

# Conjectures on Symmetric Queues in Heavy Traffic

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## 1 Introduction

We consider the queue-length process in the  $M/G/1$  queue with symmetric service discipline, which is defined as follows: with  $n$  customers in the system, the server works on the customer in position  $i$  at rate  $\gamma(n, i) \geq 0$ . We assume  $\sum_i \gamma(n, i) = 1$  for all  $n \geq 1$ , to make the service discipline work conserving. The customer arrival process is Poisson with rate  $\lambda$ . A customer arriving to a queue of size  $n$  chooses position  $i$  with probability  $\gamma(n+1, i)$ , moving each customer in position  $k \geq i$  to position  $k+1$ . Conversely, when a customer at position  $i$  departs, each customer in position  $k > i$  moves to position  $k-1$ . We refer to this system as a symmetric queue governed by  $\gamma$ . Important special cases of symmetric service disciplines are Last-Come-First-Served (LCFS), where  $\gamma(n, 1) = 1$  for all  $n \geq 1$ , and Processor Sharing (PS), which is modeled by taking  $\gamma(n, i) = 1/n$  for all  $1 \leq i \leq n$  and all  $n \geq 1$ . We refer to [4] for more background.

Symmetric service disciplines have the appealing property that the stationary distribution of the queue-length process  $(Q^\gamma(t), t \geq 0)$ , if it exists, is *insensitive* to the service-time distribution, apart from its mean  $m$ . In particular if the load  $\rho = \lambda m < 1$ , then, as  $t \rightarrow \infty$ ,  $Q^\gamma(t)$  converges weakly to a random variable  $Q^\gamma(\infty)$  which satisfies

$$P(Q^\gamma(\infty) \geq k) = \rho^k, k \geq 0. \quad (1)$$

This note states two open problems related to the queue-length *process* in heavy traffic.

## 2 Problem Statement

Assume first that the second moment of the service-time distribution is finite. Let  $r$  be a scaling parameter and let  $\lambda_r$  be a sequence of arrival rates such that  $\lambda_r \rightarrow 1/m$  and  $r(1 - \rho_r) = r(1 - \lambda_r m)$  converges to a real-valued constant. Let  $Q_r^\gamma(t), t \geq 0$  be the corresponding queue-length process in the symmetric queue governed by  $\gamma$  with

Poisson arrival rate  $\lambda_r$ . For each  $r$ , define the re-scaled queue-length process  $\hat{Q}_r^\gamma(t) = Q_r^\gamma(r^2 t)/r, t \geq 0$ .

**Problem 1:** show that  $\hat{Q}_r^\gamma(\cdot)$  converges in distribution to a reflected Brownian motion (RBM) as  $r \rightarrow \infty$ .

Our second problem relates to the case where the variance of the service-time distribution is *infinite*. To this end, let  $L$  be a slowly varying function (i.e.  $L(ax)/L(x) \rightarrow 1$  as  $x \rightarrow \infty$  for every  $a > 0$ ) and suppose that the tail  $\bar{F}$  of the service-time distribution satisfies  $\bar{F}(x) = L(x)x^{-\alpha}$  for  $\alpha \in (1, 2)$ . Let  $c_r$  be the solution of  $c_r \bar{F}(c_r) = 1/r$ . Now, for each  $r$ , define  $\hat{Q}_r^\gamma(t) = Q_r^\gamma(c_r t)/r, t \geq 0$ .

**Problem 2:** identify a candidate limit process  $Q_*^\gamma(\cdot)$  and show that  $\hat{Q}_r^\gamma(\cdot) \rightarrow Q_*^\gamma(\cdot)$  in distribution as  $r \rightarrow \infty$ .

### 3 Discussion

Problem 1 has been solved for PS in [3], [10] for general (with finite second moment) inter-arrival time distribution and a finite fourth moment assumption on the service-time distribution. The fourth moment assumption was dropped in [7] using a different method, which required Poisson arrivals. For LCFS, Problem 1 has been solved in [9] for Poisson arrivals and extended to renewal arrivals in [8].

The drift and the variance of the RBM for PS and LCFS are identical, which is no coincidence, because it is shown in [2] that the distribution of  $Q_r^\gamma(t)$  (for fixed  $t$ ) is independent of  $\gamma$ , if  $Q_r^\gamma(0) = 0$ . Given this, it is natural to conjecture that the scaling limit in Problem 1 should be independent of  $\gamma$ . It is remarkable though that this independence of  $\gamma$  may break down if the Poisson arrival assumption is relaxed. For the LCFS case, this is illustrated in [8].

To solve Problem 1, one can think of several strategies. The approach in [3], [10] is to use a detailed measure-valued description of the queueing dynamics, and hinges on the physical intuition that, in heavy traffic, the queue length becomes a deterministic function of the workload. This property, known as state-space collapse, should still hold for general  $\gamma$ . The approaches in [7] and [9] are based on connections with branching processes and excursion theory. To make such an approach work, one would first have to identify an appropriate relation between a Crump-Mode-Jagers (CMJ) branching process (see [7] and references therein for background on CMJ processes) and the service discipline  $\gamma$ . A more pragmatic way is to first prove Problem 1 for the special case of phase-type service-time distributions.

Problem 2 has been haunting me for the past 20 years. I view Problem 2 to be much harder than Problem 1. One reason is that the limiting queue-length process can no longer be a deterministic function of the heavy traffic limit of the workload process (for the latter, see [12]). Note that the distinction between finite and infinite variance is not

needed for the invariant distribution in the case: when  $\rho_r < 1$  for all  $r$ ,  $(1 - \rho_r)Q_r^\gamma(\infty)$  converges in distribution to an exponential random variable with unit mean as  $r \rightarrow \infty$ .

It is possible to determine weak convergence of  $\hat{Q}_r^\gamma(t)$  for fixed  $t$  as  $r \rightarrow \infty$ , assuming  $Q_r^\gamma(0) = 0$ , using the result in [2]. This suggests that marginal distributions of a candidate limit process  $Q_*^\gamma(\cdot)$  should be independent of  $\gamma$ . This will not be the case for the entire process, which has been determined in [1] for LCFS and in [7] for PS. In both cases, the limiting queue-length process is expressed in terms of a functional of a sequence of continuous-state branching processes, namely the height process in the LCFS case, and a Lamperti transform in the PS case; see [1] and [7] for detailed descriptions.

For PS, convergence is still an open problem, as tightness of  $\{\hat{Q}_r^{PS}(\cdot)\}_r$  has not been established. In the case where  $\gamma$  is completely general, I have no conjecture on what the candidate limit process  $Q_*^\gamma(\cdot)$  might be.

An idea which may lead to partial results of independent interest is to try and establish a relationship between the symmetric queue governed by  $\gamma$ , and some functional of a CMJ process, which should be dependent on  $\gamma$ . As mentioned in the discussion of Problem 1, such a connection would be interesting in its own right. Once such a connection has been established, it may be possible to use the technique in [5], leveraging the fact that  $\hat{Q}_r^\gamma(\cdot)$  is a regenerative process for each  $r$ .

A sufficient condition for tightness of  $\{\hat{Q}_r^{PS}(\cdot)\}_r$  has been described in [6] which is appealing, as it is stated in terms of a tail bound for the distribution of the maximum queue length during a busy cycle. Deriving such a tail bound for arbitrary symmetric service disciplines could be an interesting problem in its own right. To develop intuition, one may first approach this by assuming phase-type service times. A related tail bound for bandwidth sharing networks has been derived in [11].

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