ENTIRE FUNCTIONS OF PALEY-WIENER TYPE IN $\mathbb{C}^n$, RADON TRANSFORMS AND PROBLEMS OF HOLOMORPHIC EXTENSION

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INTRODUCTION

This thesis consists of five articles:

[A] Growth properties of Paley-Wiener functions on $\mathbb{C}^n$.
Nederl. Akad. Wetensch. Proc. A 87 , pp. 95-112 (1984).

[B] Paley-Wiener functions with prescribed indicator.
UvA Dept. of Math. Report 84-07.

[C] (with J. Korevaar) A representation of mixed derivatives with an application to the edge-of-the-wedge theorem.
Nederl. Akad. Wetensch. Proc. A 88 .... (1985).

[D] A support theorem for Radon transforms on $\mathbb{R}^n$.
Nederl. Akad. Wetensch. Proc. A 88 .... (1985).

[E] (with J. Korevaar) A lemma on mixed derivatives and a theorem on holomorphic extension. UvA Dept. of Math. Report 85-01.

Three of these papers have been published in the Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen. The latter are reproduced here with permission of the Akademie. The purpose of this introduction is to give a description of the results in these papers, to show how they are interrelated and to give some general background information as well as related results.

PALEY-WIENER FUNCTIONS

Growth properties of Paley-Wiener functions of several complex variables are the subject of papers [A] and [B]. What are Paley-Wiener functions and why would one study their growth? Paley-Wiener functions are entire functions of exponential type on $\mathbb{C}^n$ (i.e. functions $f(z)$ whose absolute value is bounded by $A \exp B \|z\|$) with the property that on the real subspace $\mathbb{R}^n$ their absolute value is bounded by a polynomial. They owe their name to the Paley-Wiener and Paley-Wiener-Schwartz theorem, cf. [PaW 34, Sc 52]. These theorems state that the entire functions of exponential type on $\mathbb{C}^n$ which are in L^2 on $\mathbb{R}^n$ (or bounded by a polynomial, respectively) are precisely the Fourier-Laplace transforms of compactly supported L^2-functions (or distributions, respectively).

The motivation for studying growth properties is that growth of holomorphic functions is closely related to their zero distribution. For holomorphic functions of one variable the simplest illustration is that for polynomials, the degree equals the number of zeros. More general relations of this kind appear through Jensen's formula and Hadamard's product theorem [Lv 80]. Precise information on the zero distribution of Fourier or Fourier-Laplace transforms is of particular interest because such knowledge can often be translated into spanning properties of subsets of function spaces, cf. the work of Beurling-Malliavin, Carleman, Korevaar, Ronkin, Zeinstra [BM 67, Ca 22, Ko 82, Ro 78, Ze 85].

INDICATORS

In the following we restrict ourselves to functions of exponential type, although similar notions and results are known for functions of arbitrary order. The maximal directional growth of a function of exponential type is measured by the indicator function

$$(*) \qquad h(z) = h_f(z) = \overline{\lim_{r\to\infty}} \frac{\log|f(rz)|}{r} \quad .$$

In the one-variable case $h(z)$ is a 1-homogeneous continuous subharmonic function, related to the classical Phragmén-Lindelöf indicator $\tau(\theta) = h(e^{i\theta})$. For a Paley-Wiener function f of one variable h_f is simply the supporting function of the support of the distribution $F^{-1}f$. In other words: $h(z) = \max\{ay, by\}$ where $[a, b]$ is the shortest interval that contains $\operatorname{supp} F^{-1}f$. Moreover, these functions have "regular growth" in the sense of Pfluger [Pf 38] and Levin [Lv 80] and also in the sense of Ahlfors-Heins [AH 49]. Both notions roughly mean that the $\overline{\lim}$ in (*) is almost a limit, i.e. there is a set A of (r , z), which is small in an appropriate sense and such that

$$\lim_{\substack{r\to\infty \\ (r,z)\notin A}} \frac{\log|f(rz)|}{r}$$

exists. In close interdependence, very much is known about the distribution of the zeros of Paley-Wiener functions of one variable. Almost all zeros lie close to the real axis and they are very regularly distributed, cf. the standard example $\sin c\, z$, and see [BM 67].

Now we turn to several variables. The indicator (*) is not sufficiently well behaved, in particular it is not a plurisubharmonic function. Therefore Lelong has introduced the regularization

$$h^*(z) = \overline{\lim_{w\to z}}\, h(w).$$

This is a 1-homogeneous plurisubharmonic function [Le 66], but it need not be continuous [Le 68]. It is bounded by $C\|\operatorname{Im} z\|$ for Paley-Wiener functions. Various types of regular growth have been defined; they imply regular distribution of the zero set, cf. [Gr 76, AR 81, GrL 85].

Paper A.

The main result of [A] is the following: A Paley-Wiener function f has regular growth if the convex hull of the support of $\mathcal{F}^{-1}f$ is a polyhedron. This observation is of interest in view of Vauthier's example of a Paley-Wiener function of irregular growth [Va 73]. The condition on the convex hull of the support is always fulfilled for Paley-Wiener functions of one variable. Similar results were also obtained by Gruman [Gr 83] and Sigurdsson [Si 84].

Ingredients of our proof are a seemingly weaker, but ultimately equivalent definition of regular growth, some results which Gruman announced in [Gr 76] and that are proved in [A], see also [GrL 85], and furthermore a slight refinement of the Plancherel-Pólya theorem [PlP 37]. The latter theorem states that the indicator f is smaller than or equal to the supporting function of the convex hull of supp $\mathcal{F}^{-1}f$, with equality on $\mathbb{S}\mathbb{R}^n$, the set of "semi-reals". The present method of proof makes it possible to obtain a bound for the size of the set where $h(z)$ and $h^*(z)$ fail to be equal. It also implies regular growth on $\mathbb{S}\mathbb{R}^n$. Finally, it is used in [D] in connection with Radon transforms.

Paper B.

Another problem related to growth properties of entire functions was solved by Kiselman and Martineau. They proved that any 1-homogeneous plurisubharmonic function u is the indicator function of some entire function of exponential type, cf. [Ki 66] and [Ma 67]. Martineau even obtained the analogous result for functions of arbitrary order. For functions of one variable this goes back to G. Pólya [Pó 29] and V. Bernstein [Be 36].

What is the situation in the case of Paley-Wiener functions? It is clear from their representation as a Fourier-Laplace transform that an extra condition of the form $|u(z)| \leq C\|\operatorname{Im} z\|$ must be imposed on u. Under this condition and assuming that u is Hölder continuous on the unit sphere, the existence of a Paley-Wiener function with indicator u could indeed be proved in [B]. Recently Sigurdsson (in his Lund thesis [Si 84]) obtained the same conclusion under different assumptions. He does not assume Hölder continuity. On the other hand he supposes that his candidate indicator function u is related to the supporting function of a convex set as in the statement of the Plancherel-Pólya theorem above. Both Sigurdsson and the author have used methods close to those of Martineau. As an application it is shown in [B] that Vauthier's example of irregular growth follows relatively easily from the present existence theorem. Martineau, Vauthier, Sigurdsson and the author all make essential use of Hörmander's L^2 - theory with weights for the $\bar{\partial}$ - problem, cf. [Hö 73].

HOLOMORPHIC EXTENSIONS

The core of the remaining papers [C,D,E] is a method to prove holomorphy or holomorphic extendability of certain functions. The central idea is that knowledge of the size of directional derivatives of a function is translated into estimates of its mixed derivatives ("Main Lemma" in [C]). These estimates then lead to converging Taylor series and thus to holomorphy properties. To put our results in perspective, let us recall two criteria for holomorphy and some theorems on holomorphic extension.

A well-known theorem of Hartogs [Ha 06] says that a function defined on an open set in $\mathbb{C}^n$ is holomorphic if (and only if) all the functions of one variable obtained by fixing (n-1)-coordinates, are holomorphic. To appreciate this result, keep in mind that a similar statement for real-analytic functions is false.

Recently Forelli [Fo 77] proved that a function defined on the unit ball, which is C^∞ at the origin and which has the property that all its slices through the origin are holomorphic as functions of one variable, is in fact holomorphic on the unit ball. The C^∞ condition cannot be disposed of, see [Ru 80].

Now as to holomorphic extendability, the first result in this direction was due to Hartogs, who shocked the mathematical world with the following result: A holomorphic function defined on a neighborhood of the boundary of a bounded domain in $\mathbb{C}^2$ (or $\mathbb{C}^n$, $n \geq 2$) extends holomorphically to the whole domain [Ha 06a]. Another classical result says that holomorphic functions, defined on a connected Reinhardt domain which contains the origin, extend to the logarithmically convex hull of that domain, e.g. [Ha 06]. In particular, Hartogs showed that a holomorphic function defined on a neighborhood of the set

$\{z : |z_j| = \lambda \cdot r_j , \lambda \in [0 , 1)\}$ extends holomorphically to the whole polydisc defined by $\{z : |z_j| < r_j\}$.

Of a somewhat different kind is the edge-of-the-wedge theorem, due to Bogoliubov [BoS 59]. The various proofs include those by F.E. Browder [Br 63], Epstein [Ep 60] and recently Bedford [Bd 74]; cf. [Ru 71]. The theorem is concerned with holomorphic functions which are defined on a multidimensional wedge, and which have appropriate boundary values when the edge of the wedge is approached. Such functions extend holomorphically not only to the reflected wedge, but also to a full neighborhood of the edge. In the one-dimensional case this is just Schwarz's reflection principle. The more surprising part of the theorem appears only in the higher dimensional case.

Paper C.

In [C] we present a straightforward proof of the edge-of-the-wedge theorem. It resembles the one by Browder. However, we employ our "Main Lemma", whereas Browder uses his more involved Hartogs-type theorem for real-analytic functions, developed in [Br 61].

Paper E.

The terminology in Forelli's article [Fo 77] strongly suggests the following question: Suppose that a function is defined on a subset of the ball consisting of some family of complex lines through the origin and that it satisfies a certain smoothness condition at the origin. If its restriction to any such complex line is a holomorphic function (of one variable), then what are the properties of this function, as far as n-dimensional holomorphy is concerned?

In [E] we treat two special cases. In the first case, where f is defined on the family consisting of those complex lines that meet the unit sphere in $\mathbb{R}^n \subset \mathbb{C}^n$, our result is that f is the restriction of a holomorphic function defined on a fairly large domain. In the second case f is defined on the family of all complex lines that meet the distinguished boundary of the unit polydisc. Here the conclusion is that f extends to the whole polydisc. The latter result refines the above mentioned theorem of Hartogs in which it was assumed that f is defined and holomorphic on a certain subdomain of the polydisc. Apart from the use of the Main Lemma, the proofs are in the spirit of Hartogs.

Paper D.

In [D] we obtain refinements of the support theorem for the Radon transformation on $\mathbb{R}^n$. The Radon transform $\hat{f}$ of a "reasonable" function f on $\mathbb{R}^n$ is defined on the set of hyperplanes in $\mathbb{R}^n$ and associates to a hyperplane H the integral $\hat{f}(H)$ of f over H. Radon transformation plays an important role in applications, cf. [He 80]. It is used to reduce partial differential equations to ordinary ones. Of truly practical importance is its relation to Röntgen Tomography. Here one considers the X-ray transform which is analogously defined, but uses lines instead of hyperplanes, and the main problem is its inversion. Integration of the X-ray transform yields the Radon transform and thus several theorems carry over from Radon- to X-ray transform, including the following support theorem:
If f is a rapidly decreasing continuous function, such that $\hat{f}(H) = 0$ for all hyperplanes H that do not meet a given bounded convex set K, then $f = 0$ outside K.

This theorem was first proved by Helgason [He 65]. Other proofs were given by Ludwig [Lw 66] and B. Weiss [We 67]. Still another proof is contained in [D]. The refinements of this theorem are twofold: First the condition "$\hat{f}(H) = 0$ for all hyperplanes that do not meet a bounded convex set" is replaced by a milder one: $\hat{f}(H)$ decreases exponentially with the distance from H to the origin and equals zero on a relatively small set of hyperplanes. Secondly it is shown that certain holes in the support of f (which is contained in K) can be detected via $\hat{f}$: If f has compact support and $\hat{f}(H) = 0$ on some open set of hyperplanes $\mathcal{H}$ then $f = 0$ on $\cup_{H \in \mathcal{H}} H$.

In the proofs we employ the Main Lemma from [C] to ascertain that the Fourier transform of f is the restriction of a Paley-Wiener function, i.e. f has compact support. Next our method for proving the Plancherel-Pólya theorem from [A] is applied once more to obtain the precise description of the support of f. Thus we see two seemingly unrelated subjects merge in a theorem belonging to a third subject.

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Growth properties of Paley-Wiener functions on $\mathbb{C}^n$

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ABSTRACT

In this paper we study growth properties of entire functions of exponential type on $\mathbb{C}^n$. We prove that a complex Fourier transform f of an L^2 function g with compact support in $\mathbb{R}^n$ has regular growth if the convex hull of the support of g is a polyhedron. The proof involves an analysis of the indicator function of f, a useful alternative description of regular growth and a lower-bounds theorem for several variables that is obtained with the aid of potential theory.

INTRODUCTION

Let f be an entire function on $\mathbb{C}$, $n(r)$ the number of zeros of f in $B(0,r)$, the ball with centre 0 and radius r, and $m(r)$ the average of log $|f|$ over the circumference $C(0,r)$. Jensen's formula reveals the close relation between the rate of growth of $m(r)$ and the speed with which $n(r)$ increases. A much more refined analysis yields that the existence of an angular density of the zeros of f is roughly equivalent to what is called regular growth of f, cf. Levin [7].

In several variables the concept of regular growth exists also and there are similar relations between regular growth of an entire function and the distribution of its zero set. Therefore one may ask for criteria for regular growth. In one variable a typical result is the following: entire functions of exponential type which are in L^2 or bounded on the real axis are of regular growth.

In this paper we will study growth properties of entire functions of exponential type on $\mathbb{C}^n$ that are in L^2 on the real subspace $\mathbb{R}^n \subset \mathbb{C}^n$. Examples on $\mathbb{C}^2$

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are

$$\frac{\sin z}{z}\cdot\frac{\sin w}{w} \text{ and } \frac{\sin^2(z^2+w^2)^{\frac{1}{2}}}{z^2+w^2}.$$

By an n-dimensional form of the Paley-Wiener theorem, such functions are Fourier transforms of L^2 functions with compact support. We will call these functions Paley-Wiener functions.

Our main result is the following:

Let $g \in L^2(\mathbb{R}^n)$ have compact support $K \subset \mathbb{R}^n$. Suppose that the convex hull of K is a polyhedron. Then

$$f(z) = \mathscr{F}g(z) = \int_{\mathbb{R}^n} e^{-i\langle z,s\rangle} g(s)ds$$

is an entire function of regular growth on $\mathbb{C}^n$.

Note that some restriction on f is necessary: an example of Vauthier [11] shows that there exist Paley-Wiener functions of irregular growth.

In section 1 we review a recent definition of regular growth and a corresponding result on growth and zero distribution. Section 2 is devoted to an analysis of the indicator function of a Paley-Wiener function. Theorem 2 is a slight refinement of the Plancherel-Pólya theorem and theorem 3 provides a precise description of the indicator function for the case where the convex hull of the support of g is polyhedral. Section 3 contains the principal results of the paper. Using potential theoretic methods we first prove a multi-dimensional "lower-bounds theorem" which may also be of independent interest. We next introduce a convenient alternate definition of regular growth which looks much weaker than the standard definition but ultimately turns out to be equivalent. The main step is in part A of theorem 5; part B of that theorem leads rather directly to the principal result, theorem 6. As an application we describe in section 4 the zero distribution of Paley-Wiener functions on $\mathbb{C}^2$ that satisfy our conditions.

1. SOME FACTS CONCERNING GROWTH

In this section we recall definitions and simple facts of regular growth and indicator functions for entire functions of exponential type.

We call a holomorphic function f on $\mathbb{C}^n$ of *exponential type* if there exist constants B, C such that

$$|f(z)| \leq Be^{C\|z\|},\ z \in \mathbb{C}^n.$$

For $z \neq 0$ the radial indicator of f is defined by:

$$h_f(z) := \overline{\lim_{r\to\infty}} \frac{\log |f(rz)|}{r}.$$

For $n=1$ this function is continuous and related to τ_f, the classical Phragmén-Lindelöf indicator function: $\tau_f(\theta) = h_f(e^{i\theta})$. For $n \geq 2$ we introduce the so-called (regularized) indicator:

$$h_f^*(z)=\overline{\lim_{w\to z}}\, h_f(w),$$

cf. Lelong [5]. It is convenient to define $h_f(0)=h_f^*(0)=0$.

When no confusion is possible, we simply write h and h^*. The indicator h^* is plurisubharmonic and positively homogeneous, i.e.

$$h^*(rz)=rh^*(z),\ r>0.$$

It need not be continuous, cf. Lelong [6]. Furthermore, except possibly for z in a very thin set, namely a set of Γ-capacity 0, $h^*(z)=h(z)$, cf. Ronkin [10]. In particular, one has $h^*(iy)=h(iy)$ for almost all unit vectors $y\in\mathbb{R}^n$.

For one variable, regular growth can be defined as follows, cf. Pfluger [8], Levin [7]: For $E\subset\mathbb{R}_{>0}$, we say that E has zero relative measure if

$$\lim_{r\to\infty}\frac{m_1(E\cap(0,r))}{r}=0.$$

Here m_1 is the Lebesgue measure on $\mathbb{R}$. By definition, f has *regular growth* in the direction $\theta\in[0,2\pi)$ if

$$\lim_{r\to\infty}{}^*\frac{\log|f(re^{i\theta})|}{r}=h(e^{i\theta})=\tau_f(\theta). \tag{1.1}$$

Here and in the sequel lim* means that the limit exists for values of r outside a suitable set of zero relative measure.

For several variables, the older definitions of regular growth were based on (1.1), cf. Gruman [1]. The following definition has the advantage of the absence of exceptional sets. Let $B(z,r)$ denote the open ball with radius r and centre z in $\mathbb{C}^n$ and $V(r)$ its volume. The Lebesgue measure on $\mathbb{C}^n$ is denoted by m. Put

$$I_f^r(z,\delta)=\frac{1}{V(\delta r)}\int_{B(rz,\delta r)}\frac{\log|f(z')|}{r}\,dm(z').$$

DEFINITION (Gruman [2]). A function f of exponential type is said to have regular growth in the direction $z\in\mathbb{C}^n\setminus\{0\}$ if for every $\varepsilon,\delta_0>0$ there exist δ with $0<\delta<\delta_0$ and $R>0$ depending on ε and δ such that for $r>R$:

$$|I_f^r(z,\delta)-h^*(z)|<\varepsilon.$$

It is said to have regular growth on a set D if it has regular growth in all directions $z\in D$ $(z\neq 0)$.

This definition leads to properties for regular growth similar to the one variable case. In particular one has, cf. [2],

PROPERTY 1. Suppose h^* is pluriharmonic in a domain D. If f has regular growth in one direction $z_0\in D$, then f has regular growth on D.

PROPERTY 2. If f grows regularly on $D\setminus F$, where D is open and F has measure zero, then f grows regularly on D.

REMARK. We will indicate proofs of these properties in section 3.

As in the one-variable case, regular growth is equivalent to regular zero distribution. This is illustrated by a version of a particular result of Gruman, cf. [2], which can be applied in our situation. We introduce some notations. Let $S=\partial B(0,1)$. For $w\in S$ put

$$T_w(\varphi)=\{z\text{: angle between the radii through } z \text{ and } w \text{ is smaller than } \varphi\}.$$

This is a cone around w. Further we set:

$$\sigma_w^r(\varphi)=\text{volume of the zero set of } f \text{ in } T_w(\varphi)\cap B(0,r)$$

$$=\frac{1}{2\pi}\int_{T_w(\varphi)\cap B(0,r)} \Delta \log |f| dm.$$

The last integral makes sense because $\log |f|$ is subharmonic, cf. section 3.1.

THEOREM 1. *Let f be an entire function of exponential type and regular growth with indicator h^*. Then the zero set of f satisfies the relation:*

$$\lim_{r\to\infty}\frac{\sigma_w^r(\varphi)}{r^{2n-1}}=\frac{1}{2\pi}\lim_{r\to\infty}\frac{1}{r^{2n-1}}\int_{T_w(\varphi)\cap B(0,r)} \Delta \log |f| dm$$

$$\stackrel{*}{=}\frac{1}{2\pi}\lim_{r\to\infty}\frac{1}{r^{2n-1}}\int_{T_w(\varphi)\cap B(0,r)} \Delta h^* dm=\frac{1}{2\pi}\int_{T_w(\varphi)\cap B(0,1)} \Delta h^* dm.$$

REMARK. The main point here is the equality $\stackrel{*}{=}$. The last equality in the theorem is an immediate consequence of the homogeneity of h^*.

As a corollary one obtains: if h^* is pluriharmonic on $T_w(\varphi_0)$, then for $\varphi<\varphi_0$

$$\lim_{r\to\infty}\frac{\sigma_w^r(\varphi)}{r^{2n-1}}=0,$$

cf. Gruman [1, 2].

2. INDICATORS OF FOURIER TRANSFORMS

We denote the support function of the convex hull of a compact set $K\subset\mathbb{R}^n$ by H_K, that is:

$$H_K(x)=\max_{t\in K}\langle x,t\rangle,\ x\in\mathbb{R}^n,$$

where, now and in the sequel, for z, w in $\mathbb{R}^n$ or $\mathbb{C}^n$:

$$\langle z,w\rangle=\sum_{j=1}^n z_j w_j.$$

Note that the support function is often defined for unit vectors x only.

We define $\mathbb{SR}^n$, the set of semi-reals in $\mathbb{C}^n$ by

$$\mathbb{SR}^n:=\{z\in\mathbb{C}^n: \exists\lambda\in\mathbb{C} \text{ and } \xi\in\mathbb{R}^n\subset\mathbb{C}^n \text{ such that } z=\lambda\cdot\xi\}.$$

Now let $g \in L^2(\mathbb{R}^n)$ have compact support K_g. We introduce

$$f(z) := \mathscr{F}g(z) = \int_{\mathbb{R}^n} e^{-i\langle z,s\rangle} g(s)ds,$$

the Fourier transform of g. The well-known Paley-Wiener theorem [10] says:

> f is an entire function of exponential type and
> $f|_{\mathbb{R}^n}$ is in L^2
> $\Leftrightarrow$
> there exists a $g \in L^2(\mathbb{R}^n)$ with compact support such that $f = \mathscr{F}g$.

Because of this theorem we will call these functions *Paley-Wiener* functions.

The following theorem, due essentially to Plancherel and Pólya [9], relates hull K_g to h_f^*.

THEOREM 2. *Let $f = \mathscr{F}g$ be a Paley-Wiener function, with radial indicator h and regularized indicator h^*. Then*

(2.1) $\quad h^*(z) \leq H_{K_g}(y)$ *where* $y = \operatorname{Im} z$,

with equality on $\mathbb{S}\mathbb{R}^n$. For $z = iy$ one has more precisely:

(2.2) $\quad h^*(iy) = H_{K_g}(y) = \overline{\lim_{\tilde{y}\to y}}\, h(i\tilde{y}),$

where the $\overline{\lim}$ is taken over values $\tilde{y} \in \mathbb{R}^n$ only.

PROOF. Since $h^*(0) = H_{K_g}(0) = 0$ we may assume $z \neq 0$. The inequality (2.1) is obvious. In the rest of the proof we shall restrict ourselves to the case $n = 2$ in order to give a more transparent exposition of the ideas.

Let us first prove (2.2). We set

$$\tilde{h}(iy) = \overline{\lim_{\tilde{y}\to y}}\, h(i\tilde{y})$$

so that in view of the homogeneity $h(iy) \leq \tilde{h}(iy) \leq h^*(iy)$.

ASSERTION. For all $y \in \mathbb{R}^2$, $y \neq 0$, $m, n, k = 0, 1, 2, \ldots$ and $\varepsilon > 0$ we have:

$$\int_{\langle y,s\rangle \geq \tilde{h}(iy)+\varepsilon} s_1^m s_2^n \langle y,s\rangle^k g(s_1,s_2)ds_1 ds_2 = 0.$$

Assuming for the moment that the assertion has been proved already, we will show how to derive (2.2) and (2.1). Since $\{s_1^m s_2^n\}$ is a spanning set for L^2 on any bounded domain, and since g has compact support, it follows from the assertion with $k = 0$ that

$$g(s_1,s_2) = 0 \text{ a.e. on } \{(s_1,s_2) : \langle y,s\rangle > \tilde{h}(iy)\}.$$

In other words

$$H_{K_g}(y) \leq \tilde{h}(iy).$$

Using inequality (2.1) with $z = iy$, relation (2.2) will follow.

Next we prove that for $z \in \mathbb{S}\mathbb{R}^n$ one has equality in (2.1). We write $z = w\xi$, $w \in \mathbb{C}$, $\xi \in \mathbb{R}^n$ and consider

$$f_\xi(w) = f(w\xi).$$

This is a function of exponential type on $\mathbb{C}$ which is bounded on the real axis. For such functions, the Phragmén-Lindelöf theorem shows that $h_\xi(u+iv) = h_\xi(iv)$, where h_ξ is the indicator of f_ξ. Hence

$$h(z) = h_\xi(w) = h_\xi(iv) = h(iy), \quad v = \operatorname{Im} w.$$

We conclude that

$$h^*(z) = \overline{\lim_{\tilde{z} \to z}}\, h(\tilde{z}) \geq \overline{\lim_{\tilde{\xi} \to \xi}}\, h_{\tilde{\xi}}(w) = \overline{\lim_{\tilde{\xi} \to \xi}}\, h_{\tilde{\xi}}(iv) = \overline{\lim_{\tilde{y} \to y}}\, h(i\tilde{y}) = H_{K_g}(y) \geq h^*(z).$$

This proves equality in (2.1) for $z = w\xi$.

PROOF OF THE ASSERTION. The proof is by induction on m and n. For every $y \in \mathbb{R}^2$, $y \neq 0$ there exists $\delta > 0$ such that for $\|y - v\| < \delta$ and all $r > r_0$ depending on v

$$|f(riv)| = \Big| \int_{\mathbb{R}^2} e^{r\langle v,s\rangle} g(s)\,ds_1\,ds_2 \Big| =$$

$$= \Big| \int_{-\infty}^{H_{K_g}(v)} e^{rt_1} dt_1 \int_{-\infty}^{\infty} g(s) J(v)\,dt_2 \Big| \leq e^{(h(iv) + \frac{1}{2}\varepsilon)r} < e^{(\tilde{h}(iy) + \varepsilon)r}.$$

Here

$$v_1 s_1 + v_2 s_2 = t_1, \quad -v_2 s_1 + v_1 s_2 = t_2 \text{ and } J(v) = \frac{1}{v_1^2 + v_2^2}$$

is the corresponding Jacobian.

Let $d\sigma$ denote arc length on the lines $\langle v, s\rangle = p$ and set:

$$p_0 = \tilde{h}(iy) + \varepsilon.$$

The one-dimensional Paley-Wiener theorem now yields that

$$\int_{\langle v,s\rangle = p} g(s)\,d\sigma = 0, \text{ for almost all } p \geq p_0,\ \|v - y\| < \delta.$$

Hence also, for $k = 1, 2, \ldots$

$$\int_{\langle v,s\rangle = p} \langle v, s\rangle^k g(s)\,d\sigma = 0 \text{ for almost all } p \geq p_0,\ \|v - y\| < \delta.$$

In particular we find:

$$\text{(2.3)} \qquad \int_{\langle v,s\rangle \geq p_0} \langle v, s\rangle^k g(s)\,ds_1\,ds_2 = 0,\ \|v - y\| < \delta,$$

which proves the assertion in case $m = n = 0$ (take $v = y$).

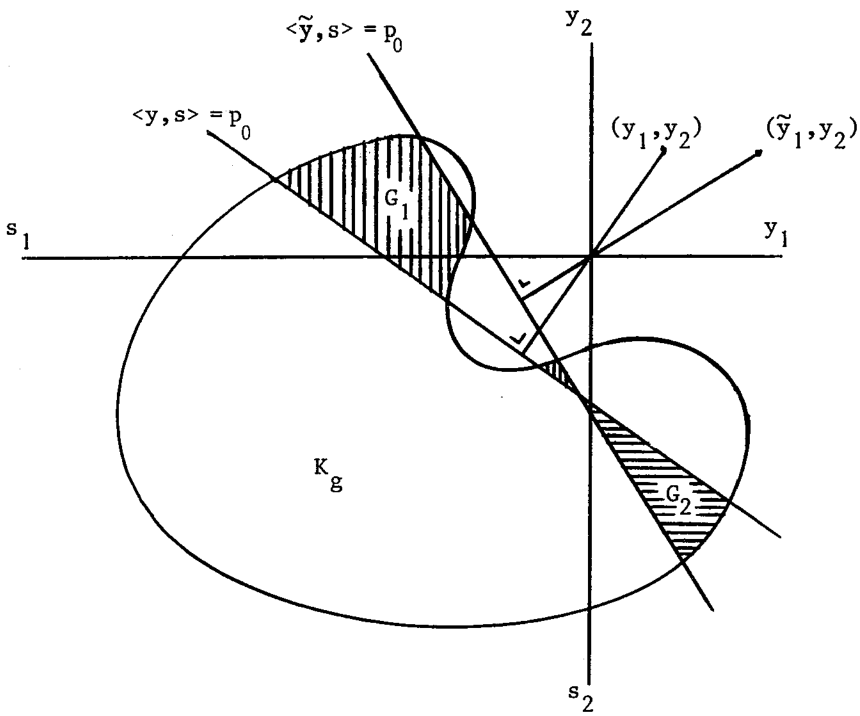
Now assume that the assertion is true for (m_0, n_0) and all k. We will prove it for $(m_0 + 1, n_0)$ and all k; then by symmetry it will also hold for $(m_0, n_0 + 1)$ and all k.

Fix $y=(y_1,y_2)\in\mathbb{R}^2$, $y\neq 0$. Let $\tilde{y}=(\tilde{y}_1,y_2)$ with $\|\tilde{y}-y\|<\delta$. Introduce the compacta:

$$G_1=K_g\cap\{s:\langle y,s\rangle\geq p_0\}\setminus\{s:\langle\tilde{y},s\rangle>p_0\},$$

$$G_2=K_g\cap\{s:\langle\tilde{y},s\rangle\geq p_0\}\setminus\{s:\langle y,s\rangle>p_0\},$$

see figure.



We observe that G_1 and G_2 depend on $\tilde{y}_1$. In fact,

(2.4) $\quad$ vol $G_i=o(1)$ if $\tilde{y}_1\to y_1$ $(i=1,2)$

and on G_i we find

(2.5) $\quad \langle\tilde{y},s\rangle=p_0+O(y_1-\tilde{y}_1)$ if $\tilde{y}_1\to y_1$

We can now make the induction step by formally differentiating with respect to y_1. For $y_1\neq\tilde{y}_1$ we write the induction hypothesis with $k+1$ instead of k:

$$(2.6)\quad \begin{cases} 0=\dfrac{1}{y_1-\tilde{y}_1}\Big[\displaystyle\int_{\langle y,s\rangle\geq p_0} s_1^{m_0}s_2^{n_0}\langle y,s\rangle^{k+1}g(s)ds_1ds_2-\int_{\langle\tilde{y},s\rangle\geq p_0} s_1^{m_0}s_2^{n_0}\langle\tilde{y},s\rangle^{k+1}g(s)ds_1ds_2\Big]\\ =\dfrac{1}{y_1-\tilde{y}_1}\displaystyle\int_{\langle y,s\rangle\geq p_0} s_1^{m_0}s_2^{n_0}(\langle y,s\rangle^{k+1}-\langle\tilde{y},s\rangle^{k+1})g(s)ds_1ds_2\\ +\dfrac{1}{y_1-\tilde{y}_1}\Big[\displaystyle\int_{G_1} s_1^{m_0}s_2^{n_0}\langle\tilde{y},s\rangle^{k+1}g(s)ds_1ds_2-\int_{G_2} s_1^{m_0}s_2^{n_0}\langle\tilde{y},s\rangle^{k+1}g(s)ds_1ds_2\Big].\end{cases}$$

We also need this formula with exponent 0 instead of $k+1$:

$$\int_{G_1} s_1^{m_0} s_2^{n_0} g(s) ds_1 ds_2 = \int_{G_2} s_1^{m_0} s_2^{n_0} g(s) ds_1 ds_2, \quad \|\tilde{y}-y\| < \delta.$$

In the expression $\int_{G_1} - \int_{G_2}$ in (2.6) we may therefore replace $\langle \tilde{y}, s\rangle^{k+1}$ by $\langle \tilde{y}, s\rangle^{k+1} - p_0^{k+1}$. After this step we let $\tilde{y}$ tend to y. Then by Lebesgue's dominated convergence theorem and the order estimates (2.4) and (2.5), the new integrals over G_1 and G_2 will tend to zero. We thus conclude from (2.6):

$$(k+1) \int_{\langle y,s\rangle \geq p_0} s_1^{m_0+1} s_2^{n_0} \langle y, s\rangle^k g(s) ds_1 ds_2 = 0. \qquad \square$$

REMARK 1. J. Korevaar suggested this proof which gives some information about the size of the sets

$$\{y \in \mathbb{R}^n : \|y\| = 1,\ h(iy) < \tilde{h}(iy) - \varepsilon\}.$$

In fact for $n=2$ it gives that these sets are finite. We note that theorem 2 as stated can also be obtained by using the classical Plancherel-Pólya theorem and the fact that $i\mathbb{R}^n$ has positive Γ-capacity in $\mathbb{C}^n$, cf. [10].

REMARK 2. Outside $\mathbb{S}\mathbb{R}^n$, there is in general no equality in (2.1). For *example*, take

$$f(z) = \frac{\cos\,(z_1^2 + z_2^2)^{\frac{1}{2}}}{z_1^2 + z_2^2 - \frac{1}{4}\pi^2}.$$

This is a Paley-Wiener function with indicator

$$h_f^*(z) = h_f(z) = |\mathrm{Im}\,(z_1^2 + z_2^2)^{\frac{1}{2}}|.$$

On $\mathbb{S}\mathbb{R}^2$, h_f coincides with $(y_1^2 + y_2^2)^{\frac{1}{2}}$, the support function of the unit disc. However, off $\mathbb{S}\mathbb{R}^2$ there is strict inequality as a small computation shows.

When we put suitable restrictions on K, the situation is totally different.

THEOREM 3. *Suppose* $g \in L^2(\mathbb{R}^n)$ *has compact support* K *with* polyhedral *convex hull. Then* $h_f^*(z) = H_K(y)$ *where* $f = \mathcal{F}g$. *In particular* h_f^* *is pluriharmonic outside a finite number of hyperplanes.*

NOTE. In case $n=1$ the condition is trivially satisfied.

Theorem 3 is an immediate consequence of the following

PROPOSITION 1. Let $\{A_k\}_{k=1,\dots,p}$ be finitely many distinct homogeneous linear forms on $\mathbb{R}^n$ and set

$$u_0(z) = \operatorname*{Max}_{k=1,\dots,p} A_k(y), \quad z = x + iy \in \mathbb{C}^n.$$

Let u be plurisubharmonic on $\mathbb{C}^n$ and such that

(A) $\quad u(z) \leq u_0(z), \quad z \in \mathbb{C}^n,$

(B) $u(z)=u_0(z),\ z\in \mathbb{S}\mathbb{R}^n$.

Then $u(z)\equiv u_0(z)$.

PROOF. Let

$$E=\{z=x+iy\in\mathbb{C}^n:\exists j,k(j\neq k) \text{ such that } A_j(y)=A_k(y)\}.$$

Observe that E is a finite union of $(2n-1)$-dimensional hyperplanes in $\mathbb{C}^n=\mathbb{R}^{2n}$. Also, E meets $i\mathbb{R}^n$ in a union of $(n-1)$-dimensional hyperplanes, and every component of $\mathbb{C}^n\setminus E$ meets $i\mathbb{R}^n\setminus E$ in an open subset of $i\mathbb{R}^n$.

We see that u_0, as a sup of finitely many pluriharmonic functions, is plurisubharmonic; also u_0 is pluriharmonic outside E. Now put

$$s(z)=u(z)-u_0(z).$$

It follows from the preceding observations that s is plurisubharmonic on $\mathbb{C}^n\setminus E$. From (A) we infer

$$s\leq 0$$

and (B) implies:

$$s\equiv 0 \text{ on } \mathbb{S}\mathbb{R}^n, \text{ in particular on } i\mathbb{R}^n.$$

We conclude that $s\equiv 0$ on every component of $\mathbb{C}^n\setminus E$ by the maximum principle for plurisubharmonic functions. Thus $u\equiv u_0$ off E. Since E has $2n$-dimensional measure zero, $u\equiv u_0$ on $\mathbb{C}^n$. □

In the next section we will use theorem 3 to prove regular growth for this kind of Paley-Wiener functions.

3. FOURIER TRANSFORMS WHICH HAVE REGULAR GROWTH

The main object in this section is to prove that Paley-Wiener functions of a certain kind have regular growth. A "lower-bounds theorem", which is of independent interest, is used to proceed from existence of a lim* to regular growth. We will also supply proofs of properties 1 and 2 of section 1.

3.1. *A lower-bounds theorem*

In this subsection we will need some potential theory. Our general reference is Hayman and Kennedy [3]. To avoid confusion about normalizations we start with a description of our main tools.

Relative to the ball $B(0,r)$ in $\mathbb{R}^n$, $n\geq 3$, we consider Green's function

$$g_r(x,\xi)=\frac{1}{\|x-\xi\|^{n-2}}-\left(\frac{r}{\|\xi\|}\cdot\frac{1}{\|x-\xi r^2/\|\xi\|^2\|}\right)^{n-2}.$$

We set σ equal to the Lebesgue measure on $\partial B(0,r)\subset\mathbb{R}^n$ and $\sigma_n=\sigma(\partial B(0,1))$. Then the harmonic measure of $E\subset\partial B(0,r)$ relative to $x\in B(0,r)$ is equal to

$$\omega_r(x,E)=\frac{1}{(2-n)\sigma_n}\int_E\frac{\partial g_r(x,\xi)}{\partial N_\xi}\,d\sigma.$$

For a subharmonic function u on a neighbourhood of $\bar{B}(0,r)$ we define the Riesz mass as

$$\mu_u = \frac{1}{(n-2)\sigma_n} \Delta u.$$

Recall that for subharmonic functions u, Δu is a positive distribution that can be viewed as a positive measure.

We will also need the following Cartan type lemma, cf. Hayman and Kennedy [3], p. 131.

LEMMA 1. Suppose μ is a positive measure on $\mathbb{R}^n$ with $\mu(\mathbb{R}^n) = \mu_0 < \infty$ and let $0 < p < q < \infty$. Then for every $m > 0$

$$\int_{\mathbb{R}^n} \|x-\xi\|^{-p} d\mu(\xi) < m,$$

outside a finite or countable union of balls $B(a_k, r_k)$ with

$$\sum_k r_k^q < C\left(\frac{\mu_0}{m}\right)^{q/p}.$$

Here C is a constant depending on p and q only. □

Now we come to a useful multidimensional version of a classical "lower-bounds theorem", cf. Levin [7], p. 21. The following result represents joint work with R. Zeinstra.

THEOREM 4 (*lower-bounds theorem*). *Let $n \geq 3$. There exists a constant $C > 0$ only depending on n, such that for any subharmonic function $u \leq 0$ defined on a neighbourhood of $\bar{B}(0,3R) \subset \mathbb{R}^n$ and for any $M \geq 2^n$ the following estimate holds:*

$$0 \geq u(x) \geq Mu(0)$$

for all $x \in B(0,R)$ outside a union of balls $B(a_\nu, \varrho_\nu)$ with

$$\sum_\nu \varrho_\nu^n \leq CR^n M^{-n/(n-2)}.$$

PROOF. We may assume $-\infty < u(0) < 0$ (or there is nothing to prove) and $R = 1$. The Poisson-Jensen-Green formula gives for $x \in B(0,2)$

$$u(x) = -\int_{B(0,2)} g_2(x,\xi) d\mu_u(\xi) + \int_{\partial B(0,2)} u(s) d\omega_2(x,s). \tag{3.1}$$

Let u^* be the harmonic function defined by the last integral in (3.1). The inequality $g(x,\xi) \leq \|x-\xi\|^{2-n}$ gives

$$u(x) \geq -\int_{B(0,2)} \|x-\xi\|^{2-n} d\mu_u(\xi) + u^*(x). \tag{3.2}$$

Since $u^*(x) \leq 0$ on $B(0,2)$, we can apply Harnack's inequality ([3], p. 35) to u^*.

Hence for $x \in B(0,1)$

$$(3.3)\qquad -u^*(x) \le \frac{(2+1)2^{n-2}}{(2-1)^{n-1}}(-u^*(0)) = -3\cdot 2^{n-2}u^*(0).$$

Because $u(0) \le u^*(0)$, (3.2) and (3.3) imply

$$u(x) \ge -\int_{B(0,2)} \|x-\xi\|^{2-n} d\mu_u(\xi) + 3\cdot 2^{n-2}u(0).$$

To estimate the integral on the right hand side, we apply lemma 1 with $m = -\tilde{M}u(0) > 0$, $\tilde{M} > 0$, $q = n$ and $p = n-2$. We then obtain

$$(3.4)\qquad u(x) \ge (\tilde{M} + 3\cdot 2^{n-2})u(0)$$

for all $x \in B(0,1)$ outside $\bigcup_\nu B(a_\nu, \varrho_\nu)$ with

$$\sum_\nu \varrho_\nu^n \le C_0 \left(\frac{\mu_u(B(0,2))}{-\tilde{M}u(0)}\right)^{n/(n-2)}.$$

It only remains to estimate $\mu_u(B(0,2))$ in terms of $u(0)$. Observe that $g_3(0,\xi) = \|\xi\|^{2-n} - 3^{2-n}$. We use a representation involving $B(0,3)$ and set $c = (2^{2-n} - 3^{2-n})^{-1}$. We thus find

$$u(0) = -\int_{B(0,3)} (\|\xi\|^{2-n} - 3^{2-n}) d\mu_u(\xi) + \int_{\partial B(0,3)} u(s) d\omega_3(0,s)$$

$$\le -\int_{B(0,2)} (\|\xi\|^{2-n} - 3^{2-n}) d\mu_u(\xi) \le -c^{-1}\mu_u(B(0,2)).$$

Hence

$$\mu_u(B(0,2)) \le -cu(0).$$

We infer that (3.4) holds outside balls $B(a_\nu, \varrho_\nu)$ with

$$\sum_\nu \varrho_\nu^n \le C_0 c^{n/(n-2)} \tilde{M}^{-n/(n-2)}.$$

We may finally choose

$$\tilde{M} = M - 3\cdot 2^{n-2} \ge \tfrac{1}{4}M > 0$$

and

$$C = C_0 c^{n/(n-2)} 4^{n/(n-2)}$$

to complete the proof. □

3.2. *P-regular growth*

For our purpose the integral type definition of regular growth (section 1) is less convenient than a definition involving sequences of points.

DEFINITION. Let f be an entire function of exponential type on $\mathbb{C}^n$. We say that f has P-regular growth in the direction $z_0 \ne 0$ if there exists a sequence $z_k \to \infty$ in $\mathbb{C}^n$ in the direction z_0, that is

$$\|z_k\| \to \infty, \ \lim_{k\to\infty} \frac{z_k}{\|z_k\|} = \frac{z_0}{\|z_0\|},$$

with the property that

$$\lim_{k\to\infty} \frac{\|z_{k+1}\|}{\|z_k\|}=1,\ h_f^*(z_0)=\lim_{k\to\infty} \frac{\log|f(z_k)|}{\|z_k\|}\|z_0\|.$$

NOTE. The relation

$$\lim_{r\to\infty}{}^* \frac{\log|f(rz_0)|}{r}=h_f^*(z_0)$$

would imply that f has P-regular growth in the direction z_0.

We will show that P-regular growth is equivalent to regular growth as defined in section 1. For this we need a consequence of Hartogs' lemma.

LEMMA 2. Let h^* be the indicator of an entire function f of exponential type on $\mathbb{C}^n$ and suppose that for z in some compact set K (which may be just a point)

$$h^*(z)\equiv 0.$$

Then for every $\varepsilon>0$ there exist a neighbourhood Ω of K and $r_0>0$ such that for $r>r_0$

$$\frac{\log|f(rz)|}{r}<\varepsilon,\ z\in\Omega.$$

PROOF. By the upper semi-continuity of h^*, we can choose compacta K_1 and K_2 with $K_2\supset K_2^0\supset K_1\supset K_1^0\supset K$ such that

$$h^*(z)<\varepsilon/2 \text{ on } K_2.$$

On K_2 we consider the family of subharmonic functions

$$u_r(z)=\frac{\log|f(rz)|}{r},\ r>1.$$

There exists a constant M such that $u_r(z)<M$ for $r>1$. Also

$$\overline{\lim_{r\to\infty}}\, u_r(z)\leq\varepsilon/2.$$

Hartogs' lemma (Hörmander [4], th. 1.6.13, p. 21) now implies that there exists $r_0>0$ such that for $r>r_0$

$$\frac{\log|f(rz)|}{r}=u_r(z)<\varepsilon/2+\varepsilon/2 \text{ on } K_1. \qquad \square$$

THEOREM 5. *Let f be an entire function of exponential type on $\mathbb{C}^n$ $(n\geq 2)$, regularized indicator h^*, that has P-regular growth in the direction $z_0\neq 0$. Then the following assertions are true:*

A: *f has regular growth in the direction z_0.*

B: *if in addition h^* is pluriharmonic on a connected neighbourhood D of z_0, then f has P-regular growth, and consequently regular growth on D.*

PROOF. Without loss of generality, suppose that $\|z_0\|=1$, $h^*(z_0)=0$ and D is convex. Thus, if h^* is pluriharmonic on D, we may assume $h^*\equiv 0$ on D, because $h^*=\operatorname{Re} g$ for some holomorphic function g on D, and we can consider fe^{-g} instead of f.

It follows from lemma 2 that for every $\varepsilon>0$ there are $\delta>0$, $r_0>0$ such that

$$\frac{\log |f(z')|}{r}<\varepsilon,\ r>r_0,\ z'\in B(rz_0, 4\delta r). \tag{3.5}$$

In case $h^*\equiv 0$ on D there exists $\delta>0$ depending on D and z_0, but not on ε, such that for every $\varepsilon>0$ there exists $r_0>0$ for which (3.5) holds.

In connection with the P-regular growth, we consider sequences $z_k\to\infty$ in $\mathbb{C}^n$. We will write $\|z_k\|=:r_k$. By the hypothesis there exists a sequence $\{z_k\}$ with

$$\lim_{k\to\infty}\frac{\log |f(z_k)|}{r_k}=0$$

such that for every large enough r we can find a $k>0$ with

$$r_{k-1}<r\le r_k,\ \frac{r_k}{r_{k-1}}<1+\tfrac{1}{4}\delta,\ \left\|\frac{z_k}{r_k}-z_0\right\|<\tfrac{1}{4}\delta.$$

Hence

$$B(rz_0,\tfrac{1}{2}\delta r)\subset B(z_k,\delta r_k).$$

We infer from the lower-bounds theorem for $\mathbb{C}^n=\mathbb{R}^{2n}$ applied to $B(z_k, 3\delta r_k)$ and (3.5) that for each $j\ge 2n$

$$\begin{cases} \left|\log |f(z)|-\varepsilon r_k\right|\le 2^j\left|\log |f(z_k)|-\varepsilon r_k\right| \\ \text{for } z\in B(z_k,\delta r_k) \text{ outside a set } S_j \text{ of measure } \le C(\delta r_k)^{2n}2^{-jn/(n-1)}. \end{cases} \tag{3.6}$$

Completion of the proof for case A. We introduce a disjoint splitting of $B(rz_0,\frac{1}{2}\delta r)\subset B(z_k,\delta r_k)$:

$$E_j=(S_{j-1}\setminus S_j)\cap B(rz_0,\tfrac{1}{2}\delta r),\ j>2n,$$

$$E_{2n}=B(rz_0,\tfrac{1}{2}\delta r)\setminus S_{2n},$$

$$E_\infty=\bigcap_{j\ge 2n} S_j\cap B(rz_0,\tfrac{1}{2}\delta r).$$

Thus E_∞ has measure zero and

$$B(rz_0,\tfrac{1}{2}\delta r)=\bigcup_{j=2n,\dots,\infty} E_j,\ \text{disjoint}.$$

Observe also that as a consequence of (3.6) we have on $E_j (j\ge 2n)$

$$\left|\log |f(z)|-\varepsilon r_k\right|\le 2^j\left|\log |f(z_k)|-\varepsilon r_k\right|, \tag{3.7}$$

$$m(E_j)\le C'(\delta r_k)^{2n}2^{-jn/(n-1)}. \tag{3.8}$$

Using the above splitting, (3.7) and (3.8) we can estimate:

$$\left|\frac{1}{V(\frac{1}{2}\delta r)}\int_{B(z_0 r,\frac{1}{2}\delta r)}\frac{\log|f(z')|}{r}\,dm(z')\right|$$

$$\leq\frac{1}{V(\frac{1}{2}\delta r)}\int_{B(z_0 r,\frac{1}{2}\delta r)}\left|\frac{\log|f(z')|-\varepsilon r_k}{r}\right|dm(z')+\varepsilon\frac{r_k}{r}$$

$$\leq\frac{1}{V(\frac{1}{2}\delta r)}\sum_{j\geq 2n}\int_{E_j}\left|\frac{\log|f(z')|-\varepsilon r_k}{r}\right|dm(z')+\varepsilon(1+\tfrac{1}{4}\delta)$$

$$\leq\frac{1}{V(\frac{1}{2}\delta r)}\sum_{j\geq 2n}C'(\delta r_k)^{2n}2^{-jn/(n-1)}\cdot 2^j\left|\frac{\log|f(z_k)|-\varepsilon r_k}{r}\right|+\varepsilon(1+\tfrac{1}{4}\delta)$$

$$\leq\tilde{C}\left|\frac{\log|f(z_k)|-\varepsilon r_k}{r_k}\right|+\varepsilon(1+\tfrac{1}{4}\delta).$$

Here $\tilde{C}$ is a constant and $V(r)=m(B(0,r))$ is as in section 1.

Since we can take ε arbitrary small and

$$\lim_{k\to\infty}\frac{\log|f(z_k)|}{r_k}=0,$$

we have proved that f has regular growth in the direction z_0.

Completion of the proof for case B. Since h^* is pluriharmonic on the cone generated by D, we may as well assume that D is a cone. Now δ need only satisfy the condition $B(z_0,4\delta)\Subset D$.

It is enough to prove P-regular growth for points z_0' in $\partial B(0,1)\cap B(z_0,\frac{1}{4}\delta)$, since we can next repeat the argument with z_0 replaced by any other point in $\partial B(0,1)\cap B(z_0,\frac{1}{4}\delta)$. We set

$$\tilde{z}_k=r_k z_0'.$$

Then

$$\tilde{z}_k\in B(r_k z_0,\tfrac{1}{4}\delta r_k)\subset B(z_k,\tfrac{1}{2}\delta r_k)$$

and

$$B(\tilde{z}_k,\tfrac{1}{2}\delta r_k)\subset B(z_k,\delta r_k).$$

On $B(z_k,\delta r_k)$ we have (3.6). We take $j>2n$ so large that S_j does not occupy the whole ball $B(\tilde{z}_k,\frac{1}{2}\delta r_k)$. Then there will be a point $z_k'\in B(\tilde{z}_k,\frac{1}{2}\delta r_k)\setminus S_j$ satisfying

$$\|z_k'-\tilde{z}_k\|\leq c\delta r_k 2^{-j/(2n-2)}$$

and

$$\left|\frac{\log|f(z_k')|}{\|z_k'\|}\right|\leq 2^j\left|\frac{\log|f(z_k)|}{r_k}\right|+3\varepsilon.$$

We finally choose $j=j_k\to\infty$ so that

$$\lim_{k\to\infty}2^{j_k}\left|\frac{\log|f(z_k)|}{r_k}\right|=0.$$

Then the sequence $\{z_k'\}$ will satisfy

$$\|z_k'\| \to \infty, \lim_{k\to\infty} \frac{\|z_{k+1}'\|}{\|z_k'\|} = 1, \lim_{k\to\infty} \frac{z_k'}{\|z_k'\|} = z_0'$$

and

$$\overline{\lim_{k\to\infty}} \left| \frac{\log |f(z_k')|}{\|z_k'\|} \right| < 3\varepsilon.$$

Since ε is arbitrary small, we obtain P-regular growth using a diagonal process. □

PROPOSITION 2. Let f be an entire function of exponential type with indicator h^*. If f has regular growth in a certain direction, it has P-regular growth in that direction.

PROOF. Suppose we have regular growth in the direction z_0 with $\|z_0\| = 1$. We may assume $h^*(z_0) = 0$. Then for every ε, $\delta_0 > 0$ there exists a number δ, $0 < \delta < \delta_0$ such that

$$\overline{\lim_{r\to\infty}} \left| \frac{1}{V(\delta r)} \int_{B(rz_0, \delta r)} \frac{\log |f(z')|}{r} \, dm(z') \right| < \varepsilon. \tag{3.9}$$

From lemma 2 we infer that there exists a number δ_0 for which

$$\overline{\lim_{r\to\infty}} \sup_{z' \in B(rz_0, \delta_0 r)} \frac{\log |f(z')|}{r} < \varepsilon. \tag{3.10}$$

It follows from (3.9) and (3.10) that

$$\overline{\lim_{r\to\infty}} \inf_{z' \in B(rz_0, \delta r)} \left| \frac{\log |f(z')|}{r} \right| < \varepsilon. \tag{3.11}$$

Let r_1 be a large enough positive number. Define a sequence $r_k \to \infty$ by

$$r_{k+1} = \left(\frac{1+\delta}{1-\delta} \right) r_k, \; k \ge 1.$$

Then the balls $B(r_k z_0, \delta r_k)$ are disjoint. Because of (3.11) we can find $z_k \in B(r_k z_0, \delta r_k)$ satisfying

$$\left| \frac{\log |f(z_k)|}{\|z_k\|} \right| < 2\varepsilon$$

and by the definition of r_k

$$\|z_k\| \to \infty, \frac{\|z_{k+1}\|}{\|z_k\|} < \left(\frac{1+\delta}{1-\delta} \right)^2, \left\| \frac{z_k}{\|z_k\|} - z_0 \right\| < \delta.$$

When we let ε and δ_0 tend to zero, we can readily construct a sequence $\{\tilde{z}_k\}$ that establishes P-regular growth. □

We have found that regular growth and P-regular growth of entire functions of exponential type are equivalent. We have also obtained a *proof of property* 1 (section 1) for regular growth.

We next indicate a *proof of property* 2 (section 1). If f has P-regular growth in the directions z_n where

(3.12) $\quad z_n \to z_0$ and $h^*(z_n) \to h^*(z_0)$,

then f has P-regular growth also in the direction z_0, as is seen by use of a diagonal process. Suppose now that f has indicator h^* on D and that f grows regularly on $D \backslash F$, where F has measure zero. Since h^* is subharmonic, $h^*(z)$ is equal to the limit of the mean values of h^* over spheres shrinking to z (use the mean-value inequality and the upper semi-continuity). Again using the upper semi-continuity of h^*, one concludes that for every $z_0 \in F$ there is a sequence $\{z_n\} \subset D \backslash F$ satisfying (3.12). Consequently f has P-regular growth on D.

Our main result is now an easy consequence of what we have done so far.

THEOREM 6. *Let $g \in L^2(\mathbb{R}^n)$ have compact support $K \subset \mathbb{R}^n$. Suppose that the convex hull of K is a polyhedron. Then $f = \mathscr{F}g$ is an entire function of regular growth on $\mathbb{C}^n$.*

PROOF. The restriction of f to any complex line in $\mathbb{S}\mathbb{R}^n$ is a Paley-Wiener function of one variable. Thus, by a theorem of Cartwright, cf. Levin [7], p. 243,

$$\lim_{r\to\infty}{}^* \frac{\log |f(rz)|}{r} = h_f(z), \; z \in \mathbb{S}\mathbb{R}^n,$$

where $h_f(z)$ is the radial indicator. It now follows from theorem 2, more specifically from (2.2) that f has P-regular growth in each direction $iy \in i\mathbb{R}^n$ (cf. the proof of property 2 above). By theorem 3, h_f^* is pluriharmonic outside a set of measure zero in $\mathbb{C}^n$ and every component of the set where h_f^* is pluriharmonic has interior points belonging to $i\mathbb{R}^n$. Hence it follows from the properties 1 and 2 that f has regular growth on $\mathbb{C}^n$.

4. APPLICATIONS AND FURTHER REMARKS

Let $g \in L^2(\mathbb{R}^2)$ have compact support $K \subset \mathbb{R}^2$ such that the convex hull of K is polygonal. As an application of our results we can determine the zero distribution of $f = \mathscr{F}g$ quite explicitly. We will keep the notations of section 1.

Let the vertices of the convex hull of K be $a_j = (\alpha_j^1, \alpha_j^2) \in \mathbb{R}^2$, $j = 1, \ldots, m$. By theorem 2

$$h_f^*(z) = H_K(y) = \operatorname*{Max}_j \langle a_j, y \rangle, \; z = x + iy \in \mathbb{C}^2.$$

Since f grows regularly (theorem 6) it follows from theorem 1 that

$$\lim_{r\to\infty} \frac{\sigma^r}{r^3} = \frac{1}{2\pi} \int_{B(0,1)} \Delta h_f^* dm > 0,$$

where σ^r is the volume of the zero set of f in $B(0, r)$.

On the other hand h_f^* is pluriharmonic outside the set

$$E=\{z=x+iy\in\mathbb{C}^2:\exists j,k(j\neq k) \text{ such that } \langle a_j,y\rangle=\langle a_k,y\rangle\}.$$

Thus if $w\in(\mathbb{C}^2\setminus E)\cap S$ and if $\varphi>0$ is chosen small enough, then we find on the cone $T_w(\varphi)$

$$\lim_{r\to\infty}\frac{\sigma_w^r(\varphi)}{r^3}=0.$$

CONCLUSION. The zero set of f is "concentrated" around E.

If $w=u+iv\in E\cap S$ then there are two possibilities which, upon appropriate adjustment of indices, may be described by

(4.1) $\langle a_1,v\rangle=\langle a_2,v\rangle>\langle a_j,v\rangle,\ j=3,\dots,m$

and

(4.2) $\langle a_1,v\rangle=\langle a_2,v\rangle=\langle a_3,v\rangle,$

respectively.

First we *examine case* (4.1). We take φ so small that

$$h_f^*(z)=\text{Max}\ (\langle a_1,y\rangle,\ \langle a_2,y\rangle) \text{ on } T_w(\varphi).$$

One may compute

(4.3) $$\lim_{r\to\infty}\frac{\sigma_w^r(\varphi)}{r^3}=\frac{1}{2\pi}\int_{T_w(\varphi)\cap B(0,1)}\Delta h_f^* dm=\frac{\|a_1-a_2\|}{3}(1-\cos\varphi).$$

Next we *consider case* (4.2). Relation (4.2) implies $v=0$, that is $w\in\mathbb{R}^2\subset\mathbb{C}^2$. For such w one obtains

(4.4) $$\lim_{r\to\infty}\frac{\sigma_w^r(\varphi)}{r^3}=\frac{1-\cos\varphi}{6}\int_{-\pi}^{\pi}\text{Max}_j\ \{\alpha_j^1\cos\theta+\alpha_j^2\sin\theta\}d\theta.$$

The computations leading to (4.3) and (4.4) involve a careful application of Green's 2nd identity, which becomes possible after introducing the definition of Δh_f^* as a distribution, and the use of polar coordinates. We leave the details to the reader.

Note that (4.3) tells us that *in contrast to the one-variable case*, the zero set of a Paley-Wiener function of several variables can have a positive density in cones around non-real directions.

REMARKS. In general, that is without restrictions on K_g, the indicator h_f^* need not be pluriharmonic anywhere, and then by theorem 1 of section 1 the zero set of f has a positive density in every cone, supposing of course that f grows regularly.

Our results can be extended to entire functions of exponential type that satisfy other boundedness conditions on $\mathbb{R}^n$, for example, functions bounded by a polynomial on $\mathbb{R}^n$. (One may represent such functions as Fourier transforms of compactly supported distributions).

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PALEY-WIENER FUNCTIONS WITH PRESCRIBED INDICATOR

by

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Introduction

Let f be an entire function of exponential type on $\mathbb{C}^n$. The (regularized) indicator function of f is defined by

$$h_f(z) = \overline{\lim_{0\neq w\to z}}\ \overline{\lim_{r\to\infty}} \frac{\log|f(rw)|}{r}, \qquad f \not\equiv 0.$$

The indicator gives information about the growth of f. It is a plurisubharmonic function and positively 1-homogeneous, that is it satisfies $h_f(rz) = rh_f(z)$, $r > 0$, cf. [8] also for generalizations to functions of arbitrary finite order.

A theorem of Martineau and Kiselman [7,5] states a kind of converse: *For every plurisubharmonic positively* 1*-homogeneous function* u *on* $\mathbb{C}^n$ *there exists an entire function of exponential type* f *such that* $h_f = u$.

In fact, Martineau proves the theorem for indicators related to functions of arbitrary order.

We will consider entire functions of exponential type which are polynomially bounded on the real subspace. We will call these functions Paley-Wiener functions. The Paley-Wiener functions occur naturally, because by the Paley-Wiener theorem [9] they are exactly the complex Fourier transforms of distributions on $\mathbb{R}^n$ with compact support. Consequently the indicator h of such a function satisfies an extra condition:

(1) $\quad |h(z)| \leq C\|\mathrm{Im}\, z\|$.

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In this report we demonstrate that the methods of Martineau lead to the following theorem which is proved in section 1. We say that a function f is Hölder continuous (with exponent η) on $E \subset \mathbb{R}^n$ if there exist $\eta, B > 0$ such that

$$|f(x) - f(y)| \leqslant B\|x - y\|^{\eta}, \qquad x,y \in E.$$

Main result:

If u *is a continuous positively homogeneous plurisubharmonic function on* $\mathbb{C}^n$ *that is Hölder continuous on the unit sphere in* $\mathbb{C}^n$ *and satisfies* (1), *then there exist an entire function of exponential type* f *and a polynomial* P *such that* $h_f = u$ *and*

$$|f(z)| \leqslant P(\|z\|) \cdot e^{h_f(z)}.$$

In general h_f need not be continuous as was shown by Lelong [6]. As an application of this theorem we simplify the proof of the following result of Vauthier [10]:

There exist entire functions of exponential type on $\mathbb{C}^2$ *which are bounded on* $\mathbb{R}^2$ *but which are not of regular growth.*

This is done in section 2.

1. A KISELMAN-MARTINEAU TYPE THEOREM.

Let $M(n)$ be the class of plurisubharmonic functions ψ on $\mathbb{C}^n$ that are Hölder continuous on the unit sphere and satisfy

$$\psi(rz) = r\psi(z), \qquad r \geq 0,$$

(1.1)

$$|\psi(z)| \leq C\|\operatorname{Im} z\| \qquad \text{for some } C.$$

We remark that indicators of Paley-Wiener functions satisfy (1.1). Next we give some examples.

a) $M(1)$ consists of functions of the form

$$\psi(z) = \operatorname{Max}\{a \operatorname{Im} z,\ b \operatorname{Im} z\}, \qquad a, b \in \mathbb{R};$$

this function is the indicator of $e^{-iaz} + e^{-ibz}$.

b) Example a) generalizes immediately to higher dimensions. Let $K \subset \mathbb{R}^n$ be compact. The following type of function is contained in $M(n)$:

$$\psi(z) = \operatorname*{Max}_{a \in K} \sum_{j=1}^{n} a_j \operatorname{Im} z_j.$$

This is the so-called support function of the convex hull of K; it is known to be Lipschitz continuous on compacta.

c) However, $M(n)$ consists of more than support functions alone if $n \geq 2$: the function

$$\left|\operatorname{Im} \sqrt{z_1^2 + z_2^2}\right|$$

is in $M(2)$. It is the indicator of $\cos \sqrt{z_1^2 + z_2^2}$. It is not a support function.

We start with two lemmas. They can be found in [8]. For the convenience of the reader and because the continuity of the functions in $M(n)$ allows us to simplify things somewhat, we will include the proofs.

For a continuous function u on a compact set K in $\mathbb{R}^n$ we define

$$T(u) = \{v:\ v \text{ is upper semi-continuous on } K,\ v \leq u,\ v \neq u\}$$

Lemma 1. Let u be a continuous function on a compact subset K of $\mathbb{R}^n$. There exists a sequence of continuous functions $u_k \in T(u)$ with the property that for every $v \in T(u)$ there exist k_o and $\delta > 0$ with

$$u_{k_o}(x) \geq v(x), \qquad x \in K$$

and

$$u_{k_o}(x) \neq u(x) \text{ implies } u_{k_o}(x) > v(x) + \delta.$$

The last inequality holds on the closure of $\{x : u_{k_o}(x) \neq u(x)\}$.

Proof: We choose positive numbers ε_m decreasing to 0. For every m there exists a finite cover of K consisting of balls $B(x_m^k)$ with centre $x_m^k \in K$ and radius ε_m. We can and do construct continuous functions u_m^k on K that satisfy

$$u_m^k = u \text{ on } K \setminus B(x_m^k),$$
$$u_m^k \geq u - \varepsilon_m,$$
$$u_m^k(x_m^k) = u(x_m^k) - \varepsilon_m.$$

The desired sequence is formed by $\{u_m^k\}$ taking first the functions u_1^k, next u_2^k, etc. Indeed, let $v \in T(u)$. First suppose that v is continuous. Then there exist $x_o \in K$, ε, $\delta > 0$ with

$$v(x) \leqslant u(x) - \varepsilon \quad \text{for} \quad \|x - x_o\| < \delta.$$

Choose m so large that $\varepsilon_m < \mathrm{Min}\{\frac{1}{2}\delta, \frac{1}{2}\varepsilon\}$. Next we can find k such that $x_o \in B(x_m^k)$. It follows that

$$\overline{B(x_m^k)} \subset \{x : \|x - x_o\| < \delta\}.$$

Hence $u_m^k > v + \frac{1}{2}\varepsilon$ on $\overline{B(x_m^k)}$ and we are done.

Next if v is upper semi-continuous, we can find continuous functions v_j such that

(1.2) $\quad v_j \downarrow v.$

Since u is continuous and $u \geqslant v$, we may replace v_j by $\mathrm{Min}\{u, v_j\}$ in (1.2). If j_o is large enough, $\mathrm{Min}\{u, v_{j_o}\} \neq u$; to finish the proof we apply the preceding argument to $v = \mathrm{Min}\{u, v_{j_o}\}$. ∎

Let S denote the unit sphere in $\mathbb{C}^n$.

Lemma 2. Let h be the indicator of an entire function of exponential type f and let v be a positively 1-homogeneous, continuous function on $\mathbb{C}^n$. If $h(z) < v(z)$ on a compact subset K of S, then for every $\varepsilon > 0$ there exists a constant C such that

$$\log|f(rz)| < C + v(rz) + \varepsilon r, \quad \text{for} \quad r > 0 \quad \text{and} \quad z \in K.$$

<u>Proof</u>: Recall Hartogs' lemma : Let $\{u_r\}$, $r > 0$, be a family of subharmonic functions on a domain $\Omega \subset \mathbb{R}^n$ and let K be a compact subset of Ω. If

a) $\{u_r\}$ is bounded from above on Ω,

b) there is a constant A such that

$$\overline{\lim_{r\to\infty}}\, u_r(z) < A \quad \text{on} \quad \Omega,$$

then for every $\varepsilon > 0$ we can find r_o such that for $r > r_o$

$$u_r(z) < A + \varepsilon \quad \text{on} \quad K,$$

cf. [4] theorem 1.6.13.

This theorem remains true if we replace the constant A by a continuous function (one uses the compactness of K). The lemma is an easy consequence of this extended Hartogs' lemma . We let Ω be a small neighbourhood of K (so that $h(z) < v(z)$ on Ω) and

$$u_r(z) = \frac{\log|f(rz)|}{r}, \qquad r \geqslant 1.$$

Since f is of exponential type the family $u_r(z)$ is uniformly bounded from above. Also

$$\overline{\lim_{r\to\infty}}\, u_r(z) \;(= h(z)) < v(z) \quad \text{on} \quad \Omega.$$

Hence, given $\varepsilon > 0$ we can find r_o such that for $r > r_o$

$$\frac{\log|f(rz)|}{r} = u_r(z) < v(z) + \varepsilon \quad \text{on} \quad K.$$

This implies the lemma. ∎

We will now formulate a special case of a very important theorem of L. Hörmander ([4] theorem 4.4.2.) on the solution of the $\bar\partial$-equation with L^2 estimates on pseudoconvex domains. For a (p,q) form

$$g = \sum_{|I|=p}{}' \sum_{|J|=q}{}' g_{I,J}\, dz^I\, d\bar z^J,$$

where Σ' means summation over strictly increasing multi-indices and the $g_{I,J}$ are measurable functions, put

$$|g|^2 = \sum_{I,J}{}' |g_{I,J}|^2.$$

<u>Theorem</u> (Hörmander). Let ϕ be a plurisubharmonic function on $\mathbb{C}^n$. For every $(p, q+1)$ form g with

$$\int_{\mathbb{C}^n} |g|^2 e^{-\phi}\, d\lambda < \infty$$

and $\bar\partial g = 0$ (as a distribution!) there is a solution u of the equation $\bar\partial u = g$ such that

$$\int_{\mathbb{C}^n} |u|^2 e^{-\phi}(1 + \|z\|^2)^{-2} d\lambda \leqslant \int_{\mathbb{C}^n} |g|^2 e^{-\phi} d\lambda. \qquad ∎$$

This theorem is crucial for our main result as well as for the results in part 2. We continue with an application to extensions of an analytic function defined on a complex linear subspace under certain growth conditions. It is a slight refinement of an application given by Hörmander, cf. [4] theorem 4.4.3.

Theorem 1. Let ρ be a plurisubharmonic function on $\mathbb{C}^n$ such that for some constants $\alpha, \beta, A, B > 0$

$$(1.3) \qquad |\rho(z) - \rho(w)| < A(\|z\| + \|w\|)^{\alpha} \|z - w\|^{\beta} + B.$$

Then there exists a constant G such that the following is true: If f is an analytic function on a complex linear subspace Σ of $\mathbb{C}^n$ of codimension K such that

$$(1.4) \qquad \int_{\Sigma} |f|^2 e^{-\rho} d\sigma < \infty,$$

where $d\sigma$ denotes Lebesgue measure on Σ, then there exists an analytic function g on $\mathbb{C}^n$ such that $g = f$ on Σ and

$$(1.5) \qquad \int_{\mathbb{C}^n} |g|^2 (1 + \|z\|^2)^{-q} e^{-\rho} d\lambda \leq G \int_{\Sigma} |f|^2 e^{-\rho} d\sigma < \infty,$$

where $q = (2 + \frac{\alpha}{\beta})K$.

Proof. The theorem is easily obtained by finitely many iterations of the special case which we will treat here, namely, the case where Σ is the hyperplane $z_n = 0$. Then f is an analytic function of $z_1, \ldots, z_{n-1}$ and we regard f as an analytic function on $\mathbb{C}^n$ which is independent of z_n.

For any smooth function ϕ on $\mathbb{C}^n$ that satisfies $\phi = 1$ on a neighbourhood of $z_n = 0$ and for any smooth function v on $\mathbb{C}^n$ the function

$$(1.6) \qquad g(z) = \phi(z) f(z) - z_n v(z)$$

is a smooth extension of f. We want g to be holomorphic, i.e. $\bar\partial g = 0$. This leads to the following $\bar\partial$ problem for v:

$$(1.7) \qquad \bar\partial v = \frac{f(z)}{z_n}\,\bar\partial\phi.$$

We will choose ϕ such that (1.7) has a solutions v that makes g an analytic extension of f which satisfies (1.5). Note that the right-hand side of (1.7) is a smooth 1-form because $\bar\partial\phi \equiv 0$ on a neighbourhood of $z_n = 0$.

We define ϕ as follows. We will write $z = (z',z_n)$. Let $\psi(w)$ be a smooth function on $\mathbb{C}$, $0 \leqslant \psi \leqslant 1$, such that ψ has support in $\overline{B(0,1)}$ and $\psi = 1$ on $B(0,\frac{1}{2})$. We put

$$\phi(z',z_n) = \psi(\|z'\|^{\alpha/\beta} z_n) \quad \text{for} \quad \|z'\| \geqslant 1$$

and we extend ϕ smoothly to $\mathbb{C}^n$ such that $\phi = 1$ in a neighbourhood of $z_n = 0$, $0 \leqslant \phi \leqslant 1$ and $\phi = 0$ for $|z_n| > 2$.
Note that

$$(1.8) \qquad |\bar\partial\phi| = O(\|z'\|^{\alpha/\beta}), \qquad \|z'\| \to \infty$$

Let $z = (z',z_n) \in \operatorname{supp} \bar\partial\phi$, $\|z'\| > 1$. Then $\frac{1}{2}\|z'\|^{-\alpha/\beta} \leqslant |z_n| \leqslant \|z'\|^{-\alpha/\beta}$, hence

$$\left|\rho(z',z_n) - \rho(z',0)\right| \leqslant A(\|z\| + \|z'\|)^{\alpha}|z_n|^{\beta} + B \leqslant 3^{\alpha}A + B, \qquad \|z'\| > 1.$$

It is clear that $|\rho(z',z_n) - \rho(z',0)|$ is bounded on the intersection of $\|z'\| \leqslant 1$ and the support of $\bar\partial\phi$, hence

$$(1.9) \qquad |\rho(z',z_n) - \rho(z',0)| \text{ is bounded on the support of } \bar\partial\phi.$$

We obtain from (1.8) and (1.9):

$$\int_{\mathbb{C}^n} \left|\frac{f(z)}{z_n}\,\bar\partial\phi\right|^2 (1+\|z\|^2)^{-\alpha/\beta}\, e^{-\rho(z)} d\lambda = \int_{\text{support }\bar\partial\phi} \ldots\ldots\ldots$$

$$\leqslant D \int_{\text{support }\bar\partial\phi} \left|\frac{f(z)}{z_n}\right|^2 (1+\|z'\|^2)^{\alpha/\beta}(1+\|z'\|^2)^{-\alpha/\beta}\, e^{-\rho(z')} d\lambda$$

$$= D \int_{\Sigma} |f(z')|^2\, e^{-\rho(z')} d\sigma \int_{\frac{1}{2}\|z'\|^{-\alpha/\beta} \leqslant |z_n| \leqslant \|z'\|^{-\alpha/\beta}} \frac{dx_n dy_n}{|z_n|^2}$$

$$= D\, 2\pi \log 2 \int_{\Sigma} |f(z')|^2\, e^{-\rho(z')} d\sigma .$$

We now apply Hörmander's theorem to obtain a solution v of (1.7) that satisfies:

$$\int_{\mathbb{C}^n} |v|^2 (1+\|z\|^2)^{-2} (1+\|z\|^2)^{-\alpha/\beta}\, e^{-\rho(z)} d\lambda$$

$$\leqslant D\, 2\pi \log 2 \int_{\Sigma} |f(z')|^2\, e^{-\rho(z')} d\sigma .$$

Combining this with (1.4), (1.6) and the fact that $|\rho(z',z_n) - \rho(z',0)|$ is bounded on the support of ϕ, (1.5) follows.

Note that the iteration is possible because $\rho(z) + c\log(1+\|z\|^2)$ is plurisubharmonic and satisfies (1.3) (perhaps with different constants A and B). ◼

<u>Lemma 3.</u> If $\psi \in M(n)$ is Hölder continuous with exponent η on the unit sphere, then ψ satisfies (1.3). One can take $\alpha = 1-\eta$, $\beta = \eta$.

<u>Proof</u>: Immediate. ◻

For any positive continuous function ϕ on $\mathbb{C}^n$ we introduce the space

$$E_\phi = \{f : f \text{ is entire in } \mathbb{C}^n \text{ and } \sup_{z\in\mathbb{C}^n}\{|f(z)|\phi(z)^{-1}\} < \infty\}.$$

For $f \in E_\phi$ we put

$$\|f\|_\phi = \sup_{z\in\mathbb{C}^n}\{|f(z)|\phi(z)^{-1}\}.$$

This turns E_ϕ into a Banach space. It is clear that for $\psi \preccurlyeq \phi$ the identity operator

$$I_{\psi,\phi} : E_\psi \to E_\phi$$

is bounded.

Next we come to the proof of the main result as stated in the introduction. We may assume $n > 1$ in view of example a) at the beginning of this section. Let $u \in M(n)$ have Hölder exponent η. We consider $u|_S$, the restriction to the unit sphere. We associate to $u|_S$ a sequence of continuous function u_m with the properties of lemma 1. The functions u_m are extended to $\mathbb{C}^n$ positively 1-homogeneously:

$$u_m(rz) := ru_m(z), \qquad r > 0, \qquad \|z\| = 1.$$

We set

$$(1.10) \quad p = (2 + {}^{\eta}/_{1-\eta})\frac{n-1}{2} + 1 + ({}^{1}/_{\eta} - 1)^{n}/_{4},$$

and put

$$H(z) := (1 + \|z\|^2)^p \, e^{u(z)}$$

and

$$v_m(z) = (1+\|z\|^2)^p\, e^{u_m(z)}.$$

<u>Assertion</u>. The operator $I_{v_m,H}$ is <u>not</u> surjective for any m.

<u>Proof</u>: Fixing m, there is a point $z_o \in \mathbb{C}^n$, $z_o \neq 0$, such that $u(z_o) > u_m(z_o)$. We will show the existence of a function $g \in E_H$ with regularized indicator h_g such that $h_g(z_o) = u(z_o)$ and $h_g \leqslant u$. This function g is clearly not in E_{v_m} (look at the points rz_o).

Let $\tau(w)$ be the restriction of u to the complex line

$$\Sigma = \{wz_o : w \in \mathbb{C}\}.$$

Then τ is a positively 1-homogeneous subharmonic function of one variable. Consequently τ is the support function for a compact convex set D in $\mathbb{C}$, in fact we have

$$D = \bigcap_{|\lambda|=1} \{w : \operatorname{Re} \lambda.\bar{w} \leqslant \tau(\lambda)\},$$

$$\tau(z) = \operatorname*{Max}_{w\in D} \operatorname{Re} z.\bar{w},$$

cf. [8]. Let $A \in D$ be such that $\tau(1) = \operatorname{Re} A$. Then $f(w) = e^{\bar{A}w}$ satisfies

$$|f(w)| \leqslant e^{\tau(w)} \quad \text{and} \quad h_f(1) = \tau(1) = u(z_o).$$

We put

$$\rho(z) = 2(u(z) + \log(1+\|z\|^2)).$$

The function ρ has Hölder exponent η on the unit sphere and (by lemma 3) satisfies (1.3) with $\alpha = 1-\eta$, $\beta = \eta$.
We observe that

$$\int_{\Sigma} |f|^2 e^{-\rho} < \infty.$$

Now we can apply theorem 1 and obtain an analytic function g with

$$g|_{\Sigma} = f$$

and

$$(1.11) \quad \int_{\mathbb{C}^n} |g|^2 e^{-\rho(z)} (1 + \|z\|^2)^{-q} d\lambda < \infty.$$

where $q = (2 + \frac{1-\eta}{\eta})(n-1)$.
Let c_n denote the volume of the unit ball in $\mathbb{C}^n$. Using the mean value property and the Schwarz inequality we infer from (1.11):

$$|g(z)| \leqslant \frac{1}{c_n r^n} \int_{B(z,r)} |g| d\lambda$$

$$(1.12) \quad \leqslant \frac{1}{c_n r^n} \operatorname{Max}_{\zeta\in B(z,r)} \{e^{\frac{1}{2}\rho(\zeta)}(1+\|\zeta\|^2)^{q/2}\} \int_{B(z,r)} |g| e^{-\frac{1}{2}\rho(\zeta)}(1+\|\zeta\|^2)^{-q/2} d\lambda$$

$$\leqslant \frac{1}{\sqrt{c_n r^n}} \operatorname{Max}_{\zeta\in B(z,r)} \{e^{\frac{1}{2}\rho(\zeta)}(1+\|\zeta\|^2)^{q/2}\} \Big(\int_{\mathbb{C}^n} |g|^2 e^{-\rho(\zeta)}(1+\|\zeta\|^2)^{-q} d\lambda \Big)^{\frac{1}{2}}$$

Finally we choose $r = \|z\|^{1-1/\eta}$. Then the Max in (1.12) can be estimated in terms of the value in z:

$$\operatorname{Max}_{\zeta\in B(z,\|z\|^{1-1/\eta})} e^{\frac{1}{2}\rho(\zeta)}(1 + \|\zeta\|^2)^{q/2} \leqslant M.\ e^{u(z)}(1 + \|z\|^2)^{q/2+1},$$

by use of lemma 3.

Combining this with (1.12), (1.11) and (1.10) we obtain that $g \in E_H$ and the proof is complete. □

<u>Proof of the main result</u>. We identify E_{v_m} with its image under $I_{v_m,H}$. Because E_{v_m} does not equal E_H, it is of the first category in E_H. This is the open mapping theorem as stated in [9] and follows from the proof of the usual open mapping theorem. We infer that

$$E_H \setminus \bigcup_m E_{v_m} \neq \emptyset .$$

The functions in $E_H \setminus \bigcup_m E_{v_m}$ have regularized indicator u. Indeed, if $f \in E_H$ has indicator $\phi \leqslant u$, $\phi \neq u$, then there exists by lemma 1 a continuous function u_m and a positive number ε such that

$$\phi(z) \leqslant u_m(z) \leqslant u(z)$$

and

$$\phi(z) < u_m(z) - \varepsilon$$

on the closure K of $\{z \in S : u_m(z) < u(z)\}$ in S. For z with $u_m(z) = u(z)$, one has

$$|f(z)|v_m(z)^{-1} \leqslant \|f\|_H .$$

On the other hand, for z with $u_m(z) < u(z)$, note that $z = rw$, $w \in K$. Then lemma 2 applied with $v(z) = u_m(z) - \varepsilon\|z\|$, K, $\frac{1}{2}\varepsilon$, gives

$$|f(z)| < Ce^{u_m(z) - \frac{1}{2}\varepsilon\|z\|}, \quad z = rw, \quad w \in K.$$

We conclude that $f \in E_{v_m}$.

Now take $f \in E_H \setminus \bigcup_m E_{v_m}$. This f has indicator u and

$$|f(z)| \leq \|f\|_H (1 + \|z\|^2)^p e^{u(z)}.$$

This proves the main result. □

2. AN APPLICATION: PALEY-WIENER FUNCTIONS CAN HAVE IRREGULAR GROWTH.

In this section we will show how our main result leads rather quickly to Vauthier's theorem mentioned in the introduction.

First we explain the notion regular growth. Let f be an entire function of exponential type on $\mathbb{C}^n$ with indicator h. For $z \in \mathbb{C}^n$, r, $\delta > 0$ we introduce

$$I_f^r(z,\delta) = \frac{1}{\operatorname{Vol} B(0,\delta r)} \int_{B(rz,\delta r)} \frac{\log|f(z')|}{r} d\lambda(z') .$$

<u>Definition</u>. A function f of exponential type on $\mathbb{C}^n$ is said to have regular growth in the "direction" $z \in \mathbb{C}^n \setminus \{0\}$ if for every ε, $\delta_o > 0$ there exists δ with $0 < \delta < \delta_o$ and $R > 0$ such that for $r > R$

$$|I_f^r(z,\delta) - h(z)| < \varepsilon.$$

It is said to have regular growth on a set D if it has regular growth in all directions $z \in D$ $(z \neq 0)$.

Regular growth of a function is related to regularity of the distribution of its zero set, cf. [1,3]. It is a well-known fact [2] that Paley-Wiener functions of one variable have regular growth. This can be generalized as follows.
Recall that Paley-Wiener functions are complex Fourier transforms of compactly supported distributions on $\mathbb{R}^n$. If the convex hull of the support of this distribution is a polyhedron, then the Fourier transform has regular growth, cf. [11]. This is true, basically because in this case the indicator is almost

everywhere pluriharmonic.

To obtain a Paley-Wiener function of irregular growth Vauthier shows the following. An entire function of exponential type can be perturbed a little, resulting in a function with the same indicator but without regular growth, if the indicator is strictly plurisubharmonic on a domain in $\mathbb{C}^n$.

Next he spends a lot of effort to obtain a Paley-Wiener function with an appropriate indicator. The latter part is simplified in the sequel. For convenience, we will work in $\mathbb{C}^2$.

Lemma 4. Let f and g be nonnegative subharmonic functions on a domain $\Omega \subset \mathbb{C}$ and let $p > 1$. Then the function

$$h(z) = (f(z)^p + g(z)^p)^{1/p}$$

is subharmonic on Ω.

Proof: Apply Jensen's inequality with the convex function $(x_1^p + x_2^p)^{1/p}$ to verify the mean value inequality for h. □

Remark. In a point z where f, g and h are C^2 one can calculate Δh explicitly. We need this only for $p = 2$:

$$\Delta h = \Delta(f^2+g^2)^{\frac{1}{2}} = \frac{f\Delta f+g\Delta g}{(f^2+g^2)^{\frac{1}{2}}} + \frac{(f_x g-g_x f)^2 + (f_y g-g_y f)^2}{(f^2+g^2)^{3/2}} . \tag{2.1}$$

This shows that h tends to be "much more" subharmonic than f or g, which is exploited in the following

Proposition 1. The function

$$h(z_1,z_2) = \{(\mathrm{Im}(z_1^2+z_2^2)^{\frac{1}{2}})^2 + (\mathrm{Im}\, z_1)^2 + (\mathrm{Im}\, z_2)^2\}^{\frac{1}{2}}$$

is in $M(2)$. It is strictly plurisubharmonic on the open set

$$\Omega = \mathbb{C}^2 \setminus (\{(z_1,z_2) : z_1\bar{z}_2 = z_2\bar{z}_1\} \cup \{(z_1,z_2) : z_1^2 + z_2^2 = 0\}).$$

Proof: For a point $z \in \mathbb{C}^2$ let Σ be any complex line through z:

$$\Sigma = \{(z_1+\lambda_1 w,\ z_2+\lambda_2 w) : w \in \mathbb{C}\}, \qquad [\lambda_1 : \lambda_2] \in \mathbb{P}^1(\mathbb{C}).$$

The function $f(z) = |\mathrm{Im}\sqrt{z_1^2+z_2^2}\,|$ and the convex function $g(z) = \|\mathrm{Im}\, z\|$ are in $M(2)$, hence $f|_\Sigma$ and $g|_\Sigma$ are subharmonic. It thus follows from lemma 4 that $h|_\Sigma$ is subharmonic, also h is Hölder continuous on the unit sphere so we infer that $h \in M(2)$. If $z \in \Omega$, h is C^∞ at z and it can be shown that $\Delta h|_\Sigma$ is positive at the point $w = 0$ for every choice of $[\lambda_1 : \lambda_2] \in \mathbb{P}^1(\mathbb{C})$. Indeed, for a line Σ with $[\lambda_1 : \lambda_2] \neq [\mathrm{Im}\, z_1 : \mathrm{Im}\, z_2]$ it follows from (2.1) that $g|_\Sigma$ has positive Laplacian at $w = 0$. In the remaining direction the result follows also from (2.1) by an explicit calculation which we omit here. ∎

The main result furnishes a Paley-Wiener function f whose indicator equals the function h of proposition 1. A perturbation of f will give a function of irregular growth. For completeness we shall give a short sketch, following Vauthier [10] part C.

Theorem 2 (Vauthier). On $\mathbb{C}^2$ there exist entire functions of exponential type, which are in L^2 on the real subspace, but which are not of regular growth.

Proof: Let z_o be a point in

$$\Omega = \mathbb{C}^2 \setminus (\{z : z_1\bar{z}_2 = z_2\bar{z}_1\} \cup \{z : z_1^2+z_2^2 = 0\}).$$

Let $B_{\frac{1}{2}}$, B_1 and B_2 be balls with centre z_o and radius $\frac{1}{2}r$, r and $2r$ respectively, where r is chosen such that $B_2 \subset\subset \Omega$. Choose numbers t_n such that

$$t_n \uparrow \infty, \quad t_n > 0, \quad t_nB_2 \cap t_{n+1}B_2 = \emptyset.$$

Later on the choice of z_o and t_n will be made more precise. We can find a C^∞ function ϕ with the following properties:

> $0 \leq \phi \leq 1$, $\quad \phi = 1$ on $\bigcup_n t_nB_1$,
> the support of ϕ is contained in $\bigcup_n t_nB_2$
> and for small $\delta > 0$ the function $(1-\delta\phi)h$ is plurisubharmonic.

The construction of ϕ goes as follows. We may assume that $t_1 = 1$. Start with a C^∞ function ϕ_o with support in B_2 and with $\phi_o = 1$ on B_1. Define ϕ by

$$\phi(z) = \phi_o(z/t_n) \quad \text{for } z \in t_nB_2, \quad n = 1,2,\ldots \text{ and}$$
$$\phi(z) = 0 \quad \text{elsewhere.}$$

As a consequence of the chain rule and the homogeneity of h we obtain that for an arbitrary second order derivative D_iD_j and for any $\delta > 0$

$$D_iD_j[(1-\delta\phi)h](z) = \frac{1}{t_n}[D_iD_j(1-\delta\phi)h](z/t_n) \quad \text{for} \quad z \in t_nB_2$$

and $\qquad = 0$ elsewhere.

Strict plurisubharmonicity of h on B_2 completes the argument. By a similar construction there also exists a C^∞ function ψ having the properties

$$0 \leqslant \psi \leqslant 1, \quad \|\bar{\partial}\psi\| \leqslant 1, \quad \psi = 1 \text{ outside } t_{2p}B_2 \text{ and for}$$
$$p > p_o, \quad \psi = 0 \text{ on } t_{2p}B_1,$$

one uses that the first order derivatives of ψ are $O(1/\|z\|)$ for $\|z\| \to \infty$.

We now set up a $\bar{\partial}$-problem:

(2.2) $\quad \bar{\partial}u = \bar{\partial}\psi f,$

where f is a Paley-Wiener function with indicator h satisfying the inequality

$$|f(z)| < (1 + \|z\|^2)^p e^{-h(z)},$$

whose existence follows from the main result.

We take as weight function

$$\rho = 2(1-\delta\phi)h + (p+3)\log(1 + \|z\|^2).$$

Since $\phi = 0$ on the support of $\bar{\partial}\psi$, we have the estimate

$$\int_{\mathbb{C}^2} \|\bar{\partial}\psi f\|^2 e^{-\rho} d\lambda \leqslant \int_{\operatorname{supp}\bar{\partial}\psi} |f|^2 e^{-\rho} d\lambda$$

$$\leqslant \int_{\mathbb{C}^2} |f|^2 e^{-(2h + (p+3)\log(1 + \|z\|^2))} d\lambda < \infty.$$

Thus, by the cited theorem of Hörmander ([4] theorem 4.4.2) we can find a solution u of (2.2) that satisfies

$$\int_{\mathbb{C}^2} |u|^2 e^{-\rho - 2\log(1+\|z\|^2)} d\lambda < \infty. \tag{2.3}$$

We put

$$g = \psi f - u.$$

As in the proof of the main result we obtain from the L^2 estimate which we have for g that g is a Paley-Wiener function. Because u is holomorphic outside the support of $\bar{\partial}\psi$ we similarly obtain from (2.3)

$$|u(z)| < C(1+\|z\|^2)^q e^{(1-\delta)h(z)} \quad \text{on } t_p B_{\frac{1}{2}}, \quad p > p_o.$$

We may assume that z_o and the sequence t_n were chosen such that

$$\lim_{n\to\infty} \frac{\log|f(t_n z_o)|}{t_n} > h(z_o)(1-\tfrac{1}{2}\delta).$$

Then it is readily seen that

$$h_g(z_o) \geq h(z_o)(1-\tfrac{1}{2}\delta).$$

On the other hand $z \in t_{2p} B_{\frac{1}{2}}$ implies

$$|g(z)| = |u(z)| < C(1+\|z\|^2)^q e^{(1-\delta)h(z)}.$$

We conclude that g has no regular growth.

Although the function g might not be in L^2 on the real subspace, g multiplied with a suitable power of

$$\frac{\sin \varepsilon z_1}{z_1} \cdot \frac{\sin \varepsilon z_2}{z_2}$$

will be. This has no consequences for the irregular growth, but the indicator changes somewhat. ∎

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A representation of mixed derivatives with an application to the edge-of-the-wedge theorem

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ABSTRACT

The authors prove a lemma which expresses the mixed derivatives of a function in $\mathbb{R}^n$ in terms of its directional derivatives of the same order in an angle. The lemma is used to derive an edge-of-the-wedge theorem for $\mathbb{C}^n$ with an explicit domain of analytic continuation. Other applications will be given in subsequent papers.

1. INTRODUCTION

In this paper we present a convenient condition for real-analyticity of continuous functions in $\mathbb{R}^n$ (section 4). The result is used to prove a form of the edge-of-the-wedge theorem which includes a description of a minimal domain of analytic continuation (section 5). Our method of proof is related to that of F.E. Browder [3], but somewhat simpler. In a subsequent article [9] one of us will prove a refinement of Helgason's support theorem for Radon transforms on $\mathbb{R}^n$ [5]. A later paper [10] will explore the relation between 1-dimensional analyticity of a function in $\mathbb{C}^n$ on a family of complex lines through the origin and n-dimensional holomorphy of the function on a neighborhood of the origin, cf. Forelli's paper [4].

Although the above results may seem unrelated, our proofs are all based on the following lemma, by which mixed derivatives can be estimated in terms of directional derivatives in an angle.

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MAIN LEMMA. *For every open subset Ω of the unit sphere S^{n-1} in $\mathbb{R}^n$, there exist a constant $B=B_\Omega$ and a family of integrable functions $\{g_\alpha\}$ with the following properties:*

$$(1.1)\qquad \int_\Omega (\omega_1 D_1+\ldots+\omega_n D_n)^{|\alpha|} g_\alpha(\omega)d\sigma=\frac{|\alpha|!}{\alpha!}D^\alpha$$

for all n-tuples $\alpha=(\alpha_1,\ldots,\alpha_n)$ of non-negative integers;

$$(1.2)\qquad \int_\Omega |g_\alpha(\omega)|d\sigma\le(B+\varepsilon)^{|\alpha|}$$

for every $\varepsilon>0$ and all α of sufficiently large height $|\alpha|$.

Here we have used the standard notations

$$D_j=\partial/\partial x_j,\ D^\alpha=D_1^{\alpha_1}\ldots D_n^{\alpha_n},$$
$$\alpha!=\alpha_1!\ldots\alpha_n!,\ |\alpha|=\alpha_1+\ldots+\alpha_n$$

while $d\sigma$ denotes the area-element of S^{n-1}.

The main lemma will be derived from the following special case for $n=2$:

LEMMA 1. *For every subinterval $(0,\lambda)$ of $(0,2\pi)$ there exist a constant $B=B_\lambda$ and a family of integrable functions $\{g_{pq}\}$ with the following properties:*

$$(1.1')\qquad \int_0^\lambda \left\{(\cos\theta)\frac{\partial}{\partial x_1}+(\sin\theta)\frac{\partial}{\partial x_2}\right\}^{p+q} g_{pq}(\theta)d\theta=\binom{p+q}{p}\frac{\partial^{p+q}}{\partial x_1^p\partial x_2^q}$$

for all non-negative integers p, q;

$$(1.2')\qquad \int_0^\lambda |g_{pq}(\theta)|d\theta\le(B+\varepsilon)^{p+q}$$

for every $\varepsilon>0$ and all p, q with sufficiently large sum $p+q$.

It is easy to see that the constants B must be ≥ 1, cf. (2.1) below. The minimal constants B_λ will form a decreasing function of λ. We prove lemma 1 with

$$(1.3)\qquad B_\lambda=\frac{3+2\sqrt{2}}{\sin\lambda}\ \text{for}\ 0<\lambda\le\pi/4,\ B_\lambda=B_{\pi/4}\ \text{for}\ \lambda>\pi/4.$$

Under translation of the interval $(0,\lambda)$, the constant B is at most doubled (lemma 2). Our constants B are not best possible; it would be of interest for the applications to have minimal values. We do have a sharp result for the case where Ω is the whole sphere (or a hemisphere): in that case B can be taken equal to 1 if we suppress the factor $|\alpha|!/\alpha!$ on the right-hand side of (1.1), see [10].

2. PROOF OF LEMMA 1

By the binomial theorem, equation (1.1′) is equivalent to the set of conditions

$$(2.1)\qquad \int_0^\lambda (\cos\theta)^{p+q-k}(\sin\theta)^k g_{pq}(\theta)d\theta=\delta_{kq},\ k=0,\ldots,p+q.$$

Taking $g_{pq}(\theta)=0$ for $\theta>\pi/4$ if necessary, we may assume that $\lambda\leq\pi/4$ so that

$$\tau=\tan\lambda\leq 1.$$

We now substitute $\theta=\arctan\tau s$. Then (2.1) takes the form

$$\tau^{k+1}\int_0^1 s^k\frac{g_{pq}(\arctan\tau s)}{(1+\tau^2s^2)^{\frac{1}{2}(p+q)+1}}\,ds=\delta_{kq},\ k=0,\dots,p+q.$$

Since the right-hand side is zero for $k\neq q$, we may replace τ^{k+1} by τ^{q+1}. Setting

$$h_{pq}(s)=\frac{\tau^{q+1}g_{pq}(\arctan\tau s)}{(1+\tau^2s^2)^{\frac{1}{2}(p+q)+1}}, \tag{2.2}$$

our system of equations reduces to

$$\int_0^1 s^k h_{pq}(s)ds=\delta_{kq},\ k=0,\dots,p+q. \tag{2.3}$$

It is convenient to introduce the linear span S_q of the powers

$$s^0,\dots,s^{q-1},s^{q+1},\dots,s^{p+q}\text{ in }L^2(0,1).$$

Equation (2.3) requires that $h_{pq}\perp S_q$ and furthermore that $\int_0^1 s^q h_{pq}=1$. We take for h_{pq} the unique element of $L^2(0,1)$ that satisfies these conditions and has minimal L^2 norm. Then h_{pq} must be a constant c times the difference between s^q and its orthogonal projection Q on S_q (hence h_{pq} is a polynomial of degree $\leq p+q$). The condition

$$1=\int_0^1 s^q h_{pq}=\int_0^1 (s^q-Q)c(s^q-Q)$$

gives the value of c and hence

$$\|h_{pq}\|_2=\{\text{distance }(s^q,S_q)\}^{-1}. \tag{2.4}$$

The above distance may be estimated rather accurately with the aid of functional analysis and Laplace integrals. However, this L^2 distance has been computed exactly by Müntz [6] and Szász [8]. Their formula gives

$$\text{distance }(s^q,S_q)=(2q+1)^{-\frac{1}{2}}\prod_{k=0,\,k\neq q}^{p+q}\frac{|k-q|}{k+q+1}$$
$$=(2q+1)^{\frac{1}{2}}\frac{q!p!q!}{(p+2q+1)!}.$$

For fixed $p+q=m$ the reciprocal $\|h_{pq}\|_2$ of the distance is maximal if $2q^2\approx m^2$, so that by a short calculation

$$\|h_{pq}\|_2\leq C^m\text{ for any }C>3+2\sqrt{2}, \tag{2.5}$$

provided $m=p+q>m_0$.

Defining g_{pq} according to (2.2), we will have (2.1) and hence (1.1′). Finally, Schwarz's inequality gives

$$
(2.6)\quad \begin{cases}
\int_0^\lambda |g_{pq}(\theta)|d\theta = \int_0^1 |g_{pq}(\arctan \tau s)| \dfrac{\tau ds}{1+\tau^2 s^2} \\
\le \|h_{pq}\|_2 \tau^{-q}\{ \int_0^1 (1+\tau^2 s^2)^{p+q} ds\}^{\frac{1}{2}} \\
\le \|h_{pq}\|_2 (1+1/\tau^2)^{\frac{1}{2}(p+q)} \le (C/\sin\lambda)^{p+q},
\end{cases}
$$

provided $p+q$ is sufficiently large, cf. (2.5). Formula (2.6) completes the proof of lemma 1. It gives (1.2′) with $B=B_\lambda$ as in (1.3); in the case $\lambda > \pi/4$ we may of course take $B_\lambda = B_{\pi/4}$.

3. DERIVATION OF THE MAIN LEMMA

We first investigate what happens in lemma 1 under rotation.

LEMMA 2. *For every interval $(a, a+\lambda)$ there exist a constant $\tilde{B}=\tilde{B}_{a\lambda}$ and a family of integrable functions $\{\tilde{g}_{pq}\}$ with the following properties:*

$$
(3.1)\quad \int_a^{a+\lambda} \{(\cos\theta)D_1 + (\sin\theta)D_2\}^{p+q}\tilde{g}_{pq}(\theta)d\theta = \binom{p+q}{p} D_1^p D_2^q
$$

for all non-negative integers p, q;

$$
(3.2)\quad \int_a^{a+\lambda} |\tilde{g}_{pq}(\theta)|d\theta \le (\tilde{B}+\varepsilon)^{p+q}
$$

for every $\varepsilon>0$ and all p, q with sufficiently large sum $p+q$.
In fact, we may take $\tilde{B}=2B$ where B is as in lemma 1.

PROOF. Substituting

$$
D_1 = \tilde{D}_1 \cos a - \tilde{D}_2 \sin a,\ D_2 = \tilde{D}_1 \sin a + \tilde{D}_2 \cos a
$$

and setting $\theta - a = t$, equation (3.1) becomes

$$
(3.3)\quad \begin{cases}
\int_0^\lambda (\tilde{D}_1 \cos t + \tilde{D}_2 \sin t)^{p+q}\tilde{g}_{pq}(a+t)dt = \\
\binom{p+q}{p} \sum\limits_{\substack{0\le j\le p\\ 0\le k\le q}} \binom{p}{j}\binom{q}{k} (\cos a)^{j+q-k}(-1)^{p-j}(\sin a)^{p-j+k}\tilde{D}_1^{j+k}\tilde{D}_2^{p+q-j-k}.
\end{cases}
$$

Now, using functions g as in lemma 1,

$$
\binom{p+q}{j+k}\tilde{D}_1^{j+k}\tilde{D}_2^{p+q-j-k} = \int_0^\lambda (\tilde{D}_1 \cos t + \tilde{D}_2 \sin t)^{p+q} g_{j+k,p+q-j-k}(t)dt.
$$

It follows that we may satisfy (3.1) by defining

$\tilde{g}_{pq}(a+t)=$

$$\binom{p+q}{p}\sum_{j,k}\frac{\binom{p}{j}\binom{q}{k}}{\binom{p+q}{j+k}}(\cos a)^{j+q-k}(-1)^{p-j}(\sin a)^{p-j+k}g_{j+k,p+q-j-k}(t).$$

With this choice of $\tilde{g}$, the inequalities (1.2′) give

$$\begin{aligned}\|\tilde{g}_{pq}\|_1 &\le \sum_{j,k}\frac{(j+k)!}{j!k!}\,\frac{(p+q-j-k)!}{(p-j)!(q-k)!}\,\|g_{j+k,p+q-j-k}\|_1\\ &\le \sum_{j,k}2^{j+k}2^{p+q-j-k}(B+\varepsilon)^{p+q}\le C^{p+q}\end{aligned}$$

for any constant $C>2(B+\varepsilon)$ and all large $p+q$. □

PROOF OF THE MAIN LEMMA. The proof is by induction with respect to the dimension. For $n=2$ the result follows from lemma 2, because Ω must then contain an interval $(a,a+\lambda)$ and we can take $g_\alpha=\tilde{g}_{pq}$ on that interval and $g_\alpha=0$ outside. We thus take $n\ge 3$ and indicate the step from $n-1$ to n. Accordingly, let Ω be an open set in S^{n-1}. We represent the points ω of S^{n-1} as follows:

$$\omega_1=\cos\theta,\ (\omega_2,\dots,\omega_n)=(\sin\theta)\omega'$$

where $0\le\theta\le\pi$ and $\omega'\in S^{n-2}$. The area element of S^{n-1} then becomes

$$d\sigma=d\theta\,(\sin\theta)^{n-2}d\sigma', \tag{3.4}$$

where $d\sigma'$ is the area element on S^{n-2}.

Next choose open subsets Ω_1 and Ω' of $(0,\pi)$ and S^{n-2} such that the points ω corresponding to $\Omega_1\times\Omega'$ form an open subset Ω_0 of Ω. The constants and functions which the main lemma (for dimensions $\le n-1$) associates with Ω_1 and Ω' will be denoted by $\tilde{B}$, $\tilde{g}$ and B^*, g^*, respectively. In order to establish (1.1), we will use functions $g_\alpha(\omega)$ that are equal to zero outside Ω_0. Representing the vector $D=(D_1,\dots,D_n)$ as (D_1,D'), the inner product $\omega\cdot D$ becomes

$$\omega\cdot D=(\cos\theta)D_1+(\sin\theta)\omega'\cdot D'.$$

The desired formula (1.1) may thus be written in the equivalent form

$$\left\{\begin{aligned}&\int_{\Omega_1\times\Omega'}\{(\cos\theta)D_1+(\sin\theta)\omega'\cdot D'\}^{\alpha_1+|\alpha'|}g_{\alpha_1,\alpha'}(\omega)(\sin\theta)^{n-2}d\theta d\sigma'\\ &=\frac{|\alpha|!}{\alpha_1!|\alpha'|!}D_1^{\alpha_1}\frac{|\alpha'|!}{\alpha'!}(D')^{\alpha'}.\end{aligned}\right. \tag{3.5}$$

Now by the result for $n=2$, cf. lemma 2,

$$(3.6)\quad \begin{cases} I(\omega') \stackrel{\text{def}}{=} \int_{\Omega_1} \{(\cos\theta)D_1+(\sin\theta)\omega'\cdot D'\}^{\alpha_1+|\alpha'|}\tilde{g}_{\alpha_1|\alpha'|}(\theta)d\theta \\ = \dfrac{|\alpha|!}{\alpha_1!|\alpha'|!} D_1^{\alpha_1}(\omega'\cdot D')^{|\alpha'|}. \end{cases}$$

Multiplying (3.6) by $g_{\alpha'}^*(\omega')$ and integrating over Ω', we obtain

$$(3.7)\qquad \int_{\Omega'} I(\omega')g_{\alpha'}^*(\omega')d\sigma' = \frac{|\alpha|!}{\alpha_1!|\alpha'|!} D_1^{\alpha_1} \int_{\Omega'} (\omega'\cdot D')^{|\alpha'|}g_{\alpha'}^*(\omega')d\sigma'.$$

Thus by the main lemma for dimension $n-1$ and our choice of functions g^*, the right-hand side of (3.7) is precisely equal to the right-hand side of (3.5).

The conclusion is that we may satisfy (3.5) or (1.1) by defining

$$(3.8)\qquad g_\alpha(\omega)(\sin\theta)^{n-2} = \tilde{g}_{\alpha_1|\alpha'|}(\theta)g_{\alpha'}^*(\omega') \text{ for } \omega\in\Omega_0.$$

For the norm of $g_\alpha(\omega)$ we then obtain, cf. (3.4),

$$\int_\Omega |g_\alpha(\omega)|d\sigma = \int_{\Omega_0} = \int_{\Omega_1} |\tilde{g}_{\alpha_1|\alpha'|}(\theta)|d\theta \int_{\Omega'} |g_{\alpha'}^*(\omega')|d\sigma'.$$

By the induction hypothesis, the right-hand side is bounded by

$$(\tilde{B}+\varepsilon)^{|\alpha|}(B^*+\varepsilon)^{|\alpha'|}$$

provided $|\alpha'|$ is large. Since $B^*\geq 1$, the weaker inequality

$$\|g_\alpha\|_1 \leq \{(\tilde{B}+\varepsilon)(B^*+\varepsilon)\}^{|\alpha|}$$

will hold for all α of large height, even when $|\alpha'|$ is small. Suitably adjusting ε, it follows that we have (1.2) with $B=B_\Omega=\tilde{B}B^*$.

4. A SUFFICIENT CONDITION FOR REAL-ANALYTICITY OF CONTINUOUS FUNCTIONS

We start with a result for C^∞ functions.

PROPOSITION 1. *Let f be a C^∞ function on a domain D in $\mathbb{R}^n$ such that, for the fixed angle spanned by a given open subset Ω of S^{n-1} and at each point of D, all directional derivatives of order k are bounded by $C^k k!$ for all large k. Then f is real-analytic on D and f has a holomorphic extension to the neighborhood*

$$(4.1)\qquad U=D+\Delta(0,1/BC)=\{z\in\mathbb{C}^n,\ z=a+b,\ a\in D,\ |b_j|<1/BC\}$$

of D in $\mathbb{C}^n$, with $B=B_\Omega$ as in the main lemma.

PROOF. We estimate the mixed derivatives of f at $a\in D$ with the aid of the main lemma:

$$(4.2)\qquad \frac{|\alpha|!}{\alpha!}|D^\alpha f(a)| = \left|\int_\Omega (\omega\cdot D)^{|\alpha|}f(a)g_\alpha(\omega)d\sigma\right| \leq C^{|\alpha|}(B+\delta)^{|\alpha|}|\alpha|!$$

for every $\delta>0$ and all α of large height. It follows that the Taylor series

$$(4.3) \qquad \sum_{\alpha\geq 0} \frac{1}{\alpha!} D^{\alpha}f(a)(x-a)^{\alpha}$$

for $f(x)$ around the point a converges throughout the open set $\square(a)$ given by $|x_j-a_j|<1/BC$, $j=1,\dots,n$. The series will converge to f on $\square(a)\cap D$, because the difference between $f(x)$ and the partial sums of the series will tend to zero there. Thus f is real-analytic on D.

The complexified series (4.3), obtained by replacing x with z, will converge throughout the polydisc $\Delta(a, 1/BC)$, hence f has a holomorphic extension to the set U of (4.1).

Now we are ready for

THEOREM 1. *Let f be a continuous function on a domain D in $\mathbb{R}^n$ such that, for the fixed angle spanned by a given open subset Ω of S^{n-1} and at each point of D, all directional derivatives of order k exist and, for large k, are bounded by $C^k k!$. Then f is real-analytic on D and f has a holomorphic extension to the neighborhood U of D in $\mathbb{C}^n$ described in* (4.1).

PROOF. Let $\{\varphi_\varepsilon\}$ be a C^∞ approximate identity relative to convolution on $\mathbb{R}^n$ such that the support of $\varphi_\varepsilon\geq 0$ belongs to the ball $|x|\leq\varepsilon$ and $\int\varphi_\varepsilon=1$. It is enough to prove the desired result for every relatively compact subdomain D' of D, because the union of the corresponding neighborhoods

$$U'=D'+\Delta(0,1/BC)$$

will be U.

For given D' we take ε less than the distance between D' and the boundary of D and we form the convolutions

$$f_\varepsilon=f*\varphi_\varepsilon$$

on D'. Observe that $f_\varepsilon\to f$ on D' as $\varepsilon\to 0$. The functions f_ε will satisfy the conditions of proposition 1 for D'. Indeed, they are of class C^∞ and for large k, the directional derivatives $D^k_\omega f_\varepsilon$ in the given angle will satisfy the inequality

$$|D^k_\omega f_\varepsilon(a)|=|(D^k_\omega f*\varphi_\varepsilon)(a)|\leq C^k k!\int\varphi_\varepsilon=C^k k!$$

at each point $a\in D'$.

It thus follows from proposition 1 that the functions f_ε have a holomorphic extension to U'. Moreover, by (4.2) applied to f_ε, the extended functions f_ε form a bounded family on every relatively compact subdomain of U'. In other words, they form a normal family on U'. Any limit function of this family will be equal to f on D' and holomorphic on U'. We conclude that f is real-analytic on D' and has a holomorphic extension to U'.

REMARK 1. Observe that in theorem 1, it would be enough to require that f be a distribution on D for which the distributional directional derivatives satisfy the given conditions. On the other hand, one needs more than just real-analyticity of f on every line, cf. the example $x_1x_2/(x_1^2+x_2^2)$ for $\mathbb{R}^2$.

REMARK 2. There are related results on the real-analyticity of separately analytic functions, cf. F.E. Browder [2], Bochnak and Siciak [1]. Their work shows that the conditions for real-analyticity in theorem 1 are much more stringent than necessary. However, the point is that theorem 1 is easy to prove and just right for certain applications, among them the one below.

5. THE EDGE-OF-THE-WEDGE THEOREM

In this section we obtain a form of the edge-of-the-wedge theorem, in which we explicitly indicate a minimal set of analytic continuation. Our proof of the theorem is related to one by Browder, cf. [3]. The main difference with Browder's proof is that he made use of his theorem on real-analyticity of functions that are separately analytic, cf. [2], whereas we use the simpler theorem 1. For general information on the edge-of-the-wedge theorem one may consult Rudin [7].

In the following D is an arbitrary domain in $\mathbb{R}^n$, where we identify $\mathbb{R}^n$ with $\mathbb{R}^n+i0$ in $\mathbb{C}^n$. We let V denote a truncated open cone in (another) $\mathbb{R}^n$:

(5.1) $\quad V=\{t\Omega, \Omega\subset S^{n-1} \text{ open}, 0<t<R\}.$

With D and V we associate the following two open sets in $\mathbb{C}^n$:

(5.2) $\quad W^+=D+iV,\ W^-=D-iV.$

Although it is not necessary, it will be assumed that Ω is connected, so that W^+ and W^- are domains in $\mathbb{C}^n$. The sets W^+ and W^- need not intersect; they are "wedges", with common "edge" D. We finally introduce the basic set

(5.3) $\quad W=W^+\cup D\cup W^-.$

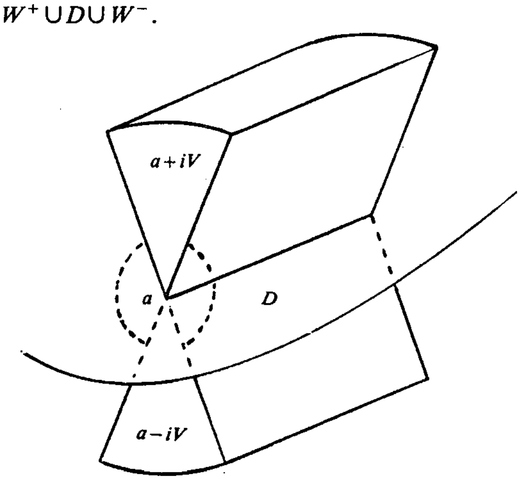

Fig. 1.

It is not easy to draw a picture even for $n=2$, but a little reflection will show that with V as in fig. 1, the set W does not contain a $\mathbb{C}^n$ neighborhood of any point $a \in D$. Indeed, the points $a+iy$ with $y \neq 0$ not in V or $-V$ are outside W. On the other hand, W does contain the intersection of a $\mathbb{C}^n$ neighborhood of a with the family of complex lines $\{z=a+s\omega,\ s \in \mathbb{C}\}$ where ω runs over Ω.

THEOREM 2. *Let D, V, W^+, W^- and W be as above. Then there exists an open neighborhood X of W in $\mathbb{C}^n$ such that every continuous function f on W which is holomorphic on W^+ and on W^- has a holomorphic extension to X. A minimal domain of analytic continuation is given by the union of W and the polydiscs $\Delta(a, r_a)$, where a runs over D and*

$$r_a = \frac{1}{B} \min\{\operatorname{dist}(a, \partial D), R\},$$

with $B=B_\Omega$ as in the main lemma.

PROOF. Let D_0 be an arbitrary relatively compact subdomain of D. We choose

$$0 < \varrho < \min\{\operatorname{dist}(D_0, \partial D), R\}$$

and define

$$V_0 = \{t\Omega, 0 < t < \varrho\}.$$

It will be sufficient to prove that every function f as in the theorem has an analytic continuation to the open set

$$U_0 = D_0 + \Delta(0, \varrho/B).$$

Indeed, the union of these sets is a neighborhood of D in $\mathbb{C}^n$ which contains the polydiscs $\Delta(a, r_a)$ (take small neighborhoods D_0 of a in $\mathbb{R}^n$ and take ϱ close to its upper bound).

With D_0 and Ω we associate the family of discs

$$\Delta_{a\omega}(\varrho) = \{z \in \mathbb{C}^n : z = a + s\omega,\ s \in \mathbb{C},\ |s| < \varrho\}$$

where a runs over D_0 and ω over Ω, while ϱ remains fixed. All these discs belong to the compact subset of W given by

$$\bar{D}_\varrho \pm i\bar{V}_0, \tag{5.4}$$

where D_ϱ stands for the ϱ-neighborhood of D_0 in $\mathbb{R}^n$.

Now let f be as in the theorem. We consider its restrictions to the discs $\Delta_{a\omega}(\varrho)$, writing

$$f_{a\omega}(s) = f(a + s\omega),\ |s| < \varrho.$$

These functions will be continuous for $|s| < \varrho$ and analytic off the real axis. Hence by application of Morera's theorem, they are analytic for $|s| < \varrho$. Since f is continuous on the compact set (5.4), $|f|$ is bounded by a constant M there.

Thus by the Cauchy-inequalities for the derivatives of $f_{a\omega}$ at the point $s=0$,

$$\left|D_\omega^k f(a)\right| = \left|\frac{\partial^k}{\partial s^k} f_{a\omega}(0)\right| \leq k!M/\varrho^k$$

for all $a \in D_0$ and all $\omega \in \Omega$.

It now follows from theorem 1 that the restriction of f to D_0 has a holomorphic extension f_0 to the open set $U_0 = D_0 + \Delta(0, \varrho/B)$ in $\mathbb{C}^n$. However, does this extension coincide with f on $U_0 \cap W$? The answer is affirmative for $U_0 \cap W^+$ because f_0 coincides with f on the discs $\Delta_{a\omega}(\varrho/B)$ (at the centers, the derivatives are the same), and the union of these discs contains an open subset of W^+. Similarly $f_0 = f$ on $U_0 \cap W^-$; by continuity, $f_0 = f$ also on $U_0 \cap D$.

We conclude that all functions f in the theorem can be extended analytically to an open set X in $\mathbb{C}^n$ which contains the union of W and the polydiscs $\Delta(a, r_a)$.

REMARK 3. A sharp estimate for the constant B_Ω in the main lemma would give information about the size of the common domain of analytic continuation X. The precise shape of X is unknown.

REMARK 4. From theorem 2 one usually derives various other forms of the edge-of-the-wedge theorem. In one strong version, the boundary values on the edge are only assumed to exist in distribution sense. Another version is a theorem on analytic continuation by reflection: Every holomorphic function f on W^+ whose imaginary part tends to zero as $y = \text{Im } z \to 0$ in V has a holomorphic extension to X; on W^-, the extension is given by the complex conjugate of $f(\bar{z})$. Cf. [7].

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A support theorem for Radon transforms on $\mathbb{R}^n$

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ABSTRACT

Let f be a rapidly decreasing continuous function on $\mathbb{R}^n$ and let $\hat{f}$ be its Radon transform. If $\hat{f}$ decreases fast enough and vanishes on a fairly large set, then it is proved that f has compact support. We also investigate the relation between the supports of f and $\hat{f}$ under these conditions. The results contain the well-known support theorem for Radon transforms on $\mathbb{R}^n$ as a special case.

1. INTRODUCTION AND RESULTS

Let f be a function on $\mathbb{R}^n$ which is integrable over every hyperplane. The Radon transform of f is the function $\hat{f}$ defined on the set of hyperplanes by $\hat{f}(H) = \int_H f$. It is convenient to parametrize the set of hyperplanes by their double cover $S^{n-1} \times \mathbb{R}$:

$$(\omega, t) \mapsto H = \{s \in \mathbb{R}^n : s \cdot \omega = t\}, \ (\omega, t) \in S^{n-1} \times \mathbb{R},$$

where S^{n-1} denotes the unit sphere in $\mathbb{R}^n$ and $s \cdot \omega = s_1\omega_1 + \ldots + s_n\omega_n$. We think of the Radon transform as defined on $S^{n-1} \times \mathbb{R}$ by

$$(1) \qquad \hat{f}(\omega, t) = \int_{s \cdot \omega = t} f(s)dm(s), \ (\omega, t) \in S^{n-1} \times \mathbb{R}.$$

In the present paper we investigate the relation between the support of f and that of $\hat{f}$. It is obvious that if f has its support in the closed ball $\bar{B}(0, R)$, the support of $\hat{f}$ is contained in $S^{n-1} \times [-R, R]$. The converse is not true in

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general, e.g. set $\mathbb{R}^2 = \mathbb{C}$ and take $f(z) = z^{-5}$ for $|z| \geq 1$, and extend f smoothly to $\mathbb{R}^2$, see [3], p. 18. In fact Zalcman recently gave an example of an entire function $f \not\equiv 0$ with the property that $|f|$ is integrable over every straight line L, while $\int_L f dm(s) = 0$, see [10].

However, if f is rapidly decreasing, i.e. for every $k \geq 0$

$$f(x) = \mathcal{O}(|x|^{-k}) \text{ for } |x| \to \infty,$$

the situation is better. We will prove the following theorems:

THEOREM 1. *Let f be a rapidly decreasing continuous function on $\mathbb{R}^n$ whose Radon transform $\hat{f}$ satisfies the growth condition*

$$(2) \qquad |\hat{f}(\omega, t)| \leq C_\omega e^{-\varepsilon_\omega t}, \ \varepsilon_\omega > 0, \ (\omega, t) \in S^{n-1} \times \mathbb{R}.$$

Suppose that for some open set $\Omega \subset S^{n-1}$ and some $R > 0$

$$\hat{f}(\omega, t) = 0, \ \omega \in \Omega, \ |t| > R.$$

Then f has compact support.

THEOREM 2. *Let f be a compactly supported continuous function on $\mathbb{R}^n$ and $\hat{f}$ its Radon transform. Then $f = 0$ on the set*

$$\{s \in \mathbb{R}^n : s \cdot \omega = t, \ (\omega, t) \notin \operatorname{supp} \hat{f}\}.$$

We will also give an *example* showing that a growth condition like (2) cannot be disposed of completely. Finally, observe that taking $\Omega = S^{n-1}$ one obtains an important special case:

SUPPORT THEOREM OF HELGASON [1, 2]. If f is a rapidly decreasing continuous function on $\mathbb{R}^n$ and $\operatorname{supp} \hat{f} \subset S^{n-1} \times [-R, R]$, then $\operatorname{supp} f \subset \bar{B}(0, R)$.

In the proof of theorem 1 we use the lemma on mixed derivatives from Korevaar-Wiegerinck [4] to show that the Fourier transform of f extends to an entire function of exponential type. Then the proof is completed with the aid of the Paley-Wiener theorem. Theorem 2 is based on a proof of the Plancherel-Polýa theorem in [8].

The proof of theorem 1 looks quite natural; nevertheless, the known proofs of the support theorem (Helgason [2], Ludwig [5], B. Weiss [7]) are all different.

For more information about the Radon transform and its applications we refer to Helgason's book [3].

I would like to thank Dr T. Koornwinder who introduced me to Radon transforms and made useful comments on an earlier draft of this paper.

2. DERIVATION OF THEOREM 1

An important tool in the proof of theorem 1 is the lemma on mixed derivatives from Korevaar-Wiegerinck [4]. It is recalled here:

MAIN LEMMA. *For every open subset Ω of the unit sphere S^{n-1} in $\mathbb{R}^n$, there exist a constant $B=B_\Omega$ and a family of integrable functions $\{g_\alpha\}$ with the following properties:*

$$(3') \qquad \int_\Omega (\omega_1 D_1 + \ldots + \omega_n D_n)^{|\alpha|} g_\alpha(\omega) d\sigma = \frac{|\alpha|!}{\alpha!} D^\alpha$$

for all n-tuples $\alpha = (\alpha_1, \ldots, \alpha_n)$ of non-negative integers;

$$(3'') \qquad \int_\Omega |g_\alpha(\omega)| d\sigma \leq (B+\varepsilon)^{|\alpha|}$$

for every $\varepsilon > 0$ and all α of sufficiently large height $|\alpha|$.

Here we have used the standard notations

$$D_j = \partial/\partial x_j, \; D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n},$$

$$\alpha! = \alpha_1! \ldots \alpha_n!, \; |\alpha| = \alpha_1 + \ldots + \alpha_n,$$

while $d\sigma$ denotes the area-element of S^{n-1}.

Next we give the

PROOF OF THEOREM 1. Since f is continuous and rapidly decreasing, we have in particular $|f(x)| \leq C(1+|x|)^{-n}$ and we conclude that f is in L^2 and that $|\hat{f}|$ is bounded by a constant M. Thus the Fourier transform $\mathscr{F}f$ is also in L^2 and using again that f is rapidly decreasing we find that $\mathscr{F}f$ is in C^∞.

Let ω be in S^{n-1}. The values of the Fourier transform $\mathscr{F}f$ on the line $\{\lambda\omega : \lambda \in \mathbb{R}\}$ through ω can be expressed very simply in terms of the Radon transform:

$$(4) \qquad (\mathscr{F}f)(\lambda\omega) = \int_{\mathbb{R}^n} f(s) e^{-i\lambda\omega \cdot s} ds = \int_{-\infty}^{\infty} \hat{f}(\omega, t) e^{-i\lambda t} dt.$$

The last equality follows from Fubini's theorem and the definition of the Radon transform (1).

By (2) the right-hand side of (4) represents (the restriction to the real line of) a holomorphic function of one variable on a neighborhood of the real axis in the complex line $\{\zeta\omega : \zeta = \lambda + i\mu \in \mathbb{C}\}$. For $\omega \in \Omega$ this function will be entire and of exponential type $\leq R$, because then the interval of integration reduces to $[-R, R]$. For these values of ω we estimate the directional derivatives of $\mathscr{F}f$ at the origin:

$$(5) \qquad \left| \frac{\partial^k \mathscr{F}f(\lambda\omega)}{\partial\lambda^k} \right|_{\lambda=0} = \left| \int_{-R}^{R} \hat{f}(\omega, t)(-it)^k dt \right| \leq 2MR^{k+1}.$$

We next consider the Taylor series for $\mathscr{F}f$ around the origin,

$$(6) \qquad \mathscr{F}f(x) \sim \sum_\alpha c_\alpha x^\alpha.$$

We estimate the coefficients with the aid of (5) and the main lemma:

$$||\alpha|!c_\alpha| = \left|\frac{|\alpha|!}{\alpha!} D^\alpha(\mathscr{F}f)(0)\right| = |\int_\Omega (\omega_1 D_1 + \ldots + \omega_n D_n)^{|\alpha|}(\mathscr{F}f)(0) g_\alpha(\omega) d\sigma|$$

$$\leqq 2MR^{|\alpha|+1}(B+\varepsilon)^{|\alpha|} \text{ if } |\alpha| \text{ is large,}$$

$$\leqq AE^{|\alpha|} \text{ with } E = R(B+\varepsilon) \text{ for all } \alpha.$$

We infer that the Taylor series in (6) defines an entire function F of exponential type on $\mathbb{C}^n$:

$$(7) \quad \begin{cases} |F(z)| = |\sum_\alpha c_\alpha z^\alpha| \leq \sum_\alpha AE^{|\alpha|} \dfrac{|z^\alpha|}{|\alpha|!} = A \sum_\alpha \dfrac{\alpha!}{|\alpha|!} \dfrac{(E|z_1|)^{\alpha_1}}{\alpha_1!} \cdots \dfrac{(E|z_n|)^{\alpha_n}}{\alpha_n!} \\ \leqq A \exp\{E\Sigma|z_j|\}. \end{cases}$$

The function F will be equal to $\mathscr{F}f$ on $\mathbb{R}^n$. Indeed, we saw that the restriction of $\mathscr{F}f$ to a real line through the origin is analytic and on such a line, $\mathscr{F}f$ and F will have the same Taylor series at the origin, cf. (6). Hence by the uniqueness theorem for analytic functions, $\mathscr{F}f$ and F are equal on every real line.

The conclusion is that F is an entire function of exponential type on $\mathbb{C}^n$ which is in L^2 on $\mathbb{R}^n$. By the Paley-Wiener theorem or its precise form the Plancherel-Pólya theorem, cf. [6] p. 171, the inverse Fourier transform of $F = \mathscr{F}f$, which equals f, has compact support.

REMARK 1. It is possible to derive Helgason's support theorem in the same manner. Indeed, for $\Omega = S^{n-1}$ the main lemma is true with $B = 1$, provided that we add a factor $|\alpha|!/\alpha!$ in (3″), cf. [4]; a proof will appear in [9]. Using this fact we now obtain the end result of (7) with $E = R(1+\varepsilon)$. By symmetry this result remains true if $F(z)$ is replaced by $F(Tz)$, where T is a rotation of S^{n-1}. Thus we obtain

$$(8) \quad |F(z)| \leqq A_T \exp\{R(1+\varepsilon)\Sigma|(Tz)_j|\}.$$

From (8) one readily derives that the special indicator function

$$h_F(\lambda; x) \overset{\text{def}}{=} \overline{\lim_{r\to\infty}} \frac{1}{r} \log |F(x + ir\lambda)|$$

of F is bounded by $R|\lambda|$ for every x and λ in $\mathbb{R}^n$. The latter function is equal to the support function of the closed ball $\bar{B}(0,R)$ and the Plancherel-Pólya theorem now shows that supp $f \subset \bar{B}(0,R)$, cf. [6], p. 171.

3. PROOF OF THEOREM 2

Let $(\omega_0, t_0) \notin \operatorname{supp} \hat{f}$ and let s_0 be a point of $\mathbb{R}^n$ such that $s_0 \cdot \omega_0 = t_0$. We will show that $f = 0$ on a neighborhood of s_0. Since supp $\hat{f}$ is closed by definition, we can find a neighborhood T of t_0 in $\mathbb{R}$ and a neighborhood U of ω_0 in

$\mathbb{R}^n$ (!) such that

$$\int_{x\cdot s=t} f(s)dm(s)=0 \text{ for } x\in U,\ t\in T.$$

From this we infer that for every non-negative integer k

$$\text{(9)} \qquad \int_{\substack{x\cdot s=t\\ t\in T}} (x\cdot s)^k f(s)ds=0,\ x\in U.$$

The crux of the proof is to derive from (9) that

$$\text{(10)} \qquad \int_{\substack{x\cdot s=t\\ t\in T}} s^\alpha (x\cdot s)^k f(s)ds=0,\ x\in U$$

for all multi-indices α and all k.

The proof of (10) is by induction on α. First we introduce some notation. Let B_0 be a ball which contains s_0 and supp f, and define

$$H_x=B_0\cap\{s\in\mathbb{R}^n \text{ such that } x\cdot s\in T\},\ x\in U.$$

Thus H_x is a neighborhood of s_0 when $x=\omega_0$. Finally set

$$G_{xy}=H_x\setminus H_y,\ x,y\in U.$$

Observe now that by the boundedness of B_0,

$$\text{(11)} \qquad \text{volume } G_{xy}=o(1) \text{ if } y\to x,$$

while on G_{xy} and G_{yx}

$$\text{(12)} \qquad y\cdot s=x\cdot s+\mathcal{O}(|y-x|) \text{ if } y\to x.$$

To make the induction step, we take $y=x+he_j$, where e_j is the j^{th} unit vector and h is sufficiently small, and write down the difference quotient of (10) with k replaced by $k+1$:

$$\text{(13)} \qquad \begin{cases} 0=\dfrac{1}{h}\left[\int_{H_y} s^\alpha (y\cdot s)^{k+1}f(s)ds-\int_{H_x} s^\alpha (x\cdot s)^{k+1}f(s)ds\right] \\ \quad =\dfrac{1}{h}\int_{H_x} s^\alpha\{(y\cdot s)^{k+1}-(x\cdot s)^{k+1}\}f(s)ds \\ \quad +\int_{G_{yx}}\dfrac{1}{h}s^\alpha (y\cdot s)^{k+1}f(s)ds-\int_{G_{xy}}\dfrac{1}{h}s^\alpha (y\cdot s)^{k+1}f(s)ds. \end{cases}$$

We also need this formula with exponent 0 instead of $k+1$:

$$\int_{G_{yx}} s^\alpha f(s)ds=\int_{G_{xy}} s^\alpha f(s)ds.$$

In the expression $\int_{G_{yx}}-\int_{G_{xy}}$ in (13) we may therefore replace $(y\cdot s)^{k+1}$ by $(y\cdot s)^{k+1}-(x\cdot s)^{k+1}$. After this step we let h tend to 0. Then by the continuity of f and the order estimates (11) and (12) the new integrals over G_{yx} and G_{xy}

will tend to zero. We thus conclude from (13)

$$(k+1)\int_{H_x} s_j s^{\alpha}(x\cdot s)^k f(s)ds=0,$$

which completes the induction proof of (10).

We finally take $k=0$ in (10), then observing that f is orthogonal to all polynomials, we conclude that $f=0$ on H_x.

REMARK 2. This proof is an adaptation of the one in [8], p. 100. The idea goes back to J. Korevaar. For a related result, involving integrals over spheres being zero, see [3], p. 16.

4. AN EXAMPLE

One might conjecture that theorem 1 remains valid if the growth condition (2) is dropped. We will now give an example to show that this is false.

Let $k(\lambda)$ be a smooth, even, function $\not\equiv 0$ on $\mathbb{R}$ whose support is contained in $[-\frac{1}{2},-1]\cup[\frac{1}{2},1]$. Define

$$(14)\qquad g(t)=\int_{-\infty}^{\infty} k(\lambda)e^{it\lambda}d\lambda.$$

We note that the function g is even and in the Schwartz class $\mathcal{S}(\mathbb{R})$, i.e. g and all its derivatives are rapidly decreasing. Also g is real-analytic because it is the restriction to $\mathbb{R}$ of an entire function of exponential type.

Let $h(\omega)$ be a smooth even function $\not\equiv 0$ on S^{n-1} such that $h=0$ on some open set $\Omega\subset S^{n-1}$. We put

$$(15)\qquad F(\omega,t)=h(\omega)g(t).$$

ASSERTION. *There exists a function f in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ such that*

$$\hat{f}(\omega,t)=F(\omega,t),\ (\omega,t)\in S^{n-1}\times\mathbb{R}.$$

Assuming this assertion for a moment, we have obtained a rapidly decreasing function f whose Radon transform $\hat{f}$ satisfies the conditions of theorem 1 except for the growth condition (2). However, f cannot have compact support, because otherwise F and hence the real-analytic function g would have compact support, which is impossible.

The assertion is a consequence of the so-called "Schwartz theorem for the Radon transform", cf. [3], p. 6, but we will give a direct proof. In view of relation (4), the desired function f should be given implicitly by the formula

$$(16a)\qquad \begin{cases} (\mathcal{F}f)(\lambda\omega)=\int_{-\infty}^{\infty}\hat{f}(\omega,t)e^{-i\lambda t}dt=\int_{-\infty}^{\infty}F(\omega,t)e^{-i\lambda t}dt \\ \qquad =h(\omega)\int_{-\infty}^{\infty}g(t)e^{-i\lambda t}dt=2\pi h(\omega)k(\lambda), \end{cases}$$

by inversion of (14). Relation (16a) also makes sense for negative values of λ, because h and k are even. In other words, we should have

$$(16b)\qquad (\mathscr{F}f)(y) = \tilde{f}(y) \stackrel{\text{def}}{=} 2\pi h\left(\frac{y}{|y|}\right) k(|y|).$$

Observe that $\tilde{f}$ is a smooth function on $\mathbb{R}^n$ with compact support (it is smooth because $k \equiv 0$ in a neighborhood of the origin). We now define

$$f = \mathscr{F}^{-1}\tilde{f}$$

so that f is in the class $\mathscr{S}$ and $\tilde{f} = \mathscr{F}f$. By 1-dimensional Fourier inversion of (4), using (16), it follows that

$$\hat{f}(\omega, t) = (2\pi)^{-1} \int_{-\infty}^{\infty} \tilde{f}(\lambda\omega) e^{it\lambda} d\lambda = h(\omega) \int_{-\infty}^{\infty} k(\lambda) e^{it\lambda} d\lambda = F(\omega, t).$$

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A LEMMA ON MIXED DERIVATIVES AND A THEOREM ON HOLOMORPHIC EXTENSION

by

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1. INTRODUCTION

In the present paper we are concerned with refinements of the holomorphy part of the following theorem of Forelli, cf. [1]:

Let f *be a function defined on the unit ball* B *in* $\mathbb{C}^n$ *such that the restriction of* f *to any complex line through* 0 *is holomorphic (or harmonic), while for every* k *there is a neighbourhood* U_k *of* 0 *on which* f *is of class* C^k. *Then* f *is holomorphic on* B *(or pluriharmonic, respectively).*

There is a similar result for all "balanced domains", cf. Rudin [4].

We will assume f defined and holomorphic only on the segments $|z| < 1$ of those complex lines through 0 that meet the real unit sphere S^{n-1}. Nevertheless we will prove that f extends to a holomorphic function on a fairly large $\mathbb{C}^n$-neighbourhood of those segments (theorem 1).

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In the two-dimensional case theorem 1 has a nice form. Let Δ denote a polydisc about 0 in $\mathbb{C}^2$ and T its distinguished boundary. If f is a function defined and holomorphic on the parts inside Δ of those complex lines through 0 which meet T and if f satisfies a smoothness condition at the origin, then f extends to a holomorphic function on Δ (theorem 2). If we assume analyticity at the origin, a similar result is true in the higher dimensional case (theorem 3).

An important element in the proof of theorem 1 is the following

MAIN LEMMA (case of a hemisphere). For the upper half $S_+^{n-1} : \omega_n \geq 0$ of the unit sphere in $\mathbb{R}^n$ and all n-tuples $\alpha = (\alpha_1,\ldots,\alpha_n)$ of non-negative integers there exist integrable functions h_α with the following properties:

$$(1.1) \qquad \int_{S_+^{n-1}} (\omega\cdot D)^{|\alpha|} h_\alpha(\omega)\,d\sigma = D^\alpha,$$

$$(1.2) \qquad \int_{S_+^{n-1}} |h_\alpha(\omega)|\,d\sigma \leq (|\alpha|+1)^{\frac{1}{2}(n-1)}.$$

Observe that the right-hand side of (1.2) is bounded by $(1+\varepsilon)^{|\alpha|}$ for all α of sufficiently large height $|\alpha|$.

Here and in the sequel we use the standard notations

$$D_j = \partial/\partial x_j, \quad D^\alpha = D_1^{\alpha_1} \ldots D_n^{\alpha_n}, \quad \omega\cdot D = \omega_1 D_1 + \ldots + \omega_n D_n,$$

$$|\alpha| = \alpha_1 + \ldots + \alpha_n, \quad \alpha! = \alpha_1! \ldots \alpha_n!,$$

while $d\sigma$ denotes the area element on S^{n-1}.

In [3] we derived a corresponding main lemma for arbitrary open subsets U of S^{n-1} and with an additional factor $|\alpha|!/\alpha!$ on the right-hand side of (1.1). The norms of the corresponding functions $g_\alpha(\omega)$ were bounded by $(B+\varepsilon)^{|\alpha|}$ for some (not optimally determined) constant $B = B_U$.

2. PROOF OF THE MAIN LEMMA

Just as in [3] we will proceed by induction relative to dimension, starting with the case $n = 2$.

LEMMA 1. <u>For every interval</u> I <u>of length</u> π <u>and every</u> $p \geqslant 0$, $q \geqslant 0$ <u>there is an integrable function</u> h_{pq} <u>on</u> I <u>with the following properties</u>:

$$\int_I \left\{(\cos\theta)\frac{\partial}{\partial x_1} + (\sin\theta)\frac{\partial}{\partial x_2}\right\}^{p+q} h_{pq}(\theta)\,d\theta = \frac{\partial^{p+q}}{\partial x_1^p \partial x_2^q}, \tag{2.1}$$

$$\int_I |h_{pq}(\theta)|\,d\theta \leqslant (p+q+1)^{\frac{1}{2}}. \tag{2.2}$$

<u>Proof</u>. Instead of $\partial/\partial x_1$ and $\partial/\partial x_2$ we may write x_1 and x_2. Substituting $x_1 = r\cos\phi$, $x_2 = r\sin\phi$, we may then put (2.1) in the equivalent form

$$\int_I \{\cos(\phi-\theta)\}^{p+q} h_{pq}(\theta)\,d\theta = \cos^p\phi\,\sin^q\phi. \tag{2.3}$$

We express $\{\cos(\phi-\theta)\}^{p+q}$ as a trigonometric polynomial in $\phi-\theta$:

$$\begin{aligned}\{\cos(\phi-\theta)\}^{p+q} &= 2^{-p-q}\{e^{i(\phi-\theta)} - e^{-i(\phi-\theta)}\}^{p+q} \\ &= 2^{-p-q}\sum_{m=0}^{p+q}\binom{p+q}{m} e^{i(p+q-2m)(\phi-\theta)}.\end{aligned} \tag{2.4}$$

The right-hand side of (2.3) is equal to a similar trigonometric polynomial in ϕ:

$$\cos^p\phi\,\sin^q\phi = 2^{-p-q}\, i^{-q} \sum_{\substack{0\leqslant j\leqslant p\\ 0\leqslant k\leqslant q}} (-1)^k \binom{p}{j}\binom{q}{k} e^{i(p-2j+q-2k)\phi}$$

(2.5)

$$= 2^{-p-q} \sum_{m=0}^{p+q} a_{p+q-2m}\, e^{i(p+q-2m)\phi},$$

where

$$a_{p+q-2m} = i^{-q} \sum_{j+k=m} (-1)^k \binom{p}{j}\binom{q}{k},$$

so that

$$(2.6)\qquad |a_{p+q-2m}| \leqslant \sum_{j+k=m} \binom{p}{j}\binom{q}{k} = \text{coeff. of } t^m \text{ in } (1+t)^p(1+t)^q = \binom{p+q}{m}.$$

Observe now that the functions

$$e^{i(p+q-2m)\theta}, \quad m \in \mathbb{Z}$$

form an orthogonal basis of $L^2(I)$. Thus any L^2 solution of (2.3) has the form

$$(2.7')\qquad h_{pq}(\theta) = \sum_{m\in\mathbb{Z}} b_{p+q-2m}\, e^{i(p+q-2m)\theta},$$

where by (2.4) and (2.5)

$$(2.7'')\qquad \pi\, 2^{-p-q}\binom{p+q}{m} b_{p+q-2m} = 2^{-p-q} a_{p+q-2m}, \quad m = 0,1,\ldots,p+q.$$

The other coefficients b_ν need not satisfy any condition beyond the convergence of $\Sigma|b_\nu|^2$. For the solution h_{pq} of minimal L^2 norm the other coefficients b_ν are to be taken equal to zero. Since by (2.6)

$$|b_{p+q-2m}| \leqslant 1/\pi, \quad m = 0,1,\ldots,p+q,$$

the solution h_{pq} of minimal L^2 norm satisfies the inequality

$$(2.8) \qquad \|h_{pq}\|_1^2 \leqslant \|h_{pq}\|_2^2 \cdot \pi = \pi \sum_{m=0}^{p+q} |b_{p+q-2m}|^2 \cdot \pi \leqslant p+q+1 .$$

(Note that also $\|h_{pq}\|_2^2 \geqslant 1$ since $|b_{p+q}| = |a_{p+q}| = 1$.) ∎

Proof of the main lemma. We use induction as in [3], section 3. The upper hemisphere $S_+^{n-1} : \omega_n \geqslant 0$ of the unit sphere in $\mathbb{R}^n$ may be parametrized by the equations

$$\omega_1 = \cos\theta_1, \quad \omega_2 = \sin\theta_1 \cos\theta_2, \quad \ldots,$$

$$\omega_{n-1} = \sin\theta_1 \ldots \sin\theta_{n-2} \cos\phi,$$

$$\omega_n = \sin\theta_1 \ldots \sin\theta_{n-2} \sin\phi, \quad 0 \leqslant \theta_j \leqslant \pi, \quad 0 \leqslant \phi \leqslant \pi.$$

For the induction step we write this as

$$\omega_1 = \cos\theta, \quad \omega_2 = (\sin\theta)\omega', \quad 0 \leqslant \theta \leqslant \pi, \quad \omega' \in S_+^{n-2}.$$

Representing the vector $D = (D_1,\ldots,D_n)$ as (D_1,D'), the inner product $\omega\cdot D$ becomes

$$\omega\cdot D = (\cos\theta)D_1 + (\sin\theta)\omega'\cdot D'.$$

Since the area elements on S^{n-1} and S^{n-2} are related by

$$d\sigma = (\sin\theta)^{n-2}\, d\theta d\sigma',$$

the desired formula (1.1) may be written in the equivalent form

$$(2.9)\qquad \int_{[0,\pi]\times S_+^{n-2}} \{(\cos\theta)D_1 + (\sin\theta)\omega'\cdot D'\}^{\alpha_1+|\alpha'|}\cdot$$

$$\cdot\ h_{\alpha_1,\alpha'}(\omega)(\sin\theta)^{n-2}\,d\theta d\sigma' = D_1^{\alpha_1}(D')^{\alpha'}.$$

By the result for $n = 2$ (lemma 1 with $I = [0,\pi]$) ,

$$(2.10)\qquad \int_0^{\pi} \{(\cos\theta)D_1 + (\sin\theta)\omega'\cdot D'\}^{\alpha_1+|\alpha'|} h_{\alpha_1|\alpha'|}(\theta)d\theta = D_1^{\alpha_1}(\omega'\cdot D')^{|\alpha'|},$$

while by the case $n-1$ of the main lemma (induction hypothesis!),

$$(2.11)\qquad \int_{S_+^{n-2}} (\omega'\cdot D')^{|\alpha'|}h_{\alpha'}(\omega')d\sigma' = (D')^{\alpha'}.$$

Combining (2.10) and (2.11), we see that we may satisfy (2.9) or (1.1) by defining

$$h_\alpha(\omega)(\sin\theta)^{n-2} = h_{\alpha_1|\alpha'|}(\theta)h_{\alpha'}(\omega').$$

We also obtain (1.2) from the induction hypothesis:

$$\|h_\alpha\|_1 = \int_{S_+^{n-1}} |h_\alpha(\omega)|d\sigma = \int_{[0,\pi]\times S_+^{n-2}} |h_{\alpha_1|\alpha'|}(\theta)||h_{\alpha'}(\omega')|d\theta d\sigma'$$

$$\leqslant (\alpha_1+|\alpha'|+1)^{\frac12}(|\alpha'|+1)^{\frac12(n-2)} \leqslant (|\alpha|+1)^{\frac12(n-1)}.\ \blacksquare$$

3. SOME RESULTS ON SEQUENCES OF POLYNOMIALS

The following two results concerning the growth of sequences of polynomials will be used in the proof of theorem 1.

LEMMA 2. Let $\{P_k(x)\}$ be a sequence of polynomials on $\mathbb{R}^n$ such that for every $x \in S^{n-1}$:

$$\overline{\lim_{k\to\infty}} \, |P_k(x)|^{1/k} \leqslant 1.$$

Then there exist an open set $U \subset S^{n-1}$ and a positive constant C such that for every k

$$\sup_{x\in U} |P_k(x)| \leqslant C^k.$$

Proof. Let

$$\phi(x) = \sup_{k\in\mathbb{N}} |P_k(x)|^{1/k}$$

and define

$$V_j := \{x \in S^{n-1} : \phi(x) \leqslant j\}, \qquad j \in \mathbb{N}.$$

Since P_k is continuous, V_j is the intersection of closed sets, $\{|P_k| \leqslant j^k\}$, hence V_j is a closed subset of S^{n-1}.

Because the function $\phi(x)$ remains finite, we have

$$\bigcup_j V_j = S^{n-1}.$$

Invoking Baire's theorem, we conclude that some set V_m contains an open set U. For $x \in U$ we thus have

$$|P_k(x)| \leqslant m^k. \qquad \blacksquare$$

In the following, let Δ be the unit polydisc in $\mathbb{C}^n$ and let T denote its distinguished boundary:

$$\Delta = \{z : |z_j| < 1\} \quad , \quad T = \{z : |z_j| = 1\} .$$

PROPOSITION 1. Let $\{P_k(z)\}$ be a sequence of polynomials on $\mathbb{C}^n$ with degree $P_k \leqslant Ck$, such that

$$\overline{\lim_{k\to\infty}} \sup_{z\in T} |P_k(z)|^{1/k} \leqslant A$$

while for every $z \in T$

$$\overline{\lim_{k\to\infty}} |P_k(z)|^{1/k} \leqslant 1.$$

Then

$$\overline{\lim_{k\to\infty}} \sup_{z\in T} |P_k(z)|^{1/k} \leqslant 1.$$

Proof. We introduce the positive plurisubharmonic functions

$$U_k(z) = |P_k(z)|^{1/k}$$

on $\bar{\Delta}$. The functions $U_k(z)$ are uniformly bounded on T, hence on $\bar{\Delta}$. For $z = (r_1 e^{i\phi_1}, \dots, r_n e^{i\phi_n})$ we have

$$(3.1) \qquad U_k(z) \leqslant \frac{1}{(2\pi)^n} \int_T \prod_{j=1}^{n} \frac{1-r_j^2}{1-2r_j\cos(\theta_j-\phi_j)+r_j^2} U_k(e^{i\theta_1}, \dots, e^{i\theta_n}) \cdot d\theta_1 \dots d\theta_n$$

This follows by repeated application of the one-dimensional Poisson formula. Applying Fatou's lemma for bounded functions to (3.1), we infer that for $z \in \Delta$

$$(3.2) \qquad \overline{\lim_{k\to\infty}}\, U_k(z) \leq 1.$$

Next we invoke Hartogs' lemma, cf. [2], p.21 to prove the uniformity of (3.2) on compacta in Δ.

It follows that for every $\varepsilon > 0$ and $0 < r < 1$ there exists k_0 such that for $k > k_0$

$$U_k(rz) \leq 1+\varepsilon \quad \text{for} \quad z \in \bar{\Delta},$$

hence

$$|P_k(rz)| \leq (1+\varepsilon)^k \quad \text{for} \quad z \in \bar{\Delta}.$$

Now let $d_k \leq ck$ denote the degree of P_k. Introduce for $z \in \bar{\Delta}$ the polynomial of one variable:

$$Q(\lambda) = P_k(\lambda z r), \qquad \lambda \in \mathbb{C}.$$

Then $Q(\lambda)$ has degree $\leq d_k$ and $Q(\lambda) \leq (1+\varepsilon)^k$ for $|\lambda| = 1$. By the maximum principle applied to $Q(\lambda)/\lambda^{d_k}$ on the complement of the unit disc, we infer that for any $z \in \bar{\Delta}$

$$|P_k(z)| = \left|\frac{Q(1/r)}{(1/r)^{d_k}}\right| (1/r)^{d_k} \leq (1+\varepsilon)^k (1/r)^{ck} \quad \text{for} \quad k > k_0.$$

Since $\varepsilon > 0$ and $r < 1$ are arbitrary, we conclude that

$$\overline{\lim_{k\to\infty}} \sup_{z\in\bar{\Delta}} |P_k(z)|^{1/k} \leq 1. \qquad \blacksquare$$

4. HOLOMORPHIC EXTENDABILITY OF FUNCTIONS WITH RESTRICTED SETS OF HOLOMORPHIC SLICES.

We introduce

$$\mathbb{S}\mathbb{R}^n := \{z \in \mathbb{C}^n : \exists \xi \in S^{n-1} \subset \mathbb{R}^n, \quad w \in \mathbb{C} \quad \text{such that} \quad z = w\xi\},$$

the set of semi-reals in $\mathbb{C}^n$. We also give the following

Definition. Let H be a real linear subspace of $\mathbb{C}^n = \mathbb{R}^{2n}$ and let f be a complex function, whose domain of definition contains an H-neighbourhood of 0. The function f is said to be of class $C^\infty\{0\}$ relative to H if for every k there is an H-neighbourhood U_k of 0 on which f is of class C^k.

We now come to the main results.

THEOREM 1. Let the function f be defined on $B \cap \mathbb{S}\mathbb{R}^n$ and let Ω be the union of the domain

$$D = \{z \in \mathbb{C}^n : \Sigma |z_i| < 1\}$$

and the images of D under the rotations which leave $\mathbb{R}^n \subset \mathbb{C}^n$ invariant. Suppose that the restriction of f to every complex line through the origin and a point of $S^{n-1} \subset \mathbb{R}^n$ is holomorphic, while f is in $C^\infty\{0\}$ relative to $\mathbb{R}^n$. Then f extends holomorphically to Ω.

Proof. By assumption we can expand the restriction of f to $\mathbb{R}^n$ in a Taylor series around 0:

$$(4.1) \qquad f(x) \sim \sum_\alpha c_\alpha x^\alpha .$$

The purpose of the proof is to show that the series in (4.1) actually represents a holomorphic function which equals f on $B \cap \mathbb{S}\mathbb{R}^n$.

For $\omega \in S^{n-1}$ the slice function

$$f_\omega(w) \overset{\text{def}}{=} f(w\omega)$$

is holomorphic on the unit disc $|w| < 1$. The power series of f_ω has the form

$$f_\omega(w) = \Sigma\, b_k(\omega) w^k,$$

where

$$(4.2) \qquad b_k(\omega) = \sum_{|\alpha|=k} c_\alpha \omega^\alpha .$$

For the coefficients $b_k(\omega)$ we have

$$(4.3) \qquad \overline{\lim_{k\to\infty}} |b_k(\omega)|^{1/k} \leqslant 1 \quad \text{for every} \quad \omega \in S^{n-1}.$$

We may now apply lemma 2 and obtain an open set $U \subset S^{n-1}$ and a constant $C > 0$ such that for every k

$$\sup_{x\in U} |b_k(x)| \leqslant C^k .$$

Next we apply the main lemma for arbitrary open subsets of S^{n-1}, cf. the introduction and [3]. We infer that for certain constants A and B and for all multi-indices α

$$(4.4) \qquad \begin{aligned} |c_\alpha| &= \left|\frac{1}{\alpha!} D^\alpha f(0)\right| = \left|\int_U \frac{1}{|\alpha|!}(\omega\cdot D)^{|\alpha|} g_\alpha(\omega)\,d\sigma\right| = \\ &= \left|\int_U b_{|\alpha|}(\omega) g_\alpha(\omega)\,d\sigma\right| \leqslant A(BC)^{|\alpha|}. \end{aligned}$$

The estimate (4.4) shows that the complexification of the Taylor series in (4.1) represents a holomorphic function g in a neighbourhood of the origin. The slices f_ω and g_ω are equal for every $\omega \in S^{n-1}$, because they are both analytic and have the same Taylor coefficients at the origin. Hence g represents a holomorphic extension of f to a full neighbourhood of the origin.

Using a more precise estimate for the coefficients c_α, we will next show that g extends to the domain D. Observe that by (4.2) and (4.4) for fixed n and all $\omega \in S^{n-1}$

$$(4.5) \qquad |b_k(\omega)| \leqslant \binom{n+k-1}{k} \max_{|\alpha|=k} |c_\alpha| \leqslant AB'^k .$$

We would like to apply proposition 1 to the polynomials b_k. To that end we define via recursion a rational map Φ_n that sends T^n, the distinguished boundary of the polydisc surjectively to the n-dimensional sphere

$$\Phi_1 : z \longmapsto \left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right),$$

and

$$\Phi_n : (z_1,\ldots,z_n) \longmapsto \left(\frac{z_1^2+1}{2z_1}, \frac{z_1^2-1}{2iz_1}\,\Phi_{n-1}(z_2,\ldots,z_n)\right).$$

Via Φ_n we can associate to a polynomial P of degree ℓ on $\mathbb{R}^{n+1}$ the polynomial

$$(4.6) \qquad Q(z) = (z_1\ldots z_n)^\ell \, P(\Phi_n(z)).$$

The polynomial Q has degree $\leqslant 2\ell n$ and it has the property that for $z \in T^n$,

$$|Q(z)| = |P\Phi_n(z)|.$$

We let $a_k(z)$ denote the polynomials in $z_1 \dots z_{n-1}$ associated to the polynomials b_k via (4.6). Since $a_k(z)$ has degree $\leq 2(n-1)k$ and since by (4.5) and (4.3)

$$\sup_{z \in T^{n-1}} |a_k(z)| = \sup_{\omega \in S^{n-1}} |b_k(\omega)| \leq AB'^k,$$

$$\overline{\lim_{k\to\infty}} |a_k(z)|^{1/k} = \overline{\lim_{k\to\infty}} |b_k(\omega)|^{1/k} \leq 1 \quad \text{for} \quad w = \Phi_n(z),\ z \in T^{n-1},$$

proposition 1 applies to $a_k(z)$. The conclusion is

$$\overline{\lim_{k\to\infty}} \sup_{\omega \in S^{n-1}} |b_k(\omega)|^{1/k} = \overline{\lim_{k\to\infty}} \sup_{z \in T^{n-1}} |a_k(z)|^{1/k} \leq 1,$$

in other words, for every $\varepsilon > 0$ there exists a constant A' such that for all k and all $\omega \in S^{n-1}$

$$|b_k(\omega)| \leq A'(1+\varepsilon)^k.$$

We next use the special version of the main lemma (section 2) and obtain similarly to (4.4):

$$|c_\alpha| \leq A'' \frac{|\alpha|!}{\alpha!}(1+\varepsilon)^{2|\alpha|}.$$

This inequality will show that the complexified series (4.1) converges to a holomorphic function throughout D, thus extending g and f to D. Indeed, it suffices to show that the series (4.1) converges pointwise which goes as follows: For $z \in D$ choose

$\varepsilon > 0$ such that $(1+\varepsilon)^2 \cdot \sum_1^n |z_j| < 1$. Then

$$|\Sigma c_\alpha z^\alpha| \leqslant A'' \sum_\alpha \frac{|\alpha|!}{\alpha!} |z_1|^{\alpha_1} \ldots |z_n|^{\alpha_n} \cdot (1+\varepsilon)^{2|\alpha|} \leqslant$$

$$\leqslant A'' \sum_k (1+\varepsilon)^{2k} \sum_{|\alpha|=k} \frac{k!}{\alpha!} |z_1|^{\alpha_1} \ldots |z_n|^{\alpha_n} =$$

$$= A'' \sum_k (1+\varepsilon)^{2k} (\sum_1^n |z_j|)^k < \infty .$$

Finally, to conclude the proof, we observe that the composition of f with any rotation that leaves $S^{n-1} \subset \mathbb{R}^n$ invariant satisfies the same hypotheses as f and thus extends to D. Equivalently, f extends to any domain obtained by applying such a rotation to D, hence f extends to Ω. ∎

In the two-dimensional case we can give a better description of the domain Ω. Observe that

$$\partial B \cap \mathbb{S}\mathbb{R}^2 = \{(e^{i\phi} \cos\theta, e^{i\phi} \sin\theta)\}$$

is topologically a two-dimensional torus; thus it looks like the distinguished boundary of a polydisc and in fact it is. This leads to the following form of theorem 1.

THEOREM 2. Let Δ be the unit polydisc in $\mathbb{C}^2$ with distinguished boundary T. Let the function f be defined on $\Delta \cap X$, where X is the union of those complex lines through the origin that meet T. Suppose that the restriction of f to any such complex line is holomorphic and that f is in $C^\infty\{0\}$ relative to the real linear subspace

$$V := \{(z,\bar{z}) : z \in \mathbb{C}\}.$$

Then f extends to a holomorphic function on Δ.

Proof. Let F denote the one to one linear transformation

$$(z,w) \longmapsto (z+iw,\ z-iw)$$

which maps $B \cap S\mathbb{R}^2$ onto $\Delta \cap X$, $\partial B \cap S\mathbb{R}^2$ onto T and $\mathbb{R}^2$ onto V:

$$F(re^{i\phi}\cos\theta,\ re^{i\phi}\sin\theta) = (re^{i(\phi+\theta)},\ re^{i(\phi-\theta)}).$$

Hence $g = f\circ F$ satisfies the conditions of theorem 1 and thus extends holomorphically to the neighbourhood Ω of $B \cap S\mathbb{R}^2$ given by theorem 1. It follows that $f = g\circ F^{-1}$ extends holomorphically to $F(\Omega)$.

To conclude the proof we compute

$$F(\Omega) = \{(z,w) : F^{-1}(z,w) \in \Omega\}$$

$$= \{(z,w) : \exists\theta \text{ with } |(z+w)\cos\theta - i(z-w)\sin\theta| + |(z-w)\cos\theta - i(z-w)\sin\theta| < 2\}$$

$$= \{(z,w) : \exists\phi \text{ with } \big||z| + |w|e^{i\phi}\big| + \big||z| - |w|e^{i\phi}\big| < 2\} = \Delta.$$ ∎

One would expect that there is a higher-dimensional analog of theorem 2. However, for $n \geqslant 3$ there is no real linear subspace available that can play the role of V. We circumvent this difficulty by imposing an analyticity condition at the origin. Instead of the main lemma we now need the identity

$$(4.7) \qquad \frac{|\alpha|!}{\alpha!} D^\alpha = (2\pi)^{-n} \int_T (\tau\cdot D)^{|\alpha|}\tau^{-\alpha}\, d\theta_1 \ldots d\theta_n,$$

where $\tau = \tau(\theta) = (e^{i\theta_1},\ldots,e^{i\theta_n})$.

THEOREM 3. *Let* Δ *be the unit polydisc in* $\mathbb{C}^n$ *with distinguished boundary* T. *Let the function* f *be defined on* $(\Delta \cap X) \cup U$, *where* U *is a neighbourhood of the origin and* X *is the union of those complex lines through the origin that meet* T. *Suppose that* f *is holomorphic on* U *and that the restriction of* f *to any complex line meeting* T *is holomorphic (as a function of one variable). Then* f *extends to a holomorphic function on* Δ.

The *proof* uses proposition 1 as well as (4.7) and is similar to that of theorem 1.

Remarks.

1. The holomorphic version of Forelli's theorem for the ball is a consequence of theorem 1. Indeed, the union of all complex rotations of the domain Δ is precisely the unit ball.

2. The hypothesis that f is of class $C^\infty\{0\}$ cannot be disposed of as the example in Rudin [4], p.63, shows.

3. Theorem 1 is optimal for the two-dimensional case, in the sense that there Ω is the largest possible domain with the property that every function which satisfies the conditions of theorem 1 extends holomorphically to Ω (Ω is the biholomorphic image of a polydisc).

4. Theorems 2 and 3 are apparently related to the well-known theorem of Hartogs which says that any holomorphic function on a connected Reinhardt domain R containing the origin extends to a holomorphic function on the logarithmically convex hull of R, cf. [2], p.36.

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SAMENVATTING

Dit proefschrift bestaat uit vijf artikelen. In de eerste twee wordt het groeigedrag bestudeerd van Paley-Wiener functies, dat zijn gehele functies van het exponentiële type in meer veranderlijken, die niet meer dan polynomiale groei hebben op $\mathbb{R}^n \subset \mathbb{C}^n$.

Het is bekend dat Paley-Wiener functies geen "regelmatige groei" behoeven te hebben. Een deelklasse waarvoor de groei wel regelmatig is, wordt gekarakteriseerd. Vervolgens wordt een existentiestelling voor Paley-Wiener functies met voorgeschreven indicator bewezen onder een milde gladheidsvoorwaarde op de indicator. Als toepassing van deze existentiestelling wordt een niet regelmatig groeiende Paley-Wiener functie geconstrueerd.

De overige drie artikelen draaien om een methode voor het bewijzen van holomorfie of holomorfe voortzetbaarheid van zekere functies. Deze methode berust op representatie en schattingen van gemengde afgeleiden in termen van richtingsafgeleiden.

Met behulp van deze methode wordt F.E. Browders bewijs van de "kant van de wig" stelling vereenvoudigd. Een andere toepassing is een verfijning van Forelli's stelling over holomorfie van functies waarvan de beperking tot complexe lijnen door de oorsprong holomorf is. Tenslotte wordt een nieuw bewijs van Helgason's dragerstelling voor de Radon-transformatie gegeven. Bovendien wordt deze stelling enigszins verscherpt. Hierbij spelen ook methoden uit het eerste deel een rol.

STELLINGEN

bij het proefschrift

Entire functions of Paley-Wiener type in $\mathbb{C}^n$, *Radon transforms and problems of holomorphic extension*

van

Johannes J.O.O. Wiegerinck

1. Laat f een snel dalende continue functie op $\mathbb{R}^n$ zijn, zodanig dat de Radon-getransformeerde $\hat{f}$ voldoet aan

$$|\hat{f}(\omega,t)| \leq C_\omega e^{-\varepsilon_\omega t}, \ \varepsilon_\omega > 0 \ , \ (\omega,t) \in S^{n-1} \times \mathbb{R} \ .$$

Veronderstel dat voor $\Omega \subset S^{n-1}$ en $R > 0$

$$\hat{f}(\omega,t) = 0 \ , \quad \omega \in \Omega \quad |t| > R.$$

Voor de conclusie dat f compacte drager heeft is dan voldoende dat Ω positieve projectieve capaciteit heeft. De conditie op Ω kan nog verder verzwakt worden.

Vgl. dit proefschrift, p. 65 en J. Siciak - Extremal plurisubharmonic functions and capacities in $\mathbb{C}^n$. Sophia Kokyuroku in Math. 14, Sophia University, Tokyo, 1982.

2. Laat $f_a(z)$ de gehele functie zijn, geassocieerd aan de logistische afbeelding $z \mapsto az(1-z)$, dat wil zeggen $f_a(z)$ voldoet aan de functionaal-vergelijking

$$f(az) = af(z)(1-f(z)), \text{ met } f(0) = 0, \ f'(0) = 1.$$

De functie f_a , $2 < a < 3$, is het eerste "natuurlijke" voorbeeld van een gehele functie die geen regelmatige groei heeft.

Vgl. H.A. Lauwerier - Entire functions for the logistic map I, II. MC/CWI report TW 228/82, AM-R 8404.

3. Laat γ een Jordan boog zijn, $d\mu$ een complexe Borelmaat op γ. De Laplace-getransformeerde

$$f(z) = \int_{\gamma} e^{z\xi}\, d\mu(\xi)$$

behoeft niet regelmatig te groeien.

Vgl. J. Clunie, W.K. Hayman - Symposium on complex analysis, Canterbury, 1973, problem 7.16, p. 175. London Math. Soc. lect. note ser. 12, Cambridge University press, Cambridge, 1974.

4. Voor elk natuurlijk getal k bestaan er gebieden Ω_k in $\mathbb{C}^2$ zodanig, dat de ruimte van kwadratisch integreerbare, holomorfe functies op Ω_k precies k-dimensionaal is.

5. Veronderstel dat $\Omega, \Omega' \Subset \mathbb{C}^n$ gebieden met C^2 rand zijn, Ω pseudoconvex en Ω' strikt pseudoconvex. Laat $\Phi : \Omega \to \Omega'$ biholomorf zijn. Dan is Φ voortzetbaar tot een continue afbeelding $\hat{\Phi} : \bar{\Omega} \to \bar{\Omega}'$, die (uniform) Hölder-continu is met betrekking tot iedere exponent $\frac{1}{2} - \varepsilon$, $\varepsilon > 0$.

Vgl. K. Diederich, I. Lieb - Konvexität in der komplexen Analysis. Birkhäuser, Basel, 1981, p. 40, theorem 4.1.

6. Als f begrensd en uniform continu is op $\mathbb{R}$, dan bestaat er een rij Paley-Wiener functies g_n op $\mathbb{C}$ zó, dat

$$\sup_{-\infty < x < \infty} |f(x) - g_n(x)| \to 0 \quad \text{als} \quad n \to \infty .$$

Deze stelling behoort met convolutie en Fourier-transformatie bewezen te worden.

Vgl. A.F. Timan - Theory of approximation of functions of a real variable. (transl.) Pergamon Press, Oxford etc., 1963.

7. Voor $z \neq \xi$ in $\mathbb{C}^2$ geldt

$$\frac{\bar{\xi}_1 - \bar{z}_1}{|\xi - z|^4} + \frac{1}{2\pi i} \int_{\mathbb{C}} \frac{1}{s} \frac{\partial}{\partial \bar{\xi}_2} \left(\frac{\bar{\xi}_2 - \bar{z}_2}{|(\xi_1 - s, \xi_2) - z|^4} \right) d\bar{s} \wedge ds = 0.$$

De bewering van *Krantz* dat het linkerlid niet holomorf is voor $|z|$ groot is onjuist. Zijn conclusie dat de Bochner-Martinelli formule het $\bar{\partial}$-probleem voor compact gedragen 1-vormen niet oplost is daarom niet gerechtvaardigd. Bovendien is zijn conclusie onjuist.

S.G. Krantz - Function theory of several complex variables, Wiley, New York, 1982, pp. 22, 23.

8. Wetenschappelijk onderzoekers behoren één en slechts één keer per jaar verslag uit te brengen over voortgang en prognoses van hun werk.