{N}$ above P_1 (resp. P_2). Schematically:

$$\begin{array}{ccc} \hat{C} & \longleftarrow & \hat{N} \\ \downarrow & & \downarrow \\ C & \longleftarrow & N \end{array} \qquad\qquad \begin{array}{ccc} \begin{pmatrix} P'' \\ P' \end{pmatrix} & \longleftarrow & \begin{pmatrix} P_1'' & P_2'' \\ P_1' & P_2' \end{pmatrix} \\ \downarrow & & \downarrow \\ P & \longleftarrow & (P_1, P_2) \end{array} .$$

Any divisor of the family parametrized by S and passing through one of the four points P_1', P_2', P_1'', P_2'' necessarily passes through a second one, other than its conjugate under the covering involution of $\hat{N}$, because any divisor of the net Λ on N which meets one of the points P_1, P_2 has to meet the other one too.

We distinguish the following curves on S:

$$\begin{aligned} \Gamma_{12} &= \text{curve of divisors } \hat{D} \geq P_1' + P_2'' \\ \Gamma_{21} &= \text{curve of divisors } \hat{D} \geq P_2' + P_1'' \\ \Gamma' &= \text{curve of divisors } \hat{D} \geq P_1' + P_2' \\ \Gamma'' &= \text{curve of divisors } \hat{D} \geq P_1'' + P_2'' \end{aligned} .$$

(9.3) LEMMA. *The following holds on* S *:*

a) $\Gamma_{12} \sim \Gamma_{21}$ *and* $\Gamma' \sim \Gamma''$ *(algebraic equivalence) ;*

b) $(\Gamma_{ij} \cdot \Gamma_{ij}) = (\Gamma_{ij} \cdot \Gamma_{ji}) = 0$ *;*

c) the curves Γ_{12}, Γ_{21}, Γ' *and* Γ'' *are smooth and connected.*

PROOF. a) As a point x describes $\hat{N}$, the curve $\Delta_x \subset S$ of divisors meeting x yields an algebraic family of divisors on S. We have $\Delta_{P_1'} = \Gamma' + \Gamma_{12}$, $\Delta_{P_2'} = \Gamma' + \Gamma_{21}$, $\Delta_{P_1''} = \Gamma'' + \Gamma_{21}$ and $\Delta_{P_2''} = \Gamma'' + \Gamma_{12}$. From this we get $\Gamma_{12} \sim \Gamma_{21}$ and $\Gamma' \sim \Gamma''$ as claimed.

b) The curves Γ_{12}, Γ_{21} don't intersect. For, intersection would imply the existence of a divisor in the family parametrized by S, of the shape $P_1' + P_2' + P_1'' + P_2'' + \ldots$ hence, projecting into Λ, the existence of a divisor $2P_1 + 2P_2 + \ldots$ in Λ. But this contradicts assumption b) of (9.1), by (8.13.b.iii). Hence $(\Gamma_{12} \cdot \Gamma_{21}) = 0$ and therefore $(\Gamma_{12} \cdot \Gamma_{12}) = (\Gamma_{21} \cdot \Gamma_{21}) = 0$, which proves b).

c) The curves Γ' and Γ'' are clearly isomorphic with one half, and Γ_{12}, Γ_{21} with the other one, of the curve of divisors of $\hat{N}$ above the pencil $\Lambda - P_1 - P_2$ of N . Smoothness is a restatement of hypothesis c) in (9.1) and connectedness follows from (8.8) , q.e.d.

(9.4) The projection mapping $f : S \longrightarrow S_0 \subset \hat{C}^{(d)}$ identifies the curves Γ_{12} and Γ_{21} along the natural isomorphism between these (double switch above P_1 and P_2) into a curve $\Gamma \subset S_0$, being bijective elsewhere.

In fact; if two divisors $\hat{D}_1$ and $\hat{D}_2$ of the family S are identified by the map f , their images D_1 and D_2 in Λ have to be identified by the map $N^{(d)} \longrightarrow C^{(d)}$. These two divisors of Λ have to belong to the pencil with base points P_1 and P_2 . By condition d) of (9.1) , D_1 and D_2 then have at least one common point outside $\{ P_1 , P_2 \}$. But, by condition c) of (9.1) , the pencil $\Lambda - P_1 - P_2$ is free from base points (cf (8.13.a.i) , hence $D_1 = D_2$.

Thus $\hat{D}_1$ and $\hat{D}_2$ ly above the same divisor of Λ . Since they have to coincide outside $\{ P_1' , P_2' , P_1'' , P_2'' \}$, it remains only to discuss their behaviour at this set. Using the schematical notation introduced above, the sums of the coefficients along the rows and columns of

$$\begin{pmatrix} P_1'' & P_2'' \\ P_1' & P_2' \end{pmatrix}$$

are the same for $\hat{D}_1$ and for $\hat{D}_2$. As remarked in the proof of (9.3.b) , at least one of the columns yields a sum equal to 1 . Suppose e.g. that this is the first one, and that $\hat{D}_1$ has coefficients

$$\begin{pmatrix} 0 & a \\ 1 & b \end{pmatrix} .$$

Then $\hat{D}_2$ either equals $\hat{D}_1$ or has coefficients

$$\begin{pmatrix} 1 & a-1 \\ 0 & b+1 \end{pmatrix} ,$$

i.e., is gotten from $\hat{D}_1$ by a double switch above P_1 and P_2 . This proves the statement at the beginning of (9.4) .

By Zariski's Main Theorem , f is a finite morphism, hence S is the normalization of S_0 . The curve Γ is smooth, and S can be viewed as the blowing up of S_0 along Γ , the curves Γ_{12} and Γ_{21} being the components of the exceptional divisor. The fact that $\Gamma_{12}\cdot\Gamma_{21} = 0$ in S tells us that Γ is an ordinary double curve of S_0 ; this is the only singularity of this surface.

(9.5) We remark, for later use, that at $D \in \Gamma \subset S_0$ it is $\dim T_{S_0}(D) = 3$, because local coordinates x,y,z can be chosen such that $\hat{\mathcal{O}}_{S_0,D}$ is isomorphic with $k[[x,y,z]]/xy$.

(9.6) Considering again both halves S' and S'' of the surface of divisors of $\hat{N}$ above Λ (see (9.1)) , and denoting by S_0' , S_0'' their respective images in $\hat{C}^{(d)}$, one proves in the same way as in (9.4) that, in $\hat{C}^{(d)}$, $S_0' \cap S_0'' = \emptyset$ holds.

The induction step

(9.7) In this subsection we shall consider the following situation. First, a diagram

$$\begin{array}{ccc} \hat{C} & \xrightarrow{\ \pi\ } & C \\ & \searrow \quad \swarrow_{f} & \\ & T & \end{array}$$

where T is a smooth (non necessarily complete) connected curve and $f : C \longrightarrow T$ is a family of complete curves parametrized by T . We assume $C_t := f^{-1}(t)$ to be smooth if $t \neq 0 \in T$ and C_0 having exactly one ordinary double point. The scheme C is assumed to be smooth and $\pi : \hat{C} \longrightarrow C$ an irreducible étale covering of degree 2 .

Secondly, we are given a T-morphism

$$\sigma : C \longrightarrow \mathbb{P}^2_T = \mathbb{P}^2 \times T$$

such that, for each $t \in T$, if $t \neq 0$ then σ_t defines on C_t a net Λ_t yielding a smooth surface of divisors in $\hat{C}_t$; for $t = 0$ we assume that, if $\hat{N} \longrightarrow N$ are the respective normalizations of $\hat{C}_0$ and C_0 , then the net induced on N by $C_0 \longrightarrow \mathbb{P}^2$ satisfies the conditions a) - d) of (9.1) .

(9.8) REMARK. Take a 1-dimensional family of admissible curves (cf (8.14)) having, for $t \neq 0$, a fixed set $M \subset \mathbb{P}^2$ of nodes and, for $t = 0$, one more singular point. Blowing up the plane along M yields a family C as above, and the projection morphism from the blowing up to $\mathbb{P}^2$ induces a morphism σ satisfying the required properties (by admissibility).

We call d the common degree of the systems Λ_t. Furthermore $S_t' \cup S_t''$ denotes the surface of divisors in $\hat{C}_t$ for $t \neq 0$, and $S' \cup S''$ stands for the surface of divisors in $\hat{N}$; S' and S'' are the halves of this surface and they are the normalizations of their respective images S_0' and S_0'' in $\hat{C}_0^{(d)}$, described in (9.4).

LEMMA. *Up to an étale base change in* T, *we may assume that there exist smooth schemes* Σ', Σ'' *and morphisms* $\Sigma' \longrightarrow T$, $\Sigma'' \longrightarrow T$ *having as fibres* S_t' *and* S_t'' *above* $t \in T$.

PROOF. Consider the d^{th} symmetric power of C (resp. $\hat{C}$) above T, $C^{(d)} = S_T^d(C)$ (resp. $\hat{C}^{(d)} = S_T^d(\hat{C})$) and the mapping $\hat{C}^{(d)} \longrightarrow C^{(d)}$ deduced from π. The map $\sigma : C \longrightarrow \mathbb{P}_T^2$ induces a natural one $\check{\mathbb{P}}_T^2 \longrightarrow C^{(d)}$ in the obvious way. We define a scheme Σ by the pullback diagram

$$\begin{array}{ccc} \Sigma & \longrightarrow & \hat{C}^{(d)} \\ \downarrow & & \downarrow \\ \check{\mathbb{P}}_T^2 & \longrightarrow & C^{(d)} \end{array} .$$

By composition, we get a morphism $\Sigma \longrightarrow T$. It follows from the definition that $\Sigma(t) = S_t' \cup S_t''$ above closed points $t \in T$. We know that $S_t' \cap S_t'' = \emptyset$ in $\hat{C}_t^{(d)}$ if $t \neq 0$ and, by (9.6), the same holds at $t = 0$.

Take the Stein factorization of $\Sigma \longrightarrow T$, yielding by the above an étale covering of degree 2 of T. Pulling back the whole situation to this cover, we may assume that the Stein factorization yields a trivial covering of T. Otherwise said: Σ splits there into two components Σ', Σ'' with the required properties, modulo smoothness of these schemes.

Fix one of them, and call it now Σ (*caution*). Call also S_t the half S_t' or S_t'' now involved, and write S instead of the corresponding S' or S''. Smoothness of Σ at a point $\hat{D}$ is equivalent to $\dim T_\Sigma(\hat{D}) = 3$. For each $t \in T$ we have, if $\hat{D} \in S_t$: $T_{S_t}(\hat{D}) = \mathrm{Ker}\,(T_\Sigma(\hat{D}) \longrightarrow T_T(t))$. Assume $t \neq 0$; since S_t is smooth, we have $\dim T_{S_t}(\hat{D}) = 2$. The curve T

being smooth, we get $\dim T_\Sigma(\hat{D}) \le 2+1=3$ and, since $\dim T_\Sigma(\hat{D}) \ge 3$, equality follows in this case. Obviously the same thing holds for $\hat{D} \in S_0$ not lying on the double locus of S_0.

It remains to consider $T_\Sigma(\hat{D})$ for $\hat{D}$ in the double locus of S_0. By (9.5), $\dim T_{S_0}(\hat{D}) = 3$. The result will follow if we show that the map $T_\Sigma(\hat{D}) \longrightarrow T_T(0)$ is zero at such $\hat{D}$. Denoting by Q_1, Q_2 the singular points of $\hat{C}_0$ (i.e., the points we called P', P" in the preceding subsection), at least one of them appears with coefficient equal to 1 in $\hat{D}$ (cf (9.3), (9.4)). Suppose thus $\hat{D} = Q_1 + \hat{D}_1$ with Q_1 not belonging to the support of $\hat{D}_1$. In view of this decomposition we get a fibre product diagram

$$\begin{array}{ccc} T_{\hat{C}(d)}(\hat{D}) & \longrightarrow & T_{\hat{C}}(Q_1) \\ \downarrow & & \downarrow \scriptstyle df(Q_1) \\ T_{\hat{C}(d-1)}(\hat{D}_1) & \longrightarrow & T_T(0) \end{array} .$$

The map we want to check to be zero is the composite map of this square, restricted to $T_\Sigma(\hat{D}) \subset T_{\hat{C}(d)}(\hat{D})$. By smoothness of $\hat{C}$ at Q_1 we have that $T_{\hat{C}}(Q) = T_{\hat{C}_0}(Q_1)$, hence $df(Q_1)=0$ and the composite map itself is zero, q.e.d.

PROPOSITION. (cf [8], p. 318) *With the choice of notations made in the proof of the preceding lemma, the inequality*

$$q(S_t) \le q(S) + 1$$

holds for $t \neq 0$.

PROOF. This is taken from (loc. cit.), our situation being topologically the same as the one dealt with there. We sketch it briefly. The normal bundles of Γ_{12} and Γ_{21} in S being trivial by (9.3), a topological model of S_t can be gotten by taking tubular neighbourhoods isomorphic with $\Gamma_{ij} \times \Delta$ (Δ = unit disk in $\mathbb{C}$) of the Γ_{ij}, throwing $\Gamma_{ij} \times \mathring{\Delta}$ away from S and pasting $\Gamma_{ij} \times \partial\Delta$ together by means of the isomorphism deduced from $\Gamma_{12} \simeq \Gamma_{21}$. Fix next a parameter value $t \neq 0$ and take a path I from t to 0. The restriction of the family Σ to I is topologically isomorphic with the cylinder of a suitable degeneration map $\rho : S_t \longrightarrow S_0$ constructed with aid of the above presentation of S_t. The cohomology sequence (we take coefficients in $\mathbb{C}$) associated to ρ leads to an exact sequence

$$(..) \longrightarrow H^q(S_0) \longrightarrow H^q(S_t) \longrightarrow H^{q-1}(\Gamma) \longrightarrow H^{q+1}(S_0) \longrightarrow (..)$$

(Γ = double locus of S_0) . Secondly, from the cohomology sequence associated to the normalization map $g : S \longrightarrow S_0$ one gets:

$$(..) \longrightarrow H^q(S_0) \longrightarrow H^q(S) \longrightarrow H^q(\Gamma) \longrightarrow H^{q+1}(S_0) \longrightarrow (..) \quad .$$

Writing out the first one at $q = 1$ yields

$$0 \longrightarrow H^1(S_0) \longrightarrow H^1(S_t) \longrightarrow H^0(\Gamma) \quad ;$$

and the second one at $q = 0,1$ yields

$$0 \longrightarrow H^0(S_0) \xrightarrow{g^*} H^0(S) \longrightarrow H^0(\Gamma) \longrightarrow H^1(S_0) \longrightarrow H^1(S) \quad .$$

Here g^* is clearly an isomorphism. hence the second sequence gives

$$0 \longrightarrow H^0(\Gamma) \longrightarrow H^1(S_0) \longrightarrow H^1(S) \quad .$$

Computing ranks, we derive: $b^1(S_t) \leq b^1(S_0) + b^0(\Gamma) \leq b^1(S) + 2b^0(\Gamma)$. By (9.3.c) the curve Γ is connected, hence $b^0(\Gamma) = 1$. Dividing by 2 we get the desired result, q.e.d.

REMARK. If the weak form of Question (8.15) had an affirmative answer, we ought to have an equality in the formula of the above proposition.

(9.9) CONCLUSION. *In the situation described in* (9.7) *, suppose that we know for both halves of the surface of divisors deduced in the normalization* $\hat{N} \longrightarrow N$ *of the situation at the origin, that their irregularity is bounded by an integer* k *. Then the irregularity of both halves of the surface of divisors at* $t \neq 0$ *is bounded by* $k + 1$.

10. THE INTERSECTION OF THREE QUADRICS

Main statement

(10.1) THEOREM. *Let* C *be a smooth plane septic,* $\hat{C} \longrightarrow C$ *a double cover, and* Λ *the linear system of line sections of* C *. If* S *is a smooth half of the surface of divisors of* $\hat{C}$ *above* Λ *, then the natural map*

$$\psi : \mathrm{Alb}(S) \longrightarrow \mathrm{Pr}(\hat{C}/C)$$

is an isogeny.

(10.2) COROLLARY (compare [27], p. 103). *Let* X *be a smooth complete intersection of three quadrics in* $\mathbb{P}^6$ *. Assume that the Fano surface* F *parametrizing the conics on* X *is smooth. Then the Abel-Jacobi mapping*

$$\mathrm{Alb}(F) \longrightarrow J(X)$$

is an isogeny.

REMARKS. For the way in which (10.2) follows from (10.1) see (8.21) or [27], p. 103 , where the idea comes from. The smoothness assumption for the Fano surface is fulfilled for a general X (ibid) . We dont know whether the Abel-Jacobi map is an isomorphism or not. Furthermore, in view of (8.16) , the statement of (10.1) is equivalent to saying that $q(S) = 14$, since $p_a(C) = 15$ in this case. The remainder of this section is devoted to the proof of this fact.

Theta characteristics on smooth plane septics

We recall ([22]) that a theta characteristic of a smooth curve C is an element ξ of the Picard group $\mathrm{Pic}(C)$ such that $2\xi = [K_C]$ holds, the latter being the canonical class of C . Theta characteristics are represented by half-canonical divisors, i.e. divisors D such that $2D \equiv K_C$. For example, if C is a plane septic and L denotes a linear section of C , then $2L$ is a naturally arising half-canonical divisor of C .

The plane septics are parametrized by a $\mathbb{P}^{35}$. We denote this space by the symbol H_7 and state:

PROPOSITION. *A general* $C \in H_7$ *has, besides the divisors of the linear system* $|2L|$ *, only a finite number of effective half-canonical divisors.*

PROOF. We content ourselves with an ad-hoc verification of this fact. An effective half-canonical divisor on a smooth $C \in H_7$ is an effective divisor D such that there exists a plane curve $\overline{C}$ of degree 4 intersecting on C the divisor $2D$. This establishes a bijective correspondence between half--canonical divisors on C and quartics $\overline{C}$ which meet C everywhere with even multiplicity. We shall say roughly that $\overline{C}$ is *everywhere tangent* to C . The proposition thus claims that a 1-dimensional algebraic family of quartics everywhere tangent to a general $C \in H_7$ must consist of double conics (i.e. conics counted twice) .

Call $H_4 \subset \mathbb{P}^{14}$ the space parametrizing plane quartics. We consider a correspondence $\Gamma \subset H_7^o \times H_4$ between smooth septics and arbitrary quartics, defined by

$$\Gamma = \{(C,\overline{C}) \in H_7^o \times H_4 \mid \overline{C} \text{ is everywhere tangent to } C\} .$$

Take an irreducible component Z of Γ projecting onto H_7^o , and consider the diagram (with obvious maps)

$$\begin{array}{ccc} H_4 & \xleftarrow{\psi} & Z \\ & & \downarrow{\scriptstyle \phi} \\ & & H_7^o \end{array} .$$

If we prove that $\dim Z \geq 36$ implies that $\psi(Z) \subset H_4$ is contained in the subvariety of double conics, we are done.

We stratify H_4 as follows:

S_0 = smooth quartics
S_1 = irreducible quartics, with $p_a = 2$
S_2 = irreducible quartics, with $p_a = 1$
S_3 = irreducible quartics, with $p_a = 0$
S_4 = line + smooth cubic
S_5 = line + irreducible singular cubic
S_6 = smooth conic + smooth conic (distinct)

S_7 = line + line + smooth conic (distinct lines)
S_8 = line + line + line + line (distinct lines)
S_9 = 2 line + smooth conic
S_{10} = 2 line + line + line (distinct lines)
S_{11} = 2 conic (smooth or not)
S_{12} = 3 line + line (distinct lines) .

These sets are all irreducible, and their respective dimensions are 14 , 13 , 12 , 11 , 11 , 10 , 10 , 9 , 8 , 7 , 6 , 5 , 4 .

Assume thus $\dim Z \geq 36$, as said before. We want to show that $\psi(Z) \subset \subset S_{11}$ holds. If this is not the case, the general member of $\psi(Z)$ would belong to an S_i , $i \neq 11$. We shall exclude all these possibilities by counting constants.

Suppose $i=0$. Take $\overline{C} \in \psi(Z) \cap S_0$ and suppose that C' , $C'' \in H_7^o$ are such that $(C',\overline{C})$, $(C'',\overline{C}) \in Z$. Say $\overline{C}\cdot C' = 2D'$, $\overline{C}\cdot C'' = 2D''$ on $\overline{C}$. Then $2(D'-D'') \equiv 0$ on $\overline{C}$, hence $D'' \equiv D' + \tau$ with $\tau \in {}_2J(\overline{C})$. By Riemann-Roch applied to $\overline{C}$ we get $\ell(D'+\tau) = 14+1-3 = 12$. So, $\dim|D'+\tau| = 11$.

We consider the mapping $f : \psi^{-1}(\overline{C}) \longrightarrow \overline{C}^{(14)}$ which assigns to a pair $(C'',\overline{C})$ as above $D'' \in \overline{C}^{(14)}$. The image $f(\psi^{-1}(\overline{C}))$ is contained in a finite union of $\mathbb{P}^{11}$'s . As for the fibres of f , suppose, with the above notations, that $D' = D''$ on $\overline{C}$. Then, if C' is given in $\mathbb{P}^2$ by $\Psi_7' = 0$ and C'' by $\Psi_7'' = 0$, we have that, for a suitable constant c , the form $\Psi_7' - c\Psi_7'' \in H^0\mathcal{O}_{\mathbb{P}^2}(7)$ goes to 0 in $\overline{C}$. From the exact sequence $0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(3) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(7) \longrightarrow \mathcal{O}_{\overline{C}}(7) \longrightarrow 0$ we derive the following one:

$$0 \longrightarrow \mathbb{C}^{10} \longrightarrow H^0\mathcal{O}_{\mathbb{P}^2}(7) \longrightarrow H^0\mathcal{O}_{\overline{C}}(7) \longrightarrow 0 \quad .$$

Hence the dimension of the fibre of f is bounded by 10 . Therefore it is $\dim \psi^{-1}(\overline{C}) \leq 10+11 = 21$. Since $\dim S_0 = 14$, we would get $\dim Z \leq 35$, contradicting our assumptions. We conclude that the general member of $\psi(Z)$ is not contained in S_0 .

Assume now $i = 1$. We shall merely point out the differences with the above case. Here we replace $\overline{C}$ by its normalization $\overline{N}$. The curve $\overline{C}$ being everywhere tangent to C implies in particular that C induces a divisor on $\overline{N}$ which has even multiplicity outside the points of $\overline{N}$ projecting into the singular locus of $\overline{C}$, i.e. a divisor which always can be put in the form $D_{sing} + 2D$, where the coefficients of D_{sing} are 0 or 1 . There are therefore only a finite number of possibilities for D_{sing} . For each such choice, the degree of D is a certain constant $d \leq 14$. If C', $C'' \in H_7^0$ are such that $(C',\overline{C})$, $(C'',\overline{C}) \in Z$, C' giving $D'_{sing} + 2D'$ and C'' giving $D''_{sing} + 2D''$, suppose that $D'_{sing} = D''_{sing}$. Then $2D' \equiv 2D''$ leads again to $D'' \equiv D' + \tau$, with $\tau \in {}_2J(\overline{N})$. Here R.R. yields $\ell(D' + \tau) = d + 1 - - 2 = d - 1 \leq 13$ (since $d > 2$, the divisor $D' + \tau$ cannot be special) . The inverse image $\psi^{-1}(\overline{C})$ breaks up in components depending on the nature of D_{sing} , and any such component is mapped into $\overline{N}^{(d)}$ for suitable d , its image lying inside a finite union of linear systems of dimension at most 12 . The fibres of this mapping are again bounded by 10 , thus $\dim \psi^{-1}(\overline{C}) \leq \leq 22$. Now $\dim S_1 = 13$ implies $\dim Z \leq 35$, contradicting our assumption.

If $i = 2$ the components of $\psi^{-1}(\overline{C})$ are shown to be of dimension at most $10 + 13 = 23$ (the allowed dimension for the linear systems we use in $\overline{N}$ has increased by 1 , since $p_a(\overline{N})$ has decreased by 1) . But $\dim S_2 = = 12$ and again $12 + 23 < 36$.

Suppose $i = 4$. Write $\overline{C} = L + \overline{C}_3$, where L is a line and $\overline{C}_3$ a smooth cubic. Say $(C',\overline{C}) \in Z$; the curve C' induces a divisor D'_L on L and a divisor $D'_{\overline{C}_3}$ on $\overline{C}_3$. These divisors must have even multiplicity at points outside the intersection of L and $\overline{C}_3$. If we write thus $D'_L = = D'_{L,sing} + 2\delta'_L$ and $D'_{\overline{C}_3} = D'_{\overline{C}_3,sing} + 2\delta'_{\overline{C}_3}$ respectively, where the multiplicities of the "singular" parts are 0 or 1 at each point, we reach again a decomposiiton of $\psi^{-1}(\overline{C})$ in components following the nature of the couple of "singular" parts. Take such a component, say X , and suppose the "δ - parts" arising from its members having degrees d_L and $d_{\overline{C}_3}$ respectively. Then we map $f : X \longrightarrow L^{(d_L)} \times \overline{C}_3^{(d_{\overline{C}_3})}$ in the obvious way. It is clear that $d_L \leq 3$ and $d_{\overline{C}_3} \leq 10$. The first projection lies inside a finite union of linear systems of dimension at most 3 and the second one of dimension at most $(10 + 1 - 1) - 1 = 9$. Therefore the image of X lies in a variety of dimension at most $3 + 9 = 12$. The dimension of the fibres of f is bounded again by 10 , because L and $\overline{C}_3$ can have no common component. So, $\dim X \leq 10 + 12 = 22$, and $\dim \psi^{-1}(\overline{C}) \leq 22$. But $\dim S_4 = 11$, hence $\dim Z \leq 33$ would follow, which is impossible.

We list the remaining cases, the proofs going in a similar way:

$i = 3$	$\dim \psi^{-1}(\overline{C}) \leq 24$	$\dim Z \leq 35$

$i = 5$	$\dim \psi^{-1}(\overline{C}) \leq 23$	$\dim Z \leq 33$
$i = 6$	$\dim \psi^{-1}(\overline{C}) \leq 24$	$\dim Z \leq 34$
$i = 7$	$\dim \psi^{-1}(\overline{C}) \leq 23$	$\dim Z \leq 32$
$i = 8$	$\dim \psi^{-1}(\overline{C}) \leq 22$	$\dim Z \leq 30$
$i = 9$	$\dim \psi^{-1}(\overline{C}) \leq 28$	$\dim Z \leq 35$
$i = 10$	$\dim \psi^{-1}(\overline{C}) \leq 27$	$\dim Z \leq 33$
$i = 12$	$\dim \psi^{-1}(\overline{C}) \leq 27$	$\dim Z \leq 31$.

The only remark to be added is the following, about multiple components. For the sake of clearness, we treat the case $i = 12$.

Say $\overline{C} = 3L_1 + L_2$ with $L_1 \neq L_2$, and suppose $(C',\overline{C}) \in Z$. Then C' induces a divisor D_1' on L_1 and a divisor D_2' on L_2 . The condition to which D_1' (resp. D_2') is subjected is to have even multiplicities at points different from $L_1 \cap L_2$. (Remark that this would not be the case for D_1' , if L_1 was counted with even multiplicity). The inverse image $\psi^{-1}(\overline{C})$ splits into components depending on the nature of the "singular" part of these divisors. Take such a component X and map it into $L_1^{(d_1)} \times L_2^{(d_2)}$ as before, where $d_i \leq 3$ depend on X . The image of this mapping has dimension bounded by $3 + 3 = 6$. To get a bound for the fibres we use the sequence $0 \longrightarrow \mathcal{O}_{\mathbb{P}^2}(5) \longrightarrow \mathcal{O}_{\mathbb{P}^2}(7) \longrightarrow \mathcal{O}_{L_1+L_2}(7) \longrightarrow 0$. By the latter, the dimension of the fibres is $\leq \dim H^0\mathcal{O}_{\mathbb{P}^2}(5) = 21$. Therefore $\dim \psi^{-1}(\overline{C}) \leq 21 + 6 = 27$, and $\dim Z \leq 31$ would follow.

This ends the proof of the Proposition.

(10.3) A theta characteristic ξ is called even if $H^0(C,\mathcal{O}_C(\xi))$ has even dimension and odd if the dimension is odd. The above proposition can be translated therefore in the following terms:

(10.4) *For a smooth plane septic, the following holds, if* C *is sufficiently general: Let* ξ *be a theta characteristic on* C , *different from* $[2L]$. *Then* $H^0(C,\mathcal{O}_C(\xi)) = 0$ *if* ξ *is even and* $\dim H^0(C,\mathcal{O}_C(\xi)) = 1$ *if* ξ *is odd.*

The surface of divisors on an étale double cover of a plane septic

(10.5) Consider again the open subvariety H_7^0 of H_7 parametrizing smooth plane septics. We cover this space by a variety X consisting of the points of order 2 in the Jacobians of these curves. The points of X can be naturally identified with the theta characteristics different from [2L] of the curves parametrized by H_7^o . This is done by attaching to $\tau \in {}_2J(C)$ the theta characteristic $\xi = \tau + [2L]$. We shall keep X identified in this way with the variety of theta characteristics (distinct from [2L]) of smooth plane septics. We shall speak therefore of even and odd covers of such septics.

By a result of Mumford ([22], p.184) this implies that X breaks up into two disconnected parts (cf also [27], p. 92) : $X_0 = \{$ even covers $\}$ and $X_1 = \{$ odd covers $\}$.

<u>PROPOSITION</u>. *The varieties* X_0 *and* X_1 *are irreducible, hence they are the irreducible components of* X .

<u>PROOF</u>. Let C be a smooth plane septic. The even covers stem (in the sense of (8.21)) from the smooth complete intersections of three quadrics in $\mathbb{P}^6$ (cf [2], (6.23)) . The odd ones stem from smooth cubic hypersurfaces of $\mathbb{P}^6$ containing a $\mathbb{P}^3 \subset \mathbb{P}^6$ (ibid, (6.27)) . Since smooth intersections of three quadrics in $\mathbb{P}^6$ define an irreducible variety and similarly do the pairs consisting of a $\mathbb{P}^3 \subset \mathbb{P}^6$ and a smooth $X_3^5 \subset \mathbb{P}^6$ containing it, we conclude that the even theta characteristics ξ with $H^0 = 0$ yield an irreducible subset X_0' of X_0 and the odd ones with $\dim H^0 = 1$ an irreducible subset X_1' of X_1 . We claim that $X_0 = \overline{X}_0'$ and $X_1 = \overline{X}_1'$. In fact: otherwise we would get an open subset in X not meeting $X_0' \cup X_1'$, which would project onto a dense subset of H_7^o ; but this contradicts (10.4) , q.e.d.

Take a smooth plane septic C . If the surface of divisors belonging to one cover $\hat{C}$ of C (and the system of linear sections of C) is smooth, then it will be so for all double covers of C . Actually it suffices that one of the halves of the surface of divisors for one cover is smooth: this follows from (8.6) , being d = 7 in our case.

The surfaces of divisors arising from the points of X_0 (resp. X_1) fit into a scheme Σ_0 above X_0 (resp. a scheme Σ_1 above X_1) . From the above we derive the existence of an open subset $U \subset H_7^o$ such that in the pullback diagrams

$$\begin{array}{ccccc} \Sigma_0 & \longrightarrow & X_0 & \longrightarrow & H_7^o \\ \uparrow & & \uparrow & & \uparrow \\ \Sigma_0|U & \xrightarrow[f_0]{} & X_0|U & \longrightarrow & U \end{array} \qquad \begin{array}{ccccc} \Sigma_1 & \longrightarrow & X_1 & \longrightarrow & H_7^o \\ \uparrow & & \uparrow & & \uparrow \\ \Sigma_1|U & \xrightarrow[f_1]{} & X_1|U & \longrightarrow & U \end{array}$$

the fibres of f_0 and f_1 yield exactly all smooth surfaces of divisors occurring for double covers of smooth septics (with respect to the system of linear sections) .

Because of the irreducibility of $X_0|U$ and $X_1|U$ we get finally two irreducible families of smooth surfaces of divisors, complete in the above sense; one for even and one for odd covers. Remark that, a priori at least, the Stein factorization of f_0 (or of f_1) could give a trivial covering of $X_0|U$ (resp. $X_1|U$) . The symmetry due to the odd degree case (cf (8.6)) allows us nevertheless to state the following

(10.6) COROLLARY. *The irregularity of a smooth half of the surface of divisors got in a double cover of a smooth plane septic* C *(with respect to the system of plane sections of* C*) depends at most on the parity of the cover, but not on the particular* C *chosen nor on the double cover of* C .

End of the proof

(10.7) We shall need the notions of (9.7) , (10.3) and (10.5) in a slightly more general setting. In order to keep things short, we just quote the following:

Suppose we are dealing with plane curves C of odd degree $d = 2k+1$ and stable in the sense of Deligne-Mumford . Here again, as in (10.5), we can identify naturally points of order 2 in J(C) with classes $\xi \in \mathrm{Pic}(C)$ such that $2\xi = [\omega_C]$ (here $\omega_C = \mathcal{O}_C(d-3)$ is the dualizing sheaf for C) , i.e. (honest) theta characteristics of C . In particular, defining as in (10.3) the parity of a theta characteristic, we have the notion of parity of double covers of such curves.

It follows from a result of Beauville ([3], Thm.(1.1)) generalizing an analogous result of Mumford for smooth curves ([22], p.184) , that this parity is invariant under deformations, i.e. if $C \longrightarrow T$ is a family of curves as above, with connected base T , and $\hat{C} \longrightarrow C$ is an étale covering of degree 2 , then the parity of the coverings $\hat{C}_t \longrightarrow C_t$ is the same for all $t \in T$.

Consider the following statement:

(σ_k) : There exists an admissible (cf (8.14)) plane septic C with k nodes and , for each parity, a double cover $\hat{C}$ of C of this parity such that the halves of the surface of divisors gotten in the normalization of $\hat{C}$ have irregularity $14-k$.

By (10.6) , if we prove (σ_0) we are done with (10.1) . We shall prove (σ_{10}) and that, for $1 \leq k \leq 10$, statement (σ_k) implies statement (σ_{k-1}) . Remark that by (8.16) the above irregularity is already known to be $\geq 14-k$, so only the reversed inequality is essential in that statement.

LEMMA. *Statement* (σ_{10}) *holds.*

PROOF. We shall show that there exists an admissible plane septic with 10 nodes as singular locus and such that the linear system induced on its normalization by the line sections is non special. Before doing so, we remark that any such curve C satisfies the requirements of (σ_{10}) . In fact, using for instance [3], Section 1 , one sees that C has double covers of both possible parities. Take an arbitrary double cover $\hat{C}$ of C , and consider the respective normalizations $\hat{N}$ and N of these two curves. By the non--speciality of the system Λ of linear sections on N , Theorem (8.19) applies (*caution:* $\hat{C}$ and C of (loc. cit.) are now $\hat{N}$ and N respectively) hence the irregularities of the halves of the surface of divisors in $\hat{N}$ above Λ equal $p_a(N) - 1 = 4$, as was to be shown.

To prove the claim at the beginning of this proof, consider a smooth curve N of genus 5 , of general moduli . The variety V of all base point free , non special g^2_7 on N is clearly irreducible and has dimension 5 . Any $g^2_7 \in V$ yields a birational map of N onto a singular plane septic C , determined upto projective isomorphisms. We see first that a general such C has nodes as only singularities (of course, this is nothing new and can be deduced in other ways, too).

In fact, if C has a point of multiplicity ≥ 3 , then there exist points P, Q, R on N such that $\dim |g^2_7 - P - Q - R| = 1$, hence $g^2_7 = |g^1_4 + P + Q + R|$. By Brill-Noether theory , N has ∞^1 g^1_4's , hence at most ∞^4 such g^2_7 will exist. Secondly, if C has a cusp, then there exists $P \in N$ such that $\dim |g^2_7 - 2P| = 1$, hence $g^2_7 = |g^1_5 + 2P|$. By Brill-Noether again, N has ∞^3 g^1_5's , hence there are at most ∞^4 g^2_7 yielding such a curve C . Finally, if C has a tacnode, then there exist $P, Q \in N$ such that $\dim |g^2_7 - P - Q| = 1$ and $|g^2_7 - 2P - 2Q| \neq \emptyset$. Thus $g^2_7 = |g^0_3 + 2P + 2Q|$ and $g^0_3 + P + Q$ is special. Given $(P,Q) \in N^{(2)}$, there are ∞^2 effective canonical divisors containing $P+Q$, hence ∞^2 choices of g^0_3 as above can be made. Therefore we obtain at most ∞^4 g^2_7 for which C has a tacnode, and a general g_7^2 yields a curve C with exactly 10 nodes as singularities.

To end with, we have to show that a general C satisfies also condition (8.14.b) . If $g_7^2 \in V$ yields a septic C with nodes as only singularities, but for which (8.14.b) fails to be true, then at least one of the following things happens:

i) $\exists\, P,Q,R \in N$ such that $|g_7^2 - 2P - 2Q - 2R| \neq \emptyset$;

ii) $\exists\, P',P'',Q,R \in N$ such that $\dim |g_7^2 - P' - P''| = 1$ and $|g_7^2 - P' - P'' - 2Q - 2R| \neq \emptyset$;

iii) $\exists\, P_i',P_i'' \in N$, $i = 1,2$ and $Q \in N$ such that $\dim |g_7^2 - P_i' - P_i''| = 1$ for $i = 1,2$ and $|g_7^2 - \Sigma P_i' - \Sigma P_i'' - 2Q| \neq \emptyset$;

iv) $\exists\, P_i',P_i'' \in N$, $i = 1,2,3$ such that $\dim |g_7^2 - P_i' - P_i''| = 1$ for $i = 1,2,3$ and $|g_7^2 - \Sigma P_i' - \Sigma P_i''| \neq \emptyset$;

v) $\exists\, P,Q \in N$ such that $|g_7^2 - 3P - 2Q| \neq \emptyset$;

vi) $\exists\, P',P'',Q \in N$ such that $\dim |g_7^2 - P' - P''| = 1$ and either $|g_7^2 - P' - 2P'' - 2Q| \neq \emptyset$ or $|g_7^2 - P' - P'' - 3Q| \neq \emptyset$;

vii) $\exists\, P_i',P_i'' \in N$, $i = 1,2$, such that $\dim |g_7^2 - P_i' - P_i''| = 1$ for $i = 1,2$ and $|g_7^2 - P_1' - P_1'' - P_2' - 2P_2''| \neq \emptyset$;

viii) $\exists\, P \in N$ such that $|g_7^2 - 4P| \neq \emptyset$;

ix) $\exists\, P',P'' \in N$ such that $\dim |g_7^2 - P' - P''| = 1$ and $|g_7^2 - P' - 3P''| \neq \emptyset$.

As above, one finds that the $g_7^2 \in N$ which satisfy to anyone of these conditions define a subvariety of V of dimension at most 4 , thereby finishing the proof of this lemma.

PROPOSITION. *For* $1 \leq k \leq 10$, *statement* (σ_k) *implies statement* (σ_{k-1}) .

PROOF. We assume (σ_k) for $k \leq 10$ fixed. Let $\overline{C}_0$ be an admissible septic with exactly k nodes, say $P_1, P_2, \ldots, P_k$, and $\hat{\overline{C}}_0$ a double cover of a given parity, such that the halves of the surface of divisors in the normalization of $\hat{\overline{C}}_0$ have irregularity $14-k$.

Consider a plane septic with singular points at $P_2, \ldots, P_k$, being smooth elsewhere, and take the pencil that it determines together with $\overline{C}_0$. By the open nature of admissibility, if we drop a finite set of parameter values, we get a family $\overline{C} \longrightarrow T$ of septics giving $\overline{C}_0$ at $t=0$ and, for $t \neq 0$ an admissible septic with exactly $k-1$ nodes, at $P_2, \ldots, P_k$.

Up to an étale base change, we may assume that the above family $\overline{C}$ has a section (one can use e.g. a line in $\mathbb{P}^2$ meeting $\overline{C}_0$ everywhere transversally to get a local multisection of $\overline{C}$; then one takes T' equal to the image of this section and pulls all back via $T' \longrightarrow T$). Consider the Picard scheme $\underline{\mathrm{Pic}}_{\overline{C}/T}$ ([13] , or [19], p. 22) . The group scheme $G = {}_2\underline{\mathrm{Pic}}_{\overline{C}/T}$ is étale above T . The (2:1) covering $\hat{\overline{C}}_0$ of $\overline{C}_0$ provides us with a point in $G(0)$. We may suppose that the component of G through that point maps isomorphically onto T (by restricting T to the image of that component and pulling back to the component itself). Since the property of $\overline{C} \longrightarrow T$ having a section is not lost in this way, we get, by (loc. cit.), p. 23 , that the section $T \longrightarrow {}_2\underline{\mathrm{Pic}}_{\overline{C}/T}$ of which we now dispose provides us with an invertible $\mathcal{O}_{\overline{C}}$ - module L such that $L \otimes L \simeq \mathcal{O}_{\overline{C}}$. Take then $\hat{\overline{C}} = \mathrm{Spec}_{\overline{C}}\,(\mathcal{O}_{\overline{C}} \oplus L)$, the module $\mathcal{O}_{\overline{C}} \oplus L$ being endowed with its naturally deduced $\mathcal{O}_{\overline{C}}$ - algebra structure. The map

$$\overline{\pi} : \hat{\overline{C}} \longrightarrow \overline{C}$$

yields an étale (2:1) covering of $\overline{C}$ inducing at $t=0$ the covering $\hat{\overline{C}}_0 \longrightarrow \overline{C}_0$ we started with. Fix a parameter $t \neq 0$ and look at the double cover $\hat{\overline{C}}_t \longrightarrow \overline{C}_t$. By (10.7) , this has the parity we have chosen at the beginning.

We claim that this situation takes care of statement (σ_{k-1}) for this parity. In fact, $\overline{C}_t$ is an admissible septic with k-1 nodes. As for the condition concerning irregularities, remark that the normalization of the above situation, say $\hat{C}_t \longrightarrow C_t$, is specialization at t of the following construction: Blowing up the plane at $P_2, \ldots, P_k$ provides us with a family $C \longrightarrow T$ and a natural morphism $C \longrightarrow \overline{C}$ over T . Form the pull-back diagram

$$\begin{array}{ccc} \hat{C} & \xrightarrow{\ \pi\ } & C \\ \downarrow & & \downarrow \\ \hat{\overline{C}} & \xrightarrow{\ \overline{\pi}\ } & \overline{C} \end{array} \quad ,$$

and consider the morphism over T :

$$\begin{array}{ccc} \hat{C} & \xrightarrow{\quad \pi \quad} & C \\ & \searrow \quad \swarrow & \\ & T & \end{array} \quad .$$

Herewith we are in the situation described in (9.7) (cf also (9.8) and (9.2)). Hence Conclusion (9.9) applies. The normalization of its specialization at the origin being exactly the normalization of the situation given by $\hat{\hat{C}}_0 \longrightarrow \overline{C}_0$, our initial hypothesis together with (9.9) says that the irregularity of the halves of the surfaces of divisors in $\hat{C}_t$ = (normalization of $\hat{\hat{C}}_t$) is bounded by $(14-k)+1 = 14-(k-1)$, q.e.d.

This ends the proof of Theorem (10.1).

11. EPILOGUE

Let F be the Fano surface of conics in a general smooth complete intersection of threee quadrics in $\mathbb{P}^6$. The computation of the irregularity $q = 14$ settles the harder part of the study of its invariants. One may ask what yield the other ones.

This question can be answered by using similar methods as in Part One. From the numerical study of the surface F, we merely quote some of the outputs:

$K_F{}^2 = 3376$,
Topological Euler characteristic $E(F) = 1760$,
$p_a(F) = 427$,
$p_g(F) = 441$ (using $q(F) = 14$).

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SAMENVATTING

Een Fano drievoud is een gladde variëteit van dimensie 3 met ampele antikanonieke klasse. Deze eis levert in lagere dimensies rationale krommen en rationale (Del Pezzo) oppervlakken. Onder het eindig aantal typen van Fano drievouden zijn sommige rationaal en sommige niet rationaal; deze rationaliteitsvraag is nog niet voor alle Fano drievouden opgelost.

In verband hiermee treedt in het complexe geval de tussenjacobiaan $J(X)$ van een Fano drievoud X op. De tussenjacobiaan is een hoofdgepolariseerde abelse variëteit die met de algebraïsche 1-cykels op X in een analoog verband staat zoals de jacobiaan van een kromme met de divisoren daarop. De algemene theorie van tussenjacobianen is nog wenig ontwikkeld, wat zeker verband houdt met reeds bekende ontmoedigende feiten. Echter, dankzij de resultaten uit [8], [28], [2], [5] o.a., is de hoop gerechtvaardigd, dat voorlopige beperking tot Fano drievouden een harmonisch geheel zou kunnen leveren.

De betrekking tussen 1-cykels op X en de tussenjacobiaan $J(X)$ wordt door het Abel-Jacobi morfisme $\phi: A^2(X) \longrightarrow J(X)$ gegeven. Voor een familie van 1-cykels op X, geparametriseerd door een variëteit F, levert dit $F \longrightarrow J(X)$ en $\mathrm{Alb}(F) \longrightarrow J(X)$, die ook wel Abel-Jacobi afbeeldingen genoemd worden. De grote lijnen van de problematiek zijn hier:

a) het bestuderen van ϕ (vgl. [4], [5]), en

b) het bestuderen van de meetkunde van $J(X)$ met behulp van geschikte families F (vgl. [8]).

Voor krommen is het Abel-Jacobi morphisme een isomorphisme (Stellingen van Abel en Jacobi), en kwestie b) levert i.h.b. de Riemann parametrizatie stelling voor de theta divisor. Hieruit volgt de Torelli stelling, die zegt dat een kromme uit zijn gepolariseerde jacobiaan terug te construeren is. Analoge resultaten zijn voor enkele typen van Fano drievouden reeds bewezen (loc. cit.). De tweede belangrijke toepassing van b) in het geval van drievouden berust op het feit ([8]) dat, als X rationaal is, de (gepolariseerde) tussenjacobiaan van X isomorf is met de (gepolariseerde) jacobiaan van een kromme. Voor sommige typen van Fano drievouden X hoopt men op deze manier uit de meetkundige studie van $J(X)$ de niet-rationaliteit van X te kunnen concluderen.

In dit proefschrift bestuderen we twee typen van Fano drievouden. In het eerste deel beschouwen we de dubbele overdekking X van $\mathbb{P}^3$ met 4^e graads vertakkingsoppervlak. De eenvoudigste krommen in X zijn de lijnen (krommen van graad 1) ; er is een 2-dimensionale familie van lijnen in X, geparametriseerd door een oppervlak F. In het algemeen is F glad. We bewijzen: als F glad is, dan is $\mathrm{Alb}(F) \longrightarrow J(X)$ een isomorfisme. Dit feit dient in kwestie b) een centrale rol te spelen: $J(X)$ kan geïnterpreteerd worden als de Picard variëteit van een oppervlak F dat met X nauw samenhangt, en waarvan de meetkunde ons redelijk goed bekend is. In hoofdstuk 6 imiteren wij de methodes van de theorie van krommen voor F vs. $J(X)$, waarbij F de plaats van de kromme inneemt en $J(X)$ die van zijn jacobiaan. Dit leidt tot een primitieve vorm van parametrizatie voor de theta divisor van $J(X)$. We hopen dat dit tot een natuurlijkere parametrizatie zal leiden; hiervoor kan een betere kennis van krommen met hoge graad in X wellicht nuttig zijn.

Bovendien, uit het isomorfisme $\mathrm{Alb}(F) \simeq J(X)$, leiden we af dat de methodes van [5] ook hier van toepassing zijn, waaruit $A^2(X) \simeq J(X)$ voor variëteiten van dit type volgt.

In het tweede deel beschouwen we het oppervlak F_1 dat de kegelsneden in een gladde volledige doorsnede X_1 van drie kwadrieken in $\mathbb{P}^6$ parametriseert. Voor een voldoend algemene X_1 is F_1 glad. Het is ons niet bekend of de Abelse variëteiten $\mathrm{Alb}(F_1)$ en $J(X_1)$ in dat geval isomorf zijn. De methodes die we in deel twee gebruiken zijn verschillend van de in deel een toegepaste. We hebben een uitspraak nodig die volgt uit het bestuderen van Prym variëteiten van krommen en een étale (2:1) overdekking daarvan. We bewijzen: uit de gladheid van F_1 volgt dat

$$\mathrm{Alb}(F_1) \longrightarrow J(X_1)$$

een isogenie is.

CURRICULUM VITAE

De schrijver van dit proefschrift werd op 20 juli 1947 in Maastricht geboren. In 1952 vestigde zijn familie zich in Barcelona. Daar bezocht hij zwitserse, franse en spaanse scholen, en behaalde hij in 1964 het Bachiller Superior diploma. In de periode 1965/70 studeerde hij wiskunde aan de Facultad de Ciencias de la Universidad de Barcelona en behaalde in 1970 het spaanse doctoraalexamen. In 1975 trad hij in contact met zijn huidige promotor Prof. dr F. Oort, om onder zijn leiding aan een nederlandse universiteit een promotieonderzoek in de algebraïsche meetkunde te verrichten. In juni 1977 legde hij het nederlandse doctoraalexamen wiskunde af, aan het Math. Instituut van de Universiteit van Amsterdam.

In oktober 1977 trad hij bij het Math. Instituut van de R.U. Utrecht als tijdelijk wetenschappelijk medewerker in dienst, waar dit proefschrift tot stand kwam.

STELLINGEN

behorende bij het proefschrift *Abel-Jacobi isogenies for certain types of Fano threefolds* van *G.E. Welters*.

1.

Het (Hodge) (2,2) vermoeden voor variëteiten van dimensie 4 impliceert het (2,2) vermoeden voor alle variëteiten. Algemener: Het (i,i) vermoeden voor alle $i < p$ en het (p,p) vermoeden voor variëteiten van dimensie 2p impliceren samen het (p,p) vermoeden.

2.

Zij X een algemeen hyperoppervlak van $\mathbb{P}^4$, van graad 4. De variëteit van kegelsneden die op X liggen is een glad oppervlak, waarvan de topologische Euler karakteristiek gelijk is aan 172704.

(A. Collino, J.P. Murre, G.E. Welters, On the family of conics lying on a quartic threefold, Rend.Sem.Mat.Univ.Pol. Torino 38 (1980) 151-181)

3.

Een stelling van P.A. Griffiths over hyperoppervlakken van $\mathbb{P}^n$ zegt i.h.b. voor $n = 4$: "Zij $X \subset \mathbb{P}^4$ een voldoende algemeen hyperoppervlak van graad ≥ 5; dan is de Abel-Jacobi afbeelding $A^2(X) \longrightarrow J(X)$ nul".
Dit geldt algemener, o.a. ook voor volledige doorsneden $(d_1, \ldots, d_r)$ in $\mathbb{P}^{r+3}$ met $\Sigma d_i \geq r+4$, en voor cyklische overdekkingen van $\mathbb{P}^3$ van graad k met een diskriminant van graad $k \cdot m$, waarbij $(k-1) \cdot m \geq 4$.

(P.A. Griffiths, On the periods of certain rational integrals I,II, Annals of Math. 90 (1969) 460-541; p.p. 506-508)

4.

De Prym variëteit van een étale (2:1) overdekking van een gladde kromme over een algebraïsch afgesloten lichaam k bestaat - als hoofdgepolariseerde abelse variëteit - ook als $kar(k) = 2$.

(D. Mumford, Prym varieties I, Contributions to Analysis, Academic Press, 1974)

5.

Zij R een noetherse ring, I een ideaal van R, en M (resp. N) een R- (resp. R/I-) moduul van eindig type. Voor alle $k \in \mathbb{Z}$, $k \geq 0$, geldt :

$$\mathrm{Ext}^k_R(M,N) \simeq \varinjlim_n \mathrm{Ext}^k_{R/I^n}(M/I^nM,N).$$

6.

Zij M een C^∞-manifold en X een C^∞-vektorveld op M, zodat de verzameling $N = \{ p \in M \mid X(p) = 0 \} \subset M$ een submanifold van M is. We kiezen voor ieder $p \in N$ een lokaal coördinaten systeem $x_1,\ldots,x_m$ van M zodat N in deze omgeving geschreven wordt als $x_1 = \ldots = x_{\upsilon(p)} = 0$. We schrijven

$$X = \Sigma\, a_i \frac{\partial}{\partial x_i} \quad , \quad \mu(p) = \det \left(\frac{\partial a_i}{\partial x_j}(p)\right)_{1 \le i,j \le \upsilon(p)} .$$

De operator ι_X op de globale differentiaalvormen $A(M)$ van M levert een complex, en de restrictie afbeelding induceert een morfisme

$$f : H_*(A(M),\iota_X) \longrightarrow A(N) .$$

Er geldt: f is een isomorfisme d.e.s.d.a. $\mu(p) \neq 0 \quad \forall p \in N$.

7.

De veel geprezen intuïtie van de klassieke italiaanse algebraïsch-meetkundigen is aan het begrip "generico" te danken.

8.

"Art. 16.1. Het proefschrift dient aan het einde een bondig geschreven curriculum van de auteur(s) te bevatten. 2. Het curriculum dient te bevatten: a. de gebooretdatum; b. de geboorteplaats; c.het tijdvak waarin voorbereidend onderwijs werd genoten en de aard van het onderwijs; (...)."

Het openbaar maken van gegevens die geen verband houden met het proefschrift of zijn totstandkoming zou echter vrijwillig moeten zijn.

(Algemeen Promotiereglement van de Rijksuniversiteit te Utrecht, 1980)

9.

Het is typisch, dat onze maatschappij geen uitgebreid dossier over de machtverhouding ten opzichte van de mammoet concerns ter diskussie weet te brengen.

10.

In de volgende lijst treedt één woord op, dat geen nederlands is:
Arrangement, bouillon, cachet, caissière, conciërge, detachement, douairière, echec, employé, forfaitaire, hausse, malaise, manoeuvre, nuance, quarantaine, quitte, remise, retour, surprise, vliegreiziger.

Utrecht, 13 april 1981