

An upper bound on the two-arms exponent for critical percolation on \mathbb{Z}^d

J. van den Berg^a and D. G. P. van Engelenburg^{b,1}

^a*CWI and VU University Amsterdam, The Netherlands. E-mail: J.van.den.Berg@cwi.nl*

^b*University of Vienna, Austria. E-mail: diderik.van.engelenburg@univie.ac.at*

Received 21 October 2020; revised 15 December 2020; accepted 1 February 2021

Abstract. Consider critical site percolation on \mathbb{Z}^d with $d \geq 2$. Cerf (*Ann. Probab.* **43** (2015) 2458–2480) pointed out that from classical work by Aizenman, Kesten and Newman (*Comm. Math. Phys.* **111** (1987) 505–532) and Gandolfi, Grimmett and Russo (*Comm. Math. Phys.* **114** (1988) 549–552) one can obtain that the two-arms exponent is at least $1/2$. The paper by Cerf slightly improves that lower bound.

Except for $d = 2$ and for high d , no *upper* bound for this exponent seems to be known in the literature so far (not even implicitly). We show that the distance- n two-arms probability is at least $cn^{-(d^2+4d-2)}$ (with $c > 0$ a constant which depends on d), thus giving an upper bound $d^2 + 4d - 2$ for the above mentioned exponent.

Résumé. Nous considérons la percolation critique par sites dans \mathbb{Z}^d pour $d \geq 2$. Cerf (*Ann. Probab.* **43** (2015) 2458–2480) a montré, en s'appuyant sur des travaux classiques dûs à Aizenman, Kesten et Newman (*Comm. Math. Phys.* **111** (1987) 505–532) et Gandolfi, Grimmett and Russo (*Comm. Math. Phys.* **114** (1988) 549–552), que l'exposant critique à deux-bras est au moins égal à $1/2$. L'article de Cerf améliore cette borne inférieure.

Sauf dans le cas $d = 2$ ou en grande dimension, il ne semble pas que la littérature contienne une borne *supérieure* pour cette exposant (pas même de manière implicite). Nous prouvons que la probabilité de “deux bras à distance n ” est au moins $cn^{-(d^2+4d-2)}$ (où $c > 0$ est une constante qui peut dépendre de la dimension d). Ainsi, l'exposant mentionné est inférieur ou égal à $d^2 + 4d - 2$.

MSC2020 subject classifications: Primary 60K35; secondary 82B43

Keywords: Critical percolation; Critical exponent

1. Introduction and main results

Consider nearest-neighbour site percolation with parameter p on \mathbb{Z}^d , with $d \geq 2$. Each site is, independently of the others, open with probability p and closed with probability $1 - p$. The corresponding product measure is denoted by P_p . We use the standard notation $\theta(p)$ for the probability that a given site belongs to an infinite open cluster, and $p_c = p_c(d) := \sup\{p \in [0, 1] : \theta(p) = 0\}$ for the critical probability.

Let $\Lambda(n)$ be the box $[-n, n]^d$, and let $\partial\Lambda(n)$ be its boundary $\{x \in \Lambda(n) : \|x\| = n\}$, where $\|x\| := \max_{1 \leq i \leq d} |x_i|$. Following [4], we use the notation “two-arms(0, n)” for the event that in $\Lambda(n)$ there are two distinct open clusters connecting neighbors of 0 to $\partial\Lambda(n)$. More precisely, it is the event that there are open paths π and π' with the following properties: π connects a neighbor of 0 with $\partial\Lambda(n)$; π' connects a neighbor of 0 with $\partial\Lambda(n)$; and there is no open path in $\Lambda(n)$ connecting π with π' .

In two dimensions, two-arms events are, somewhat confusing, in fact events involving *four arms* with alternating ‘colour’, and have been widely studied since Kesten’s classical work [10].

¹This paper arose from a part of the master thesis work at the University of Amsterdam and the VU University Amsterdam by DvE in the period September 2019–February 2020, supervised by JvdB.

In general dimensions two-arms events are related to the uniqueness of the infinite cluster. Indeed, it is straightforward that

$$\begin{aligned} & \lim_{n \rightarrow \infty} P_p(\text{two-arms}(0, n)) \\ &= P_p(\exists \text{ two neighbours of } 0 \text{ that belong to distinct infinite open clusters}), \end{aligned}$$

which is known to be 0 ('uniqueness of the infinite cluster'). The paper [3] by Burton and Keane gives the most elegant proof of uniqueness of the infinite cluster, but does not give a bound for the speed of convergence above. Cerf [4] pointed out that from earlier uniqueness proofs (in [1] and [8]) one can obtain that in any dimension the two-arms exponent is at least $1/2$. One of Cerf's main results in the above mentioned paper is a clever computation which slightly improves that bound. More precisely, he showed that

$$\limsup_{n \rightarrow \infty} \frac{\log P_{p_c}(\text{two-arms}(0, n))}{\log n} \leq -\frac{2d^2 + 3d - 3}{4d^2 + 5d - 5},$$

i.e. the two-arms exponent is at least $(2d^2 + 3d - 3)/(4d^2 + 5d - 5)$.

To our knowledge, the literature, so far, does not provide *upper* bounds on the two-arms exponent, except for $d = 2$ and for 'high' d . Our main result, Theorem 1.1 below, gives the upper bound $d^2 + 4d - 2$. Somewhat remarkably, one of the main ingredients of the proof of our theorem is an intermediate result by Cerf which he used for his above mentioned lower bound for the exponent.

Our upper bound on the exponent (as well as Cerf's lower bound) is essentially only relevant for 'intermediate' dimensions: For site percolation on the triangular lattice the exact value $5/4$ has been proved for the exponent (see [14]), and this is also believed to hold for other nice $2D$ lattices. Recently, in [7] an upper bound $4/3$ was proved for bond percolation on the square lattice. In [12] it has been proved that for 'high' dimensions the exponent is 4, and this is believed to hold for all $d \geq 7$.

Two-arms events are also (at least potentially) important for the study of one of the main conjectures in percolation theory, namely that $\theta(p_c) = 0$ (this conjecture has been proved for $d = 2$ and for high d only). See the last paragraph of Section 1 in [4].

Some other examples where results on two-arms probabilities are important, are works about upper bounds for the percolation correlation length near criticality ([6]) and about the convergence rate in a central limit theorem for minimal spanning trees [5].

We now state our main result more precisely, where (here and throughout the rest of the paper) *we use the shorthand notation* \mathbb{P} *for* P_{p_c} .

Theorem 1.1. *There is a constant $c > 0$, which depends only on the dimension d , such that for all $n \geq 1$,*

$$\mathbb{P}(\text{two-arms}(0, n)) \geq cn^{-(d^2+4d-2)}. \quad (1)$$

A crucial part in the proof of Theorem 1.1 is the following proposition, which may also be interesting in itself. We first introduce more notation. For $m \leq n$, we write $A_2(m, n)$ for the event that in $\Lambda(n)$ there are two distinct open clusters connecting $\Lambda(m)$ with $\partial\Lambda(n)$.

If $W \subset \mathbb{Z}^d$ and $U, V \subset W$, we use the notation $\{U \leftrightarrow V\}$ for the event that there is an open path from U to V , and the notation $\{U \leftrightarrow V \text{ in } W\}$ for the event that there is an open path from U to V of which every site is in W .

Proposition 1.2. *For all $M \geq 2$ there is a $\delta = \delta(d, M) > 0$ and a sequence of integers $1 = n_1 < n_2 < \dots$ such that $\max_i \frac{n_{i+1}}{n_i} \leq 8M$, and*

$$\mathbb{P}(A_2(n_i, Mn_i)) \geq \delta, \quad \text{for all } i. \quad (2)$$

We postpone the proof of Proposition 1.2 to Section 2, and now show how Theorem 1.1 follows from that proposition, combined with (a straightforward generalization of) the following result by Cerf and a lower bound for 'restricted' one-arm probabilities.

Lemma 1.3 (Case $l = 2n$ and $p = p_c$ of Corollary 7.3 in [4]). *Let $d \geq 2$. There is a $C = C(d) > 0$ such that for all $n \geq 1$,*

$$\mathbb{P}(A_2(n, 3n)) \leq \frac{Cn^{4d-2}\mathbb{P}(\text{two-arms}(0, n))}{\min_{x, y \in \partial\Lambda(n)} \mathbb{P}(x \leftrightarrow y \text{ in } \Lambda(2n))}. \quad (3)$$

Proof of Theorem 1.1 from Proposition 1.2. We first prove a slightly weaker version (with a larger exponent) and then point out how to obtain the actual theorem. Choose $M = 3$. Let $n \geq 1$. Take i such that $n \leq n_i \leq 24n (= 8Mn)$. By (2) and (3) we have,

$$\delta \leq \mathbb{P}(A_2(n_i, 3n_i)) \leq \frac{C(n_i)^{4d-2} \mathbb{P}(\text{two-arms}(0, n_i))}{\min_{x,y \in \partial \Lambda(n_i)} \mathbb{P}(x \leftrightarrow y \text{ in } \Lambda(2n_i))}.$$

By Lemma 6.1 in [4], the denominator in the expression above is at least some constant (which depends on d only) times $(n_i)^{-2d(d-1)}$. Hence

$$\begin{aligned} \mathbb{P}(\text{two-arms}(0, n)) &\geq \mathbb{P}(\text{two-arms}(0, n_i)) \\ &\geq C' \delta (n_i)^{-((4d-2)+2d(d-1))} \\ &\geq C'' n^{-((4d-2)+2d(d-1))}, \end{aligned} \tag{4}$$

where C' and C'' are positive constants which depend only on d . This proves a slightly weaker form of Theorem 1.1: with exponent $(4d-2) + 2d(d-1) = 2d^2 + 2d - 2$ instead of $d^2 + 4d - 2$.

The exponent in Theorem 1 can be obtained by using Lemma 1.4 below (which generalizes Lemma 1.3 and can be proved in the same way as the mentioned Corollary 7.3 in [4]), and Lemma 1.5 below (which is an improvement in [2] of Cerf's Lemma 6.1 mentioned above).

Lemma 1.4. *For all $M \geq 2$ there is a $c_1 > 0$ such that for all $n \geq 1$,*

$$\mathbb{P}(A_2(n, Mn)) \leq \frac{c_1 n^{4d-2} \mathbb{P}(\text{two-arms}(0, n))}{\min_{x,y \in \partial \Lambda(n)} \mathbb{P}(x \leftrightarrow y \text{ in } \Lambda((M-1)n))}. \tag{5}$$

Lemma 1.5 (Corollary 1.2 in [2]). *There is a constant $c_2 = c_2(d) > 0$ such that for all $n \geq 1$,*

$$\min_{x,y \in \partial \Lambda(n)} \mathbb{P}(x \leftrightarrow y \text{ in } \Lambda(9n)) \geq c_2 n^{-d^2}. \tag{6}$$

By now choosing $M = 10$, and repeating the same argument which led to the earlier mentioned weaker version of Theorem 1.1 (but now using Lemma's 1.4 and 1.5 instead of Lemma 1.3 and Cerf's Lemma 6.1, respectively) we get, for some constant $c > 0$,

$$\mathbb{P}(\text{two-arms}(0, n)) \geq cn^{-((4d-2)+d^2)} = cn^{-(d^2+4d-2)},$$

which completes the proof of Theorem 1.1. □

Remark. From the above it is clear that there are potentially at least two strategies to improve the exponent in Theorem 1.1: By improving the exponent $4d - 2$ in (3) (and in its generalization (5)); or by further improving Lemma 1.5 (which itself, as mentioned before, was an improvement of Lemma 6.1 in [4]).

2. Proof of Proposition 1.2

We will use the following lemma, where we use the notation $V(k, m)$ for the event that there is an open crossing in the d -direction of the rectangle $[0, m]^{d-1} \times [0, k]$. (Typically we will use this with $k < m$, so that the crossing is in the 'easy' direction).

Subsets of \mathbb{Z}^d will sometimes be denoted in a somewhat informal way; for instance we will simply write $\{z_d = 0\}$ for the set $\{z \in \mathbb{Z}^d : z_d = 0\}$.

Lemma 2.1. *For all $M \geq 2$ there is an $\varepsilon = \varepsilon(M, d) > 0$ such that, for all $n \geq 2$ the following holds:*

$$\mathbb{P}(A_2(4n, 4Mn)) \geq \varepsilon, \tag{7}$$

or

$$\mathbb{P}(V(16Mn, 16M^2n)) \leq 1 - \varepsilon. \tag{8}$$

Proof of Lemma 2.1. The proof is by contradiction. Suppose the lemma is not true. Then there is an $M \geq 2$ such that for all $\varepsilon > 0$ there is an $n \geq 2$ with

$$\mathbb{P}(A_2(4n, 4Mn)) < \varepsilon, \quad (9)$$

and

$$\mathbb{P}(V(16Mn, 16M^2n)) > 1 - \varepsilon. \quad (10)$$

Fix such an M and choose ε very small (how small will become clear later). We will show that we get a contradiction. Take n such that (9) and (10) hold. First, observe that the event $V(16Mn, 16M^2n)$ is contained in the event that the ‘face’ $[0, 16M^2n]^{d-1} \times \{0\}$ is connected to the hyperplane $\{z_d = 16Mn\}$ by an open path in the half-space $\{z_d \geq 0\}$. By partitioning the above mentioned ‘face’ in ‘subfaces’ of length $2n$, and using a version of the standard ‘square-root trick’ (see e.g. [9]), we get that the probability of having an open path in the half-space $\{z_d \geq 0\}$ from $[0, 2n]^{d-1} \times \{0\}$ to the hyperplane $\{z_d = 16Mn\}$ is ‘large’, and hence that $\mathbb{P}(\Lambda(n) \leftrightarrow \partial\Lambda(16Mn))$ is ‘large.’ More precisely,

$$\mathbb{P}(\Lambda(n) \leftrightarrow \partial\Lambda(16Mn)) \geq 1 - \varepsilon^a, \quad (11)$$

for some $a = a(M, d) > 0$.

We proceed the proof with a renormalization argument. For $x \in \mathbb{Z}^d$, we use the notation $\Lambda(x; n)$ for the box obtained by translating $\Lambda(n)$ over the vector x . Note that if $x, y \in \mathbb{Z}^d$ are neighbours, then

$$\Lambda(nx; 4n) \supset \Lambda(ny; n), \quad (12)$$

and

$$\Lambda(ny; 16Mn) \supset \Lambda(nx; 4Mn). \quad (13)$$

Let, for each $x \in \mathbb{Z}^d$, $W(x)$ be the random set of vertices $\{z \in \Lambda(nx; n) : z \leftrightarrow \partial\Lambda(nx; 16Mn)\}$.

Definition 2.2. Call a point $x \in \mathbb{Z}^d$ *good* if the following holds:

$$W(x) \neq \emptyset \quad (14)$$

and any two open paths connecting $\Lambda(nx; 4n)$ with $\partial\Lambda(nx; 4Mn)$ are connected by an open path in $\Lambda(nx; 4Mn)$.

So, for each site x ,

$$\begin{aligned} \mathbb{P}(x \text{ is good}) &= \mathbb{P}(0 \text{ is good}) \\ &\geq \mathbb{P}(\Lambda(n) \leftrightarrow \partial\Lambda(16Mn)) - \mathbb{P}(A_2(4n, 4Mn)) \\ &\geq 1 - \varepsilon^a - \varepsilon, \end{aligned} \quad (15)$$

where the last inequality comes from (9) and (11). Note that the last expression converges to 1 as $\varepsilon \rightarrow 0$.

Further, suppose that two neighbours x and y are both good. Then, by definition, $W(x) \neq \emptyset$ and $W(y) \neq \emptyset$. Moreover, if $v \in W(x)$ and $z \in W(y)$ we have (again by definition), that $v \leftrightarrow \partial\Lambda(nx; 16Mn)$ and $z \leftrightarrow \partial\Lambda(ny; 16Mn)$. The corresponding path for v obviously crosses the annulus between $\Lambda(nx; 4n)$ and $\partial\Lambda(nx; 4Mn)$. By (12) and (13) the corresponding path for z also crosses that annulus. Hence by the second property in the definition of *good* we have that v and z are connected by an open path. Concluding, if two neighbours x and y are both good, then $W(x)$ and $W(y)$ are non-empty and every point of $W(x)$ has an open path to every point of $W(y)$. This gives immediately the following:

Observation 2.3. If there is an infinite path of good points, then there is an infinite open cluster.

The remaining part of the proof of Lemma 2.1 is quite standard and we summarize it as follows: By taking ε sufficiently small, we can make $\mathbb{P}(x \text{ is good})$ as large as we want (by (15)). Moreover, the process $(x \text{ is good} : x \in \mathbb{Z}^d)$ is finite-dependent (and the ‘dependence range’ depends only on M , which we have fixed). Hence, using general domination results in [13], we can take ε so small that the above mentioned process dominates a supercritical Bernoulli site percolation process. Finally, again using that whether or not x is *good* depends only on the states (open/closed) of the vertices in $\Lambda(nx; 16Mn)$, the function $p \rightarrow P_p(x \text{ is good})$ is continuous, and hence the above domination also holds for some p smaller than p_c . So, by Observation 2.3, $\theta(p) > 0$ for some $p < p_c$.

This is the desired contradiction and completes the proof of Lemma 2.1. \square

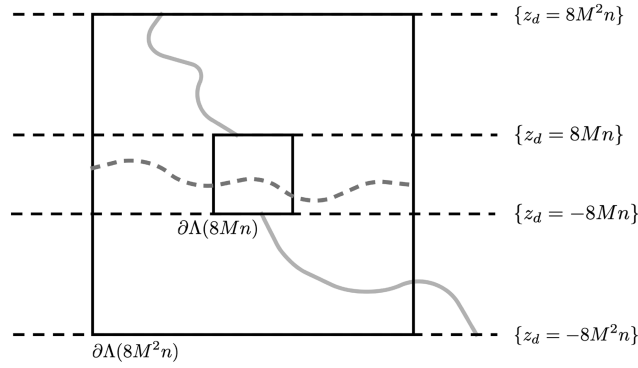


Fig. 1. The events E_1 , F_2 and F_3 imply the event $A_2(8Mn, 8M^2n)$.

2.1. Completion of the proof of Proposition 1.2

Fix an $M \geq 2$. Take $\varepsilon = \varepsilon(M)$ as in Lemma 2.1. Let $n \geq 2$. Suppose that (8) holds. Then (as is easy to see by a standard argument) there is an $\varepsilon^* = \varepsilon^*(\varepsilon) > 0$ such that the event $V(16Mn - 2, 16M^2n)$ has probability at most $1 - \varepsilon^*$. (We need this tiny modification of the event $V(16Mn, 16M^2n)$ to get independence of the events E_1 , F_2 and F_3 defined below). Hence with probability $\geq \varepsilon^*$ there is a ‘barrier’ of closed sites in the box $[-8M^2n, 8M^2n]^{d-1} \times [-8Mn, 8Mn]$ which separates top from bottom of that box (and which itself does not touch that top and bottom). Call this event E_1 .

See Figure 1. Let E_2 be the event that there is an open crossing in the d -direction of the box $[-8M^2n, 8M^2n]^{d-1} \times [8Mn, 8M^2n]$, and E_3 the analog of this event for the box obtained by reflection in the hyperplane $\{z_d = 0\}$. Note that the annulus between $\Lambda(8Mn)$ and $\partial\Lambda(8M^2n)$ is the union of $2d$ ‘copies’ (obtained by rotations and translations) of the box in the definition of E_2 , and that each path from $\Lambda(8Mn)$ to $\partial\Lambda(8M^2n)$ crosses at least one of those boxes in the ‘easy’ direction. Further, by a classical argument going back to the work by Kesten [11], the probability that there is such an open path is larger than some constant $\tilde{\varepsilon} = \tilde{\varepsilon}(M, d) > 0$. Hence, $\mathbb{P}(E_2) = \mathbb{P}(E_3) \geq \tilde{\varepsilon}/(2d)$. By partitioning $[-8M^2n, 8M^2n]^{d-1} \times [8Mn, 8M^2n]$ in translates of $[-8Mn, 8Mn]^{d-1} \times [8Mn, 8M^2n]$, we get that there is a constant $\varepsilon' = \varepsilon'(M, d) > 0$ such that $\mathbb{P}(F_2) = \mathbb{P}(F_3) > \varepsilon'$, where the event F_2 is defined by

$$F_2 = \left\{ \begin{array}{l} [-8Mn, 8Mn]^{d-1} \times [8Mn, 8M^2n] \\ \text{in the half-space } \{z_d \geq 8Mn\} \end{array} \right\}$$

and F_3 is the event obtained from F_2 by reflection in the hyperplane $\{z_d = 0\}$.

It is easy to see that the event $A_2(8Mn, 8M^2n)$ contains the event $E_1 \cap F_2 \cap F_3$ (see again Figure 1, where the red structure indicates the above mentioned ‘barrier’, and the two green paths represent the events F_2 and F_3), and hence, by independence,

$$\mathbb{P}(A_2(8Mn, 8M^2n)) \geq \mathbb{P}(E_1 \cap F_2 \cap F_3) \geq \varepsilon^*(\varepsilon')^2. \tag{16}$$

Concluding, we have that for each $n \geq 2$, (7) or (16) holds. From this, Proposition 1.2 follows immediately.

References

- [1] M. Aizenman, H. Kesten and C. M. Newman. Uniqueness of the infinite cluster and continuity of connectivity functions for short- and long-range percolation. *Comm. Math. Phys.* **111** (1987) 505–532. MR0901151
- [2] J. van den Berg and H. Don. A lower bound for point-to-point connection probabilities in critical percolation. *Electron. Commun. Probab.* **25** (2020) 47. MR4125794 <https://doi.org/10.1214/20-ecp326>
- [3] R. M. Burton and M. Keane. Density and uniqueness in percolation. *Comm. Math. Phys.* **121** (1989) 501–505. MR0990777
- [4] R. Cerf. A lower bound on the two-arms exponent for critical percolation on the lattice. *Ann. Probab.* **43** (2015) 2458–2480. MR3395466 <https://doi.org/10.1214/14-AOP940>
- [5] S. Chatterjee and S. Sen. Minimal spanning trees and Stein’s method. *Ann. Appl. Probab.* **27** (2017) 1588–1645. MR3678480 <https://doi.org/10.1214/16-AAP1239>
- [6] H. Duminil-Copin, G. Kozma and V. Tassion. Upper bounds on the percolation correlation length, 2020. Available at arXiv:1902.03207v2.
- [7] H. Duminil-Copin, I. Manolescu and V. Tassion. Planar random-cluster model: fractal properties of the critical phase, 2020. Available at arXiv:2007.14707.
- [8] A. Gandolfi, G. R. Grimmett and L. Russo. On the uniqueness of the infinite open cluster in the percolation model. *Comm. Math. Phys.* **114** (1988) 549–552. MR0929129

- [9] G. R. Grimmett. *Percolation*, 2nd edition. Springer, Berlin, 1999. MR1707339 <https://doi.org/10.1007/978-3-662-03981-6>
- [10] H. Kesten. The critical probability of bond percolation on the square lattice equals $\frac{1}{2}$. *Comm. Math. Phys.* **74** (1980) 41–59. MR0575895
- [11] H. Kesten. Analyticity properties and power law estimates in percolation theory. *J. Stat. Phys.* **25** (1981) 717–756. MR0633715 <https://doi.org/10.1007/BF01022364>
- [12] G. Kozma and A. Nachmias. Arm exponents in high dimensional percolation. *J. Amer. Math. Soc.* **24** (2011) 375–409. MR2748397 <https://doi.org/10.1090/S0894-0347-2010-00684-4>
- [13] T. M. Liggett, R. H. Schonmann and A. M. Stacey. Domination by product measures. *Ann. Probab.* **25** (1997) 71–95. MR1428500 <https://doi.org/10.1214/aop/1024404279>
- [14] S. Smirnov and W. Werner. Critical exponents for two-dimensional percolation. *Math. Res. Lett.* **8** (2001) 729–744. MR1879816 <https://doi.org/10.4310/MRL.2001.v8.n6.a4>