SIMULTANEOUS APPROXIMATIONS IN TRANSCENDENTAL NUMBER THEORY

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ACADEMISCH PROEFSCHRIFT

TER VERKRIJGING VAN DE GRAAD VAN
DOCTOR IN DE WISKUNDE EN NATUURWETENSCHAPPEN
AAN DE UNIVERSITEIT VAN AMSTERDAM
OP GEZAG VAN DE RECTOR MAGNIFICUS
DR. G. DEN BOEF
HOOGLERAAR IN DE FACULTEIT
DER WISKUNDE EN NATUURWETENSCHAPPEN
IN HET OPENBAAR TE VERDEDIGEN
IN DE AULA DER UNIVERSITEIT
(TIJDELIJK IN DE LUTHERSE KERK, INGANG SINGEL 411, HOEK SPUI)
OP WOENSDAG 15 MAART 1978 DES NAMIDDAGS TE 3.00 UUR PRECIES

DOOR

ALEX BIJLSMA

GEBOREN TE ROTTERDAM

1978
MATHEMATISCH CENTRUM, AMSTERDAM

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VOORWOORD

Het onderzoek waarvan dit proefschrift verslag uitbrengt, is verricht aan het Mathematisch Instituut van de Universiteit van Amsterdam. Mijn belangstelling voor simultane approximaties werd gewekt door de voordracht over dit onderwerp die dr. P.L. Cijsouw in 1976 aan het Mathematisch Centrum te Amsterdam hield; deze voordracht maakte deel uit van het door prof. dr. R. Tijdeman geleide seminarium "Applications of the theory of Gel'fond-Baker to elementary number theoretic problems".

Ik dank allen die op enigerlei wijze hebben bijgedragen aan het ontstaan van dit proefschrift. In het bijzonder dank ik dr. H. Jager en dr. P.L. Cijsouw, respectievelijk promotor en coreferent, voor hun onafgebroken belangstelling en waardevolle adviezen. Het Mathematisch Centrum ben ik dankbaar voor zijn bereidheid het proefschrift uit te geven; de afdeling Reproductie voor de technische uitvoering.

CONTENTS

1. INTRODUCTION

1.1. DEFINITION. Let $\mathcal{F}$ be a field containing the rational numbers. An element ξ of $\mathcal{F}$ is called *algebraic* if there is a polynomial $P \neq 0$ with rational coefficients such that $P(\xi) = 0$; ξ is called *transcendental* if ξ is not algebraic.

Transcendental number theory is the branch of number theory that studies the relationship between transcendental and algebraic elements of the field of complex numbers, and of certain other fields such as the one introduced in 6.3 below. Many theorems in transcendental number theory have the following form: a set X of n-tuples of complex numbers and a complex function f, defined on X, are given and it is stated that for all $(a_1,\ldots,a_n)$ that belong to X the set $\{a_1,\ldots,a_n,f(a_1,\ldots,a_n)\}$ contains at least one transcendental number. The two most famous examples of such theorems are:

1.2. THEOREM (Hermite-Lindemann). *Let* $a \neq 0$ *be a complex number. Then the set* $\{a,\exp(a)\}$ *contains at least one transcendental number.*

PROOF. [Waldschmidt 1974], Théorème 3.1.1. The original memoirs are [Hermite 1873] and [Lindemann 1882]. □

1.3. THEOREM (Gel'fond-Schneider). *Let* a *and* b *be complex numbers; suppose that* $a \neq 0$ *and that* b *is not rational. Let* $\ell(a) \neq 0$ *be a value of the logarithm of* a *and put* $a^b := \exp(b\ell(a))$. *Then the set* $\{a,b,a^b\}$ *contains at least one transcendental number.*

PROOF. [Waldschmidt 1974], Théorème 2.1.1. The original memoirs are [Gel'fond 1934] and [Schneider 1934]. □

The theory of approximation measures provides a quantitative generalization to theorems of this type. This is achieved by introducing the concepts of degree and height of an algebraic number; instead of merely stating that some number is transcendental, we are enabled to estimate how well it can be approximated by algebraic numbers of given degree and height.

1.4. DEFINITION. *(a)* Let P be a polynomial in one variable with complex coefficients. Then the *height* of P, denoted h(P), is defined as the maximum of the absolute values of the coefficients of P.

(b) Let F be a field containing the rational numbers; assume $\xi \in F$ is algebraic. The *minimal polynomial* of ξ is defined as the unique irreducible polynomial P with rational integer coefficients and positive leading coefficient that satisfies $P(\xi) = 0$.

(c) Let F be a field containing the rational numbers; assume $\xi \in F$ is algebraic. The *degree* of ξ, denoted $dg(\xi)$, is the degree of the minimal polynomial of ξ.

(d) Let F be a field containing the rational numbers; assume $\xi \in F$ is algebraic. The *height* of ξ, denoted $h(\xi)$, is the height of the minimal polynomial of ξ.

An approximation measure for a transcendental complex number a is a positive function g of two positive integer variables, such that for all d and H the inequality

$$|a - \alpha| > g(d,H)$$

is satisfied for all α belonging to the set of algebraic complex numbers of degree at most d and height at most H (note that for fixed d and H this set is finite). Again, many theorems are known of the following form: a set X of n-tuples of algebraic complex numbers and a complex function f, defined on X, are given and it is stated that for all $(a_1,\dots,a_n)$ that belong to X the number $f(a_1,\dots,a_n)$ possesses a certain approximation measure g. In such results, however, the symmetry between $a_1,\dots,a_n$ and $f(a_1,\dots,a_n)$ is lost; the object of the following chapters is to investigate whether analogous theorems that preserve this symmetry can be derived. Let $a_1,\dots,a_n$ be transcendental complex numbers; a simultaneous approximation measure for the set $\{a_1,\dots,a_n\}$ is a positive function g of two positive integer variables, such that for all d and H the inequality

$$\max(|a_1-\alpha_1|,\dots,|a_n-\alpha_n|) > g(d,H) \tag{1.1}$$

is satisfied for all algebraic complex numbers $\alpha_1,\dots,\alpha_n$ of degree at most d and height at most H. The results we desire are of the following form: a set X of n-tuples of complex numbers and a complex function f, defined on X, are given and it is stated that for all $(a_1,\dots,a_n)$ that belong to X the set $\{a_1,\dots,a_n,f(a_1,\dots,a_n)\}$ possesses a certain simultaneous approximation measure g. In order to simplify the problem, we shall limit ourselves to approximation by algebraic numbers of fixed degree; in other words, only the dependence of g on the second variable will be considered. Moreover, we relax the definition of a simultaneous approximation measure in that we allow that (1.1) is not satisfied for finitely many n-tuples $(\alpha_1,\dots,\alpha_n)$; if necessary, the validity of (1.1) may be extended to all $(\alpha_1,\dots,\alpha_n)$ by multiplying $g(d,H)$ by a sufficiently large factor that depends only on $a_1,\dots,a_n,d$ and not on H.

The generalization of the Hermite-Lindemann theorem to a result of this form has essentially been achieved; for instance, Cijsouw [1975] proved that for every positive number ε, positive integer d and complex number a, there exist only finitely many pairs (α,β) of algebraic complex numbers of degree at most d such that

$$\max(|a-\alpha|,|\exp(a)-\beta|) < \exp(-\log^2 H \log_2^{-1+\varepsilon} H),$$

where $H = \max(2,h(\alpha),h(\beta))$. (The notation $\log_2$ is an abbreviation for loglog.)

The case of the Gel'fond-Schneider theorem presents more difficulties. After some initial results (see [Ricci 1935], [Franklin 1937]) the following statement appeared in [Schneider 1957]: suppose ε is a positive real number, d is a positive integer and a and b are complex numbers such that $a \neq 0$, $a \neq 1$ and b is not rational; suppose that $\ell(a)$ is a value of the logarithm of a. Then there exist only finitely many triples (α,β,γ) of algebraic complex numbers of degree at most d such that

$$\max(|a-\alpha|,|b-\beta|,|a^b-\gamma|) < \exp(-\log^{5+\varepsilon} H),$$

where $H = \max(h(\alpha),h(\beta),h(\gamma))$ and $a^b = \exp(b\ell(a))$. Bundschuh [1973] remarked that in Schneider's proof a condition like "β not rational" is needed and tried to prove a theorem without such a restriction. His assertion is that, in the situation described above, there are only finitely many triples (α,β,γ) with

$$\max(|a-\alpha|,|b-\beta|,|a^b-\gamma|) < \exp(-\log^4 H \log_2^{-2+\varepsilon} H),$$

where $H = \max(2, h(\alpha), h(\beta), h(\gamma))$. However, there is an error at the beginning of the proof of his Satz 2a, so that his result, too, is only valid under some extra assumption.

Earlier Šmelev [1971] had proved that, in the situation described above, only finitely many triples (α,β,γ) with β not rational have the property

$$\max(|a-\alpha|, |b-\beta|, |a^b-\gamma|) < \exp(-\log^4 H \log_2^3 H).$$

Cijsouw and Waldschmidt [1977] recently improved upon the above results by demonstrating the following theorem.

1.5. <u>THEOREM</u>. *Let* ε *be a positive real number,* d *a positive integer. Let* a *and* b *be complex numbers with* $a \neq 0$*; let* $\ell(a) \neq 0$ *be a value of the logarithm of* a *and put* $a^b := \exp(b\ell(a))$*. Then there are only finitely many triples* (α,β,γ)*, with* β *not rational, of algebraic complex numbers of degree at most* d *for which*

$$\max(|a-\alpha|, |b-\beta|, |a^b-\gamma|) < \exp(-\log^3 H \log_2^{1+\varepsilon} H),$$

where $H = \max(2, h(\alpha), h(\beta), h(\gamma))$.

<u>PROOF</u>. [Cijsouw-Waldschmidt 1977], Theorem 2. □

In [Bijlsma 1977], the present author showed that from all these theorems the condition that β not be rational cannot be omitted. More precisely, the following was proved:

1.6. <u>THEOREM</u>. *For any fixed positive integer* κ*, there exist irrational real numbers* a *and* b *with* $0 < a < 1$ *and* $0 < b < 1$ *such that for infinitely many triples* (α,β,γ) *of rational numbers*

$$\max(|a-\alpha|, |b-\beta|, |a^b-\gamma|) < \exp(-\log^\kappa H),$$

where $H = \max(h(\alpha), h(\beta), h(\gamma))$.

<u>PROOF</u>. [Bijlsma 1977], Theorem 1. □

In theorems like 1.5, the occurrence of a condition upon the number β is undesirable and one would naturally want to replace it by a condition upon the given number b. This is certainly possible: for instance, it is quite easy to see that the estimate in 1.5 holds for arbitrary triples (α,β,γ) if one assumes that, for real b, the convergents p_n/q_n of the continued fraction expansion of b satisfy

$$q_{n+1} \ll \exp(\log^3 q_n), \quad n \to \infty.$$

(Note that the real numbers b for which this condition is not fulfilled, are U*-numbers (see [Schneider 1957], III §3) and thus form a set of Lebesgue measure zero.) A sharper result in the same direction is given by the next theorem.

1.7. THEOREM. *Let* ε *be a positive real number,* d *a positive integer. Let* a *and* b *be complex numbers; suppose that* $a \neq 0$ *and that* b *is not rational. Let* $\ell(a) \neq 0$ *be a value of the logarithm of* a *and put* $a^b := \exp(b\ell(a))$. *If* b *is not real, or if* b *is real such that the convergents* p_n/q_n *of the continued fraction expansion of* b *satisfy*

$$(1.2) \qquad q_{n+1} \ll \exp(q_n^3), \quad n \to \infty,$$

there are only finitely many triples (α,β,γ) *of algebraic complex numbers of degree at most* d *with*

$$\max(|a-\alpha|,|b-\beta|,|a^b-\gamma|) < \exp(-\log^3 H \log_2^{1+\varepsilon} H),$$

where $H = \max(2,h(\alpha),h(\beta),h(\gamma))$.

PROOF. [Bijlsma 1977], Theorem 2. □

The purpose of Chapter 2 below is to make this last assertion still more precise: it will be proved that the sufficient condition (1.2) may be sharpened to

$$\limsup_{n\to\infty} \frac{\log q_{n+1}}{q_n^3 \log^{4+\varepsilon} q_n} = 0,$$

while a necessary condition is given by

$$\limsup_{n\to\infty} \frac{\log q_{n+1}}{q_n^3 \log^{4+\varepsilon} q_n} < \infty.$$

In Chapter 3 we consider the question whether these conditions on b may be replaced by analogous assumptions as to the nature of a.

Chapter 4 is based on the following results of Fel'dman [1964] and Šmelev [1970]:

1.8. THEOREM. *Let* ε *be a positive real number,* d *a positive integer. Let* a *and* b *be complex numbers; suppose that* $a \neq 0$ *and that* b *is algebraic but not rational. Let* $\ell(a)$ *be a value of the logarithm of* a*; put* $a^b :=$ $\exp(b\ell(a))$. *Let* P *be an irreducible polynomial of two variables, with rational integer coefficients, of degree at least one in each variable, such that* $P(0,0) \neq 0$ *and* $P(1,1) \neq 0$. *If* $P(a,a^b) = 0$, *there exist only finitely many algebraic complex numbers* α *of degree at most* d *for which*

$$|a - \alpha| < \exp(-\log^2 H \log_2^{-1+\varepsilon} H),$$

where $H = \max(2,h(\alpha))$.

PROOF. [Fel'dman 1964]. □

1.9. THEOREM. *Let* ε *be a positive real number,* d *a positive integer. Let* a *and* b *be complex numbers; suppose that* $a \neq 0$, $a \neq 1$, a *is algebraic and* b *is not rational. Let* $\ell(a)$ *be a value of the logarithm of* a*; put* $a^b :=$ $\exp(b\ell(a))$. *Let* P *be a polynomial of two variables, with rational integer coefficients, of degree at least one in each variable, such that* P *is not divisible by a polynomial containing only the first variable. If* $P(b,a^b) = 0$, *there exist only finitely many algebraic complex numbers* β *of degree at most* d *with*

$$|b - \beta| < \exp(-\log^3 H \log_2^{5+\varepsilon} H),$$

where $H = \max(2,h(\beta))$.

PROOF. [Šmelev 1970], Theorem 1. □

In Chapter 4 below, 1.8 and 1.9 are generalized to obtain simultaneous approximation measures for subsets of $\{a,b,P(a,b,a^b)\}$; we also consider the case that $\ell(a) = 1$ and the case where P does not remain fixed.

Now let $a_1,\ldots,a_n,b_1,\ldots,b_n$ be complex numbers such that none of $a_1,\ldots,a_n$ is zero; let $\ell_1(a_1),\ldots,\ell_n(a_n)$ be non-zero values of the logarithms of $a_1,\ldots,a_n$ respectively and put

$$\underline{a}^{\underline{b}} := \exp(b_1\ell_1(a_1)+\ldots+b_n\ell_n(a_n)).$$

Wallisser [1973] derived a simultaneous approximation measure for $\{b_1,\ldots,b_n\}$ in case $a_1,\ldots,a_n$ and $\underline{a}^{\underline{b}}$ are algebraic; Bundschuh [1975] gave a simultaneous approximation measure for $\{b_1,\ldots,b_n,\underline{a}^{\underline{b}}\}$ in case $a_1,\ldots,a_n$ are algebraic (however, this proof depends on the erroneous statement in [Bundschuh 1973]). Again, Cijsouw and Waldschmidt [1977] improved upon these results. For non-zero b_0, they established simultaneous approximation measures for $\{b_0,b_1,\ldots,b_n,\underline{a}^{\underline{b}}\exp(b_0)\}$ in case of algebraic $a_1,\ldots,a_n$ and for $\{a_1,\ldots,a_n,b_0,b_1,\ldots,b_n,\underline{a}^{\underline{b}}\exp(b_0)\}$. Furthermore, they proved the following theorems.

1.10. <u>THEOREM</u>. *Let* ε *be a positive real number,* d *a positive integer. Let* $a_1,\ldots,a_n,b_1,\ldots,b_n$ *be complex numbers such that* $a_1,\ldots,a_n$ *are algebraic and none of* $a_1,\ldots,a_n$ *is zero; let* $\ell_1(a_1),\ldots,\ell_n(a_n)$ *be non-zero values of the logarithms of* $a_1,\ldots,a_n$ *respectively and put*

$$\underline{a}^{\underline{b}} := \exp(b_1\ell_1(a_1)+\ldots+b_n\ell_n(a_n)).$$

Then there are only finitely many $(n+1)$*-tuples* $(\beta_1,\ldots,\beta_n,\gamma)$ *of algebraic complex numbers of degree at most* d *such that* $1,\beta_1,\ldots,\beta_n$ *are linearly independent over the rationals and*

$$\max(|b_1-\beta_1|,\ldots,|b_n-\beta_n|,|\underline{a}^{\underline{b}}-\gamma|) < \exp(-\log^2 H \log_2^{\varepsilon} H),$$

where $H = \max(2,h(\beta_1),\ldots,h(\beta_n),h(\gamma))$.

<u>PROOF</u>. [Cijsouw-Waldschmidt 1977], Theorem 3 and the remark following Theorem 4. □

1.11. <u>THEOREM</u>. *Let* ε *be a positive real number,* d *a positive integer. Let* $a_1,\ldots,a_n,b_1,\ldots,b_n$ *be complex numbers such that none of* $a_1,\ldots,a_n$ *is zero; let* $\ell_1(a_1),\ldots,\ell_n(a_n)$ *be non-zero values of the logarithms of*

$a_1,\dots,a_n$ *respectively and put* $\underline{a}^{\underline{b}} := \exp(b_1\ell_1(a_1)+\dots+b_n\ell_n(a_n))$. *Then there are only finitely many* $(2n+1)$*-tuples* $(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n,\gamma)$ *of algebraic complex numbers of degree at most* d *such that* $1,\beta_1,\dots,\beta_n$ *are linearly independent over the rationals and*

$$\max(|a_1-\alpha_1|,\dots,|a_n-\alpha_n|,|b_1-\beta_1|,\dots,|b_n-\beta_n|,|\underline{a}^{\underline{b}}-\gamma|) <$$

$$\exp(-\log^{n+2}H \log_2^{1+\varepsilon}H),$$

where $H = \max(2,h(\alpha_1),\dots,h(\alpha_n),h(\beta_1),\dots,h(\beta_n),h(\gamma))$.

PROOF. [Cijsouw-Waldschmidt 1977], Theorem 5. □

Because of the unnatural condition on the numbers $\beta_1,\dots,\beta_n$, 1.10 and 1.11 cannot be called simultaneous approximation measures in the strict sense. Again, it would be preferable if it could be replaced by a condition on the given numbers. This is achieved in Chapter 5: there analogues of 1.10 and 1.11 are proved in which the condition that $1,\beta_1,\dots,\beta_n$ be linearly independent over the rationals is replaced by conditions depending on $a_1,\dots,a_n$ and their logarithms and on $b_1,\dots,b_n$. In the case of 1.10, it turns out to be sufficient to demand that $\ell_1(a_1),\dots,\ell_n(a_n)$ or $1,b_1,\dots,b_n$ be linearly independent over the rationals. In the case of 1.11, it is sufficient if there exists a $C > 1$ such that for rational integer $x_0,\dots,x_n$ that are not all zero we have

$$(1.3)\qquad |x_0+x_1b_1+\dots+x_nb_n| \geq C^{-1}\exp(-X^{(2n+4)/(n^2+n)}\log^{-1-2/(n+1)}X),$$

where $X = \max(2,|x_0|,\dots,|x_n|)$. The case that $b_1,\dots,b_n$ are fixed algebraic numbers is also considered; the results of this chapter enable us to derive many-variable analogues to the theorems in Chapter 4.

Finally, Chapter 6 gives an analogue to 1.7 in which the complex numbers have been replaced by an algebraically closed, complete field with a non-archimedean valuation.

2. A TEST FOR THE SIMULTANEOUS APPROXIMABILITY OF a, b AND a^b

As stated in the Introduction, the purpose of the present chapter is to prove a sharpened version of 1.7. This is achieved in 2.24 below; before that, however, we need to give a number of auxiliary results. First of all we introduce some definitions to simplify the phrasing of what follows.

2.1. DEFINITION. *(a)* By $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ we denote the sets of all positive integers, of all rational integers, and of all rational, real and complex numbers respectively. By $]a,b[$ we denote the open, by $[a,b]$ the closed interval from a to b.

(b) By $\mathbb{A}$ we denote the set of all algebraic complex numbers, by $\mathbb{A}_d$ the set of all algebraic complex numbers of degree at most d.

(c) If F is some field, $F[X_1,\ldots,X_n]$ denotes the ring of all polynomials in n variables with coefficients from F; instead of $F[X_1]$ we write $F[X]$.

2.2. DEFINITION. *(a)* Let F be a field containing $\mathbb{Q}$; assume $\eta \in F$ is algebraic. Then η is called an *algebraic integer* if its minimal polynomial has leading coefficient 1.

(b) Let F be a field containing $\mathbb{Q}$; assume $\eta \in F$ is algebraic. The *denominator* of η, denoted $\mathrm{den}(\eta)$, is the least $m \in \mathbb{N}$ such that $m\eta$ is an algebraic integer.

(c) Let F be a field containing $\mathbb{Q}$; assume $\eta \in F$, $\eta' \in F$ are algebraic. Then η and η' are called *conjugate* if η and η' have the same minimal polynomial.

(d) For $\eta \in \mathbb{A}$, by $\lceil\overline{\eta}\rceil$ we denote the maximum of the absolute values of the conjugates of η in $\mathbb{A}$.

2.3. DEFINITION. *(a)* A field F is called a *normal extension of* $\mathbb{Q}$ if $\mathbb{Q} \subset F$ and if every $\eta \in F$ that is algebraic of degree d has exactly d conjugates in F; it is called a *finite* normal extension of $\mathbb{Q}$ if it is a normal extension of $\mathbb{Q}$ and has finite degree over $\mathbb{Q}$.

(b) If F is a finite normal extension of $\mathbb{Q}$, $\mathrm{Gal}(F/\mathbb{Q})$ denotes the set of automorphisms σ of F that satisfy $\sigma(\xi) = \xi$ for all $\xi \in \mathbb{Q}$.

2.4. LEMMA. *(a) Let F be an algebraically closed field containing $\mathbb{Q}$; as-*

sume $\eta_1,\ldots,\eta_n \in F$ *are algebraic. Then there exists a finite normal extension* F' *of* $\mathbb{Q}$ *such that* $F' \subset F$ *and* F' *contains* $\eta_1,\ldots,\eta_n$.

(b) Let F *be a finite normal extension of* $\mathbb{Q}$. *Then* $\eta,\eta' \in F$ *are conjugate if and only if there exists a* $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$ *with* $\sigma(\eta) = \eta'$.

PROOF. [Van der Waerden 1966], §§ 41, 57. □

2.5. LEMMA. *(a) Let* F *be a field containing* $\mathbb{Q}$*; assume* $\eta \in F$ *is algebraic. Then* $\mathrm{den}(\eta) \leqq h(\eta)$.

(b) For $\eta \in \mathbb{A}$ *we have* $\lceil\eta\rceil \leqq h(\eta) + 1$.

(c) For $\eta_1,\ldots,\eta_n \in \mathbb{A}$ *we have* $\lceil\eta_1+\ldots+\eta_n\rceil \leqq \lceil\eta_1\rceil + \ldots + \lceil\eta_n\rceil$ *and* $\lceil\eta_1\ldots\eta_n\rceil \leqq \lceil\eta_1\rceil\ldots\lceil\eta_n\rceil$.

PROOF. *(a)* [Schneider 1957], Hilfssatz 2.

(b) [Schneider 1957], Hilfssatz 1.

(c) According to 2.4*(a)*, $\mathbb{C}$ contains a finite normal extension F of $\mathbb{Q}$ containing $\eta_1,\ldots,\eta_n$; put $G := \mathrm{Gal}(F/\mathbb{Q})$. According to 2.4*(b)*,

$$\forall\xi \in F\colon \lceil\xi\rceil = \max_{\sigma\in G} |\sigma(\xi)|.$$

Thus, if $\eta := \eta_1 + \ldots + \eta_n$, we have

$$\lceil\eta\rceil = \max_{\sigma\in G} |\sigma(\eta)| = \max_{\sigma\in G} |\sigma(\eta_1)+\ldots+\sigma(\eta_n)| \leqq$$

$$\max_{\sigma\in G} |\sigma(\eta_1)| + \ldots + \max_{\sigma\in G} |\sigma(\eta_n)| = \lceil\eta_1\rceil + \ldots + \lceil\eta_n\rceil$$

and similarly $\lceil\eta_1\ldots\eta_n\rceil \leqq \lceil\eta_1\rceil\ldots\lceil\eta_n\rceil$. □

2.6. LEMMA. *For* $\eta \in \mathbb{A}$ *we have either* $\eta = 0$ *or*

$$|\eta| \geqq \exp(-2dg(\eta)\max(\log\lceil\eta\rceil,\log \mathrm{den}(\eta))).$$

PROOF. [Waldschmidt 1974], (1.2.3). □

2.7. LEMMA (Siegel). *Let* F *be a field such that* $\mathbb{Q} \subset F \subset \mathbb{C}$ *and* F *has degree* d *over* $\mathbb{Q}$. *Suppose* $a_{i,j} \in F$ *for* $i = 1,\ldots,n$ *and* $j = 1,\ldots,m$ *such that*

$$\lceil a_{i,j}\rceil \leqq A,\ i = 1,\ldots,n,\ j = 1,\ldots,m.$$

If $n > dm$, *there exist* $x_1,\dots,x_n \in \mathbb{Z}$, *not all zero, such that*

$$\sum_{i=1}^{n} a_{i,j}x_i = 0,\ j = 1,\dots,m,$$

and

$$\max(|x_1|,\dots,|x_n|) \leq (\sqrt{2}\ nA)^{dm/(n-dm)}.$$

PROOF. [Waldschmidt 1974], Lemme 1.3.1. □

The next lemma gives a connexion between the rate of growth of an entire function and its number of zeros. Results of such a type are fundamental in transcendence proofs.

2.8. LEMMA. *Let* f *be an entire function. Suppose* $S,T \in \mathbb{N}$, $A,R \in \mathbb{R}$ *such that* $A > 2$ *and* $R \geq 2S$. *If* $f^{(t)}(s) = 0$ *for* $s = 0,\dots,S-1$ *and* $t = 0,\dots,T-1$, *it follows that*

$$\max_{|z|\leq R} |f(z)| \leq 2 \max_{|z|\leq AR} |f(z)| \left(\frac{2}{A}\right)^{ST}.$$

PROOF. [Cijsouw 1974], Lemma 7. □

2.9. DEFINITION. A *branch of the logarithm* is a function ℓ, defined on a set $K \subset \mathbb{C}\setminus\{0\}$, such that ℓ is holomorphic on K and $\exp(\ell(z)) = z$ for all $z \in K$.

Now we give two lemmas about vanishing linear forms in the logarithms of algebraic numbers.

2.10. LEMMA. *Suppose* $d \in \mathbb{N}$, K *a compact subset of the complex plane not containing* 0, ℓ_1 *and* ℓ_2 *branches of the logarithm, defined on* K, *such that* ℓ_1 *does not take the value* 0. *Then only finitely many pairs* $(\alpha,\gamma) \in K^2$ *of algebraic numbers of degree at most* d *have the property that a* $\beta \in \mathbb{Q}$ *exists with*

$$\beta\ell_1(\alpha) - \ell_2(\gamma) = 0$$

and

$$h(\beta) \geqq \log H,$$

where $H = \max(h(\alpha), h(\gamma))$.

PROOF. [Bijlsma 1977], Lemma 3. □

2.11. LEMMA. *For every* $d \in \mathbb{N}$, *there exists an effectively computable* $C > 1$, *depending only on* d, *with the following property. Let* K *be a compact subset of the complex plane not containing* 0, ℓ_1 *and* ℓ_2 *branches of the logarithm, defined on* K, *such that* ℓ_1 *does not take the value* 0. *Then only finitely many pairs* $(\alpha,\gamma) \in K^2$ *of algebraic numbers of degree at most* d *have the property that a* $\beta \in \mathbb{Q}$ *exists with*

$$\beta\ell_1(\alpha) - \ell_2(\gamma) = 0$$

and

$$C \log H < B \log B, \tag{2.1}$$

where $H = \max(h(\alpha), h(\gamma))$ *and* $B = h(\beta)$.

PROOF. I. Suppose the assertion of the lemma to be false. Let C be some real number greater than 1; additional restrictions on the choice of C will be formulated at later stages of the proof. Let $K, \ell_1, \ell_2, \alpha, \beta, \gamma$ satisfy the conditions of the lemma. By $c_1, c_2, \ldots$ we shall denote real numbers greater than 1 that depend only on d. Throughout the proof we shall assume that H is sufficiently large in terms of d, K, ℓ_1, ℓ_2 and C, which will lead to a contradiction. It is clear that $|\ell_1(\alpha)|$ is bounded above and below by positive constants depending on K and ℓ_1; thus we may assume

$$\log^{-1/2} H < |\ell_1(\alpha)| < \log^{1/2} H. \tag{2.2}$$

Similarly, $|\ell_2(\gamma)|$ is bounded above by a constant depending on K and ℓ_2; thus we may assume

$$|\ell_2(\gamma)| < \log^{1/2} H. \tag{2.3}$$

Define $L := B - 1$. We introduce the auxiliary function

$$\Phi(z) := \sum_{\lambda_1=0}^{L} \sum_{\lambda_2=0}^{L} p(\lambda_1,\lambda_2)\alpha^{\lambda_1 z}\gamma^{\lambda_2 z}, \quad z \in \mathbb{C},$$

where

$$\alpha^{\lambda_1 z} = \exp(\lambda_1 z \ell_1(\alpha)), \quad \gamma^{\lambda_2 z} = \exp(\lambda_2 z \ell_2(\gamma))$$

and where $p(\lambda_1,\lambda_2)$ are rational integers to be determined later. We have

$$\Phi^{(t)}(z) = \ell_1^t(\alpha) \sum_{\lambda_1=0}^{L} \sum_{\lambda_2=0}^{L} p(\lambda_1,\lambda_2)(\lambda_1+\lambda_2\beta)^t \alpha^{\lambda_1 z}\gamma^{\lambda_2 z},$$

$$z \in \mathbb{C}, \ t \in \mathbb{N} \cup \{0\}.$$

Now put $a := \mathrm{den}(\alpha)$, $b := \mathrm{den}(\beta)$, $c := \mathrm{den}(\gamma)$,

$$S := [\tfrac{1}{2}d^{-1}B^{1/2}\log^{1/2}B \ \log^{-1/2}H],$$

$$T := [\tfrac{1}{2}d^{-1}B^{3/2}\log^{-1/2}B \ \log^{1/2}H]$$

and consider the system of linear equations

$$(ac)^{Ls}b^t\ell_1^{-t}(\alpha)\Phi^{(t)}(s) = 0, \quad s = 0,\dots,S-1, \quad t = 0,\dots,T-1.$$

These are ST equations in the $(L+1)^2$ unknowns $p(\lambda_1,\lambda_2)$; the coefficients are algebraic integers and lie in a field with degree at most d^2 over $\mathbb{Q}$. The absolute values of the conjugates of the coefficients are less than or equal to

$$(ac)^{LS}b^T\max(1,|\lambda_1+\lambda_2\beta|^T)\max(1,\overline{|\alpha|}^{LS})\max(1,\overline{|\gamma|}^{LS}) \leqq$$

$$H^{4LS}B^{2T}L^T c_1^{LS+T} \leqq \exp(c_2 B^{3/2}\log^{1/2}B \ \log^{1/2}H).$$

As $(L+1)^2 > \frac{1}{2}B^2 > d^2ST$, there exists, by 2.7, a non-trivial choice for the $p(\lambda_1,\lambda_2)$, such that

(2.4) $$\Phi^{(t)}(s) = 0, \quad s = 0,\dots,S-1, \quad t = 0,\dots,T-1,$$

while

$$P := \max_{\lambda_1,\lambda_2} |p(\lambda_1,\lambda_2)| \leq$$
$$(c_3L^2\exp(c_2B^{3/2}\log^{1/2}B\ \log^{1/2}H))^{d^2ST/((L+1)^2-d^2ST)} \leq$$
$$\exp(c_4B^{3/2}\log^{1/2}B\ \log^{1/2}H).$$

II. For $k \in \mathbb{N} \cup \{0\}$ we put $T_k := 2^kT$; suppose

$$2^k \leq B^{1/4}\log^{1/4}B\ \log^{-1/4}H.$$

Then, for our special choice of the $p(\lambda_1,\lambda_2)$ we have

(2.5) $$\Phi^{(t)}(s) = 0,\ s = 0,\dots,S-1,\ t = 0,\dots,T_k-1.$$

This is proved by induction; for $k = 0$ the assertion is precisely (2.4). Now suppose that (2.5) holds for some k, while

$$2^{k+1} \leq B^{1/4}\log^{1/4}B\ \log^{-1/4}H.$$

By 2.8 we have

(2.6) $$\max_{|z|\leq 2S} |\Phi(z)| \leq 2 \max_{|z|\leq 2AS} |\Phi(z)| \left(\frac{2}{A}\right)^{ST_k},$$

where $A = \log^{1/2}H$. Here (2.2) and (2.3) show

(2.7) $$\max_{|z|\leq 2AS} |\Phi(z)| \leq (L+1)^2P\exp(4LS\log H) \leq$$
$$\exp(c_5B^{3/2}\log^{1/2}B\ \log^{1/2}H).$$

Furthermore,

$$\left(\frac{2}{A}\right)^{ST_k} \leq \exp(-c_6^{-1}B^2\log_2 H) \leq \exp(-c_6^{-1}B^2\log B),$$

the last estimate by 2.10. Now (2.1) gives

$$c_5B^{3/2}\log^{1/2}B\ \log^{1/2}H < c_5C^{-1/2}B^2\log B;$$

therefore, if we choose C so large that $c_5 C^{-1/2} < c_6^{-1}$, substitution in (2.6) shows

$$\max_{|z| \leqq 2S} |\Phi(z)| \leqq \exp(-c_7^{-1} B^2 \log B).$$

Using the formula

$$\Phi^{(t)}(s) = \frac{t!}{2\pi i} \int_{|z-s|=1} \frac{\Phi(z)}{(z-s)^{t+1}}\, dz,$$

we see that for $s = 0,\ldots,S-1$, $t = 0,\ldots,T_{k+1}-1$,

$$|\Phi^{(t)}(s)| \leqq \exp(t \log t - c_7^{-1} B^2 \log B) \leqq$$

$$\exp(c_8 B^{7/4} \log^{3/4} B \log^{1/4} H - c_7^{-1} B^2 \log B).$$

Now (2.1) gives

$$c_8 B^{7/4} \log^{3/4} B \log^{1/4} H < c_8 C^{-1/4} B^2 \log B;$$

therefore, if we choose C so large that $c_8 C^{-1/4} < c_7^{-1}$, we may conclude

$$(2.8) \qquad |\Phi^{(t)}(s)| \leqq \exp(-c_9^{-1} B^2 \log B).$$

However, $\ell_1^{-t}(\alpha)\Phi^{(t)}(s)$ is algebraic and for $s = 0,\ldots,S-1$, $t = 0,\ldots,T_{k+1}-1$ we have

$$\mathrm{dg}(\ell_1^{-t}(\alpha)\Phi^{(t)}(s)) \leqq d^2,$$

$$\mathrm{den}(\ell_1^{-t}(\alpha)\Phi^{(t)}(s)) \leqq (ac)^{LS_b^{T_{k+1}}} \leqq H^{2LS_B^{T_{k+1}}} \leqq$$

$$\exp(c_{10} B^{7/4} \log^{3/4} B \log^{1/4} H),$$

$$\overline{\left|\ell_1^{-t}(\alpha)\Phi^{(t)}(s)\right|} \leqq (L+1)^2 P H^{2LS_B^{T_{k+1}}} L^{T_{k+1}} c_{11}^{LS+T_{k+1}} \leqq$$

$$\exp(c_{12} B^{7/4} \log^{3/4} B \log^{1/4} H)$$

(here (2.1) is used), so, by 2.6, either $\Phi^{(t)}(s) = 0$ or

$$|\ell_1^{-t}(\alpha)\Phi^{(t)}(s)| \geqq \exp(-c_{13}B^{7/4}\log^{3/4}B\ \log^{1/4}H).$$

In the latter case, (2.2) shows that

$$(2.9) \qquad |\Phi^{(t)}(s)| \geqq \exp(-c_{14}B^{7/4}\log^{3/4}B\ \log^{1/4}H) >$$

$$\exp(-c_{14}C^{-1/4}B^2\log B).$$

Combining (2.8) and (2.9), and choosing C so large that $c_{14}C^{-1/4} < c_9^{-1}$, gives $\Phi^{(t)}(s) = 0$ for $s = 0,\dots,S-1$, $t = 0,\dots,T_{k+1}-1$. This completes the proof of (2.5).

III. Now let k be the *largest* natural number with

$$2^k \leqq B^{1/4}\log^{1/4}B\ \log^{-1/4}H.$$

From (2.5) it follows that

$$\Phi^{(t)}(s) = 0,\ s = 0,\dots,S-1,\ t = 0,\dots,T_k-1.$$

Once more apply 2.8; this gives (2.6) again and (2.7) also remains unchanged, but from the maximality of k we now get

$$\left(\frac{2}{A}\right)^{ST_k} \leqq \exp(-c_{15}^{-1}B^{9/4}\log^{1/4}B\ \log^{-1/4}H\ \log_2 H) \leqq$$

$$\exp(-c_{15}^{-1}B^{9/4}\log^{5/4}B\ \log^{-1/4}H),$$

the last estimate by 2.10. Using (2.1) we find

$$c_5B^{3/2}\log^{1/2}B\ \log^{1/2}H < c_5C^{-3/4}B^{9/4}\log^{5/4}B\ \log^{-1/4}H;$$

if C is chosen so large that $c_5C^{-3/4} < c_{15}^{-1}$, we see

$$\max_{|z|\leqq 2S} |\Phi(z)| \leqq \exp(-c_{16}^{-1}B^{9/4}\log^{5/4}B\ \log^{-1/4}H).$$

Now

$$\Phi^{(t)}(0) = \frac{t!}{2\pi i} \int_{|z|=1} \frac{\Phi(z)}{z^{t+1}}\,dz,$$

whence for $t = 0,1,\ldots,(L+1)^2-1$ we have

$$|\Phi^{(t)}(0)| \leqq \exp(t \log t - c_{16}^{-1}B^{9/4}\log^{5/4}B \log^{-1/4}H) \leqq$$

$$\exp(c_{17}B^2\log B - c_{16}^{-1}B^{9/4}\log^{5/4}B \log^{-1/4}H).$$

By (2.1) we know

$$c_{17}B^2\log B < c_{17}C^{-1/4}B^{9/4}\log^{5/4}B \log^{-1/4}H;$$

if C is chosen so large that $c_{17}C^{-1/4} < c_{16}^{-1}$, we conclude

(2.10) $$|\Phi^{(t)}(0)| \leqq \exp(-c_{18}^{-1}B^{9/4}\log^{5/4}B \log^{-1/4}H),\ t = 0,1,\ldots,(L+1)^2-1.$$

For these values of t we have

$$dg(\ell_1^{-t}(\alpha)\Phi^{(t)}(0)) = 1,$$

$$den(\ell_1^{-t}(\alpha)\Phi^{(t)}(0)) \leqq B^{(L+1)^2} \leqq \exp(c_{19}B^2\log B),$$

$$\overline{|\ell_1^{-t}(\alpha)\Phi^{(t)}(0)|} \leqq (L+1)^2 P B^{(L+1)^2} L^{(L+1)^2} c_{20}^{(L+1)^2} \leqq$$

$$\exp(c_{21}B^2\log B)$$

(once more (2.1) is used), so according to 2.6 either $\Phi^{(t)}(0) = 0$ or

$$|\ell_1^{-t}(\alpha)\Phi^{(t)}(0)| \geqq \exp(-c_{22}B^2\log B).$$

In the latter case, (2.2) shows that

(2.11) $$|\Phi^{(t)}(0)| \geqq \exp(-c_{23}B^2\log B) >$$

$$\exp(-c_{23}C^{-1/4}B^{9/4}\log^{5/4}B \log^{-1/4}H).$$

Combining (2.10) and (2.11), and choosing C so large that $c_{23}C^{-1/4} < c_{18}^{-1}$, gives

$$\Phi^{(t)}(0) = 0, \quad t = 0,\ldots,(L+1)^2-1.$$

IV. For $t = 0,\ldots,(L+1)^2-1$ we now have

$$\sum_{\lambda_1=0}^{L} \sum_{\lambda_2=0}^{L} p(\lambda_1,\lambda_2)(\lambda_1+\lambda_2\beta)^t = 0. \tag{2.12}$$

As the $p(\lambda_1,\lambda_2)$ are not all zero, it follows that the coefficient matrix of the system (2.12), which is of the Vandermonde type, must be singular. From this we deduce the existence of $\lambda_1,\lambda_2,\lambda_1',\lambda_2' \in \{0,\ldots,L\}$ with $\lambda_1 + \lambda_2\beta = \lambda_1' + \lambda_2'\beta$, or

$$\beta = \frac{\lambda_1' - \lambda_1}{\lambda_2 - \lambda_2'}.$$

This gives

$$B = h(\beta) \leq L = B - 1,$$

so we get a contradiction. □

2.12. <u>DEFINITION</u>. For $\varepsilon > 0$, $b \in \mathbb{R}\setminus\mathbb{Q}$ we define

$$\omega(\varepsilon,b) := \limsup_{n\to\infty} \frac{\log q_{n+1}}{q_n^3 \log^{4+\varepsilon} q_n},$$

where p_n/q_n are the convergents of the continued fraction expansion of b.

2.13. <u>LEMMA</u> (Bertrand's postulate). *If* $n \geq 1$, *there is at least one prime* p *such that* $n < p \leq 2n$.

<u>PROOF</u>. [Hardy-Wright 1960], Theorem 418. □

2.14. <u>LEMMA</u>. *If* $b \in \mathbb{R}$, $v \in \mathbb{Z}$, $w \in \mathbb{Z}\setminus\{0\}$ *and*

$$\left|b - \frac{v}{w}\right| < \frac{1}{2w^2},$$

the number v/w *is a convergent of the continued fraction expansion of* b.

PROOF. [Perron 1954], Satz 2.11. □

2.15. LEMMA. *Suppose* $b \in \mathbb{R} \setminus \mathbb{Q}$. *By* $(p_n/q_n)_{n=0}^{\infty}$ *we denote the sequence of convergents of the continued fraction expansion of* b; *here* $p_n \in \mathbb{Z}$, $q_n \in \mathbb{N}$, $(p_n,q_n) = 1$. *Then*

$$|b - \frac{p_n}{q_n}| \geq \frac{1}{q_n(q_n+q_{n+1})}, \quad n \in \mathbb{N} \cup \{0\}.$$

PROOF. [Perron 1954], § 13, (12). □

2.16. LEMMA. *Let* $\varepsilon,\lambda > 0$ *be given. Then there exists an irrational number* b *with* $0 < b < 1$ *and* $\lambda \leq \omega(\varepsilon,b) \leq 5\lambda$.

PROOF. I. By induction we define a sequence $(\beta_n)_{n=1}^{\infty}$ of rational numbers in the interval $]0,1[$. Take $\beta_1 = 1/w_1$, where w_1 is a natural number depending on λ, to be determined later. Now suppose β_n has already been chosen; put $\beta_n = v_n/w_n$, where $v_n,w_n \in \mathbb{N}$ and $(v_n,w_n) = 1$. Consider the partition

$$D = \left(0,\frac{1}{w_{n+1}},\frac{2}{w_{n+1}},\ldots,\frac{w_{n+1}-1}{w_{n+1}},1\right)$$

of the interval $]0,1[$, where w_{n+1} is the smallest prime number greater than $\exp(4\lambda w_n^3 \log^{4+\varepsilon} w_n)$. By 2.13, it is clear that

$$\exp(4\lambda w_n^3 \log^{4+\varepsilon} w_n) < w_{n+1} \leq 2\exp(4\lambda w_n^3 \log^{4+\varepsilon} w_n). \tag{2.13}$$

The width of the partition D is less than $\exp(-4\lambda w_n^3 \log^{4+\varepsilon} w_n)$; thus there exists a $v_{n+1} \in \{1,\ldots,w_{n+1}-1\}$ with

$$|\beta_n - \frac{v_{n+1}}{w_{n+1}}| < \exp(-4\lambda w_n^3 \log^{4+\varepsilon} w_n). \tag{2.14}$$

Now define β_{n+1} as v_{n+1}/w_{n+1}. Note that $(v_{n+1},w_{n+1}) = 1$ because w_{n+1} is prime and $1 \leq v_{n+1} < w_{n+1}$. If w_1 is chosen sufficiently large, (2.13) ensures that the sequence $(w_n)_{n=1}^{\infty}$ is strictly increasing.

II. The sequence $(\beta_n)_{n=1}$ has the property

$$\forall m > n: |\beta_m - \beta_n| < \exp(-2\lambda w_n^3 \log^{4+\varepsilon} w_n). \tag{2.15}$$

To prove this, put $I_k := \{x \in \mathbb{R} : |\beta_k - x| < \exp(-2\lambda w_k^3 \log^{4+\varepsilon} w_k)\}$. Then (2.15) can be written as $\forall m > n$: $\beta_m \in I_n$. By (2.14) $\beta_m \in I_{m-1}$, so it is sufficient to prove $\forall k > n$: $I_k \subset I_{k-1}$. Take $x \in I_k$, which means $|\beta_k - x| < \exp(-2\lambda w_k^3 \log^{4+\varepsilon} w_k)$. Then, by (2.13) and (2.14),

$$|\beta_{k-1} - x| \leqq |\beta_k - x| + |\beta_{k-1} - \beta_k| <$$

$$\exp(-2\lambda w_k^3 \log^{4+\varepsilon} w_k) + \exp(-4\lambda w_{k-1}^3 \log^{4+\varepsilon} w_{k-1}) <$$

$$\exp(-2\lambda \exp(12\lambda w_{k-1}^3)) + \exp(-4\lambda w_{k-1}^3 \log^{4+\varepsilon} w_{k-1}) <$$

$$2 \exp(-4\lambda w_{k-1}^3 \log^{4+\varepsilon} w_{k-1}) < \exp(-2\lambda w_{k-1}^3 \log^{4+\varepsilon} w_{k-1})$$

if w_1 is chosen sufficiently large in relation to λ, so $x \in I_{k-1}$.

III. From (2.15) we see that $(\beta_n)_{n=1}^{\infty}$ is a Cauchy sequence; it converges to a limit, which we shall call b. Then $b \in [0,1]$ and

$$\text{(2.16)} \qquad \forall n: |b - \beta_n| \leqq \exp(-2\lambda w_n^3 \log^{4+\varepsilon} w_n).$$

By 2.14, this implies that β_n is a convergent of the continued fraction expansion of b, say

$$\beta_n = \frac{p_{k_n}}{q_{k_n}}.$$

Thus b is irrational. Now suppose $\omega(\varepsilon,b) < \lambda$. By 2.15 and the definition of $\omega(\varepsilon,b)$, we then have for sufficiently large n,

$$|b - \beta_n| > \frac{1}{q_{k_n}(q_{k_n} + q_{k_n+1})} \geqq \frac{1}{q_{k_n}(q_{k_n} + \exp(\lambda q_{k_n}^3 \log^{4+\varepsilon} q_{k_n})} \geqq$$

$$\exp(-2\lambda q_{k_n}^3 \log^{4+\varepsilon} q_{k_n}) = \exp(-2\lambda w_n^3 \log^{4+\varepsilon} w_n),$$

which contradicts (2.16).

If, on the other hand, $\omega(\varepsilon,b) > 5\lambda$, there are infinitely many convergents p_k/q_k of b with

$$(2.17) \qquad q_{k+1} > \exp(5\lambda q_k^3 \log^{4+\varepsilon} q_k).$$

As all the β_n are convergents of b and $(w_n)_{n=1}^{\infty}$ is increasing, for every k satisfying (2.17) we can find an n_k with

$$w_{n_k} \leq q_k < q_{k+1} \leq w_{n_k+1};$$

thus

$$w_{n_k+1} \geq q_{k+1} > \exp(5\lambda q_k^3 \log^{4+\varepsilon} q_k) \geq \exp(5\lambda w_{n_k}^3 \log^{4+\varepsilon} w_{n_k}),$$

which contradicts (2.13). □

2.17. DEFINITION. Suppose $\varepsilon > 0$, $d \in \mathbb{N}$. Then the set $P_d(\varepsilon)$ is defined as the set of all $b \in \mathbb{C}$ such that there exist an $a \in \mathbb{C} \setminus \{0\}$, a value $\ell(a) \neq 0$ of the logarithm of a, and infinitely many triples $(\alpha,\beta,\gamma) \in \mathbb{A}_d^3$ with

$$\max(|a-\alpha|,|b-\beta|,|a^b-\gamma|) < \exp(-\log^3 H \log_2^{1+\varepsilon} H),$$

where $H = \max(2,h(\alpha),h(\beta),h(\gamma))$ and $a^b = \exp(b\ell(a))$.

2.18. LEMMA. *For every* $d \in \mathbb{N}$, $n \in \mathbb{N}$, *there exists an effectively computable* $C > 1$, *depending only on* d *and* n, *with the following property. Suppose* $\alpha_1,\ldots,\alpha_n \in \mathbb{A}_d \setminus \{0\}$, $\beta_0,\ldots,\beta_n \in \mathbb{A}_d$; *let* $\ell_1(\alpha_1),\ldots,\ell_n(\alpha_n)$ *be values of the logarithms of* $\alpha_1,\ldots,\alpha_n$ *respectively. For* $1 \leq j \leq n$, *let* $A_j \geq 6$ *be an upper bound for* $h(\alpha_j)$ *and* $\exp(|\ell_j(\alpha_j)|)$. *Put*

$$\Lambda := \beta_0 + \beta_1\ell_1(\alpha_1) + \ldots + \beta_n\ell_n(\alpha_n).$$

Then either $\Lambda = 0$ *or*

$$|\Lambda| > \exp(-C\Omega \log \Omega'(\log B + \log \Omega)),$$

where $\Omega = (\log A_1)\ldots(\log A_n)$, $\Omega' = (\log A_1)\ldots(\log A_{n-1})$, $B = \max(6,h(\beta_0),\ldots,h(\beta_n))$.

PROOF. [Baker 1977]; see also [Cijsouw-Waldschmidt 1977], Theorem 1 and the remark following it. □

2.19. LEMMA. *Suppose* $\alpha_1,\dots,\alpha_n \in \mathbb{A}\setminus\{0\}$ *; let* $\ell_1(\alpha_1),\dots,\ell_n(\alpha_n)$ *be values of the logarithms of* $\alpha_1,\dots,\alpha_n$ *respectively. Then* $\ell_1(\alpha_1),\dots,\ell_n(\alpha_n)$ *are linearly independent over* $\mathbb{Q}$ *if and only if they are linearly independent over* $\mathbb{A}$.

PROOF. [Baker 1967], Theorem 1. □

2.20. LEMMA. *For every* $d \in \mathbb{N}$ *, there exists an effectively computable* $C > 1$*, depending only on* d*, such that, if* $\varepsilon > 0$*, no number* $b \in \mathbb{R}\setminus\mathbb{Q}$ *with* $\omega(\varepsilon,b) < C^{-1}$ *belongs to* $P_d(\varepsilon)$.

PROOF. Suppose $b \in \mathbb{R}\setminus\mathbb{Q}$ belongs to $P_d(\varepsilon)$. This gives the existence of an $a \neq 0$, a value $\ell(a) \neq 0$ of the logarithm of a, and infinitely many triples $(\alpha,\beta,\gamma) \in \mathbb{A}_d^3$ with

$$\max(|a-\alpha|,|b-\beta|,|a^b-\gamma|) < \exp(-\log^3 H \log_2^{1+\varepsilon} H), \tag{2.18}$$

where $a^b = \exp(b\ell(a))$ and $H = \max(h(\alpha),h(\beta),h(\gamma))$.

Let (α,β,γ) be such a triple and assume that H is large enough in terms of $d,a,b,\ell(a)$ and ε. Let $\bar\ell$ be a branch of the logarithm, defined on a disk K_1, centred at a, such that $\bar\ell(a) = \ell(a)$. Then

$$|\ell(a) - \bar\ell(\alpha)| < \exp(-\log^3 H \log_2^{1+2\varepsilon/3} H);$$

from $\ell(a) \neq 0$ we thus get $\bar\ell(\alpha) \neq 0$. Let ℓ^* be a branch of the logarithm, defined on a disk K_2, centred at a^b, such that

$$|b\ell(a) - \ell^*(\gamma)| < \exp(-\log^3 H \log_2^{1+2\varepsilon/3} H).$$

Together with (2.18), these formulas show

$$|\beta\bar\ell(\alpha) - \ell^*(\gamma)| < \exp(-\log^3 H \log_2^{1+\varepsilon/3} H).$$

In case $\beta\bar\ell(\alpha) - \ell^*(\gamma) \neq 0$, 2.18 would imply

$$|\beta\bar\ell(\alpha) - \ell^*(\gamma)| > \exp(-\log^3 H \log^{1+\varepsilon/3} H),$$

which is a contradiction. Therefore $\beta\bar\ell(\alpha) - \ell^*(\gamma) = 0$, so $\bar\ell(\alpha)$ and

$\ell^*(\gamma)$ are linearly dependent over $\mathbb{A}$; 2.19 now states that these numbers must also be linearly dependent over $\mathbb{Q}$. This gives the existence of $\xi,\eta \in \mathbb{Q}$, not both zero, such that $\xi\bar{\ell}(\alpha) + \eta\ell^*(\gamma) = 0$. Here $\eta \neq 0$ because $\bar{\ell}(\alpha) \neq 0$, so

$$\beta = \frac{\ell^*(\gamma)}{\bar{\ell}(\alpha)} = -\frac{\xi}{\eta} \in \mathbb{Q}.$$

Applying 2.11 then shows that

$$\log H \geq c^{-1}B \log B,$$

where $B = h(\beta)$ and $c > 1$ depends only on d.

Put $q := \mathrm{den}(\beta)$; then q must tend to infinity with H, so $\log H \geq c^{-1}q \log q \geq q$ for H large enough. This gives

$$(2.19) \qquad |b - \beta| < \exp(-\log^3 H \log_2^{1+\varepsilon} H) < \exp(-c^{-3}q^3 \log^{4+\varepsilon} q);$$

thus, by 2.14, β is a convergent of the continued fraction expansion of b, say $\beta = p_n/q_n$. By 2.15 we have, if $\omega(\varepsilon,b) < C^{-1}$, and n is large enough,

$$(2.20) \qquad |b - \beta| > \frac{1}{q_n(q_n+q_{n+1})} \geq \frac{1}{q_n(q_n+\exp(C^{-1}q_n^3\log^{4+\varepsilon} q_n))} > \exp(-2C^{-1}q^3\log^{4+\varepsilon} q).$$

Comparison of (2.19) and (2.20) gives a contradiction if $2C^{-1} < c^{-3}$. □

2.21. LEMMA. *All rational numbers* ξ *satisfy the following inequality:*

$$\mathrm{den}(\xi) \leq h(\xi) \leq \max(1,|\xi|)\mathrm{den}(\xi).$$

PROOF. If $|\xi| \leq 1$, we have $h(\xi) = \mathrm{den}(\xi)$; otherwise $h(\xi) = |\xi|\mathrm{den}(\xi)$. □

2.22. LEMMA. *Suppose* $b \in \mathbb{R}\setminus\mathbb{Q}$. *By* $(p_n/q_n)_{n=0}^{\infty}$ *we denote the sequence of convergents of the continued fraction expansion of* b; *here* $p_n \in \mathbb{Z}$, $q_n \in \mathbb{N}$, $(p_n,q_n) = 1$. *Then*

$$|b - \frac{p_n}{q_n}| < \frac{1}{q_n q_{n+1}}, \; n \in \mathbb{N} \cup \{0\}.$$

<u>PROOF</u>. [Perron 1954], Satz 2.10. □

2.23. <u>LEMMA</u>. *For every* $M \in \mathbb{N}$, *there exists an effectively computable* $C > 1$, *depending only on* M, *such that, if* $\varepsilon > 0$ *and* $d \in \mathbb{N}$, *every number* $b \in \mathbb{R} \setminus \mathbb{Q}$ *with* $|b| \leqq M$ *and* $\omega(\varepsilon, b) > C$ *belongs to* $\mathcal{P}_d(\varepsilon)$.

<u>PROOF</u>. I. Let $C > 1$ be some number depending only on M, to be determined later; suppose $\varepsilon > 0$, $|b| \leqq M$ and $\omega(\varepsilon, b) > C$. By $c_1, c_2, \ldots$ we shall denote real numbers greater than 1 that depend only on M. By 2.21 and 2.22, we have for infinitely many $n \in \mathbb{N}$,

$$|b - \frac{p_n}{q_n}| < \frac{1}{q_n q_{n+1}} < \frac{1}{\exp(Cq_n^3 \log^{4+\varepsilon} q_n)} \leqq$$
$$\exp(-c_1^{-1} C h^3(p_n/q_n) \log^{4+\varepsilon} h(p_n/q_n)).$$

Consequently, there exists a sequence $(\beta_n)_{n=1}^{\infty}$ of rational numbers with the properties

$$(2.21) \qquad |b - \beta_n| < \exp(-c_1^{-1} C h^3(\beta_n) \log^{4+\varepsilon} h(\beta_n)),$$

$$(2.22) \qquad h(\beta_{n+1}) > \exp(c_1^{-1} C h^3(\beta_n) \log^{4+\varepsilon} h(\beta_n)).$$

Furthermore, it is no restriction to assume that $h(\beta_1)$ is greater than some bound depending only on M.

II. We now use induction to define a sequence $(\alpha_n)_{n=1}^{\infty}$ of rational numbers in the interval $]0,1[$ with the properties

$$(2.23) \qquad |\alpha_n - \alpha_{n+1}| < \exp(-2C^{1/2} \log^3 h(\alpha_n) \log_2^{1+\varepsilon} h(\alpha_n)),$$

$$(2.24) \qquad h(\alpha_{n+1}) > h^2(\alpha_n),$$

$$(2.25) \qquad h(\alpha_n) < (2h(\beta_n))^{2h(\beta_n)},$$

$$(2.26) \qquad \alpha_n^{1/\mathrm{den}(\beta_n)} \in \mathbb{Q}.$$

Choose

$$\alpha_1 := 2^{-\mathrm{den}(\beta_1)}.$$

Now suppose $\alpha_1,\dots,\alpha_n$ have already been chosen and possess the desired properties. By 2.13, there is a prime number u with

$$\mathrm{den}(\beta_{n+1})h(\beta_{n+1}) \leqq u \leqq 2\ \mathrm{den}(\beta_{n+1})h(\beta_{n+1}).$$

We observe

$$\frac{u}{\mathrm{den}(\beta_{n+1})} \geqq h(\beta_{n+1}) \geqq \exp(-c_1^{-1}Ch^3(\beta_n)\log^{4+\varepsilon}h(\beta_n)).$$

If C is chosen large enough in terms of c_1, we may write

$$(2.27)\qquad c_1^{-1}Ch^3(\beta_n)\log^{4+\varepsilon}h(\beta_n) >$$

$$16C^{1/2}h^3(\beta_n)\log^3(2h(\beta_n))(\log(2h(\beta_n)) + \log_2(2h(\beta_n)))^{1+\varepsilon};$$

by (2.25) the right hand member of (2.27) is greater than

$$2C^{1/2}\log^3 h(\alpha_n)\log_2^{1+\varepsilon}h(\alpha_n),$$

so

$$(2.28)\qquad \frac{u}{\mathrm{den}(\beta_{n+1})} > \exp(2C^{1/2}\log^3 h(\alpha_n)\log_2^{1+\varepsilon}h(\alpha_n)).$$

Write $w := \mathrm{den}(\beta_{n+1})$ and consider the partition

$$D = \left(0,\frac{1}{u^w},\frac{2^w}{u^w},\dots,\frac{(u-1)^w}{u^w},1\right)$$

of the interval $]0,1[$. Take $t \in \{0,\dots,u-1\}$. Then

$$\frac{(t+1)^w}{u^w} - \frac{t^w}{u^w} \leqq \frac{w(t+1)^{w-1}}{u^w} \leqq \frac{w}{u};$$

therefore the width of the partition D does not exceed w/u.

By (2.28), the interval

$$\{x \in\ ]0,1[:\ |\alpha_n - x| < \exp(-2C^{1/2}\log^3 h(\alpha_n)\log_2^{1+\varepsilon}h(\alpha_n))\}$$

has a length greater than w/u, so that this interval contains at least one of the points of D. This proves the existence of a $t \in \{1,\dots,u-1\}$ with

$$|\alpha_n - \frac{t^w}{u^w}| < \exp(-2C^{1/2}\log^3 h(\alpha_n)\log_2^{1+\varepsilon} h(\alpha_n)).$$

If one defines $\alpha_{n+1} := (t/u)^w$, (2.23) is satisfied, $\alpha_{n+1} \in]0,1[$ and furthermore $\alpha_{n+1}^{1/w} \in \mathbb{Q}$. Finally,

$$h(\alpha_{n+1}) = u^w \leqq (2\ \mathrm{den}(\beta_{n+1})h(\beta_{n+1}))^{\mathrm{den}(\beta_{n+1})} \leqq$$

$$(2h^2(\beta_{n+1}))^{h(\beta_{n+1})} \leqq (2h(\beta_{n+1}))^{2h(\beta_{n+1})},$$

and, by 2.21,

$$h(\alpha_{n+1}) = u^w \geqq (\mathrm{den}(\beta_{n+1})h(\beta_{n+1}))^{\mathrm{den}(\beta_{n+1})} \geqq$$

$$(c_2^{-1}h^2(\beta_{n+1}))^{c_2^{-1}h(\beta_{n+1})} = \exp(2c_2^{-1}h(\beta_{n+1})\log(c_2^{-1}h(\beta_{n+1}))) \geqq$$

$$\exp(2c_2^{-1}\exp(Ch^3(\beta_n));$$

here (2.22) is used and it is supposed that $h(\beta_1)$ is large enough in terms of c_1 and c_2. As $h^2(\alpha_n) < \exp(4h(\beta_n)\log(2h(\beta_n)))$, this proves $h(\alpha_{n+1}) > h^2(\alpha_n)$. The construction of the sequence $(\alpha_n)_{n=1}^{\infty}$ is now completed.

III. The sequence $(\alpha_n)_{n=1}^{\infty}$ has the property

$$\text{(2.29)} \qquad \forall m > n: |\alpha_m - \alpha_n| < \exp(-C^{1/2}\log^3 h(\alpha_n)\log_2^{1+\varepsilon} h(\alpha_n)).$$

To prove this, put $i_k := C^{1/2}\log^3 h(\alpha_k)\log_2^{1+\varepsilon} h(\alpha_k)$ and $I_k := \{x \in \mathbb{R} : |\alpha_k - x| < \exp(-i_k)\}$. Then (2.29) can be written as $\forall m > n$: $\alpha_m \in I_n$. By (2.23), $\alpha_m \in I_{m-1}$, so it is sufficient to prove $\forall k > n$: $I_k \subset I_{k-1}$. Take $x \in I_k$, which means $|\alpha_k - x| < \exp(-i_k)$. Then, by (2.23) and (2.24),

$$|\alpha_{k-1} - x| \leqq |\alpha_k - x| + |\alpha_{k-1} - \alpha_k| <$$

$$\exp(-i_k) + \exp(-2i_{k-1}) <$$

$$\exp(-6i_{k-1}) + \exp(-2i_{k-1}) < 2\exp(-2i_{k-1}) < \exp(-i_{k-1}),$$

if $h(\beta_1)$ is sufficiently large, so $x \in I_{k-1}$.

IV. From (2.29) we see that $(\alpha_n)_{n=1}^{\infty}$ is a Cauchy sequence; it converges to a limit, which we shall call a. Then

$$\text{(2.30)} \qquad \forall n\colon\ |a - \alpha_n| \leqq \exp(-C^{1/2}\log^3 h(\alpha_n)\log_2^{1+\varepsilon} h(\alpha_n)).$$

If we define

$$\gamma_n := \alpha_n^{\beta_n},$$

we have $\gamma_n \in \mathbb{Q}$ and $\log h(\gamma_n) < c_3 \log h(\alpha_n)$. Furthermore, from (2.26) it follows that $h(\beta_n) < c_4 \log h(\alpha_n)$. From this we conclude that, if C is large enough in terms of c_1, c_3 and c_4, (2.21) and (2.30) give

$$\max(|a-\alpha_n|, |b-\beta_n|) \leqq \exp(-C^{1/4}\log^3 H_n \log_2^{1+\varepsilon} H_n),$$

where $H_n = \max(2, h(\alpha_n), h(\beta_n), h(\gamma_n))$. As the function $(x,y) \mapsto x^y$ is continuously differentiable on every compact subset of $]0,1[\ \times\ ]-M-1, M+1[$, there is a $\lambda > 1$, depending only on a and b, such that

$$\forall n\colon\ |a^b - \gamma_n| < \lambda \max(|a-\alpha_n|, |b-\beta_n|).$$

Consequently, if n is large enough in terms of C and λ, we have

$$\max(|a-\alpha_n|, |b-\beta_n|, |a^b-\gamma_n|) < \exp(-\log^3 H_n \log_2^{1+\varepsilon} H_n),$$

which proves the lemma. □

2.24. <u>THEOREM</u>. *Suppose* $\varepsilon > 0$, $d \in \mathbb{N}$, $b \in \mathbb{C}$.

(i) If $b \notin \mathbb{R}$, *then* $b \notin P_d(\varepsilon)$.

(ii) If $b \in \mathbb{R} \setminus \mathbb{Q}$ *and* $\omega(\varepsilon, b) = 0$, *then* $b \notin P_d(\varepsilon)$.

(iii) If $b \in \mathbb{R} \setminus \mathbb{Q}$ *and* $\omega(\varepsilon, b) = \infty$, *then* $b \in P_d(\varepsilon)$.

(iv) If $b \in \mathbb{Q}$, *then* $b \in P_d(\varepsilon)$.

(v) In case $b \in \mathbb{R} \setminus \mathbb{Q}$ *with* $0 < \omega(\varepsilon, b) < \infty$, *no general assertion is possible.*

PROOF. In case $b \in \mathbb{Q}$, taking $z \in \mathbb{C}$ such that

$$|z - \zeta| < \exp(-\log^4 h(\zeta))$$

for infinitely many $\zeta \in \mathbb{A}_d$ (which is possible by 1.6) and writing

$$a = z^{\mathrm{den}(b)}; \quad (\alpha,\beta,\gamma) = (\zeta^{\mathrm{den}(b)}, b, \zeta^{b\,\mathrm{den}(b)})$$

shows that $b \in P_d(\varepsilon)$; this proves *(iv)*.

The assertion *(i)* is a trivial consequence of 1.5. We may therefore assume that $b \in \mathbb{R} \setminus \mathbb{Q}$, in which case $\omega(\varepsilon,b)$ is defined. According to 2.20, there exists an effectively computable $c_0 > 1$ such that if $\omega(\varepsilon,b) < c_0^{-1}$, then $b \notin P_d(\varepsilon)$; this proves *(ii)*. By 2.16, it is also clear that there exists a $b \in \mathbb{R} \setminus \mathbb{Q}$ with $0 < \omega(\varepsilon,b) < c_0^{-1}$ and so there is a b with $0 < \omega(\varepsilon,b) < \infty$ that does not belong to $P_d(\varepsilon)$.

Furthermore, 2.23 asserts that for every $M \in \mathbb{N}$, there exists an effectively computable $c_M > 1$ such that, if $|b| \leq M$ and $\omega(\varepsilon,b) > c_M$, then $b \in P_d(\varepsilon)$. In case $\omega(\varepsilon,b) = \infty$ we may choose $M = |b|$ and *(iii)* is proved. Again by 2.16, it is clear that for any $M \in \mathbb{N}$ there exists a b with $|b| \leq M$ and $c_M + 1 < \omega(\varepsilon,b) < \infty$; so there is a b with $0 < \omega(\varepsilon,b) < \infty$ that does belong to $P_d(\varepsilon)$. □

3. THE NON-INTERCHANGEABILITY OF a AND b

Let $\varepsilon > 0$, $d \in \mathbb{N}$ be given. In Chapter 2 our aim was to establish a criterion to decide for which $b \in \mathbb{C}$ there exist an $a \in \mathbb{C}\setminus\{0\}$ and a value $\ell(a) \neq 0$ of the logarithm of a such that for infinitely many triples $(\alpha,\beta,\gamma) \in \mathbb{A}_d^3$ the inequality

$$(3.1) \qquad \max(|a-\alpha|,|b-\beta|,|a^b-\gamma|) < \exp(-\log^3 H \log_2^{1+\varepsilon} H)$$

is satisfied (here H denotes the maximum of 2 and the heights of α, β and γ, and $a^b = \exp(b\ell(a))$).

Let us now consider the converse problem, namely to investigate for which $a \in \mathbb{C}\setminus\{0\}$ there exist a $b \in \mathbb{C}$ and a value $\ell(a) \neq 0$ of the logarithm of a, such that for infinitely many triples $(\alpha,\beta,\gamma) \in \mathbb{A}_d^3$ inequality (3.1) is satisfied.

3.1. <u>LEMMA</u>. *If* $\eta \in \mathbb{A}$ *and* $k \in \mathbb{N}$, *the following assertions hold:*

$$(3.2) \qquad \overline{|\eta^k|} = \overline{|\eta|}^k,$$

$$(3.3) \qquad \mathrm{den}(\eta) \leq \mathrm{den}(\eta^k) \leq \mathrm{den}^k(\eta).$$

<u>PROOF</u>. I. According to 2.4*(a)*, $\mathbb{C}$ contains a finite normal extension F of $\mathbb{Q}$ that contains η; put $G := \mathrm{Gal}(F/\mathbb{Q})$. According to 2.4*(b)*,

$$\forall \xi \in F: \overline{|\xi|} = \max_{\sigma \in G} |\sigma(\xi)|.$$

Therefore

$$\overline{|\eta^k|} = \max_{\sigma\in G} |\sigma(\eta^k)| = \left(\max_{\sigma\in G} |\sigma(\eta)|\right)^k = \overline{|\eta|}^k.$$

II. If $m := \mathrm{den}(\eta^k)$, the number $m\eta^k$ is an algebraic integer. The same must then be true for $m^{k-1}m\eta^k = (m\eta)^k$; therefore there is a monic polynomial $P \in \mathbb{Z}[X]$ such that $P((m\eta)^k) = 0$. This shows that $m\eta$ is an algebraic integer.

III. The last inequality in (3.3) is trivial. □

3.2. <u>LEMMA</u>. *Suppose* $\eta \in \mathbb{A}$. *Then*

$$h(\eta) \leqq (2\ \mathrm{den}(\eta)\max(1, \overline{|\eta|}))^{dg(\eta)}.$$

<u>PROOF</u>. [Cijsouw-Tijdeman 1973], Lemma 1. □

3.3. <u>LEMMA</u>. *Suppose* $d,j,k \in \mathbb{N}$ *with* $j \leqq k$, $\eta \in \mathbb{A}_d$. *Then*

$$h(\eta^j) \leqq 2^{2d}h^{2d}(\eta^k).$$

<u>PROOF</u>. It is clear that $\mathrm{den}(\eta^j) \leqq \mathrm{den}(\eta^k)$. Indeed, if $m := \mathrm{den}(\eta^k)$, the number $m\eta^k$ is an algebraic integer. The same must then be true for $m^{k-j}(m\eta^k)^j = (m\eta^j)^k$; therefore there is a monic polynomial $P \in \mathbb{Z}[X]$ such that $P((m\eta^j)^k) = 0$. This shows that $m\eta^j$ is an algebraic integer.

Applying 3.1 and 3.2, we find

$$h(\eta^j) \leqq (2\ \mathrm{den}(\eta^j)\max(1, \overline{|\eta^j|}))^{dg(\eta^j)} \leqq$$

$$(2\ \mathrm{den}(\eta^k)\max(1, \overline{|\eta^k|}))^d \leqq (2h(\eta^k)(h(\eta^k)+1))^d \leqq$$

$$2^{2d}h^{2d}(\eta^k). \quad \square$$

3.4. <u>LEMMA</u>. *For each* $a \in \mathbb{A}\setminus\{0\}$, *there exists an effectively computable* $C > 1$, *depending only on* a, *with the following property: if* $\beta \in \mathbb{Q}$ *and* $\gamma \in \mathbb{A}$, *while* $\ell_1(a)$, $\ell_2(\gamma)$ *are values of the logarithms of* a *and* γ *respectively with*

$$(3.4) \qquad \beta\ell_1(a) - \ell_2(\gamma) = 0,$$

it follows that

$$\log h(\gamma) < C\ dg(\gamma)h(\beta).$$

<u>PROOF</u>. By $c_1, c_2, \ldots$ we shall denote real numbers greater than 1 depending only on a. Let $\beta, \gamma, \ell_1(a)$ and $\ell_2(\gamma)$ be such that (3.4) is satisfied; put $\beta = v/w$, where $v \in \mathbb{Z}$, $w \in \mathbb{N}$, $(v,w) = 1$. If $P = \Sigma_{j=0}^d a_j X^j$ is the minimal polynomial of γ, the minimal polynomial of $1/\gamma$ is either $P^* := \Sigma_{j=0}^d a_{d-j}X^j$ or $-P^*$. Thus $h(\gamma) = h(1/\gamma)$; replacing γ if necessary by $1/\gamma$

and observing that $-\ell_2(\gamma)$ is a value of the logarithm of $1/\gamma$ will ensure that $v \geq 0$. We have

$$\exp(w\ell_2(\gamma)) = \exp^w(\ell_2(\gamma)) = \gamma^w$$

and

$$\exp(w\ell_2(\gamma)) = \exp(w\beta\ell_1(a)) = \exp(v\ell_1(a)) = \exp^v(\ell_1(a)) = a^v;$$

applying 3.1 gives

$$\overline{|\gamma|} = \overline{|a|}^{v/w}$$

and

$$\operatorname{den}(\gamma) \leq \operatorname{den}^v(a).$$

According to 3.2,

$$h(\gamma) \leq (2\operatorname{den}(\gamma)\max(1,\overline{|\gamma|}))^d \leq (2\operatorname{den}^v(a)\max(1,\overline{|a|}^{v/w}))^d \leq$$

$$(c_1^v c_2^{v/w})^{c_3 d} \leq c_4^{dv} \leq c_4^{dh(\beta)}. \quad \square$$

3.5. <u>LEMMA</u>. *Suppose* $d \in \mathbb{N}$, $a \in \mathbb{A}\setminus\{0\}$, $\ell_1(a) \neq 0$ *a value of the logarithm of* a. *Suppose* K *is a compact subset of the complex plane, not containing* 0, ℓ_2 *a branch of the logarithm defined on* K. *Then only finitely many rational numbers* β *have the property that an algebraic number* $\gamma \in K$ *of degree at most* d *exists with*

$$(3.5) \qquad \beta\ell_1(a) - \ell_2(\gamma) = 0.$$

<u>PROOF</u>. Suppose the assertion of the lemma to be false. Let β be a rational number and $\gamma \in K$ algebraic with $dg(\gamma) \leq d$ such that (3.5) holds. By c_1, $c_2, \ldots$ we shall denote real numbers greater than 1 depending only on d, a, $K, \ell_1(a)$ and ℓ_2; we assume that $h(\beta)$ is greater than such a number, which will lead to a contradiction. It is no restriction to assume $dg(a) \leq d$.

By 2.11, we see that either

$$(3.6)\qquad \log \max(h(a),h(\gamma)) \geqq c_1^{-1} h(\beta) \log h(\beta)$$

or

$$(3.7)\qquad \max(h(a),h(\gamma)) \leqq c_2.$$

As there are only finitely many rationals of the form $\ell_2(\gamma)/\ell_1(a)$, where $dg(\gamma) \leq d$ and $h(\gamma) \leqq c_2$, we may henceforth assume that (3.6) holds and even that

$$(3.8)\qquad \log h(\gamma) \geqq c_1^{-1} h(\beta) \log h(\beta).$$

On the other hand, by 3.4 we see that

$$(3.9)\qquad \log h(\gamma) < c_3 h(\beta).$$

Comparing (3.8) and (3.9) gives a contradiction if $h(\beta)$ is sufficiently large. □

Now we are able to solve the problem stated in the introduction to this chapter. If there are infinitely many $\alpha \in \mathbb{A}_d$ with

$$(3.10)\qquad |a - \alpha| < \exp(-\log^3 A \log_2^{1+\varepsilon} A),$$

where $A = h(\alpha)$, taking $b = \beta = 1$, $\gamma = \alpha$ shows that (3.1) holds for infinitely many triples (α,β,γ). Suppose, on the other hand, that infinitely many such triples satisfy (3.1), yet for only finitely many $\alpha \in \mathbb{A}_d$ inequality (3.10) holds. If H is taken large enough, (3.1) then implies that $\alpha = a$. Let K be a closed disk in the complex plane, centred at a^b, that does not contain 0. It is clear that there exists a branch ℓ^* of the logarithm, defined on K, such that $\ell^*(a^b) = b\ell(a)$. From (3.1) we deduce that $\gamma \in K$ and that the inequality

$$|\beta\ell(a) - \ell^*(\gamma)| < \exp(-\log^3 H \log_2^{1+\varepsilon/2} H)$$

holds. If it were the case that $\beta\ell(a) - \ell^*(\gamma) \neq 0$, 2.18 would imply

$$|\beta\ell(a) - \ell^*(\gamma)| > \exp(-\log^2 H \log_2^{1+\varepsilon/2} H),$$

which gives a contradiction. Therefore

$$(3.11) \qquad \beta\ell(a) - \ell^*(\gamma) = 0.$$

We have now proved that $\ell(a)$ and $\ell^*(\gamma)$ are linearly dependent over the field of all algebraic numbers; using 2.19, we find that these numbers must also be linearly dependent over $\mathbb{Q}$. In other words, there are $\xi,\eta \in \mathbb{Q}$, not both zero, such that $\xi\ell(a) + \eta\ell^*(\gamma) = 0$. Here $\eta \neq 0$ because $\ell(a) \neq 0$, so

$$\beta = \frac{\ell^*(\gamma)}{\ell(a)} = -\frac{\xi}{\eta} \in \mathbb{Q}.$$

Using 3.5, we see that in the triples (α,β,γ) satisfying (3.1) occur only finitely many values of β. From (3.11) we see that the number of values of γ must also be finite and we arrive at a contradiction. Thus we have proved

3.6. <u>THEOREM</u>. *Suppose* $\varepsilon > 0$, $d \in \mathbb{N}$, $a \in \mathbb{C}\setminus\{0\}$. *Then the following two assertions are equivalent:*

(i) There exist a $b \in \mathbb{C}$ *and a value* $\ell(a) \neq 0$ *of the logarithm of* a, *such that for infinitely many triples* (α,β,γ) *of algebraic numbers of degree at most* d

$$\max(|a-\alpha|,|b-\beta|,|a^b-\gamma|) < \exp(-\log^3 H \log_2^{1+\varepsilon} H),$$

where $H = \max(2,h(\alpha),h(\beta),h(\gamma))$ *and* $a^b := \exp(b\ell(a))$.

(ii) There exist infinitely many $\alpha \in \mathbb{A}_d$ *such that*

$$|a - \alpha| < \exp(-\log^3 A \log_2^{1+\varepsilon} A),$$

where $A = \max(2,h(\alpha))$. □

A more interesting problem occurs if we demand that the triples (α,β,γ) contain infinitely many values of β, or, in other words, that $b \notin \mathbb{Q}$.

3.7. <u>DEFINITION</u>. Suppose $\varepsilon > 0$, $d \in \mathbb{N}$. Then the set $P_d^*(\varepsilon)$ is defined as the set of all $a \in \mathbb{C}\setminus\{0\}$ such that there exist a $b \in \mathbb{C}\setminus\mathbb{Q}$, a value

$\ell(a) \neq 0$ of the logarithm of a, and infinitely many triples $(\alpha,\beta,\gamma) \in \mathbb{A}_d^3$ with

$$\max(|a-\alpha|,|b-\beta|,|a^b-\gamma|) < \exp(-\log^3 H \log_2^{1+\varepsilon} H),$$

where $H = \max(2,h(\alpha),h(\beta),h(\gamma))$ and $a^b = \exp(b\ell(a))$.

In this case the condition (3.10) turns out to be no longer sufficient and there appears an extra condition on the nature of the numbers α occurring in (3.10).

3.8. THEOREM. *For every fixed* $d \in \mathbb{N}$, *there exists an effectively computable* $C > 1$ *with the following property. If* $\varepsilon > 0$, $a \in \mathbb{C}\setminus\{0\}$ *and* ℓ_1 *is a branch of the logarithm, defined on a neighbourhood of* a, *such that for infinitely many* $k \in \mathbb{N}$ *an* $\alpha \in \mathbb{A}$, *depending on* k, *exists with*

$$(3.12) \qquad \exp(k^{-1}\ell_1(\alpha)) \in \mathbb{A}_d\setminus\{1\},$$

$$(3.13) \qquad |a - \alpha| < \exp(-C \log^3 A \log_2^{1+\varepsilon} A),$$

where $A = \max(2,h(\alpha))$, *the number* a *necessarily belongs to* $P_d^*(\varepsilon)$.

PROOF. I. Let $C > 1$ be given and suppose $\varepsilon > 0$, $a \in \mathbb{C}\setminus\{0\}$; let $(k_n)_{n=1}^{\infty}$ and $(\alpha_n)_{n=1}^{\infty}$ be sequences from $\mathbb{N}$ and $\mathbb{A}$, respectively, such that

$$\lim_{n\to\infty} k_n = \infty,$$

$$(3.14) \qquad \forall n: \exp(k_n^{-1}\ell_1(\alpha_n)) \in \mathbb{A}_d\setminus\{1\},$$

$$(3.15) \qquad \forall n: |a - \alpha_n| < \exp(-C \log^3 A_n \log_2^{1+\varepsilon} A_n),$$

where $A_n = \max(2,h(\alpha_n))$. Put $\theta_n := \exp(k_n^{-1}\ell_1(\alpha_n))$; then $\theta_n \in \mathbb{A}_d\setminus\{1\}$ and

$$\alpha_n = \theta_n^{k_n}.$$

Furthermore, 3.3 states

$$h(\theta_n) \leq 2^{2d}h^{2d}(\alpha_n) \leq 2^{2d}A_n^{2d}.$$

From this it follows that

$$\lim_{n\to\infty} A_n = \infty;$$

indeed, the opposite would imply that there is an $\alpha \in \mathbb{A}$ such that for infinitely many pairs (k,m) of natural numbers with $k \neq m$

$$\exp(k^{-1}\ell_1(\alpha)) = \exp(m^{-1}\ell_1(\alpha)) \neq 1, \tag{3.16}$$

so

$$\left(\frac{1}{k} - \frac{1}{m}\right)\ell_1(\alpha) \equiv 0 \pmod{2\pi i}. \tag{3.17}$$

By choosing $\min(k,m)$ in such a way that the left hand member of (3.17) has absolute value less than 2π, one proves that $\ell_1(\alpha) = 0$ and thus $\exp(k^{-1}\ell_1(\alpha)) = 1$, which contradicts (3.16).

By taking partial sequences if necessary, we can ensure that

$$\forall n\colon\ k_{n+1} > \exp(C \log^3 A_n \log_2^{1+\varepsilon} A_n), \tag{3.18}$$

$$\forall n\colon\ A_{n+1} > A_n^2. \tag{3.19}$$

Furthermore, it is no restriction to assume that A_1 is sufficiently large in relation to C.

II. Put $\beta_1 := k_1^{-1}$; the sequence $(\beta_n)_{n=1}^{\infty}$ of rational numbers from the interval $]0,1[$ will be defined inductively. Suppose $\beta_1,\ldots,\beta_n$ have been chosen; by (3.18) we see that there is a $j_{n+1} \in \{1,\ldots,k_{n+1}-1\}$ with

$$0 < \left|\beta_n - \frac{j_{n+1}}{k_{n+1}}\right| < \exp(-C \log^3 A_n \log_2^{1+\varepsilon} A_n). \tag{3.20}$$

Define $\beta_{n+1} := j_{n+1}/k_{n+1}$. It follows from (3.20) that $(\beta_n)_{n=1}^{\infty}$ is a Cauchy sequence, and that its limit b satisfies

$$\forall n\colon\ |b - \beta_n| \leq \exp(-\tfrac{1}{2}C \log^3 A_n \log_2^{1+\varepsilon} A_n). \tag{3.21}$$

To prove this, it will be sufficient to show

$$(3.22)\qquad \forall m > n:\ |\beta_m - \beta_n| < \exp(-\tfrac{1}{2}C \log^3 A_n \log_2^{1+\varepsilon} A_n).$$

Put $i_r := \frac{1}{2}C \log^3 A_r \log_2^{1+\varepsilon} A_r$, $I_r := \{x \in \mathbb{R} : |\beta_r - x| < \exp(-i_r)\}$. Then (3.22) can be written as $\forall m > n$: $\beta_m \in I_n$. By (3.20), $\beta_m \in I_{m-1}$, so it is sufficient to prove $\forall r > n$: $I_r \subset I_{r-1}$. Take $x \in I_r$, which means $|\beta_r - x| < \exp(-i_r)$. Then, by (3.19) and (3.20),

$$|\beta_{r-1} - x| \leqq |\beta_r - x| + |\beta_{r-1} - \beta_r| < \exp(-i_r) + \exp(-2i_{r-1}) <$$

$$\exp(-8i_{r-1}) + \exp(-2i_{r-1}) < 2\exp(-2i_{r-1}) < \exp(-i_{r-1})$$

if A_1 is sufficiently large in relation to C. This proves (3.22).

III. Define $\gamma_n := \exp(\beta_n \ell_1(\alpha_n)) = \exp(j_n k_n^{-1} \ell_1(\alpha_n))$; then

$$\gamma_n = \theta_n^{j_n} \in \mathbb{A}_d \setminus \{1\}.$$

By (3.15) and (3.21), it is clear that the sequences $(\alpha_n)_{n=1}^{\infty}$ and $(\gamma_n)_{n=1}^{\infty}$ are bounded and remain uniformly away from 0; applying 2.10, together with the fact that $\lim_{n\to\infty} A_n = \infty$ shows $h(\beta_n) \leqq \log A_n$ for sufficiently large n. From (3.20) one sees that $(\beta_n)_{n=1}^{\infty}$ does not become eventually constant, so that $|b - \beta_n| > 0$ for infinitely many n; by (3.21) and 2.14, b must be irrational.

IV. By 3.3,

$$h(\gamma_n) = h(\theta_n^{j_n}) \leqq 2^{2d} h^{2d}(\theta_n^{k_n}) \leqq 2^{2d} A_n^{2d} \leqq A_n^{3d};$$

therefore

$$H_n := \max(2, h(\alpha_n), h(\beta_n), h(\gamma_n)) \leqq A_n^{3d}$$

for almost all n; hence, if C is sufficiently large in relation to d, (3.15) and (3.21) imply

$$\max(|a-\alpha_n|, |b-\beta_n|) < \exp(-2 \log^3 H_n \log_2^{1+\varepsilon} H_n)$$

for almost all n.

As the function $(x,y) \mapsto \exp(y\ell_1(x))$ is continuously differentiable on a compact neighbourhood of (a,b), there exists a λ, independent of n, such that

$$|\exp(b\ell_1(a)) - \exp(\beta_n \ell_1(\alpha_n))| < \lambda \max(|a-\alpha_n|, |b-\beta_n|)$$

and therefore

$$|\exp(b\ell_1(a)) - \gamma_n| < \exp(-\log^3 H_n \log_2^{1+\varepsilon} H_n)$$

for almost all n. Thus $a \in P_d^*(\varepsilon)$. □

3.9. THEOREM. *If* $\varepsilon > 0$, $d \in \mathbb{N}$ *and* $a \in \mathbb{C} \setminus \{0\}$ *belongs to* $P_d^*(\varepsilon)$, *there is a branch* ℓ_1 *of the logarithm, defined on a neighbourhood of* a, *such that for infinitely many* $k \in \mathbb{N}$ *an* $\alpha \in \mathbb{A}$, *depending on* k, *exists with*

$$\text{(3.23)} \qquad \exp(k^{-1}\ell_1(\alpha)) \in \mathbb{A}_d \setminus \{1\},$$

$$\text{(3.24)} \qquad |a - \alpha| < \exp(-\log^3 A \log_2^{1+\varepsilon} A),$$

where $A = \max(2, h(\alpha))$.

PROOF. I. Suppose $b \in \mathbb{C} \setminus \mathbb{Q}$ and $\ell(a) \neq 0$ a value of the logarithm of a, such that there exist infinitely many triples (α,β,γ) of algebraic numbers of degree at most d satisfying

$$\text{(3.25)} \qquad \max(|a-\alpha|, |b-\beta|, |a^b-\gamma|) < \exp(-\log^3 H \log_2^{1+\varepsilon} H),$$

where $H = \max(2, h(\alpha), h(\beta), h(\gamma))$ and $a^b = \exp(b\ell(a))$. From this it is immediately clear that (3.24) holds; we now proceed to prove (3.23).

Let $\bar{\ell}, \ell^*$ be branches of the logarithm, defined on disks K_1, K_2, centred at a and a^b respectively, such that $\bar{\ell}(a) = \ell(a)$ and $\ell^*(a^b) = b\ell(a)$. Let (α,β,γ) be a triple satisfying (3.25); we suppose H to be greater than a certain bound depending on ε,d,a,b and $\ell(a)$. As $a \neq 0$ and $a^b \neq 0$, we may assume $\alpha \neq 0$ and $\gamma \neq 0$. Clearly

$$\text{(3.26)} \qquad |\ell(a) - \bar{\ell}(\alpha)| < \exp(-\log^3 H \log_2^{1+2\varepsilon/3} H),$$

(3.27) $\qquad |b\ell(a) - \ell^*(\gamma)| < \exp(-\log^3 H \log_2^{1+2\varepsilon/3} H)$;

from $\ell(a) \neq 0$ we thus get $\bar{\ell}(\alpha) \neq 0$. As a consequence of (3.26), (3.27) and

$$|b - \beta| < \exp(-\log^3 H \log_2^{1+\varepsilon} H)$$

we have

$$|\beta\bar{\ell}(\alpha) - \ell^*(\gamma)| < \exp(-\log^3 H \log_2^{1+\varepsilon/3} H).$$

If it were the case that $\beta\bar{\ell}(\alpha) - \ell^*(\gamma) \neq 0$, 2.18 would imply

$$|\beta\bar{\ell}(\alpha) - \ell^*(\gamma)| > \exp(-\log^3 H \log_2^{1+\varepsilon/3} H),$$

which is a contradiction. Therefore $\beta\bar{\ell}(\alpha) - \ell^*(\gamma) = 0$.

II. We have just proved that $\bar{\ell}(\alpha)$ and $\ell^*(\gamma)$ are linearly dependent over the field of all algebraic numbers; using 2.19 we find that these numbers must also be linearly dependent over $\mathbb{Q}$. In other words, there are $\xi, \eta \in \mathbb{Q}$, not both zero, such that $\xi\bar{\ell}(\alpha) + \eta\ell^*(\gamma) = 0$. Here $\eta \neq 0$ because $\bar{\ell}(\alpha) \neq 0$, so

$$\beta = \frac{\ell^*(\gamma)}{\bar{\ell}(\alpha)} = -\frac{\xi}{\eta} \in \mathbb{Q}.$$

Put $\beta = v/w$, where $v \in \mathbb{Z}$, $w \in \mathbb{N}$, $(v,w) = 1$. Let $x,y \in \mathbb{Z}$ be defined by $xv + yw = 1$ and take $\theta := \gamma^x\alpha^y$; clearly $\theta \in \mathbb{A}_d$ and

$$\theta = \exp(x\ell^*(\gamma)+y\bar{\ell}(\alpha)) = \exp(xvw^{-1}\bar{\ell}(\alpha)+y\bar{\ell}(\alpha)) = \exp(w^{-1}\bar{\ell}(\alpha)).$$

As $\bar{\ell}(\alpha)$ is bounded and unequal to zero, we have now proved

$$\exp(w^{-1}\bar{\ell}(\alpha)) \in \mathbb{A}_d \setminus \{1\}$$

if w is sufficiently large. That the number w occurring in this formula tends to infinity with H, immediately results from (3.25) and the fact that $b \notin \mathbb{Q}$. □

It could be asked whether the condition (3.23) on the special nature of the numbers α approximating a might be dispensed with if the approximation measure itself is strengthened. That, at least in cade $d = 1$, such a result does not even hold when the right hand side of (3.24) is replaced by a very rapidly decreasing function of the height, follows from 3.12 below.

3.10. LEMMA (Dirichlet's Theorem). *If* $a,b \in \mathbb{Z}$ *such that* $b \neq 0$ *and* $(a,b) = 1$, *the residue class* $\{a+bn: n \in \mathbb{Z}\}$ *contains infinitely many prime numbers.*

PROOF. [Apostol 1976], Theorem 7.9. □

3.11. LEMMA. *Let* $(a_n)_{n=1}^{\infty}$ *be a sequence of positive integers, suppose* $a_0 \in \mathbb{Z}$. *Define*

$$p_0 := a_0;\ p_1 := a_0a_1 + 1;\ p_{n+1} := a_{n+1}p_n + p_{n-1},\ n \in \mathbb{N};$$

$$q_0 := 1;\ q_1 := a_1;\ q_{n+1} := a_{n+1}q_n + q_{n-1},\ n \in \mathbb{N}.$$

Then for each $n \in \mathbb{N} \cup \{0\}$ *we have* $p_n \in \mathbb{Z}$, $q_n \in \mathbb{N}$, $(p_n,q_n) = 1$; *and there exists a number* $b \in \mathbb{R}\setminus\mathbb{Q}$ *such that* $(p_n/q_n)_{n=0}^{\infty}$ *is the sequence of convergents of the continued fraction expansion of* b.

PROOF. The first assertion follows from Satz 2.1, the second one from Satz 2.6 of [Perron 1954]. □

3.12. THEOREM. *For any* $f: \mathbb{N} \to]0,\infty[$ *and* $\varepsilon > 0$, *there exists an* $a \in \mathbb{R}\setminus\{0\}$ *such that infinitely many rational numbers* ξ *satisfy*

$$|a - \xi| < f(h(\xi)),$$

while $a \notin P_1^*(\varepsilon)$.

PROOF. I. By induction we define two sequences $(a_n)_{n=0}^{\infty}$ and $(q_n)_{n=0}^{\infty}$ of non-negative integers. Take $a_0 := 0$, $q_0 := 1$ and let $a_1 = q_1$ be a prime number greater than $1/f(1)$. Now suppose $a_0,\dots,a_n$ and $q_0,\dots,q_n$ have already been defined in such a way that $q_1,\dots,q_n$ are prime numbers and $q_k > 1/f(q_{k-1})$ for $k = 1,\dots,n$. According to 3.10, the residue class $\{q_{n-1}+xq_n: x \in \mathbb{Z}\}$ contains infinitely many prime numbers, including one greater than $1/f(q_n)$. This prime number we shall call q_{n+1}, the corresponding value of x will be a_{n+1}.

The sequence $(q_n)_{n=0}^{\infty}$ we have now constructed has the property that $q_n > 1/f(q_{n-1})$ and q_n prime for $n \in \mathbb{N}$, while $(q_n)_{n=0}^{\infty}$ and $(a_n)_{n=0}^{\infty}$ are connected by

$$(3.28) \qquad q_0 = 1;\ q_1 = a_1;\ q_{n+1} = a_{n+1}q_n + q_{n-1},\ n \in \mathbb{N}.$$

Using 3.11 we see that the numbers q_n are the denominators of the convergents of the continued fraction expansion of some irrational number a.

II. Let p_n be the numerator of the n-th convergent of a. According to 3.11,

$$p_0 = 0;\ p_1 = 1;\ p_{n+1} = a_{n+1}p_n + p_{n-1},\ n \in \mathbb{N}.$$

Comparison with (3.28) shows that $\forall n$: $p_n < q_n$, so we have $h(p_n/q_n) = q_n$. Now 2.22 states that

$$|a - \frac{p_n}{q_n}| < \frac{1}{q_nq_{n+1}} < \frac{1}{q_{n+1}} < f(q_n);$$

thus, if we define $\xi_n := p_n/q_n$,

$$|a - \xi_n| < f(h(\xi_n)),\ n \in \mathbb{N}.$$

III. Now suppose $a \in P_1^*(\varepsilon)$; according to 3.9, there is a branch ℓ_1 of the logarithm such that, for infinitely many positive integers k, an algebraic number α exists with

$$(3.29) \qquad \exp(k^{-1}\ell_1(\alpha)) \in \mathbb{Q}\setminus\{1\},$$

$$(3.30) \qquad |a - \alpha| < \exp(-\log^3 A \log_2^{1+\varepsilon} A),$$

where $A = h(\alpha)$. By 2.14, (3.30) implies that α is a convergent of the continued fraction expansion of a; thus $\exists n$: $\alpha = \xi_n = p_n/q_n$, where q_n is prime. On the other hand, from (3.29) it follows that $\alpha = \theta^k$ with $\theta \in \mathbb{Q}\setminus\{1\}$, which gives a contradiction if $k > 1$. □

4. SIMULTANEOUS APPROXIMATION MEASURES FOR SUBSETS OF $\{a,b,P(a,b,a^b)\}$

The purpose of this chapter is to generalize 1.8 and 1.9 in order to obtain simultaneous approximation measures for subsets of $\{a,b,P(a,b,a^b)\}$, where P is a polynomial with rational integer coefficients. As a further generalization, our results will not depend on the precise coefficients of P, but only on the degree and height of P. Our main theorem is 4.12; the other theorems in this chapter treat various special cases. In 4.8 and 4.10 we consider the case that a is a fixed algebraic number, in 4.11 the case that this is true for b. In 4.7 we assume that $a^b = \exp(b)$; this result was independently proved by Väänänen [19..], who used the method of Siegel and Šidlovskiĭ.

4.1. LEMMA. *Suppose* $P_0 \in \mathbb{Z}[X]$ *, its height and degree bounded by* H_0 *and* n_0 *respectively, with* $n_0 \leqq \log H_0$. *If there exist a real number* $\lambda > 6$ *and a transcendental number* ξ *such that*

$$|P_0(\xi)| < \exp(-n_0 \log H_0),$$

there also exist a positive integer $s \leqq n_0$ *and an irreducible divisor* $P_1 \in \mathbb{Z}[X]$ *of* P_0*, its height and degree bounded by* $\exp(s^{-1}\log H_0 + 2s^{-1}n_0)$ *and* $s^{-1}n_0$ *respectively, satisfying*

$$|P_1(\xi)| < \exp(-s^{-1}(\lambda-6)n_0 \log H_0).$$

PROOF. The assertion is proved as Lemma VI' in Chapter III of [Gel'fond 1952] under the extra assumption that the coefficients of P_0 have greatest common divisor 1. This, however, is no restriction; indeed, if this greatest common divisor is $m > 1$, the polynomial $m^{-1}P_0$ has still height and degree bounded by H_0 and n_0 respectively, while $|m^{-1}P_0(\xi)| < |P_0(\xi)|$. Moreover, any irreducible divisor of $m^{-1}P_0$ also divides P_0. □

4.2. LEMMA. *Suppose* $P_1 \in \mathbb{Z}[X]$ *separable, its height bounded by* H_1*, its degree exactly* k. *Let* $a > 0$ *be the leading coefficient of* P_1 *and* $\theta_1,\ldots,\theta_k$ *its roots; let* ξ *be an arbitrary complex number. Then*

$$|P_1(\xi)| \geqq a \exp(-2k^2 - k \log H_1) \min_{i=1,\ldots,k} |\xi - \theta_i|.$$

PROOF. [Fel'dman 1951], Lemma 5. □

4.3. DEFINITION. By $S_d^{(n)}$ we denote the set of all polynomials $P \in \mathbb{Z}[X_1,\dots,X_n]$ such that the degree of P in each variable does not exceed d, while for all $(\xi_1,\dots,\xi_{n-1}) \in \mathbb{A}_d^{n-1}$ the degree of $P(\xi_1,\dots,\xi_{n-1},X) \in \mathbb{A}[X]$ is at least one. Instead of $S_d^{(2)}$ we usually write S_d.

4.4. LEMMA. *Let F be a finite normal extension of $\mathbb{Q}$; suppose $\xi \in F$. Then $\xi \in \mathbb{Q}$ if and only if $\sigma(\xi) = \xi$ for all $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$.*

PROOF. [Van der Waerden 1966], § 57. □

4.5. LEMMA. *Suppose $d,n \in \mathbb{N}$, $a_1,\dots,a_n \in \mathbb{C}$, $b \in \mathbb{C}\setminus\mathbb{A}$. Then there exist effectively computable $C_1,\dots,C_5$, greater than 1, depending only on $d,n,a_1,\dots,a_n$ and b, such that the following holds. Suppose $\alpha_1,\dots,\alpha_n \in \mathbb{A}_d$, $\gamma \in \mathbb{A}_d$, $P \in S_d^{(n+1)}$ with*

$$H := \max(h(\alpha_1),\dots,h(\alpha_n),h(\gamma),h(P)) \geq C_1, \tag{4.1}$$

$$\max(|a_1-\alpha_1|,\dots,|a_n-\alpha_n|,|P(a_1,\dots,a_n,b)-\gamma|) < \exp(-C_2\log H). \tag{4.2}$$

Then there exists a number $\beta \in \mathbb{A}$ of degree at most C_3 and height at most $\exp(C_4\log H)$ with

$$|b-\beta|^{C_5} < \max(|a_1-\alpha_1|,\dots,|a_n-\alpha_n|,|P(a_1,\dots,a_n,b)-\gamma|).$$

PROOF. Suppose $C_1,C_2 > 1$, $\alpha_1,\dots,\alpha_n \in \mathbb{A}_d$, $\gamma \in \mathbb{A}_d$, $P \in S_d^{(n+1)}$ are such that (4.1) and (4.2) are satisfied. By $c_1,c_2,\dots$ we shall denote real numbers greater than 1 depending only on $d,n,a_1,\dots,a_n$ and b; throughout the proof we shall without further mention assume that C_1 and C_2 are greater than such a number. Put

$$U := -\log\max(|a_1-\alpha_1|,\dots,|a_n-\alpha_n|,|P(a_1,\dots,a_n,b)-\gamma|).$$

Remark that the transcendency of b ensures that U is well-defined; from (4.2) it follows that $U > C_2\log H$.

Denote $d_j := \mathrm{dg}(\alpha_j)$, $m_j := \mathrm{den}(\alpha_j)$ for $j = 1,\dots,n$; denote $d_0 := \mathrm{dg}(\gamma)$, $m_0 := \mathrm{den}(\gamma)$, $D := d_0d_1\dots d_n$. Then $D \leq c_1$, $\max(m_0,m_1,\dots,m_n) \leq H$. Let R_0 be the polynomial

$$R_0 := \prod_{\delta_0=1}^{d_0} \cdots \prod_{\delta_n=1}^{d_n} (P(\alpha_1^{(\delta_1)},\ldots,\alpha_n^{(\delta_n)},X) - \gamma^{(\delta_0)}),$$

where

$$\alpha_j = \alpha_j^{(1)},\ \alpha_j^{(2)},\ldots,\ \alpha_j^{(d_j)}$$

are the conjugates of α_j for $j = 1,\ldots,n$, and

$$\gamma = \gamma^{(1)},\ \gamma^{(2)},\ldots,\ \gamma^{(d_0)}$$

are the conjugates of γ. Now, if $F \subset \mathbb{C}$ is a finite normal extension of $\mathbb{Q}$ containing $\alpha_1,\ldots,\alpha_n$ and γ (such an extension always exists by 2.4*(a)*) and $\sigma \in \mathrm{Gal}(F/\mathbb{Q})$, the coefficients of R_0 are clearly invariant under σ; this implies that $R_0 \in \mathbb{Q}[X]$, by 4.4. Consequently, the polynomial $R := m_0^D m_1^{dD} \ldots m_n^{dD} R_0$ has rational integer coefficients; the degree of this polynomial is bounded by $dD \leqq c_2$ and its height by

$$H^D(H^{dD})^n c_3(c_4H(H+1)^{dn}+H+1)^D \leqq \exp(c_5 \log H).$$

Furthermore, we have

$$|P(\alpha_1,\ldots,\alpha_n,b) - \gamma| \leqq$$

$$|P(\alpha_1,\ldots,\alpha_n,b) - P(a_1,\ldots,a_n,b)| + |P(a_1,\ldots,a_n,b) - \gamma| \leqq$$

$$c_6H \max_{j=1,\ldots,n} |a_j - \alpha_j| + |P(a_1,\ldots,a_n,b) - \gamma| < \exp(-c_7^{-1}U)$$

and

$$|m_0^D m_1^{dD} \ldots m_n^{dD} \prod_{\delta_0\delta_1\ldots\delta_n>1} (P(\alpha_1^{(\delta_1)},\ldots,\alpha_n^{(\delta_n)},b) - \gamma^{(\delta_0)})| \leqq$$

$$H^D(H^{dD})^n(c_8H(H+1)^{dn}+H+1)^{D+1} \leqq \exp(c_9 \log H) < \exp(c_{10}C_2^{-1}U);$$

thus

$$|R(b)| < \exp(-c_7^{-1}U+c_{10}C_2^{-1}U) < \exp(-c_{11}^{-1}U).$$

By the definition of $S_d^{(n+1)}$, R cannot be a constant.

Now apply 4.1 with $P_0 = R$, $n_0 = c_2$, $H_0 = \exp(c_5 \log H)$, $\lambda = c_2^{-1}c_5^{-1}c_{11}^{-1}U \log^{-1}H$, $\xi = b$. We deduce the existence of a positive integer $s \leq c_2$ and an irreducible divisor $Q \in \mathbb{Z}[X]$ of R, such that the degree of Q is bounded by $s^{-1}c_2 \leq c_2$ and its height by $\exp(s^{-1}c_5 \log H + 2s^{-1}c_2) \leq \exp(c_{12}\log H)$, while

$$|Q(b)| < \exp(-s^{-1}(\lambda-6)c_2c_5 \log H) =$$

$$\exp(-s^{-1}c_{11}^{-1}U + 6s^{-1}c_2c_5 \log H) <$$

$$\exp(-s^{-1}c_{11}^{-1}U + 6s^{-1}c_2c_5C_2^{-1}U) < \exp(-c_{13}^{-1}U).$$

Apply 4.2 with $P_1 = Q$, $H_1 = \exp(c_{12}\log H)$, $\xi = b$. Note that Q is separable because Q is irreducible. Let $\theta_1,\ldots,\theta_k$ be the zeros of Q; then $k \leq c_2$. We find

$$\min_{i=1,\ldots,k} |b - \theta_i| \leq \exp(2k^2 + kc_{12}\log H)|Q(b)| \leq$$

$$\exp(2c_2^2 + c_2c_{12}C_2^{-1}U - c_{13}^{-1}U) < \exp(-c_{14}^{-1}U).$$

Moreover, for $i = 1,\ldots,k$ we have $h(\theta_i) \leq \exp(c_{12}\log H)$, $dg(\theta_i) \leq c_2$; thus there exists an algebraic number β of degree at most c_2 and height at most $\exp(c_{12}\log H)$ with

$$|b - \beta| < \exp(-c_{14}^{-1}U).$$

The lemma follows by taking $C_3 = c_2$, $C_4 = c_{12}$, $C_5 = c_{14}$. □

4.6. LEMMA. *Suppose* $d, M \in \mathbb{N}$. *Then there exists an effectively computable* $C > 1$, *depending only on* d *and* M, *such that only finitely many pairs* $(\beta,\gamma) \in \mathbb{A}_d^2$ *with* $|\beta| \leq M$ *satisfy*

$$|\exp(\beta) - \gamma| < \exp(-C \log^2 H \log_2^{-1} H),$$

where $H = \max(2, h(\beta), h(\gamma))$.

PROOF. [Cijsouw 1975], Theorem 2. □

4.7. THEOREM. *Suppose* $\varepsilon > 0$, $d \in \mathbb{N}$, $b \in \mathbb{C} \setminus \{0\}$. *Then there are only finitely many triples* $(\beta,\gamma,P) \in \mathbb{A}_d^2 \times S_d$ *such that*

$$(4.3) \qquad \max(|b-\beta|,|P(b,\exp(b)-\gamma|) < \exp(-\log^2 H \log_2^{-1+\varepsilon} H),$$

where $H = \max(2,h(\beta),h(\gamma),h(P))$.

PROOF. I. Suppose the assertion of the theorem to be false; from this a contradiction will be derived. By $c_1, c_2, \ldots$ we shall denote real numbers greater than 1 depending only on ε, d and b.

First we show that under this assumption exp(b) must be transcendental. Suppose $(\beta,\gamma,P) \in \mathbb{A}_d^2 \times S_d$ such that (4.3) is satisfied; assume that H is sufficiently large in terms of ε, d and b. Suppose also that exp(b) is algebraic; as b is a value of the logarithm of exp(b), application of 2.18 shows that either $\beta - b = 0$ or

$$(4.4) \qquad |\beta - b| > \exp(-\log^{1+\varepsilon} H).$$

It is no restriction to assume that ε is so small that (4.3) and (4.4) cannot hold simultaneously; therefore we need only consider the case that $\beta - b = 0$. However, in this case the number b is algebraic but unequal to 0; thus exp(b) is transcendental by 1.2.

II. Apply 4.5; this gives the existence of an algebraic number η of degree at most c_1 and height at most $\exp(c_2 \log H)$ satisfying

$$(4.5) \qquad |\exp(b) - \eta| < \exp(-c_3^{-1} \log^2 H \log_2^{-1+\varepsilon} H),$$

which contradicts 4.6. □

4.8. THEOREM. *Suppose* $\varepsilon > 0$, $d \in \mathbb{N}$, $a \in \mathbb{A} \setminus \{0\}$, $b \in \mathbb{C} \setminus \mathbb{Q}$. *Let* $\ell(a) \neq 0$ *be a value of the logarithm of* a. *Then there are only finitely many triples* $(\beta,\gamma,P) \in \mathbb{A}_d \times S_d$ *such that*

$$(4.6) \qquad \max(|b-\beta|,|P(b,a^b)-\gamma|) < \exp(-\log^2 H \log_2^{\varepsilon} H),$$

where $H = \max(2,h(\beta),h(\gamma),h(P))$ *and* $a^b = \exp(b\ell(a))$.

PROOF. I. Suppose the assertion of the theorem to be false; from this a contradiction will be derived. By $c_1, c_2, \dots$ we shall denote real numbers greater than 1 depending only on ε, d, a, b and $\ell(a)$.

First we show that a^b must be transcendental. Let K be a closed disk in the complex plane, centred at a^b, that does not contain 0. It is clear that there exists a branch ℓ^* of the logarithm, defined on K, such that

$$(4.7) \qquad b\ell(a) = \ell^*(a^b).$$

Suppose $(\beta,\gamma,P) \in \mathbb{A}_d^2 \times S_d$ such that (4.6) is satisfied; assume that H is sufficiently large in terms of ε, d, a, b and $\ell(a)$. From (4.6) and (4.7) it is clear that

$$|\beta\ell(a) - \ell^*(a^b)| < \exp(-\log^2 H \log_2^{\varepsilon/2} H).$$

Now suppose that a^b is algebraic. If it were the case that $\beta\ell(a) - \ell^*(a^b) \neq 0$, 2.18 would imply

$$|\beta\ell(a) - \ell^*(a^b)| > \exp(-\log^{1+\varepsilon} H),$$

which gives a contradiction, because we may assume ε to be arbitrarily small. Therefore $\beta\ell(a) - \ell^*(a^b) = 0$ and so, by (4.7), we find that $\beta = b$; accordingly, $b \in \mathbb{A} \setminus \mathbb{Q}$ and so $a^b \in \mathbb{C} \setminus \mathbb{A}$ by 1.3.

II. Apply 4.5; this gives the existence of an algebraic number η of degree at most c_1 and height at most $\exp(c_2 \log H)$ satisfying

$$(4.8) \qquad |a^b - \eta| < \exp(-c_3^{-1} \log^2 H \log_2^{\varepsilon} H).$$

From (4.7) and (4.8) we then deduce that $\eta \in K$ and that the inequality

$$|\beta\ell(a) - \ell^*(\eta)| < \exp(-\log^2 H \log_2^{\varepsilon/2} H)$$

holds, which, by 2.18, implies $\beta\ell(a) - \ell^*(\eta) = 0$.

III. We have proved that $\ell(a)$ and $\ell^*(\eta)$ are linearly dependent over $\mathbb{A}$; using 2.19 we find that these numbers must also be linearly dependent over $\mathbb{Q}$. In other words, there are $\xi_1, \xi_2 \in \mathbb{Q}$, not both zero, such that $\xi_1\ell(a) + \xi_2\ell^*(\eta) = 0$. Here $\xi_2 \neq 0$ because $\ell(a) \neq 0$, so

$$\beta = \frac{\ell^*(\eta)}{\ell(a)} = -\frac{\xi_1}{\xi_2} \in \mathbb{Q}.$$

Using 3.5 we see that $h(\beta) < c_4$; from (4.6) it then follows that $b = \beta \in \mathbb{Q}$. This contradicts the conditions of the theorem. □

4.9. LEMMA. *Suppose* $d \in \mathbb{N}$, $a \in \mathbb{A}$. *Then there exists an effectively computable* $C > 1$, *depending only on* d *and* a, *such that only finitely many* $\alpha \in \mathbb{A}_d$ *satisfy*

$$|a - \alpha| < \exp(-C \log H),$$

where $H = h(\alpha)$.

PROOF. [Brauer 1929], Satz 1. □

4.10. THEOREM. *Suppose* $\varepsilon > 0$, $d \in \mathbb{N}$, $a \in \mathbb{A}\setminus\{0\}$, $b \in \mathbb{C}$, $P \in S_d$. *Let* $\ell(a) \neq 0$ *be a value of the logarithm of* a. *Then there are only finitely many pairs* $(\beta,\gamma) \in \mathbb{A}_d^2$ *such that*

$$\max(|b-\beta|,|P(b,a^b)-\gamma|) < \exp(-\log^2 H \log_2^{\varepsilon} H),$$

where $H = \max(2,h(\beta),h(\gamma))$ *and* $a^b = \exp(b\ell(a))$.

PROOF. The proof is completely analogous to that of 4.8, up to the end of part I, where the Gel'fond-Schneider theorem was used to establish the transcendency of a^b. This time, having proved that $\beta = b$, we deduce that $H = h(\gamma)$ and that $P(b,a^b)$ is algebraic, while from (4.6) we see that

$$|P(b,a^b) - \gamma| < \exp(-\log^2 H \log_2^{\varepsilon} H),$$

which contradicts 4.9. Therefore the assumption that a^b be algebraic is disproved, it follows that $b \notin \mathbb{Q}$ and we may proceed as in the case of 4.8. □

Thus, in case P is taken fixed, the condition that $b \notin \mathbb{Q}$ may be dropped from 4.8. In the same way it is possible to remove the restriction $b \neq 0$ from 4.7.

4.11. THEOREM. *Suppose* $\varepsilon > 0$, $d \in \mathbb{N}$, $a \in \mathbb{C}\setminus\{0\}$, $b \in \mathbb{A}\setminus\mathbb{Q}$. *Let* $\ell(a) \neq 0$

be a value of the logarithm of a. *Then there are only finitely many triples* $(\alpha,\gamma,P) \in \mathbb{A}_d^2 \times S_d$ *with*

$$\max(|a-\alpha|,|P(a,a^b)-\gamma|) < \exp(-\log^2 H \log_2^{2+\varepsilon} H), \tag{4.9}$$

where $H = \max(2,h(\alpha),h(\gamma),h(P))$ *and* $a^b = \exp(b\ell(a))$.

PROOF. I. Suppose the assertion of the theorem to be false; from this a contradiction will be derived. By $c_1,c_2,\ldots$ we shall denote real numbers greater than 1 depending only on ε,d,a,b and $\ell(a)$.

First we show that a^b must be transcendental. Let $\bar{\ell},\ell^*$ be branches of the logarithm, defined on disks K_1,K_2, centred at a and a^b respectively, such that

$$\ell(a) = \bar{\ell}(a), \tag{4.10}$$

$$b\ell(a) = \ell^*(a^b). \tag{4.11}$$

Suppose $(\alpha,\gamma,P) \in \mathbb{A}_d^2 \times S_d$ such that (4.9) is satisfied; assume that H is sufficiently large in terms of ε,d,a,b and $\ell(a)$. From (4.10) and (4.11) it is clear that $\alpha \in K_1$ and

$$|b\bar{\ell}(\alpha) - \ell^*(a^b)| < \exp(-\log^2 H \log_2^{2+\varepsilon/2} H).$$

Now suppose that a^b is algebraic. If it were the case that $b\bar{\ell}(\alpha) - \ell^*(a^b) \neq 0$, 2.18 would imply

$$|b\bar{\ell}(\alpha) - \ell^*(a^b)| > \exp(-\log^{1+\varepsilon} H),$$

which gives a contradiction because we may assume ε to be arbitrarily small. Therefore $b\bar{\ell}(\alpha) - \ell^*(a^b) = 0$ and so, by (4.10) and (4.11), we find $\alpha = a$; accordingly, the number a is algebraic and so $a^b \in \mathbb{C}\setminus\mathbb{A}$ by 1.3.

II. Apply 4.5; this gives the existence of an algebraic number η of degree at most c_1 and height at most $\exp(c_2\log H)$ satisfying

$$|a^b - \eta| < \exp(-c_3^{-1}\log^2 H \log_2^{2+\varepsilon} H). \tag{4.12}$$

From (4.11) and (4.12) we deduce that $\eta \in K_2$ and that the inequality

$$|b\ell(a) - \ell^*(\eta)| < \exp(-\log^2 H \log_2^{2+\varepsilon/2} H)$$

holds; comparison with (4.9) and (4.10) gives

$$|b\bar{\ell}(\alpha) - \ell^*(\eta)| < \exp(-\log^2 H \log_2^{2+\varepsilon/3} H).$$

If it were the case that $b\bar{\ell}(\alpha) - \ell^*(\eta) \neq 0$, 2.18 would imply

$$|b\bar{\ell}(\alpha) - \ell^*(\eta)| > \exp(-\log^2 H \log_2^{2+\varepsilon/3} H),$$

which gives a contradiction. Therefore $b\bar{\ell}(\alpha) - \ell^*(\eta) = 0$.

III. We have proved that $\bar{\ell}(\alpha)$ and $\ell^*(\eta)$ are linearly dependent over $\mathbb{A}$; using 2.19 we find that these numbers must also be linearly dependent over $\mathbb{Q}$. In other words, there are $\xi_1, \xi_2 \in \mathbb{Q}$, not both zero, such that $\xi_1\bar{\ell}(\alpha) + \xi_2\ell^*(\eta) = 0$. Here $\xi_2 \neq 0$ because, by (4.10), $\bar{\ell}(\alpha) \neq 0$, so

$$b = \frac{\ell^*(\eta)}{\bar{\ell}(\alpha)} = -\frac{\xi_1}{\xi_2} \in \mathbb{Q}.$$

This contradicts the conditions of the theorem. □

4.12. <u>THEOREM</u>. *Suppose* $\varepsilon > 0$, $d \in \mathbb{N}$, $a \in \mathbb{C}\setminus\{0\}$, $b \in \mathbb{C}$. *Let* $\ell(a) \neq 0$ *be a value of the logarithm of* a. *If* $b \in \mathbb{R}$, *or if* $b \in \mathbb{R}\setminus\mathbb{Q}$ *such that the convergents* p_n/q_n *of the continued fraction expansion of* b *satisfy*

$$(4.13) \qquad q_{n+1} \ll \exp(q_n^3),\ n \to \infty,$$

there are only finitely many quadruples $(\alpha,\beta,\gamma,P) \in \mathbb{A}_d^3 \times S_d^{(3)}$ *such that*

$$(4.14) \qquad \max(|a-\alpha|, |b-\beta|, |P(a,b,a^b)-\gamma|) < \exp(-\log^3 H \log_2^{1+\varepsilon} H),$$

where $H = \max(2, h(\alpha), h(\beta), h(\gamma), h(P))$ *and* $a^b = \exp(b\ell(a))$.

<u>PROOF</u>. I. Suppose that b has the required properties, but that nevertheless there exist infinitely many quadruples $(\alpha,\beta,\gamma,P) \in \mathbb{A}_d^3 \times S_d^{(3)}$ that satisfy (4.14).

From this it follows that a^b must be transcendental. If both a and b are algebraic, this is precisely the Gel'fond-Schneider theorem 1.3. Now suppose that at least one of a and b is transcendental; then (4.14) implies that there are infinitely many pairs $(\alpha,\beta) \in \mathbb{A}_d^2$ that satisfy

$$\max(|a-\alpha|,|b-\beta|) < \exp(-\log^3 H \log_2^{1+\varepsilon} H),$$

where $H \geqq \max(2,h(\alpha),h(\beta))$. If a^b is algebraic, this contradicts 1.7.

II. Apply 4.5; this gives the existence of infinitely many triples (α,β,η) of algebraic numbers, their degrees bounded by some constant depending only on ε,d,a,b and $\ell(a)$ such that

$$\max(|a-\alpha|,|b-\beta|,|a^b-\eta|) < \exp(-\log^3 H' \log_2^{1+\varepsilon/2} H'),$$

where $H' \geqq \max(2,h(\alpha),h(\beta),h(\eta))$. Once again we get a contradiction with 1.7. □

It is clear that the use of 2.24 instead of 1.7 would replace condition (4.13) by the slightly sharper $\omega(\varepsilon,b) = 0$.

5. MANY-VARIABLE ANALOGUES

In the present chapter we give analogues of 1.10 and 1.11 in which the condition that $1, \beta_1, \ldots, \beta_n$ be linearly independent over $\mathbb{Q}$ is replaced by conditions on $a_1, \ldots, a_n, b_1, \ldots, b_n$. As proved in 5.5 and 5.7, the estimate of 1.10 holds whenever $\ell_1(a_1), \ldots, \ell_n(a_n)$ or $1, b_1, \ldots, b_n$ are linearly independent over $\mathbb{Q}$. In 5.8 and 5.10 it is proved that a similar result can be obtained in case the rôles of $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ in 1.10 are interchanged; here we need the additional restriction that $b_1, \ldots, b_n$ may not all be rational. Under these conditions, it is also possible to prove many-variable analogues of 4.8 and 4.11; this is done in 5.11 and 5.12. Finally, in 5.15 we give an analogue of 1.11 in which the condition that $1, \beta_1, \ldots, \beta_n$ be linearly independent is replaced by the strong linear independence condition (1.3). In 5.13 it is shown that 5.15 may also be regarded as a generalization of the type of theorem of which 1.7 is an example; 5.16 uses 5.15 to give a many-variable analogue of 4.12.

5.1. LEMMA. *Suppose* $d, n \in \mathbb{N}$ *; then there exists an effectively computable* $C > 1$, *depending only on* d *and* n, *such that for all* n*-tuples* $(\eta_1, \ldots, \eta_n)$ *of algebraic complex numbers of height at most* $H \geq 2$ *the following assertions hold:*

$$dg(\eta_1 + \ldots + \eta_n) \leq d \Rightarrow h(\eta_1 + \ldots + \eta_n) \leq H^C,$$

$$dg(\eta_1 \ldots \eta_n) \leq d \Rightarrow h(\eta_1 \ldots \eta_n) \leq H^C.$$

PROOF. Suppose $dg(\eta_1 + \ldots + \eta_n) \leq d$; put $\eta := \eta_1 + \ldots + \eta_n \in \mathbb{A}_d$. The minimal polynomial of η is

$$a \prod_{j=1}^{k} (X - \eta^{(j)}),$$

where $a \in \mathbb{N}$ and where $\eta^{(1)}, \ldots, \eta^{(k)}$ are the different conjugates of η (thus $k \leq d$). By 2.5*(c)* we have, for $j = 1, \ldots, k$,

$$|\eta^{(j)}| \leq \overline{|\eta|} \leq \overline{|\eta_1|} + \ldots + \overline{|\eta_n|} \leq n(H+1).$$

According to 2.4*(a)*, $\mathbb{C}$ contains a finite normal extension F of $\mathbb{Q}$, such that $\eta_1, \ldots, \eta_n \in F$. Put $G := \mathrm{Gal}(F/\mathbb{Q})$. By 2.4*(b)*,

$$a \leq \prod_{j=1}^{k} \operatorname{den}(\eta^{(j)}) \leq \left(\max_{\sigma\in G} \operatorname{den} \sigma(\eta)\right)^k \leq$$

$$\max_{\sigma\in G} \left(\operatorname{den} \sigma(\eta_1)\cdot\ldots\cdot\operatorname{den} \sigma(\eta_n)\right)^k \leq$$

$$\max_{\sigma\in G} \operatorname{den}^k\sigma(\eta_1)\cdot\ldots\cdot \max_{\sigma\in G} \operatorname{den}^k\sigma(\eta_n) \leq H^{dn}.$$

The first assertion of the lemma follows from the trivial fact that the height of a product of polynomials is less than the product of their heights times a constant depending only on their number and their degrees.

The second assertion is proved analogously, with the obvious replacements. □

5.2. LEMMA. *Suppose* $d,n \in \mathbb{N}$. *Then there exists an effectively computable* $C > 1$, *depending only on* d *and* n, *with the following property. Suppose* $\alpha_1,\ldots,\alpha_n \in \mathbb{A}_d$ *are multiplicatively dependent; then there exist* $t_1,\ldots,t_n \in \mathbb{Z}$, *not all zero, such that*

$$\alpha_1^{t_1}\ldots\alpha_n^{t_n} = 1$$

and

$$\max(|t_1|,\ldots,|t_n|) < C \log^{n-1} H,$$

where $H = \max(2,h(\alpha_1),\ldots,h(\alpha_n))$.

PROOF. [Van der Poorten-Loxton 1977], Theorem 1. □

5.3. LEMMA. *Suppose* $R > 0$, $\omega_1,\ldots,\omega_\ell \in \mathbb{C}$, $z_0 \in \mathbb{C}$. *Let* $P_1,\ldots,P_\ell \in \mathbb{C}[X]$ *denote non-trivial polynomials of degree at most* d. *Define*

$$\Phi(z) := \sum_{k=1}^{\ell} P_k(z)\exp(\omega_k z),\ z \in \mathbb{C}.$$

If Φ *is not identically zero, the number of zeros of* Φ *in* $\{z \in \mathbb{C}: |z - z_0| \leq R\}$ *does not exceed*

$$3\ell(d+1) - 3 + 4R \max(|\omega_1|,\ldots,|\omega_\ell|).$$

PROOF. [Tijdeman 1971], Theorem 1, Corollary. □

The following lemma is a generalization of 2.10 and 2.11 to an arbitrary number of logarithms; it has been brought to our attention that Waldschmidt [19..], in an as yet unpublished paper, proved a slightly sharper version of the second assertion of the lemma.

5.4. LEMMA. *Suppose* $d,n \in \mathbb{N}$, K *a compact subset of the complex plane not containing* 0, $\ell_1,\ldots,\ell_n$ *branches of the logarithm, defined on* K. *Then there exist effectively computable* $C, H_0 > 1$, *depending only on* d,n,K *and* $\ell_1,\ldots,\ell_n$, *with the following property. Let* $\alpha_1,\ldots,\alpha_n \in K$ *be algebraic of degree at most* d *and heights at most* $H_1,\ldots,H_n$ *respectively, such that* $\ell_1(\alpha_1),\ldots,\ell_n(\alpha_n)$ *are linearly dependent over* $\mathbb{Q}$. *If* $H \geqq \max(H_0,H_1,\ldots,H_n)$, *then there exist* $x_1,\ldots,x_n \in \mathbb{Z}$, *not all zero, such that*

$$x_1\ell_1(\alpha_1) + \ldots + x_n\ell_n(\alpha_n) = 0$$

while

$$|x_j| \leqq C \log_2^{n+1} H \prod_{i\neq j} \log H_i,\ j = 1,\ldots,n.$$

In case $H_1 = \ldots = H_n$, *this inequality may be sharpened to*

$$|x_j| \leqq C \log^{n-1} H \log_2 H,\ j = 1,\ldots,n.$$

PROOF. I. Let $\alpha_1,\ldots,\alpha_n \in K$ be algebraic of degree at most d and heights at most $H_1,\ldots,H_n$ respectively, such that $\ell_1(\alpha_1),\ldots,\ell_n(\alpha_n)$ are linearly dependent over $\mathbb{Q}$; let H be a number greater than $\max(H_1,\ldots,H_n)$. By c_1, $c_2,\ldots$ we shall denote real numbers greater than 1 that depend only on d,n,K and $\ell_1,\ldots,\ell_n$. By C we shall denote some number greater than 1; additional restrictions on the choice of C will be formulated at later stages of the proof. Throughout the proof we shall assume that H is sufficiently large in terms of $d,n,K,\ell_1,\ldots,\ell_n$ and C.

II. First we shall prove that there exist $u_1,\ldots,u_n \in \mathbb{Z}$, not all zero, their absolute values bounded by $c_1\log^{2n-3}H$, such that

$$u_1\ell_1(\alpha_1) + \ldots + u_n\ell_n(\alpha_n) = 0.$$

It is clear that there exist $r_1,\ldots,r_n \in \mathbb{Z}$, not all zero, such that

(5.1) $$r_1 \ell_1(\alpha_1) + \ldots + r_n \ell_n(\alpha_n) = 0.$$

By 5.2, there exist $s_1,\ldots,s_n \in \mathbb{Z}$, not all zero, their absolute values bounded by $c_2 \log^{n-1} H$, such that

$$\alpha_1^{s_1} \ldots \alpha_n^{s_n} = 1;$$

suppose for instance that $s_n \neq 0$. Thus there is a $s_{n+1} \in \mathbb{Z}$ with

(5.2) $$s_1 \ell_1(\alpha_1) + \ldots + s_n \ell_n(\alpha_n) + s_{n+1} \cdot 2\pi i = 0.$$

As $|\ell_j(\alpha_j)| \leqq c_3$ for $j = 1,\ldots,n$, we have $|s_{n+1}| \leqq c_4 \log^{n-1} H$. If $s_{n+1} = 0$, there is nothing left to prove; assume $s_{n+1} \neq 0$. Comparison of (5.1) and (5.2) yields

$$(r_1 - \frac{r_n}{s_n} s_1) \ell_1(\alpha_1) + \ldots + (r_{n-1} - \frac{r_n}{s_n} s_{n-1}) \ell_{n-1}(\alpha_{n-1}) -$$
$$\frac{r_n}{s_n} s_{n+1} \cdot 2\pi i = 0.$$

Again by 5.2, there exist $t_1,\ldots,t_{n-1} \in \mathbb{Z}$, not all zero, their absolute values bounded by $c_5 \log^{n-2} H$, such that

$$\alpha_1^{t_1} \ldots \alpha_{n-1}^{t_{n-1}} = 1.$$

Thus there is a $t_n \in \mathbb{Z}$ with

(5.3) $$t_1 \ell_1(\alpha_1) + \ldots + t_{n-1} \ell_{n-1}(\alpha_{n-1}) + t_n \cdot 2\pi i = 0.$$

As $|\ell_j(\alpha_j)| \leqq c_3$ for $j = 1,\ldots,n-1$, we have $|t_n| \leqq c_6 \log^{n-2} H$. If $t_n = 0$, there is nothing left to prove; assume $t_n \neq 0$. Comparison of (5.2) and (5.3) yields

$$(s_1 t_n - s_{n+1} t_1) \ell_1(\alpha_1) + \ldots + (s_{n-1} t_n - s_{n+1} t_{n-1}) \ell_{n-1}(\alpha_{n-1}) +$$
$$s_n t_n \ell_n(\alpha_n) = 0.$$

If we define $u_j := s_j t_n - s_{n+1} t_j$ for $j = 1,\ldots,n-1$ and $u_n := s_n t_n$, the numbers $u_1,\dots,u_n$ possess all the desired properties; in particular $u_n \neq 0$ because $s_n \neq 0$ and $t_n \neq 0$.

III. For convenience's sake, in the remainder of the proof we shall assume that $u_n \neq 0$ and write $\beta_j := -u_j/u_n$ for $j = 1,\dots,n-1$; we have

$$\beta_1 \ell_1(\alpha_1) + \dots + \beta_{n-1}\ell_{n-1}(\alpha_{n-1}) - \ell_n(\alpha_n) = 0$$

and

$$B := \max(h(\beta_1),\dots,h(\beta_{n-1})) \leqq c_1 \log^{2n-3} H. \tag{5.4}$$

Define $\Omega := (\log H_1)\dots(\log H_n)$ and

$$\Omega_j := \prod_{i \neq j} \log H_i$$

for $j = 1,\dots,n$. Define

$$L_0 := [C^{1+1/n}\Omega \log_2^{n+2} H \log_2^{-n-1} A] - 1;$$

$$L_j := [C\Omega_j \log_2^{n+1} H \log_2^{-n} A] - 1,\ j = 1,\dots,n,$$

where

$$A := \begin{cases} H \text{ if } H_1 = \dots = H_n, \\ e^e \text{ otherwise.} \end{cases}$$

We introduce the auxiliary function

$$\Phi(z) := \sum_{\lambda_0=0}^{L_0} \dots \sum_{\lambda_n=0}^{L_n} p(\lambda_0,\dots,\lambda_n) z^{\lambda_0} \alpha_1^{\lambda_1 z} \dots \alpha_n^{\lambda_n z},\ z \in \mathbb{C},$$

where

$$\alpha_j^{\lambda_j z} = \exp(\lambda_j z \ell_j(\alpha_j)),\ j = 1,\dots,n,$$

and where $p(\lambda_0,\dots,\lambda_n)$ are rational integers to be determined later. We have

$$\Phi^{(t)}(z) = \sum_{\underline{\tau}} \frac{t!}{\tau_0!\dots\tau_{n-1}!} \ell_1^{\tau_1}(\alpha_1)\dots\ell_{n-1}^{\tau_{n-1}}(\alpha_{n-1})\Phi_{\underline{\tau}}(z), \tag{5.5}$$

where the summation ranges over all n-tuples $\underline{\tau} = (\tau_0,\ldots,\tau_{n-1})$ of non-negative integers satisfying $\tau_0 + \ldots + \tau_{n-1} = t$, and where

$$\Phi_{\underline{\tau}}(z) = \sum_{\lambda_0=0}^{L_0} \ldots \sum_{\lambda_n=0}^{L_n} p(\lambda_0,\ldots,\lambda_n) \frac{\lambda_0!}{(\lambda_0-\tau_0)!} z^{\lambda_0-\tau_0} \alpha_1^{\lambda_1 z} \ldots \alpha_n^{\lambda_n z} \times \prod_{\nu=1}^{n-1} (\lambda_\nu+\lambda_n\beta_\nu)^{\tau_\nu}.$$

Now put $a_j := \mathrm{den}(\alpha_j)$ for $j = 1,\ldots,n$, $b_j = \mathrm{den}(\beta_j)$ for $j = 1,\ldots,n-1$, $S := [C^{1/n}\log_2^2 H \log_2^{-1} A]$, $T := [\frac{1}{2}d^{-1}C^{1+1/n}\Omega \log_2^{n+2} H \log_2^{-n-1} A]$, and consider the system of linear equations

$$a_1^{L_1 s} \ldots a_n^{L_n s} b_1^{\tau_1} \ldots b_{n-1}^{\tau_{n-1}} \Phi_{\underline{\tau}}(s) = 0, \; s = 0,\ldots,S-1, \; \underline{\tau} \in V(T),$$

where $V(T)$ is the set of all n-tuples $\underline{\tau} = (\tau_0,\ldots,\tau_{n-1})$ of non-negative integers satisfying $\tau_0 + \ldots + \tau_{n-1} \leqq T - 1$. These are fewer than ST^n equations in the $(L_0+1)\ldots(L_n+1)$ unknowns $p(\lambda_0,\ldots,\lambda_n)$; the coefficients are algebraic integers in the number field $\mathbb{Q}(\alpha_1,\ldots,\alpha_n)$ of degree at most d^n. The absolute values of the conjugates of the coefficients are less than or equal to

$$a_1^{L_1 S} \ldots a_n^{L_n S} \max{}^T(b_1,\ldots,b_{n-1}) L_0! S^{L_0} \Big(\prod_{j=1}^{n} \max(1, \overline{|\alpha_j|}^{L_j S}) \Big) \times$$

$$\max_{\nu=1,\ldots,n-1} \max(1, |\lambda_\nu+\lambda_n\beta_\nu|^T) \leqq$$

$$c_7^{LS+T} L_0^{L_0} S^{L_0} L^T B^{2T} \prod_{j=1}^{n} H_j^{2L_j S} \leqq \exp(c_8 C^{1+1/n}\Omega \log_2^{n+3} H \log_2^{-n-1} A);$$

here (5.4) is used and $L := \max(L_1,\ldots,L_n)$. As

$$(L_0+1)\ldots(L_n+1) \geqq \tfrac{1}{2}C^{n+1+1/n}\Omega^n \log_2^{n^2+2n+2} H \log_2^{-n^2-n-1} A \geqq 2d^n ST^n,$$

2.7 states that there is a non-trivial choice for the $p(\lambda_0,\ldots,\lambda_n)$ such that

$$(5.6) \qquad \Phi_{\underline{\tau}}(s) = 0, \; s = 0,\ldots,S-1, \; \underline{\tau} \in V(T),$$

while

$$P := \max_{\lambda_0,\ldots,\lambda_n} |p(\lambda_0,\ldots,\lambda_n)| \leq$$

$$c_9(L_0+1)\ldots(L_n+1)\exp(c_8C^{1+1/n}\Omega \log_2^{n+3}H \log_2^{-n-1}A) \leq$$

$$\exp(c_{10}C^{1+1/n}\Omega \log_2^{n+3}H \log_2^{-n-1}A).$$

IV. Define $K := [n^2\log_2 H/\log 2] + 1$. For $k = 0,\ldots,K$ we put $S_k := 2^k S$, $T_k := T - k[T/2K]$. Then, for our special choice of the $p(\lambda_0,\ldots,\lambda_n)$, we have

$$\Phi_{\underline{\tau}}(s) = 0,\quad s = 0,\ldots,S_k-1,\quad \underline{\tau} \in V(T_k). \tag{5.7}$$

This is proved by induction; for $k = 0$ the assertion is precisely (5.6). Now suppose that (5.7) holds for some $k \leq K - 1$. Then, for $\underline{\tau} \in V(T_{k+1})$ and $m \leq [T/2K]$ we have

$$\Phi_{\underline{\tau}}^{(m)}(z) = \sum_{\underline{\mu}} \frac{m!}{\mu_0!\ldots\mu_{n-1}!} \ell_1^{\mu_1}(\alpha_1)\ldots\ell_{n-1}^{\mu_{n-1}}(\alpha_{n-1})\Phi_{\underline{\tau}+\underline{\mu}}(z),\quad z \in \mathbb{C},$$

where the summation ranges over all n-tuples $\underline{\mu} = (\mu_0,\ldots,\mu_{n-1})$ of non-negative integers satisfying $\mu_0 + \ldots + \mu_{n-1} = m$, and where $\underline{\tau} + \underline{\mu} = (\tau_0+\mu_0,\ldots,\tau_{n-1}+\mu_{n-1})$. Clearly $\underline{\tau} + \underline{\mu} \in V(T_k)$ and thus, by the induction hypothesis,

$$\Phi_{\underline{\tau}}^{(m)}(s) = 0,\quad s = 0,\ldots,S_k-1,\quad \underline{\tau} \in V(T_{k+1}),\ m \leq [T/2K].$$

By 2.8 we have, for $\underline{\tau} \in V(T_{k+1})$,

$$\max_{|z|\leq S_{k+1}} |\Phi_{\underline{\tau}}(z)| \leq 2 \max_{|z|\leq 3S_{k+1}\log A} |\Phi_{\underline{\tau}}(z)| \left(\frac{2}{3\log A}\right)^{S_k[T/2K]}. \tag{5.8}$$

Here

$$\max_{|z|\leq 3S_{k+1}\log A} |\Phi_{\underline{\tau}}(z)| \leq$$
$$c_{11}^{L_0+LS_{k+1}\log A + T_{k+1}}(L_0+1)\ldots(L_n+1)PL_0!(S_{k+1}\log A)^{L_0}(BL)^{T_{k+1}} \leq$$

$$\exp(2^k c_{12}C^{1+1/n}\Omega \log_2^{n+3}H \log_2^{-n-1}A)$$

and

$$\left(\frac{2}{3\log A}\right)^{S_k[T/2K]} \leqq \exp(-2^k c_{13}^{-1} C^{1+2/n} \Omega \log_2^{n+3} H \log_2^{-n-1} A);$$

therefore, if we choose C so large that $c_{13}^{-1} C^{1/2n} > c_{12}$, substitution in (5.8) shows that

$$\max_{\substack{|z| \leqq S_{k+1} \\ \underline{\tau} \in V(T_{k+1}),}} |\Phi_{\underline{\tau}}(z)| \leqq \exp(-2^k c_{14}^{-1} C^{1+3/2n} \Omega \log_2^{n+3} H \log_2^{-n-1} A),$$

and thus

$$(5.9) \qquad |\Phi_{\underline{\tau}}(s)| \leqq \exp(-2^k c_{14}^{-1} C^{1+3/2n} \Omega \log_2^{n+3} H \log_2^{-n-1} A),$$

$$s = 0,\ldots,S_{k+1}-1,\ \underline{\tau} \in V(T_{k+1}).$$

However, $\Phi_{\underline{\tau}}(s)$ is algebraic and for $s = 0,\ldots,S_{k+1}-1$, $\underline{\tau} \in V(T_{k+1})$ we have

$$\mathrm{dg}(\Phi_{\underline{\tau}}(s)) \leqq d^n,$$

$$\mathrm{den}(\Phi_{\underline{\tau}}(s)) \leqq a_1^{L_1 s} \ldots a_n^{L_n s} b_1^{\tau_1} \ldots b_{n-1}^{\tau_{n-1}} \leqq H_1^{L_1 S_{k+1}} \ldots H_n^{L_n S_{k+1}} B^{T_{k+1}} \leqq$$

$$\exp(2^k c_{15} C^{1+1/n} \Omega \log_2^{n+3} H \log_2^{-n-1} A),$$

$$\overline{|\Phi_{\underline{\tau}}(s)|} \leqq c_{16}^{LS_{k+1}+T_{k+1}} (L_0+1)\ldots(L_n+1) P L_0! S_{k+1}^{L_0} (BL)^{T_{k+1}} \times$$

$$H_1^{L_1 S_{k+1}} \ldots H_n^{L_n S_{k+1}} \leqq \exp(2^k c_{17} C^{1+1/n} \Omega \log_2^{n+3} H \log_2^{-n-1} A),$$

so, by 2.6, either $\Phi_{\underline{\tau}}(s) = 0$ or

$$(5.10) \qquad |\Phi_{\underline{\tau}}(s)| \geqq \exp(-2^k c_{18} C^{1+1/n} \Omega \log_2^{n+3} H \log_2^{-n-1} A).$$

Combining (5.9) and (5.10), and choosing C so large that $c_{14}^{-1} C^{1/2n} > c_{18}$, gives $\Phi_{\underline{\tau}}(s) = 0$ for $s = 0,\ldots,S_{k+1}-1$, $\underline{\tau} \in V(T_{k+1})$. This completes the proof of (5.7).

V. Taking $k = K$ in (5.7) yields

$$\Phi_{\underline{\tau}}(s) = 0,\ s = 0,\ldots,S_K-1,\ \underline{\tau} \in V(T_K).$$

Substitution in (5.5) shows that

$$\Phi^{(t)}(s) = 0,\ s = 0,\ldots,S_K-1,\ t = 0,\ldots,T_K-1.$$

Thus the number of zeros of Φ in $W := \{z \in \mathbb{C}\colon |z| \leqq S_K-1\}$ is at least

$$S_K T_K \geqq 2^{[n^2\log_2 H/\log 2]} ST \geqq \tfrac{1}{2} ST \log^{n^2} H \geqq$$

$$\tfrac{1}{4} d^{-1} C^{1+2/n} \Omega \log^{n^2} H \log_2^{n+4} H \log_2^{-n-2} A.$$

However, if Φ is not identically zero, from 5.3 we see that the number of zeros of Φ in W does not exceed

$$3(L_0+1)\ldots(L_n+1) + c_{19} S_K L \leqq$$

$$3C^{n+1+1/n}\Omega^n \log_2^{n^2+2n+2} H \log_2^{-n^2-n-1} A + c_{19} 2^{n^2\log_2 H/\log 2 + 1} SL \leqq$$

$$3C^{n+1+1/n}\Omega \log^{n^2-n} H \log_2^{n^2+2n+2} H \log_2^{-n^2-n-1} A +$$

$$c_{20} C^{1+1/n} \Omega \log^{n^2} H \log_2^{n+3} H \log_2^{-n-1} A \leqq$$

$$c_{21} C^{1+1/n} \Omega \log^{n^2} H \log_2^{n+3} H \log_2^{-n-1} A.$$

In this case, comparison of the two estimates for the number of zeros of Φ in W yields

$$\tfrac{1}{4} d^{-1} C^{1+2/n} \Omega \log^{n^2} H \log_2^{n+4} H \log_2^{-n-2} A \leqq$$

$$c_{21} C^{1+1/n} \Omega \log^{n^2} H \log_2^{n+3} H \log_2^{-n-1} A,$$

so

$$C \leqq (4dc_{21}\log_2^{-1} H \log_2 A)^n \leqq c_{22}.$$

If C is sufficiently large, there is a contradiction; thus Φ is identically zero. As the $p(\lambda_0,\ldots,\lambda_n)$ are not all zero, it follows that two of the frequencies of Φ must be equal. This shows that there exist two non-identical n-tuples $(\lambda_1,\ldots,\lambda_n)$ and $(\lambda_1',\ldots,\lambda_n')$ of non-negative integers that satisfy

$$\lambda_1\ell_1(\alpha_1) + \ldots + \lambda_n\ell_n(\alpha_n) = \lambda_1'\ell_1(\alpha_1) + \ldots + \lambda_n'\ell_n(\alpha_n),$$

while

$$\max(\lambda_j,\lambda_j') \leqq L_j.$$

Taking $x_j := \lambda_j - \lambda_j'$ proves the lemma. □

5.5. THEOREM. *Suppose* $\varepsilon > 0$, $d \in \mathbb{N}$, $a_1,\ldots,a_n \in \mathbb{A}\setminus\{0\}$, $b_1,\ldots,b_n \in \mathbb{C}$, $\ell_1(a_1),\ldots,\ell_n(a_n)$ *values of the logarithms of* $a_1,\ldots,a_n$ *respectively such that* $\ell_1(a_1),\ldots,\ell_n(a_n)$ *are linearly independent over* $\mathbb{Q}$. *Put*

$$\underline{a}^{\underline{b}} := \exp(b_1\ell_1(a_1)+\ldots+b_n\ell_n(a_n)).$$

Then there are only finitely many $(n+1)$*-tuples* $(\beta_1,\ldots,\beta_n,\gamma) \in \mathbb{A}_d^{n+1}$ *for which*

$$\max(|b_1-\beta_1|,\ldots,|b_n-\beta_n|,|\underline{a}^{\underline{b}}-\gamma|) < \exp(-\log^2 H \log_2^{\varepsilon} H), \tag{5.11}$$

where $H = \max(2,h(\beta_1),\ldots,h(\beta_n),h(\gamma))$.

PROOF. I. Suppose the assertion of the theorem to be false; from this a contradiction will be derived. Let $(\beta_1,\ldots,\beta_n,\gamma)$ be an $(n+1)$-tuple satisfying (5.11). By $c_1,c_2,\ldots$ we shall denote real numbers greater than 1 depending only on $\varepsilon,d,n,a_1,\ldots,a_n,b_1,\ldots,b_n$ and $\ell_1(a_1),\ldots,\ell_n(a_n)$; we assume H to be greater than such a number. Let ℓ be a branch of the logarithm, defined on a disk K, centred at $\underline{a}^{\underline{b}}$, such that $\ell(\underline{a}^{\underline{b}}) = b_1\ell_1(a_1) + \ldots + b_n\ell_n(a_n)$. From (5.11) we deduce that $\gamma \in K$ and that the inequality

$$|\beta_1\ell_1(a_1) + \ldots + \beta_n\ell_n(a_n) - \ell(\gamma)| < \exp(-\log^2 H \log_2^{\varepsilon/2} H)$$

holds. If it were the case that $\beta_1\ell_1(a_1) + \ldots + \beta_n\ell_n(a_n) - \ell(\gamma) \neq 0$, 2.18 would imply

$$|\beta_1\ell_1(a_1) + \dots + \beta_n\ell_n(a_n) - \ell(\gamma)| > \exp(-\log^2 H \log_2^{\varepsilon/2} H),$$

which gives a contradiction. Therefore

$$\beta_1\ell_1(a_1) + \dots + \beta_n\ell_n(a_n) - \ell(\gamma) = 0. \tag{5.12}$$

II. We have now proved that $\ell_1(a_1),\dots,\ell_n(a_n),\ell(\gamma)$ are linearly dependent over $\mathbb{A}$; using 2.19 we find that these numbers must also be linearly dependent over $\mathbb{Q}$. By 5.4, there exist $x_1,\dots,x_{n+1} \in \mathbb{Z}$, not all zero, such that $x_1\ell_1(a_1) + \dots + x_n\ell_n(a_n) + x_{n+1}\ell(\gamma) = 0$, while $|x_{n+1}| \leqq c_1\log_2^{n+2} H$. Adding x_{n+1} times (5.12) gives

$$(x_1+\beta_1 x_{n+1})\ell_1(a_1) + \dots + (x_n+\beta_n x_{n+1})\ell_n(a_n) = 0.$$

III. By the conditions of the theorem $\ell_1(a_1),\dots,\ell_n(a_n)$ are linearly independent over $\mathbb{Q}$; 2.19 states that then $\ell_1(a_1),\dots,\ell_n(a_n)$ are linearly independent over $\mathbb{A}$. Thus for all $j \in \{1,\dots,n\}$ we have $x_j + \beta_j x_{n+1} = 0$. Now $x_{n+1} = 0$ would imply $x_1 = \dots = x_{n+1} = 0$, which we have assumed to be not the case; therefore $x_{n+1} \neq 0$. From this it follows that for all $j \in \{1,\dots,n\}$ we have

$$\beta_j = -\frac{x_j}{x_{n+1}}.$$

By 2.21 this implies $h(\beta_j) \leqq c_2\text{den}(\beta_j) \leqq c_2|x_{n+1}|$, and thus

$$h(\beta_j) \leqq c_3\log_2^{n+2} H, \quad j = 1,\dots,n. \tag{5.13}$$

Apply 5.1 with $\eta_j = \exp(\beta_j\ell_j(a_j))$. We have

$$dg(\eta_j) \leqq dg(a_j)\text{den}(\beta_j) \leqq c_4\log_2^{n+2} H,$$

so, by 3.4,

$$h(\eta_j) \leqq \exp(c_5 dg(\eta_j)h(\beta_j)) \leqq \exp(c_6\log_2^{2n+4} H),$$

and thus

(5.14) $$h(\gamma) \leq \exp(c_7 \log_2^{2n+4} H).$$

However, (5.13) and (5.14) show that

$$H = \max(2, h(\beta_1), \ldots, h(\beta_n), h(\gamma)) \leq \exp(c_8 \log_2^{2n+4} H),$$

which gives a contradiction for sufficiently large H. □

5.6. LEMMA. *Suppose* $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{A} \setminus \{0\}$ *such that* $1, \beta_1, \ldots, \beta_n$ *are linearly independent over* $\mathbb{Q}$, $\ell_1(\alpha_1), \ldots, \ell_n(\alpha_n)$ *non-zero values of the logarithms of* $\alpha_1, \ldots, \alpha_n$ *respectively. Then* $\exp(\beta_1 \ell_1(\alpha_1) + \ldots + \beta_n \ell_n(\alpha_n))$ *is transcendental.*

PROOF. The lemma is proved as Theorem 2 of [Baker 1967] under the extra assumption that none of $\alpha_1, \ldots, \alpha_n$ equals 1. However, this is not used anywhere in the proof. □

5.7. THEOREM. *Suppose* $\varepsilon > 0$, $d \in \mathbb{N}$, $a_1, \ldots, a_n \in \mathbb{A} \setminus \{0\}$, $b_1, \ldots, b_n \in \mathbb{C}$ *such that* $1, b_1, \ldots, b_n$ *are linearly independent over* $\mathbb{Q}$, $\ell_1(a_1), \ldots, \ell_n(a_n)$ *non-zero values of the logarithms of* $a_1, \ldots, a_n$ *respectively. Put*

$$\underline{a}^{\underline{b}} := \exp(b_1 \ell_1(a_1) + \ldots + b_n \ell_n(a_n)).$$

Then there are only finitely many $(n+1)$*-tuples* $(\beta_1, \ldots, \beta_n, \gamma) \in \mathbb{A}_d^{n+1}$ *for which*

(5.15) $$\max(|b_1 - \beta_1|, \ldots, |b_n - \beta_n|, |\underline{a}^{\underline{b}} - \gamma|) < \exp(-\log^2 H \log_2^{\varepsilon} H),$$

where $H = \max(2, h(\beta_1), \ldots, h(\beta_n), h(\gamma))$.

PROOF. In case $n = 1$, the assertion of the theorem follows from 5.5. We now proceed by induction and assume that the assertion has been proved for all natural numbers less than n, while it does not hold for n itself; from this a contradiction will be derived. Let $(\beta_1, \ldots, \beta_n, \gamma)$ be an $(n+1)$-tuple satisfying (5.15). By $c_1, c_2, \ldots$ we shall denote real numbers greater than 1 depending only on $\varepsilon, d, n, a_1, \ldots, a_n, b_1, \ldots, b_n$ and $\ell_1(a_1), \ldots, \ell_n(a_n)$; we suppose H to be greater than such a number. If $\ell_1(a_1), \ldots, \ell_n(a_n)$ are linearly independent over $\mathbb{Q}$, the assertion is proved in 5.5; therefore we may suppose that there exist $\eta_1, \ldots, \eta_n \in \mathbb{Q}$, not all zero, their heights bounded by a constant c_1, such that $\eta_1 \ell_1(a_1) + \ldots + \eta_n \ell_n(a_n) = 0$. It is no restriction to assume $\eta_n \neq 0$; thus we may write

$$\ell_n(a_n) = -\sum_{j=1}^{n-1} \frac{\eta_j}{\eta_n} \ell_j(a_j).$$

If, for $j = 1,\ldots,n-1$, we define

$$(5.16) \qquad b_j' := b_j - \frac{\eta_j}{\eta_n} b_n, \quad \beta_j' := \beta_j - \frac{\eta_j}{\eta_n} \beta_n,$$

we have

$$\prod_{j=1}^{n-1} \exp(b_j' \ell_j(a_j)) = \underline{a}^{\underline{b}}$$

and, for $j = 1,\ldots,n-1$,

$$|b_j' - \beta_j'| \leq |b_j - \beta_j| + \left|\frac{\eta_j}{\eta_n}\right| \cdot |b_n - \beta_n| <$$

$$\left(1+\left|\frac{\eta_j}{\eta_n}\right|\right) \exp(-\log^2 H \log_2^{\varepsilon} H).$$

Furthermore, the numbers $1, b_1', \ldots, b_{n-1}'$ are linearly independent over $\mathbb{Q}$, as may be seen from (5.16). The numbers $\beta_1', \ldots, \beta_{n-1}'$ are algebraic of degree at most d^2, while their heights are bounded by $\exp(c_2 \log H)$ (here 5.1 is used). Thus

$$(5.17) \qquad \max(|b_1'-\beta_1'|, \ldots, |b_{n-1}'-\beta_{n-1}'|, |\underline{a}^{\underline{b}}-\gamma|) < \exp(-\log^2 H' \log_2^{\varepsilon/2} H'),$$

where $H' := \exp(c_2 \log H) \geq \max(h(\beta_1'), \ldots, h(\beta_{n-1}'), h(\gamma))$. From the induction hypothesis it now follows that

$$(5.18) \qquad \max(h(\beta_1'), \ldots, h(\beta_{n-1}'), h(\gamma)) \leq c_3;$$

from (5.17) and (5.18) we see that

$$\beta_1' = b_1', \ \ldots, \ \beta_{n-1}' = b_{n-1}', \ \gamma = \underline{a}^{\underline{b}}.$$

Thus $b_1', \ldots, b_{n-1}'$ and $\underline{a}^{\underline{b}}$ are algebraic; however, by means of 5.6 this leads to a contradiction with the linear independence of $1, b_1', \ldots, b_{n-1}'$. □

From the proof it is clear that the theorem remains valid when the condition that $1,b_1,\ldots,b_n$ be linearly independent over $\mathbb{Q}$ is replaced by the condition $\underline{a}^{\underline{b}} \notin \mathbb{A}_d$.

5.8. THEOREM. *Suppose* $\varepsilon > 0$, $d \in \mathbb{N}$, $a_1,\ldots,a_n \in \mathbb{C}\setminus\{0\}$, $b_1,\ldots,b_n \in \mathbb{A}$ *such that* $1,b_1,\ldots,b_n$ *are linearly independent over* $\mathbb{Q}$, $\ell_1(a_1),\ldots,\ell_n(a_n)$ *non-zero values of the logarithms of* $a_1,\ldots,a_n$ *respectively. Put*

$$\underline{a}^{\underline{b}} := \exp(b_1\ell_1(a_1)+\ldots+b_n\ell_n(a_n)).$$

Then there are only finitely many $(n+1)$*-tuples* $(\alpha_1,\ldots,\alpha_n,\gamma) \in \mathbb{A}_d^{n+1}$ *for which*

$$\max(|a_1-\alpha_1|,\ldots,|a_n-\alpha_n|,|\underline{a}^{\underline{b}}-\gamma|) < \exp(-\log^{n+1}H \log_2^{2+\varepsilon}H), \tag{5.19}$$

where $H = \max(2,h(\alpha_1),\ldots,h(\alpha_n),h(\gamma))$.

PROOF. Suppose the assertion of the theorem to be false; from this a contradiction will be derived. Let $(\alpha_1,\ldots,\alpha_n,\gamma)$ be an $(n+1)$-tuple satisfying (5.19); assume that H is sufficiently large in terms of $\varepsilon,d,n,a_1,\ldots,a_n$, $b_1,\ldots,b_n$ and $\ell_1(a_1),\ldots,\ell_n(a_n)$. Let $\bar{\ell}_1,\ldots,\bar{\ell}_n$ and ℓ be branches of the logarithm, defined on disks $K_1,\ldots,K_n$ and K, centred at $a_1,\ldots,a_n$ and $\underline{a}^{\underline{b}}$ respectively, such that

$$\bar{\ell}_j(a_j) = \ell_j(a_j),\ j = 1,\ldots,n, \tag{5.20}$$

$$\ell(\underline{a}^{\underline{b}}) = b_1\ell_1(a_1) + \ldots + b_n\ell_n(a_n). \tag{5.21}$$

From (5.19), (5.20) and (5.21) we deduce that $\alpha_j \in K_j$ and $\bar{\ell}_j(\alpha_j) \neq 0$ for $j = 1,\ldots,n$, that $\gamma \in K$ and that the inequality

$$|b_1\bar{\ell}_1(\alpha_1) + \ldots + b_n\bar{\ell}_n(\alpha_n) - \ell(\gamma)| < \exp(-\log^{n+1}H \log_2^{2+\varepsilon/2}H)$$

holds. If it were the case that $b_1\bar{\ell}_1(\alpha_1) + \ldots + b_n\bar{\ell}_n(\alpha_n) - \ell(\gamma) \neq 0$, 2.18 would imply

$$|b_1\bar{\ell}_1(\alpha_1) + \ldots + b_n\bar{\ell}_n(\alpha_n) - \ell(\gamma)| > \exp(-\log^{n+1}H \log_2^{2+\varepsilon/2}H),$$

which gives a contradiction. Therefore

$$b_1\bar{\ell}_1(\alpha_1) + \ldots + b_n\bar{\ell}_n(\alpha_n) - \ell(\gamma) = 0$$

and thus $\exp(b_1\bar{\ell}_1(\alpha_1)+\ldots+b_n\bar{\ell}_n(\alpha_n))$ is algebraic. This contradicts 5.6 above. □

5.9. LEMMA. *Suppose* $d \in \mathbb{N}$, $b \in \mathbb{Q}$. *Then there is an effectively computable* $C > 1$, *depending only on* d *and* b, *with the following property: if* $\alpha, \gamma \in \mathbb{A}_d$, *while* $\ell_1(\alpha), \ell_2(\gamma)$ *are values of the logarithms of* α *and* γ *respectively with*

$$(5.22) \qquad b\ell_1(\alpha) - \ell_2(\gamma) = 0,$$

it follows that

$$\log h(\gamma) < C \log \max(2, h(\alpha)).$$

PROOF (cf. 3.4). Let $\alpha, \gamma, \ell_1(\alpha)$ and $\ell_2(\gamma)$ be such that (5.22) is satisfied; put $b = v/w$, where $v \in \mathbb{Z}$, $w \in \mathbb{N}$, $(v,w) = 1$. If $P = \Sigma_{j=0}^{d} a_j X^j$ is the minimal polynomial of γ, the minimal polynomial of $1/\gamma$ is either $P^* := \Sigma_{j=0}^{d} a_{d-j} X^j$ or $-P^*$. Thus $h(\gamma) = h(1/\gamma)$; replacing γ if necessary by $1/\gamma$ and observing that $-\ell_2(\gamma)$ is a value of the logarithm of $1/\gamma$ shows that it is sufficient to prove the lemma in case $v \geqq 0$. We have

$$\exp(w\ell_2(\gamma)) = \exp^w(\ell_2(\gamma)) = \gamma^w$$

and

$$\exp(w\ell_2(\gamma)) = \exp(wb\ell_1(\alpha)) = \exp(v\ell_1(\alpha)) = \exp^v(\ell_1(\alpha)) = \alpha^v;$$

applying 3.1 gives

$$\overline{|\gamma|} = \overline{|\alpha|}^{v/w}$$

and

$$\mathrm{den}(\gamma) \leqq \mathrm{den}^v(\alpha).$$

According to 3.2,

$$h(\gamma) \leq (2\ \mathrm{den}(\gamma)\max(1,\overline{|\gamma|}))^{dg(\gamma)} \leq (2\ \mathrm{den}^v(\alpha)\max(1,\overline{|\alpha|}^{v/w}))^d \leq$$

$$(2h^v(\alpha)(h(\alpha)+1)^{v/w})^d. \quad \square$$

5.10. THEOREM. *Suppose* $\varepsilon > 0$, $d \in \mathbb{N}$, $a_1,\dots,a_n \in \mathbb{C}\setminus\{0\}$, $b_1,\dots,b_n \in \mathbb{A}$, *not all rational,* $\ell_1(a_1),\dots,\ell_n(a_n)$ *values of the logarithms of* $a_1,\dots,a_n$ *respectively such that* $\ell_1(a_1),\dots,\ell_n(a_n)$ *are linearly independent over* $\mathbb{Q}$. *Define*

$$\underline{a}^{\underline{b}} := \exp(b_1\ell_1(a_1)+\dots+b_n\ell_n(a_n)).$$

Then there are only finitely many $(n+1)$*-tuples* $(\alpha_1,\dots,\alpha_n,\gamma) \in \mathbb{A}_d^{n+1}$ *for which*

$$(5.23) \qquad \max(|a_1-\alpha_1|,\dots,|a_n-\alpha_n|,|\underline{a}^{\underline{b}}-\gamma|) < \exp(-\log^{n+1}H \log_2^{2+\varepsilon}H),$$

where $H = \max(2,h(\alpha_1),\dots,h(\alpha_n),h(\gamma))$.

PROOF. I. In case $n = 1$, the assertion of the theorem follows from 5.8. We now proceed by induction and assume that the assertion has been proved for all natural numbers less than n, while it does not hold for n itself; from this a contradiction will be derived. Let $(\alpha_1,\dots,\alpha_n,\gamma)$ be an $(n+1)$-tuple satisfying (5.23). By $c_1,c_2,\dots$ we shall denote real numbers greater than 1 depending only on $\varepsilon,d,n,a_1,\dots,a_n,b_1,\dots,b_n$ and $\ell_1(a_1),\dots,\ell_n(a_n)$; we suppose H to be greater than such a number. If $1,b_1,\dots,b_n$ are linearly independent over $\mathbb{Q}$, the assertion is proved in 5.8; therefore we may suppose that there exist $\eta_0,\dots,\eta_n \in \mathbb{Q}$, not all zero, their heights bounded by a constant c_1, such that $\eta_0 + \eta_1 b_1 + \dots + \eta_n b_n = 0$. As $\eta_1 = \dots = \eta_n = 0$ would imply $\eta_0 = 0$, it is no restriction to assume $\eta_n \neq 0$; thus we may write

$$(5.24) \qquad b_n = -\frac{\eta_0}{\eta_n} - \sum_{j=1}^{n-1} \frac{\eta_j}{\eta_n} b_j.$$

Let ℓ and $\bar{\ell}_n$ be branches of the logarithm, defined on disks K_1 and K_2, centred at $\underline{a}^{\underline{b}}$ and a_n respectively, such that

$$(5.25)\qquad \ell(\underline{a}^{\underline{b}}) = b_1\ell_1(a_1) + \ldots + b_n\ell_n(a_n),$$

$$(5.26)\qquad \bar{\ell}_n(a_n) = \ell_n(a_n).$$

From (5.23) and (5.26) we deduce that $\alpha_n \in K_2$; from (5.23) and (5.25) we deduce that $\gamma \in K_1$ and that the inequality

$$(5.27)\qquad |b_1\ell_1(a_1) + \ldots + b_n\ell_n(a_n) - \ell(\gamma)| < \exp(-\log^{n+1}H \log_2^{2+\varepsilon/2}H)$$

holds. For $j = 1,\ldots,n-1$ we define

$$a_j' := a_j\exp(-\frac{\eta_j}{\eta_n}\ell_n(a_n)),\quad \ell_j^*(a_j') := \ell_j(a_j) - \frac{\eta_j}{\eta_n}\ell_n(a_n),$$

$$(a_j')^{b_j} := \exp(b_j\ell_j^*(a_j')).$$

Then the numbers $\ell_1^*(a_1'),\ldots,\ell_{n-1}^*(a_{n-1}')$ are non-zero values of the logarithms of $a_1',\ldots,a_{n-1}'$ respectively, and are linearly independent over $\mathbb{Q}$. We also define

$$\gamma' := \gamma\exp(\frac{\eta_0}{\eta_n}\bar{\ell}_n(\alpha_n)),\quad \ell^*(\gamma') := \ell(\gamma) + \frac{\eta_0}{\eta_n}\bar{\ell}_n(\alpha_n).$$

Then $\ell^*(\gamma')$ is a value of the logarithm of γ'. Using (5.24) we obtain

$$|(a_1')^{b_1}\ldots(a_{n-1}')^{b_{n-1}} - \gamma'| \leqq$$

$$c_2|b_1\ell_1^*(a_1') + \ldots + b_{n-1}\ell_{n-1}^*(a_{n-1}') - \ell^*(\gamma')| =$$

$$c_2|\sum_{j=1}^{n-1} b_j(\ell_j(a_j)-\frac{\eta_j}{\eta_n}\ell_n(a_n)) - \ell(\gamma) - \frac{\eta_0}{\eta_n}\bar{\ell}_n(\alpha_n)| \leqq$$

$$c_2|\sum_{j=1}^{n-1} b_j(\ell_j(a_j)-\frac{\eta_j}{\eta_n}\ell_n(a_n)) - \frac{\eta_0}{\eta_n}\ell_n(a_n) - \ell(\gamma)| +$$

$$c_2|\frac{\eta_0}{\eta_n}(\ell_n(a_n)-\bar{\ell}_n(\alpha_n))| \leqq$$

$$c_2|b_1\ell_1(a_1) + \ldots + b_n\ell_n(a_n) - \ell(\gamma)| + c_3|a_n - \alpha_n| \leqq$$

$$c_4\exp(-\log^{n+1}H \log_2^{2+\varepsilon/2}H),$$

the last inequality by (5.23) and (5.27). Finally, we define

$$\alpha_j' := \alpha_j \exp(-\frac{\eta_j}{\eta_n}\bar{\ell}_n(\alpha_n)),\ j = 1,\dots,n-1.$$

Then, for $j = 1,\dots,n-1$, we have

$$|a_j' - \alpha_j'| \leqq c_5|a_j - \alpha_j| + c_6|a_n - \alpha_n| \leqq c_7\exp(-\log^{n+1}H \log_2^{2+\varepsilon}H).$$

The numbers $\alpha_1',\dots,\alpha_{n-1}',\gamma'$ are algebraic of degree at most $d^2|\eta_n| \leqq c_8$, while their heights are bounded by $\exp(c_9\log H)$ (here 5.1 and 5.9 are used). Thus

$$(5.28)\qquad \max(|a_1'-\alpha_1'|,\dots,|a_{n-1}'-\alpha_{n-1}'|,|(a_1')^{b_1}\dots(a_{n-1}')^{b_{n-1}}-\gamma'|) <$$

$$\exp(-\log^{n+1}H'\log_2^{2+\varepsilon/2}H'),$$

where $H' := \exp(c_9\log H) \geqq \max(h(\alpha_1'),\dots,h(\alpha_{n-1}'),h(\gamma'))$. From the induction hypothesis it now follows that

$$(5.29)\qquad \max(h(\alpha_1'),\dots,h(\alpha_{n-1}'),h(\gamma')) \leqq c_{10};$$

from (5.28) and (5.29) we see that

$$(5.30)\qquad \alpha_1' = a_1',\ \dots,\ \alpha_{n-1}' = a_{n-1}',\ \gamma' = (a_1')^{b_1}\dots(a_{n-1}')^{b_{n-1}}.$$

II. From (5.30) we see that

$$(5.31)\qquad b_1\ell_1^*(\alpha_1') + \dots + b_{n-1}\ell_{n-1}^*(\alpha_{n-1}') - \ell^*(\gamma') = 0.$$

Thus $\ell_1^*(\alpha_1'),\dots,\ell_{n-1}^*(\alpha_{n-1}'),\ell^*(\gamma')$ are linearly dependent over $\mathbb{A}$; using 2.19 we find that these numbers must also be linearly dependent over $\mathbb{Q}$. In other words, there are $\xi_1,\dots,\xi_n \in \mathbb{Q}$, not all zero, such that $\xi_1\ell_1^*(\alpha_1') + \dots + \xi_{n-1}\ell_{n-1}^*(\alpha_{n-1}') + \xi_n\ell^*(\gamma') = 0$. Adding ξ_n times (5.31) gives

$$(5.32)\qquad (\xi_1+\xi_n b_1)\ell_1^*(\alpha_1') + \dots + (\xi_{n-1}+\xi_n b_{n-1})\ell_{n-1}^*(\alpha_{n-1}') = 0.$$

By (5.30), $\ell_1^*(\alpha_1'),\ldots,\ell_{n-1}^*(\alpha_{n-1}')$ are linearly independent over $\mathbb{Q}$; 2.19 states that then $\ell_1^*(\alpha_1'),\ldots,\ell_{n-1}^*(\alpha_{n-1}')$ are linearly independent over $\mathbb{A}$. Thus for all $j \in \{1,\ldots,n-1\}$ we have $\xi_j + \xi_n b_j = 0$. Now $\xi_n = 0$ would imply $\xi_1 \ell_1^*(\alpha_1') + \ldots + \xi_{n-1}\ell_{n-1}^*(\alpha_{n-1}') = 0$ and thus $\xi_1 = \ldots = \xi_n = 0$, which we have assumed to be not the case; therefore $\xi_n \neq 0$. From this it follows that for all $j \in \{1,\ldots,n-1\}$ we have

(5.33) $$b_j = -\frac{\xi_j}{\xi_n} \in \mathbb{Q}.$$

Substitution in (5.24) gives $b_n \in \mathbb{Q}$, which gives a contradiction with the conditions of the theorem. □

5.11. <u>THEOREM</u>. *Suppose* $\varepsilon > 0$, $d \in \mathbb{N}$, $a_1,\ldots,a_n \in \mathbb{A}\setminus\{0\}$, $b_1,\ldots,b_n \in \mathbb{C}$, *not all rational,* $\ell_1(a_1),\ldots,\ell_n(a_n)$ *non-zero values of the logarithms of* $a_1,\ldots,a_n$ *respectively such that*

(5.34) $1,b_1,\ldots,b_n$ *linearly independent over* $\mathbb{Q}$

or

(5.35) $\ell_1(a_1),\ldots,\ell_n(a_n)$ *linearly independent over* $\mathbb{Q}$.

Define

$$\underline{a}^{\underline{b}} := \exp(b_1\ell_1(a_1)+\ldots+b_n\ell_n(a_n)).$$

Then there are only finitely many $(n+2)$*-tuples* $(\beta_1,\ldots,\beta_n,\gamma,P) \in \mathbb{A}_d^{n+1} \times S_d^{(n+1)}$ *with*

(5.36) $$\max(|b_1-\beta_1|,\ldots,|b_n-\beta_n|,|P(b_1,\ldots,b_n,\underline{a}^{\underline{b}})-\gamma|) < \exp(-\log^2 H \log_2^{\varepsilon} H),$$

where $H = \max(2,h(\beta_1),\ldots,h(\beta_n),h(\gamma),h(P))$.

<u>PROOF</u>. I. Suppose the assertion of the theorem to be false; from this a contradiction will be derived. Let $(\beta_1,\ldots,\beta_n,\gamma,P)$ be an $(n+2)$-tuple satisfying (5.36). By $c_1,c_2,\ldots$ we shall denote real numbers greater than 1

depending only on $\varepsilon, d, n, a_1, \ldots, a_n, b_1, \ldots, b_n$ and $\ell_1(a_1), \ldots, \ell_n(a_n)$; we assume H to be greater than such a number. First we show that $\underline{a}^{\underline{b}}$ must be transcendental. Suppose this number to be algebraic; then 5.5 and 5.7, together with the inequality

$$(5.37) \qquad \max(|b_1-\beta_1|, \ldots, |b_n-\beta_n|, |\underline{a}^{\underline{b}}-\underline{a}^{\underline{b}}|) < \exp(-\log^2 H \log_2^{\varepsilon} H),$$

where $H \geq \max(h(\beta_1), \ldots, h(\beta_n), h(\underline{a}^{\underline{b}}))$, imply

$$\max(h(\beta_1), \ldots, h(\beta_n), h(\underline{a}^{\underline{b}})) < c_1.$$

Together with (5.37) this shows $\beta_j = b_j$ for $j = 1, \ldots, n$. Accordingly, the numbers $b_1, \ldots, b_n$ are algebraic. If (5.34) holds, the transcendency of $\underline{a}^{\underline{b}}$ follows from 5.6. Now suppose that (5.35) holds. Define

$$\ell(\underline{a}^{\underline{b}}) := b_1\ell_1(a_1) + \ldots + b_n\ell_n(a_n);$$

then $\ell(\underline{a}^{\underline{b}})$ is a value of the logarithm of $\underline{a}^{\underline{b}}$ and $\ell_1(a_1), \ldots, \ell_n(a_n), \ell(\underline{a}^{\underline{b}})$ are linearly dependent over $\mathbb{A}$. Using 2.19, we find that these numbers must also be linearly dependent over $\mathbb{Q}$. In other words, there are $\xi_1, \ldots, \xi_{n+1} \in \mathbb{Q}$, not all zero, such that $\xi_1\ell_1(a_1) + \ldots + \xi_n\ell_n(a_n) + \xi_{n+1}\ell(\underline{a}^{\underline{b}}) = 0$. Thus

$$(\xi_1+\xi_{n+1}b_1)\ell_1(a_1) + \ldots + (\xi_n+\xi_{n+1}b_n)\ell_n(a_n) = 0.$$

By (5.35) and 2.19, this implies $\xi_j + \xi_{n+1}b_j = 0$ for all $j \in \{1, \ldots, n\}$. Now $\xi_{n+1} = 0$ would imply $\xi_1 = \ldots = \xi_{n+1} = 0$, which we have assumed to be not the case; therefore $\xi_{n+1} \neq 0$. From this it follows that for all $j \in \{1, \ldots, n\}$ we have

$$b_j = -\frac{\xi_j}{\xi_{n+1}} \in \mathbb{Q},$$

which contradicts the conditions of the theorem.

II. Apply 4.5; this gives the existence of an algebraic number η of degree at most c_2 and height at most $\exp(c_3 \log H)$ satisfying

$$|\underline{a}^{\underline{b}} - \eta| < \exp(-c_4^{-1}\log^2 H \log_2^{\varepsilon} H).$$

As H may be taken arbitarily large, there exist infinitely many (n+1)-tuples $(\beta_1,\dots,\beta_n,\gamma)$ of algebraic numbers of degree at most c_5 satisfying

$$\max(|b_1-\beta_1|,\dots,|b_n-\beta_n|,|\underline{a}^{\underline{b}}-\gamma|) < \exp(-\log^2 H' \log_2^{\varepsilon/2} H'),$$

where $H' \geq \max(2,h(\beta_1),\dots,h(\beta_n),h(\gamma))$. This contradicts 5.5 or 5.7. □

5.12. THEOREM. *Suppose* $\varepsilon > 0$, $d \in \mathbb{N}$, $a_1,\dots,a_n \in \mathbb{C}\setminus\{0\}$, $b_1,\dots,b_n \in \mathbb{A}$, *not all rational,* $\ell_1(a_1),\dots,\ell_n(a_n)$ *non-zero values of the logarithms of* $a_1,\dots,a_n$ *respectively such that* (5.34) *or* (5.35) *holds. Define*

$$\underline{a}^{\underline{b}} := \exp(b_1\ell_1(a_1)+\dots+b_n\ell_n(a_n)).$$

Then there are only finitely many (n+2)*-tuples* $(\alpha_1,\dots,\alpha_n,\gamma,P) \in \mathbb{A}_d^{n+1} \times S_d^{(n+1)}$ *with*

(5.38) $$\max(|a_1-\alpha_1|,\dots,|a_n-\alpha_n|,|P(a_1,\dots,a_n,\underline{a}^{\underline{b}})-\gamma|) < \exp(-\log^{n+1} H \log_2^{2+\varepsilon} H)$$

where $H = \max(2,h(\alpha_1),\dots,h(\alpha_n),h(\gamma),h(P))$.

PROOF. I. Suppose the assertion of the theorem to be false; from this a contradiction will be derived. Let $(\alpha_1,\dots,\alpha_n,\gamma,P)$ be an (n+2)-tuple satisfying (5.38). By $c_1,c_2,\dots$ we shall denote real numbers greater than 1 depending only on $\varepsilon,d,n,a_1,\dots,a_n,b_1,\dots,b_n$ and $\ell_1(a_1),\dots,\ell_n(a_n)$; we assume H to be greater than such a number. First we show that $\underline{a}^{\underline{b}}$ must be transcendental. Suppose this number to be algebraic; then 5.8 and 5.10, together with the inequality

(5.39) $$\max(|a_1-\alpha_1|,\dots,|a_n-\alpha_n|,|\underline{a}^{\underline{b}}-\underline{a}^{\underline{b}}|) < \exp(-\log^{n+1} H \log_2^{2+\varepsilon} H),$$

where $H \geq \max(h(\alpha_1),\dots,h(\alpha_n),h(\underline{a}^{\underline{b}}))$, imply

$$\max(h(\alpha_1),\dots,h(\alpha_n),h(\underline{a}^{\underline{b}})) \leq c_1.$$

Together with (5.39) this shows $\alpha_j = a_j$ for $j = 1,\dots,n$. Accordingly, the numbers $a_1,\dots,a_n$ are algebraic. If (5.34) holds, the transcendency of $\underline{a}^{\underline{b}}$ follows from 5.6. Now suppose that (5.35) holds. Define

$$\ell(\underline{a}^{\underline{b}}) := b_1\ell_1(a_1) + \ldots + b_n\ell_n(a_n);$$

then $\ell(\underline{a}^{\underline{b}})$ is a value of the logarithm of $\underline{a}^{\underline{b}}$ and $\ell_1(a_1),\ldots,\ell_n(a_n),\ell(\underline{a}^{\underline{b}})$ are linearly dependent over $\mathbb{A}$. Using 2.19 we find that these numbers must also be linearly dependent over $\mathbb{Q}$. In other words, there are $\xi_1,\ldots,\xi_{n+1} \in \mathbb{Q}$, not all zero, such that $\xi_1\ell_1(a_1) + \ldots + \xi_n\ell_n(a_n) + \xi_{n+1}\ell(\underline{a}^{\underline{b}}) = 0$. Thus

$$(\xi_1+\xi_{n+1}b_1)\ell_1(a_1) + \ldots + (\xi_n+\xi_{n+1}b_n)\ell_n(a_n) = 0.$$

By (5.35) and 2.19, this implies $\xi_j + \xi_{n+1}b_j = 0$ for all $j \in \{1,\ldots,n\}$. Now $\xi_{n+1} = 0$ would imply $\xi_1 = \ldots = \xi_{n+1} = 0$, which we have assumed to be not the case; therefore $\xi_{n+1} \neq 0$. From this it follows that for all $j \in \{1,\ldots,n\}$ we have

$$b_j = -\frac{\xi_j}{\xi_{n+1}} \in \mathbb{Q},$$

which contradicts the conditions of the theorem.

II. Apply 4.5; this gives the existence of an algebraic number η of degree at most c_2 and height at most $\exp(c_3\log H)$ satisfying

$$|\underline{a}^{\underline{b}} - \eta| < \exp(-c_4^{-1}\log^{n+1}H\,\log_2^{2+\varepsilon}H).$$

As H may be taken arbitrarily large, there exist infinitely many (n+1)-tuples $(\alpha_1,\ldots,\alpha_n,\eta)$ of algebraic numbers of degree at most c_5 satisfying

$$\max(|a_1-\alpha_1|,\ldots,|a_n-\alpha_n|,|\underline{a}^{\underline{b}}-\eta|) < \exp(-\log^{n+1}H'\log_2^{2+\varepsilon/2}H'),$$

where $H' \geqq \max(2,h(\alpha_1),\ldots,h(\alpha_n),h(\eta))$. This contradicts 5.8 or 5.10. □

That a result similar to 5.5, 5.7, 5.8 and 5.10 does not hold without some extra condition if $a_1,\ldots,a_n,b_1,\ldots,b_n$ and $\underline{a}^{\underline{b}}$ are approximated simultaneously, is evident from 1.6. One approach to obtain such a condition was demonstrated by Wüstholz [1976, 19..], who assumed $a_1,\ldots,a_n$ to be U*-numbers with a sufficiently dense sequence of algebraic numbers converging to them; here, however, we shall frame a condition that is closely analogous to the one we found for n = 1. Indeed, in case n = 1, 1.7 states that *(i)* in 5.13 is a sufficient condition; in order to generalize this to arbitrary values of n, it is necessary to bring it into the form *(ii)* in 5.13.

5.13. LEMMA. *For* $b \in \mathbb{C}$ *the following assertions are equivalent:*

(i) $b \notin \mathbb{R}$ *or* $b \in \mathbb{R} \setminus \mathbb{Q}$ *such that the convergents* p_n/q_n *of the continued fraction expansion of* b *satisfy*

$$q_{n+1} << \exp(q_n^3),\ n \to \infty; \tag{5.40}$$

(ii) there exists a $C > 1$ *such that for all* $x_0, x_1 \in \mathbb{Z}$ *that are not both zero we have*

$$|x_0 + x_1 b| \geqq C^{-1}\exp(-|x_1|^3). \tag{5.41}$$

PROOF. I. Suppose *(i)* holds. First we consider the case $b \notin \mathbb{R}$; we have $b = b_1 + b_2 i$ with $b_2 \neq 0$. Now for all $(x_0, x_1) \in \mathbb{Z}^2$ we have

$$|x_0 + x_1 b| = |x_0 + x_1 b_1 + x_1 b_2 i| \geqq |x_1 b_2|$$

and therefore *(ii)* holds.

Now consider the case that $b \in \mathbb{R} \setminus \mathbb{Q}$ such that the convergents p_n/q_n of the continued fraction expansion of b satisfy (5.40). Then there is a number c_1 such that for all n the inequality

$$q_{n+1} \leqq c_1 \exp(q_n^3)$$

holds. Suppose *(ii)* is not true, thus for all $c_2 > 1$ there exist $x_0, x_1 \in \mathbb{Z}$, not both zero, such that

$$|x_0 + x_1 b| < c_2^{-1}\exp(-|x_1|^3). \tag{5.42}$$

If c_2 is chosen large enough, (5.42) implies $x_1 \neq 0$ and

$$|x_0 + x_1 b| < \frac{1}{2|x_1|},$$

and thus, by 2.14, $-x_0/x_1$ is a continued fraction convergent of b, say

$-x_0/x_1 = p_n/q_n$, so $q_n \leqq |x_1|$. Then, by 2.15,

$$|x_0 + x_1 b| = |x_1|\cdot|b - \frac{p_n}{q_n}| \geq |x_1|\frac{1}{q_n(q_n+q_{n+1})} >$$

$$\frac{q_n}{2q_nq_{n+1}} = \frac{1}{2q_{n+1}} \geqq \frac{1}{2c_1}\exp(-q_n^3) \geqq \frac{1}{2c_1}\exp(-|x_1|^3).$$

If c_2 is chosen sufficiently large in terms of c_1, this contradicts (5.42).

II. Suppose that *(ii)* holds. If $b \in \mathbb{Q}$, we immediately get a contradiction; if $b \notin \mathbb{R}$, there is nothing left to prove. Thus we may assume that $b \in \mathbb{R}\setminus\mathbb{Q}$. Let p_n/q_n denote the convergents of the continued fraction expansion of b and suppose that (5.40) does not hold, i.e. for all c_3 there exists an $n \in \mathbb{N}$ such that

$$q_{n+1} \geqq c_3\exp(q_n^3).$$

For such an n we have, by 2.22,

$$|q_n b - p_n| < \frac{1}{q_{n+1}} \leqq c_3^{-1}\exp(-q_n^3),$$

which contradicts (5.41) if c_3 is chosen sufficiently large in terms of C. □

5.14. <u>LEMMA</u>. *Suppose* $d,n \in \mathbb{N}$, K *a compact subset of the complex plane not containing* 0, $\ell_1,\ldots,\ell_n$ *branches of the logarithm, defined on* K. *Then there exist effectively computable* $C,H_0 > 1$, *depending only on* d,n,K *and* $\ell_1,\ldots,\ell_n$, *with the following property. Let* $\alpha_1,\ldots,\alpha_n \in K$ *be algebraic of degree at most* d *and height at most* H, *such that* $\ell_n(\alpha_n)$ *is linearly dependent of* $\ell_1(\alpha_1),\ldots,\ell_{n-1}(\alpha_{n-1})$ *over* $\mathbb{A}$. *If* $H \geqq H_0$, *there exist* $x_1,\ldots,x_n \in \mathbb{Z}$, *not all zero, such that*

$$x_1\ell_1(\alpha_1) + \ldots + x_n\ell_n(\alpha_n) = 0,$$

while $x_n \neq 0$ *and*

$$\max(|x_1|,\ldots,|x_n|) < C\log^{n-1}H\log_2 H.$$

<u>PROOF</u>. Choose a linearly independent subset $\ell_{k_1}(\alpha_{k_1}),\ldots,\ell_{k_m}(\alpha_{k_m})$ of

$\ell_1(\alpha_1),\ldots,\ell_{n-1}(\alpha_{n-1})$ such that all of $\ell_1(\alpha_1),\ldots,\ell_{n-1}(\alpha_{n-1})$ can be expressed as a linear combination of $\ell_{k_1}(\alpha_{k_1}),\ldots,\ell_{k_m}(\alpha_{k_m})$ with algebraic coefficients. As $\ell_n(\alpha_n)$ is linearly dependent of $\ell_1(\alpha_1),\ldots,\ell_{n-1}(\alpha_{n-1})$ over $\mathbb{A}$, $\ell_n(\alpha_n)$ is also linearly dependent of $\ell_{k_1}(\alpha_{k_1}),\ldots,\ell_{k_m}(\alpha_{k_m})$ over $\mathbb{A}$. By 2.19 and 5.4, there exist $y_1,\ldots,y_{m+1} \in \mathbb{Z}$, not all zero, with

$$y_1\ell_{k_1}(\alpha_{k_1}) + \ldots + y_m\ell_{k_m}(\alpha_{k_m}) + y_{m+1}\ell_n(\alpha_n) = 0$$

and $\max(|y_1|,\ldots,|y_{m+1}|) < c \log^m H \log_2 H$, where c depends only on d,m,K and the branches of the logarithm involved. If $y_{m+1} = 0$, this would imply $y_1\ell_{k_1}(\alpha_{k_1}) + \ldots + y_m\ell_{k_m}(\alpha_{k_m}) = 0$ and thus, by the linear independence of $\ell_{k_1}(\alpha_{k_1}),\ldots,\ell_{k_m}(\alpha_{k_m})$, it would follow that $y_1 = \ldots = y_{m+1} = 0$, which we have assumed to be not the case. Therefore $y_{m+1} \neq 0$. □

5.15. THEOREM. *Suppose* $\varepsilon > 0$, $d \in \mathbb{N}$, $a_1,\ldots,a_n \in \mathbb{C}\setminus\{0\}$, $b_1,\ldots,b_n \in \mathbb{C}$, $\ell_1(a_1),\ldots,\ell_n(a_n)$ *non-zero values of the logarithms of* $a_1,\ldots,a_n$ *respectively. Put*

$$\underline{a}^{\underline{b}} := \exp(b_1\ell_1(a_1)+\ldots+b_n\ell_n(a_n)).$$

Suppose that there is a $C > 1$ *such that for all* $x_0,\ldots,x_n \in \mathbb{Z}$, *not all zero, we have*

$$|x_0+x_1b_1+\ldots+x_nb_n| \geq C^{-1}\exp(-X^{(2n+4)/(n^2+n)}\log^{-1-2/(n+1)}X),$$

where $X = \max(2,|x_0|,\ldots,|x_n|)$. *Then there are only finitely many* $(2n+1)$-*tuples* $(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\gamma)$ *of algebraic numbers of degree at most* d *for which*

$$(5.43) \qquad \max(|a_1-\alpha_1|,\ldots,|a_n-\alpha_n|,|b_1-\beta_1|,\ldots,|b_n-\beta_n|,|\underline{a}^{\underline{b}}-\gamma|) <$$
$$\exp(-\log^{n+2}H \log_2^{1+\varepsilon}H),$$

where $H = \max(2,h(\alpha_1),\ldots,h(\alpha_n),h(\beta_1),\ldots,h(\beta_n),h(\gamma))$.

PROOF. I. Suppose the assertion of the theorem to be false; from this a contradiction will be derived. Let $(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\gamma)$ be a $(2n+1)$-tuple satisfying (5.43). By $c_1,c_2,\ldots$ we shall denote real numbers greater than 1 depending only on $\varepsilon, d, n, a_1, \ldots, a_n, b_1, \ldots, b_n, C$ and $\ell_1(a_1), \ldots, \ell_n(a_n)$. Throughout the proof we shall without further mention assume that H is sufficiently large in terms of these numbers.

Let $\bar{\ell}_1, \ldots, \bar{\ell}_n$ and ℓ be branches of the logarithm, defined on disks $K_1, \ldots, K_n$ and K, centred at $a_1, \ldots, a_n$ and $\underline{a}^{\underline{b}}$ respectively, such that

$$(5.44) \qquad \bar{\ell}_j(a_j) = \ell_j(a_j), \; j = 1, \ldots, n,$$

$$(5.45) \qquad \ell(\underline{a}^{\underline{b}}) = b_1\ell_1(a_1) + \ldots + b_n\ell_n(a_n).$$

II. First we show that it does not restrict the generality of the proof to assume that $b_1\ell_1(a_1) + \ldots + b_n\ell_n(a_n) \neq 0$. Indeed, assume that this special case of the theorem has been proved and that $b_1\ell_1(a_1) + \ldots + b_n\ell_n(a_n) = 0$. The conditions of the theorem imply that $n \geq 2$ and $b_n \neq 0$; by (5.43), we may suppose $\beta_n \neq 0$. Define

$$b_j' := -\frac{b_j}{b_n}, \; \beta_j' := -\frac{\beta_j}{\beta_n}, \; a_j^{b_j'} := \exp(b_j'\ell_j(a_j)), \; j = 1, \ldots, n-1,$$

and $\gamma' := \alpha_n$. Then $\beta_1', \ldots, \beta_{n-1}'$ are algebraic of degree at most d^2 and height at most $\exp(c_1 \log H)$, while

$$|b_j' - \beta_j'| \leq c_2 \exp(-\log^{n+2} H \log_2^{1+\varepsilon} H), \; j = 1, \ldots, n-1.$$

Now $b_1'\ell_1(a_1) + \ldots + b_{n-1}'\ell_{n-1}(a_{n-1}) - \ell_n(a_n) = 0$ and therefore

$$|a_1^{b_1'} \ldots a_{n-1}^{b_{n-1}'} - \gamma'| \leq$$

$$c_3|b_1'\ell_1(a_1) + \ldots + b_{n-1}'\ell_{n-1}(a_{n-1}) - \bar{\ell}_n(\alpha_n)| \leq$$

$$c_3|b_1'\ell_1(a_1) + \ldots + b_{n-1}'\ell_{n-1}(a_{n-1}) - \ell_n(a_n)| +$$

$$c_3|\ell_n(a_n) - \bar{\ell}_n(\alpha_n)| = c_3|\ell_n(a_n) - \bar{\ell}_n(\alpha_n)| \leq$$

$$c_4 \exp(-\log^{n+2} H \log_2^{1+\varepsilon} H),$$

the last inequality by (5.43) and (5.44). Thus

$$\max(|a_1-\alpha_1|,\ldots,|a_{n-1}-\alpha_{n-1}|,|b_1'-\beta_1'|,\ldots,|b_{n-1}'-\beta_{n-1}'|,$$

$$|a_1^{b_1'}\ldots a_{n-1}^{b_{n-1}'}-\gamma'|) < \exp(-\log^{n+2}H'\log_2^{1+\varepsilon/2}H'),$$

where $H' := \exp(c_1\log H) \geqq \max(h(\alpha_1),\ldots,h(\alpha_{n-1}),h(\beta_1'),\ldots,h(\beta_{n-1}'),h(\gamma'))$. Furthermore $b_1'\ell_1(a_1) + \ldots + b_{n-1}'\ell_{n-1}(a_{n-1}) = \ell_n(a_n) \neq 0$; and for $x_0,\ldots,x_{n-1} \in \mathbb{Z}$ with $\max(2,|x_0|,\ldots,|x_{n-1}|) = X$ we have

$$|x_0+x_1b_1'+\ldots+x_{n-1}b_{n-1}'| = \frac{1}{|b_n|}|x_0b_n-x_1b_1-\ldots-x_{n-1}b_{n-1}| \geqq$$

$$c_5^{-1}C^{-1}\exp(-X^{(2n+4)/(n^2+n)}\log^{-1-2/(n+1)}X).$$

As

$$\frac{2n+4}{n^2+n} < \frac{2(n-1)+4}{(n-1)^2+n-1},$$

from the special case of the theorem we deduce

$$\max(h(\alpha_1),\ldots,h(\alpha_{n-1}),h(\beta_1'),\ldots,h(\beta_{n-1}'),h(\gamma')) \leqq c_6.$$

Together with (5.43) this shows that

$$a_1 = \alpha_1,\ \ldots,\ a_n = \alpha_n,$$

and thus the numbers $a_1,\ldots,a_n$ are algebraic. From (5.43) and 5.7 we deduce

$$\max(h(\beta_1),\ldots,h(\beta_n),h(\gamma)) \leqq c_7;$$

so it is seen that the general case of the theorem follows from the special one. Henceforth we shall assume that $b_1\ell_1(a_1) + \ldots + b_n\ell_n(a_n) \neq 0$.

III. From (5.43), (5.44) and (5.45) we deduce that $\alpha_j \in K_j$ and $\bar{\ell}_j(\alpha_j) \neq 0$ for $j = 1,\ldots,n$, that $\gamma \in K$ and $\ell(\gamma) \neq 0$ and that the inequality

$$|\beta_1\bar{\ell}_1(\alpha_1) + \ldots + \beta_n\bar{\ell}_n(\alpha_n) - \ell(\gamma)| < \exp(-\log^{n+2}H\log_2^{1+\varepsilon/2}H)$$

holds. If it were the case that $\beta_1\bar{\ell}_1(\alpha_1) + \ldots + \beta_n\bar{\ell}_n(\alpha_n) - \ell(\gamma) \neq 0$, 2.18 would imply

$$|\beta_1\bar{\ell}_1(\alpha_1) + \ldots + \beta_n\bar{\ell}_n(\alpha_n) - \ell(\gamma)| > \exp(-\log^{n+2}H \log_2^{1+\varepsilon/2}H),$$

which gives a contradiction. Therefore

$$(5.46) \qquad \beta_1\bar{\ell}_1(\alpha_1) + \ldots + \beta_n\bar{\ell}_n(\alpha_n) - \ell(\gamma) = 0.$$

It will now be our object to prove the existence of $x_0,\ldots,x_n \in \mathbb{Z}$, not all zero, such that

$$(5.47) \qquad x_0 + x_1\beta_1 + \ldots + x_n\beta_n = 0,$$

while

$$(5.48) \qquad \max(|x_0|,\ldots,|x_n|) \leqq c_8\log^{(n^2+n)/2}H \log_2^n H.$$

In order to do so, we suppose that no non-trivial (n+1)-tuple $(x_0,\ldots,x_n)$ satisfies both (5.47) and (5.48) and proceed by a method of descent. Define E as the set of all $m \in \{1,\ldots,n\}$ with the property that there exist $r_0^{(1)},\ldots,r_n^{(1)},r_0^{(2)},\ldots,r_n^{(2)},\ldots,r_0^{(m)},\ldots,r_n^{(m)} \in \mathbb{Z}$ and a collection $\{k_1,\ldots,k_m\}$ of m different numbers from $\{1,\ldots,n\}$ such that

$$(5.49) \qquad (r_0^{(1)}+r_1^{(1)}\beta_1+\ldots+r_n^{(1)}\beta_n)\bar{\ell}_{k_1}(\alpha_{k_1}) + \ldots +$$
$$(r_0^{(m)}+r_1^{(m)}\beta_1+\ldots+r_n^{(m)}\beta_n)\bar{\ell}_{k_m}(\alpha_{k_m}) = 0;$$

$$(5.50) \qquad \max\{|r_i^{(j)}|: 0 \leqq i \leqq n \wedge 1 \leqq j \leqq m\} \leqq$$
$$c_9\log^{(n^2+n-m^2+m)/2}H \log_2^{n-m+1}H;$$

$$(5.51) \qquad \exists j \in \{1,\ldots,m\}: r_0^{(j)}+r_1^{(j)}\beta_1+\ldots+r_n^{(j)}\beta_n \neq 0;$$

$$(5.52) \qquad r_0^{(1)}\bar{\ell}_{k_1}(\alpha_{k_1}) + \ldots + r_0^{(m)}\bar{\ell}_{k_m}(\alpha_{k_m}) \neq 0.$$

IV. First we investigate whether $n \in E$. From (5.46) and 5.14 it follows that there exist $s_1,\ldots,s_{n+1} \in \mathbb{Z}$ with

$$(5.53)\qquad s_1\bar{\ell}_1(\alpha_1) + \ldots + s_n\bar{\ell}_n(\alpha_n) + s_{n+1}\ell(\gamma) = 0,$$

while $\max(|s_1|,\ldots,|s_{n+1}|) \leqq c_{10}\log^n H \log_2 H$ and $s_{n+1} \neq 0$. Comparison of (5.46) and (5.53) yields

$$(s_1+s_{n+1}\beta_1)\bar{\ell}_1(\alpha_1) + \ldots + (s_n+s_{n+1}\beta_n)\bar{\ell}_n(\alpha_n) = 0.$$

Thus (5.49) and (5.50) are satisfied with $m = n$; as $s_{n+1} \neq 0$ and $\ell(\gamma) \neq 0$, the condition corresponding to (5.52) is also fulfilled. To prove that $n \in E$ it is now sufficient to show that (5.51) holds. However, if this were not the case, we would have $s_1 + s_{n+1}\beta_1 = 0$ and (5.47) and (5.48) would follow. Thus $n \in E$.

V. Now let $m \in \{2,\ldots,n\}$ be such that $m \in E$; then, in particular, the logarithms $\bar{\ell}_{k_1}(\alpha_{k_1}),\ldots,\bar{\ell}_{k_m}(\alpha_{k_m})$ occurring in (5.49) are linearly dependent over $\mathbb{A}$. According to 2.19 and 5.4, there exist $t_1,\ldots,t_m \in \mathbb{Z}$, not all zero, with

$$(5.54)\qquad t_1\bar{\ell}_{k_1}(\alpha_{k_1}) + \ldots + t_m\bar{\ell}_{k_m}(\alpha_{k_m}) = 0$$

and

$$(5.55)\qquad \max(|t_1|,\ldots,|t_m|) \leqq c_{11}\log^{m-1} H \log_2 H.$$

It is no restriction to assume $t_m \neq 0$. Comparison of (5.49) and (5.54) yields $u_1\bar{\ell}_{k_1}(\alpha_{k_1}) + \ldots + u_{m-1}\bar{\ell}_{k_{m-1}}(\alpha_{k_{m-1}}) = 0$, where

$$u_j = r_0^{(j)}t_m + r_1^{(j)}t_m\beta_1 + \ldots + r_n^{(j)}t_m\beta_n -$$

$$r_0^{(m)}t_j - r_1^{(m)}t_j\beta_1 - \ldots - r_n^{(m)}t_j\beta_n,\; j = 1,\ldots,m-1.$$

Now there are two possibilities: either $u_1,\ldots,u_{m-1}$ are all zero, or at least one of them is not zero. In the first case we get $m - 1$ vanishing linear combinations of $1,\beta_1,\ldots,\beta_n$, the absolute values of whose coefficients are bounded by

$$\max\{|r_i^{(j)}t_m - r_i^{(m)}t_j| : 0 \leqq i \leqq n \wedge 1 \leqq j \leqq m-1\},$$

which, by (5.50) and (5.55), is less than

$$c_{12}\log^{(n^2+n-m^2+m)/2+m-1}H\ \log_2^{n-m+1+1}H =$$

$$c_{12}\log^{(n^2+n-(m-1)^2+(m-1))/2}H\ \log_2^{n-(m-1)+1}H.$$

As $(n^2+n-(m-1)^2+(m-1))/2 \leq (n^2+n)/2$ and $n-(m-1)+1 \leq n$, we have proved (5.47) and (5.48), unless all the coefficients in all of these linear combinations are zero, i.e.

$$\forall i \in \{0,\dots,n\}\ \forall j \in \{1,\dots,m-1\}: r_i^{(j)}t_m - r_i^{(m)}t_j = 0.$$

But this would imply

$$r_0^{(1)}\bar{\ell}_{k_1}(\alpha_{k_1}) + \dots + r_0^{(m)}\bar{\ell}_{k_m}(\alpha_{k_m}) =$$

$$\frac{t_1}{t_m} r_0^{(m)}\bar{\ell}_{k_1}(\alpha_{k_1}) + \dots + \frac{t_{m-1}}{t_m} r_0^{(m)}\bar{\ell}_{k_{m-1}}(\alpha_{k_{m-1}}) + r_0^{(m)}\bar{\ell}_{k_m}(\alpha_{k_m}) =$$

$$\frac{r_0^{(m)}}{t_m}(t_1\bar{\ell}_{k_1}(\alpha_{k_1})+\dots+t_m\bar{\ell}_{k_m}(\alpha_{k_m})) = 0$$

by (5.54), which gives a contradiction with (5.52).

Thus there only remains the second case, where

$$\exists j \in \{1,\dots,m-1\}: r_0^{(j)}t_m + r_1^{(j)}t_m\beta_1 + \dots + r_n^{(j)}t_m\beta_n -$$

$$r_0^{(m)}t_j - r_1^{(m)}t_j\beta_1 - \dots - r_n^{(m)}t_j\beta_n \neq 0,$$

so the conditions corresponding to (5.49), (5.50) and (5.51) hold for m - 1. To prove the statement corresponding to (5.52) we note that

$$(r_0^{(1)}t_m-r_0^{(m)}t_1)\bar{\ell}_{k_1}(\alpha_{k_1}) + \dots + (r_0^{(m-1)}t_m-r_0^{(m)}t_{m-1})\bar{\ell}_{k_{m-1}}(\alpha_{k_{m-1}}) =$$

$$t_m(r_0^{(1)}\bar{\ell}_{k_1}(\alpha_{k_1})+\dots+r_0^{(m-1)}\bar{\ell}_{k_{m-1}}(\alpha_{k_{m-1}})) -$$

$$r_0^{(m)}(t_1\bar{\ell}_{k_1}(\alpha_{k_1})+\dots+t_{m-1}\bar{\ell}_{k_{m-1}}(\alpha_{k_{m-1}})) =$$

$$t_m(r_0^{(1)}\bar{\ell}_{k_1}(\alpha_{k_1})+\dots+r_0^{(m)}\bar{\ell}_{k_m}(\alpha_{k_m})) -$$

$$r_0^{(m)}(t_1\bar{\ell}_{k_1}(\alpha_{k_1})+\ldots+t_m\bar{\ell}_{k_m}(\alpha_{k_m})) =$$

$$t_m(r_0^{(1)}\bar{\ell}_{k_1}(\alpha_{k_1})+\ldots+r_0^{(m)}\bar{\ell}_{k_m}(\alpha_{k_m})) \neq 0;$$

here (5.52), (5.54) and the fact that $t_m \neq 0$ are used. This proves that $m - 1 \in E$.

VI. Repeating this process, we deduce that $1 \in E$. As the $\bar{\ell}_j(\alpha_j)$ are non-zero, this contradicts (5.49) and (5.51). Thus our assumption that no non-trivial (n+1)-tuple $(x_0,\ldots,x_n)$ satisfies (5.47) and (5.48) was erroneous.

Let $(x_0,\ldots,x_n) \in \mathbb{Z}^{n+1}$ be a non-trivial solution of (5.47) and (5.48). Then

$$|x_0 + x_1b_1 + \ldots + x_nb_n| \leqq$$

$$|x_0 + x_1\beta_1 + \ldots + x_n\beta_n| + \max(|x_1|,\ldots,|x_n|) \times$$

$$\max(|b_1-\beta_1|,\ldots,|b_n-\beta_n|) <$$

$$c_8\log^{(n^2+n)/2}H \log_2^n H \exp(-\log^{n+2}H \log_2^{1+\varepsilon}H) \leqq$$

$$\exp(-\log^{n+2}H \log_2^{1+\varepsilon/2}H).$$

Define

$$X := \max(\exp((n^2+3n)/(2n+4)),|x_0|,\ldots,|x_n|).$$

As the function

$$x \mapsto x^{(2n+4)/(n^2+n)}\log^{-1-2/(n+1)}x$$

is increasing for $x \geqq \exp((n^2+3n)/(2n+4))$, (5.48) shows that

$$X^{(2n+4)/(n^2+n)}\log^{-1-2/(n+1)}X \leqq c_{13}\log^{n+2}H \log_2 H$$

and thus

$$|x_0 + x_1b_1 + \dots + x_nb_n| \leqq$$

$$\exp(-c_{13}^{-1}X^{(2n+4)/(n^2+n)}\log^{-1-2/(n+1)}X \log_2^{\varepsilon/2}H) <$$

$$C^{-1}\exp(-X^{(2n+4)/(n^2+n)}\log^{-1-2/(n+1)}X),$$

if H is sufficiently large. This contradicts the conditions of the theorem. □

5.16. THEOREM. *Suppose* $\varepsilon > 0$, $d \in \mathbb{N}$, $a_1,\dots,a_n \in \mathbb{C}\setminus\{0\}$, $b_1,\dots,b_n \in \mathbb{C}$, $\ell_1(a_1),\dots,\ell_n(a_n)$ *non-zero values of the logarithms of* $a_1,\dots,a_n$ *respectively. Put*

$$\underline{a}^{\underline{b}} := \exp(b_1\ell_1(a_1)+\dots+b_n\ell_n(a_n)).$$

Suppose that there is a $C > 1$ *such that for all* $x_0,\dots,x_n \in \mathbb{Z}$, *not all zero, we have*

$$(5.56)\qquad |x_0+x_1b_1+\dots+x_nb_n| \geqq C^{-1}\exp(-X^{(2n+4)/(n^2+n)}\log^{-1-2/(n+1)}X),$$

where $X = \max(2,|x_0|,\dots,|x_n|)$. *Then there are only finitely many* $(2n+2)$*-tuples* $(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n,\gamma,P) \in \mathbb{A}_d^{2n+1} \times S_d^{(2n+1)}$ *such that*

$$(5.57)\qquad \max(|a_1-\alpha_1|,\dots,|a_n-\alpha_n|,|b_1-\beta_1|,\dots,|b_n-\beta_n|,$$

$$|P(a_1,\dots,a_n,b_1,\dots,b_n,\underline{a}^{\underline{b}})-\gamma|) < \exp(-\log^{n+2}H \log_2^{1+\varepsilon}H),$$

where $H = \max(2,h(\alpha_1),\dots,h(\alpha_n),h(\beta_1),\dots,h(\beta_n),h(\gamma),h(P))$.

PROOF. I. Suppose the assertion of the theorem to be false; from this a contradiction will be derived. Let $(\alpha_1,\dots,\alpha_n,\beta_1,\dots,\beta_n,\gamma,P)$ be a $(2n+2)$-tuple satisfying (5.57). By $c_1,c_2,\dots$ we shall denote real numbers greater than 1 depending only on $\varepsilon,d,n,a_1,\dots,a_n,b_1,\dots,b_n$ and $\ell_1(a_1),\dots,\ell_n(a_n)$; we assume H to be greater than such a number.

First we show that $\underline{a}^{\underline{b}}$ must be transcendental. Suppose this number to be algebraic; then 5.15, together with (5.56) and the inequality

$$(5.58)\qquad \max(|a_1-\alpha_1|,\dots,|a_n-\alpha_n|,|b_1-\beta_1|,\dots,|b_n-\beta_n|,|\underline{a}^{\underline{b}}-\underline{a}^{\underline{b}}|) <$$

$$\exp(-\log^{n+2} H \log_2^{1+\varepsilon} H),$$

where $H \geqq \max(h(\alpha_1),\ldots,h(\alpha_n),h(\beta_1),\ldots,h(\beta_n),h(\underline{a}^{\underline{b}}))$, implies

$$\max(h(\alpha_1),\ldots,h(\alpha_n),h(\beta_1),\ldots,h(\beta_n),h(\underline{a}^{\underline{b}})) < c_1.$$

Together with (5.58) this shows $\alpha_j = a_j$ and $\beta_j = b_j$ for $j = 1,\ldots,n$. Accordingly, the numbers $a_1,\ldots,a_n,b_1,\ldots,b_n$ are algebraic. As a consequence of (5.56), moreover, $1,b_1,\ldots,b_n$ are linearly independent over $\mathbb{Q}$; the transcendency of $\underline{a}^{\underline{b}}$ now follows from 5.6.

II. Apply 4.5; this gives the existence of an algebraic number η of degree at most c_2 and height at most $\exp(c_3 \log H)$ satisfying

$$|\underline{a}^{\underline{b}} - \eta| < \exp(-c_4^{-1} \log^{n+2} H \log_2^{1+\varepsilon} H).$$

As H may be taken arbitrarily large, there exist infinitely many $(2n+1)$-tuples $(\alpha_1,\ldots,\alpha_n,\beta_1,\ldots,\beta_n,\gamma)$ of algebraic numbers of degree at most c_5 satisfying

$$\max(|a_1-\alpha_1|,\ldots,|a_n-\alpha_n|,|b_1-\beta_1|,\ldots,|b_n-\beta_n|,|\underline{a}^{\underline{b}}-\eta|) <$$

$$\exp(-\log^{n+2} H' \log_2^{1+\varepsilon/2} H'),$$

where $H' \geqq \max(2,h(\alpha_1),\ldots,h(\alpha_n),h(\beta_1),\ldots,h(\beta_n),h(\gamma))$. This contradicts 5.15. $\square$

6. THE p-ADIC CASE

6.1. <u>DEFINITION</u>. *(a)* Let F be a field. A function $v: F \to \mathbb{R}$ is called a *non-archimedean valuation* of F if it satisfies the following requirements:

(i) $\forall \eta \in F$: $v(\eta) \geq 0$;
(ii) $\forall \eta \in F$: $v(\eta) = 0 \Leftrightarrow \eta = 0$;
(iii) $\forall \eta, \eta' \in F$: $v(\eta\eta') = v(\eta)v(\eta')$;
(iv) $\forall \eta, \eta' \in F$: $v(\eta+\eta') \leq \max(v(\eta), v(\eta'))$.

(b) Let F be a field, v a non-archimedean valuation of F. The function $(\eta,\eta') \mapsto v(\eta-\eta')$ is called the *distance function* induced by v.

A trivial consequence of *(i)-(iv)*, which we shall frequently use, is that for all $\eta, \eta' \in F$ such that $v(\eta) \neq v(\eta')$ we have $v(\eta+\eta') = \max(v(\eta), v(\eta'))$.

The distance function induced by v makes it possible to consider F as a metric space. Thus one can ask whether or not F is complete, and if not, construct the completion of F.

6.2. <u>LEMMA</u>. *(a) Let F be a field, v a non-archimedean valuation of F, F' the completion of F with respect to the distance function induced by v. Then there exists exactly one non-archimedean valuation w of F' such that $w|F = v$; and F' is complete with respect to the distance function induced by w.*

(b) Let F be a field, v a non-archimedean valuation of F; suppose that F is complete with respect to the distance function induced by v. Let F' be the algebraic closure of F. Then there exists exactly one non-archimedean valuation w of F' such that $w|F = v$.

(c) Let p be a prime number. For $\xi \in \mathbb{Q}$ we define $v_p(\xi) := p^{-m}$, where $\xi = p^m \frac{x}{y}$, $m \in \mathbb{Z}$, $x \in \mathbb{Z}$, $y \in \mathbb{N}$, $p \nmid x$, $p \nmid y$. Then the function v_p is a non-archimedean valuation of $\mathbb{Q}$.

(d) Let p be a prime number, v_p the non-archimedean valuation of $\mathbb{Q}$ defined in (c). Let w_p be the unique non-archimedean valuation of the algebraic closure of the completion of $\mathbb{Q}$ with respect to the distance function induced by v_p satisfying $w_p|\mathbb{Q} = v_p$. Then the completion of this field with respect to the distance function induced by w_p is algebraically closed.

PROOF. *(a)* [Amice 1975], Proposition 2.3.1. *(b)* [Amice 1975], Proposition 2.6.10. *(c)* Trivial. *(d)* [Amice 1975], Corollaire 2.7.3. □

From now on p will denote a fixed prime number; the purpose of this chapter is to prove an analogue to 1.7 in which the complex numbers have been replaced by the algebraically closed, complete field described in 6.2*(d)*.

6.3. DEFINITION. *(a)* Let v_p be the non-archimedian valuation of $\mathbb{Q}$ defined in 6.2*(c)*, w_p the unique non-archimedian valuation of the algebraic closure of the completion of $\mathbb{Q}$ with respect to the distance function induced by v_p satisfying $w_p|\mathbb{Q} = v_p$. Then the completion of this field with respect to the distance function induced by w_p is denoted by $\mathbb{C}_p$.

(b) Let v_p be the non-archimedian valuation of $\mathbb{Q}$ defined in 6.2*(c)*; take $\eta \in \mathbb{C}_p$. Then $|\eta|_p$, also called the *p-adic valuation* of η, denotes the value at η of the unique non-archimedian valuation of $\mathbb{C}_p$ satisfying $|\xi|_p = v_p(\xi)$ for all $\xi \in \mathbb{Q}$.

(c) For $a \in \mathbb{C}_p$, $R \in \mathbb{R}$, $R \geqq 0$ we write

$$\mathcal{B}(a,R) := \{z \in \mathbb{C}_p : |z - a|_p < R\},$$

$$\overline{\mathcal{B}}(a,R) := \{z \in \mathbb{C}_p : |z - a|_p \leqq R\}.$$

(d) If $f = \sum_{k=0}^{\infty} a_k X^k$ is a power series with coefficients in $\mathbb{C}_p$, while $R \in \mathbb{R}$, $R \geqq 0$, we denote

$$|f|_R := \sup\{|a_k|_p R^k : k \in \mathbb{N} \cup \{0\}\}.$$

6.4. LEMMA. *Let* M,R *be positive real numbers. Consider* $f(z) = \sum_{k=0}^{\infty} a_k z^k$, *where* $a_0, a_1, \ldots \in \mathbb{C}_p$*; suspose the power series converges for all* $z \in \mathcal{B}(0,R)$. *Let* $r \in \mathcal{B}(0,R)$ *be such that* $|f(z)|_p < M$ *for all* $z \in \mathbb{C}_p$ *with* $|z|_p = |r|_p$. *Then*

$$\left|\frac{f^{(n)}(z)}{n!}\right|_p < \frac{M}{|r|_p^n},\ n \in \mathbb{N} \cup \{0\},\ z \in \mathcal{B}(0,|r|_p).$$

PROOF. [Adams 1966], Appendix, Theorem 9. □

6.5. LEMMA. *Let* R *be a positive real number. Consider* $f(z) = \Sigma_{k=0}^{\infty} a_k z^k$, *where* $a_0, a_1, \ldots \in \mathbb{C}_p$*; suppose the power series converges for all* $z \in \mathcal{B}(0,R)$. *Take* $r \in \mathcal{B}(0,R)$ *and put* $R' := |r|_p$. *Then* $|f|_{R'}$ *is finite and*

$$\sup_{|z|_p \leq R'} |f(z)|_p = |f|_{R'}. \tag{6.1}$$

PROOF. As $|r|_p < R$, the series $\Sigma_{k=0}^{\infty} a_k r^k$ converges, and consequently $|a_k|_p |r|_p^k$ takes a maximum for some k; thus the right hand member of (6.1) is finite. If $|z|_p \leq |r|_p$, we have

$$|f(z)|_p = \left|\sum_{k=0}^{\infty} a_k z^k\right|_p \leq \max_{k \geq 0} |a_k|_p |r|_p^k;$$

so

$$\sup_{|z|_p \leq |r|_p} |f(z)|_p \leq \max_{k \geq 0} |a_k|_p |r|_p^k.$$

In particular, the left hand member is finite.

Application of 6.4 shows that for every $k \in \mathbb{N} \cup \{0\}$ and every $\varepsilon > 0$,

$$|a_k|_p |r|_p^k = \left|\frac{f^{(k)}(0)}{k!}\right| |r|_p^k < \sup_{|z|_p \leq |r|_p} |f(z)|_p + \varepsilon.$$

Therefore

$$\max_{k \geq 0} |a_k|_p |r|_p^k \leq \sup_{|z|_p \leq |r|_p} |f(z)|_p,$$

which proves the lemma. □

6.6. LEMMA. *Let* R *be a positive real number,* $a \in \mathbb{C}_p$. *Consider* $f(z) = \Sigma_{k=0}^{\infty} a_k (z-a)^k$, *where* $a_0, a_1, \ldots \in \mathbb{C}_p$*; suppose the power series converges for all* $z \in \mathcal{B}(a,R)$. *Take* $b \in \mathcal{B}(a,R)$. *Then*

$$f(z) = \sum_{k=0}^{\infty} b_k (z-b)^k, \quad z \in \mathcal{B}(a,R),$$

where

$$b_k = \sum_{n=k}^{\infty} a_k \binom{n}{k} (b-a)^{n-k}, \quad k \in \mathbb{N} \cup \{0\}.$$

PROOF. The proof is based on the obvious fact that a series in $\mathbb{C}_p$ satisfies Cauchy's condition if and only if the general term tends to zero, and therefore convergence is conserved under rearrangement of terms. For $z \in \mathcal{B}(a,R)$ we have

$$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n = \sum_{n=0}^{\infty} a_n(z-b+b-a)^n =$$
$$\sum_{n=0}^{\infty} a_n \sum_{k=0}^{n} \binom{n}{k}(z-b)^k(b-a)^{n-k} = \sum_{k=0}^{\infty} (z-b)^k \sum_{n=k}^{\infty} a_n\binom{n}{k}(b-a)^{n-k} =$$
$$\sum_{k=0}^{\infty} b_k(z-b)^k. \quad \square$$

6.7. <u>LEMMA</u>. *Let* R *be a positive real number,* $a \in \mathbb{C}_p$. *Consider* $f(z) = \Sigma_{k=0}^{\infty} a_k(z-a)^k$, *where* $a_0, a_1, \ldots \in \mathbb{C}_p$; *suppose the power series converges for all* $z \in \mathcal{B}(a,R)$. *Take* $b \in \mathcal{B}(a,R)$ *and suppose that* $f(b) = 0$. *Then there exist* $b_0, b_1, \ldots \in \mathbb{C}_p$ *such that*

$$f(z) = (z-b) \sum_{k=0}^{\infty} b_k(z-a)^k, \quad z \in \mathcal{B}(a,R),$$

and

$$|b_k|_p \leq \sup_{n\geq k+1} |a_n|_p |b-a|_p^{n-k-1}, \quad k \in \mathbb{N} \cup \{0\}.$$

<u>PROOF</u>. Applying 6.6 twice, we see that for $z \in \mathcal{B}(a,R)$ we have

$$f(z) = \sum_{k=0}^{\infty} a_k(z-a)^k = \sum_{k=0}^{\infty} (z-b)^k \sum_{n=k}^{\infty} a_n\binom{n}{k}(b-a)^{n-k} =$$
$$\sum_{k=1}^{\infty} (z-b)^k \sum_{n=k}^{\infty} a_n\binom{n}{k}(b-a)^{n-k} =$$
$$(z-b) \sum_{k=0}^{\infty} (z-b)^k \sum_{n=k+1}^{\infty} a_n\binom{n}{k+1}(b-a)^{n-k-1} =$$
$$(z-b) \sum_{k=0}^{\infty} (z-a)^k \sum_{j=k}^{\infty} \sum_{n=j+1}^{\infty} a_n\binom{n}{j+1}(b-a)^{n-j-1}\binom{j}{k}(a-b)^{j-k} =$$
$$(z-b) \sum_{k=0}^{\infty} b_k(z-a)^k,$$

where

$$b_k = \sum_{j=k}^{\infty} \sum_{n=j+1}^{\infty} a_n\binom{n}{j+1}(b-a)^{n-j-1}\binom{j}{k}(a-b)^{j-k} =$$

$$\sum_{n=k+1}^{\infty} a_n(b-a)^{n-k-1} \sum_{j=k}^{n-1} (-1)^{j-k}\binom{j}{k}\binom{n}{j+1},$$

so

$$|b_k|_p \leqq \sup_{n\geqq k+1} |a_n|_p |b-a|_p^{n-k-1}. \quad \square$$

6.8. LEMMA. *Consider the power series* $f = \sum_{k=0}^{\infty} a_k X^k$ *with coefficients in* $\mathbb{C}_p$. *Take* $0 < R' < R$ *and suppose* $|f|_R < \infty$. *Then* f *converges on* $\overline{B}(0,R')$ *to a function, which we shall again denote by* f. *If* f *has at least* h *zeros on* $\overline{B}(0,R')$, *multiplicities included, we have*

$$|f|_{R'} \leqq \left(\frac{R'}{R}\right)^h |f|_R.$$

PROOF. The first assertion of the lemma is trivial; we now proceed to prove the second one. Let $\alpha \in \overline{B}(0,R')$ be such that $f(\alpha) = 0$. By 6.7, we may write

$$f(z) = (z-\alpha) \sum_{k=0}^{\infty} b_k z^k, \; z \in \overline{B}(0,R'),$$

with

$$|b_k|_p \leqq \sup_{n\geqq k+1} |a_n|_p |\alpha|_p^{n-k-1} \leqq \sup_{n\geqq k+1} |a_n|_p R^{n-k-1};$$

thus

$$\sup_{k\geqq 0} |b_k|_p R^k \leqq \sup_{k\geqq 0} \sup_{n\geqq k+1} |a_n|_p R^{n-1} = \frac{1}{R} \sup_{n\geqq 1} |a_n|_p R^n \leqq \frac{1}{R} \sup_{n\geqq 0} |a_n|_p R^n = \frac{1}{R}|f|_R.$$

Define $g := \sum_{k=0}^{\infty} b_k X^k$; then

$$(6.2) \qquad |g|_R = \sup_{k\geqq 0} |b_k|_p R^k \leqq \frac{1}{R}|f|_R < \infty.$$

On the other hand, for $z \in \overline{B}(0,R')$ we have $f(z) = (z-\alpha)g(z)$; from the unicity of power series development we conclude

$$a_k = b_{k-1} - \alpha b_k, \; k \geqq 0,$$

where $b_{-1} := 0$. Thus

$$|a_k|_p \leqq \max(|b_{k-1}|_p, |\alpha|_p|b_k|_p), \ k \geqq 0,$$

and

$$|f|_R = \sup_{k\geqq 0} |a_k|_p R^k \leqq \sup_{k\geqq 0} R^k \max(|b_{k-1}|_p, |\alpha|_p|b_k|_p) =$$

$$\max(\sup_{k\geqq 0} |b_{k-1}|_p R^k, \ |\alpha|_p \sup_{k\geqq 0} |b_k|_p R^k) =$$

$$\max(\sup_{k\geqq 0} |b_k|_p R^{k+1}, \ |\alpha|_p \sup_{k\geqq 0} |b_k|_p R^k) =$$

$$\max(R|g|_R, |\alpha|_p|g|_R) = R|g|_R,$$

so

$$(6.3) \qquad |f|_R \leqq R|g|_R.$$

From (6.2) and (6.3) we see that $f = (X-\alpha)g$ with

$$|g|_R = \frac{1}{R}|f|_R < \infty.$$

Similarly it is proved that

$$|g|_{R'} = \frac{1}{R'}|f|_{R'}.$$

If $\alpha_1, \ldots, \alpha_h$ are the zeros of f on $\overline{B}(0,R')$, repeating the argument gives

$$f = \prod_{j=1}^{h} (X-\alpha_j)\cdot f^*,$$

with

$$|f^*|_R = \frac{1}{R^h}|f|_R < \infty, \ |f^*|_{R'} = \frac{1}{R'^h}|f|_{R'}.$$

Thus

$$|f|_{R'} = R'^h|f^*|_{R'} \leqq R'^h|f^*|_R = \left(\frac{R'}{R}\right)^h|f|_R. \quad \square$$

6.9. <u>COROLLARY</u>. *Let* R *be a positive real number. Consider* $f(z) = \Sigma_{k=0}^{\infty} a_k z^k$, *where* $a_0, a_1, \ldots \in \mathbb{C}_p$*; suppose the power series converges for all* $z \in \mathcal{B}(0,R)$. *If* $|r_1|_p < |r_2|_p < R$ *and* f *has at least* h *zeros in* $\overline{\mathcal{B}}(0,|r_1|_p)$, *multiplicities included, we have*

$$\sup_{|z|_p \leq |r_1|_p} |f(z)|_p \leq \left(\frac{|r_1|_p}{|r_2|_p}\right)^h \sup_{|z|_p \leq |r_2|_p} |f(z)|_p.$$

6.10. <u>LEMMA</u>. *The set of all* $z \in \mathbb{C}_p$ *for which* $\Sigma_{n=0}^{\infty} z^n/n!$ *converges, is exactly* $\mathcal{B}(0,p^{-1/(p-1)})$. *The set of all* $z \in \mathbb{C}_p$ *for which* $\Sigma_{n=1}^{\infty} (-1)^{n-1} z^n/n$ *converges, is exactly* $\mathcal{B}(0,1)$.

<u>PROOF</u>. [Amice 1975], Proposition 3.5.5(v). □

6.11. <u>DEFINITION</u>. For $z \in \mathcal{B}(0,p^{-1/(p-1)})$ define $e_p(z) := \Sigma_{n=0}^{\infty} z^n/n!$.
For $z \in \mathcal{B}(1,1)$ define $\ell_p(z) := \Sigma_{n=1}^{\infty} (-1)^{n-1}(z-1)^n/n$.

6.12. <u>LEMMA</u>. *(a) For* $z_1, z_2 \in \mathcal{B}(0,p^{-1/(p-1)})$ *we have* $e_p(z_1+z_2) = e_p(z_1)e_p(z_2)$.
(b) For $z \in \mathcal{B}(0,p^{-1/(p-1)})$ *we have* $|e_p(z) - 1|_p = |z|_p$.
(c) For $z \in \mathcal{B}(0,p^{-1/(p-1)})$ *we have* $\ell_p(e_p(z)) = z$, $e_p'(z) = e_p(z)$.
(d) For $z_1, z_2 \in \mathcal{B}(1,1)$ *we have* $\ell_p(z_1 z_2) = \ell_p(z_1) + \ell_p(z_2)$.
(e) For $z \in \mathcal{B}(1,p^{-1/(p-1)})$ *we have* $|\ell_p(z)|_p = |z - 1|_p$.
(f) For $z \in \mathcal{B}(1,p^{-1/(p-1)})$ *we have* $e_p(\ell_p(z)) = z$.

<u>PROOF</u>. *(a)* [Amice 1975], Proposition 3.5.5(ii).
(b) For $n \in \mathbb{N}$, $n \geq 2$, $z \in \mathcal{B}(0,p^{-1/(p-1)})$ we have

$$\left|\sum_{k=2}^{n} \frac{z^k}{k!}\right|_p \leq \max_{k=2,\ldots,n} \frac{|z|_p^k}{|k!|_p} < \max_{k=2,\ldots,n} |z|_p \frac{p^{-(k-1)/(p-1)}}{p^{-(k-1)/(p-1)}};$$

here we use the inequality

$$|k!|_p \geq p^{-(k-1)/(p-1)},$$

which follows from Lemme 3.5.6 of [Amice 1975]. We conclude that

$$\left|\sum_{k=1}^{n} \frac{z^k}{k!}\right|_p = |z|_p;$$

thus

$$|e_p(z) - 1|_p = |\sum_{k=1}^{\infty} \frac{z^k}{k!}|_p = \lim_{n\to\infty} |\sum_{k=1}^{n} \frac{z^k}{k!}|_p = |z|_p.$$

(c) follows from *(b)* and the formal identity of the power series involved.

(d) [Amice 1975], Proposition 3.5.5(iii).

(e) Similar to *(b)*, observing that $|k|_p \geqq |k!|_p$.

(f) follows from *(e)* and the formal identity of the power series involved. □

We remark that in 1.4*(c)*, 1.4*(d)* and 2.2*(b)*, we have already defined $dg(\eta)$, $h(\eta)$ and $den(\eta)$ for all algebraic $\eta \in \mathbb{C}_p$. In order to give p-adic analogue of our earlier reasoning, it is also necessary to define $\lceil\eta\rceil$.

6.13. <u>DEFINITION</u>. Let $\eta \in \mathbb{C}_p$ be algebraic with minimal polynomial P. Then $\lceil\eta\rceil$ denotes the maximum of the absolute values of the zeros of P in $\mathbb{C}$.

6.14. <u>LEMMA</u>. *Let F be a finite normal extension of $\mathbb{Q}$; suppose $\eta \in F$ has minimal polynomial $P \in \mathbb{Z}[X]$. Then $\eta' \in \mathbb{C}$ has minimal polynomial P if and only if there exists an injective homomorphism $\sigma\colon F \to \mathbb{C}$ such that $\sigma(\eta) = \eta'$ and $\sigma(\xi) = \xi$ for all $\xi \in \mathbb{Q}$.*

<u>PROOF</u>. [Van der Waerden 1966], § 41. □

6.15. <u>LEMMA</u>. *For algebraic $\eta_1,\ldots,\eta_n \in \mathbb{C}_p$ we have*

$$\lceil\eta_1+\ldots+\eta_n\rceil \leqq \lceil\eta_1\rceil + \ldots + \lceil\eta_n\rceil, \quad \lceil\eta_1\ldots\eta_n\rceil \leqq \lceil\eta_1\rceil\ldots\lceil\eta_n\rceil.$$

<u>PROOF</u>. According to 2.4*(a)*, there exists a finite normal extension F of $\mathbb{Q}$ such that $F \subset \mathbb{C}_p$ and F contains $\eta_1,\ldots,\eta_n$. Let G be the set of all injective homomorphisms $\sigma\colon F \to \mathbb{C}$ such that $\sigma(\xi) = \xi$ for all $\xi \in \mathbb{Q}$; according to 6.14 we have

$$\forall\xi \in F\colon \lceil\xi\rceil = \max_{\sigma\in G} |\sigma(\xi)|.$$

Thus, if $\eta := \eta_1 + \ldots + \eta_n$, we have

$$\lceil\eta\rceil = \max_{\sigma\in G} |\sigma(\eta)| = \max_{\sigma\in G} |\sigma(\eta_1)+\ldots+\sigma(\eta_n)| \leqq$$

$$\max_{\sigma\in G} |\sigma(\eta_1)| + \ldots + \max_{\sigma\in G} |\sigma(\eta_n)| = \lceil\eta_1\rceil + \ldots + \lceil\eta_n\rceil$$

and similarly $\lceil\eta_1\ldots\eta_n\rceil \leqq \lceil\eta_1\rceil\ldots\lceil\eta_n\rceil$. □

6.16. LEMMA. *Let* $\eta \in \mathbb{C}_p$ *be an algebraic integer. Then* $|\eta|_p \leqq 1$.

PROOF. Suppose $\eta \in \mathbb{C}_p$ is an algebraic integer such that $|\eta|_p > 1$. There exist $a_0, a_1, \ldots, a_{d-1} \in \mathbb{Z}$ such that

$$\eta^d + a_{d-1}\eta^{d-1} + \ldots + a_1\eta + a_0 = 0.$$

Now for $j = 0,1,\ldots,d-1$ we have $a_j \in \mathbb{Z}$ and so $|a_j|_p \leqq 1$; thus

$$|a_j\eta^j|_p \leqq |\eta|_p^j < |\eta|_p^d.$$

Therefore

$$0 = |\eta^d + a_{d-1}\eta^{d-1} + \ldots + a_1\eta + a_0|_p = |\eta|_p^d > 1,$$

and we have arrived at a contradiction. □

6.17. LEMMA. *Let* $\eta \in \mathbb{C}_p$ *be algebraic of degree at most* d. *Then either* $\eta = 0$ *or*

$$|\eta|_p \geqq \mathrm{den}^{-d}(\eta)\lceil\eta\rceil^{-d}.$$

PROOF. Put $\eta' := \mathrm{den}(\eta)\cdot\eta$; then η' is an algebraic integer. As $|\mathrm{den}(\eta)|_p \leqq 1$, we have $|\eta|_p \geqq |\eta'|_p$. Let $a \in \mathbb{Z}$ be the constant term in the minimal polynomial of η'. Then a is the product of the zeros of this polynomial in $\mathbb{C}_p$ (apart from a possible factor -1). By 6.16, we may conclude $|\eta'|_p \geqq |a|_p$. But a is also plus or minus the product of the zeros of this polynomial in $\mathbb{C}$; thus $|a| \leqq \lceil\eta'\rceil^d$. Let F be a finite normal extension of $\mathbb{Q}$ such that $F \subset \mathbb{C}_p$ and F contains η. Then, by 6.14,

$$\lceil\eta'\rceil = \lceil\mathrm{den}(\eta)\cdot\eta\rceil = \sup_\sigma |\sigma(\mathrm{den}(\eta)\cdot\eta)| = \mathrm{den}(\eta)\cdot\sup_\sigma |\sigma(\eta)| =$$

$$\mathrm{den}(\eta)\lceil\eta\rceil,$$

where σ runs through the injective homomorphisms of F into $\mathbb{C}$ that leave $\mathbb{Q}$ fixed. The lemma now follows upon observing that $|a|_p \geqq |a|^{-1}$. □

6.18. LEMMA. *Let F be a field containing $\mathbb{Q}$; assume $\eta \in F$ is algebraic. Let the minimal polynomial of η have leading coefficient* a, *height* H, *degree* d. *Then, for each* $j \in \mathbb{N} \cup \{0\}$, *there exist* $a_0^{(j)},\ldots,a_{d-1}^{(j)} \in \mathbb{Z}$ *such that*

$$(a\eta)^j = a_0^{(j)} + a_1^{(j)}\eta + \ldots + a_{d-1}^{(j)}\eta^{d-1},$$

while

$$\max(|a_0^{(j)}|,\ldots,|a_{d-1}^{(j)}|) \leqq (2H)^j.$$

PROOF. [Baker 1975], § 2.3. □

6.19. LEMMA. *Suppose* $d \in \mathbb{N}$, $0 < \delta < 1$, $0 < \varepsilon < 1$. *Put* $A :=$ $\{z \in \mathbb{C}_p : \delta < |z - 1|_p < p^{-1/(p-1)}\}$. *Then only finitely many pairs* $(\alpha,\gamma) \in A^2$ *of algebraic numbers of degree at most* d *have the property that a* $\beta \in \mathbb{Q}$ *exists with*

$$\beta\ell_p(\alpha) - \ell_p(\gamma) = 0$$

and

$$(6.4) \qquad B \log^{-1-\varepsilon} B \geqq \log H,$$

where $H = \max(h(\alpha),h(\gamma))$ and $B = \max(2,h(\beta))$.

PROOF. I. Suppose $\alpha,\gamma \in A$, $\beta \in \mathbb{Q}$, such that the conditions of the lemma are fulfilled. By $c_1,c_2,\ldots$ we shall denote real numbers greater than 1 depending only on p,d,δ and ε; we suppose that H is greater than such a number, which will lead to a contradiction.

Define $L := B - 1$. We introduce the auxiliary function

$$\Phi(z) := \sum_{\lambda_1=0}^{L} \sum_{\lambda_2=0}^{L} q(\lambda_1,\lambda_2)\alpha^{\lambda_1 z}\gamma^{\lambda_2 z}, \quad z \in \overline{B}(0,1),$$

where

$$\alpha^{\lambda_1 z} = e_p(\lambda_1 z\ell_p(\alpha)), \quad \gamma^{\lambda_2 z} = e_p(\lambda_2 z\ell_p(\gamma))$$

and where $q(\lambda_1,\lambda_2)$ are rational integers to be determined later. We have

$$\Phi^{(t)}(z) = \ell_p^t(\alpha) \sum_{\lambda_1=0}^{L} \sum_{\lambda_2=0}^{L} q(\lambda_1,\lambda_2)(\lambda_1+\lambda_2\beta)^t \alpha^{\lambda_1 z} \gamma^{\lambda_2 z},$$

$z \in \overline{B}(0,1),\ t \in \mathbb{N} \cup \{0\}.$

Now put $S := [\frac{1}{2}d^{-1}\log^{1+\varepsilon/2}B]$, $T := [\frac{1}{2}d^{-1}B^2\log^{-1-\varepsilon/2}B]$ and consider the system of linear equations

$$(6.5) \qquad \Phi^{(t)}(ps) = 0,\ s = 0,\dots,S-1,\ t = 0,\dots,T-1$$

in the $(L+1)^2$ unknowns $q(\lambda_1,\lambda_2)$; we shall show that it has a non-trivial integer solution. Multiplying (6.5) by $(ac)^{pLs}b^t\ell_p^{-t}(\alpha)$, where a,b,c are the leading coefficients in the minimal polynomials of α, β and γ respectively, gives

$$\sum_{\lambda_1=0}^{L} \sum_{\lambda_2=0}^{L} q(\lambda_1,\lambda_2)(\lambda_1+\lambda_2\beta)^t b^t (a\alpha)^{\lambda_1 ps}(c\gamma)^{\lambda_2 ps} \times$$
$$a^{(L-\lambda_1)ps}c^{(L-\lambda_2)ps} = 0.$$

Substitution of

$$(\lambda_1+\lambda_2\beta)^t = \sum_{\tau=0}^{t} \binom{t}{\tau}\lambda_1^{t-\tau}(\lambda_2\beta)^\tau$$

transforms this into

$$\sum_{\lambda_1=0}^{L} \sum_{\lambda_2=0}^{L} q(\lambda_1,\lambda_2)(a\alpha)^{\lambda_1 ps}(c\gamma)^{\lambda_2 ps} a^{(L-\lambda_1)ps} c^{(L-\lambda_2)ps} \times$$
$$\sum_{\tau=0}^{t} \binom{t}{\tau}\lambda_1^{t-\tau}\lambda_2^\tau (b\beta)^\tau b^{t-\tau} = 0.$$

According to 6.18, for each $j \in \mathbb{N} \cup \{0\}$, there exist $a_0^{(j)},\dots,a_{d-1}^{(j)}$, $c_0^{(j)},\dots,c_{d-1}^{(j)} \in \mathbb{Z}$, their absolute values bounded by $(2H)^j$, such that

$$(a\alpha)^j = a_0^{(j)} + a_1^{(j)}\alpha + \dots + a_{d-1}^{(j)}\alpha^{d-1},$$

$$(c\gamma)^j = c_0^{(j)} + c_1^{(j)}\gamma + \dots + c_{d-1}^{(j)}\gamma^{d-1}.$$

Our set of equations thus becomes

$$\sum_{\delta_1=0}^{d-1}\sum_{\delta_2=0}^{d-1} A(\delta_1,\delta_2,s,t)\alpha^{\delta_1}\gamma^{\delta_2} = 0,$$

where

$$A(\delta_1,\delta_2,s,t) = \sum_{\lambda_1=0}^{L}\sum_{\lambda_2=0}^{L}\sum_{\tau=0}^{t} q(\lambda_1,\lambda_2)a^{(L-\lambda_1)ps}a_{\delta_1}^{(\lambda_1 ps)} \times c^{(L-\lambda_2)ps}c_{\delta_2}^{(\lambda_2 ps)}\binom{t}{\tau}(b\lambda_1)^{t-\tau}\lambda_2^{\tau}(b\beta)^{\tau}.$$

We see that (6.5) is certainly satisfied if $A(\delta_1,\delta_2,s,t) = 0$ for $\delta_1 = 0,\ldots,d-1$; $\delta_2 = 0,\ldots,d-1$; $s = 0,\ldots,S-1$; $t = 0,\ldots,T-1$. These are d^2ST linear equations in the $(L+1)^2$ unknowns $q(\lambda_1,\lambda_2)$ with rational integer coefficients, whose absolute values are at most

$$T(c_1H)^{4pLS}2^T L^T B^T \leqq \exp(c_2B^2\log^{-\varepsilon/2}B)$$

(here (6.4) is used). As $(L+1)^2 < \frac{1}{2}B^2 < d^2ST$, 2.7 states that there is a non-trivial choice for the $q(\lambda_1,\lambda_2)$, such that (6.5) holds, while

$$Q := \max_{\lambda_1,\lambda_2} |q(\lambda_1,\lambda_2)| \leqq$$
$$(c_3L^2\exp(c_2B^2\log^{-\varepsilon/2}B))^{d^2ST/((L+1)^2-d^2ST)} \leqq$$
$$\exp(c_4B^2\log^{-\varepsilon/2}B).$$

II. For $k \in \mathbb{N} \cup \{0\}$ we put $T_k := 2^kT$. Then, for our special choice of the $q(\lambda_1,\lambda_2)$, we have

$$\Phi^{(t)}(s) = 0,\ s = 0,\ldots,S-1,\ t = 0,\ldots,T_k-1. \tag{6.6}$$

This is proved by induction; for $k = 0$ the assertion is precisely (6.5). Now suppose that (6.6) holds for some k; thus Φ has at least ST_k zeros in $\overline{\mathcal{B}}(0,p^{-1})$. Moreover, as $|\alpha - 1|_p$ and $|\gamma - 1|_p$ are less than $p^{-1/(p-1)}$, the power series for Φ converges, by 6.12, on some $\mathcal{B}(0,r)$ with $r > 1$. Now 6.9 states that

$$\sup_{|z|_p\leqq p^{-1}} |\Phi(z)|_p \leqq p^{-ST_k} \sup_{|z|_p\leqq 1} |\Phi(z)|_p.$$

Here

$$\sup_{|z|_p \leq 1} |\Phi(z)|_p \leq 1,$$

as is evident from 6.12 and the fact that $q(\lambda_1,\lambda_2) \in \mathbb{Z}$. Furthermore

$$p^{-ST_k} \leq \exp(-2^k c_5^{-1} B^2).$$

Thus

$$\sup_{|z|_p \leq p^{-1}} |\Phi(z)|_p \leq \exp(-2^k c_5^{-1} B^2);$$

by 6.4 this implies

$$|\Phi^{(t)}(ps)|_p \leq p^t \exp(-2^k c_5^{-1} B^2), \quad s,t \in \mathbb{N} \cup \{0\},$$

and so

(6.7) $$|\Phi^{(t)}(ps)|_p \leq \exp(-2^k c_6^{-1} B^2), \quad s \in \mathbb{N} \cup \{0\}, t = 0,\dots,T_{k+1}-1.$$

However, $\ell_p^{-t}(\alpha)\Phi^{(t)}(ps)$ is algebraic and for $s = 0,\dots,S-1$, $t = 0,\dots,T_{k+1}-1$ we have

$$\mathrm{dg}(\ell_p^{-t}(\alpha)\Phi^{(t)}(ps)) \leq d^2,$$

$$\mathrm{den}(\ell_p^{-t}(\alpha)\Phi^{(t)}(ps)) \leq (ac)^{pLs} b^t \leq H^{2pLS} B^{T_{k+1}} \leq$$

$$\exp(2^k c_7 B^2 \log^{-\varepsilon/2} B);$$

$$\overline{|\ell_p^{-t}(\alpha)\Phi^{(t)}(ps)|} \leq Q c_8^{LS+T_{k+1}} L^{T_{k+1}+2} H^{2pLS} B^{T_{k+1}} \leq$$

$$\exp(2^k c_9 B^2 \log^{-\varepsilon/2} B).$$

Applying 6.17 gives $\Phi^{(t)}(ps) = 0$ or

$$|\ell_p^{-t}(\alpha)\Phi^{(t)}(ps)|_p \geq \exp(-2^k c_{10} B^2 \log^{-\varepsilon/2} B).$$

Furthermore, by 6.12,

$$|\ell_p^t(\alpha)|_p = |\alpha - 1|_p^t \geqq \delta^t \geqq \exp(-2^k c_{11} B^2 \log^{-1-\varepsilon/2} B),$$

so in the latter case we have

$$(6.8) \qquad |\Phi^{(t)}(ps)|_p \geqq \exp(-2^k c_{12} B^2 \log^{-\varepsilon/2} B).$$

Combining (6.7) and (6.8) gives $\Phi^{(t)}(ps) = 0$ for $s = 0,\ldots,S-1$, $t = 0,\ldots,T_{k+1}-1$. This completes the proof of (6.6).

III. From (6.6) it follows that $\Phi^{(t)}(0) = 0$ for $t = 0,\ldots,(L+1)^2-1$, in other words, that

$$(6.9) \qquad \sum_{\lambda_1=0}^{L} \sum_{\lambda_2=0}^{L} q(\lambda_1,\lambda_2)(\lambda_1+\lambda_2\beta)^t = 0.$$

As the $q(\lambda_1,\lambda_2)$ are not all zero, it follows that the coefficient matrix of the system (6.9), which is of the Vandermonde type, must be singular. From this we deduce the existence of $\lambda_1,\lambda_2,\lambda_1',\lambda_2' \in \{0,\ldots,L\}$ with $\lambda_1 + \lambda_2\beta = \lambda_1' + \lambda_2'\beta$, or

$$\beta = \frac{\lambda_1' - \lambda_1}{\lambda_2 - \lambda_2'}.$$

This gives

$$B = h(\beta) \leqq L = B - 1,$$

so we get the desired contradiction. □

6.20. <u>LEMMA</u>. *Let* R *be a positive real number,* $a \in \mathbb{C}_p$. *Consider* $f(z) = \sum_{k=0}^{\infty} a_k(z-a)^k$, *where* $a_0,a_1,\ldots \in \mathbb{C}_p$; *suppose the power series converges for all* $z \in B(a,R)$. *Take* $b,r \in B(a,R)$ *and define*

$$g(z) := \frac{f(z) - f(b)}{z - b}, \quad z \in B(a,R) \setminus \{b\}.$$

Then g *is bounded on* $\overline{B}(a,|r-a|_p) \setminus \{b\}$.

<u>PROOF</u>. For $z \in B(a,R)$ we have

$$f(z) - f(b) = \sum_{k=0}^{\infty} a'_k(z-a)^k,$$

where $a'_0 := a_0 - \Sigma_{k=0}^{\infty} a_k(b-a)^k$ and $a'_k := a_k$ for $k \geq 1$. According to 6.7,

$$g(z) = \frac{f(z) - f(b)}{z - b} = \sum_{k=0}^{\infty} b_k(z-a)^k, \; z \in B(a,R) \setminus \{b\},$$

with

$$|b_k|_p \leq \sup_{n \geq k+1} |a'_n|_p |b - a|_p^{n-k-1}, \; k \in \mathbb{N} \cup \{0\}.$$

Thus

$$\sup_{k \geq 0} |b_k|_p |r - a|_p^k = \sup_{k \geq 0} \sup_{n \leq k+1} |a'_n|_p |b - a|_p^{n-k-1} |r - a|_p^k \leq$$

$$\sup_{k \geq 0} \sup_{n \geq k+1} |a'_n|_p \max(|b-a|_p, |r-a|_p)^{n-1} =$$

$$\sup_{n \geq 1} |a_n|_p \max(|b-a|_p, |r-a|_p)^{n-1}.$$

As $\max(|b-a|_p, |r-a|_p) < R$, the right hand member is finite. Application of 6.5 shows that g is bounded on $\overline{B}(a, |r-a|_p) \setminus \{b\}$. □

6.21. <u>LEMMA</u>. *For every* $d \in \mathbb{N}$, $n \in \mathbb{N}$, *there exists an effectively computable* $C > 1$, *depending only on* d *and* n, *with the following property. Suppose* $\alpha_1, \ldots, \alpha_n \in B(1,1)$ *algebraic of degree at most* d, $\beta_0, \ldots, \beta_{n-1} \in \overline{B}(0,1)$ *algebraic of degree at most* d, $\beta_n \in \mathbb{C}_p$ *algebraic of degree at most* d *with* $|\beta_n|_p = 1$. *Put* $\Lambda := \beta_0 + \beta_1 \ell_p(\alpha_1) + \ldots + \beta_n \ell_p(\alpha_n)$. *Then either* $\Lambda = 0$ *or*

$$|\Lambda| > \exp(-C\Omega \log \Omega'(\log B + \log \Omega)),$$

where $\Omega = (\log A_1) \ldots (\log A_n)$, $\Omega' = (\log A_1) \ldots (\log A_{n-1})$, $A_j = \max(16, h(\alpha_j))$ *for* $j = 1, \ldots, n$, $B = \max(6, h(\beta_0), \ldots, h(\beta_n))$.

<u>PROOF</u>. See the remark preceding Theorem 3 in [Van der Poorten 1977]. □

6.22. <u>LEMMA</u>. *Suppose* $\alpha_1, \ldots, \alpha_n \in B(1,1)$ *algebraic. Then* $\ell_p(\alpha_1), \ldots, \ell_p(\alpha_n)$ *are linearly independent over* $\mathbb{Q}$ *if and only if they are linearly independent over the field of all algebraic numbers in* $\mathbb{C}_p$.

PROOF. [Brumer 1967], Theorem 1. □

6.23. THEOREM. *Suppose* $\varepsilon > 0$, $d \in \mathbb{N}$, $a \in B(1,p^{-1/(p-1)}) \setminus \{1\}$, $b \in \overline{B}(0,1)$. *Suppose that there exists a* $C > 1$ *such that for all* $x_0, x_1 \in \mathbb{Z}$ *that are not both zero we have*

$$(6.10) \qquad |x_0 + x_1 b|_p \geq C^{-1}\exp(-X^3\log^{-2}X),$$

where $X = \max(2,|x_0|,|x_1|)$. *Then there are only finitely many triples* $(\alpha,\beta,\gamma) \in \mathbb{C}_p^3$ *of algebraic numbers of degree at most* d *with*

$$\max(|a-\alpha|_p, |b-\beta|_p, |a^b-\gamma|_p) < \exp(-\log^3 H \log_2^{1+\varepsilon} H),$$

where $H = \max(2,h(\alpha),h(\beta),h(\gamma))$ *and* $a^b = e_p(\ell_p(a))$.

PROOF. I. Let (α,β,γ) be a triple satisfying the conditions of the theorem; we suppose H to be greater than a certain bound depending only on ε,d,a and b. This will lead to a contradiction. From 6.20 we see

$$(6.11) \qquad |\ell_p(a) - \ell_p(\alpha)|_p < \exp(-\log^3 H \log_2^{1+\varepsilon/2} H),$$

$$(6.12) \qquad |b\ell_p(a) - \ell_p(\gamma)|_p < \exp(-\log^3 H \log_2^{1+\varepsilon/2} H).$$

As a consequence of (6.11), (6.12) and

$$(6.13) \qquad |b - \beta|_p < \exp(-\log^3 H \log_2^{1+\varepsilon} H)$$

we have

$$|\beta\ell_p(\alpha) - \ell_p(\gamma)|_p < \exp(-\log^3 H \log_2^{1+\varepsilon/3} H).$$

If it were the case that $\beta\ell_p(\alpha) - \ell_p(\gamma) \neq 0$, 6.21 would imply

$$|\beta\ell_p(\alpha) - \ell_p(\gamma)|_p > \exp(-\log^3 H \log_2^{1+\varepsilon/3} H),$$

which is a contradiction. Therefore $\beta\ell_p(\alpha) - \ell_p(\gamma) = 0$.

II. We have just proved that $\ell_p(\alpha)$ and $\ell_p(\gamma)$ are linearly dependent over the field of all algebraic numbers in $\mathbb{C}_p$; using 6.22, we find that these numbers must also be linearly dependent over $\mathbb{Q}$. In other words, there are $\xi, \eta \in \mathbb{Q}$, not both zero, such that $\xi \ell_p(\alpha) + \eta \ell_p(\gamma) = 0$. As

$$|a - \alpha|_p < \exp(-\log^3 H \log_2^{1+\varepsilon} H) < |a - 1|_p,$$

we have

$$(6.14) \qquad |\alpha - 1|_p < |a - 1|_p < p^{-1/(p-1)}.$$

Thus, by 6.12, $\ell_p(\alpha) \neq 0$ and we may conclude

$$\beta = \frac{\ell_p(\gamma)}{\ell_p(\alpha)} = -\frac{\xi}{\eta} \in \mathbb{Q}.$$

Using 6.19 with $\delta = \frac{1}{2}|a - 1|_p$ gives that $\log H > B \log^{-1-\varepsilon/10} B$, where $B = \max(2, h(\beta))$. Then, a fortiori, $\log H > B^{1/2}$, so $\log_2 H > \frac{1}{2}\log B$; thus

$$\exp(-\log^3 H \log_2^{1+\varepsilon/3} H) < \exp(-\tfrac{1}{4}B^3 \log^{-2+\varepsilon/30} B) <$$

$$\exp(-B^3 \log^{-2+\varepsilon/31} B).$$

From (6.13) we see that

$$|b - \beta|_p < \exp(-B^3 \log^{-2+\varepsilon/31} B),$$

and so, if we put $\beta = -x_0/x_1$ with $x_0 \in \mathbb{Z}$, $x_1 \in \mathbb{N}$, $(x_0, x_1) = 1$,

$$(6.15) \qquad |x_0 + x_1 b|_p = |x_1|_p |b - \beta|_p \leq |b - \beta|_p < \exp(-B^3 \log^{-2+\varepsilon/31} B),$$

where $B = \max(2, |x_0|, |x_1|)$. From (6.10) we know thay b cannot be rational; from (6.13) it then follows that B tends to infinity with H. Therefore (6.15) contradicts (6.10). □

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ponential function (to appear).

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INDEX OF SPECIAL SYMBOLS

SAMENVATTING

In dit proefschrift wordt bewezen dat voor bepaalde functies van twee of meer complexe variabelen niet tegelijkertijd de functiewaarde en de argumenten zodanig kunnen worden benaderd door algebraïsche getallen van begrensde graad, dat de fout, uitgedrukt in de hoogte van die algebraïsche getallen, binnen een zekere grens valt. De beschouwde functies zijn samengesteld uit polynomen, de exponentiële functie en de logaritme; ten opzichte van de reeds bekende resultaten onderscheiden de hier gegeven stellingen zich vooral door het ontbreken van beperkende voorwaarden op de algebraïsche getallen waarmee wordt benaderd. In het laatste hoofdstuk wordt een analoge theorie opgezet voor de completering van de algebraïsche afsluiting van het lichaam der p-adische getallen.

STELLINGEN

I

Zij $\varepsilon > 0$, $d \in \mathbb{N}$. Bijna alle $b \in \mathbb{Z}_p$, in de zin van de Haar-maat, hebben de volgende eigenschap: als $a \in \mathbb{C}_p$ met $0 < |a-1|_p < p^{-1/(p-1)}$, bestaan er slechts eindig veel drietallen $(\alpha,\beta,\gamma) \in \mathbb{C}_p^3$ van algebraïsche getallen van graad ten hoogste d met

$$\max(|a-\alpha|_p, |b-\beta|_p, |a^b-\gamma|_p) < \exp(-\log^3 H \log_2^{1+\varepsilon} H),$$

waar H het maximum van 2 en de hoogten van α, β en γ voorstelt, en $a^b = e_p(b\ell_p(a))$. Niet alle $b \in \mathbb{Z}_p$ hebben echter deze eigenschap.

Dit proefschrift, 6.21.
A. Bijlsma, *On the simultaneous approximation of* a, b *and* a^b, Compositio Math. 35 (1977) 99-111.
V. Jarník, *Sur les approximations diophantiques des nombres* p-*adiques*, Rev.Ci.(Lima) 47 (1945) 489-505.

II

De uitspraak van 2.11 van dit proefschrift wordt onwaar zodra het rechterlid van (2.1) door een wezenlijk sneller stijgende functie van B wordt vervangen.

III

In Satz 1' van [Halász 1968] wordt het bestaan geëist van een functie $L:]0,\infty[\to \mathbb{C}\setminus\{0\}$, zó, dat

$$\lim_{x\to\infty} \frac{L(\lambda x)}{L(x)} = 1 \text{ uniform voor } \lambda \in [1,2].$$

Met behulp van een methode van Korevaar, Van Aardenne-Ehrenfest en De Bruijn [1949] en Balkema [1973] kan men bewijzen dat voor elke Lebesgue-meetbare L die voldoet aan

$$\lim_{x\to\infty} \frac{L(\lambda x)}{L(x)} = 1, \ \lambda \in [1,2],$$

de convergentie uniform is.

J. Korevaar, T. van Aardenne-Ehrenfest & N.G. de Bruijn, *A note on slowly oscillating functions,* Nieuw Arch.Wisk. (2) 23 (1949) 77-86.
A.A. Balkema, *Monotone transformations and limit laws,* MC Tracts 45, Mathematisch Centrum, Amsterdam, 1973.
A. Halász, *Uber die Mittelwerte multipikativer zahlentheoretischer Funktionen,* Acta Math. Acad.Sci.Hungar. 19 (1968) 365-403.

IV

Voor elk natuurlijk getal κ bestaan er positieve reële getallen a_1, a_2, zó, dat $\log a_1$ en $\log a_2$ lineair onafhankelijk over $\mathbb{Q}$ zijn, terwijl voor oneindig veel viertallen $(\alpha_1, \alpha_2, x_1, x_2) \in \mathbb{Q}^2 \times \mathbb{Z}^2$ geldt

$$\max(|a_1-\alpha_1|, |a_2-\alpha_2|, |x_1 \log a_1 + x_2 \log a_2|) < \exp(-\log^{\kappa} H),$$

waar H het maximum van de hoogten van $\alpha_1, \alpha_2, x_1, x_2$ voorstelt en x_1, x_2 niet beide 0 zijn. Hieruit volgt dat Satz 4 van [Wüstholz 1976] niet juist is.

G. Wüstholz, *Simultane Approximationen, die mit Potenzen bzw. Logarithmen zusammenhängen,* proefschrift, Freiburg, 1976.

V

Veronderstel dat κ en λ reële getallen groter dan 1 zijn. Als $\lambda \geq \kappa$, kan men een $a \in \mathbb{C}$ en een rij $(\alpha_n)_{n=1}^{\infty}$ van algebraïsche complexe getallen van begrensde graad aangeven, zó, dat

$$\forall n \in \mathbb{N} : |a - \alpha_n| < \exp(-\log^{\kappa} A_n)$$

en

$$\limsup_{n\to\infty} \frac{\log\log A_{n+1}}{\log\log A_n} \leq \lambda,$$

waar A_n de hoogte van α_n voorstelt. Als echter $\lambda < \kappa - 1$, is dit onmogelijk.

G. Wüstholz, *Linearformen in Logarithmen von U-Zahlen mit ganzzahligen Koeffizienten* (ter publicatie voorgedragen).

VI

Zij R een positief reëel getal. Beschouw $f(z) = \Sigma_{k=0}^{\infty} a_k z^k$, waar $a_0, a_1, \ldots \in \mathbb{C}_p$; veronderstel dat de machtreeks convergeert voor elke $z \in \mathbb{C}_p$ met $|z|_p <$ R. Zij $\varepsilon > 0$ en $0 < |r_1|_p < |r_2|_p < R$. Veronderstel dat er $c_1, \ldots, c_n \in \mathbb{C}_p$ bestaan, alle verschillend, met $|c_i|_p < |r_1|_p$ voor $i = 1, \ldots, n$, zó, dat

$$|f^{(j)}(c_i)|_p < \varepsilon, \; i = 1, \ldots, n, \; j = 0, \ldots, m-1.$$

Zij $\rho := \min(\{|c_i - c_j|_p : 1 \leqq i < j \leqq n\} \cup \{p^{-1/(p-1)}\})$. Dan geldt

$$\sup_{|z|_p \leqq |r_1|_p} |f(z)|_p \leqq \left(\frac{|r_1|_p}{|r_2|_p}\right)^{mn} \sup_{|z|_p \leqq |r_2|_p} |f(z)|_p + \left(\frac{|r_1|_p}{\rho}\right)^{mn} \varepsilon.$$

VII

Zij d een discriminant. Schrijf $d = \delta^2 \Delta$ met $\delta \in \mathbb{N}$ en Δ een fundamentaaldiscriminant. De verzameling F wordt ingeval $d < 0$ verkregen door uit elke equivalentieklasse van de primitieve, positieve binaire kwadratische vormen met geheel-rationale coëfficiënten en discriminant d precies één element te kiezen, ingeval $d > 0$ door uit elke equivalentieklasse van de primitieve binaire kwadratische vormen met geheel-rationale coëfficiënten en discriminant d precies één element met positieve eerste coëfficiënt te kiezen. Zij nu n een natuurlijk getal met $(n, \delta) = 1$. Geef met $\rho_d(n)$ het aantal geordende drietallen (Q, x, y) aan met $Q \in F$, $x \in \mathbb{Z}$, $y \in \mathbb{Z}$ en $Q(x,y) = n$, zó, dat ingeval $d > 0$ is voldaan aan

$$y \geqq 0, \; D_1 Q(x,y) > \frac{t_0}{u_0} y,$$

waar (t_0, u_0) de kleinste positieve oplossing van de vergelijking van Pell $t^2 - du^2 = 4$ is. Dan is

$$\rho_d(n) = w_d \sum_{\nu | n} \left(\frac{d}{\nu}\right),$$

waar $w_d = 1$ als $d > 0$, $w_d = 6$ als $d = -3$, $w_d = 4$ als $d = -4$, $w_d = 2$ als $d \leqq -7$.

A. Bijlsma, *Representation of a natural number by positive binary quadratic forms of a*

given discriminant, Nieuw Arch.Wisk. (3) 23 (1975) 105-114.

VIII

In tegenstelling tot hetgeen Heilbronn [1934] verklaart, is noch stelling V, noch het bijzondere geval daarvan dat men verkrijgt door voor d een negatieve fundamentaaldiscriminant te kiezen, bewezen door Dirichlet.

H. Heilbronn, *On the class-number in imaginary quadratic fields*, Quart.J.Math.Oxford Ser. 5 (1934) 150-160.

IX

Zij $(\lambda_n)_{n=1}^{\infty}$ een rij positieve reële getallen waarvoor geldt

$$\lim_{n\to\infty} \frac{\lambda_n}{n} = \infty, \quad \liminf_{n\to\infty} (\lambda_{n+1}-\lambda_n) > 0.$$

Zij $(d_n)_{n=1}^{\infty}$ een rij complexe getallen en veronderstel dat de algemene Dirichletreeks $\Sigma_{n=1}^{\infty} d_n \exp(-\lambda_n s)$ voor elke $s \in \mathbb{C}$ absoluut convergeert; geef de som aan met $f(s)$. Voor elke verzameling $X \subset \mathbb{R}$ noteren we

$$M(\sigma;X) := \sup_{t\in X} |f(\sigma+it)|.$$

Dan is $M(\sigma;X)$ eindig voor elke $\sigma \in \mathbb{R}$, $X \subset \mathbb{R}$. Zij nu K een compact deel van $\mathbb{R}$, $\delta > 0$, $\varepsilon > 0$. Dan bestaat er een σ_1 zó dat voor elke $\sigma < \sigma_1$ geldt

$$M(\sigma;K) \leqq \min_{a\in K} M(\sigma;[a,a+\delta])M^{\varepsilon}(2\sigma;\mathbb{R}).$$

P. Turán, *Eine neue Methode in der Analysis und deren Anwendungen*, Akadémiai Kiadó, Budapest, 1953.

X

Veronderstel $k_1,\ldots,k_n,\ell_1,\ldots,\ell_n \in \mathbb{R}$ met $k_1 > k_2 \geqq \ldots \geqq k_n$ en $\ell_1 > \ell_2 \geqq \ldots \geqq \ell_n$; zij σ een niet-identieke permutatie van $\{1,\ldots,n\}$. Dan is $k_1\ell_1 + \ldots + k_n\ell_n > k_1\ell_{\sigma(1)} + \ldots + k_n\ell_{\sigma(n)}$.