J. Nonlinear Sci. Vol. 14: pp. 237–278 (2004)
DOI: 10.1007/s00332-004-0583-3

Journal of
Nonlinear
Science

Stabilization by Slow Diffusion in a Real Ginzburg-Landau System

A. Doelman,[1,2] G. Hek,[1] and N. Valkhoff[1]

[1] Korteweg–deVries Instituut, Universiteit van Amsterdam, Plantage Muidergracht 24, 1018 TV Amsterdam, The Netherlands
e-mail:doelman@science.uva.nl

[2] Centrum voor Wiskunde en Informatica, P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

Received May 19, 2003; accepted January 28, 2004
Online publication April 5, 2004
Communicated by E. Knobloch

Summary. The Ginzburg-Landau equation is essential for understanding the dynamics of patterns in a wide variety of physical contexts. It governs the evolution of small amplitude instabilities near criticality. It is well known that the (cubic) Ginzburg-Landau equation has various unstable solitary pulse solutions. However, such localized patterns have been observed in systems in which there are two competing instability mechanisms. In such systems, the evolution of instabilities is described by a Ginzburg-Landau equation coupled to a diffusion equation.

In this article we study the influence of this additional diffusion equation on the pulse solutions of the Ginzburg-Landau equation in light of recently developed insights into the effects of slow diffusion on the stability of pulses. Therefore, we consider the limit case of slow diffusion, i.e., the situation in which the additional diffusion equation acts on a long spatial scale. We show that the solitary pulse solution of the Ginzburg-Landau equation persists under this coupling. We use the Evans function method to analyze the effect of the slow diffusion and to show that it acts as a control mechanism that influences the (in)stability of the pulse. We establish that this control mechanism can indeed stabilize a pulse when higher order nonlinearities are taken into account.

1. Introduction

It is well known that homoclinic pulse solutions to scalar (spatially homogeneous) reaction-diffusion equations in one (unbounded) spatial dimension are unstable [17]. In the context of systems of reaction-diffusion equations, pulse solutions can certainly be stable. A famous example is the FitzHugh-Nagumo system, which exhibits a stable traveling pulse solution [19]. From a perturbative point of view, this pulse solution can be seen as a combination of two scalar front solutions in a singular limit. Since scalar traveling fronts can be stable, it is natural to expect that the "double front" FitzHugh-Nagumo pulses can also be stable. Recently, however, it has been shown that singularly perturbed systems may also have asymptotically stable pulse solutions that, in the singular limit, merge with an unstable pulse solution of a scalar limit system [8], [7]. The stabilization of the originally unstable scalar pulse is caused by the effects of coupling the scalar equation to a diffusion equation in which the diffusion acts on a long spatial scale. In other words, coupling a scalar equation to another "slow" diffusion equation introduces a "control mechanism" that may remove the unstable eigenvalue of the linearized stability problem associated with the homoclinic pulse in the scalar limit problem. It has been shown in [8], [7] that this a priori counterintuitive control mechanism is responsible for the existence of stable pulse solutions in well-studied systems such as the Gray-Scott and Gierer-Meinhardt models.

The Ginzburg-Landau equation,

$$A_t = \alpha_1 A_{xx} + \alpha_2 A + \alpha_3 |A|^2 A, \tag{1.1}$$

with $\alpha_i \in \mathbb{C}$, $i = 1, 2, 3$, $A(x,t) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{C}$, can be seen as a normal form that describes the leading order behavior of small perturbations in "marginally unstable" systems of nonlinear partial differential equations defined on unbounded domains [26]. It is relevant for understanding the dynamics of "instabilities" in a wide variety of physical contexts. The Ginzburg-Landau equation couples in a very natural way to a (real) diffusion equation,

$$\begin{cases} A_t &= \alpha_1 A_{xx} + \alpha_2 A + \alpha_3 |A|^2 A + \mu A B, \\ \beta_0 B_t &= \beta_1 B_{xx} + G(B, B_x, |A|^2), \end{cases} \tag{1.2}$$

with $\mu, \beta_0, \beta_1 \in \mathbb{R}$, $B(x,t) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, and G a (real) function of B, B_x, $|A|^2$ that is in general nonlinear. Coupled equations like (1.2) appear in systems with two competing instability mechanisms, one that is governed by a Ginzburg-Landau equation, and one governed by a (real) diffusion equation. Such a competition of instability mechanisms is a very natural phenomenon that for instance occurs in systems with a neutrally stable mode at $k_c = 0$. Coupled equations of the type (1.2) appear in binary fluid convection [29], [30], biological/chemical systems [4], geophysical morphodynamics [24], systems with symmetries [3], etc. Even though most derivations of equations of the type (1.2) are formal and often not completely consistent, it is clear that models of the type (1.2) play an important role in the description of systems with competing instability mechanisms. Moreover, equations such as (1.2) also appear beyond the context of marginally unstable modes and serve as phenomenological models [35]. We refer to [31] for a survey of the appearance and relevance of systems of the form (1.2) and to Remarks 1.1 and 1.2 for a discussion on the background and the structure of (1.2).

The Ginzburg-Landau equation (1.1) has an extremely rich variety of solutions (see for instance [36], [5] and the references therein). Many of these solutions are not stable. For instance, the only stable stationary solutions of the real supercritical Ginzburg-Landau equation, i.e., $\alpha_i \in \mathbb{R}$ with $\alpha_1 > 0, \alpha_2 > 0, \alpha_3 < 0$ in (1.1), are the Eckhaus stable plane waves and a simple front solution; all quasiperiodic and homoclinic solutions are unstable [6], [14]. Nevertheless, some of the a priori unstable solutions, such as several types of homoclinic pulses, seem to be observable, and thus stable (see [29], [30] and the references therein, especially for observations in binary fluid convection). This would be a contradiction when the observed structure is really governed by (1.1). However, the observations are done in physical systems with competing instability mechanisms. Thus, the dynamics of patterns cannot be described sufficiently by (1.1); it has to be coupled to a diffusion equation. In other words, equations of the type (1.2) should give a better description of the observed stable phenomena [29], [30].

In this paper we investigate whether unstable pulse solutions of the uncoupled Ginzburg-Landau equation (1.1) can be stabilized by the coupling of (1.1) to a diffusion equation. In other words, we study whether the above-described "control mechanism" is able to explain the observations of stable pulses in systems of the type (1.2) that seem to correspond to unstable pulses of the (uncoupled) Ginzburg-Landau equation. Here, we do not study (1.2) in its most general setting, but make some explicit choices. As a first step towards understanding the dynamics of (1.2), we assume that the coefficients α_i, $i = 1, 2, 3$ are real. Note that this is not a very strong assumption; the coefficients of the Ginzburg-Landau equation are for instance real when the underlying physical system has a (spatial) reflection symmetry [26]. We next reduce the three-component model (1.2)—recall that $A(x, t) \in \mathbb{C}$—to a two-component model by restricting to the case $A(x, t) \in \mathbb{R}$. Furthermore, as in [29], [30], we focus on the situation in which the Ginzburg-Landau equation describes a subcritical bifurcation—note that the observations of stable pulselike patterns in binary fluid convection are also done in the subcritical regime [29], [30]. The uncoupled scalar limit equation can now be scaled into

$$A_t = A_{xx} - (1 - \mu b_0)A + A^3, \tag{1.3}$$

with b_0 a constant. Here, we have kept the B-dependence of this limit explicitly visible by replacing $\alpha_2 A + \mu AB$ in (1.2) by $-(1 - \mu b_0)A$, without any further scaling. This equation has an unstable standing pulse solution (if $\mu b_0 < 1$), which is, of course, also an unstable pulse solution of the associated complex version of (1.3). In (1.3) we have implicitly assumed that B approaches the constant value b_0 in the uncoupled scalar limit. This is implied by the idea of "coupling to slow diffusion," as will be explained in more detail later on. We therefore assume that the diffusion coefficient of B is much larger than that of A, i.e., we introduce $0 < \varepsilon \ll 1$ and consider ε^{-2} as diffusion coefficient in the B-equation. This choice is elaborated in Remark 1.3. Note that this ε is not related to the small parameter that appears in the derivation of the modulation equation (1.2). Finally, we choose β and $G(B, B_x, |A|^2)$ explicitly,

$$\begin{cases} A_t &= A_{xx} - A + A^3 + \mu AB, \\ \varepsilon^2 \tau B_t &= \varepsilon^{-2} B_{xx} - \alpha \varepsilon^2 B + \sigma B_x + \nu A^2 + \beta A^2 B. \end{cases} \tag{1.4}$$

The terms B and A^2 (originally $|A|^2$) are the first to appear in the derivation process as sketched in Remark 1.2. However, we have also introduced B_x to break the reversibility symmetry (so that we may expect to see traveling pulses) and $A^2 B$ ($|A|^2 B$) as a representative of higher order nonlinear terms in G. Note that various nonlinear terms appear in the literature; we choose $A^2 B$ to represent the possible effects of these terms. We have made rather explicit choices for the relative magnitudes of the coefficients of the terms in the B-equation. The reason for this is mostly mathematical, for the chosen ratio of the magnitudes of the coefficients is the "significant degeneration" in the terminology of singular perturbation theory [12]. In other words, in this scaling the effects of the various terms are balanced. Other scalings do not generate new phenomena, but can be obtained by taking a relevant limit in this "significant" scaling. We refer to Remark 1.3 for a physical interpretation and motivation of the scalings.

Note that equation (1.4) can indeed be seen as a small perturbation of the scalar limit system (1.3). The B equation can be written as $B_{xx} = \mathcal{O}(\varepsilon^2)$. Thus, since B must be bounded (recall that B represents a perturbation), it follows (formally) that B must be constant in the limit $\varepsilon \to 0$ (at least in the original underlying system on a long spatial scale), i.e., $B \to b_0$ as $\varepsilon \to 0$ (1.3).

The main goal of this paper is to study the persistence and stability of the (unstable) homoclinic pulse solution of the (uncoupled, real) Ginzburg-Landau equation (1.3) as solution of the full system (1.4). The stability analysis centers around the question, Can the unstable pulse be stabilized by the control mechanism introduced by the slow B-diffusion? Here we only consider the pulse in (1.3) with $A \geq 0$ for all x, i.e., we do not pay attention to its negative (and symmetrical) counterpart (the analysis is identical).

The persistence problem corresponds to searching for (slowly) traveling waves in system (1.4). Thus, we introduce $\xi := x - Ct$ and reduce (1.4) to a four-dimensional singularly perturbed ODE. We put $C = \varepsilon^2 c$; the magnitude ε^2 of the wave speed is determined by the (forthcoming) analysis. The ODE for $(a, v, b, d) = (A, A_\xi, B, B_\xi/\varepsilon)$ is now given by

$$\begin{cases} a' &= v, \\ v' &= a - a^3 - \mu ab - \varepsilon^2 cv, \\ b' &= \varepsilon d, \\ d' &= \varepsilon[\alpha\varepsilon^2 b - \varepsilon(\sigma + \tau\varepsilon^4 c)d - \nu a^2 - \beta ba^2], \end{cases} \tag{1.5}$$

where the $'$ represents differentiation with respect to ξ. A physically relevant pulse solution satisfies $\lim_{x\to\pm\infty} |A(x,t)| = \lim_{x\to\pm\infty} |B(x,t)| = 0$. Therefore, we look for homoclinic solutions $\gamma_h(\xi)$ to (1.5) that satisfy $\lim_{\xi\to\pm\infty} \gamma_h(\xi) = (0,0,0,0)$, where $S = (0,0,0,0)$ is a fixed point of (1.5). Note that this does not contradict the convergence of B to b_0 as $\varepsilon \to 0$ (since this is for x restricted to a fixed interval of $\mathcal{O}(1)$ length). By the methods of geometric singular perturbation theory [13], [20] the existence of such homoclinic solutions, and thus the persistence of the limit pulse, is established. The homoclinic solution $\gamma_h(\xi)$ corresponds to the (traveling) pulse solution $(A_h(x,t), B_h(x,t))$ of (1.4) that is given in Figure 1. We first consider the system for $\beta = 0$ and show for all $\alpha > 0$ and $\mu \neq 0$ that there is a unique homoclinic orbit $\gamma_h(\xi)$ that merges with the scalar pulse in the limit $\varepsilon \to 0$. The associated pulse solution to (1.4) travels with a uniquely determined wave speed if $\sigma \neq 0$ (Theorem 2.1). In the case $\beta \neq 0$, the situation is less transparent: There are open regions in parameter space in which two different orbits $\gamma_h(\xi)$ exist, while there are other regions where no homoclinic orbit can exist (Theorem 2.2). These regions are separated by manifolds at which homoclinic saddle node bifurcations take place. Thus, the inclusion of higher order nonlinear terms of G in (1.2) has a nontrivial effect on the persistence problem.

The stability analysis of the (traveling) pulses is based on the Evans function and can be studied by the recently developed method that decomposes the Evans function $\mathcal{D}(\lambda, c, \varepsilon)$

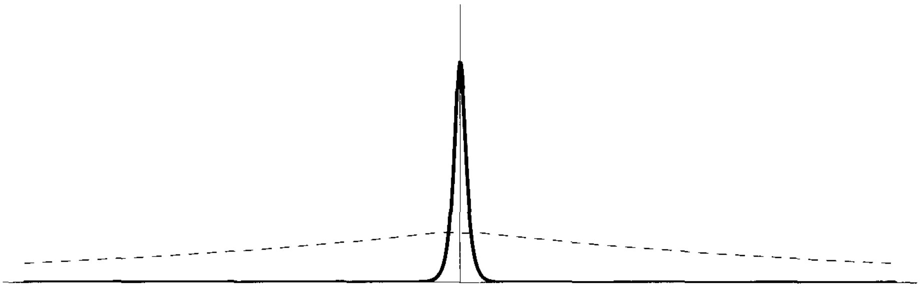

Fig. 1. The homoclinic pulse solution $(A_h(x,t), B_h(x,t))$ of (1.4) as function of x (for $\varepsilon = 0.01, \sigma = 0, \alpha = 4, \beta = 0, \mu = 1, \nu = \frac{1}{3}$). Note that $B_h(x,t)$ (dashed line) is constant—at leading order—on intervals of $\mathcal{O}(1)$ length, i.e., a length that is of the order of the width of the $A_h(x,t)$ spike.

into a product of two transmission functions, $t_1(\lambda)$ and $t_2(\lambda)$ [8], [7]. The stability of the pulse is in essence determined by the zeroes of the slow transmission function $t_2(\lambda, c, \varepsilon)$. This function can be computed explicitly with the use of hypergeometric functions. This enables us to explicitly obtain all eigenvalues of the stability problem for a given combination of the parameters $(\tau, \alpha, \sigma, \mu, \nu, \beta)$ (for ε small enough and $\alpha > 0$).

As for the persistence problem, we first focus on the simplest case, $\beta = 0$. We find that the coupling of the slow diffusion equation indeed has the expected $\mathcal{O}(1)$ effect on the eigenvalues of the stability problem associated with the pulse in the scalar limit (1.3). However, the control mechanism cannot bring the unstable eigenvalue of the scalar limit problem into the stable (complex) half plane. This is in essence due to the uniqueness result for $\beta = 0$ (Theorem 2.1). Since bifurcations (of homoclinic orbits) are impossible in the existence problem, there cannot be a double eigenvalue at $\lambda = 0$. In terms of the Evans function, if $\beta = 0$, we can show that $t_2(0, c, \varepsilon) \neq 0$ for all possible parameter combinations (Lemma 4.2)—note that the eigenvalue at $\lambda = 0$ that is related to the translation invariance corresponds to a zero of the fast transmission function t_1. Thus, the eigenvalue of the reduced limit system cannot cross through $\lambda = 0$. This is similar to the situation in equations of Gray-Scott or Gierer-Meinhardt type [8], [7]. However, for these systems there is a second eigenvalue in the unstable half plane which merges with $\lambda = 0$ in the singular limit $\varepsilon \to 0$. It is shown in [8], [7] that these two eigenvalues can merge and become a pair of complex conjugate eigenvalues. This pair can cross the imaginary axis, so that the pulses in the Gray-Scott/Gierer-Meinhardt system can indeed become stable through a Hopf bifurcation. This scenario is not possible in (1.4) for $\beta = 0$, since there is always only one eigenvalue in the unstable half plane. Thus, the Hopf stabilization scenario is also impossible. We conclude that the pulses are always unstable if $\beta = 0$ (Theorem 4.4).

The β coefficient represents higher order nonlinearities in the function G in (1.2). These nonlinear terms have a major impact on the dynamics of (1.2). The β-term induces saddle-node bifurcations in the persistence problem (Theorem 2.2, Lemma 2.5). In the context of the stability analysis we find, as expected, that these saddle-node bifurcations coincide with zeroes of t_2 (Lemma 4.5). The saddle-node bifurcations provide an additional mechanism by which a pulse may be stabilized. At a saddle-node bifurcation two pulses are created. From the point of view of the stability analysis these pulses differ in the sense that one initially has a small positive eigenvalue, the other a small negative eigenvalue. Thus, together with the positive eigenvalue associated with the scalar limit pulse, the pulse with the small unstable eigenvalue can have a pair of unstable eigenvalues. A related relevant new class of pulses are those pulses that do not merge with a $\beta = 0$ pulse, but diverge, in the limit $\beta \to 0$. It is possible to explicitly compute the spectrum of the eigenvalue problem associated with the stability of these pulses for $0 < \varepsilon \ll \beta \ll 1$; such pulses have two unstable real eigenvalues (Lemma 4.6). We show that by moving away from the $0 < \varepsilon \ll \beta \ll 1$ limit, such a pair of unstable eigenvalues can become a complex conjugate pair of eigenvalues that, eventually, moves by a Hopf bifurcation into the stable half space (Theorem 4.7).

Thus we conclude that, as in the Gray-Scott/Gierer-Meinhardt cases, the slow diffusion control mechanism is indeed able to stabilize an unstable pulse of the (scalar, real) subcritical Ginzburg-Landau equation. However, we immediately remark that our analysis also shows that this control mechanism is a rather subtle one. We found that to make the mechanism effective we need the higher order terms in the nonlinearity G of (1.2) and the associated bifurcations.

The paper is organized as follows. In Section 2 we study the persistence problem. The stability analysis and the decomposition of the Evans function is developed in Section 3 and applied in Section 4. Section 4 also contains a brief discussion of the results.

Remark 1.1. The Ginzburg-Landau equation is a modulation equation that describes the dynamics of small perturbations at values of the control (or bifurcation) parameter R close to the critical value R_c [26]. It can be derived, under very general conditions, if the linear stability analysis of a "trivial background state" at $R = R_c$ indicates the existence of "marginally unstable" waves of the form $e^{i(k_c x+\omega_c t)}$. However, if $(k_c, \omega_c) = (0, 0)$, the derivation process has to be adapted and one finds that the dynamics of small perturbations are governed by a modulation equation that is a real, scalar, reaction-diffusion equation ([33] and the references therein). Note that $(k_c, \omega_c) = (0, 0)$ appears naturally, for instance when the background state undergoes a transcritical bifurcation, i.e., $(k_c, \omega_c) = (0, 0)$ is not a degeneration, but a generic phenomenon. In case of competing instabilities there are two distinct marginally unstable waves for R close to R_c, one with (k_c^1, ω_c^1) and one with (k_c^2, ω_c^2). Mathematically speaking this is a codimension 2 situation that occurs naturally in many physical systems. The dynamics of small perturbations are now governed by a coupled system of modulation equations. If both $(k_c^i, \omega_c^i) \neq (0, 0)$ and there are no resonances, then the system consists of two coupled Ginzburg-Landau equations (see [11] and the references therein). If either one, but not both, $(k_c^i, \omega_c^i) = (0, 0)$, then one can derive a system of the type (1.2).

Remark 1.2. The (formal) derivation of equations of the types (1.2) and (1.4) is done by a classical multiple-scales analysis in which the equations appear through the application of a solvability condition. We do not intend to go into the details of this process here; however, we make a couple of remarks on the structure of the equations in (1.2)/(1.4). The leading order input of B into the A equation is by the interaction of the critical (k_c, ω_c)-mode associated with A with the critical $(0, 0)$-mode associated with B. This implies that the coupling occurs as a quadratic AB-term in the Ginzburg-Landau component of (1.2).

Vice versa, A enters at leading order into the $(0, 0)$-mode of B through a $|A|^2$-term. For a consistent derivation of equations of the type (1.2)/(1.4) it is necessary to carefully order the magnitudes of the various terms that appear (such as $|A|^2$, B_{xx}, $|A|^2B$, B^2, but also terms we have not included in our choice of G in (1.2), such as $|A|_x^2$, A_{xx}, etc.—see also [31] and Remark 1.4). Recall that the ε introduced in (1.4) is assumed to be $\mathcal{O}(1)$ with respect to the small parameter of the derivation process, and thus, from the derivational point of view, all coefficients in (1.4) are of the same order. This is not consistent with the derivation process where, for instance, the $|A|^2B$-term appears at a higher order than the $|A|^2$-term. In fact, apart from the case in which there is a conservation law ([25] and Remark 1.4), it is debatable whether a fully consistent derivation of equations of the type (1.2)/(1.4) is possible at all. The situation here is in essence the same as that of the (uncoupled) Ginzburg-Landau equation with other, noncubic, nonlinearities, such as the quintic-cubic Ginzburg-Landau equation ([36], [5], but also [29], [30] for a system of the type (1.2) with a Ginzburg-Landau equation that includes quintic terms). Like (1.2)/(1.4), these equations cannot be derived in a consistent way as modulation equations (unless one imposes additional, nongeneric constraints on the underlying system). Nevertheless, the quintic Ginzburg-Landau equations have shown their relevance in understanding pattern-generating systems. For a similar reason "inconsistent" systems of the type (1.2)/(1.4) appear all through the literature (see the above-mentioned references).

Remark 1.3. The assumption that the B equation is a slow diffusion equation has a direct interpretation in the underlying physical system. In fact, the magnitude of the diffusion coefficient in a modulation equation describes the natural spatial scale of the evolution associated with the instability (see also [33]). Thus, the assumption of slow diffusion in (1.4) has in essence the same meaning in the underlying system: The instabilities associated with B generated at $(k_c, \omega_c) = (0, 0)$ evolve on a longer spatial scale than those of A at $(k_c, \omega_c) \neq (0, 0)$. The choice of the $\mathcal{O}(\varepsilon^2)$ magnitude of the linear B-term in the B-equation is related to the fact that the $(0, 0)$-mode is neutrally stable in many applications. A perfectly neutrally stable B-mode would have no B-term, i.e., $\alpha = 0$ in (1.4). Thus, the assumption that the coefficient of the B-term is $\mathcal{O}(\varepsilon^2)$ approximates the neutrally stable case. Note that the perfectly neutrally stable case $\alpha = 0$ can also be studied by the methods of this paper, but is technically much more involved (see Remark 2.3 and Section 4.4). Finally, the $\mathcal{O}(\varepsilon^2)$ magnitude of β_0 (1.2) is mostly motivated by the observations in the existence and stability analysis. For instance, at this magnitude the transitions from stable to unstable, and vice versa, occur. It means for the underlying system that the perturbations associated with the $(0, 0)$-mode evolve faster in time than those of the (k_c, ω_c)-mode.

Remark 1.4. In the presence of a conservation law one can derive (in a consistent way) an equation like (1.2) with a G that is not covered by (1.2), $G = \tilde{\beta}_2|A|_{xx}^2$ [25], so that the B-equation can be written as $B_t = (\tilde{\beta}_1 B + \tilde{\beta}_2 A^2)_{xx}$. It has been shown recently [28] that there can be stable periodic pulse patterns with a long spatial period in such a system of modulation equations in the supercritical case (i.e., $A - A^3$ instead of $-A + A^3$ in (1.4)). Thus, the results of [28] are similar to the results in this paper. However, the motivation differs slightly, since there is no pulse solution to $A = 0$ in the (real scalar) supercritical case, which implies that the results of [28] cannot be seen as persistence and stabilization results. Moreover, due to the different character of the B-equation, the existence and stability analysis in [28] differs essentially from the analysis here.

Remark 1.5. The unstable pulse solution of the subcritical real Ginzburg-Landau limit system (1.3) corresponds to an element of the family of Hocking-Stewartson solutions of the general complex Ginzburg-Landau equation (1.1) [18]. The (in)stability of the family of Hocking-Stewartson solutions has been studied in [1]. Other results on the instability of pulses in the Ginzburg-Landau equation can be found in [14] and [21], [22]. However, it should be noted that the latter two papers consider a cubic-quintic Ginzburg-Landau equation in the nearly integrable nonlinear Schrödinger limit; as a consequence, the pulses do not have $\mathcal{O}(1)$ unstable eigenvalues.

2. Persistence

In this section, we study the existence of traveling and stationary pulse solutions to (1.4), or equivalently (1.5), using the methods developed in [10], [7], [16]. The four dimensional "fast" system (1.5) can alternatively be written as the slow system

$$\begin{cases} \varepsilon\dot{a} &= v, \\ \varepsilon\dot{v} &= a - a^3 - \mu ab - \varepsilon^2 cv, \\ \dot{b} &= d, \\ \dot{d} &= \alpha\varepsilon^2 b - \varepsilon(\sigma + \tau\varepsilon^4 c)d - \nu a^2 - \beta ba^2. \end{cases} \tag{2.1}$$

Here the dot denotes the derivative with respect to the slow variable $\xi^* = \xi/\varepsilon$. Putting $\varepsilon = 0$ in (1.5), we obtain the associated fast reduced system, which is given by

$$\begin{cases} a' &= v, \\ v' &= (1 - \mu b_0)a - a^3, \\ b' &= 0, \\ d' &= 0. \end{cases} \tag{2.2}$$

Hence solutions satisfy $b \equiv b_0$, $d \equiv d_0$. For any μ and b_0 such that $1 - \mu b_0 > 0$, system (2.2) has a positive and a negative homoclinic orbit $(\pm a_0(\xi; b_0), \pm v_0(\xi; b_0))$, satisfying $a_0 \geq 0$ and $\lim_{\xi\to\pm\infty}(a_0(\xi; b_0), v_0(\xi; b_0)) = (0, 0)$. Here

$$a_0(\xi; b_0) = \sqrt{2(1 - \mu b_0)}\operatorname{sech}(\sqrt{1 - \mu b_0}\,\xi), \qquad v_0(\xi; b_0) = a_0'(\xi; b_0). \tag{2.3}$$

This orbit corresponds to pulse solutions $A(x, t) = a_0(\xi; b_0)$, with $\lim_{\xi\to\pm\infty} A = 0$, in the uncoupled Ginzburg-Landau equation (1.3). This pulse may persist in various forms as a solution to the full equation, such as wave trains of multipulses [10], [7]. Here, we restrict our attention to the most simple form, the solutions that satisfy $\lim_{x\to\pm\infty} |A(x, t)| = \lim_{x\to\pm\infty} |B(x, t)| = 0$. Therefore, we look for homoclinic solutions $\gamma_h(\xi)$ to (1.5) that satisfy $\lim_{\xi\to\pm\infty} \gamma_h(\xi) = (0, 0, 0, 0)$, where we define S as the fixed point $(0, 0, 0, 0)$ of (1.5).

Our existence (or persistence) results are summarized in the following two theorems and illustrated by Figure 2. The first Theorem considers the simpler case, $\beta = 0$.

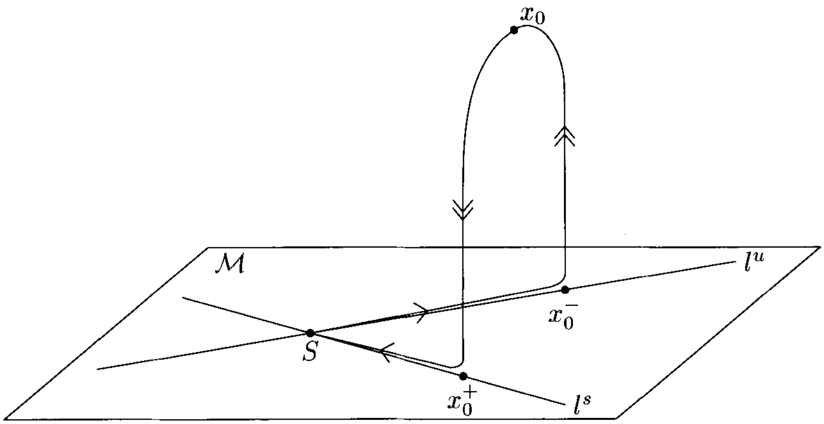

Fig. 2. A sketch of a solution (a_h, v_h, b_h, d_h) homoclinic to S, which corresponds to a traveling pulse solution $(A_h(\xi), B_h(\xi)) = (a_h(\xi), b_h(\xi))$.

Theorem 2.1. *Let $\beta = 0, \alpha > 0, \mu \neq 0, \sigma \in \mathbb{R}$, and $\nu \in \mathbb{R}$. For every $\varepsilon > 0$ sufficiently small, there is a b_0 satisfying $1 - \mu b_0 > 0$ with a corresponding wave speed $C = \varepsilon^2 c = \varepsilon^2 c_0 + \mathcal{O}(\varepsilon^3)$ such that system* (1.5) *has a solution $\gamma_h(\xi) = (a_h(\xi), v_h(\xi), b_h(\xi), d_h(\xi))$ that is homoclinic to $S = (0,0,0,0)$. This solution satisfies $|a_h(\xi; b_0) - a_0(\xi; b_0)| = \mathcal{O}(\varepsilon)$, $|v_h(\xi; b_0) - v_0(\xi; b_0)| = \mathcal{O}(\varepsilon)$ uniformly on $\mathbb{R}$ with a_0, v_0 as in (2.3), and $b_h(\xi) = b_0 + \mathcal{O}(\varepsilon)$ for $\xi = \mathcal{O}(1)$. The orbit $\gamma_h(\xi)$ corresponds to a traveling pulse solution $(A_h(\xi), B_h(\xi)) = (a_h(\xi), b_h(\xi))$ of (1.4) with* $\lim_{\xi \to \pm\infty}(A_h(\xi), B_h(\xi)) = (0,0)$. *There are two distinct cases to consider:*

(i) $\sigma = 0$: the solution is a standing wave solution, i.e., $c \equiv 0$, with

$$b_0 = \frac{2\nu}{\alpha}\left[-\mu\nu + \sqrt{\mu^2\nu^2 + \alpha}\,\right], \tag{2.4}$$

(ii) $\sigma \neq 0$: the solution is a traveling wave with

$$b_0 = \frac{4c}{\mu(4c - 3\sigma)} \quad \text{and} \quad c_0 = \frac{3\sigma\mu\nu}{\sigma^2 + 4\alpha}\left(-2\mu\nu - \sqrt{4\mu^2\nu^2 + \sigma^2 + 4\alpha}\right).$$

Thus, for every choice of the parameters (with $\alpha > 0$, $\mu \neq 0$), there is exactly one standing or traveling (single) pulse solution to (1.4) that can be seen as the continuation of the (standing) pulse of the uncoupled Ginzburg-Landau equation. In fact, this solution is recovered explicitly if $\nu = 0$, for which the Ginzburg-Landau equation has no influence on the B-equation (and we thus find $b_0 = c = 0$). The pulse $\gamma_h(\xi)$ is sketched as homoclinic orbit in phase space in Figure 2, its counterpart $(A_h(\xi), B_h(\xi))$ is given as the solution to the PDE (1.4) in Figure 1.

The case $\beta \neq 0$ is more complex. To avoid cumbersome computations, we only consider stationary orbits.

Theorem 2.2. *Let $\beta \neq 0$, $\alpha > 0$, $\mu \neq 0$, $\mu \in \mathbb{R}$, and let $\varepsilon > 0$ be sufficiently small. Consider $c = \sigma = 0$. There are open regions in $(\mu, \alpha, \nu, \beta)$-space in which system* (1.5) *possesses* 0, 1, *or* 2 *(different) orbits* $\gamma_h(\xi)$ *homoclinic to* $S = (0, 0, 0, 0)$. *The boundary between the regions with* 0 *or* 2 *types of homoclinic orbits is formed by a codimension* 1 *manifold of homoclinic saddle-node bifurcations.*

For each solution $\gamma_h(\xi) = (a_h(\xi), v_h(\xi), b_h(\xi), d_h(\xi))$ *the approximate value of* b_0 *is determined by an explicit polynomial (see (2.19)). The fast coordinates* $a_h(\xi)$, $v_h(\xi)$ *satisfy* $|a_h(\xi) - a_0(\xi; b_0)| = \mathcal{O}(\varepsilon)$, $|v_h(\xi) - v_0(\xi; b_0)| = \mathcal{O}(\varepsilon)$ *uniformly on* $\mathbb{R}$ *with* a_0, v_0 *as in (2.3);* $b_h(\xi) = b_0 + \mathcal{O}(\varepsilon)$ *for* $\xi = \mathcal{O}(1)$. *The orbit* $\gamma_h(\xi)$ *corresponds to a traveling pulse solution* $(A_h(\xi), B_h(\xi)) = (a_h(\xi), b_h(\xi))$ *of (1.4) with* $\lim_{x\to\pm\infty}(A_h(\xi), B_h(\xi)) = (0, 0)$.

Thus, the unperturbed homoclinic pulses of the uncoupled Ginzburg-Landau equation may not persist, or may persist in two different forms when $\beta \neq 0$. In Section 2.5 we will give a more precise description of this situation. Sections 2.1 to 2.4 are devoted to the proof of both theorems using a geometrical singular perturbation approach [13], [20].

2.1. The Reduced Fast and Slow Systems

The fast reduced system (2.2) is in essence a two-parameter (b_0 and d_0) family of planar, integrable systems (Figure 3). Since we are interested in homoclinic orbits, we assume throughout this paper that

$$1 - \mu b_0 > 0. \tag{2.5}$$

In that case, (2.2) possesses three two-dimensional invariant manifolds $\{v_0 = 0, a_0 = 0\}$, $\{v_0 = 0, a_0 = -\sqrt{1-\mu b_0}\}$, and $\{v_0 = 0, a_0 = +\sqrt{1-\mu b_0}\}$. Only the first one, $\mathcal{M}_0 := \{(a_0, v_0, b_0, d_0) \mid a_0 = v_0 = 0\}$, is filled with hyperbolic equilibria, and is thus a normally hyperbolic invariant manifold; $\mathcal{M}_0$ has three-dimensional stable and unstable manifolds $W^s(\mathcal{M}_0)$ and $W^u(\mathcal{M}_0)$, which are the unions of the two-parameter families of one-dimensional stable and unstable manifolds of the saddle points $(a, v, b_0, d_0) = (0, 0, b_0, d_0) \in \mathcal{M}_0$. For each (b_0, d_0) satisfying (2.5) $\mathcal{M}_0$ is connected to itself by a homoclinic orbit (2.3). The family of homoclinic orbits forms a homoclinic manifold $\mathcal{H}$ in which $W^s(\mathcal{M}_0)$ and $W^u(\mathcal{M}_0)$ merge. Inside every homoclinic orbit the plane is filled with periodic orbits (Figure 3). Geometric singular perturbation theory [13], [20] guarantees that if ε is sufficiently small, there exists a locally invariant manifold $\mathcal{M}_\varepsilon$ for

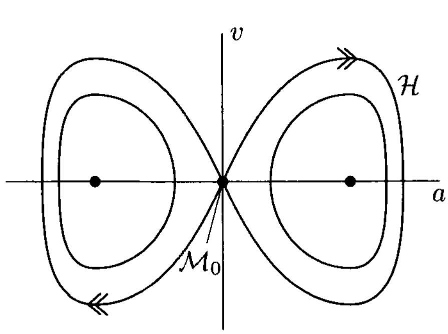

Fig. 3. The (a_0, v_0) phase space of the fast subsystem.

the perturbed system (1.5) which is diffeomorphic to and $\mathcal{O}(\varepsilon)$ C^1-close to $\mathcal{M}_0$. In the current case it is clear that $\mathcal{M}_0$ is still invariant under (1.5) and therefore $\mathcal{M}_\varepsilon = \mathcal{M}_0$.

In addition, Fenichel theory states that for $0 < \varepsilon \ll 1$ the manifold $\mathcal{M}_\varepsilon$ has three-dimensional stable and unstable manifolds $W^s(\mathcal{M}_\varepsilon)$ and $W^u(\mathcal{M}_\varepsilon)$. These manifolds are $\mathcal{O}(\varepsilon)$ close and diffeomorphic to their counterparts $W^s(\mathcal{M}_0)$ and $W^u(\mathcal{M}_0)$.

The flow on $\mathcal{M}_\varepsilon$ for the full system (1.5) can be determined by substituting $(a, v) = (0, 0)$. The resulting two-dimensional system is linear and has one equilibrium point $(b, d) = (0, 0)$, which is S in the four-dimensional system. It is given by

$$\begin{aligned} b' &= \varepsilon d, \\ d' &= \varepsilon[-\varepsilon(\sigma + \tau c\varepsilon^4)d + \alpha\varepsilon^2 b]. \end{aligned} \tag{2.6}$$

If S is a saddle point within $\mathcal{M}_\varepsilon$, construction of solutions homoclinic to S may be possible. The eigenvalues of the linear system (2.6) are given, at leading order, by $\lambda^2 + \varepsilon^2\sigma\lambda - \alpha\varepsilon^4 = 0$; hence, with $\Gamma := \varepsilon^{-2}\lambda$, by

$$\Gamma_\pm = \frac{1}{2}\left[-\sigma \pm \sqrt{\sigma^2 + 4\alpha}\,\right]. \tag{2.7}$$

Thus, the restriction of S to $\mathcal{M}_\varepsilon$ is a saddle point if $\alpha > 0$, which is assumed for the remainder of this section. The stable and unstable manifolds of S (restricted to $\mathcal{M}_\varepsilon$) are given by

$$l^{u,s} = W^{u,s}(0,0)\,|_{\mathcal{M}_\varepsilon} = \{(b, d) \mid d = \varepsilon[\Gamma_\pm + \mathcal{O}(\varepsilon^4)]b\}. \tag{2.8}$$

Remark 2.3. As noted in Remark 1.3, there are many systems in which the mode associated with B is neutrally (un)stable [29]. Neutral stability corresponds to $\alpha = 0$ in (1.4), and thus in (1.5) and (2.6). As a consequence, the flow on $\mathcal{M}_\varepsilon$ degenerates. In the system described by (2.6) one cannot construct homoclinic orbits to the critical point $(0, 0)$ on $\mathcal{M}_\varepsilon$, since it has either no stable or no unstable manifold (depending on the sign of σ). However, as soon as one includes higher order terms of $G(B, B_x, |A|^2)$ into this system, the structure of $(0, 0)$ on $\mathcal{M}_\varepsilon$ becomes less singular. A natural nonlinear term to include in the neutral case is B^2. Then, $(0, 0)$ indeed has a stable and an unstable manifold on $\mathcal{M}_\varepsilon$ (which are tangent in $(0, 0)$). As a consequence, we can use the methods developed in this section to construct a homoclinic solution to $(0, 0, 0, 0)$ in (1.5). Since this is technically a bit more involved, we do not consider this case here explicitly. Nevertheless, we conclude that the homoclinic pulse of the uncoupled Ginzburg-Landau equation can also persist in the neutral case with $\alpha = 0$. Note, however, that in this case the pulses will not decay exponentially but algebraically as $\xi \to \infty$; see Section 4.4.

2.2. Persistent Fast Connections

In the reduced fast system, we found that the three-dimensional stable and unstable manifolds of $\mathcal{M}_0$ merge and form a homoclinic manifold $\mathcal{H}$. In the general case, where $0 < \varepsilon \ll 1$, the stable and unstable manifolds $W^s(\mathcal{M}_\varepsilon)$ and $W^u(\mathcal{M}_\varepsilon)$ will no longer merge, but may intersect in one or more two-dimensional surfaces. If homoclinic orbits to the point S exist, they must lie on one of these intersections. Since the system (1.5) is an $\mathcal{O}(\varepsilon)$ perturbation of an integrable system with periodic orbits inside $\mathcal{H}$, components of $W^u(\mathcal{M}_\varepsilon)$ and $W^s(\mathcal{M}_\varepsilon)$ inside $\mathcal{H}$ wind around $(a, v) = (\sqrt{1-\mu b_0}, 0)$ and intersect the hyperplane $\{v = 0\}$ several times.

Adiabatic Melnikov theory [32] provides a measure Δ as a function of b_0 and d_0 that determines the $\mathcal{O}(\varepsilon)$ distance between the first intersections of $W^u(\mathcal{M}_\varepsilon)$ and $W^s(\mathcal{M}_\varepsilon)$ with $\{v = 0\}$. For system (1.5), the measure Δ is at leading order given by

$$\Delta(b_0, d_0) = \int_{-\infty}^{\infty} \mu d_0 a_0(\xi; b_0) v_0(\xi; b_0) \xi \; d\xi,$$

where $(a_0(\xi; b_0), v_0(\xi; b_0), b_0, d_0) = \gamma_0(\xi; b_0, d_0)$ are the homoclinic solutions for the $\varepsilon = 0$ system given by (2.3). A simple zero (b_0, d_0) of Δ corresponds to an orbit $\gamma(\xi)$ in a transverse intersection of $W^s(\mathcal{M}_\varepsilon)$ and $W^u(\mathcal{M}_\varepsilon)$ (see for instance [7] for more details), which is therefore biasymptotic to $\mathcal{M}_\varepsilon$. This orbit is in leading order given by $\gamma_0(\xi; b_0, d_0)$ during its fast circuit away from $\mathcal{M}_\varepsilon$.

Solving $\Delta(b_0, d_0) = 0$, we find $d_0 = 0$ at leading order, so a transverse intersection of $W^s(\mathcal{M}_\varepsilon)$ and $W^u(\mathcal{M}_\varepsilon)$ occurs near $d_0 = 0$. To obtain the next term in the expansion of d_0, we determine a higher order correction of $\Delta(b_0, d_0)$,

$$\Delta(b_0, d_0) = \int_{-\infty}^{\infty} \varepsilon c_0 v_0^2(\xi; b_0) + \mu d_0 a_0(\xi; b_0) v_0(\xi; b_0) \xi \; d\xi, \tag{2.9}$$

in which we have assumed that $d_0 = \mathcal{O}(\varepsilon)$ (see [7]). Hence, $\Delta_1(b_0, d_0) = 0$ yields either

$$d_0 = \frac{2}{3}\frac{c_0}{\mu}(1 - \mu b_0)\varepsilon, \quad \text{with} \quad \mu \neq 0, \tag{2.10}$$

or

$$b_0 = 1/\mu, \quad \text{with} \quad \mu \neq 0, \tag{2.11}$$

or

$$c_0 = 0, \quad \text{with} \quad \mu = 0 \tag{2.12}$$

(all at leading order). Relations (2.10) and (2.11) both represent a one-parameter family or two-dimensional manifold of orbits that are biasymptotic to $\mathcal{M}_\varepsilon$. In the next section we will find that (2.11) cannot yield homoclinic orbits to S. In situation (2.12) the fast flow (equations for a and v) is decoupled and possesses the usual homoclinic orbit $(a_0(\xi), v_0(\xi)) = (\sqrt{2}\text{sech}\ \xi, a_0'(\xi))$ to $(a = 0, v = 0)$. Given this solution, the equations for b and d can be solved and the behavior of solutions for $|\xi| \to \infty$ can be checked. Alternatively, the methods in Section 2.3 give pairs $(b_0 = \mathcal{O}(1), d_0 = \mathcal{O}(\varepsilon))$ for which $(b(\xi), d(\xi)) \to (0, 0)$ as $|\xi| \to \infty$. However, since the behavior of the fast pulses is not influenced by the slow field, these GL-pulses will remain unstable. Therefore, pulses satisfying (2.12) will not be considered here.

2.3. *Takeoff and Touchdown Curves*

We have determined two one-parameter families of orbits biasymptotic to $\mathcal{M}_\varepsilon$. However, our aim is to find orbits homoclinic to $S = (0, 0, 0, 0)$. The construction so far is only based on the fast dynamics of (1.5). We now turn our attention to slow segments of the biasymptotic orbits.

For any orbit $\gamma(\xi, x_0)$ with $x_0 = \gamma(0, x_0) \in W^u(\mathcal{M}_\varepsilon) \cap W^s(\mathcal{M}_\varepsilon) \cap \{v = 0\}$, Fenichel theory implies that there exist two orbits $\gamma^+ = \gamma^+(\xi, x_0^+) \subset \mathcal{M}_\varepsilon$ and $\gamma^- = \gamma^-(\xi, x_0^-) \subset \mathcal{M}_\varepsilon$, such that $\|\gamma(\xi, x_0) - \gamma^+(\xi, x_0^+)\|$ is exponentially small in ε when $\xi \geq \mathcal{O}(\frac{1}{\varepsilon})$ and $\|\gamma(\xi, x_0) - \gamma^-(\xi, x_0^-)\|$ is exponentially small in ε when $-\xi \geq \mathcal{O}(\frac{1}{\varepsilon})$. A homoclinic orbit $\gamma_h(\xi)$ to S must either tend to S via a strong stable or unstable manifold, or via its stable and unstable manifolds l^u and l^s within $\mathcal{M}_\varepsilon$. The former is the case in situation (2.12). In the latter, generic case, $||\gamma_h(\xi, x_0) - l^{u,s}||$ is exponentially small in ε for $\xi \geq \mathcal{O}(\frac{1}{\varepsilon})$. Whether such an orbit γ_h exists depends on the positions of the base points $x_0^\pm = \gamma^\pm(0, x_0^\pm) \in \mathcal{M}_\varepsilon$.

We define the curves $T_o \subset \mathcal{M}_\varepsilon$ (takeoff) and $T_d \subset \mathcal{M}_\varepsilon$ (touchdown) as

$$T_o := \bigcup_{x_0} \{x_0^- = \gamma^-(0, x_0^-)\} \qquad \text{and} \qquad T_d := \bigcup_{x_0} \{x_0^+ = \gamma^+(0, x_0^+)\},$$

where the unions are over all $x_0 \in W^s(\mathcal{M}_\varepsilon) \cap W^u(\mathcal{M}_\varepsilon) \cap \{v = 0\}$. The takeoff set T_o represents the collection of base points of all of the Fenichel fibers in $W^u(\mathcal{M}_\varepsilon)$ that are asymptotic to $\mathcal{M}_\varepsilon$ as $\xi \to \infty$. Similarly, T_d represents the set of base points of the fibers in $W^s(\mathcal{M}_\varepsilon)$ that are asymptotic to $\mathcal{M}_\varepsilon$ as $\xi \to -\infty$.

Detailed information about the positions of T_o and T_d is given by the relation between $x_0 = \gamma(0, x_0) = (a_0, 0, b_0, d_0)$ and its base points $x_0^- = (a_0^-, v_0^-, b_0^-, d_0^-)$ and x_0^+, respectively, or in fact by the relation between b_0 and $b_0^\pm$, d_0 and $d_0^\pm$. Thus we determine the sets T_o and T_d by measuring the change in b and d of $\gamma(\xi, x_0) \subset W^u(\mathcal{M}_\varepsilon) \cap W^s(\mathcal{M}_\varepsilon)$ during half a circuit through the fast field. The accumulated change in d during a circuit can be measured by integrating d' along the orbit $\gamma(\xi, x_0)$ until the orbit settles down near $\mathcal{M}_\varepsilon$. Since it takes $\mathcal{O}(|\log\varepsilon|)$ "time" to leave an $O(\varepsilon)$ neighborhood of $\mathcal{M}_\varepsilon$, make a fast loop, and return to the neighborhood of $\mathcal{M}_\varepsilon$, we integrate up to $\mathcal{O}(|\log\varepsilon|)$. For the positive and negative half circuits we thus take, with some arbitrary $k > 0$ independent of ε,

$$\Delta^- d = -\int_{k\log\varepsilon}^{0} d'|_{\gamma(\xi;x_0)}\, d\xi \qquad \text{and} \qquad \Delta^+ d = \int_{0}^{-k\log\varepsilon} d'|_{\gamma(\xi;x_0)}\, d\xi,$$

respectively. For $\gamma(\xi; x_0)$ with $x_0 = (a_0, 0, b_0, d_0)$ we find, by (1.5),

$$\begin{aligned}
\Delta d &= \Delta^+ d + \Delta^- d \\
&= \int_{k\log\varepsilon}^{-k\log\varepsilon} d'|_{\gamma(\xi;x_0)}\, d\xi \\
&= -\varepsilon \int_{k\log\varepsilon}^{-k\log\varepsilon} (\nu a^2 + \beta b a^2)|_{\gamma(\xi;x_0)}\, d\xi + \mathcal{O}(\varepsilon^2|\log\varepsilon|) \\
&= -\varepsilon \int_{k\log\varepsilon}^{-k\log\varepsilon} (\nu + \beta b_0) a_0^2(\xi; b_0)\, d\xi + \mathcal{O}(\varepsilon^2|\log\varepsilon|) \\
&= -\varepsilon \int_{-\infty}^{\infty} (\nu + \beta b_0) a_0^2(\xi; b_0)\, d\xi + \mathcal{O}(\varepsilon^{1+2K}) + \mathcal{O}(\varepsilon^2|\log\varepsilon|),
\end{aligned}$$

where we have used that $\gamma(\xi, x_0)$ can be approximated by the unperturbed homoclinic orbit $\gamma_0(\xi, x_0) = (a_0(\xi; b_0), v_0(\xi; b_0), b_0, d_0)$ (2.3) during the fast circuit. Moreover, it follows from (2.3) that $K = k\sqrt{1-\mu b_0} > 0$ (see also the explicit computation below (2.13)). Note that the above argument also implies that $\Delta^- d = \Delta^+ d = (1/2)\Delta d$ up to terms of $\mathcal{O}(\varepsilon^2|\log\varepsilon|)$. Thus, we conclude by (2.3), and by choosing k large enough, that

$$\begin{aligned}\Delta d &= -2\varepsilon(\nu + \beta b_0)\sqrt{1-\mu b_0}\int_{-\infty}^{\infty}\operatorname{sech}^2(\chi)\,d\chi + \mathcal{O}(\varepsilon^2|\log\varepsilon|)\\ &= -4\varepsilon(\nu+\beta b_0)\sqrt{1-\mu b_0} + \mathcal{O}(\varepsilon^2|\log\varepsilon|).\end{aligned} \tag{2.13}$$

Similarly we find

$$\Delta b = \int_{k\log\varepsilon}^{-k\log\varepsilon} b'|_{\gamma(\xi,x_0)}\,d\xi = \varepsilon\int_{k\log\varepsilon}^{-k\log\varepsilon}\big(d_0 + \mathcal{O}(\varepsilon)\big)\,d\xi = -2k\varepsilon\log\varepsilon\big(d_0 + \mathcal{O}(\varepsilon)\big).$$

For a homoclinic orbit $\gamma_h(\xi)$ to S, Δd and Δb should account for the "jump" from l^u to l^s. If a fast loop $\gamma(\xi, x_0)$ originates at $\xi = 0$ in a point x_0 given by (2.11), so in a point on the curve $(a_0(0), 0, b_0{=}1/\mu, d_0)$, then Δd is at most $\mathcal{O}(\varepsilon^2|\log\varepsilon|)$ and Δb is at most $\mathcal{O}(\varepsilon|\log\varepsilon|)$, depending on the d coordinate of x_0. Because the distance (in d) between l^u and l^s near $b = 1/\mu$ is $\mathcal{O}(\varepsilon)$, this cannot give rise to a homoclinic loop. Therefore, (2.11) cannot yield any homoclinic orbit to S, and we only have to consider loops prescribed by (2.10). Near the line $d_0 = \frac{2}{3}\frac{c_0}{\mu}(1-\mu b_0)\varepsilon (= \mathcal{O}(\varepsilon))$, the coordinate d_0 is also $\mathcal{O}(\varepsilon)$, and $\Delta b = \mathcal{O}(\varepsilon^2|\log\varepsilon|)$. Therefore a jump from l^u to l^s is possible. We now know that we made the right choice for the scaling of the wave speed c, when we started the traveling wave analysis.

The takeoff and touchdown curves are now obtained by correcting (2.10) for the change in d during half an excursion through the fast field,

$$\begin{aligned}T_o &:= \left\{(b_0^-, d_0^-) = (b_0, d_0) \mid d_0 = \varepsilon\left(\frac{2}{3}\frac{c_0}{\mu}(1-\mu b_0) + 2(\nu+\beta b_0)\sqrt{1-\mu b_0}\right)\right\},\\ T_d &:= \left\{(b_0^+, d_0^+) = (b_0, d_0) \mid d_0 = \varepsilon\left(\frac{2}{3}\frac{c_0}{\mu}(1-\mu b_0) - 2(\nu+\beta b_0)\sqrt{1-\mu b_0}\right)\right\},\end{aligned}$$

up to corrections (in d_0) of $\mathcal{O}(\varepsilon^2|\log\varepsilon|)$. The takeoff and touchdown curves are illustrated in Figure 4.

2.4. Homoclinic Pulse Solutions

We recall that a generic homoclinic orbit to S satisfies $||\gamma_h(\xi, x_0) - l^{u,s}|| = \mathcal{O}(e^{-\kappa/\varepsilon})$ for $|\xi| > \mathcal{O}(1/\varepsilon)$ and some $\kappa > 0$. Therefore, its takeoff and touchdown points $S_o = x_0^-$ and $S_d = x_0^+$ must lie on l^u and l^s, respectively (see Figure 2), and thus in the intersections $T_o \cap l^u$ and $T_d \cap l^s$ (Figure 4). The b-coordinates of S_o and S_d have to be equal at leading order since $\Delta b = \mathcal{O}(\varepsilon^2|\log\varepsilon|)$ during an excursion through the fast field. Therefore the

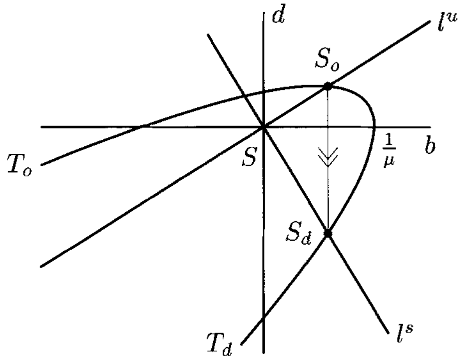

Fig. 4. The takeoff and touchdown curves T_o and T_d, which merge at $b_0 = 1/\mu$, and their intersections with l^u and l^s (in $\mathcal{M}_\varepsilon$). The wave speed c is chosen in such a way that a homoclinic loop is possible, as in Lemma 2.4.

leading order term b_0 of the homoclinic orbit has to fulfill both equations

$$\begin{aligned} T_o \cap l^u; \qquad \Gamma_+ b_0 &= \frac{2}{3}\frac{c_0}{\mu}(1-\mu b_0) + 2(\nu + \beta b_0)\sqrt{1-\mu b_0}, \\ T_d \cap l^s; \qquad \Gamma_- b_0 &= \frac{2}{3}\frac{c_0}{\mu}(1-\mu b_0) - 2(\nu + \beta b_0)\sqrt{1-\mu b_0}, \end{aligned} \tag{2.14}$$

for a certain wave speed c_0, where $\Gamma_\pm$ is given by (2.7).

Lemma 2.4. *If $\mu \neq 0$, $\beta = 0$, and $\alpha > 0$, there are two kinds of solution pairs (b_0, c_0) to system* (2.14)*:*

(i) If $\sigma = 0$,

$$b_0 = \frac{2\nu}{\alpha}\left(-\mu\nu + \sqrt{\mu^2\nu^2 + \alpha}\right) \quad \text{and} \quad c_0 = 0.$$

(ii) If $\sigma \neq 0$, then $4c_0 - 3\sigma \neq 0$ and

$$b_0 = \frac{4c_0}{\mu(4c_0 - 3\sigma)} \quad \text{and} \quad c_0 = \frac{3\sigma\mu\nu}{\sigma^2 + 4\alpha}\left(-2\mu\nu - \sqrt{4\mu^2\nu^2 + \sigma^2 + 4\alpha}\right).$$

If $\varepsilon > 0$ is small enough, the corresponding intersections $T_o \cap l^u$ and $T_d \cap l^s$ are transversal.

Proof. We solve system (2.14) by first solving the sum of the equations and then plugging the resulting solutions into their difference. The sum is given by $(\Gamma_+ + \Gamma_- + \frac{4}{3}c_0)b_0 - \frac{4}{3}\frac{c_0}{\mu} = 0$, which, by (2.7), reduces to $(4c_0 - 3\sigma)b_0 - 4c_0/\mu = 0$. We distinguish two cases:

Case 1. If $4c_0 - 3\sigma = 0$, this implies that $c_0 = 0$ and therefore $\sigma = 0$.

Case 2. If $4c_0 - 3\sigma \neq 0$, this implies that $b_0 = 4c_0/\mu(4c_0 - 3\sigma)$.

The difference of the equations in (2.14) is given by $(\Gamma_+ - \Gamma_-)b_0 = 4\nu\sqrt{1-\mu b_0}$, which reduces to $\sqrt{\sigma^2+4\alpha}\, b_0 = 4\nu\sqrt{1-\mu b_0}$. Note that

$$\text{sign}(b_0) = \text{sign}(\nu). \tag{2.15}$$

Taking squares, we obtain

$$(\sigma^2+4\alpha)b_0^2 + 16\mu\nu^2 b_0 - 16\nu^2 = 0; \tag{2.16}$$

we will use (2.15) to eliminate improper roots of this equation.

1. Plugging case 1, $c_0 = \sigma = 0$, into (2.16), we obtain $\alpha b_0^2 + 4\mu\nu^2 b_0 - 4\nu^2 = 0$ and thus $b_{0\pm} = \frac{2\nu}{\alpha}\left(-\mu\nu \pm \sqrt{\mu^2\nu^2+\alpha}\right)$. Because (2.15) has to hold, we can easily see that
$$b_0 = \frac{2\nu}{\alpha}\left(-\mu\nu + \sqrt{\mu^2\nu^2+\alpha}\right),$$
which gives us case *(i)* of Lemma 2.4. In this case the intersections $T_o \cap l^u$ and $T_d \cap l^s$ are transversal at leading order in ε if $\pm\alpha \neq \mp\nu\mu/\sqrt{1-\mu b_0}$. Equality can only be obtained if $\mu\nu < 0$ and $b_0 = \frac{1}{\mu} - \frac{\nu^2\mu}{\alpha^2}$, which is not the case in the intersection points. We conclude that the intersections are transversal at leading order and that they are therefore transversal for small enough ε.
2. If $4c_0 - 3\sigma \neq 0$, plugging $b_0 = 4c_0/\mu(4c_0 - 3\sigma)$ into (2.16) yields the equality $(\sigma^2+4\alpha)c_0^2 + 12\mu^2\nu^2\sigma c_0 - 9\mu^2\nu^2\sigma^2 = 0$ and therefore
$$c_0^\pm(\sigma) = \frac{3\mu\nu\sigma}{\sigma^2+4\alpha}\left(-2\mu\nu \pm \sqrt{4\mu^2\nu^2+\sigma^2+4\alpha}\right).$$
It is clear that $\text{sign}(c_0^\pm) = \pm\text{sign}(\mu\nu\sigma)$. Since $\alpha > 0$ we have $c_0^\pm(\sigma) = \frac{3}{4}\sigma$ if and only if $\sigma = 0$, which means that $\sigma \neq 0$ is equivalent to $4c_0 - 3\sigma \neq 0$. We also have
$$\lim_{\sigma\to\infty} c_0^\pm(\sigma) = \pm 3\mu\nu, \qquad \lim_{\sigma\to-\infty} c_0^\pm(\sigma) = \mp 3\mu\nu,$$
which means that $c_0^\pm(\sigma)$ is of order 1 in the limit $\sigma \to \pm\infty$. Therefore $c_0(\sigma) < \frac{3}{4}\sigma$ if $\sigma > 0$ and $c_0(\sigma) > \frac{3}{4}\sigma$ if $\sigma < 0$, or in other words,
$$\text{sign}(4c_0^\pm(\sigma) - 3\sigma) = -\text{sign}(\sigma). \tag{2.17}$$
Thus
$$\text{sign}(b_0) = \text{sign}\left(\frac{4c_0^\pm}{\mu(4c_0^\pm - 3\sigma)}\right) = \mp\text{sign}\left(\frac{\mu\nu\sigma}{\mu\sigma}\right) = \mp\text{sign}(\nu).$$
Because (2.15) has to hold, we can conclude
$$c_0 = c_0^- = \frac{3\mu\nu\sigma}{\sigma^2+4\alpha}\left(-2\mu\nu - \sqrt{4\mu^2\nu^2+\sigma^2+4\alpha}\right).$$
This proves case *(ii)*. Again it can be checked that the intersections $T_o \cap l^u$ and $T_d \cap l^s$ are transversal at leading order, so that we may conclude transversality for $\varepsilon > 0$ small enough. □

Note that the condition $1 - \mu b_0 > 0$ (2.5) is fulfilled in both cases *(i)* and *(ii)* of the above Lemma. For example, in case *(ii)* we have $\sigma \neq 0$, and by (2.17),

$$1 - \mu b_0 = 1 - \frac{4c_0}{4c_0 - 3\sigma} = \frac{-3\sigma}{4c_0 - 3\sigma} > 0.$$

Lemma 2.4 is an important ingredient in the proof of Theorem 2.1. The idea of the proof is to formally construct a singular homoclinic orbit that consists of two slow parts in $\mathcal{M}_0$ with a middle fast part given by an unperturbed homoclinic orbit. The essential idea behind singular perturbation theory is that persistence of this singular orbit for small $\varepsilon > 0$ can be established by showing that the singular structure corresponds to transversal intersections of pairs of manifolds.

Proof of Theorem 2.1. Let $\beta = 0, \alpha > 0, \mu \neq 0, \mu \in \mathbb{R}, \sigma \in \mathbb{R}$ all be fixed and $\varepsilon > 0$ sufficiently small. For any b_0 satisfying $1 - \mu b_0 > 0$ and arbitrary d_0, the unperturbed system (2.2) has homoclinic solutions $\gamma_0(\xi; b_0, d_0) = (a_0(\xi; b_0), v_0(\xi; b_0), b_0, d_0)$ with a_0 and v_0 as in (2.3) with $\lim_{|\xi| \to \infty}(a_0(\xi), v_0(\xi)) = (0, 0)$.

By the Fenichel and Melnikov arguments (Sections 2.1 and 2.2), the manifold of homoclinic solutions $\gamma_0(\xi; b_0, d_0)$ is perturbed to manifolds $W^u(\mathcal{M}_\varepsilon)$ and $W^s(\mathcal{M}_\varepsilon)$ that intersect transversally. In fact, it follows from the explicit $\mathcal{O}(\varepsilon^2)$ expression (2.9) for the distance between $W^u(\mathcal{M}_\varepsilon)$ and $W^s(\mathcal{M}_\varepsilon)$ that the angle between $W^u(\mathcal{M}_\varepsilon)$ and $W^s(\mathcal{M}_\varepsilon)$ near $W^u(\mathcal{M}_\varepsilon) \cap W^s(\mathcal{M}_\varepsilon)$ is $> C\varepsilon^2$ for some $C > 0$. The intersection $W^u(\mathcal{M}_\varepsilon) \cap W^s(\mathcal{M}_\varepsilon)$ is represented by a family of orbits $\gamma(\xi; x_0) = \gamma(\xi; b_0) = (a(\xi; b_0), v(\xi; b_0), b(\xi; b_0), d(\xi; b_0))$ with $\gamma(0) = x_0 \equiv (a(0), 0, b_0, \frac{2c_0}{3\mu}(1 - \mu b_0)\varepsilon + \mathcal{O}(\varepsilon^2))$ that are biasymptotic to $\mathcal{M}_\varepsilon$. Since the fast dynamics of (1.5) is an $\mathcal{O}(\varepsilon)$ perturbation of (2.2), these orbits satisfy $|a(\xi; b_0) - a_0(\xi; b_0)|$, $|v(\xi; b_0) - v_0(\xi; b_0)|$, $|b(\xi) - b_0| = \mathcal{O}(\varepsilon)$ as long as $|\xi| = \mathcal{O}(1)$, or in other words, the fast part of $\gamma(\xi; b_0)$ is $\mathcal{O}(\varepsilon)$ close to $\gamma_0(\xi)$.

We first consider case *(ii)* of Theorem 2.1, i.e., case *(ii)* of Lemma 2.4 with $\sigma \neq 0$. For each orbit $\gamma(\xi; x_0)$, Fenichel theory yields the existence of points $x_0^+ \in T_d$ and $x_0^- \in T_o$, and orbits $\gamma^\pm(\xi; x_0^\pm)$ on $\mathcal{M}_\varepsilon$, such that $\|\gamma(\xi; x_0) - \gamma^\pm(\xi; x_0^\pm)\| \to 0$ as $\xi \to \pm\infty$. In order to construct a singular homoclinic orbit, x_0^+ and x_0^- need to satisfy $x_0^+ \in l^s \cap T_d$ and $x_0^- \in l^u \cap T_o$. Thus, if we can construct a singular orbit consisting of pieces of l^u and l^s connected by a jump through the fast field, we may conclude that it persists for small enough $\varepsilon > 0$ (since all relevant intersections, $T_o \cap l^u$, $W^u(\mathcal{M}_\varepsilon) \cap W^s(\mathcal{M}_\varepsilon)$, and $T_d \cap l^s$ are transversal; see also Lemma 2.4).

Lemma 2.4 gives a $c_0 = c_0(\alpha, \sigma, \mu, \mu)$ for which the intersections of l^s and T_d and of l^u and T_o occur at the same value of $b_0 = b_0(c_0, \alpha, \sigma, \mu, \nu)$. Since $\Delta b = \mathcal{O}(\varepsilon^2 |\log \varepsilon|)$ (Section 2.3), we can indeed construct a singular orbit that persists as the orbit $\gamma_h(\xi) := \gamma(\xi; b_0)$, with $\xi = x - Ct$, that is homoclinic to $S = (0, 0, 0, 0)$.

If $\sigma = c = 0$, systems (1.4) and (1.5) are reversible. Hence, the symmetry

$$\xi \to -\xi, \qquad v \to -v, \qquad d \to -d \tag{2.18}$$

maps l^u to l^s and T_o to T_d (and vice versa). Note that this condition is fulfilled in case *(i)* of Lemma 2.4. Thus, an intersection $l^u \cap T_o$ automatically corresponds to a (symmetric) homoclinic orbit $\gamma_h \in W^u(S) \cap W^s(S)$ since all intersections of manifolds involved are transversal. This proves case *(i)* of the Theorem.

In both cases, the orbit γ_h satisfies $a_h'(\xi) = v(\xi) > 0$ for $\xi < 0$ and $a_h'(\xi) = v(\xi) < 0$ for $\xi < 0$, and moreover, by Section 2.3, $||\gamma_h(\xi, x_0) - l^{u,s}|| = \mathcal{O}(e^{-\kappa/\varepsilon})$ for $|\xi| \geq \mathcal{O}(\frac{1}{\varepsilon})$ and some $\kappa > 0$. This implies that the approximations $a(\xi; b_0) = a_0(\xi; b_0) + \mathcal{O}(\varepsilon)$, $v(\xi; b_0) = v_0(\xi; b_0) + \mathcal{O}(\varepsilon)$ can be extended to the slow parts of $\gamma_h(\xi)$ so that they are uniform on $\mathbb{R}$.

In the setting of system (1.4) the orbit $\gamma_h(\xi)$ corresponds to a pulse solution $(A_h(\xi), B_h(\xi)) = (a_h(\xi), b_h(\xi))$ that satisfies $\lim_{x\to\pm\infty}(A_h(\xi), B_h(\xi)) = (0, 0)$ and that travels with speed $C = \varepsilon^2 c$ which is at leading order given by Lemma 2.4. □

2.5. *Pulse solutions for $\beta \neq 0$*

The introduction of higher order nonlinearities in $G(B, B_x, |A|^2)$ in (1.2), i.e., by our specific choice $\beta \neq 0$ in (1.4), merely increases the algebraic complexity of the persistence problem. The essence of the geometrical structure developed in the preceding subsections is not influenced by the term $\beta A^2 B$, or in general, other higher order nonlinear terms (such as B^2; see Remark 2.3).

As in the proof of Theorem 2.1/Lemma 2.4, we only need to determine the intersection $l^u \cap T_o$ and $l^s \cap T_d$ and check transversality. An orbit γ_h in $W^u(S) \cap W^s(S)$ corresponds to a solution b_0 of both equations in (2.14); thus, one expects that γ_h again selects a (uniquely determined) wave speed c.

For simplicity, we do not consider the most general case, i.e., the equivalent of case *(ii)* in Lemma 2.4. Instead, we assume that $\sigma = 0$ (case *(i)*), so that $c = 0$ and (2.14) can be written as a cubic polynominal in b_0,

$$4\mu\beta^2 b_0^3 + (\alpha + 8\mu\nu\beta - 4\beta^2)b_0^2 + (4\mu\nu^2 - 8\nu\beta)b_0 - 4\nu^2 = 0, \tag{2.19}$$

where we again must be careful, since we might have introduced improper solutions by eliminating the square root. For future reference we note that equation (2.19) has two types of solutions: solutions $b_0(\beta)$ that correspond to pulse solutions that merge in the limit $\beta \to 0$ with a pulse solution of the $\beta = 0$ case, or solutions for which $b_0(\beta)$ diverges. The latter solutions are for $0 < \varepsilon \ll \beta \ll 1$, i.e., β must be positive, given by

$$b_0(\beta) = -\frac{\alpha}{4\mu}\frac{1}{\beta^2} + \mathcal{O}\left(\frac{1}{\beta}\right). \tag{2.20}$$

Next, we again note that (1.4) and (1.5) has become a reversible system by setting $c = \sigma = 0$. As a consequence, we can again use the symmetry (2.18) to conclude that any intersection of l^u and T_o corresponds directly to a symmetric homoclinic orbit to S. Hence, the essence of the proof of Theorem 2.2 has been reduced to studying the cubic polynomial (2.19). Note that we can consider $\mu > 0$, due to the symmetry

$$\mu \to -\mu, \qquad \nu \to -\nu, \qquad b \to -b, \qquad d \to -d.$$

Lemma 2.5. *If $\beta \neq 0$, $c = \sigma = 0$, and $\alpha > 0$, the number of intersections between l^u and T_o depends on the parameters μ, ν, β in the following way.*

(i) *If $\mu > 0$ and $\beta < 0$, then for each α there exists one intersection at b_0.*

(ii) *If $\mu\nu > 0$ and $\beta > 0$, then for each α there exist two intersections; one with positive b_0 and one with negative b_0.*

(iii) *If $\mu > 0$, $\nu < 0$, and $0 < \beta < -\mu\nu$, then there exists an $\alpha^* > 0$ such that there are no intersections if $\alpha < \alpha^*$; l^u and T_o are tangent at $b_0 < 0$ if $\alpha = \alpha^*$; and there exist two intersections, both with b_0 negative, if $\alpha > \alpha^*$.*

(iv) *If $\mu > 0$, $\nu < 0$, and $\beta > -\mu\nu$, then there exist α_1^* and α_2^* with $0 < \alpha_1^* < \alpha_2^*$ such that there exist two intersections with $b_0 > 0$ if $\alpha < \alpha_1^*$; l^u and T_o are tangent if $\alpha = \alpha_1^*$; there are no intersections if $\alpha_1^* < \alpha < \alpha_2^*$; l^u and T_o are again tangent if $\alpha = \alpha_2^*$; and there are two intersections, both with $b_0 < 0$, if $\alpha > \alpha_2^*$.*

See Figure 5 for a picture of each of the above cases.

Proof. The proof is by inspection of the domain of T_o, the intersection of T_o with the d-axis, the sign of T_o, and the behavior of T_o for $b \to |\infty|$. □

As was already noted, the proof of Theorem 2.2 now follows from the symmetry (2.18), the Fenichel and Melnikov arguments, and the above Lemma. Transversality of the intersecting manifolds and curves can always be ensured for small enough ε, unless α is close to α^* or $\alpha_{1,2}^*$. The several subcases of the Lemma describe the open regions in $(\mu, \alpha, \nu, \beta)$-space in which system (1.4) possesses 0, 1, or 2 (different) orbits homoclinic to $S = (0, 0, 0, 0)$. In cases *(iii)* and *(iv)* the curves T_o and l^u (T_d and l^s) have tangencies at first order for $\alpha = \alpha_1^*(\mu, \nu, \beta)$, in which two elements of $l^u \cap T_o$ are created or annihilated. It can be shown that these correspond to bifurcations $\tilde{\alpha}_1^* = \alpha_1^* + \mathcal{O}(\varepsilon)$ in which two homoclinic orbits to S are created or annihilated. Hence, at $\alpha = \tilde{\alpha}_1^*$ a saddle-node bifurcation of homoclinic orbits takes place. Since they are of codimension 1 type, they generically also occur when varying one of the other parameters μ, ν, β while keeping the others fixed. In Section 4.3 we will vary β for fixed α, μ, ν and recover the saddle-node bifurcations for some $\beta = \beta^*$.

3. Linear Stability Analysis

In this section we study the stability of the homoclinic pulse solutions $(A_h(\xi), B_h(\xi))$ of (1.4) by the Evans function method [2]. First, we transform system (1.4) into a system in which these pulses are stationary by replacing (x, t) by the traveling coordinates (ξ, t),

$$\begin{aligned} -c\varepsilon^2 A_\xi + A_t &= -A + A^3 + \mu AB + A_{\xi\xi}, \\ -\varepsilon^4 \tau c B_\xi + \varepsilon^2 \tau B_t &= -\alpha\varepsilon^2 B + \nu A^2 + \beta BA^2 + \sigma B_\xi + \varepsilon^{-2} B_{\xi\xi}. \end{aligned} \tag{3.1}$$

Next, we linearize around $(A_h(\xi), B_h(\xi))$ and, with a small abuse of notation, reintroduce $a(\xi)$ and $b(\xi)$ by

$$\begin{aligned} A(\xi, t) &= A_h(\xi) + a(\xi)e^{\lambda t}, \\ B(\xi, t) &= B_h(\xi) + b(\xi)e^{\lambda t}, \end{aligned}$$

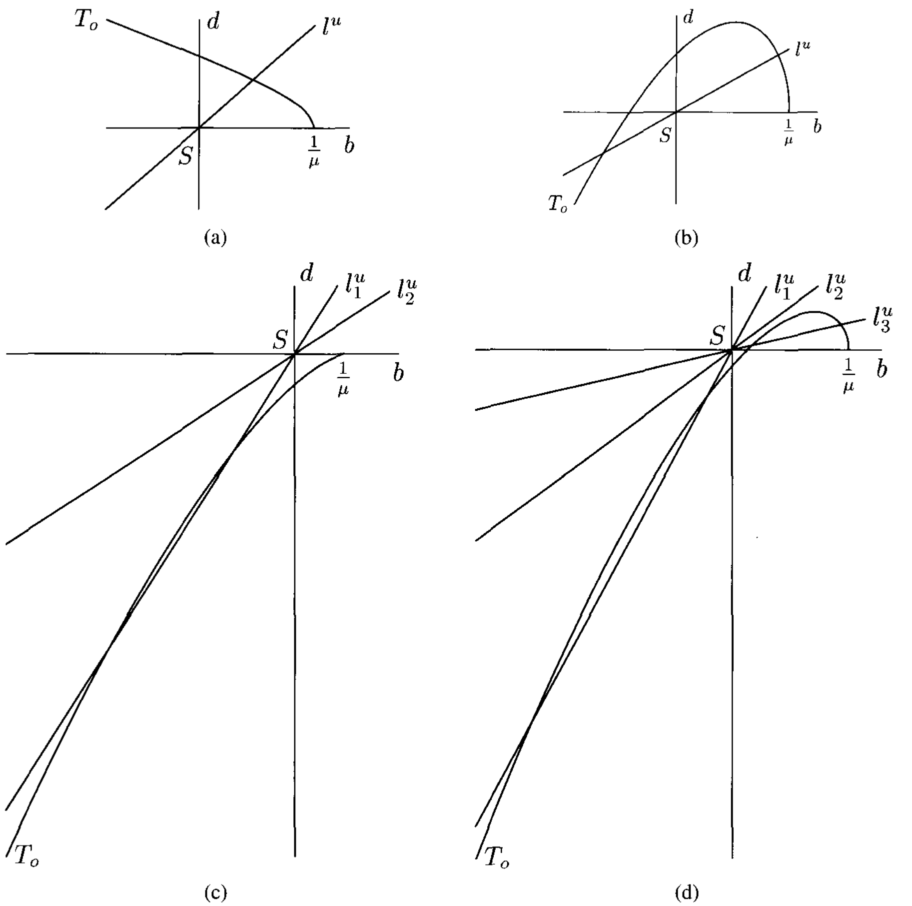

Fig. 5. The different intersection situations of T_o and l^u of all four cases in Lemma 2.5. (a) Case (i): For each $\alpha > 0$ there is always one intersection. (b) Case (ii): For each $\alpha > 0$ there are two intersections. (c) Case (iii): Variation of the curve l^u with α: the curve l_2^u with $0 < \alpha < \alpha^*$ and l_1^u with $\alpha > \alpha^*$ such that there are two intersections; α^* is the only bifurcation value. (d) Case (iv): The curve l_1^u with $\alpha > \alpha_2^*$ such that there are two intersections both with $b_0 < 0$; the curve l_2^u with $\alpha_1^* < \alpha < \alpha_2^*$; the curve l_3^u with $0 < \alpha < \alpha_1^*$ such that there are two intersections both with $b_0 > 0$.

where $(A(\xi, t), B(\xi, t))$ is a solution of (3.1). The linearized equations for $(a(\xi), b(\xi))$ read

$$\begin{aligned} -c\varepsilon^2 a_\xi + \lambda a &= -a + 3A_h^2 a + \mu A_h b + \mu B_h a + a_{\xi\xi}, \\ -\tau\varepsilon^4 c b_\xi + \varepsilon^2 \tau \lambda b &= -\alpha\varepsilon^2 b + 2\nu A_h a + \beta A_h B_h a + \beta A_h^2 b + \sigma b_\xi + \varepsilon^{-2} b_{\xi\xi}. \end{aligned} \tag{3.2}$$

This system can be written as a four-dimensional system of (first order) ODEs by introducing $\phi(\xi) = (a(\xi), v(\xi), b(\xi), d(\xi))^{\top}$

$$\phi_\xi(\xi) = \mathcal{M}(\xi; \lambda, c, \varepsilon)\phi(\xi), \tag{3.3}$$

where $\mathcal{M}$ is given by

$$\mathcal{M}(\xi;\lambda,c,\varepsilon) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ [1-3A_h^2-\mu B_h]+\lambda & -c\varepsilon^2 & -\mu A_h & 0 \\ 0 & 0 & 0 & \varepsilon \\ -2[\beta A_h B_h+\nu A_h]\varepsilon & 0 & [\tau\lambda+\alpha]\varepsilon^3-\beta A_h^2\varepsilon & -[\tau c\varepsilon^4+\sigma]\varepsilon^2 \end{pmatrix}. \tag{3.4}$$

Taking the limit $|\xi| \to \infty$ in $\mathcal{M}(\xi;\lambda,c,\varepsilon)$ yields the constant coefficient matrix

$$\mathcal{M}_\infty(\lambda,c,\varepsilon) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \lambda+1 & -c\varepsilon^2 & 0 & 0 \\ 0 & 0 & 0 & \varepsilon \\ 0 & 0 & [\tau\lambda+\alpha]\varepsilon^3 & -[\tau c\varepsilon^4+\sigma]\varepsilon^2 \end{pmatrix}, \tag{3.5}$$

where we used that $\lim_{|\xi|\to\infty} A_h(\xi) = \lim_{|\xi|\to\infty} B_h(\xi) = 0$. The eigenvalues of the martrix $\mathcal{M}_\infty(\lambda,c,\varepsilon)$ are

$$\begin{aligned} \Lambda_{1,4}(\lambda,c,\varepsilon) &= \frac{1}{2}\left[-\varepsilon^2 c \pm \sqrt{\varepsilon^4 c^2+4(\lambda+1)}\,\right], \\ \Lambda_{2,3}(\lambda,c,\varepsilon) &= \frac{\varepsilon^2}{2}\left[-(\tau\varepsilon^4 c+\sigma) \pm \sqrt{(\tau c\varepsilon^4+\sigma)^2+4(\tau\lambda+\alpha)}\,\right]. \end{aligned} \tag{3.6}$$

The associated eigenvectors are given by

$$E_{1,4}(\lambda,c,\varepsilon) = (1, \Lambda_{1,4}(\lambda,c,\varepsilon), 0, 0)^{\mathrm{T}},\; E_{2,3}(\lambda,c,\varepsilon) = (0,0,1,\Lambda_{2,3}(\lambda,c,\varepsilon)/\varepsilon)^{\mathrm{T}}. \tag{3.7}$$

The essential spectrum σ_e of the linear eigenvalue problem (3.2), or (3.3), is determined by values of λ for which either one of the eigenvalues $\Lambda_j(\lambda)$ of $\mathcal{M}_\infty(\lambda)$ is purely imaginary, i.e., for λ such that $\Lambda_j(\lambda) = ik$ for some $k \in \mathbb{R}$ [17]. Hence, σ_e consists of the curves

$$\begin{aligned} \lambda_1(k) &= -(1+k^2)+\varepsilon^2 cki, \\ \lambda_2(k) &= \frac{1}{\tau}(-\alpha-\varepsilon^{-4}k^2+[\sigma k+\varepsilon^4\tau ck]i), \end{aligned}$$

with $k \in \mathbb{R}$ (see Figure 6). Thus, we may conclude that the stability of the pulse $(A_h(\xi), B_h(\xi))$ is completely determined by the discrete spectrum of the eigenvalue problem (3.2)/(3.3) (recall that $\alpha, \tau > 0$).

We denote the complement of the essential spectrum by

$$\mathcal{C}_e := \mathbb{C} \setminus \sigma_e = \mathcal{C}_e^+ \cup \mathcal{C}_e^-. \tag{3.8}$$

Here, $\mathcal{C}_e^+$ is that part of $\mathcal{C}_e$ that is to the right of σ_e, i.e., $\mathcal{C}_e^+$ is the (open, unbounded) connected component of $\mathcal{C}_e$ that includes the halfplane $\{\mathrm{Re}(\lambda) \geq 0\}$ and that is bounded by (parts of) σ_e (note that $\mathcal{C}_e^-$ consists of several disjoint components; see Figure 6). For $\lambda \in \mathcal{C}_e^+$ (and not $\mathcal{O}(\varepsilon^4)$ close to its boundary), the eigenvalues $\Lambda_j(\lambda)$ of $\mathcal{M}_\infty(\lambda)$ can be ordered

$$\mathrm{Re}\Lambda_1(\lambda,c,\varepsilon) > \mathrm{Re}\Lambda_2(\lambda,c,\varepsilon) > 0 > \mathrm{Re}\Lambda_3(\lambda,c,\varepsilon) > \mathrm{Re}\Lambda_4(\lambda,c,\varepsilon).$$

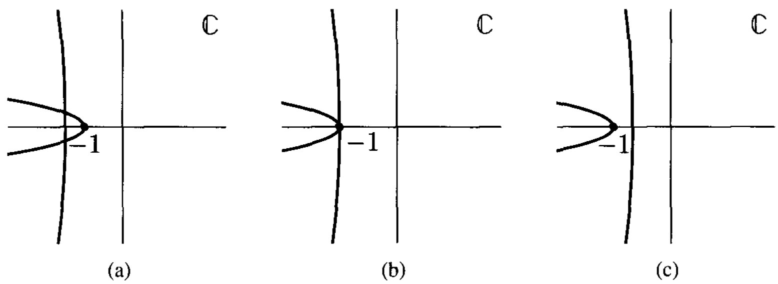

Fig. 6. The essential spectrum of the linear eigenvalue problem associated with (3.2) in the complex plane: (*a*) for $0 < \tau < \alpha$; (*b*) for $\tau = \alpha$; (*c*) for $0 < \alpha < \tau$.

3.1. The Evans Function and Its Decomposition

The discrete spectrum of eigenvalue problem (3.3), and thus the stability of the traveling pulse $(A_h(\xi), B_h(\xi))$, can be determined by the Evans function $\mathcal{D}(\lambda, c, \varepsilon)$. In this section, we give a brief exposition of the characteristics of the Evans function approach in (singularly perturbed) reaction-diffusion equations; we refer to [2], [15], [8], [7] for the details of the proofs of the statements and/or Lemmas in this section.

The Evans function $\mathcal{D}(\lambda, c, \varepsilon)$ is in essence a Wronskian, i.e., the determinant of a (fundamental) matrix of four independent solutions $\phi_j(\xi; \lambda, c, \varepsilon)$, $j = 1, \cdots, 4$, of the four-dimensional linear system (3.3). Since $\mathcal{M}(\xi; \lambda)$ converges to the constant coefficient matrix $\mathcal{M}_\infty(\lambda)$ as $\xi \to \pm\infty$, we can prescribe the behavior of the solutions $\phi_j(\xi; \lambda, c, \varepsilon)$ for $\xi \to \pm\infty$ in terms of the eigenvalues and eigenvectors $\Lambda_j(\lambda)$ and $E_j(\lambda)$ of $\mathcal{M}_\infty(\lambda)$ ((3.6),(3.7)).

Lemma 3.1. *For all $\lambda \in \mathcal{C}_e^+$ (3.8), there exist four independent solutions $\phi_j(\xi; \lambda, c, \varepsilon)$ of (3.3), $j = 1, \cdots, 4$, such that*

$$\lim_{\xi \to -\infty} \phi_{1,2}(\xi; \lambda) e^{-\Lambda_{1,2}(\lambda)\xi} = E_{1,2}(\lambda), \qquad \lim_{\xi \to +\infty} \phi_{3,4}(\xi; \lambda) e^{-\Lambda_{3,4}(\lambda)\xi} = E_{3,4}(\lambda).$$

Thus, $\phi_1(\xi; \lambda, c, \varepsilon)$ and $\phi_2(\xi; \lambda, c, \varepsilon)$ span the two-dimensional space $\Phi_-(\xi; \lambda, c, \varepsilon)$ of solutions of (3.3) that approach $(0, 0, 0, 0)^{\mathrm{T}}$ as $\xi \to -\infty$, while $\phi_3(\xi; \lambda, c, \varepsilon)$ and $\phi_4(\xi; \lambda, c, \varepsilon)$ span the two-dimensional space $\Phi_+(\xi; \lambda, c, \varepsilon)$ of solutions that approach $(0, 0, 0, 0)^{\mathrm{T}}$ as $\xi \to \infty$. The functions $\phi_1(\xi; \lambda)$ and $\phi_4(\xi; \lambda)$ converge on the fast spatial scale (since $\Lambda_{1,4}(\lambda) = \mathcal{O}(1)$), and are determined uniquely by their behavior at $-\infty$, respectively $+\infty$. As a consequence, we can define the first transmission function $t_1(\lambda, c, \varepsilon)$ by the behavior of $\phi_1(\xi; \lambda)$ as $\xi \to +\infty$:

$$\lim_{\xi \to +\infty} \phi_1(\xi; \lambda, c, \varepsilon) e^{-\Lambda_1(\lambda, c, \varepsilon)\xi} = t_1(\lambda, c, \varepsilon) E_1(\lambda, c, \varepsilon). \tag{3.9}$$

This transmission function is an analytic function of λ for $\lambda \in \mathcal{C}_e^+$ [2], [8]. To determine $\phi_2(\xi; \lambda, c, \varepsilon)$ (and likewise $\phi_3(\xi; \lambda, c, \varepsilon)$) uniquely we need to impose a condition on its behavior as $\xi \to +\infty$. As $\phi_1(\xi; \lambda, c, \varepsilon)$, a solution $\phi_2(\xi; \lambda, c, \varepsilon)$ of (3.3) will in general also diverge as $\xi \to +\infty$ on the fast spatial scale (3.9); however, it can be shown that there is one solution $\phi_2(\xi; \lambda, c, \varepsilon)$ that only increases at the slow exponential rate (associated with $\Lambda_2(\lambda)$) as $\xi \to +\infty$ [8], [7]:

Lemma 3.2. *For $\lambda \in \mathcal{C}_e^+$ such that $t_1(\lambda) \neq 0$ there exists a (unique) solution $\phi_2(\xi; \lambda, c, \varepsilon)$ of (3.3), and a second transmission function $t_2(\lambda)$, such that*

$$\lim_{\xi \to +\infty} \phi_2(\xi; \lambda, c, \varepsilon) e^{-\Lambda_2(\lambda, c, \varepsilon)\xi} = t_2(\lambda, c, \varepsilon) E_2(\lambda, c, \varepsilon). \tag{3.10}$$

Both $\phi_2(\xi; \lambda)$ and $t_2(\lambda)$ may have a simple pole at the zeroes of $t_1(\lambda)$ (see [7], [8] and Section 3.2). Nevertheless, the Evans function $\mathcal{D}(\lambda, \varepsilon)$ can be defined by

$$\mathcal{D}(\lambda, \varepsilon) = \det[\phi_1(\xi), \phi_2(\xi), \phi_3(\xi), \phi_4(\xi)] e^{-\int_0^\xi \mathrm{Tr}(\mathcal{M}(\eta; \lambda)) d\eta}, \tag{3.11}$$

$\mathcal{D}(\lambda, \varepsilon)$ does not depend on ξ and is an analytic function of λ (for $\lambda \in \mathcal{C}_e^+$) [2]. This can be understood by noticing that the space $\Phi_-(\xi; \lambda, c, \varepsilon)$ is spanned by two analytic functions $\phi_1(\xi; \lambda)$ and $\tilde{\phi}_2(\xi; \lambda)$ as defined in Lemma 3.1. Thus, a standard, analytical Evans function $\tilde{\mathcal{D}}(\lambda)$ can be defined as in (3.11) with $\phi_2(\xi; \lambda)$ replaced by $\tilde{\phi}_2(\xi; \lambda)$. By construction, the function $\phi_2(\xi; \lambda) \in \Phi_-(\xi; \lambda, c, \varepsilon)$ as defined in Lemma 3.2 can be written as $\phi_2(\xi; \lambda) = \tilde{\phi}_2(\xi; \lambda) + C(\lambda)\phi_1(\xi; \lambda)$, with $C(\lambda)$ such that it has a singularity at the poles of $\phi_2(\xi; \lambda)$ and $t_2(\lambda)$. Due to the determinant in the definition of $\mathcal{D}$ (3.11), it follows that $\mathcal{D}(\lambda) = \tilde{\mathcal{D}}(\lambda)$ for all $\lambda \in \mathcal{C}_e^+$ for which $t_1(\lambda) \neq 0$. Hence, the singularities of $\mathcal{D}(\lambda)$ are removable and $\mathcal{D}(\lambda) = \tilde{\mathcal{D}}(\lambda)$ for all $\lambda \in \mathcal{C}_e^+$.

As a consequence, $\mathcal{D}(\lambda)$ can be seen as a classical eigenfunction (for instance in the sense of [2]). An eigenfunction of (3.3) converges to 0 for both $\xi \to -\infty$ and $\xi \to +\infty$; hence it must be an element of both spaces $\Phi_-(\xi; \lambda)$ and $\Phi_+(\xi; \lambda)$. In other words, the solutions $\phi_1(\xi; \lambda)$, $\phi_2(\xi; \lambda)$, $\phi_3(\xi; \lambda)$, and $\phi_4(\xi; \lambda)$ cannot be independent if λ is an eigenvalue, so that $\mathcal{D}(\lambda, \varepsilon) = 0$ at an eigenvalue of (3.3). More precisely, there is a one-to-one correspondence between zeroes of $\mathcal{D}(\lambda, \varepsilon)$ and eigenvalues of (3.3), counting multiplicities [2].

Since

$$\mathrm{Tr}(\mathcal{M}(\eta; \lambda)) = \mathrm{Tr}(\mathcal{M}_\infty(\lambda)) = \sum_{j=1}^{4} \Lambda_j(\lambda),$$

the Evans function $\mathcal{D}(\lambda, \varepsilon)$ can be decomposed into a product of the transmission functions $t_1(\lambda, \varepsilon)$ and $t_2(\lambda, \varepsilon)$,

$$\begin{aligned}
\mathcal{D}(\lambda, c, \varepsilon) &= \lim_{\xi \to \infty} \det[\phi_1(\xi), \phi_2(\xi), \phi_3(\xi), \phi_4(\xi)] e^{-\int_0^\xi \mathrm{Tr}(\mathcal{M}_\infty(x)) dx} \\
&= \lim_{\xi \to \infty} \det[\phi_1(\xi) e^{-\Lambda_1 \xi}, \phi_2(\xi) e^{-\Lambda_2 \xi}, \phi_3(\xi) e^{-\Lambda_3 \xi}, \phi_4(\xi) e^{-\Lambda_4 \xi}] \\
&= \det[t_1(\lambda, c, \varepsilon) E_1(\lambda, c, \varepsilon), t_2(\lambda, c, \varepsilon) E_2(\lambda, c, \varepsilon), E_3(\lambda, c, \varepsilon), E_4(\lambda, c, \varepsilon)] \\
&= \varepsilon t_1(\lambda, c, \varepsilon) t_2(\lambda, c, \varepsilon) \sqrt{[\varepsilon^4 c^2 + 4(\lambda + 1)][(\tau c \varepsilon^4 + \sigma)^2 + 4(\tau\lambda + \alpha)]}.
\end{aligned}$$

Therefore the eigenvalues of (3.3) are determined by the zeroes of $t_1(\lambda, c, \varepsilon)$ and $t_2(\lambda, c, \varepsilon)$. Moreover, the zeroes of $t_1(\lambda, c, \varepsilon)$ are strongly related to the eigenvalues associated with the stability problem of the homoclinic pulse of the fast reduced limit problem, i.e., the uncoupled Ginzburg-Landau equation (1.3),

$$(\mathcal{L}_0 - \lambda)a = a_{\xi\xi} + [3a_0^2(\xi) - (1 - \mu b_0 + \lambda)]a = 0, \tag{3.12}$$

where $a_0(\xi)$ (2.3), respectively b_0, is the leading order approximation of $A_h(\xi)$, resp. $B_h(\xi)$, in the fast field (Theorem 2.1). By introducing the new independent variable $\chi = \sqrt{1 - \mu b_0}\xi$,

$$\tilde{\lambda} = \frac{\lambda}{1 - \mu b_0}, \qquad w(\chi) = \frac{-\sqrt{1 - \mu b_0}}{\mu} a(\xi), \qquad w_h(\chi) = \frac{1}{\sqrt{1 - \mu b_0}} a_0(\xi), \tag{3.13}$$

where $w_h(\chi)$ is the positive homoclinic solution of

$$W_{\chi\chi} = W - W^3, \qquad w_h(\chi) = \sqrt{2}\operatorname{sech}\chi, \tag{3.14}$$

system (3.12) transforms into

$$(\tilde{\mathcal{L}}_0 - \tilde{\lambda})w = w_{\chi\chi} + [3w_h^2(\chi) - (1 + \tilde{\lambda})]w = 0. \tag{3.15}$$

Note that the scaling of $w(\chi)$ in (3.13) is at this point irrelevant; however it will become meaningful when we consider an inhomogeneous version of (3.12). Using hypergeometric functions, it is possible to determine the eigenvalues and eigenfunctions of this equation explicitly; see Remark 3.7. The two eigenvalues are $\tilde{\lambda}_0^f = 3$ and $\tilde{\lambda}_1^f = 0$. The associated eigenfunction $w_j^f(\chi)$, $j = 0, 1$, can be expressed in terms of $w_h(\chi)$ and $\dot{w}_h(\chi)$; $w_0^f(\chi)$ is even and $w_1^f(\chi) = \dot{w}_h(\chi)$ is odd as a function of χ. Thus, we conclude

Lemma 3.3. *The eigenvalues λ_0^f and λ_1^f of the fast reduced stability problem (3.12) are given by $\lambda_0^f = 3(1 - \mu b_0) > 0$ and $\lambda_1^f = 0$.*

This lemma establishes the well-known fact that the pulse solution of the uncoupled subcritical Ginzburg-Landau equation (1.3) is indeed unstable. The zeroes of $t_1(\lambda)$ can now be determined by a winding number argument [2], [15], [8], [7].

Lemma 3.4. *The transmission function $t_1(\lambda)$ has two simple zeroes for $\lambda \in C_e$, $\lambda_0(\varepsilon), \lambda_1(\varepsilon) \in \mathbb{R}$ (i.e., $t_1(\lambda) \neq 0$ for $\lambda \neq \lambda_{0,1}(\varepsilon)$); $\lambda_1(\varepsilon) \equiv \lambda_1^f = 0$ and $\lim_{\varepsilon\to 0} \lambda_0(\varepsilon) = \lambda_0^f > 0$.*

The zero $\lambda_1(\varepsilon) = 0$ of course corresponds to the eigenfunction of (3.3) given by the derivative of the pulse $(A_h(\xi), B_h(\xi))$. Although the fast transmission function $t_1(\lambda)$ has a real positive root at $\lambda_0(\varepsilon)$, this does not imply that $\mathcal{D}(\lambda_0(\varepsilon), c, \varepsilon) = 0$, since we will see in the next section that the slow transmission function $t_2(\lambda, c, \varepsilon)$ has a pole (of order one) at $\lambda = \lambda_0(\varepsilon)$. Hence, Lemma 3.4 does not yield a positive eigenvalue of (3.3), so that one cannot conclude from this lemma that stabilization by slow diffusion is impossible.

3.2. The NLEP Approach

We will use the NLEP approach that has been developed in [8], [7] to determine the zeroes of the slow transmission function $t_2(\lambda, c, \varepsilon)$. These zeroes can be called "slow-fast eigenvalues" since they exist due to the interactions of the fast field and the slow field. Thus, unlike the zeroes of $t_1(\lambda, c, \varepsilon)$, the zeroes of $t_2(\lambda, c, \varepsilon)$ are not related to eigenvalues of the fast reduced limit problem. In order to study the interactions between slow and fast effects, we need to define the fast spatial region more accurately. Therefore, we introduce the interval

$$I_f = \left[-1/\sqrt{\varepsilon}, 1/\sqrt{\varepsilon} \right]. \tag{3.16}$$

The choice of the boundaries of I_f is not very relevant; any choice will be suitable as long as it is in the transition zone (or "matching region" in the terminology of matched asymptotics) between the slow ξ^*- and the fast ξ-scale, i.e., so that $|\xi^*| \ll 1$ and $|\xi| \gg 1$ on the boundary of I_f.

To determine $t_2(\lambda, c, \varepsilon)$, we need to study the fundamental solution $\phi_2(\xi; \lambda) = (a_2(\xi; \lambda), v_2(\xi; \lambda), b_2(\xi; \lambda), d_2(\xi; \lambda))^\top$ (Lemma 3.1). By the limiting behavior of ϕ_2 $(\xi; \lambda)$ for $\xi \to \pm\infty$ (Lemma 3.1 and 3.2), by the structure of $E_2(\lambda)$ and $E_3(\lambda)$ (3.7), and by the (fast) exponential decay of $A_h(\xi)$, we conclude from (3.2) that $a_2(\xi)$ and $v_2(\xi)$ are exponentially small outside I_f, and that up to exponentially small terms, the equation for $(b_2(\xi; \lambda), d_2(\xi; \lambda))^\top$ is of constant coefficients type outside I_f (see also [8], [7]). Thus we conclude

Lemma 3.5. *For all $\lambda \in \mathcal{C}_e^+$ such that $t_1(\lambda, \varepsilon) \neq 0$ there exist $\mathcal{O}(1)$ constants $C_-, C_+ > 0$ and a third (meromorphic) transmission function $t_3(\lambda, \varepsilon)$ such that*

$$\phi_2(\xi; \lambda) = \begin{cases} E_2(\lambda)e^{\Lambda_2(\lambda)\xi} + \mathcal{O}(e^{C_-\xi}) & \text{for} \quad \xi < -\frac{1}{\sqrt{\varepsilon}} \\ t_2(\lambda)E_2(\lambda)e^{\Lambda_2(\lambda)\xi} + t_3(\lambda)E_3(\lambda)e^{\Lambda_3(\lambda)\xi} + \mathcal{O}(e^{-C_+\xi}) & \text{for} \quad \xi > \frac{1}{\sqrt{\varepsilon}} \end{cases}. \tag{3.17}$$

Note that $t_3(\lambda)$ governs the decay of $\phi_2(\xi; \lambda)$ on the long spatial scale (which has not been incorporated in Lemma 3.2). Since $b_{\xi\xi} = \mathcal{O}(\varepsilon^2)$ in I_f, it follows that $b_2(\xi)$ does not change during the passage of $\phi_2(\xi; \lambda)$ of I_f. The b-components of $E_{2,3}(\lambda)$ are equal to 1 (3.7), so that

$$t_2(\lambda, c, \varepsilon) + t_3(\lambda, c, \varepsilon) = 1 + \mathcal{O}(\sqrt{\varepsilon}). \tag{3.18}$$

We will determine a second relation between $t_2(\lambda, c, \varepsilon)$ and $t_3(\lambda, c, \varepsilon)$ by determining the change of the $d_2(\xi; \lambda)$-coefficient, i.e., $b_\xi(\xi; \lambda)$, of $\phi_2(\xi; \lambda)$ over the fast field I_f. Since we now know that $b_2(\xi; \lambda) = 1$ at leading order in I_f, we note that the b-equation decouples from the full system (3.2). For $\xi \in I_f$ we can furthermore approximate $B_h(\xi)$ by b_0 and $A_h(\xi)$ by $a_0(\xi)$ (see Theorems 2.1 and 2.2). As a consequence, we obtain at leading order an inhomogeneous version of the fast reduced limit problem (3.12),

$$(\mathcal{L}_0 - \lambda)a = a_{\xi\xi} + [3a_0^2(\xi) - (1 - \mu b_0 + \lambda)]a = -\mu a_0(\xi) \tag{3.19}$$

(for $\xi \in I_f$). Thus, if we consider (3.19) for $\xi \in \mathbb{R}$, we know that for all $\lambda \in \mathcal{C}_e^+$ such that $\lambda \neq \lambda_j^f$, $j = 0, 1$ (the eigenvalues of the fast reduced linear stability problem, Lemma 3.3) we can define $a_{in}(\xi;\lambda,c)$ as the unique bounded solution of (3.19). Using $a_{in}(\xi;\lambda,c)$, we deduce from (3.2) that the leading order behavior of $b_2(\xi)$ over I_f is governed by

$$b_{\xi\xi} = -\varepsilon^2[2(\nu+\beta b_0)a_0 a_{in} + \beta a_0^2],$$

where we have (again) approximated $A_h(\xi)$ by $a_0(\xi)$, $B_h(\xi)$ by b_0, and $b(\xi)$ by 1 (and thus b_ξ by 0). Hence, we have the following explicit leading order approximation for the change in $b_\xi(\xi;\lambda)$ over I_f,

$$\begin{aligned}\Delta_{\text{fast}} b_\xi &= \int_{-1/\sqrt{\varepsilon}}^{1/\sqrt{\varepsilon}} b_{\xi\xi}\, d\xi \\ &= -\varepsilon^2 \int_{-\infty}^{\infty} \left[2(\nu+\beta b_0)a_0(\xi)a_{in}(\xi) + \beta a_0^2(\xi)\right] d\xi + \mathcal{O}(\varepsilon^2\sqrt{\varepsilon}).\end{aligned} \tag{3.20}$$

Outside I_f, $\phi_2(\xi;\lambda)$ is prescribed by Lemma 3.5. Since we thus have a leading order approximation for $\phi_2(\xi;\lambda)$ on the boundaries of I_f, we can compute a second expression for the change in $b_\xi(\xi;\lambda)$ over I_f,

$$\begin{aligned}\Delta_{\text{slow}} b_\xi &= t_2(\lambda)\Lambda_2(\lambda) + t_3(\lambda)\Lambda_3(\lambda) - \Lambda_2(\lambda) + \mathcal{O}(e^{-C/\sqrt{\varepsilon}}) \\ &= \varepsilon^2(t_2(\lambda)-1)\sqrt{\sigma^2+4(\tau\lambda+\alpha)} + \mathcal{O}(\varepsilon^4),\end{aligned} \tag{3.21}$$

where we have used (3.6) and (3.18). The fast "jump" $\Delta_{\text{fast}} b_\xi$ must of course "match" the slow "jump" $\Delta_{\text{slow}} b_\xi$, which gives us the desired expression for the transmission function $t_2(\lambda,c,\varepsilon)$,

$$t_2(\lambda,c,\varepsilon) = 1 - \frac{1}{\sqrt{\sigma^2+4(\tau\lambda+\alpha)}} \int_{-\infty}^{\infty} [2(\nu+\beta b_0)a_0 a_{in} + \beta a_0^2]\, d\xi + \mathcal{O}(\sqrt{\varepsilon}). \tag{3.22}$$

The inhomogeneous problem (3.19) can—in general—only be solved when the operator $(\mathcal{L}_0 - \lambda)$ is invertible. Therefore, it is clear from the combination of (3.19) and (3.22) that $t_2(\lambda)$ will—in general—have a (simple) pole near the eigenvalues $\lambda_{0,1}^f$ of the fast reduced limit problem. However, since $\mathcal{D}(\lambda)$ is analytic [2], the product $t_1(\lambda)t_2(\lambda)$ cannot have poles. Hence the poles of $t_2(\lambda)$ must coincide with the zeroes $\lambda_{0,1}(\varepsilon)$ of $t_1(\lambda)$ (which are asymptotically close to $\lambda_{0,1}^f$, Lemma 3.4). Note that this is consistent with the fact that $t_2(\lambda)$ could only be defined for λ such that $t_1(\lambda) \neq 0$ (Lemma 3.2).

However, as is usual in these systems [8], [7], $t_2(\lambda)$ does not have a pole at $\lambda = 0$. To see this, we first transform (3.19) for $a_{in}(\xi)$ into a simpler form, using the transformations (3.13),

$$(\tilde{\mathcal{L}}_0(\chi) - \tilde{\lambda})w = w_{\chi\chi} + [3w_h^2(\chi) - (1+\tilde{\lambda})]w = w_h(\chi). \tag{3.23}$$

The solution $a_{in}(\xi)$ of (3.19) corresponds to the (uniquely determined bounded) solution $w = w_{in}(\chi;\tilde{\lambda})$ of (3.23)—see Section 3.3, where we will determine $w_{in}(\chi;\tilde{\lambda})$ explicitly. The inhomogeneous problem cannot be solved at an eigenvalue λ_j^f of $\tilde{\mathcal{L}}_0$, unless the inhomogeneous term satisfies a solvability condition,

$$\int_{-\infty}^{\infty} w_h(\chi) w_j^f(\chi)\, d\chi = 0. \tag{3.24}$$

Because $w_h(\chi)$ is even as a function of χ, and $w_1^f(\chi) = \dot{w}_h(\chi)$ is odd, we conclude that $t_2(\lambda)$ does not have a pole at $\lambda = 0$, and thus that $\mathcal{D}(0, c, \varepsilon) \equiv 0$ (since $t_1(0, c, \varepsilon) \equiv 0$, Lemma 3.4). On the other hand, the solvability condition (3.24) can clearly not be satisfied for $\lambda = \lambda_0^f$—recall that both $w_0^f(\chi)$ and $w_h(\chi)$ are positive. Thus, $t_2(\lambda)$ has a pole at $\lambda_0(\varepsilon) = \lambda_0^f$, at leading order, that cancels the zero of $t_1(\lambda)$—see once again [8], [7] for more details. We conclude that $\mathcal{D}(\lambda)$ does not have a zero at $\lambda_0(\varepsilon)$.

In other words, the coupling of the slow B-equation to the Ginzburg-Landau A-equation has a significant impact on the stability of the traveling pulse $(A_h(\xi), B_h(\xi))$, since the $\mathcal{O}(1)$ unstable eigenvalue λ_0^f of the stability problem associated with the fast reduced limit problem, i.e., the uncoupled Ginzburg-Landau equation (1.3), is canceled by the interactions between the slow and the fast effects. Thus, the full system (1.4) indeed is a singular perturbation of the reduced system (1.3), since the $\mathcal{O}(1)$ effect on the stability of the pulse is induced by the $\mathcal{O}(\varepsilon)$ terms in (1.4). Furthermore, we may conclude that the stability of the pulse is determined by the zeroes of the slow transmission function $t_2(\lambda, c, \varepsilon)$.

Lemma 3.6. *Let $\lambda \in \mathcal{C}_e^+$, $\lambda \neq 0$. Then λ is a zero of $\mathcal{D}(\lambda, c, \varepsilon)$ if and only if $t_2(\lambda, c, \varepsilon) = 0$.*

Note that this result also implies that the eigenvalue problem (3.3) has a double eigenvalue at $\lambda = 0$ if $t_2(0) = 0$. We will see in the next section that a zero of $t_2(\lambda)$ is directly related to a saddle-node bifurcation of homoclinic orbits in the existence problem (Theorem 2.2).

3.3. A Reduction to Hypergeometric Functions

Lemma 3.6 establishes that the stability of the pulse $(A_h(\xi), B_h(\xi))$ is determined by the zeroes of the explicit expression (3.22). At this point, however, (3.22) is not fully explicit, since we do not have an expression for the entry $a_{in}(\xi; \lambda)$, the bounded solution of (3.19). In this section, we will derive a fully explicit expression for $t_2(\lambda, c, 0)$.

First, we use (3.13) to express the leading order approximation $t_2(\lambda, c, 0)$ of $t_2(\lambda, c, \varepsilon)$ (3.22) in terms of $w_h(\chi)$ (3.14), $w_{in}(\chi; \tilde{\lambda})$ (3.23), and $\tilde{\lambda}$,

$$\begin{aligned} t_2(\tilde{\lambda}, c, 0) &= 1 + \frac{2\mu}{\sqrt{\sigma^2 + 4(\tau\tilde{\lambda}(1-\mu b_0) + \alpha)}} \\ &\quad \times \left[\frac{\nu + \beta b_0}{\sqrt{1-\mu b_0}} \int_{-\infty}^{\infty} w_h w_{in} d\chi - \frac{\beta}{2\mu}\sqrt{1-\mu b_0} \int_{-\infty}^{\infty} w_h^2 d\chi \right]. \end{aligned} \tag{3.25}$$

Since $\int_{-\infty}^{\infty} w_h^2 d\chi = 4$ (3.14), we only need to find an expression for

$$R(\tilde{\lambda}) = \int_{-\infty}^{\infty} w_h(\chi) w_{in}(\chi; \tilde{\lambda}) d\chi. \tag{3.26}$$

We define $P \in \mathbb{C}$ and $F(\chi)$ by

$$P = +\sqrt{1 + \tilde{\lambda}}, \qquad w(\chi) = F(\chi)(w_h(\chi))^P, \tag{3.27}$$

so that the homogeneous equation (3.15) associated with (3.23) transforms into

$$\ddot{F} + 2P\frac{\dot{w}_h}{w_h}\dot{F} - \frac{1}{2}(P+3)(P-2)w_h^2 F = 0.$$

We introduce the new independent variable z by

$$z = \frac{1}{2}\left(1 - \frac{\dot{w}_h(\chi)}{w_h(\chi)}\right), \tag{3.28}$$

so that

$$1 - 2z = \frac{\dot{w}_h}{w_h}, \qquad w_h^2 = 8z(1-z), \qquad \frac{d}{d\chi} = 2z(1-z)\frac{d}{dz},$$

and obtain a hypergeometric differential equation for $F(z)$,

$$z(1-z)F'' + (1-2z)(P+1)F' - (P+3)(P-2)F = 0. \tag{3.29}$$

Note that this equation is symmetric with respect to $z \to 1-z$ (which corresponds through (3.28) with the reversibility symmetry of (3.15)). Standard theory (see, for instance, [27]) yields two independent solutions, the hypergeometric functions

$$F_1(z) = F(P+3, P-2, P+1, z), \qquad F_2(z) = z^{-P}F(3, -2, 1-P, z), \tag{3.30}$$

where

$$F(3, -2, 1-P, z) = \frac{(1-P)(2-P) - 6(2-P)z + 12z^2}{(1-P)(2-P)}. \tag{3.31}$$

The function $F_1(1-z)$ is, by the symmetry, also a solution of (3.29), and can therefore be expressed in terms of $F_1(z)$ and $F_2(z)$. It follows (see [27], [8]) that

$$F_1(1-z) = M(P)z^{-P}F(3, -2, 1-P, z), \tag{3.32}$$

with

$$M(P) = \frac{\Gamma(P+1)\Gamma(P)}{\Gamma(P+3)\Gamma(P-2)}. \tag{3.33}$$

Thus, by combining (3.30), (3.31), and (3.32), we see that both fundamental solutions of (3.29), are very simple hypergeometric functions: $F_1(z)$ and $F_2(z)$, are quadratic polynomials in z divided by $(1-z)^P$, respectively z^P. To solve the inhomogeneous problem (3.23), we replace $F(z)$ by $G(z)$, with

$$F(z) = 2^{-\frac{1}{2}(1+3P)}G(z), \tag{3.34}$$

in (3.27), so that (3.23) transforms into

$$z(1-z)G'' + (1-2z)(P+1)G' - (P+3)(P-2)G = [z(1-z)]^{-\frac{1}{2}(1+P)}. \tag{3.35}$$

We can solve this inhomogeneous equation by using the two independent solutions of the homogeneous equation and apply the variation of constants method; i.e., we introduce the functions $g_1(z)$ and $g_2(z)$ and set

$$G(z) = g_1(z)H_1(z) + g_2(z)H_2(z), \tag{3.36}$$

with

$$H_1(z) = F_1(z), \qquad H_2(z) = F_1(1-z), \tag{3.37}$$

(3.30), (3.32). Since equation (3.35) is also symmetric, we can set $g_2(z) = g_1(1-z)$. We obtain

$$g_1(z) = \frac{1}{PM(P)}\left(\int_0^z [\zeta(1-\zeta)]^{\frac{1}{2}(P-1)} H_1(1-\zeta)d\zeta + g_0\right), \tag{3.38}$$

with

$$g_0 = -\int_0^1 [\zeta(1-\zeta)]^{\frac{1}{2}(P-1)} H_1(1-\zeta)d\zeta \tag{3.39}$$

(see [7] for more details). We can now return to $R(\tilde{\lambda})$ (3.26), or equivalently $R(P)$. It follows by (3.27), (3.28), (3.34), (3.36), (3.38), and (3.39) that

$$\begin{aligned} R(P) &= \int_0^1 G(z,P)[z(1-z)]^{\frac{1}{2}(P-1)}dz \\ &= 2\int_0^1 g_1(z)H_1(z)[z(1-z)]^{\frac{1}{2}(P-1)}dz \\ &= -\frac{2}{PM(P)}\int_0^1\int_z^1 [\zeta(1-\zeta)]^{\frac{1}{2}(P-1)}H_1(1-\zeta)d\zeta \\ &\qquad \times H_1(z)[z(1-z)]^{\frac{1}{2}(P-1)}dz. \end{aligned} \tag{3.40}$$

Recall that $H_1(z)$, i.e., $F_1(z)$ (3.37), is a quadratic polynomial in z divided by $(1-z)^P$. With this expression for $R(P)$, we have also derived an explicit formula for $t_2(P, c, 0)$,

$$\begin{aligned} t_2(P,c,0) &= 1 + \frac{2\mu}{\sqrt{\sigma^2 + 4(\tau(1-\mu b_0)(P^2-1)+\alpha)}} \\ &\qquad \times \left(\frac{\nu + \beta b_0}{\sqrt{1-\mu b_0}}R(P) - \frac{2\beta}{\mu}\sqrt{1-\mu b_0}\right). \end{aligned} \tag{3.41}$$

Remark 3.7. The results on the eigenvalues of the fast reduced limit problems (3.12)/(3.15) formulated in Lemma 3.3 can be obtained directly from (3.30), (3.31), and (3.32). An eigenfunction $w(\chi)$ of (3.15) corresponds to a solution $F(P)$ of (3.29) that is bounded at both $z = 0$ and $z = 1$ (since $\lim_{\chi\to\pm\infty} w(\chi) = 0$). It follows from (3.31) and (3.32) that $F_2(z)$ can be made regular only at $z = 0$ for $P = 1$ and $P = 2$. Hence, by (3.27), the eigenvalues of (3.15) are given by $\tilde{\lambda} = 0$ and $\tilde{\lambda} = 3$. Note that we can likewise obtain explicit expressions for the associated eigenfunctions.

4. The Stability of the Homoclinic Patterns

In Section 2 we established the existence of pulse solutions to system (1.4). Section 3 was devoted to the linear stability analysis of these solutions. We found that their stability is determined only by the discrete spectrum, and we are able to control the eigenvalues corresponding to the fast reduced limit problem. By the NLEP approach we obtained an expression for $t_2(P(\lambda))$, and we proved that this transmission function has a pole that exactly coincides with the positive root $\lambda_0(\varepsilon)$ of $t_1(\lambda, c, \varepsilon)$. Hence the equation $t_2(P(\lambda)) = 0$ gives the eigenvalues of the stability problem except for the trivial eigenvalue $\lambda_1 = 0$, which is a zero of $t_1(\lambda)$. Since t_2 depends on $R(P)$, the understanding of $R(P)$ is crucial for the stability analysis. In the following (sub)section we state some analytic results on $R(P)$. Furthermore we used *Mathematica* to evaluate $R(P)$ and solve $t_2(P(\lambda)) = 0$, for various values of the parameters. The result for $\beta = 0$ is stated in Section 4.2, and those for $\beta \neq 0$ are given in Section 4.3.

4.1. Some Asymptotic Results

As stated above, the understanding of $R(P)$ is crucial for analyzing the behavior of the eigenvalues. The following Lemma gives some analytical results on R.

Lemma 4.1. *Let $R(P)$ be as given in (3.26)/(3.40), then*
(i) $R(P = 1) = 1$;
(ii) $R(P)$ is given by $R(P) = \frac{R_2}{P-2} + \mathcal{O}(1)$ for some constant $R_2 \neq 0$ near $P = 2$;
(iii) $R(P) = -\frac{4}{P^2} + \mathcal{O}(\frac{1}{P^4})$ for $P \gg 1$.

Proof.
(i) A particular solution of (3.35) is given by

$$G_p(z, P = 1) = \frac{1}{6z(1-z)}. \tag{4.1}$$

However, this solution is singular in $z = 0$ and $z = 1$, so that it cannot correspond to $w_{in}(\chi)$. To determine the solution $G_b(z, P = 1)$ that corresponds to the bounded solution $w_{in}(\chi)$, we use the fact that general solutions of (3.35) are of the form

$$G(z, 1) = c_1 F_1(z, 1) + c_2 F_2(z, 1) + G_p(z, 1).$$

Here $F_1(z, P = 1)$ and $F_2(z, P = 1)$ are two independent solutions of the corresponding homogeneous equation, and c_1 and c_2 are constants. It is easy to check that $F_1(z, 1) = 1 - 2z$ (3.30) is a bounded solution of the homogeneous equation. Thus F_1 alone cannot cancel the singularities in G_p. The second, independent solution F_2 (3.30) we already found is not well-defined for $P = 1$, i.e., $\tilde{\lambda} = 0$. An alternative independent solution can be obtained by applying variation of constants

$$F_2(z, 1) = F_1(z, 1) \int^z u(x)dx$$

to (3.29). This yields

$$\frac{u'}{u} = \frac{-2[6z^2 - 6z + 1]}{(1-2z)z(1-z)},$$

and thus

$$u(z) = \frac{c}{(1-2z)^2 z^2 (1-z)^2},$$

with c a constant. This determines a second independent solution $F_2(z, 1)$:

$$F_2(z, 1) = (1 - 2z)\left[\tilde{c}_1\left(6\ln z - \frac{1}{z} - 6\ln(1-z) + \frac{1}{1-z} + \frac{8}{1-2z}\right) + \tilde{c}_2\right], \quad (4.2)$$

with constants $\tilde{c}_1$ and $\tilde{c}_2$. Note that F_2 is singular in both $z = 0$ and $z = 1$. With the choice $\tilde{c}_1 = 1/6$, we obtain a solution of the inhomogeneous equation for $P = 1$ with only logarithmic singularities:

$$\begin{aligned} G_b(z, 1) &= \tilde{c}F_1(z, 1) + 1/6F_2(z, 1) + G_p(z, 1) \\ &= C(1 - 2z) + (1 - 2z)[\ln z - \ln(1 - z)] + 2, \end{aligned}$$

for constants $\tilde{c}$ and C. $G_b(z, 1)$ corresponds to the bounded solutions

$$w_{in}(\chi, \tilde{\lambda} = 0) = C\dot{w}_h + \frac{1}{4}\dot{w}_h \ln\left(\frac{w_h - \dot{w}_h}{w_h + \dot{w}_h}\right) + \frac{1}{2}w_h. \quad (4.3)$$

Substituting this into (3.26), we find $R(P = 1) = 1$.

(ii) This observation follows immediately from the fact that the operator $\tilde{\mathcal{L}}_0$—see (3.15) and (3.23)—has a simple eigenvalue at $\tilde{\lambda} = 3$, i.e., $P = 2$.

(iii) The leading order behavior of $R(P)$ for $P^2 \gg 1$ can be obtained from (3.23) and (3.26). We decompose $w_{in}(\chi; P^2)$ into $w_{in}(\xi; P) = \frac{1}{P^2}w_1(\chi) + \frac{1}{P^4}w_r(\chi; P)$ and substitute this into (3.23). It follows that $w_1(\chi) = -w_h(\chi)$ and that $w_r(\chi; P) = \mathcal{O}(1)$ with respect to the small parameter $1/P^2$. The expansion of w_{in} can now be used to evaluate the leading order behavior of $R(P)$ by (3.26). □

The analytical results we stated for $R(P)$ in the above lemma are confirmed by Figure 7, which is obtained by evaluating (3.40) with *Mathematica*. Using $R(P = 1) = 1$ in (3.41), the leading order expression of $t_2(P, c, \varepsilon)$ is given by

$$t_2(P = 1, c, 0) = 1 + \frac{1}{\sqrt{\sigma^2 + 4\alpha}}\left(\frac{2\mu(\nu + \beta b_0)}{\sqrt{1 - \mu b_0}} - 4\beta\sqrt{1 - \mu b_0}\right). \quad (4.4)$$

Recall that $P = 1$ corresponds to the critical eigenvalue $\tilde{\lambda} = 0$.

4.2. The Case $\beta = 0$

In Theorem 2.1 we found conditions under which a pulse solution $(A_h(x), B_h(x))$ to System (1.4) exists. In Section 3 we studied the linear stability of these pulses and found that the eigenvalues of this eigenvalue problem are given as zeroes of $t_2(P(\lambda))$, except for $\lambda_1 = 0$. Note that if we plug b_0, as found in case *(i)* and case *(ii)* of Theorem 2.1, into the expression (3.41) for $t_2(P(\lambda))$, the parameters μ and ν only appear as a product, so that it is natural to consider the product $m = \mu\nu$ as one parameter. Therefore, we study the eigenvalues as functions of m.

First, we establish that eigenvalues cannot pass through 0 and that there always is a unique real unstable eigenvalue for m small enough.

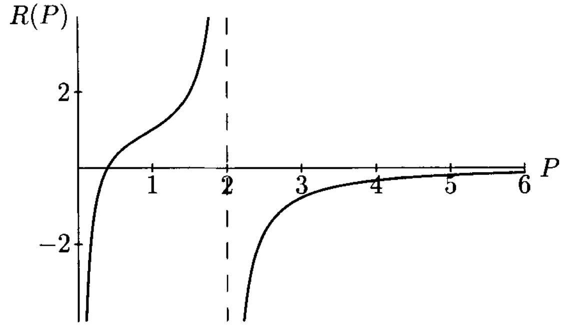

Fig. 7. The R(P)-graph.

Lemma 4.2. *For* $\beta = 0$*,* t_2 *satisfies* $t_2(\lambda = 0, c, \varepsilon) \neq 0$*.*

Proof. For $\beta = 0$ the equation $t_2 = 0$ reads

$$t_2(\lambda = 0, c, 0) = t_2(P = 1, c, 0) = 1 + \frac{2\mu\nu}{\sqrt{\sigma^2 + 4\alpha\sqrt{1 - \mu b_0}}} = 0,$$

with b_0 given by cases *(i)* and *(ii)* of Theorem 2.1. Inspection of both cases shows that no (allowed) parameter combinations exist for which $t_2(\lambda = 0, c, 0) = 0$ has a solution. □

Lemma 4.3. *Let* $\beta = 0$*. There exists a* $m_u > 0$ *such that the eigenvalue problem (3.2)/(3.3) has exactly one real unstable eigenvalue* $\lambda_u(m, \varepsilon) > 0$ *for all m with* $|m| < m_u$*. Moreover,* $\lim_{m \to 0} \lambda_u(m, \varepsilon) = 3$ *(at leading order in* ε*).*

Proof. A similar result has been proved in [7] by a topological winding number argument (Theorem 5.1). Here, we use the explicit expression (3.41) for $t_2(P)$ in combination with Lemma 4.1 *(ii)*. It follows from (3.41) that $t_2(P) = 1 + C(P)mR(P)$ (at leading order in ε)—recall that m = $\mu\nu$ and that $\beta = 0$—where $C(P)$ is a smooth function of P (for $P > 1$, i.e., $\lambda > 0$ (3.13), (3.27)). Since $R(P)$ has a singularity near $P = 2$ (Lemma 4.1 *(ii)*), we conclude that $t_2(P)$ must change sign at $P = P_u(m, \varepsilon)$ near 2 if $|m|$ is small enough, and that $\lim_{m \to 0} P_u(m, \varepsilon) = 2 + \mathcal{O}(\sqrt{\varepsilon})$. Hence, the eigenvalue $\lambda_u(m, \varepsilon)$ that corresponds to $P_u(m, \varepsilon)$ exists. Since $R(P)$ only has one singularity in the right half plane—recall that a singularity of $R(P)$ corresponds to eigenvalues of the fast reduced stability problem (3.15)—and since $R(P)$ is bounded for $P \gg 1$ (Lemma 4.1 *(iii)*), it follows that $\lambda_u(m, \varepsilon)$ is the unique real unstable eigenvalue. By Lemma 2.4 we know that $\mu b_0 \to 0$ as $m \to 0$, and thus we conclude by (3.13) and (3.27) that $\lim_{m \to 0} \lambda_u(m, \varepsilon) = 3 + \mathcal{O}(\sqrt{\varepsilon})$. □

The latter result shows that the stability problem behaves like a regular perturbed system as m approaches 0, and that the full problem has an eigenvalue that is asymptotically close to that of the reduced system. In other words, the unstable pulse of the uncoupled Ginzburg-Landau equation (1.3) cannot be stabilized by the coupling to the slow field for

$|m| \ll 1$. However, evaluation of (3.41) by *Mathematica* shows that the zero $\lambda_u(m, \varepsilon)$ of $t_2(\lambda)$ moves away from that of the singular limit systems as $|m|$ increases (see below and Figure 8), so that we may indeed conclude that the coupling to the slow diffusion equation in (1.4) is singular, since it has an $\mathcal{O}(1)$ effect on the eigenvalue $\lambda_u(m, \varepsilon)$. Thus, the "control mechanism" of coupling to slow diffusion (see the Introduction) might indeed be strong enough to stabilize the pulse.

Nevertheless, this is not the case. This is in essence due to Lemma 4.2, which does not allow $\lambda_u(m, \varepsilon)$ to move to the stable half plane by passing through $\lambda = 0$. The pulse $(A_h(\xi), B_h(\xi))$ can, however, a priori also be stabilized by a Hopf bifurcation. This is the route by which similar pulses in the Gray-Scott and the Gierer-Meinhardt equations are stabilized [8], [7]. In this scenario, $\lambda_u(m, \varepsilon)$ must merge with a second real unstable eigenvalue into a pair of complex conjugate eigenvalues. This pair can then move into the stable half plane through a Hopf bifurcation. The question is, of course, whether such a second unstable eigenvalue exists.

In general there can indeed be a second (real) eigenvalue, $\lambda_e(m, \varepsilon) < 0$. Varying m, this eigenvalue can appear from, or disappear into, the essential spectrum (that is in the left half plane) by an edge bifurcation (Figure 8). Edge bifurcations are studied systematically in the context of perturbed integrable systems in [22], [23] and can be analyzed in singularly perturbed reaction-diffusion problems by the decomposition of the Evans function in combination with the NLEP approach as sketched in Section 3 (see [9] for more details). However, $\lambda_e(m, \varepsilon)$ cannot cross to the positive half plane (Lemma 4.2), and so it cannot merge with $\lambda_u(m, \varepsilon)$. In principle, there could be a second edge bifurcation in which another eigenvalue $\lambda_{e,2}(m, \varepsilon)$ is created. The two eigenvalues $\lambda_{e,2}(m, \varepsilon)$ and $\lambda_e(m, \varepsilon)$ could form a pair of complex conjugated eigenvalues and leave the real axis. Such a pair could cross the imaginary axis. However, a scenario that involves Hopf bifurcations cannot change the parity of the number of eigenvalues in the right half plane. Hence, if there is an odd number of unstable eigenvalues for a certain value of m, the pulse cannot be stabilized through a mechanism that is based on eigenvalues that appear from the essential spectrum.

The following Theorem establishes that the $(A_h(\xi), B_h(\xi))$ pulse can never be stabilized if we do not consider higher order terms in the equation for B (in (1.4)).

Theorem 4.4. *Let $\beta = 0$, $\alpha > 0$, $\tau > 0$, $\mu \neq 0$, $\nu, \sigma \in \mathbb{R}$, and let $\varepsilon > 0$ be sufficiently small. The traveling pulse solution $(A_h(\xi), B_h(\xi))$ of (1.4)—as given by Theorem 2.1—is unstable.*

Proof. The proof is completely based on the fact that we have an explicit formula for $t_2(\lambda)$ (3.41). This expression has already been used to derive Lemmas 4.2 and 4.3. Thus, we know that there is exactly one unstable real eigenvalue in the eigenvalue problem associated with the stability of the pulse $(A_h(\xi), B_h(\xi))$ for $|m|$ small enough; i.e., for $|m| = |\mu\nu|$ small enough, $(A_h(\xi), B_h(\xi))$ is indeed unstable.

As m moves away from 0, we also know that eigenvalues cannot cross through $\lambda = 0$ (Lemma 4.2). Therefore, the only way for the real eigenvalue $\lambda_u(m)$ to gain stability as m is varied is via the Hopf bifurcation scenario of the Gray-Scott and Gierer-Meinhardt pulses [8], [7], i.e. to make $\lambda_u(m)$ merge with another real eigenvalue and to let the resulting pair of complex conjugate eigenvalues cross the imaginary axis. Lemma 4.2

also implies that we do not have to pay attention to eigenvalues that enter the unstable half plane from the stable half plane as $|m|$ increases (eigenvalues that may be created by edge bifurcations). These eigenvalues can only come in pairs and can thus only remove an even number of unstable eigenvalues from the unstable half plane (by going back again). For the same reason we do not have to pay attention to the possible existence of unstable eigenvalues with nonzero imaginary parts: These eigenvalues also come in pairs and can only bring an even number of eigenvalues to the stable half plane as m is varied (note that Lemma 4.3 does not exclude the possibility of having unstable eigenvalues with $\mathrm{Im}(\lambda) \neq 0$ for $|m|$ small).

Hence, we may restrict ourselves to studying $t_2(\lambda, \varepsilon)$ for $\lambda > 0$. We may also neglect the correction terms (in ε), so that we only need to consider $t_2(P, c, 0)$ (3.41) for $P > 1$ (3.27), (3.13). As in the proof of Lemma 4.3, we note that $t_2(P, 0)$ can be written as $1 + mC(P; m)R(P)$, where $C(P; m)$ is a smooth bounded function of P (since $P > 1$) and $R(P)$ has one singularity (at $P = 2$ – Lemma 4.1). Thus, $t_2(P)$ also has only one singularity. It is also clear from (3.41) and Lemma 4.1 *(iii)* that $t_2(P, c, 0) \to 1$ as $P \to \infty$ for m fixed. Hence, for any fixed m, eigenvalues cannot escape to ∞. This implies that for any fixed m the number of (nondegenerate) zeroes of $t_2(P) = 1 + mC(P; m)R(P)$ can only be odd, since zeroes can only be created/annihilated as m is varied through tangencies of the t_2-curve with the P-axis (which correspond to pairs of conjugate eigenvalues becoming real/being created). Hence, for finite m, the Gray-Scott/Gierer-Meinhardt Hopf bifurcation mechanism cannot stabilize the pulse.

Finally, we consider the limit $|m| \gg 1$. It follows from Lemma 2.4 *(i)* and *(ii)* that it is possible to determine explicit expressions for μb_0 as asymptotic series in $1/|m|$. These expressions can be used to compute an explicit leading order expression for $t_2(P)$ (again as asymptotic series in $1/|m|$). As a consequence, we can compute approximations for the zeroes of $t_2(P)$ for $|m| \gg 1$. We refrain from giving the computational details here (the analysis is very similar to that in the proof of Lemma 4.6). It follows that there is exactly one real unstable eigenvalue for $|m| \gg 1$ (see also Figure 8).

Thus, we conclude that the eigenvalue problem (3.2)/(3.3) has at least one unstable real eigenvalue for any m. Hence, $(A_h(\xi), B_h(\xi))$ is spectrally unstable (and unstable in the nonlinear sense, by the standard methods of [17]). □

As an example, we determine the eigenvalues of the stability problem associated with the pulse solutions of (1.4) (with $\beta = 0$) by evaluating $t_2(P(\lambda))$ and solving the equation $t_2(\lambda) = 0$ with the help of *Mathematica*. We consider only the case of standing pulses, i.e., $c = \sigma = 0$ or case *(i)* of Theorem 2.1. We find that there are three cases to consider: (a) $\alpha/\tau < 1$, (b) $\alpha/\tau = 1$, and (c) $\alpha/\tau > 1$ (see Figure 8). This distinction is caused by the character of the edge bifurcation or, in other words, the behavior of the "new" eigenvalue $\lambda_e(m, \varepsilon) < 0$.

In case (a), $\alpha/\tau < 1$, there are edge bifurcations at $m = 0$ and $m = m_{\mathrm{edge}} > 0$, with $\lambda_e(0) = \lambda_e(m_{edge}) = -\alpha/\tau$. In case (b), the degeneration $\alpha/\tau = 1$, there appears an eigenvalue $\lambda_e(m)$ with $\lambda_e(0) = -1$ at $m = 0$ as m is increased, as well as when m is decreased; in case (c), $\alpha/\tau > 1$, there are edge bifurcations at $m = 0$ and $m = m_{\mathrm{edge}} < 0$, with $\lambda_e(0) = \lambda_e(m_{\mathrm{edge}}) = -1$. In the limits $\lambda_e(m, \varepsilon)$ near σ_e and $|m| \gg 1$, a leading order approximation of $\lambda_e(m)$ can be determined explicitly (using (3.41) and the methods developed in [9]). For instance, it follows that $\lambda_e(m) \uparrow 0$ as $m \to +\infty$ (Figure 8).

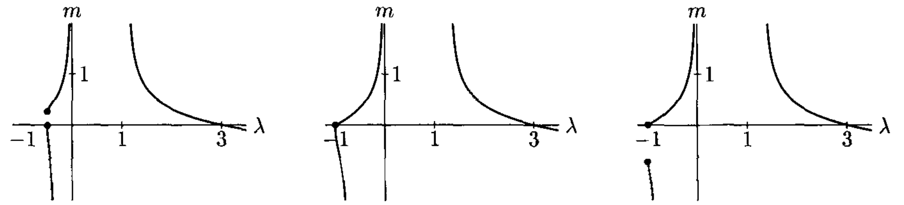

Fig. 8. The zeroes of $t_2(P(\lambda))$, corresponding to the eigenvalues $\lambda_0(m)$ and $\lambda_e(m)$ for $\beta = 0$ and $c = \sigma = 0$: (a) $\alpha/\tau < 1$, (b) $\alpha/\tau = 1$, and (c) $\alpha/\tau > 1$. In each case the left endpoints of the curves coincide with the edge of the essential spectrum.

We conclude that there can be at most three eigenvalues in the case $\beta = \sigma = 0$; they are $\lambda_u(m) > 0$, the trivial eigenvalue $\lambda_1(m) \equiv 0$, and $\lambda_e(m) < 0$; $\lambda_u(m)$ exists for all m, while $\lambda_e(m)$ may have merged with σ_e for some values of m (when $\alpha \neq \tau$).

4.3. Stability analysis for $\beta \neq 0$

Recall that in Section 2 we established the existence of open regions for which system (1.4) possesses zero, one, or two (different) pulse solutions for $\beta \neq 0$ (Theorem 2.2). This result was based on the different cases described in Lemma 2.5. In this subsection we use expression (3.41) and *Mathematica* to determine the eigenvalues of the associated stability problem for various parameter combinations. For simplicity, we consider only the case $\sigma = 0$, as in Section 2.5. We find also that for $\beta \neq 0$ many of the pulse solutions $(A_h(x;\beta), B_h(x;\beta))$ to (1.4) are unstable. However, we also show that there are parameter combinations (with $\beta \neq 0$) for which the pulse solutions can be stable.

For $\beta = 0$ we found that the only zero-eigenvalue is the trivial eigenvalue, $\lambda_1(m) \equiv 0$, that corresponds to the translation symmetry of the system (Lemma 4.2). For $\beta \neq 0$ we have again $\lambda_1(\beta) \equiv 0$ for all parameters; however, in this case there are parameter combinations for which there is an additional eigenvalue $\lambda_{sn}(\beta = \beta^*) = 0$, i.e., a solution of $t_2(P = 1) = 0$.

Lemma 4.5. *Let* $\beta \neq 0$ *and* $\sigma = 0$. *A saddle-node bifurcation of homoclinic solutions occurs exactly at a zero of* $t_2(\lambda = 0)$, *and vice versa, a second zero-eigenvalue of* $D(\lambda)$ *corresponds to a saddle-node bifurcation.*

Thus, this Lemma states that there is a one-to-one correspondence between the second zero-eigenvalue of t_2 with the saddle-node bifurcations at $\alpha = \alpha^*$ or $\beta = \beta^*$ in Lemma 2.5 and Theorem 2.2.

Proof. In Section 2.5 we found that an intersection at b_0 between l^u and T_d immediately gives rise to a homoclinic orbit. It is clear that the saddle-node bifurcation occurs when l^u and T_d are also tangent at b_0. This yields (at leading order)

$$\sqrt{\alpha} b_0 = 2(\nu + \beta b_0)\sqrt{1 - \mu b_0}, \qquad \sqrt{\alpha} = 2\beta\sqrt{1 - \mu b_0} - \frac{\mu(\nu + \beta b_0)}{\sqrt{1 - \mu b_0}}. \tag{4.5}$$

The second equation, i.e., the tangency condition, clearly is equivalent with the equation $t_2(P = 1, c = 0, 0) = 0$ given by

$$1 + \frac{1}{2\sqrt{\alpha}}\left(\frac{2\mu(\nu + \beta b_0)}{\sqrt{1 - \mu b_0}} - 4\beta\sqrt{1 - \mu b_0}\right) = 0, \tag{4.6}$$

since $R(P = 1) = 1$. □

Apart from the saddle-node bifurcation, a pulse $(A_h(x;\beta), B_h(x;\beta))$ can as a function of β also diverge as $\beta \downarrow 0$; see (2.20). The eigenvalues of the associated stability problem can be determined explicitly in this limit.

Lemma 4.6. *Let $0 < \varepsilon \ll \beta \ll 1$ and let the pulse $(A_h(x;\beta), B_h(x;\beta))$ be described by (2.20). Apart from the trivial eigenvalue $\lambda_1(\beta) \equiv 0$, the linearized stability problem has two eigenvalues, $\lambda_d(\beta) = \frac{5\alpha}{4\tau} + \mathcal{O}(\beta) > 0$ and $\lambda_u(\beta) = \frac{3\alpha}{4}\frac{1}{\beta^2} + \mathcal{O}(\frac{1}{\beta}) \gg 1$.*

In other words, we have found that the pattern $(A_h(x;\beta), B_h(x;\beta))$ with b_0 given by (2.20) for $0 < \beta$ small, has two unstable eigenvalues.

Proof. Substitution of the approximation (2.20) into (3.41) yields (for $0 < \varepsilon \ll \beta \ll 1$),

$$t_2(P, 0, 0) = 1 - \beta\frac{R(P) + 2 + \mathcal{O}(\beta)}{\sqrt{(\tau + \mathcal{O}(\beta))(P^2 - 1) + 4\beta^2}}. \tag{4.7}$$

Thus, $t_2(P) = 1 + \mathcal{O}(\beta)$, i.e., $t_2(P) \neq 0$, unless $P^2 - 1 = \mathcal{O}(\beta^2)$ or $P = 2 + \mathcal{O}(\beta)$ (see also below). The latter case is associated with the $\mathcal{O}(1)$ unstable eigenvalue λ_0^f of the fast reduced-limit system that is the origin of the pole of $R(P)$ at $P = 2$ (Figure 7). We see by Lemma 4.1 *(ii)* that $t_2(P)$ must change sign for P $\mathcal{O}(\beta)$ close to 2. Hence, there is an eigenvalue $\tilde{\lambda}_u = 3 + \mathcal{O}(\beta)$ (3.27), i.e., by (3.13) and (2.20), $\lambda_u(\beta) = \frac{3\alpha}{4}\frac{1}{\beta^2} + \mathcal{O}(\frac{1}{\beta}) \gg 1$.

The other zero of t_2 can be obtained by setting $P^2 = 1 + P_2\beta^2$, so that by Lemma 4.1 *(i)* and (4.7),

$$t_2(P_2) = 1 - \frac{3 + \mathcal{O}(\beta)}{\sqrt{\tau P_2 + 4 + \mathcal{O}(\beta)}}.$$

Hence, $t_2(P_2) = 0$ for $P_2 = \frac{5}{\tau} + \mathcal{O}(\beta)$, which corresponds by (3.27), (3.13), and (2.20) to $\lambda_d(\beta) = \frac{5\alpha}{4\tau} + \mathcal{O}(\beta)$.

Finally we note that the behavior of $t_2(P)$ is controlled by Lemma 4.1 *(iii)* for $P \gg 1$, so that also for $P \gg 1$ $t_2(P) = 1 + \mathcal{O}(\beta)$. Moreover, due to the sign of $R(P)$ as $P \downarrow 0$, the singularity $R(P)$ at $P = 0$ (Figure 7) cannot generate a (stable) eigenvalue (note that this singularity will not play a role for $\alpha < \tau$, since $P = 0$ will then be to the left of the tip of the essential spectrum; see Figure 6). Thus, we may indeed conclude that t_2 can only be 0 at $\lambda_d(\beta)$ and $\lambda_u(\beta)$. □

These two lemmas establish two mechanisms by which the number of unstable eigenvalues can become even (and > 1), so that a pulse $(A_h(\xi;\beta), B_h(\xi;\beta))$ may stabilize through the Hopf-bifurcation scenario (by first merging into pairs of complex conjugate eigenvalues). In the case of Lemma 4.6 this is obvious. In the case of the saddle-node bifurcation, Lemma 4.5, it is because at $\beta = \beta^*$ two pulses are created with initially identical spectra and two zero eigenvalues $\lambda_1 \equiv 0$ and $\lambda_{sn}(\beta^*) = 0$. As β moves away from β^*, $\lambda_{sn}(\beta)$ will become negative for one of the pulses and positive for the other. Thus, either one of the pulses will have an even number of unstable eigenvalues.

We indeed observed that a pulse may be stabilized by the Hopf scenario by varying the parameters (see Theorem 4.7 below). We did not perform a fully extensive study through the $(\alpha, \beta, \mu, \nu, \tau)$-parameter space. Instead, the observations below are based on exploring typical collections of parameter combinations. As in [8], [7], we combined evaluation of the explicit expression for $t_2(P)$ (using *Mathematica*) with the asymptotic analysis of relevant limits (for instance $|m| \ll 1$, $|\beta| \gg 1$, etc.).

We study the parameter space by dividing it into the four cases as described in Lemma 2.5; see also Figure 5. In each of these cases we choose β as main parameter.

Case 1, Lemma 2.5. There is one pulse solution, which can be seen as a continuation of the unstable pulse at $\beta = 0$. As in the $\beta = 0$ case, this pulse has one unstable eigenvalue $\lambda_u(\beta, m)$ that can be related to $\lambda_0^f > 0$, the unstable eigenvalue associated with the fast reduced-limit problem. See Figure 9(a).

Case 2, Lemma 2.5. There exist two pulse solutions for every $\beta > 0$, one with b_0 positive and the other with b_0 negative. Pulses with b_0 positive are similar to those of Case 1 and $\beta = 0$. Again, there is always one unstable eigenvalue $\lambda_u(\beta, m)$; see Figure 9(a). The pulse solutions with negative b_0 behave in ways that are essentially different. These pulses are of the type described by Lemma 4.6. Thus, for $\beta > 0$ small there are two positive real eigenvalues. If β is increased, the eigenvalues merge at some value $\beta = \beta_{c_1}$ and indeed form a pair of complex conjugate eigenvalues (with positive real part). At first the real part of these eigenvalues decays. If β is increased further, the real part will increase again and eventually the complex pair merges at some value $\beta = \beta_{c_2}$. At that point two positive real eigenvalues reappear. These real eigenvalues persist as β

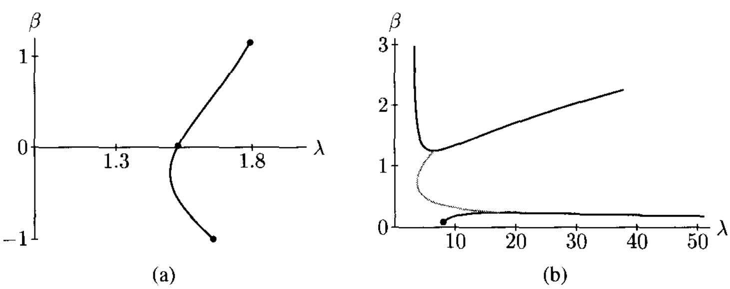

Fig. 9. The eigenvalues as function of β for pulses in Case 1 and 2 of Lemma 2.5. (a) The eigenvalues of a pulse with $\alpha = \tau = \mu = \nu = 1$ and $b_0 > 0$; for $-1 < \beta < 0$, this is Case 1, for $\beta > 0$ Case 2. (b) The eigenvalues of a pulse in Case 2, with $\alpha = 3$, $\tau = 1/2$, $\mu = 1/3$, $\nu = 1/3$, and $b_0 < 0$. For β small there are two unstable eigenvalues $\lambda_d(\beta)$ and $\lambda_u(\beta)$; see Lemma 4.6. The theoretical limit is given by $\lim_{\beta\to 0} \lambda_d(\beta) = \frac{5\alpha}{4\tau} + \mathcal{O}(\varepsilon) = 7.5 + \mathcal{O}(\varepsilon)$. The grey line is the real part of the complex conjugate eigenvalues.

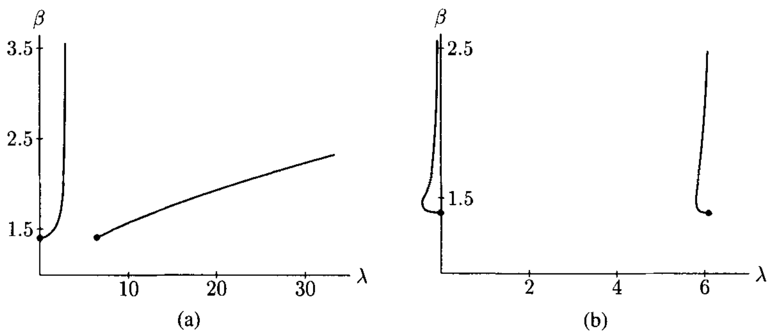

Fig. 10. The eigenvalues of two pulses of Case 3 of Lemma 2.5, both with $\alpha = 4$, $\tau = 1/2$, $\mu = 1/4$, $\nu = -1/4$, and $b_0 > 0$, for $\beta > \beta^* \approx 1.40$ in (a) the pulse with smallest $b_0 > 0$ which always has two unstable eigenvalues $\lambda_{sn}(\beta)$ and $\lambda_u(\beta)$, and in (b) the pulse with largest $b_0 > 0$, which always has one unstable eigenvalue $\lambda_u(\beta)$ and one stable eigenvalue $\lambda_{sn}(\beta)$.

is increased even further; see Figure 9(b). We have not been able to find parameters for which the complex pair of eigenvalues that is formed at $\beta = \beta_{c_1}$ crosses the imaginary axis, i.e., in which the pulse becomes stable (at least for a bounded β-interval). However, the possibility that such pulses exist cannot be excluded.

Case 3, Lemma 2.5. There are two saddle-node bifurcations, one after which two pulse solutions are created with negative b_0 and one after which two pulses are created with positive b_0. We only describe the behavior of the pulses with $b_0 > 0$, since those with $b_0 < 0$ behave as described in Case 4 below. The pulse that has the smallest value of b_0 is the one for which the eigenvalue $\lambda_{sn}(\beta)$ is positive. Moreover, for each $\beta \geq \beta^*$ there exists a second positive real eigenvalue $\lambda_u(\beta)$ (which can be related to λ_0^f). However, these eigenvalues do not merge; they both remain real and positive; see Figure 10(a). The other pulse solution that is created at the saddle-node bifurcation also has an unstable eigenvalue $\lambda_u(\beta)$, which cannot be stabilized since $\lambda_{sn}(\beta) < 0$ for $\beta \geq \beta^*$ (there are no other real positive eigenvalues); see Figure 10(b).

Case 4, Lemma 2.5. We consider $\beta > -\mu\nu > 0$ and decrease β below $-\mu\nu$—which is in fact Case 3—so that we can combine Lemmas 4.5 and 4.6. At $\beta = \beta^*$ there is a saddle-node bifurcation after which two pulses appear, both with $b_0 < 0$. The pulse with $\lambda_{sn}(\beta) < 0$ can become stable, and the pulse with $\lambda_{sn}(\beta) > 0$ will remain unstable. The pulse with $\lambda_{sn}(\beta) < 0$ is stabilized by the Hopf scenario for the two unstable eigenvalues in the limit $\beta \ll 1$. Thus, $\lambda_{sn}(\beta) < 0$ does not play a role in the stabilization process (in fact, it disappears into the essential spectrum at a certain value of β).

Theorem 4.7. *Consider (for example) $\alpha = 12.5$, $\mu = 0.5$, $\nu = -0.5$, and $\tau = 1$. The pulse $(A_h(\xi;\beta), B_h(\xi;\beta))$—that is, for $0 < \varepsilon \ll \beta \ll 1$ described by (2.20) and Lemma 4.6—exists for $\beta \in (0, \beta^*)$ and merges with another pulse solution at $\beta = \beta^*$ in a saddle-node bifurcation of homoclinic solutions of the type described in Case 4 of Lemma 2.5.*

As β increases, the two eigenvalues $\lambda_d(\beta) > 0$ and $\lambda_u(\beta) > 0$ of the associated linearized stability problem (3.2)—Lemma 4.6—merge, and become a pair of complex conjugate eigenvalues. At $\beta = \beta_{Hopf}$, this pair crosses through the imaginary axis. The pulse $(A_h(x), B_h(x))$ is asymptotically stable as a solution of (1.4) for $\beta \in (\beta_{Hopf}, \beta^*)$.

The proof of this Theorem follows immediately from the theory developed in the preceding subsections. In fact, the situation is in essence identical to that of the stability proofs of the pulses in the Gray-Scott [8] and Gierer-Meinhardt [7] systems. In all three situations there is a fully explicit control on the eigenvalues in an asymptotic limit (here, $0 < \varepsilon \ll \beta \ll 1$, Lemma 4.6). In this limit, there are two unstable eigenvalues; one, $\lambda_u(\beta)$, associated with the unstable eigenvalue λ_0^f of a singular limit, and one, $\lambda_d(\beta)$, related to a possible singularity of t_2 near $P = 1$ (in fact, the asymptotic behavior is a bit less straightforward here, due to the β-dependence of the relation between P and λ). The Hopf-bifurcation scenario followed by $\lambda_u(\beta)$ and $\lambda_d(\beta)$ as β increases can be traced by the expression (3.41) for $t_2(P; \beta)$. This way, the "fate" of the eigenvalue $\lambda_{sn}(\beta)$ can also be followed. There is an edge bifurcation at $\beta_{edge} < \beta^*$ at which $\lambda_{sn}(\beta)$ "pops" out of the essential spectrum; $\lambda_{sn}(\beta) < 0$ for all $\beta \in (\beta_{edge}, \beta^*)$.

Note that we have thus established the spectral stability of $(A_h(\xi; \beta), B_h(\xi; \beta))$ for $\beta \in (\beta_{Hopf}, \beta^*)$. The asymptotic (or nonlinear) stability follows immediately, since the linear operator associated with the full system (3.2) is clearly sectorial [17].

The exact behavior of the eigenvalues $\lambda_u(\beta)$, $\lambda_d(\beta)$, and $\lambda_{sn}(\beta)$ is shown in Figure 11. In Figure 11(a) the real part of λ has been plotted as a function of β; in Figure 11(b) the track of the eigenvalues through the complex plane is shown. Note that for the choice of parameters of Theorem 4.7, the width of the β-region of stability is relatively small. The region (β_{Hopf}, β^*) can be enlarged by varying the parameters. However, one has to be careful, since other parameter combinations can yield a different bifurcation scenario, as we show in Figure 12. In this scenario, the positive $\lambda_{sn}(\beta)$ is connected to $\lambda_u(\beta)$ or $\lambda_d(\beta)$. As in Case 2/Figure 9(b), $\lambda_u(\beta)$ and $\lambda_d(\beta)$ do not cross the imaginary axis, but instead reappear as a pair of unstable real eigenvalues (after being complex conjugate for a certain β-range).

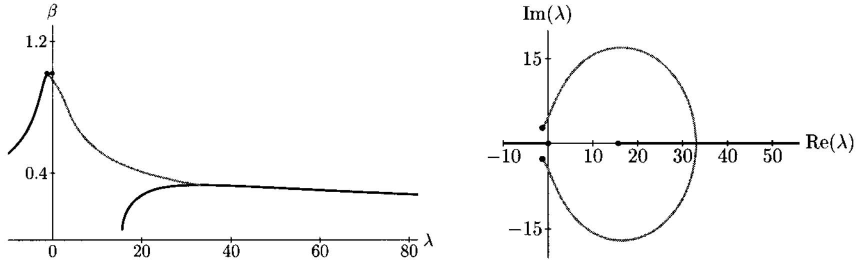

Fig. 11. The eigenvalues of the pulse solutions in Case 4 of Lemma 2.5, both with $\alpha = 12.5$, $\tau = 1$, $\mu = 1/2$, and $\nu = -1/2$: (a) Two unstable eigenvalues, $\lambda_d(\beta)$ and $\lambda_u(\beta)$, merge and become stable. In Lemma 4.6 we found the theoretical limit $\lim_{\beta\to 0} \lambda_d(\beta) = \frac{5\alpha}{4\tau} + \mathcal{O}(\varepsilon) = 15.625 + \mathcal{O}(\varepsilon)$. The black line to the left is the eigenvalue $\lambda_{sn}(\beta)$ that appears at the saddle-node bifurcation $\beta = \beta^*$. (b) The path of the eigenvalues through the complex plane as function of β.

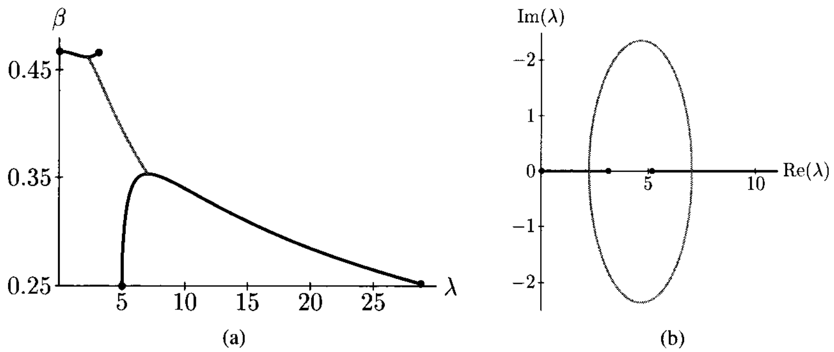

Fig. 12. The eigenvalues of a pulse in Case 4 of Lemma 2.5, with $\alpha = 4$, $\tau = 1$, $\mu = 1/2$, $\nu = -1/2$, and thus $b_0 < 0$: (a) the eigenvalues of the pulse b_0 farthest from zero; (b) the path of the eigenvalues through the complex plane as a function of β of the same pulse as in (a).

4.4. Discussion

The main goal of this paper has been to understand analytically, and in an asymptotic limit, how an additional B-diffusion equation in systems of the type (1.2)/(1.4) is able to stabilize unstable localized structures in an (uncoupled) Ginzburg-Landau equation. Such a stabilization effect has been encountered in various settings—see the Introduction. Therefore, we focused on a model system that is as simple as possible, i.e., we on purpose did not consider an explicit system from the literature. In this system, (1.4) with $\beta = 0$ (and possibly $\sigma = 0$), we deduced through an Evans function analysis that the slow B-equation indeed has an $\mathcal{O}(1)$ influence on the stability characteristics of the pulses that correspond to those of the uncoupled limit. However, due to the relative simplicity of this system (i.e., the lack of nontrivial bifurcational behavior in the existence problem), we found that the pulses nevertheless remain unstable. Adding a higher order nonlinearity, i.e., setting $\beta \neq 0$ in (1.4), introduced various mechanisms by which the pulse could be stabilized. Due to the structure of both the persistence and stability problem, we may conclude that the slow diffusion "control mechanism" can indeed explain the existence of stable pulses in equations of the type (1.2)/(1.4), if G includes a higher order nonlinearity.

Of course, the present analysis is only a first step towards understanding the stabilization of patterns in coupled Ginzburg-Landau/diffusion equations such as (1.2), i.e., in systems with interacting instability mechanisms. Here, we mention two essential next steps (which both are the topic of a work in progress).

The first is to consider a complex Ginzburg-Landau equation instead of the reduction to a real-valued amplitude A. Especially in the case of a supercritical Ginzburg-Landau equation with complex coefficients, the persistence question alone is already highly nontrivial (the associated ODE is five-dimensional). There is an extremely rich family of unstable localized solutions in this case [36], [5]; many of these could, a priori, be stabilized by slow diffusion.

A second important issue, which has only been considered very briefly in the present paper, is the fact that the B-term in equation (1.2) represents an instability that is often associated with a marginally (un)stable mode (see the Introduction). As was already noted in Remarks 1.3 and 2.3, α in (1.4) should be 0 in such systems, while it has been assumed throughout this paper that $\alpha > 0$. It has been indicated in Remark 2.3 that the persistence problem is—in essence—covered by the methods developed in Section 2, as long as one includes the higher order nonlinearity B^2 in the B-equation of (1.4). However, in order to study the stability of these pulses, one has to take into account that the rate of decay of these pulses (as $|\xi| \to \infty$) is no longer exponential if $\alpha = 0$. The exponential decay of the pulses is an essential ingredient of the construction and the decomposition of the Evans function in Section 3. Nevertheless, a method to study the stability of algebraically decaying pulses has been developed recently [34]. It is expected that the stabilization of pulses in the marginally (un)stable $\alpha = 0$ case can be studied by a combination of the ideas presented in [34] and the decomposition techniques for the Evans function presented here.

Acknowledgments

The authors acknowledge the input of T. Kapitula through stimulating discussions and thank B. Sandstede for constructive comments on an earlier version of the manuscript. This work was supported by NWO grant MEERVOUD 632000.002.

References

[1] A. Afendikov and T. Bridges. Instability of the Hocking-Stewartson pulse and its implications for three-dimensional Poiseuille flow. *Proc. Roy. Soc. Lond. A*, **457**:257–272, 2001.

[2] J. Alexander, R. Gardner, and C. Jones. A topological invariant arising in the stability analysis of travelling waves. *J. Reine Angew. Math.*, **410**:167–212, 1990.

[3] P. Coullet and S. Fauve. Propagative phase dynamics for systems with Galilean invariance. *Phys. Rev. Lett.*, **55**:2857–2859, 1985.

[4] G. Dewel, S. Métens, M. F. Hilali, P. Borckmans, and C. B. Price. Resonant patterns through coupling with a zero mode. *Phys. Rev. Lett.*, **74**:4647–4650, 1995.

[5] A. Doelman. Breaking the hidden symmetry in the Ginzburg-Landau equation. *Physica D*, **97**(4):398–428, 1996.

[6] A. Doelman, R. A. Gardner, and C. K. R. T. Jones. Instability of quasiperiodic solutions of the Ginzburg-Landau equation. *Proc. Roy. Soc. Edinburgh Sect. A*, **125**(3):501–517, 1995.

[7] A. Doelman, R. A. Gardner, and T. J. Kaper. Large stable pulse solutions in reaction-diffusion equations. *Indiana Univ. Math. J.*, **50**(1):443–507, 2001.

[8] A. Doelman, R. A. Gardner, and T. J. Kaper. A stability index analysis of 1-D patterns of the Gray-Scott model. *Mem. Am. Math. Soc.*, **155**(737):xii; 64, 2002.

[9] A. Doelman, D. Iron, and Y. Nishiura. Destabilization of fronts in a class of bi-stable systems. To appear in *SIAM J. Math. Anal.*, 2004.

[10] A. Doelman, T. J. Kaper, and P. A. Zegeling. Pattern formation in the one-dimensional Gray-Scott model. *Nonlinearity*, **10**(2):523–563, 1997.

[11] A. Doelman and V. Rottschäfer. Singularly perturbed and nonlocal modulation equations for systems with interacting instability mechanisms. *J. Nonlinear Sci.*, **7**(4):371–409, 1997.

[12] W. Eckhaus. *Asymptotic analysis of singular perturbations*, vol. 9 of *Studies in Mathematics and Its Applications*. North-Holland Publishing Co., Amsterdam, 1979.

[13] N. Fenichel. Geometric singular perturbation theory for ordinary differential equations. *J. Diff. Eq.*, **31**(1):53–98, 1979.

[14] T. Gallay. Periodic patterns and traveling fronts for the Ginzburg-Landau equation. In *Proceedings of the IUTAM/ISIMM Symposium on Structure and Dynamics of Nonlinear Waves in Fluids (Hannover, 1994)*, vol. 7 of *Adv. Ser. Nonlinear Dynam.*, pp. 230–238, World Scientific Publishing Co., River Edge, NJ, 1995.

[15] R. Gardner and C. K. R. T. Jones. Stability of travelling wave solutions of diffusive predator-prey systems. *Trans. Am. Math. Soc.*, **327**(2):465–524, 1991.

[16] G. Hek. Fronts and pulses in a class of reaction-diffusion equations: A geometric singular perturbation approach. *Nonlinearity*, **14**(1):35–72, 2001.

[17] D. Henry. *Geometric theory of semilinear parabolic equations*, vol. 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1981.

[18] L. M. Hocking and K. Stewartson. On the nonlinear response of a marginally unstable plane parallel flow to a two-dimensional disturbace. *Proc. Roy. Soc. Lond. A*, **326**:289–313, 1972.

[19] C. K. R. T. Jones. Stability of the travelling wave solution of the FitzHugh-Nagumo system. *Trans. Am. Math. Soc.*, **286**(2):431–469, 1984.

[20] C. K. R. T. Jones. Geometric singular perturbation theory. In *Dynamical systems (Montecatini Terme, 1994)*, vol. 1609 of *Lecture Notes in Math.*, pp. 44–118. Springer, Berlin, 1995.

[21] T. Kapitula. Stability criterion for bright solitary waves of the perturbed cubic-quintic Schrödinger equation. *Physica D*, **116**(1–2):95–120, 1998.

[22] T. Kapitula and B. Sandstede. Stability of bright solitary-wave solutions to perturbed nonlinear Schrödinger equations. *Physica D*, **124**:58–103, 1998.

[23] T. Kapitula and B. Sandstede. Edge bifurcations for near integrable systems via Evans function techniques. *SIAM J. Math. Anal.*, **33**:1117–1143, 2002.

[24] N. L. Komarova and A. C. Newell. Nonlinear dynamics of sand banks and sand waves. *J. Fluid Mech.*, **415**:285–321, 2000.

[25] P. C. Matthews and S. M. Cox. Pattern formation with a conservation law. *Nonlinearity*, **13**(4):1293–1320, 2000.

[26] A. Mielke. The Ginzburg-Landau equation in its role as a modulation equation. In *Handbook of dynamical systems,* Vol. 2, pp. 759–834. North-Holland, Amsterdam, 2002.

[27] P. M. Morse and H. Feshbach. *Methods of theoretical physics.* 2 vol. McGraw-Hill Book Co., Inc., New York, 1953.

[28] J. Norbury, J. Wei, and M. Winter. Existence and stability of singular patterns in a Ginzburg-Landau equation coupled with a mean field. *Nonlinearity*, **15**(6):2077–2096, 2002.

[29] H. Riecke. Self-trapping of traveling-wave pulses in binary mixture convection. *Phys. Rev. Lett.*, **68**:301–304, 1992.

[30] H. Riecke. Solitary waves under the influence of a long-wave mode. *Physica D*, **92**(1-2):69–94, 1996.

[31] H. Riecke. Localized structures in pattern-forming systems. In *Pattern formation in continuous and coupled systems* (Minneapolis, MN, 1998), vol. 115 of *IMA Vol. Math. Appl.*, pp. 215–229. Springer, New York, 1999.

[32] C. Robinson. Sustained resonance for a nonlinear system with slowly varying coefficients. *SIAM J. Math. Anal.*, **14**(5):847–860, 1983.

[33] V. Rottschäfer and A. Doelman. On the transition from the Ginzburg-Landau equation to the extended Fisher-Kolmogorov equation. *Physica D*, **118**(3–4):261–292, 1998.

[34] B. Sandstede and A. Scheel. Evans function and blow-up methods in critical eigenvalue problems. To appear in *Discr. Cont. Dyn. Sys.*, 2004.

[35] L. Tsimring and I. Aranson. Localised and cellular patterns in a vibrated granular layer. *Phys. Rev. Lett.*, **79**:213–216, 1997.

[36] W. van Saarloos and P. C. Hohenberg. Fronts, pulses, sources and sinks in generalized complex Ginzburg-Landau equations. *Physica D*, **56**(4):303–367, 1992.