Fairness vs Welfare: a Hybrid Congestion Aftermarket

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ABSTRACT

We consider network flow congestion management modelled after electricity distribution networks. The desired consumption or production of the agents that populate such networks are determined by a higher-level (e.g. national) market mechanism, but this can lead to congestion locally. We first consider congestion solutions in the form of curtailment independent of the price set by the higher-level market. Congestion solutions of this type that satisfy properties of fairness are described in the literature. We contrast these fair solutions with curtailment solutions that maximize total welfare, and we present an algorithmic mechanism that computes such maximal welfare solutions. We then combine the two approaches to compute hybrid congestion solutions where agents can choose to either claim their fair share or to participate in a welfare-maximizing aftermarket. We incentivize aftermarket participation with an individually rational pricing scheme, while offering agents' fair shares at the higher-level price. Our aftermarket solution provides a budget balanced alternative to locational marginal pricing that gives agents the choice to claim their fair share at a fair price.

CCS CONCEPTS

• **Networks** → **Network resources allocation**; • **Hardware** → *Smart grid*; • **Computing methodologies** → *Multi-agent systems.*

KEYWORDS

fairness, congestion management, resource allocation, mechanisms

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1 INTRODUCTION

With the changes in the electric grid brought about by the energy transition new challenges arise, many of which concern flexibility of users [7, 19, 21, 27]. A central challenge is that of grid congestion, which traditionally only occurs at the transmission system level, but now also occurs at the local distribution system level [26]. With the rapidly increasing penetration of distributed and renewable energy resources these problems can be expected to extend into the future [4], especially since research indicates that neither grid expansion nor storage are solutions on the short to mid term [9, 24].

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A popular congestion management approach at the transmission system level is locational marginal pricing (LMP), which is part of the modern standardized market design [15]. LMP determines the area price based on the marginal cost of producers that can actually deliver in that area within the transmission network capacity constraints. Nevertheless, since the price set at the transmission system level does not reflect local distribution grid constraints, local congestion management is required [20]. LMP can fill this role as well, implementing what is essentially scarcity pricing, i.e. raising local prices until users sufficiently reduce their network usage. However, LMP is not budget balanced and does little to utilize local flexibility.

In this paper we propose alternative mechanisms for local congestion management. We model capacity constrained distribution networks as rooted weighted trees whose edge weights represent line capacities and whose root represents a connection to a higher-level network, e.g. the transmission system. Such non-cyclic graphs accurately model the active radial structure of most distribution networks [23]. The vertices are populated by agents that represent prosumers: users of the network that can both consume and/or produce power. Based on the price $\hat{p}$ set by the higher-level market mechanism, these agents have desired prosumptions that may cause congestion in the local network. Our congestion management mechanisms follow the common approach to active power congestion management, which is curtailment [2, 12, 22, 25, 26]. Curtailment entails allocating to agents a prosumption that is a reduction of their desired prosumption in order to resolve congestion.

Mechanisms for local congestion management can be designed to focus on different aspects, the most straightforward being economic efficiency as expressed by utilitarian welfare. Mechanisms such as LMP aim to achieve this through price signals that result in allocation of capacity to the most competitive agents. However, such solutions do not consider another aspect that has become prominent in energy networks: fairness [10]. Fairness deserves explicit consideration because, when resolving grid congestion, the question arises how congestion solutions affect different users, not just individually but also relative to each other. Since energy has become a basic need for full participation in modern society, this issue of fairness between users must be addressed [6].

In this paper we consider both approaches. On the one hand, congestion solutions should be fair to all users of the network. On the other hand, congestion implies a market limitation and has economic consequences for the users. Users may differ in how they value fairness versus welfare, which suggests that congestion solutions ideally are able to reconcile these differing viewpoints. However, since fair congestion solutions consider relative prosumptions instead of individual demand curves, it is unlikely that a fair curtailment solution also maximizes the total (utilitarian) welfare.

We first consider congestion solutions in the form of curtailment that maximize the agents' total welfare. We present an algorithmic mechanism that computes the maximal welfare curtailment allocation that is feasible for all agents at the price $\hat{p}$ set by the higher-level market mechanism. Our algorithm considers the demand curves that the agents submit to the higher-level market mechanism to determine maximal welfare, and is purely a curtailment solution in the sense that it does not send price signals.

For fair congestion solutions we turn to the literature [11, 16]. Rather than defining fairness ourselves we work with the abstract concept of agents' 'fair shares' that can be determined by any fair congestion solution of choice. With these generalised fair shares we then go on to propose an algorithmic mechanism that computes a hybrid congestion solution that combines fairness and welfare. Our mechanism provides agents with the choice to either claim their fair share or to participate in welfare maximization, resulting in a hybrid congestion solution that focuses on the goals of the two sets of agents in parallel to each other.

Finally, we provide incentives to organize this hybrid solution. We present a pricing scheme for the welfare maximization aspect of our hybrid solution that lets us define a congestion aftermarket. Our congestion aftermarket operates on top of the fair shares and the higher-level price $\hat{p}$, letting agents trade portions of their allocated fair shares at locally marginal prices. This principle of local aftermarket prices bears similarities to LMP, with the two most important differences being that our aftermarket is budget balanced and does not expose agents that choose to claim their fair share to congestion prices. Our aftermarket incentivizes participation as we prove that it is individually rational for agents to participate, but does so without imposing economical consequences on agents that choose not to participate. Since agents are always free to claim their fair share at the higher-level price $\hat{p}$, we argue that our hybrid congestion management mechanism constitutes a fair mechanism.

The main contribution of this paper is our congestion management solution, provided in algorithmic form, that both allows individual agents to claim their fair share and simultaneously maximizes welfare for the other agents in our novel congestion aftermarket.

2 RELATED WORK

Integral network management frameworks, such as that proposed by [14], generally rely on congestion pricing like LMP. [5] and [8] address congestion with demand side management which also usually relies on price signals. [17, 18] focuses on the distribution market operator to implement settlement or penalties for congestion management. [13] consider fairness in energy rates with respect to how the burden of network overhead costs are divided over prosumers. [20] propose time-slot auctions for EV charging to resolve congestion and promote fairness among asymmetric parties. [1] address fairness in EV charging through curtailment with a fair allocation that is found by optimizing a fair objective function under feasibility constraints. [11] consider different notions of fairness for greedy local matching under capacity constraints. [3] evaluate which factors in power networks, specifically PV, should be subject to fairness considerations. In comparison to the discussed work, we propose opt-in fairness alongside welfare maximization and emphasise self-contained local market resolution of congestion.

3 PRELIMINARIES

In this paper we consider congestion in tree graphs modelled on electricity distribution grids. Given a situation where the users of the network collectively cause congestion, we seek solutions for this congestion. In particular, we investigate both fair and welfare maximizing solutions. Fair solutions can be given as curtailment of users, while welfare solutions often involve a market mechanism.

We start by modelling commodity flow in congested tree networks populated by agents. Our model resembles a standard source-sink flow network with edge constraints, except that we have multiple sources and sinks in the form of agents. The agents act as either consumers or producers of the commodity based on a demand (or supply) curve that they submit to a higher-level (e.g. national) market. The price $\hat{p}$ set by this market then determines each agent's desired prosumption. These desired prosumptions may cause congestion in the local network that we consider. We will compute allocations that resolve congestion given such an initial situation. We can have two objectives for these allocations: fair division of capacity over the agents, or welfare expressed by demand curves.

Let a **congestion tree** $T = (V, E, A)$ be a rooted weighted tree (V, E) with a set of agents A located at the vertices. Let a virtual edge at the root r represent the connection to a virtual parent that represents an external network. Let the edge weights be strictly positive, representing flow capacities, and denote the weight of an edge between vertex $v \in V$ and its parent as the **capacity** C_v.

Let supply and demand both be represented by **prosumption**; respectively by negative and positive prosumption. Let an agent's prosumption induce a matching flow from the external network to the vertex of that agent. Using this terminology, a maximal production is synonymous with a minimal (negative) flow.

Let each agent $a \in A$ submit a **demand curve** $d_a(p)$ which is a strictly monotonically decreasing function of price p. A positive demand $d_a(p) > 0$ indicates a consumer, while a negative demand $d_a(p) < 0$ indicates a producer. Given a price $\hat{p}$ let $A^+ \subseteq A$ and $A^- \subseteq A$ denote the sets of consumers and producers, respectively.

The inverse relation of the demand curve expresses the marginal price or **marginal** $m_a(q)$ of an agent a, which is a strictly monotonically decreasing function of its prosumption q. Note that the marginal $m_a(q)$ of a production, i.e. $q < 0$, represents a marginal cost. The **welfare** $W_a(p, q)$ of an agent a is then given by its prosumption surplus $\int_0^q m_a(x) - p \, dx$. Note that for a producer this expression takes the equivalent form $\int_q^0 p - m_a(x) \, dx$.

Since the flow in the entire network is induced by the prosumptions of all individual agents, it will be convenient to work with **allocations** $Y : A \to \mathbb{R}$ that allocate a prosumption to each agent. Because we consider trees, an allocation Y then defines the flow over each edge as the sum of prosumptions in the subtree under that edge. This way, the **root flow** $F(Y)$ is given by the sum of allocated prosumptions over all agents, i.e. $F(Y) = \sum_{a \in A} Y(a)$.

Congestion occurs when commodity flow induced by an allocation Y, e.g. of desired prosumptions $Y(a) = d_a(\hat{p})$ $(a \in A)$ given a price $\hat{p}$, results in the tree's edge capacities being exceeded. An allocation Y on a congestion tree $T = (V, E, A)$ is **congestion free** if for each vertex v the root flow $F_v(Y)$ of the subtree $T_v = (V_v, E_v, A_v)$ with root v does not exceed the capacity C_v, i.e. $|F_v(Y)| \leq C_v$. An allocation Y is **desire compatible** if each agent a is allocated a prosumption between zero and its desire $d_a(\hat{p})$. An allocation Y is **feasible** if it is both congestion free and desire compatible. Finally, we say that a root flow f for a congestion tree T is feasible if there exists a feasible allocation Y on T with $F(Y) = f$.

We define a **congestion solution** Y as a feasible and Pareto allocation on T. We impose these restrictions since a non-feasible allocation does not satisfy the boundary conditions of the congestion problem, and if an allocation is not Pareto then it is possible to improve the allocation for an agent while still resolving congestion.

Congestion only occurs under certain circumstances, and when no congestion occurs a computed congestion solution will simply allocate the desired prosumptions. In contrast to the uncertainty of congestion occurring, the agents always participate in the higher-level market through their submitted demand curves. Thus, if congestion occurs relatively infrequently, agents' higher-level market participation dominates their participation in congestion management mechanisms. Therefore, if the higher-level market is incentive compatible we assume that agents truthfully submit their demand curves. Independent of this assumption curtailment solutions are **individually rational** since, at the same price $\hat{p}$, any prosumption less than the desired prosumption still has positive surplus.

4 FAIR SHARE CONGESTION MANAGEMENT

When, for a price $\hat{p}$, congestion occurs in a congestion tree $T = (V, E, A)$ due to agents' desired prosumptions $d_a(\hat{p})$, there are often different congestion solutions Y possible. As stated before, an allocation Y must at least be feasible and Pareto to qualify as a congestion solution. However, an allocation Y may be required to have additional desirable properties. One such property is that of fairness, which may uniquely determine the allocation Y.

A fair congestion solution Y_{fair} allocates to each agent a 'fair share' of the available capacity. Previous work considers different notions of fairness for this setting [11]. Unique fair allocations are also provided for similar settings, such as by [16]. For the present work it is only important that a fair allocation is unique, feasible, and Pareto. Going forward, when we refer to 'the fair allocation' or 'the fair shares' we will refer to the egalitarian fair allocation discussed in [16]. However, any other notion of fairness that constitutes a unique feasible and Pareto allocation on T, such as those from [11], can be substituted for egalitarian fairness.

5 MAXIMAL WELFARE SOLUTIONS

As opposed to fair congestion solutions we may aim to find congestion solutions in the form of allocations that maximize the welfare of the agents. The problem can be formulated as follows: given a congestion tree $T = (V, E, A)$ and a price $\hat{p}$, find a feasible allocation Y that maximizes the total welfare $\sum_{a \in A} W_a(\hat{p}, Y(a))$.

This problem decomposes into local division problems. Consider a congestion tree $T = (V, E, A)$ consisting of a single vertex r. When consumption congestion (i.e. flow is positive and exceeds capacity) occurs at r then the available capacity C_r has to be divided over the consumers $a \in A^+$. The available consumption capacity C^+ that is to be divided is given by the capacity C_r increased by the (maximal) production of the producers $a \in A^-$, which can meet local demand, i.e. $C^+ = C_r + \sum_{a \in A^-} -d_a(\hat{p})$. This means that the aggregated consumption $\sum_{a \in A^+} Y(a)$ must equal C^+ for the allocation to be Pareto.

Looking at the welfare of the consumers, we see that

$$\begin{aligned}\sum_{a \in A^+} W(\hat{p}, Y(a)) &= \sum_{a \in A^+} \left(\int_0^{Y(a)} m_a(x) - \hat{p}\, dx \right) \\ &= \sum_{a \in A^+} \left(\int_0^{Y(a)} m_a(x)\, dx \right) - \hat{p} \cdot \sum_{a \in A^+} Y(a) \qquad (1) \\ &= \sum_{a \in A^+} \left(\int_0^{Y(a)} m_a(x)\, dx \right) - \hat{p} \cdot C^+.\end{aligned}$$

Equation (1) tells us that optimization of the total welfare among consumers does not depend on the price $\hat{p}$, because it represents a constant factor independent of the division. Hence, the local problem reduces to finding a feasible allocation Y that maximizes

$$\sum_{a \in A^+} \int_0^{Y(a)} m_a(x)\, dx \qquad \text{s.t.} \sum_{a \in A^+} Y(a) = C^+ \qquad (2)$$

The above optimization problem is a standard market efficiency optimization problem, the solution to which is an allocation that minimizes the difference between agents' marginals $m_a(Y(a))$ at their allocated prosumptions. Indeed, when for two consumers a and b we have $m_a(Y(a)) > m_b(Y(b))$, that means consumer a can obtain more welfare from a marginal unit of consumption than consumer b does. Therefore, the allocation Y may be improved in total welfare by shifting an amount of consumption x from consumer b to consumer a such that $m_a(Y(a)+x) = m_b(Y(b)-x)$. A market obtains the unique solution by setting a single (scarcity) price p so that the demands sum up to the available consumption capacity, i.e. p such that $\sum_{a \in A^+} d_a(p) = C^+$. This is the principle under which locational marginal pricing (LMP) works; increasing the local price at r for all agents there reduces consumption while ensuring equal marginals among prosumers.

Our approach bears similarities to locational marginal pricing in that we compute allocations that exactly divide the available consumption capacity C^+ by setting a single marginal p among the consumers. However, we merely use this to compute the allocations and do not alter the actual price $\hat{p}$. As stated earlier, the price $\hat{p}$ does not affect the composition of the maximal welfare allocation.

In our setting of congestion trees with multiple nodes, we must account for possible congestion in subtrees. As such, we may not be able to find a feasible allocation Y for which all consumers' marginals are equal. To solve this problem we will compute feasible welfare maximizing allocations for both minimal and maximal local root flows recursively on subtrees of T. These minimal and maximal local flow allocations then define agent-specific bounds of feasibility for the subtrees of T. The initial agent-specific bounds prior to consideration of subtrees are 0 and the agent's desired prosumption $d_a(\hat{p})$. This is, respectively, because consumers should not be allocated production and vice versa, and because the price $\hat{p}$ makes it so that any units of prosumption in excess of the agent's desired prosumption $d_a(\hat{p})$ are negative welfare for that agent.

The agent-specific bounds are used to bound the agents' demand curves. See Figure 1 for a visual representation. The resulting bounded demand curves $\overline{d}_a$ let us determine prosumption levels based on marginals p, within the constraints of feasibility. Given a set of bounded demand curves $\overline{d}_a$ for the consumers $a \in A^+$, we

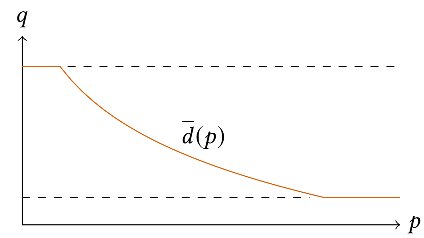

Figure 1: A bounded demand curve $\overline{d}$ (red). The dashed lines indicate the maximum and minimum feasible demands.

can look at the aggregated bounded demand curve to determine a marginal p such that $\sum_{a \in A^+} \overline{d}_a(p) = C^+$. This way we find the consumer allocation $Y(a) = \overline{d}_a(p)$ $(a \in A^+)$ that feasibly divides the available consumption capacity C^+ over the consumers A^+ and maximizes their total welfare. In this case of consumption congestion the producers would be allocated maximal production within their individual bounds, i.e $Y(a) = \overline{d}_a(\hat{p})$ $(a \in A^-)$.

5.1 Maximal Welfare Congestion Algorithm

Consider a congestion tree $T = (V, E, A)$ and a price $\hat{p}$. The recursive algorithm presented here in Algorithm 1 computes for each agent $a \in A$ a unique lower bound l_a and upper bound u_a for which the following is an invariant of subtrees:

Theorem 1. *Given a congestion tree $T = (V, E, A)$ and a price $\hat{p}$, there exists a unique set of lower bounds $\{l_a\}_{a \in A}$ and upper bounds $\{u_a\}_{a \in A}$ such that for any feasible root flow f, a feasible allocation $Y_f^\dagger$ uniquely maximizes the welfare among feasible allocations Y_f with root flow f if and only if $Y_f^\dagger$ is given by either*

$$Y_f^\dagger(a) = \begin{cases} \min(\max(d_a(p), l_a), u_a) & a \in A^+ \\ l_a & a \in A^- \end{cases} \tag{3}$$

for all $a \in A$ or

$$Y_f^\dagger(a) = \begin{cases} u_a & a \in A^+ \\ \min(\max(d_a(p), l_a), u_a) & a \in A^- \end{cases} \tag{4}$$

for all $a \in A$, for some marginal p.

Proof. The proof is by induction. For the induction basis, consider a tree T with only one vertex r. For the agents $a \in A$ we define initial lower and upper bounds l'_a and u'_a that ensure desire compatibility, i.e. 0 and $d_a(p)$ ordered such that $l'_a \leq u'_a$. Consider a feasible root flow$f$. We can formulate three properties of the feasible allocation $Y_f^\dagger$ that maximizes the welfare among feasible allocations with root flow f:

- $Y_f^\dagger$ must be bounded by the bounds l'_a and u'_a to be feasible, i.e. $l'_a \leq Y_f^\dagger(a) \leq u'_a$ $(a \in A)$,
- Consumers must be allocated maximal consumption or producers must be allocated maximal production (i.e. minimal flow). If not, then we can increase both consumption and production by some amount, which means $Y_f^\dagger$ is not Pareto,
- Consumer and producer welfare is maximal when the difference between respective agents' marginals is minimal, as discussed earlier in Section 5.

From these three properties we can see that $Y_f^\dagger$ is given by either

$$Y_f^\dagger(a) = \begin{cases} \min(\max(d_a(p), l'_a), u'_a) & a \in A^+ \\ l'_a & a \in A^- \end{cases} \tag{5}$$

for all $a \in A$ or

$$Y_f^\dagger(a) = \begin{cases} u'_a & a \in A^+ \\ \min(\max(d_a(p), l'_a), u'_a) & a \in A^- \end{cases} \tag{6}$$

for all $a \in A$, for some marginal p. Here, the first property imposes the minimum of l_a and maximum of u_a on all agents $a \in A$, the second property requires allocating either l_a to all producers $a \in A^-$ or u_a to all consumers $a \in A^+$, and the third property leads to a single marginal p across all other agents.

Now consider $Y_{f_{\max}}^\dagger$ and $Y_{f_{\min}}^\dagger$ for the maximal and minimal feasible root flows $f_{\max}$ and $f_{\min}$. By Equations (5) and (6) for all agents, if $f \leq f'$, then $Y_f^\dagger(a) \leq Y_{f'}^\dagger(a)$ $(a \in A)$. Thus, for every feasible root flow f we have $Y_{f_{\min}}^\dagger(a) \leq Y_f^\dagger(a) \leq Y_{f_{\max}}^\dagger(a)$ $(a \in A)$. This means we can take $l_a = Y_{f_{\min}}^\dagger(a)$ $(a \in A)$ and $u_a = Y_{f_{\max}}^\dagger(a)$ $(a \in A)$ to obtain the unique bounds described by Theorem 1.

For the induction step, assume that the theorem holds for all subtrees T_c of T with c a child of the root vertex r. Thus for all agents a not at the root r we have lower and upper bounds l'_a and u'_a by the induction hypothesis. For agents a at the root r we again define initial lower and upper bounds l'_a and u'_a that ensure desire compatibility, i.e. 0 and $d_a(p)$ ordered such that $l'_a \leq u'_a$.

From here we follow the same argumentation as for the induction basis. Since an allocation that feasibly maximizes the welfare on T must also maximize the welfare on each subtree for that subtree's root flow, a maximal welfare allocation on T must be bounded by the bounds l'_a and u'_a. So again, for any feasible root flow f, $Y_f^\dagger$ is given by either Equation (5) or Equation (6) for some marginal p (minimize differences across subtrees). The unique bounds are again obtained by taking $l_a = Y_{f_{\min}}^\dagger(a)$ $(a \in A)$ and $u_a = Y_{f_{\max}}^\dagger(a)$ $(a \in A)$. □

Corollary 2. *For $\{l_a\}_{a \in A}$ and $\{u_a\}_{a \in A}$ as in Theorem 1,*

$$Y_{wel}(a) = \begin{cases} u_a & a \in A^+ \\ l_a & a \in A^- \end{cases} \tag{7}$$

is the unique feasible allocation that maximizes the total welfare among all feasible allocations on T.

Proof. Since Equation (7) is of the form of Equation (3) for some marginal p (and of the form of Equation (4) for some other marginal p), Y_{wel} is feasible. In addition, Y_{wel} uniquely maximizes the prosumption of consumers within the bounds (u_a) and uniquely minimizes the prosumption of producers within the bounds (l_a). Thus Y_{wel} is the unique feasible maximal surplus allocation. □

Our algorithm essentially implements the proof of Theorem 1.

Each recursion step of the algorithm considers a different subtree T_v of T. We will denote the unique lower and upper bounds defined by Theorem 1 for direct subtrees T_c of T_v, i.e. with c a child of vertex v, as l'_a and u'_a respectively for agents $a \in A_c$. In addition, for agents a located at the vertex v, we will similarly denote the initial bounds as l'_a and u'_a. This way, when considering T_v, we have l'_a, u'_a $(a \in A_v)$ denoting the bounds that ensure desire compatibility for all agents and ensure congestion freeness on all strict subtrees of T_v. As such, the information contained in these bounds l'_a, u'_a lets us focus exclusively on the root capacity C_v of T_v when computing the unique bounds l_a, u_a defined for T_v by Theorem 1.

Algorithm 1 presents the algorithmic mechanism that computes the unique bounds l_a, u_a for the congestion tree T, and with them, by Corollary 2, the feasible allocation that maximizes the total welfare on T. On Lines 1 to 8 we compute the bounds l'_a, u'_a for all agents $a \in A$; in Lines 1 to 3 for agents at the root by setting the initial bounds of 0 and their desired prosumption, and in Lines 4 to 8 for all other agents through recursion on subtrees.

On Lines 9 and 10 we introduce a help function for clarity. This function takes any value x and 'bounds' it by the summed l'_a and summed u'_a of a set of agents $B \subset A$. The result is that any value x bounded by this function can be computed as the sum (aggregate) of a set of values that are feasibly allocated to the agents $a \in B$. In essence, the bounding function lets us apply the information contained in the l'_a, u'_a in a simple and straightforward way.

On Lines 11 to 14 we aggregate the agents' demand curves into an aggregated bounded demand and supply curve. The aggregated bounded demand curve indicates for each marginal p what the combined demand of the consumers is. Because of the bounds, these aggregated demands are guaranteed to not cause feasibility or congestion issues in strict subtrees of T. See also Figure 1.

On Lines 15 to 22 we compute the bounds u_a $(a \in A)$ that constitute the unique maximal welfare allocation $Y^{\dagger}_{f_{\max}}$ that feasibly maximizes the root flow. For the consumers, the aggregated demand can at most equal the capacity C_r plus the maximal production $-\sum_{a \in A^-} l'_a$, which means a maximal positive flow as seen on Line 15. If the aggregated demand does not exceed this, i.e. there is no consumption congestion, then the maximal consumption is simply $\sum_{a \in A^+} u'_a$. This last case is caught by the bounded function.

On Line 16, since the aggregated demand is a continuous decreasing (thus invertible) function that takes values between $\sum_{a \in A^+} l'_a$ and $\sum_{a \in A^+} u'_a$, we can select a marginal p that corresponds to the determined maximum positive flow. Then on Lines 17 and 18 we compute the upper bounds u_a for individual consumersa $a \in A^+$ by breaking down the aggregated demand at the selected marginal p.

For the producers, the aggregated supply must at least match the maximal consumption $\sum_{a \in A^+} u'_a$ minus the capacity C_r, which means a minimal negative flow as seen on Line 19. Again, the bounded function on Line 19 catches the no congestion case.

On Lines 20 to 22 we select the marginal p that corresponds to the minimal supply (i.e. maximal negative flow) and subsequently compute u_a for all producers, analogous to Lines 16 to 18.

Analogously to the bounds u_a $(a \in A)$ before, on Lines 23 to 30 we compute bounds l_a $(a \in A)$ that constitute the unique maximal welfare allocation $Y^{\dagger}_{f_{\min}}$ that feasibly minimizes the root flow.

With the output of Algorithm 1, the maximal welfare allocation Y_{wel} on T can now be found by taking for each agent $a \in A$ their most extreme bound, i.e. for each consumer $a \in A^+$ their upper bound u_a and for each producer $a \in A^-$ their lower bound l_a.

Algorithm 1: MaxWelfare $(T, \hat{p})$

Input: *A congestion tree $T = (V, E, A)$ and a price $\hat{p}$*
Output: *The unique l_a and u_a for all $a \in A$*

// Initialize with agent desires
1 **for** *agents a at root r* **do**
2 $\quad l'_a \leftarrow \min(0, d_a(\hat{p}))$
3 $\quad u'_a \leftarrow \max(0, d_a(\hat{p}))$
// Recursion on child vertices
4 **for** *children c of root r* **do**
5 $\quad$ lowers, uppers $\leftarrow$ `maxwelfare`$(T_c, \hat{p})$
6 $\quad$ **for** *agents $a \in A_c$* **do**
7 $\quad\quad l'_a \leftarrow$ lowers$[a]$
8 $\quad\quad u'_a \leftarrow$ uppers$[a]$
// Now we have l'_a and u'_a $\forall a \in A$
// Add a bounding function for clarity:
9 **Function** bounded(x, B)
$\quad$ **Input:** *A value $x \in \mathbb{R}$ and a subset of agents $B \subset A$*
$\quad$ **Output:** *The value closest to x between the combined lower and upper bounds of the agent(s) in B*
10 $\quad$ **return** $\min(\max(x, \sum_{a \in B} l'_a), \sum_{a \in B} u'_a))$
// Aggregate bounded demand curves:
11 **Function** demand(p)
12 $\quad$ **return** $\sum_{a \in A^+}$ bounded$(d_a(p), a)$
13 **Function** supply(p)
14 $\quad$ **return** $\sum_{a \in A^-}$ bounded$(d_a(p), a)$
// Compute maximum flow values in two steps:
// Compute positive u_a values
15 maxposflow $\leftarrow$ bounded$(C_r - \sum_{a \in A^-} l'_a, A^+)$
16 Select marginal p s.t. demand(p) = maxposflow
17 **for** $a \in A^+$ **do**
18 $\quad u_a \leftarrow$ bounded$(d_a(p), a)$
// Compute negative u_a values
19 maxnegflow $\leftarrow$ bounded$(C_r - \sum_{a \in A^+} u'_a, A^-)$
20 Select marginal p s.t. supply(p) = maxnegflow
21 **for** $a \in A^-$ **do**
22 $\quad u_a \leftarrow$ bounded$(d_a(p), a)$
// Compute minimum flow values in two steps:
// Compute negative l_a values
23 minnegflow $\leftarrow$ bounded$(-C_r - \sum_{a \in A^+} u'_a, A^-)$
24 Select marginal p s.t. supply(p) = minnegflow
25 **for** $a \in A^-$ **do**
26 $\quad l_a \leftarrow$ bounded$(d_a(p), a)$
// Compute positive l_a values
27 minposflow $\leftarrow$ bounded$(-C_r - \sum_{a \in A^-} l'_a, A^+)$
28 Select marginal p s.t. demand(p) = minposflow
29 **for** $a \in A^+$ **do**
30 $\quad l_a \leftarrow$ bounded$(d_a(p), a)$
// Now we have l_a and u_a $\forall a \in A$
31 **return** $\{l_a\}_{a \in A}, \{u_a\}_{a \in A}$

THEOREM 3. *Given a congestion tree $T = (V, E, A)$ and a price $\hat{p}$, Algorithm 1 computes the unique feasible maximal welfare allocation Y_{wel} described by Corollary 2.*

PROOF. We showed how Algorithm 1 computes the bounds l_a, u_a described by Theorem 1. The maximal welfare allocation Y_{wel} on T is then found by simply taking for each consumer $a \in A^+$ their upper bound u_a and for each producer $a \in A^-$ their lower bound l_a, as described by Corollary 2. □

6 A HYBRID SOLUTION WITH A CHOICE

In Sections 4 and 5 respectively, we introduced the fair allocation Y_{fair} and the maximal welfare allocation Y_{wel}, both of which are congestion solutions (i.e. feasible and Pareto). Being curtailment solutions, the price per unit that the agents trade their allocated prosumptions for is the higher-level market price $\hat{p}$.

Each agent carries a private preference for one of the two solutions. Such preferences may be principled or based on individual circumstance. Where the fair solution provides agents with prosumptions that are fair relative to each other at the cost of welfare, the maximal welfare solution rewards economical efficiency at the cost of some welfare among less efficient agents. This last situation can be observed with LMP as well, where scarcity pricing pushes some less competitive agents out of the market.

In this section we propose a way for the two solutions to exist in parallel, and for individual agents to choose in which one they want to participate. We do this by allowing agents to either "claim their fair share" or participate in a congestion aftermarket. More specifically, we first curtail agents with the fair allocation Y_{fair}. Then the subset $A^{fair} \subset A$ of agents that indicate that they want to participate in the fair solution are allocated as determined by Y_{fair}. Subsequently, we compute a feasible maximal welfare allocation for the remaining agents, with the prosumption already allocated to the agents in A^{fair} fixed. This gives us a hybrid allocation Y_{hyb}.

Algorithm 2 presents a modified version of Algorithm 1 that computes the hybrid solution Y_{hyb}. The modification is small: for the agents $a \in A^{fair}$ that choose to claim their fair share, we initialize both their bounds at this fair share on Lines 2 to 4. For the other agents Algorithm 2 then proceeds identical to Algorithm 1.

Algorithm 2: Hybrid $(T, \hat{p}, Y_{fair}, A^{fair})$

Input: *A congestion tree $T = (V, E, A)$, a price $\hat{p}$, a congestion solution Y_{fair} and a subset of agents A^{fair}*
Output: *Unique l_a and u_a for all $a \in A$*

`// Initialize with fair shares or agent desires`
1 **for** *agents a at root r* **do**
2   **if** $a \in A^{fair}$ **then**
3     $l'_a \leftarrow Y_{fair}(a)$
4     $u'_a \leftarrow Y_{fair}(a)$
5   **else**
6     $l'_a \leftarrow \min(0, d_a(\hat{p}))$
7     $u'_a \leftarrow \max(0, d_a(\hat{p}))$
`// Recursion on child vertices`
8 **for** *children c of root r* **do**
9   lowers, uppers $\leftarrow$ Hybrid$(T_c, \hat{p}, Y_{fair}, A^{fair})$
10   **for** *agents $a \in A_c$* **do**
11     $l'_a \leftarrow$ lowers$[a]$
12     $u'_a \leftarrow$ uppers$[a]$
`// Now we have` l'_a `and` u'_a $\forall a \in A$
13 From here proceeds identical to Algorithm 1

THEOREM 4. *Given a congestion tree $T = (V, E, A)$, a price $\hat{p}$, a fair congestion solution Y_{fair}, and a subset of agents A^{fair}, Algorithm 2 computes the unique feasible allocation Y_{hyb} that allocates the fair share $Y_{fair}(a)$ to agents $a \in A^{fair}$ and maximizes the total welfare of the agents $a \in A \setminus A^{fair}$ given the fair shares allocated to A^{fair}.*

PROOF. Algorithm 2 initializes $l'_a = u'_a = Y_{fair}(a)$ for agents $a \in A^{fair}$. If $l'_a = u'_a$ for an agent a then $l'_a = l_a = u_a = u'_a$ because of the use of the bounded function when computing l_a and u_a. Therefore, the lower and upper bounds of such an agent will stay constant through recursive steps of the algorithm. As a result, $Y_{hyb}(a) = Y_{fair}(a)$ for $a \in A^{fair}$.

For agents $a \notin A^{fair}$, the bounds l_a and u_a are computed identically to Algorithm 1. Since the fair shares $Y_{fair}(a)$ claimed by agents $a \in A^{fair}$ are part of the feasible allocation Y_{fair}, we know that fixing these prosumptions does not render it impossible to find a feasible allocation. In other words, for agents $a \notin A^{fair}$, Algorithm 2 can be regarded as Algorithm 1 on a congestion tree with its capacities adjusted for the fair shares $Y_{fair}(a)$ claimed by agents $a \in A^{fair}$. Thus, Y_{hyb} maximizes the total welfare among agents $a \notin A^{fair}$ given that $Y_{hyb}(a) = Y_{fair}(a)$ for $a \in A^{fair}$. □

The computation of the hybrid solution Y_{hyb} from a fair allocation Y_{fair} gives rise to the **difference allocation** Y_{diff}:

$$Y_{diff}(a) = Y_{hyb}(a) - Y_{fair}(a) \qquad a \in A. \tag{8}$$

This difference allocation indicates how the prosumptions allocated to maximize welfare deviate from the fair shares. Accordingly, $Y_{diff}(a) = 0$ ($a \in A^{fair}$). Y_{diff} essentially tells us how units of prosumption are transferred between agents relative to their fair shares, and thus will form the basis for the congestion aftermarket.

For some agents $a \in A$ the change $Y_{diff}(a)$ from their fair share $Y_{fair}(a)$ to $Y_{hyb}(a)$ moves them away from their desired prosumption $d_a(\hat{p})$. In order to incentivize these agents to still choose to participate in welfare maximization we can implement a pricing scheme. In Section 7 we will lay out the specifics of such a pricing scheme, including computation of agent-specific prices p_a. What is important is that we can interpret the difference allocation Y_{diff} as a congestion aftermarket in the following way. Each agent $a \in A$ gets to trade its allocated fair share $Y_{fair}(a)$ at the higher-level market price $\hat{p}$. Subsequently, agents $a \in A$ can choose to enter the competitive congestion aftermarket to trade an amount of prosumption equal to $Y_{diff}(a)$ at a certain price p_a (defined in Section 7). This may mean either selling a portion of their allocated fair share or purchasing additional prosumption from other agents participating in the aftermarket.

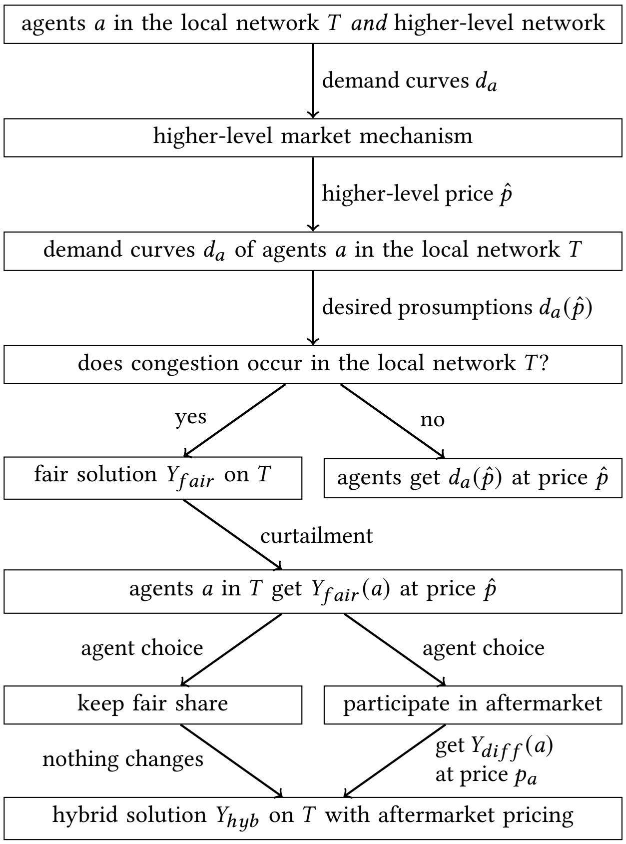

Figure 2: Visualization of the construction of a hybrid congestion solution: a fair curtailment solution with optional participation in a welfare-maximizing priced aftermarket.

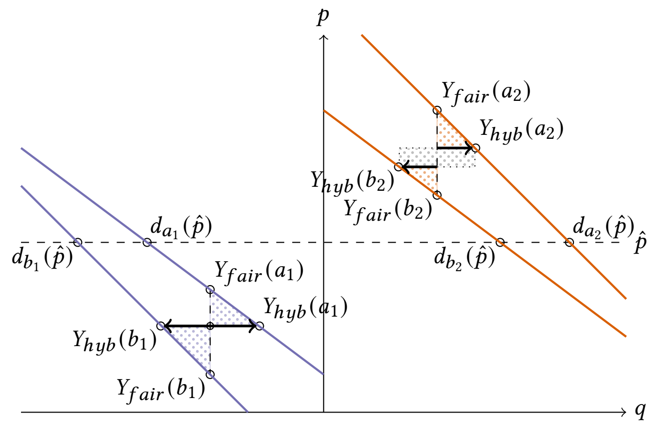

Figure 3: Two examples of transitions from higher-level market to fair curtailment to aftermarket, shown on marginal functions of two producers a_1, b_1 (blue, left) that share a connection and two consumers a_2, b_2 (red, right) that share a connection. Indicated values refer to q-coordinates. Desires of all four agents have equal (on a horizontal line) marginals, namely the price $\hat{p}$. Subsequently, the producers are allocated an equal (on a vertical line) fair share as indicated, as are the consumers. Finally, the producers return to equal (on a horizontal line) marginals in the aftermarket. The consumers, however, are constrained by some intermediate capacity in this example, so in the aftermarket their marginals only approach each other. The black arrows indicate the sign and magnitude of aftermarket trades Y_{diff}, and the hatched areas indicate the prosumers' aftermarket surplus. The efficiency gap between the consumers is shown in gray, with any price between the two marginals being acceptable for both parties.

Note that when no congestion occurs, then both Y_{fair} and Y_{hyb} simply allocate the desired prosumptions $d_a(p)$ to all agents $a \in A$. This means that Y_{diff} is zero and hence that the congestion aftermarket does not exist. In other words, the congestion aftermarket only serves to let agents efficiently divide the available capacity among themselves when congestion occurs. Importantly, the aftermarket approach does not interfere in the higher-level market mechanism when no congestion occurs.

Figure 2 visualizes the different steps taken to arrive at the hybrid congestion solution Y_{hyb} with a congestion aftermarket. First the interaction between the agents' demand curves d_a, the higher-level market and its price $\hat{p}$, and the desired prosumptions $d_a(\hat{p})$ is indicated. We then turn our attention to the local network T where the desired prosumptions $d_a(\hat{p})$ may cause congestion. If no congestion occurs in T, the higher-level market mechanism can operate as intended in T. If, however, congestion does occur in T we must curtail the desired prosumptions $d_a(\hat{p})$, for which we use a fair congestion solution Y_{fair}. Now, based on agents' private preferences, agents $a \in A$ either claim their fair share $Y_{fair}(a)$ at price $\hat{p}$ or participate in a welfare-maximizing congestion aftermarket. The resulting hybrid solution Y_{hyb} provides every agent $a \in A$ with their fair share $Y_{fair}(a)$ at the price $\hat{p}$, but on top of that provides those agents $a \notin A^{fair}$ that chose to participate in the aftermarket with a prosumption $Y_{diff}(a)$ traded at a certain (averaged) price p_a.

Figure 3 shows how first welfare is maximized by market clearing, then fair shares are allocated to resolve congestion, and finally welfare is increased again through the congestion aftermarket.

7 AN AFTERMARKET PRICING SCHEME

In Section 6 we discussed a congestion aftermarket where units of prosumption are traded according to a difference allocation Y_{diff} at individual prices p_a, without specifying these prices. In this section we present an explicit pricing scheme for the congestion aftermarket. This pricing scheme will ensure budget balance and individual rationality. Individual rationality means that it will be economically beneficial for agents to participate in the aftermarket, independent of whether they buy or sell units of prosumption there.

Our goal for this pricing scheme is to put prices on trades of prosumption between agents that are given by the difference allocation Y_{diff}. By putting prices on bilateral trades rather than on purchases and sales individually, we will automatically satisfy budget balance. For individual rationality, the price for a trade should be such that both agents get positive surplus out of each unit transferred between them in the trade. If the post-aftermarket

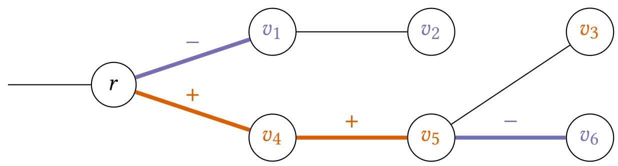

Figure 4: A congestion tree T with three congestion regions. Edges from v towards r marked with + indicate $F_v(Y) = C_v$ (red) while − indicates $F_v(Y) = -C_v$ (blue). Vertices v_1, v_4, v_5, v_6 are intermediate obstructions, but only v_1, v_4, v_6 are bottlenecks. $\{v_1, v_2\}$ and $\{v_6\}$ are negative congestion regions while $\{v_3, v_4, v_5\}$ is a positive congestion region. $\{r\}$ is uncongested.

marginals $m_a(Y_{hyb}(a))$ and $m_b(Y_{hyb}(b))$ of an aftermarket buyer a and seller b are equal, then we can put a price equal to that marginal on the trade between the two agents. In this case the aftermarket operates as regular unconstrained market-clearing.

However, due to the capacity constraints, not all trades attain maximal efficiency. For example, consider two consumers a and b that are curtailed by Y_{fair} because of a congestion they both contributed to. Now, consumer a would like to purchase a number of units x from consumer b because consumer b has a smaller marginal $m_b(Y_{fair}(b)) < m_a(Y_{fair}(a))$. Ideally, this number of units x is such that the two consumers' marginals become equal, i.e. $m_b(Y_{fair}(b) - x) = m_a(Y_{fair}(a) + x)$, and consumer a would pay a price per unit equal to that marginal. However, in our example we have this trade cause new congestion on one of the edges between the vertices where consumers a and b are located. Thus, a reduced number of units is traded. This means that after the trade, the marginal of consumer a will still be higher than that of consumer b, i.e. $m_b(Y_{fair}(b)) < m_b(Y_{hyb}(b)) < m_a(Y_{hyb}(a)) < m_a(Y_{fair}(a))$. For both consumers, any price between the two marginals results in a positive surplus. The gap indicates an economic inefficiency caused by the constraints, shown in Figure 3 for agents a_2 and b_2.

To deal with pricing under these capacity constraints we need to consider three aspects. First, we need to determine which trades can happen in which part of the network. Second, we need to determine what price to put on a trade when multiple prices yield a mutually beneficial trade. Third, we need to determine which agents trade with which agents and in what quantity.

For the **first aspect**, we look at what we call bottlenecks and congestion regions. In short, a congestion tree $T = (V, E, A)$ consists of alternating positive and negative congestion regions separated by congestion bottlenecks, and aftermarket trades are confined to these congestion regions. There may also be a single uncongested region containing the root r of T where all agents a are allocated their desired prosumption $d_a(\hat{p})$ by both Y_{hyb} and Y_{fair}, which therefore does not play a role in the aftermarket. See also Figure 4.

Definition 5. Given a congestion tree $T = (V, E, A)$, a price $\hat{p}$, and a congestion solution Y, we say that a vertex v is a positive **intermediate obstruction** if $F_v(Y) = C_v$ and there exists a consumer a with $Y(a) \neq d_a(\hat{p})$ at a vertex $u \in V_v$ such that for all vertices $w \neq v$ on the path from u to v, $F_w(Y) < C_w$. Analogously for a negative intermediate obstruction.

Definition 6. Given a congestion tree $T = (V, E, A)$, a price $\hat{p}$, and a congestion solution Y, we say that a positive intermediate obstruction v is a positive **bottleneck** if not the closest other intermediate obstruction on the root path of v is also a positive intermediate obstruction. Analogously for a negative bottleneck.

Definition 7. Given a congestion tree $T = (V, E, A)$, a price $\hat{p}$, and a congestion solution Y, we say that a subgraph R^+ of T is a positive **congestion region** if it is a connected component of the forest obtained from T by removing all edges between bottlenecks and their parents, and it contains a positive bottleneck. Analogously for a negative congestion region R^-.

Lemma 8. Given a congestion tree $T = (V, E, A)$, a price $\hat{p}$, and a congestion solution Y, all producers in a positive congestion region R^+ are allocated their desired prosumption. i.e. $Y(a) = d_a(\hat{p})$ for all $a \in A^-$ in R^+. Analogously for consumers.

Proof. Assume, without loss of generality, that there is a positive congestion region R^+ and a producer $a \in A^-$ with $Y(a) \neq d_a(\hat{p})$ located at a vertex u in R^+. Consider, on the root path of u, the intermediate obstruction v closest to u. Such v exists and is in R^+ because a congestion region contains a bottleneck that is on the root path of every vertex in that congestion region. Since a positive congestion region, by definition, contains no negative intermediate obstructions, v is a positive intermediate obstruction. Hence there exists a vertex u' with a consumer $b \in A^+$ for which $Y(b) \neq d_b(\hat{p})$ such that, on the root path of u', v is the vertex closest to u' that is at positive capacity (i.e. $F_v(Y) = C_v$). Therefore, since v is the closest intermediate obstruction to both u and u' on their root paths, there exists an $\epsilon > 0$ such that $Y(a)$ and $Y(b)$ can be feasibly decreased and increased, respectively, by ϵ without causing congestion on the root paths of u and u', and thus anywhere in T. We conclude that the congestion solution Y is not Pareto, which is a contradiction. □

Lemma 9. Given a congestion tree $T = (V, E, A)$ and a price $\hat{p}$, bottlenecks are independent of the congestion solution Y.

Proof. The proof is by induction. For the induction basis, consider without loss of generality a positive bottleneck v with no other bottlenecks in its subtree T_v. Since $F_v(Y) = C_v$, a congestion solution Y' for which v is not a bottleneck must have $F_v(Y') < C_v$. Since, by Lemma 8, $Y(a) = d_a(\hat{p})$ $(a \in A_v^-)$, it must also be that $Y'(a) = d_a(\hat{p})$ $(a \in A_v^-)$ for Y' to be Pareto. Thus it must be that $\sum_{a \in A_v^+} Y'(a) < \sum_{a \in A_v^+} Y(a)$. Now we consider two cases.

If there exist no other bottlenecks on the root path of v, then v connects to an uncongested region R where $Y(a) = d_a(\hat{p})$ (a in R). In this case, Y' cannot feasibly allocate more to any agent a in R than Y does, i.e. $Y'(a) = Y(a)$ (a in R). But Y' allocates less to consumers $a \in A_v^+$ than Y does, so Y' is not Pareto.

If there do exist other bottlenecks on the root path of v, then among these the closest to v must be a negative bottleneck u. u is in a negative congestion region R^- that v connects to. Since $F_u(Y) = -C_u$ and $F_v(Y') < F_v(Y)$, it must be that Y' allocates more to agents in R^- than Y does, i.e. $\sum_{a \in B} Y'(a) > \sum_{a \in B} Y(a)$ for B the set of agents in R^- (agents $a \in A_u \setminus (A_v \cup B)$ are in subtrees of positive bottlenecks in T_u). But because, by Lemma 8, $Y(a) = d_a(\hat{p})$ $(a \in B^+)$, this means that $\sum_{a \in B^-} Y'(a) > \sum_{a \in B^-} Y(a)$. So in this case not only are the consumers $a \in A_v^+$ now allocated less (less consumption) by Y' than by Y, also the producers $a \in B^-$ in R^- are allocated more (less production) by Y' than by Y, which means that Y' is not Pareto.

For the induction step, assume that the lemma holds for all bottlenecks other than v in T_v. Hence we know that $F_u(Y') = F_u(Y)$ for any of these other bottlenecks u, for all congestion solutions Y'. Therefore the can follow the same argumentation as in the induction basis for the congestion region R^+ containing v (instead of T_v). ◻

With these definitions and lemmas we formalized the fact that congestion trees consist of alternating positive and negative congestion regions invariant across congestion solutions, which are the only allocations that we are interested in. Differences between congestion solutions only appear among consumers in positive congestion regions and producers in negative congestion regions.

Lemma 10. The aftermarket trades Y_{diff} in the subtree T_v of a bottleneck v result in a net-zero root flow, i.e. $F_v(Y_{diff}) = 0$.

Proof. Since Y_{diff} is the difference between two congestion solutions, Y_{hyb} and Y_{fair}, this follows from Lemma 9. ◻

Lemma 11. The aftermarket trades Y_{diff} of all producers in positive congestion regions R^+ are zero, i.e. $Y_{diff}(a) = 0$ for producers $a \in A^-$ in R^+. Analogously for consumers.

Proof. This follows from Lemmas 8 and 10. ◻

Lemmas 10 and 11 show how aftermarket trades are confined to congestion regions, and that consumers only trade with other consumers (within the same positive congestion region) while producers only trade with other producers (within the same negative congestion region). Consequently, both consumers and producers only interact in one of two ways in the aftermarket: they either buy from, or sell to, agents with the same prosumption sign (±).

For the **second aspect** (what price to put on a trade when multiple prices yield a mutually beneficial trade), we distinguish between two types of aftermarket participants which we call strainers and relievers. The strainers' aftermarket trades Y_{diff} are aligned with their prosumption (i.e. $d_a(\hat{p}) > 0 < Y_{diff}(a)$ or $d_a(\hat{p}) < 0 > Y_{diff}(a)$ for a strainer a), moving them closer to their desired prosumption. Because their prosumption was curtailed by Y_{fair} to resolve congestion, these movements strain the line capacities. To relieve this strain on capacity, the relievers accept aftermarket trades that move them further away from their desired prosumption (i.e. $d_a(\hat{p}) > 0 > Y_{diff}(a)$ or $d_a(\hat{p}) < 0 < Y_{diff}(a)$ for a reliever a). In a positive congestion region R^+, the strainers are consumers $a \in A^+$ in R^+ that buy (i.e. $d_a(\hat{p}) > 0 < Y_{diff}(a)$) and the relievers are consumers $a \in A^+$ in R^+ that sell (i.e. $d_a(\hat{p}) > 0 > Y_{diff}(a)$), while in a negative congestion region R^- the strainers are producers $a \in A^-$ in R^- that sell (i.e. $d_a(\hat{p}) < 0 > Y_{diff}(a)$) and the relievers are producers $a \in A^-$ in R^- that buy (i.e. $d_a(\hat{p}) < 0 < Y_{diff}(a)$).

This leads us to choose the marginal of the strainer as the price for every trade, for two reasons. Firstly and objectively, in the aftermarket a strainer a attains, with $Y_{fair}(a) + Y_{diff}(a)$, at most its desired prosumption $d_a(\hat{p})$ for which the marginal is $\hat{p}$ for all agents, while a reliever a is bounded by a prosumption of zero for which the marginal $m_a(0)$ exclusively depends on the submitted demand curve. This makes the strainer's marginal the most consistent choice of price since it always reflects the real marginal value in the aftermarket at its vertex and not a bounded value. Secondly and subjectively, we established that the role of the relievers is to enable additional prosumption for strainers by essentially resolving some congestion. Since any price between the marginals of the reliever and strainer is acceptable for a trade, we may want to maximally reward the role that works to resolve congestion by setting the price at the strainer's marginal (instead of at for example the midpoint).

For the **third aspect** (which agents trade with which agents and in what quantity), we look at the matching of supply and demand in the aftermarket. When a congestion region contains multiple strainers and relievers in the aftermarket, it is not yet clear which strainer trades with which reliever. Note that this matching merely labels indistinguishable units whose flows are already determined by Y_{diff}. Since strainers always pay a price equal to their marginals, the matching does not impact them. For relievers, however, it can matter which strainer they are said to trade with. To resolve the ambiguity, we simply proportionally match all strainers and relievers that could trade with each other. As a result, relievers' prices are set at the proportional average of the accessible strainers' marginals.

We implement this proportional matching recursively on subtrees, which, by Lemma 10, results in matching within congestion regions. This recursive approach where relievers and strainers in the aftermarket are maximally matched locally within each subtree before moving up the tree to larger subtrees (i.e. greedy local-first matching), is equivalent to our goal of matching every reliever with every strainer accessible to them. These are equivalent because if there is a difference in marginals between strainers in a congestion region, then there must be an intermediate obstruction between them that necessitates local matching or else Y_{hyb} would not maximize welfare. If there is no difference between the strainer's marginals then all matchings with relievers are equivalent.

Algorithm 3 presents our pricing scheme in accordance with the three discussed aspects. With information about recursively traded quantities on subtrees (Lines 1 to 8) we can identify the quantity that each agent will still trade outside strict subtrees of T. On Line 11 we compute these untraded quantities for each agent $a \in A$. The untraded quantities tell us the total remaining aftermarket demand and supply in the subtree T, computed on Lines 12 and 13. The maximal matching of supply and demand, given on Line 14, determines the portions q'_a of the untraded quantities that will be traded within T among agents $a \in A$. Note that if the root r of T is a bottleneck then, by Lemma 10, $F_r(Y_{diff}) = 0$, i.e. all aftermarket supply and demand in T is matched and untraded quantities are 0.

On Line 15 we check if any trades can be made in T. If not, we simply return the prices p_a and quantities q_a. Otherwise, we can assign new quantities traded in T. On Lines 16 to 19 we proportionally assign quantities of supply and demand. For the supply or demand side or both, the newly traded quantities q'_a equal the previously untraded quantities $\mathtt{untraded}[a]$, i.e. we maximally match aftermarket supply and demand within T.

On Lines 20 and 21 we identify which agents are strainers and which are relievers. Having identified the strainers, we can compute a single price p_{rel} for the the reliever side of the newly matched trades in T. Since we want to proportionally match each reliever with each strainer, we compute this price as the proportional average of the strainers' prices p_a, i.e. their marginals $m_a(Y_{hyb}(a))$.

Algorithm 3: PricingScheme (T, Y_{hyb}, Y_{fair})

Input: *A congestion tree $T = (V, E, A)$ and the two congestion solutions Y_{hyb} and Y_{fair}*
Output: *For each $a \in A$, a price p_a and a quantity traded q_a*

`// Initialize prices and quantities`
1 **for** *agents a at root r* **do**
2 $\quad p_a \leftarrow 0$
3 $\quad q_a \leftarrow 0$

`// Recursion on child vertices`
4 **for** *children c of root r* **do**
5 $\quad$ prices, quantities $\leftarrow$ PricingScheme(T_c, Y_{hyb}, Y_{fair})
6 $\quad$ **for** *agents $a \in A_c$* **do**
7 $\quad\quad p_a \leftarrow$ prices$[a]$
8 $\quad\quad q_a \leftarrow$ quantities$[a]$

`// Infer total aftermarket quantities`
9 **for** *agents $a \in A$* **do**
10 $\quad Y_{diff}(a) \leftarrow Y_{hyb}(a) - Y_{fair}(a)$

`// Identify still untraded quantities (of` Y_{diff}`)`
11 untraded $\leftarrow \{Y_{diff}(a) - q_a\}_{a \in A}$

`// Total still untraded demand and supply`
12 demand $\leftarrow \sum_{a \in A} \max(0, \text{untraded}[a])$
13 supply $\leftarrow \sum_{a \in A} \min(0, \text{untraded}[a])$

`// Maximally match untraded demand and supply`
14 matching $\leftarrow \min(\text{demand}, -\text{supply})$

15 **if** matching $\neq 0$ **then**
$\quad$ `// Proportionally assign new trade quantities`
16 $\quad$ **for** $a \in A$ *with* untraded$[a] > 0$ **do**
17 $\quad\quad q'_a \leftarrow \frac{\text{matching}}{\text{demand}} \cdot \text{untraded}[a]$
18 $\quad$ **for** $a \in A$ *with* untraded$[a] < 0$ **do**
19 $\quad\quad q'_a \leftarrow \frac{\text{matching}}{-\text{supply}} \cdot \text{untraded}[a]$
$\quad$ `// Identify strainers by trades aligned with prosumption, and relievers as complement`
20 $\quad$ strainers $\leftarrow \{a \in A \mid \text{untraded}[a] \cdot Y_{hyb}(a) > 0\}$
21 $\quad$ relievers $\leftarrow \{a \in A \mid \text{untraded}[a] \neq 0\} \setminus$ strainers
$\quad$ `// Proportional average of strainers' marginals sets relief price of new trades`
22 $\quad p_{\text{rel}} \leftarrow \left(\sum_{a \in \text{strainers}} q'_a \cdot m_a(Y_{hyb}(a))\right) / \sum_{a \in \text{strainers}} q'_a$
$\quad$ `// Update price and quantity with new trades`
23 $\quad$ **for** $a \in$ strainers **do**
24 $\quad\quad p_a \leftarrow m_a(Y_{hyb}(a))$
25 $\quad\quad q_a \leftarrow q'_a + q_a$
26 $\quad$ **for** $a \in$ relievers **do**
27 $\quad\quad p_a \leftarrow (p_{\text{rel}} \cdot q'_a + p_a \cdot q_a)/(q'_a + q_a)$
28 $\quad\quad q_a \leftarrow q'_a + q_a$
29 **return** $\{p_a\}_{a \in A}, \{q_a\}_{a \in A}$

We are now ready to set a price p_a for each agent $a \in A$. For the strainers, on Lines 23 and 24, the price p_a always equals their marginal $m_a(Y_{hyb}(a))$ as discussed before. For the relievers, on Lines 26 and 27, the price of the new trades q'_a in T is set by the price p_{rel} (taking the weighted average of new and previous trades).

The output of the pricing scheme presented in Algorithm 3 on the full congestion tree $T = (V, E, A)$ on which Y_{fair} and Y_{hyb} are defined is, for each agent $a \in A$, a price p_a and a quantity traded q_a. This final quantity traded q_a equals the predetermined aftermarket trade $Y_{diff}(a)$. The price p_a is the price at which the agent a trades the quantity q_a on the congestion aftermarket, with the total amount paid given by $q_a \cdot p_a = Y_{diff}(a) \cdot p_a$.

For the hybrid allocation Y_{hyb} as a whole, each agent $a \in A$ receives its fair share of prosumption $Y_{fair}(a)$ at a market clearing price $\hat{p}$, and on top of that optionally trades $Y_{diff}(a)$ prosumption in the congestion aftermarket at a price p_a. The total payment for the final prosumption $Y_{hyb}(a)$ is given by $Y_{fair}(a) \cdot \hat{p} + Y_{diff}(a) \cdot p_a$.

Theorem 12. *Given the allocations Y_{hyb}, Y_{fair}, and their difference Y_{diff} on the congestion tree $T = (V, E, A)$, Algorithm 3 computes prices p_a for the aftermarket trades $Y_{diff}(a)$ $(a \in A)$ such that*

$$\sum_{a \in A} Y_{diff}(a) \cdot p_a = 0 \quad \textit{(budget balance)}$$

$$\int_{Y_{fair}(a)}^{Y_{hyb}(a)} m_a(x) - p_a \, dx \geq 0 \ (a \in A) \quad \textit{(individual rationality).}$$

Proof. The aftermarket is easily seen to be budget balanced since we showed how Algorithm 3 computes the reliever prices by aggregating and exactly distributing the constrainer prices. We also showed how we chose constrainer prices to always equal their marginals in Y_{hyb}, and how reliever prices are consequently equal to or better than their marginals in Y_{hyb}. Therefore, each unit traded between agents in the aftermarket has positive prosumption surplus for both strainer and reliever, resulting in individual rationality. □

Note that since aftermarket trades only occur between agents in the same congestion region, as stated by Lemma 10, Theorem 12 also holds for any subtree T_v of a bottleneck v.

8 FULL AFTERMARKET PARTICIPATION

Since it is individually rational for agents to participate in the congestion aftermarket, it may be that all agents choose to participate in it. In this case, since A^{fair} is empty, $Y_{hyb} = Y_{wel}$. Moreover, since the aftermarket is budget-balanced and Y_{wel} maximizes total welfare, Y_{hyb} with the aftermarket also maximizes the total welfare. The difference between the two is the distribution of the welfare among the agents. The aftermarket, relative to Y_{wel}, increases welfare for some agents while decreasing it for some others.

The deciding factor in how the welfare is redistributed among the agents between Y_{wel} and Y_{hyb} with the aftermarket is the fair allocation Y_{fair}. If $Y_{fair} = Y_{wel}$ then nothing is traded in the aftermarket and welfare is distributed identically among agents both with and without aftermarket, but any other Y_{fair} results in a redistribution of welfare relative to Y_{wel}. What is interesting to note is that the choice of fair shares through Y_{fair} thus translates to a choice of welfare distribution among the agents, even if no agent claims their fair share and all agents participate in the aftermarket.

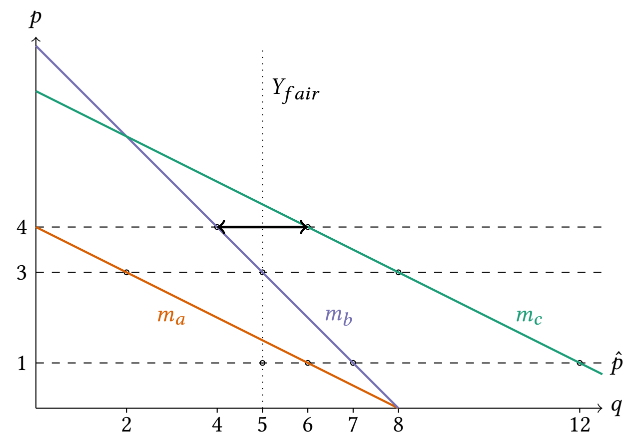

Figure 5: An example congestion problem. Shown are marginal functions for consumers a (red), b (blue), and c (green) with simple linear demand curves $d_a(p) = 8-2p$, $d_b(p) = 8-p$, and $d_c(p) = 14-2p$. These consumers share a total capacity of 15, and the higher-level price $\hat{p}$ is currently 1. The consumers' demands at $p = 1$ lead to a total demand of $6+7+12 = 25$, exceeding the capacity by 10. LMP sets the local congestion price at 3 to reduce total demand to $2+5+8 = 15$. An equitable solution allocates a total consumption of $5+5+5 = 15$. A subsequent aftermarket trade at $p = 4$, in which consumer a does not participate, allocates a total consumption of $5+4+6 = 15$.

9 NUMERICAL EXAMPLE

Figure 5 and Table 1 show a simple numerical example illustrating the benefits of our hybrid solution over straightforward LMP.

10 CONCLUSIONS

In this paper we considered congestion management in systems modelled after electricity distribution networks. In particular, we considered the welfare expressed by agents' demand curves in relation to quantities allocated by congestion-resolving curtailment mechanisms. We differentiated between congestion solutions focused on fairness and those focused on maximizing welfare.

We presented an algorithmic mechanism to find such welfare-maximizing congestion solutions for tree networks populated by both consumers and producers. These model e.g. common prosumer-oriented low- and medium-voltage electricity networks. We then went on to propose a hybrid congestion solution that provides agents with the choice between fairness and welfare maximization. We argued that giving agents the choice to claim a fair share of the available capacity at the original higher-level market clearing price is sufficient to constitute a fair congestion management mechanism. In such a mechanism we can let agents that do not choose to claim their fair share engage in welfare-maximizing activity amongst themselves. We achieved this choice-based hybrid congestion solution by applying our welfare-maximizing mechanism after locking in the fair shares of agents that decided to claim their fair share.

We then went on to define the welfare-maximizing part of our hybrid congestion solution as a congestion aftermarket by presenting a pricing scheme for the changes relative to the fair solution.

	Y	Y_{LMP}	Y_{fair}	Y_{hyb}	Y'_{hyb}
price p	1	3	1	1 & 4	1 & 3
consumption$_a$	6	2	5	5	2
consumption$_b$	7	5	5	4	5
consumption$_c$	12	8	5	6	8
consumption$_{\text{total}}$	**25**	**15**	**15**	**15**	**15**
payment$_a$	6	6	5	5	-4
payment$_b$	7	15	5	1	5
payment$_c$	12	24	5	9	14
payment$_{\text{total}}$	**25**	**45**	**15**	**15**	**15**
surplus$_a$	9	1	8.75	8.75	11
surplus$_b$	24.5	12.5	22.5	23	22.5
surplus$_c$	36	16	23.75	24	26
surplus$_{\text{total}}$	**69.5**	**29.5**	**55**	**55.75**	**59.5**

Table 1: Values corresponding to the example congestion problem from Figure 5. Allocation Y is the result of the higher-level market and is the preferred outcome, but causes congestion since its total consumption of 25 exceeds the capacity of 15. The LMP solution Y_{LMP} simply raises the price to find the efficient allocation of 2, 5, 8. However, the total payment now exceeds the cost of the total consumption in the higher-level market, causing a budget imbalance of 30 (i.e. the consumers pay the network). The simple equity allocation Y_{fair} curtails all three consumers to 5 but keeps the original price $\hat{p}$ and is therefore budget balanced. However, Y_{fair} can be improved in terms of efficiency. Our aftermarket, in which agent a chooses not to participate, allows an extra trade between agents b and c at a price $p = 4$ which increases the efficiency while maintaining budget balance (i.e. the consumers pay each other). Finally, Y'_{hyb} shows full aftermarket participation (at a price $p = 3$). As we can see, this results in the maximally efficient allocation 2, 5, 8 that was also found by LMP. However, the aftermarket has avoided congestion pricing and drastically reduced consumer's costs. The difference in surplus between full aftermarket participation Y'_{hyb} and Y_{LMP} is exactly the 30 congestion overpayment. Notice that for Y'_{hyb} agent a bought 5 units at $\hat{p} = 1$ each and sold 3 units at $p = 3$ each in the aftermarket, netting an *income* of 4. A different notion of fairness, e.g. proportional, would affect aftermarket payments, but Y'_{hyb} would always allocate 2, 5, 8.

We showed that this pricing scheme makes participation in the aftermarket an individually rational choice, and defines in a budget-balanced aftermarket. As a result, in contrast to popular congestion management mechanisms such as locational marginal pricing (LMP) where scarcity prices generate income for the mechanism, our hybrid solution gives agents the option of receiving a fair share at a non-scarcity price while still incentivizing welfare maximization through participation in a budget-balanced internal market.

Our Theorems 4 and 12, supported by Algorithms 2 and 3 respectively, provide local prosumer networks of arbitrary size with a way of becoming autarkic in their congestion management by offering internally-defined fair shares in parallel with a completely internal congestion aftermarket that feasibly maximizes welfare.

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