A Simple Optimal Contention Resolution Scheme for Uniform Matroids

Danish Kashaev * Richard Santiago †

Abstract

A common approach to tackle a combinatorial optimization problem is to first solve a continuous relaxation and then round the obtained fractional solution. For the latter, the framework of contention resolution schemes (or CR schemes), introduced by Chekuri, Vondrak, and Zenklusen, is a general and successful tool. A CR scheme takes a fractional point x in a relaxation polytope, rounds each coordinate x_i independently to get a possibly non-feasible set, and then drops some elements in order to satisfy the independence constraints. Intuitively, a CR scheme is c-balanced if every element i is selected with probability at least $c \cdot x_i$.

It is known that general matroids admit a $(1-1/e)$-balanced CR scheme, and that this is (asymptotically) optimal. This is in particular true for the special case of uniform matroids of rank one. In this work, we provide a simple and explicit monotone CR scheme with a balancedness of $1 - \binom{n}{k} \left(1 - \frac{k}{n}\right)^{n+1-k} \left(\frac{k}{n}\right)^k$, and show that this is optimal. As n grows, this expression converges from above to $1 - e^{-k}k^k/k!$. While this asymptotic bound can be obtained by combining previously known results, these require defining an exponential-sized linear program, as well as using random sampling and the ellipsoid algorithm. Our procedure, on the other hand, has the advantage of being simple and explicit. Moreover, this scheme generalizes into an optimal CR scheme for partition matroids.

1 Introduction

Contention resolution schemes were introduced by Chekuri, Vondrak, and Zenklusen [5] as a tool for submodular maximization under various types of constraints. A set function $f : 2^N \to \mathbb{R}$ is submodular if for any two sets $A \subseteq B \subseteq N$ and an element $v \notin B$, the corresponding marginal gains satisfy $f(A \cup \{v\}) - f(A) \geq f(B \cup \{v\}) - f(B)$. Submodular functions are a classical object in combinatorial optimization and operations research [11]. Given a finite ground set N, an independence family $\mathcal{I} \subseteq 2^N$ and a submodular set function $f : 2^N \mapsto \mathbb{R}$, the problem consists of (approximately) solving $\max_{S \in \mathcal{I}} f(S)$.

A successful technique to tackle this problem in recent years has been the relaxation and rounding approach. It consists of first relaxing the discrete problem into a continuous version $\max_{x \in P_\mathcal{I}} F(x)$, where $F : [0,1]^N \mapsto \mathbb{R}$ is a suitable extension of f, and $P_\mathcal{I}$ is a relaxation polytope of the independence family $\mathcal{I}$. One can assume here that $P_\mathcal{I}$ is the convex hull of all the independent sets, where every $S \in \mathcal{I}$ is naturally seen as an element in $\{0,1\}^N$. The first step of the relaxation and rounding approach then approximately solves $\max_{x \in P_\mathcal{I}} F(x)$ to obtain a fractional point $x \in P_\mathcal{I}$.

*Danish.Kashaev@cwi.nl, Centrum Wiskunde & Informatica, Amsterdam.
†rtorres@ethz.ch, ETH Zürich, Switzerland.

In order to get a feasible solution to the original problem, we then need to round this fractional point into an integral and feasible (i.e., independent) one while keeping the objective value as high as possible. Contention resolution schemes are a powerful tool to tackle this problem, and have found other applications outside of submodular maximization.

At the high level, given a fractional point x, the procedure first generates a random set $R(x)$ by independently including each element i with probability x_i. Since $R(x)$ might not necessarily belong to $\mathcal{I}$, the contention resolution scheme then removes some elements from it in order to get an independent set. We denote the support of a point x by $\mathrm{supp}(x) := \{i \in N \mid x_i > 0\}$. A CR scheme is then formally defined as follows.

Definition 1.1 (CR scheme)**.** $\pi = (\pi_x)_{x \in P_{\mathcal{I}}}$ is a c-balanced *contention resolution scheme* for the polytope $P_{\mathcal{I}}$ if for every $x \in P_{\mathcal{I}}$, π_x is an algorithm that takes as input a set $A \subseteq \mathrm{supp}(x)$ and outputs an independent set $\pi_x(A) \in \mathcal{I}$ contained in A such that

$$\mathbb{P}\Big[i \in \pi_x(R(x)) \mid i \in R(x)\Big] \geq c \quad \forall i \in \mathrm{supp}(x).$$

Moreover, a contention resolution scheme is *monotone* if for any $x \in P_{\mathcal{I}}$:

$$\mathbb{P}[i \in \pi_x(A)] \geq \mathbb{P}[i \in \pi_x(B)] \quad \text{for any } i \in A \subseteq B \subseteq \mathrm{supp}(x).$$

A c-balanced contention resolution scheme ensures that every element in the random set $R(x)$ is kept with probability at least c. The goal when designing CR schemes is thus to maximize such c, that we call the balancedness.

Moreover, monotonicity is a desirable property for a c-balanced CR scheme to have, since one can then get approximation guarantees for the constrained submodular maximization problem via the relaxation and rounding approach (see [5] for more details).

By presenting a variety of CR schemes for different constraints, the work in [5] gives improved approximation algorithms for linear and submodular maximization problems under matroid, knapsack, matchoid constraints, as well as their intersections. There has also been work done for getting CR schemes for different types of independence families [4, 9], or by having the elements of the random set $R(x)$ arrive in an online fashion [2, 8, 10, 1]. In this work, we restrict our attention to matroid constraints and the offline setting (i.e., we know the full set $R(x)$ in advance).

A monotone CR scheme with a balancedness of $1 - (1 - 1/n)^n$ for the uniform matroid of rank one is given in [6, 7], where it is also shown that this is optimal. That is, there is no c-balanced CR scheme for the uniform matroid of rank one with $c > 1 - (1 - 1/n)^n$. The work of [5] extends this result by proving the existence of a monotone $1 - (1 - 1/n)^n$-balanced CR scheme for any matroid. This requires defining an exponential-sized linear program and using its dual. The existence argument can then be turned into an efficient procedure by using random sampling and the ellipsoid algorithm to construct a CR scheme with a balancedness of $1 - 1/e$, which is asymptotically optimal.

By using a reduction from the previously mentioned existence proof ([5, see Theorem 4.3]) and a recent result from [3], one can prove the existence of a $(1 - e^{-k}k^k/k!)$-balanced CR scheme for the uniform matroid of rank k (i.e., cardinality constraints). The main drawback of this approach is, however, its lack of simplicity. In a different context, another procedure which may be interpreted as a monotone CR scheme for cardinality constraints with a slightly worse balancedness of $1 - 1/\sqrt{k+3}$ is shown in [2].

1.1 Our contributions

Our main result is to provide a simple, explicit, and optimal monotone CR scheme for the uniform matroid of rank k on n elements, with a balancedness of $c(k,n) := 1 - \binom{n}{k}\left(1 - \frac{k}{n}\right)^{n+1-k}\left(\frac{k}{n}\right)^k$. This result is encapsulated in Theorem 2.1 (balancedness), Theorem 2.4 (optimality), and Theorem 2.5 (monotonicity). This generalizes the balancedness factor of $1 - (1 - 1/n)^n$ given in [6, 7] for the rank one (i.e., $k = 1$) case. Moreover, for a fixed value of k, we have that $c(k,n)$ converges from above to $1 - e^{-k}k^k/k!$. While it is possible to prove the existence of a $(1 - e^{-k}k^k/k!)$-balanced CR scheme by combining results from [5, 3], these require defining an exponential-sized linear program and using its dual. In addition, to turn this existence proof into an actual algorithm, one needs to use random sampling and the ellipsoid method. The advantage of our CR scheme is thus that it is a very simple and explicit procedure. Moreover, our balancedness is an explicit formula which also depends on n (the number of elements) in addition to k, and $c(k,n) > 1 - e^{-k}k^k/k!$ for every fixed n. We also discuss how the above CR scheme for uniform matroids naturally generalizes to partition matroids.

1.2 Preliminaries on matroids

This section provides a brief background on matroids. A *matroid* $\mathcal{M}$ is a pair $(N, \mathcal{I})$ consisting of a ground set N and a non-empty family of *independent sets* $\mathcal{I} \subseteq 2^N$ which satisfy:

- If $A \in \mathcal{I}$ and $B \subseteq A$, then $B \in \mathcal{I}$.
- If $A \in \mathcal{I}$ and $B \in \mathcal{I}$ with $|A| > |B|$, then $\exists\, i \in A \backslash B$ such that $B \cup \{i\} \in \mathcal{I}$.

Given a matroid $\mathcal{M} = (N, \mathcal{I})$ its *rank function* $r : 2^N \to \mathbb{R}_{\geq 0}$ is defined as $r(A) = \max\{|S| : S \subseteq A,\ S \in \mathcal{I}\}$. Its *matroid polytope* is given by $P_{\mathcal{I}} := \operatorname{conv}(\{\mathbf{1}_S : S \in \mathcal{I}\}) = \{x \in \mathbb{R}^N_{\geq 0} : x(A) \leq r(A),\ \forall A \subseteq N\}$, where $x(A) := \sum_{i \in A} x_i$. Note that this implies $x_i \in [0,1]$ for every $i \in N$.

The next two classes of matroids are of special interest for this work.

Example 1.1 (Uniform matroid)**.** The uniform matroid of rank k on n elements $U_n^k := (N, \mathcal{I})$ is the matroid whose independent sets are all the subsets of the ground set of cardinality at most k. That is, $\mathcal{I} := \{A \subseteq N : |A| \leq k\}$. Its matroid polytope is $P_{\mathcal{I}} = \{x \in [0,1]^N : x(N) \leq k\}$.

Example 1.2 (Partition matroid)**.** Partition matroids are a generalization of uniform matroids. Suppose the ground set is partitioned into k blocks: $N = D_1 \uplus \cdots \uplus D_k$ and each block D_i has a certain capacity $d_i \in \mathbb{Z}_{\geq 0}$. The independent sets are then defined to be $\mathcal{I} := \{A \subseteq N : |A \cap D_i| \leq d_i,\ \forall i \in \{1, \ldots, k\}\}$. The matroid polytope in this case is $P_{\mathcal{I}} = \{x \in [0,1]^N : x(D_i) \leq d_i,\ \forall i \in \{1, \ldots, k\}\}$. The uniform matroid U_n^k is simply a partition matroid with one block N and one capacity k. Moreover, the restriction of a partition matroid to each block D_i is a uniform matroid of rank d_i on the ground set D_i.

2 An optimal monotone contention resolution scheme for uniform matroids

We assume throughout this whole section that $n \geq 2$ and that $k \in \{1, \ldots, n-1\}$. We denote by $P_{\mathcal{I}}$ the matroid polytope of U_n^k. For any point $x \in P_{\mathcal{I}}$, let $R(x)$ be the random set satisfying

$\mathbb{P}[i \in R(x)] = x_i$ independently for each coordinate. If the size of $R(x)$ is at most k, then $R(x)$ is already an independent set and the CR scheme returns it. If however $|R(x)| > k$, then the CR scheme returns a random subset of k elements by making the probabilities of each subset of k elements depend linearly on the coordinates of the original point $x \in P_{\mathcal{I}}$. More precisely, given an arbitrary $x \in P_{\mathcal{I}}$, for any set $A \subseteq \text{supp}(x)$ with $|A| > k$ and any subset $B \subseteq A$ of size k, we define

$$q_A(B) := \frac{1}{\binom{|A|}{|B|}} \Big(1 + \bar{x}(A \setminus B) - \bar{x}(B)\Big), \tag{2.1}$$

where we use the following notation: $\bar{x}(A) := \frac{1}{|A|} x(A) = \frac{1}{|A|} \sum_{i \in A} x_i$. We then define a randomized CR scheme π for U_n^k as follows.

Algorithm 2.1 (CR scheme π for U_n^k)**.** We are given a point $x \in P_{\mathcal{I}}$ and a set $A \subseteq \text{supp}(x)$.

- If $|A| \leq k$, then $\pi_x(A) = A$.
- If $|A| > k$, then for every $B \subseteq A$ with $|B| = k$, $\pi_x(A) = B$ with probability $q_A(B)$.

One can check that this CR scheme is well-defined, i.e., that q_A is a valid probability distribution.

Lemma 2.1. *The above procedure π is a well-defined CR scheme. That is, $\forall x \in P_{\mathcal{I}}$ and $A \subseteq \text{supp}(x)$, we have $q_A(B) \geq 0$ and $\sum_{B \subseteq A, |B|=k} q_A(B) = 1$.*

Proof. Since $\bar{x}(A \setminus B) \in [0,1]$ and $\bar{x}(B) \in [0,1]$, it directly follows from the definition (2.1) that $q_A(B) \geq 0$. In order to prove the second claim, we need the equality

$$\sum_{B \subseteq A, |B|=k} x(B) = \binom{|A|-1}{k-1} x(A) \tag{2.2}$$

that we derive the following way:

$$\begin{aligned}
\sum_{B \subseteq A, |B|=k} x(B) &= \sum_{B \subseteq A, |B|=k} \sum_{i \in A} x_i \mathbf{1}_{\{i \in B\}} = \sum_{i \in A} x_i \sum_{B \subseteq A, |B|=k} \mathbf{1}_{\{i \in B\}} \\
&= \sum_{i \in A} x_i \, |\{B \subseteq A \mid |B| = k, i \in B\}| = \binom{|A|-1}{k-1} x(A).
\end{aligned}$$

Hence,

$$\begin{aligned}
\sum_{B \subseteq A, |B|=k} q_A(B) &= \sum_{B \subseteq A, |B|=k} \frac{1}{\binom{|A|}{k}} \left(1 + \frac{x(A \setminus B)}{|A|-k} - \frac{x(B)}{k}\right) \\
&= 1 + \frac{1}{\binom{|A|}{k}} \sum_{B \subseteq A, |B|=k} \left(\frac{x(A)}{|A|-k} - \frac{x(B)}{|A|-k} - \frac{x(B)}{k}\right) \\
&= 1 + \frac{1}{\binom{|A|}{k} \left(|A|-k\right)} \sum_{B \subseteq A, |B|=k} \left(x(A) - \frac{|A|}{k} x(B)\right) \\
&= 1 + \frac{1}{\binom{|A|}{k} \left(|A|-k\right)} \left(\binom{|A|}{k} x(A) - \binom{|A|}{k} x(A)\right) = 1.
\end{aligned}$$

∎

$n \setminus k$	1	2	3	4	9	99	999
2	0.75						
3	0.704	0.852					
4	0.684	0.813	0.895				
5	0.672	0.793	0.862	0.918			
10	0.651	0.759	0.813	0.850	0.961		
100	0.633	0.732	0.779	0.810	0.874	0.996	
1000	0.632	0.730	0.776	0.809	0.869	0.962	0.999

Table 1: Numerical values for the balancedness $c(k, n)$ of Theorem 2.1

An important object in the design of CR schemes are the marginals. These are heavily used in [4] to design an optimal CR scheme for bipartite matchings. We provide an explicit formula for the marginals of our procedure in the following lemma. We postpone its proof to the appendix.

Lemma 2.2. *The marginals of the CR scheme defined in Algorithm 2.1 satisfy:*

$$\mathbb{P}[e \in \pi_x(A)] = \frac{k - x_e}{|A|} + \frac{x(A \setminus e)}{|A|(|A| - 1)}.$$

We now state our main result.

Theorem 2.1. *Algorithm 2.1 is a c-balanced CR scheme for the uniform matroid of rank k on n elements, where $c = 1 - \binom{n}{k} \left(1 - \frac{k}{n}\right)^{n+1-k} \left(\frac{k}{n}\right)^k$.*

Since we use the above expression often throughout this section, we denote it by

$$c(k, n) := 1 - \binom{n}{k} \left(1 - \frac{k}{n}\right)^{n+1-k} \left(\frac{k}{n}\right)^k.$$

We note that setting $k = 1$ gives $c(1, n) = 1 - (1 - 1/n)^n$, which matches the optimal balancedness for U_n^1 provided in [6, 7]. This converges to $1 - 1/e$ when n gets large. In addition, the balancedness factor improves as k grows (see Table 1).

Proposition 2.1. *For a fixed k, the limit of $c(k, n)$ as n tends to infinity is*

$$\lim_{n \to \infty} c(k, n) = 1 - e^{-k} \frac{k^k}{k!}.$$

Proof. We use Stirling's approximation, which states that:

$$n! \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n. \tag{2.3}$$

This means that these two quantities are asymptotic, i.e., their ratio tends to 1 if we tend n to infinity. By (2.3), we get

$$\frac{n!}{(n-k)!} \sim \sqrt{2\pi n} \left(\frac{n}{e}\right)^n \frac{1}{\sqrt{2\pi(n-k)}} \left(\frac{e}{n-k}\right)^{n-k} = e^{-k} \frac{n^n}{(n-k)^{n-k}} \sqrt{\frac{n}{n-k}}.$$

Using the above expression leads to the desired result:

$$1 - c(k,n) = \binom{n}{k} \left(1 - \frac{k}{n}\right)^{n+1-k} \left(\frac{k}{n}\right)^k = \frac{k^k}{k!} \frac{n!}{(n-k)!} \frac{(n-k)^{n+1-k}}{n^{n+1}}$$
$$\sim e^{-k} \frac{k^k}{k!} \frac{n-k}{n}\sqrt{\frac{n}{n-k}} = e^{-k} \frac{k^k}{k!} \sqrt{\frac{n-k}{n}} \sim e^{-k} \frac{k^k}{k!}. \qquad \square$$

2.1 Outline of the proof of Theorem 2.1

Throughout this whole section on uniform matroids, we fix an arbitrary element $e \in N$. In order to prove Theorem 2.1, we need to show that for every $x \in P_{\mathcal{I}}$ with $x_e > 0$ we have $\mathbb{P}[e \in \pi_x(R(x)) \mid e \in R(x)] \geq c(k,n)$. This is equivalent to showing that for every $x \in P_{\mathcal{I}}$ with $x_e > 0$ we have

$$\mathbb{P}[e \notin \pi_x(R(x)) \mid e \in R(x)] \leq 1 - c(k,n). \tag{2.4}$$

We now introduce some definitions and notation that will be needed. For any $B \subseteq A \subseteq N$, let $p_A(B) := \mathbb{P}[R_A(x) = B] = \prod_{i \in B} x_i \prod_{i \in A \setminus B}(1 - x_i)$, where $R_A(x)$ is the random set obtained by rounding each coordinate of $x|_A$ in the reduced ground set A to one independently with probability x_i. Note that $p_N(B) = \mathbb{P}[R(x) = B]$. We do not write the dependence on $x \in P_{\mathcal{I}}$ for simplicity of notation. We mainly work on the set $N \setminus \{e\}$. For this reason, we define $S := N \setminus \{e\}$. Note that $|S| = n - 1$; we use this often in our arguments.

With the above notation we can rewrite the probability in (2.4) in a more convenient form. For any $x \in P_{\mathcal{I}}$ satisfying $x_e > 0$, we get

$$\begin{aligned}\mathbb{P}\Big[e \notin \pi_x(R(x)) \mid e \in R(x)\Big] &= \sum_{A \subseteq S} \mathbb{P}[e \notin \pi_x(R(x)) \mid R_S(x) = A, e \in R(x)]\, \mathbb{P}[R_S(x) = A \mid e \in R(x)] \\ &= \sum_{A \subseteq S, |A| \geq k} \mathbb{P}\Big[e \notin \pi_x(R(x)) \mid R(x) = A \cup e\Big]\, p_S(A) \\ &= \sum_{A \subseteq S, |A| \geq k} p_S(A) \sum_{B \subseteq A, |B| = k} q_{A \cup e}(B).\end{aligned}$$

The obtained expression is a multivariable function of the variables $x_1, \ldots, x_n$, since $p_S(A)$ and $q_{A \cup e}(B)$ depend on those variables as well. We denote it as follows.

$$G(x) := \sum_{A \subseteq S, |A| \geq k} p_S(A) \sum_{B \subseteq A, |B| = k} q_{A \cup e}(B). \tag{2.5}$$

One then has that for proving Theorem 2.1 it is enough to show the following.

Theorem 2.2. *Let $G(x)$ and $c(k,n)$ be as defined above. Then $\max_{x \in P_{\mathcal{I}}} G(x) = 1 - c(k,n)$. Moreover, the maximum is attained at the point $(x_1, \ldots, x_n) = (k/n, \ldots, k/n) \in P_{\mathcal{I}}$.*

Indeed, Theorem 2.2 implies that for every $x \in P_{\mathcal{I}}$ we have $G(x) \leq 1 - c(k,n)$, with equality holding if $x = (k/n, \ldots, k/n)$. In particular, for any $x \in P_{\mathcal{I}}$ satisfying $x_e > 0$, we get

$$G(x) = \mathbb{P}\Big[e \notin \pi_x(R(x)) \mid e \in R(x)\Big] \leq 1 - c(k,n),$$

which proves Theorem 2.1 by (2.4).

Notice that for the conditional probability to be well defined, we need the assumption that $x_e > 0$. However, in our case $G(x)$ is simply a multivariable polynomial function of the n variables $x_1, \ldots, x_n$ and is thus also defined when $x_e = 0$. We may therefore forget the conditional probability and simply treat Theorem 2.2 as a multivariable maximization problem over a bounded domain. We now state the outline of the proof for Theorem 2.2.

We first maximize $G(x)$ over the variable x_e, and get an expression depending only on the x-variables in S. This is done in Section 2.2. We then maximize the above expression over the unit hypercube $[0,1]^S$ (see Section 2.3). Finally, we combine the above two results to show that the maximum in Theorem 2.2 is attained at the point $x_i = k/n$ for every $i \in N$; this is done in Section 2.4.

2.2 Maximizing over the variable x_e

The matroid polytope of U_n^k is given by $P_{\mathcal{I}} = \{x \in [0,1]^N : x(N) \le k\}$. We define a new polytope by removing the constraint $x_e \le 1$ from $P_{\mathcal{I}}$:

$$\widetilde{P}_{\mathcal{I}} := \{x \in \mathbb{R}^N_{\ge 0} : x(N) \le k \text{ and } x_i \le 1 \ \ \forall i \in S\}.$$

Clearly, $P_{\mathcal{I}} \subseteq \widetilde{P}_{\mathcal{I}}$.

We now present the main result of this section, where we consider the maximization problem $\max\{G(x) \mid x \in \widetilde{P}_{\mathcal{I}}\}$ and maximize $G(x)$ over the variable x_e while keeping all the other variables (x_i for every $i \in S$) fixed to get an expression depending only on the x-variables in S.

Lemma 2.3. *For every $x \in \widetilde{P}_{\mathcal{I}}$,*

$$G(x) \le \sum_{A \subseteq S, |A| = k} p_S(A)\Big(1 - \bar{x}(A)\Big). \tag{2.6}$$

Moreover, equality holds when $x_e = k - x(S)$.

Proof.

$$\begin{aligned} G(x) &= \sum_{A \subseteq S, |A| \ge k} p_S(A) \sum_{B \subseteq A, |B| = k} q_{A \cup e}(B) \\ &= \sum_{A \subseteq S, |A| \ge k} p_S(A) \sum_{B \subseteq A, |B| = k} \frac{1}{\binom{|A|+1}{k}} \Big(1 + \bar{x}\Big((A \setminus B) \cup e\Big) - \bar{x}(B)\Big) \\ &= \sum_{A \subseteq S, |A| \ge k} p_S(A) \frac{1}{\binom{|A|+1}{k}} \sum_{B \subseteq A, |B| = k} \left(1 + \frac{x(A \setminus B) + x_e}{|A| - k + 1} - \frac{x(B)}{k}\right). \end{aligned} \tag{2.7}$$

We now maximize this expression with respect to the variable x_e over $\widetilde{P}_{\mathcal{I}}$ while keeping all the other variables fixed. Since this is a linear function of x_e and the coefficient of x_e is positive, the maximal value will be $x_e = k - x(S)$ in order to satisfy the constraint $x(N) \le k$. Note that this was the reason for the definition of $\widetilde{P}_{\mathcal{I}}$, since $k - x(S)$ might not necessarily be smaller than 1. We thus plug-in $x_e = k - x(S)$ in (2.7) and write an inequality to emphasize that the derivation holds for any $x \in \widetilde{P}_{\mathcal{I}}$.

$$(2.7) \le \sum_{A\subseteq S,|A|\ge k} p_S(A)\frac{1}{\binom{|A|+1}{k}} \sum_{B\subseteq A,|B|=k}\left(1+\frac{x(A\setminus B)+k-x(S)}{|A|-k+1}-\frac{x(B)}{k}\right)$$
$$= \sum_{A\subseteq S,|A|\ge k} p_S(A)\frac{1}{\binom{|A|+1}{k}} \sum_{B\subseteq A,|B|=k}\left(1+\frac{k-x(S\setminus A)-x(B)}{|A|-k+1}-\frac{x(B)}{k}\right)$$
$$= \sum_{A\subseteq S,|A|\ge k} p_S(A)\frac{1}{\binom{|A|+1}{k}} \sum_{B\subseteq A,|B|=k}\left(\frac{|A|+1}{|A|-k+1}-\frac{x(S\setminus A)}{|A|-k+1}-\frac{|A|+1}{k(|A|-k+1)}x(B)\right). \tag{2.8}$$

Notice the only part which depends on B in the last summation is $x(B)$. By using Equation (2.2) and noticing that $\sum_{B\subseteq A,|B|=k} 1 = \binom{|A|}{k}$, we get

$$(2.8) = \sum_{A\subseteq S,|A|\ge k} p_S(A)\frac{1}{\binom{|A|+1}{k}}\frac{1}{|A|-k+1}\left(\binom{|A|}{k}(|A|+1)-\binom{|A|}{k}x(S\setminus A)-\frac{|A|+1}{k}\binom{|A|-1}{k-1}x(A)\right)$$
$$= \sum_{A\subseteq S,|A|\ge k} p_S(A)\frac{1}{\binom{|A|+1}{k}}\frac{1}{|A|-k+1}\binom{|A|}{k}\left(|A|+1-x(S\setminus A)-\frac{|A|+1}{|A|}x(A)\right)$$
$$= \sum_{A\subseteq S,|A|\ge k} \frac{p_S(A)}{|A|+1}\left(|A|+1-x(S\setminus A)-\frac{|A|+1}{|A|}x(A)\right)$$
$$= \sum_{A\subseteq S,|A|\ge k} p_S(A)\left(1-\frac{x(S\setminus A)}{|A|+1}-\frac{x(A)}{|A|}\right). \tag{2.9}$$

Now, note that by definition of the term $p_S(A)$, we have

$$x_i\, p_S(A) = (1-x_i)\, p_S(A\cup i) \quad \text{for any } i\in S\setminus A. \tag{2.10}$$

We compute the middle term in (2.9) by plugging in (2.10) and the change of variable $B := A\cup i$.

$$\sum_{A\subseteq S,|A|\ge k} \frac{1}{|A|+1} p_S(A)\, x(S\setminus A) = \sum_{A\subseteq S,|A|\ge k}\sum_{i\in S}\frac{1}{|A|+1}\, x_i\, p_S(A)\, \mathbf{1}_{\{i\notin A\}}$$
$$= \sum_{i\in S}\sum_{A\subseteq S,|A|\ge k}\frac{1}{|A|+1}\,(1-x_i)\, p_S(A\cup i)\,\mathbf{1}_{\{i\notin A\}}$$
$$= \sum_{i\in S}\sum_{B\subseteq S,|B|\ge k+1}\frac{1}{|B|}\,(1-x_i)\, p_S(B)\,\mathbf{1}_{\{i\in B\}}$$
$$= \sum_{B\subseteq S,|B|\ge k+1}\frac{1}{|B|}\, p_S(B)\sum_{i\in S}\mathbf{1}_{\{i\in B\}} - \sum_{B\subseteq S,|B|\ge k+1}\frac{1}{|B|}\, p_S(B)\sum_{i\in S} x_i\mathbf{1}_{\{i\in B\}}$$
$$= \sum_{B\subseteq S,|B|\ge k+1} p_S(B) - \sum_{B\subseteq S,|B|\ge k+1}\frac{p_S(B)}{|B|}\, x(B)$$
$$= \sum_{B\subseteq S,|B|\ge k+1} p_S(B)\left(1-\frac{x(B)}{|B|}\right)$$

$$= \sum_{A\subseteq S, |A|\geq k+1} p_S(A)\left(1-\frac{x(A)}{|A|}\right). \tag{2.11}$$

We finally plug-in (2.11) into (2.9) and use $\sum_{A\subseteq S,|A|\geq k} = \sum_{A\subseteq S,|A|\geq k+1} + \sum_{A\subseteq S,|A|=k}$ to get

$$(2.9) = \sum_{A\subseteq S,|A|=k} p_S(A)\left(1-\frac{x(A)}{|A|}\right) = \sum_{A\subseteq S,|A|=k} p_S(A)\Big(1-\bar{x}(A)\Big).$$

Notice that the only place where we used an inequality was from (2.7) to (2.8). Hence equality holds when $x_e = k - x(S)$. □

2.3 Maximizing $h_S^k : [0,1]^S \mapsto \mathbb{R}$

In this section, we turn our attention into maximizing the right-hand side expression in (2.6) over the unit hypercube $[0,1]^S$. In fact, we work with the following function instead:

$$h_S^k(x) := \sum_{A\subseteq S,\, |A|=k} p_S(A)(k - x(A)).$$

A plot of $h_S^k(x)$ for $S = \{1,2\}$ and $k = 1,2$ is presented in Figure 1. Note that the above function is simply the right hand side of (2.6) multiplied by k. Hence, maximizing one or the other is equivalent.

Theorem 2.3. *Let $n \geq 2$, so that $|S| = n-1 \geq 1$ and $k \in \{1,\ldots,n-1\}$. Then the function $h_S^k(x)$ attains its maximum over the unit hypercube $[0,1]^S$ at the point $(k/n,\ldots,k/n)$ with value*

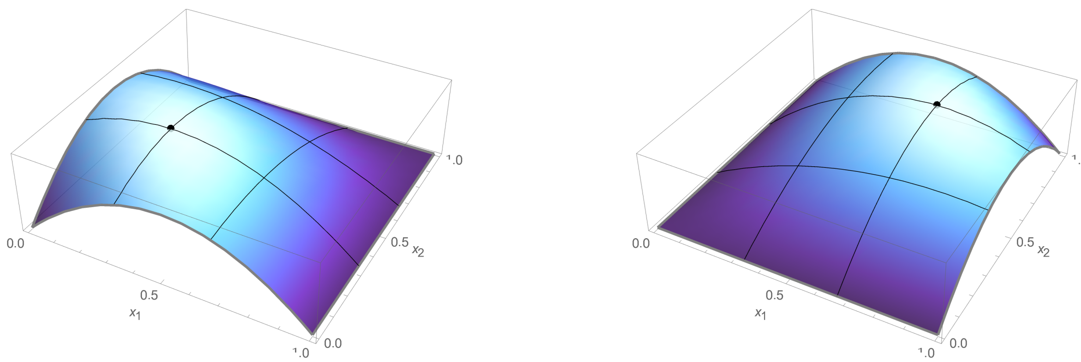
$$h_S^k\Big(k/n,\ldots,k/n\Big) = k\binom{n}{k}\left(1-\frac{k}{n}\right)^{n+1-k}\left(\frac{k}{n}\right)^k = k\Big(1-c(k,n)\Big).$$

For simplicity, we denote this maximum value by:

$$\alpha(k,n) := k\binom{n}{k}\left(1-\frac{k}{n}\right)^{n+1-k}\left(\frac{k}{n}\right)^k.$$

Notice that $h_S^0(x) = h_S^n(x) = 0$ for any $x \in [0,1]^S$. Hence Theorem 2.3 holds for $k = 0$ and $k = n$ as well. Moreover, the function $h_S^k(x)$ also satisfies an interesting duality property: $h_S^k(x) = h_S^{n-k}(1-x)$.

In order to prove Theorem 2.3, we first show that h has a unique extremum (in particular a local maximum) in the interior of $[0,1]^S$ at the point $(k/n,\ldots,k/n)$ — see Proposition 2.2. We then use induction on n to show that any point in the boundary of $[0,1]^S$ has a lower function value than $h_S^k(k/n,\ldots,k/n)$. Since our function is continuous over a compact domain, it attains a maximum. That maximum then has to be attained at $(k/n,\ldots,k/n)$ by the two arguments above. That is, the unique extremum cannot be a local minimum or a saddle point. Otherwise, since there are no more extrema in the interior and the function is continuous, the function would increase in some direction leading to a point in the boundary having higher value. For completion, in the appendix, we present another proof showing local maximality that relies on the Hessian matrix.



(a) $S = \{1, 2\}, k = 1$ (b) $S = \{1, 2\}, k = 2$

Figure 1: Plot of $h_S^k(x)$ for $S = \{1, 2\}$. The maximum is attained at $x_1 = x_2 = 1/3$ in (a) and at $x_1 = x_2 = 2/3$ in (b).

Proposition 2.2. *For any $k \in \{1, \dots, n-1\}, h_S^k(x)$ has a unique extremum in the interior of the unit hypercube $[0,1]^S$ at the point $(k/n, \dots, k/n)$.*

For proving Proposition 2.2 we need the following lemma. We leave its proof to the appendix.

Lemma 2.4. *The following holds for any $x \in [0,1]^S$:*

$$h_S^k(x) = \sum_{i=0}^{k-1} Q_S^i(x) \big(x(S) - i\big)$$

where

$$Q_S^k(x) := \sum_{A \subseteq S, |A| = k} p_S(A).$$

The above formula actually holds for h_A^k with any $A \subseteq N$. We use this in Section 2.5 with $A = N$. We are now able to prove Proposition 2.2.

Proof of Proposition 2.2. Let $k \in \{1, \dots, n-1\}$. To find the extrema of $h_S^k : [0,1]^S \mapsto \mathbb{R}$, we want to solve $\nabla h_S^k(x) = 0$. We thus first need to compute the partial derivatives $\frac{\partial h_S^k(x)}{\partial x_i}$ for every $i \in S$. Note that for a set $A \subseteq S$ such that $i \in A$, we have

$$\begin{aligned}
\frac{\partial}{\partial x_i} p_S(A)(k - x(A)) &= (k - x(A)) \prod_{j \in A \setminus i} x_j \prod_{j \in S \setminus A} (1 - x_j) - p_S(A) \\
&= (k - x(A)) \prod_{j \in A \setminus i} x_j \prod_{j \in S \setminus A} (1 - x_j) - x_i \prod_{j \in A \setminus i} x_j \prod_{j \in S \setminus A} (1 - x_j) \\
&= (k - x(A))\, p_{S \setminus i}(A \setminus i) - x_i\, p_{S \setminus i}(A \setminus i) = p_{S \setminus i}(A \setminus i)\Big(k - x(A \setminus i) - 2x_i\Big).
\end{aligned}$$

For a set $A \subseteq S$ such that $i \notin A$, we get

$$\frac{\partial}{\partial x_i} p_S(A)(k - x(A)) = -(k - x(A)) \prod_{j \in A} x_j \prod_{j \in (S \setminus A) \setminus i} (1 - x_j) = -p_{S \setminus i}(A)\Big(k - x(A)\Big).$$

We then have

$$\begin{aligned}
\frac{\partial h_S^k(x)}{\partial x_i} &= \frac{\partial}{\partial x_i}\sum_{\substack{A\subseteq S\\|A|=k}} p_S(A)(k-x(A)) = \sum_{\substack{A\subseteq S\\|A|=k\\i\in A}} \frac{\partial}{\partial x_i} p_S(A)(k-x(A)) + \sum_{\substack{A\subseteq S\\|A|=k\\i\notin A}} \frac{\partial}{\partial x_i} p_S(A)(k-x(A))\\
&= \sum_{\substack{A\subseteq S\\|A|=k\\i\in A}} p_{S\setminus i}(A\setminus i)\Big(k-x(A\setminus i)-2x_i\Big) - \sum_{\substack{A\subseteq S\\|A|=k\\i\notin A}} p_{S\setminus i}(A)\Big(k-x(A)\Big)\\
&= \sum_{\substack{B\subseteq S\setminus i\\|B|=k-1}} p_{S\setminus i}(B)\Big(k-x(B)-2x_i\Big) - \sum_{\substack{A\subseteq S\setminus i\\|A|=k}} p_{S\setminus i}(A)\Big(k-x(A)\Big)\\
&= \sum_{\substack{A\subseteq S\setminus i\\|A|=k-1}} p_{S\setminus i}(A)\Big(k-1-x(A)+1-2x_i\Big) - h_{S\setminus i}^k(x)\\
&= \sum_{\substack{A\subseteq S\setminus i\\|A|=k-1}} p_{S\setminus i}(A)\Big(k-1-x(A)\Big) + (1-2x_i)\sum_{\substack{A\subseteq S\setminus i\\|A|=k-1}} p_{S\setminus i}(A) - h_{S\setminus i}^k(x)\\
&= (1-2x_i)Q_{S\setminus i}^{k-1}(x) - \Big(h_{S\setminus i}^k(x) - h_{S\setminus i}^{k-1}(x)\Big)\\
&= (1-2x_i)Q_{S\setminus i}^{k-1}(x) - Q_{S\setminus i}^{k-1}(x)\Big(x(S\setminus i)-(k-1)\Big) = Q_{S\setminus i}^{k-1}(x)\Big(k-x(S)-x_i\Big)
\end{aligned}$$

where we use Lemma 2.4 in the last line. From the above it follows that $\nabla h_S^k(x)=0$ if and only if

$$Q_{S\setminus i}^{k-1}(x)\Big(k-x(S)-x_i\Big)=0 \quad \forall i\in S. \tag{2.12}$$

Notice that

$$\begin{aligned}
Q_{S\setminus i}^{k-1}(x)=0 \iff \sum_{\substack{A\subseteq S\setminus i\\|A|=k-1}} p_{S\setminus i}(A)=0 &\iff p_{S\setminus i}(A)=0 \quad \forall A\subseteq S\setminus i, |A|=k-1\\
&\iff \prod_{j\in A} x_j \prod_{j\in(S\setminus i)\setminus A}(1-x_j)=0 \quad \forall A\subseteq S, |A|=k-1.
\end{aligned}$$

We can see this implies that such a solution lies on the boundary of $[0,1]^S$, since there exists an index $j\in S$ such that $x_j=0$ or $x_j=1$. Since we are focusing on extrema in the interior, we may disregard that solution. Hence, by (2.12), we have $x_i=k-x(S)$ for all $i\in S$. By setting $x_i=t$ for every $i\in S$, we get

$$t=k-(n-1)t \iff t=k/n \iff x_i=k/n \quad \forall i\in S.$$

Therefore, $h_S^k(x)$ has a unique extremum in the interior of $[0,1]^S$ at the point $(k/n,\dots,k/n)$. □

In order to prove Theorem 2.3, we need one additional lemma; we leave its proof to the appendix.

Lemma 2.5. *The following holds for any $n\geq 2$ and $k\in\{1,\dots,n-1\}$:*

$$\alpha(k,n) > \alpha(k-1,n-1), \tag{2.13}$$

$$\alpha(k,n) > \alpha(k,n-1). \tag{2.14}$$

Proof of Theorem 2.3. We prove the statement by induction on $n \geq 2$. The base case corresponds to $n = 2$ and $k = 1$. In this case, we get $S = \{1\}$ and $h_S^k(x) = x_1(1 - x_1)$. It is easy to see that this is a parabola which attains its maximum at the point $x_1 = 1/2$ over the unit interval $[0, 1]$. Moreover the function value at that point is $1/4 = \alpha(1, 2)$.

We now prove the induction step. Let $n \geq 3$ and $k \in \{1, \ldots, n-1\}$, and assume by induction hypothesis that the statement holds for any $2 \leq n' < n$ and $k \in \{1, 2, \ldots, n'-1\}$.

By Proposition 2.2, $h_S^k(x)$ has a unique extremum in the interior of $[0, 1]^S$ at the point $(k/n, \ldots, k/n)$. We first show that the function $h_S^k(x)$ evaluated at that point is indeed equal to $\alpha(k, n)$.

$$\begin{aligned} h_S^k(k/n, \ldots, k/n) &= \binom{n-1}{k}\left(\frac{k}{n}\right)^k \left(1 - \frac{k}{n}\right)^{n-1-k}\left(k - k\frac{k}{n}\right) = k\binom{n-1}{k}\left(\frac{k}{n}\right)^k\left(1 - \frac{k}{n}\right)^{n-k} \\ &= k\frac{n-k}{n}\binom{n}{k}\left(\frac{k}{n}\right)^k\left(1 - \frac{k}{n}\right)^{n-k} = \alpha(k, n). \end{aligned}$$

We next show that any point on the boundary of $[0, 1]^S$ has a lower function value than $\alpha(k, n)$. A point $x \in [0, 1]^S$ lies on the boundary if there exists $i \in S$ such that $x_i = 0$ or $x_i = 1$.

- Suppose there exists $i \in S$ such that $x_i = 0$. For any set $A \subseteq S$ containing i, we get $p_S(A) = 0$. Hence:

$$h_S^k(x) = \sum_{A \subseteq S, |A| = k} p_S(A)(k - x(A)) = \sum_{A \subseteq S \setminus i, |A| = k} p_{S \setminus i}(A)(k - x(A)) = h_{S \setminus i}^k(x).$$

  If $k = n - 1$, then $h_{S \setminus i}^k(x) = 0$. We then clearly get $h_S^k(x) = h_{S \setminus i}^k(x) = 0 < \alpha(k, n)$. If $k < n - 1$, then by induction hypothesis and Lemma 2.5,

$$h_S^k(x) = h_{S \setminus i}^k(x) \leq \alpha(k, n-1) < \alpha(k, n).$$

- Suppose there exists $i \in S$ such that $x_i = 1$. For any set $A \subseteq S$ not containing i, we get $p_S(A) = 0$. Hence:

$$\begin{aligned} h_S^k(x) &= \sum_{A \subseteq S, |A| = k} p_S(A)(k - x(A)) = \sum_{\substack{A \subseteq S, |A| = k \\ i \in A}} p_S(A)(k - x(A)) \\ &= \sum_{\substack{A \subseteq S, |A| = k \\ i \in A}} p_{S \setminus i}(A \setminus i)(k - 1 - x(A \setminus i)) = \sum_{\substack{A \subseteq S \setminus i \\ |A| = k-1}} p_{S \setminus i}(A)(k - 1 - x(A)) = h_{S \setminus i}^{k-1}(x). \end{aligned}$$

  If $k = 1$, then $h_{S \setminus i}^{k-1}(x) = 0$. We then clearly get $h_S^k(x) = h_{S \setminus i}^{k-1}(x) = 0 < \alpha(k, n)$. If $k > 1$, then by induction hypothesis and Lemma 2.5,

$$h_S^k(x) = h_{S \setminus i}^{k-1}(x) \leq \alpha(k - 1, n - 1) < \alpha(k, n).$$

Since our function is continuous over a compact domain, it attains a maximum. By using continuity, together with the facts that $(k/n, \ldots, k/n)$ is the unique extremum in the interior, and that it has higher function value than any point at the boundary, it follows that $(k/n, \ldots, k/n)$ must be a global maximum. This completes the proof. $\square$

2.4 Proof of Theorem 2.1

We now have all the ingredients to prove Theorem 2.2 and, therefore, Theorem 2.1. The two main building blocks for the proof are Lemma 2.3 and Theorem 2.3.

Proof of Theorem 2.2. By Lemma 2.3, we get that for any $x \in P_{\mathcal{I}}$ (since $P_{\mathcal{I}} \subseteq \widetilde{P}_{\mathcal{I}}$),

$$G(x) \leq \sum_{A \subseteq S,\, |A|=k} p_S(A)(1-\bar{x}(A)). \tag{2.15}$$

Moreover, for every $x \in P_{\mathcal{I}}$ satisfying $x_e = k - x(S)$, equality holds in (2.15).
By Theorem 2.3, we get that for any $x \in P_{\mathcal{I}}$,

$$\sum_{A \subseteq S,\, |A|=k} p_S(A)(1-\bar{x}(A)) \leq 1 - c(k,n). \tag{2.16}$$

Equality holds in (2.16) if $x_i = k/n$ for every $i \in S$. This holds because the above expression does not depend on x_e, and the projection of the polytope $P_{\mathcal{I}}$ to the S coordinates is included in the unit hypercube $[0,1]^S$.

Therefore, by combining (2.15) and (2.16), we get that $G(x) \leq 1 - c(k,n)$ for every $x \in P_{\mathcal{I}}$. Moreover, for the point $x_i = k/n$ for every $i \in N$, equality holds: $G(k/n, \ldots, k/n) = 1 - c(k,n)$. Indeed, (2.15) holds with equality because $x_e = k - x(S)$ is satisfied (since $k - x(S) = k - (n-1)k/n = k/n$), and (2.16) also holds with equality because $x_i = k/n$ for every $i \in S$. □

2.5 Optimality

In this section, we argue that a balancedness of $c(k,n)$ is in fact optimal for U_n^k.

Theorem 2.4. *There does not exist a c-balanced CR scheme for the uniform matroid of rank k on n elements satisfying* $c > 1 - \binom{n}{k}\left(1-\frac{k}{n}\right)^{n+1-k}\left(\frac{k}{n}\right)^k$.

The proof uses a similar argument to the one used for U_n^1 in [5]. It relies on computing the value $\mathbb{E}\big[r(R(x)\big]$, i.e., the expected rank of the random set $R(x)$. However, for values of $k > 1$, the argument becomes more involved than the one presented in [5]. Our proof uses Lemma 2.4.

Corollary 2.1 (of Lemma 2.4)**.** *Let $x \in P_{\mathcal{I}}$ be the point $x_i = k/n$ for all $i \in N$. Then* $h_N^k(x) = \sum_{i=0}^{k-1} Q_N^i(x)(k-i)$.

Proof of Theorem 2.4. Let π be an arbitrary c-balanced CR scheme for U_n^k, and fix the point x such that $x_i = \frac{k}{n}$ for every $i \in N$. Clearly, $x \in P_{\mathcal{I}} = \{x \in [0,1]^N : x_1 + \cdots + x_n \leq k\}$. Let $R(x)$ be the random set satisfying $\mathbb{P}[i \in R(x)] = x_i$ for each i independently, and denote by $I := \pi_x(R(x))$ the set returned by the CR scheme π. By definition of a CR scheme, we have $\mathbb{E}[|I|] \leq \mathbb{E}\big[r(R(x))\big]$ and $\mathbb{E}[|I|] = \sum_{i \in N} \mathbb{P}[i \in I] \geq \sum_{i \in N} c\, x_i = \frac{nck}{n} = ck$. It follows that $c \leq \mathbb{E}\big[r(R(x))\big]/k$. Moreover, recall that

$$\mathbb{P}[|R(x)| = i] = \sum_{A \subseteq N,\, |A|=i} p_N(A) = Q_N^i(x). \tag{2.17}$$

We then have

$$\mathbb{E}[r(R(x))] = \sum_{i=0}^{k} i\, \mathbb{P}[r(R(x)) = i] = \sum_{i=0}^{k-1} i\, \mathbb{P}\big[|R(x)| = i\big] + k\, \mathbb{P}\big[|R(x)| \geq k\big]$$

$$
\begin{aligned}
&= \sum_{i=0}^{k-1} i\, \mathbb{P}\big[|R(x)| = i\big] + k \left(1 - \mathbb{P}\big[|R(x)| \leq k-1\big]\right) \\
&= k + \sum_{i=0}^{k-1} i\, \mathbb{P}\big[|R(x)| = i\big] - k \sum_{i=0}^{k-1} \mathbb{P}\big[|R(x)| = i\big] = k - \sum_{i=0}^{k-1} (k-i) Q_N^i(x) = k - h_N^k(x) \\
&= k - \sum_{A \subseteq N, |A|=k} p_N(A)(k - x(A)) = k - \sum_{A \subseteq N, |A|=k} \left(\frac{k}{n}\right)^k \left(1 - \frac{k}{n}\right)^{n-k} \left(k - k\frac{k}{n}\right) \\
&= k \left(1 - \binom{n}{k} \left(1 - \frac{k}{n}\right)^{n+1-k} \left(\frac{k}{n}\right)^k\right).
\end{aligned}
$$

where the last two equalities in the third line follow by (2.17) and Corollary 2.1 respectively. Combining this with the bound $c \leq \mathbb{E}\big[r(R(x))\big]/k$ leads to the desired result. □

2.6 Monotonicity

We next argue that Algorithm 2.1 is a monotone CR scheme. This is a desirable property for CR schemes, since they can then be used to derive approximation guarantees for constrained submodular maximization problems. We need Lemma 2.2, i.e., the marginals of the CR scheme.

Theorem 2.5. *Algorithm 2.1 is a monotone CR scheme for U_n^k. That is, for every $x \in P_{\mathcal{I}}$ and $e \in A \subseteq B \subseteq supp(x)$, we have $\mathbb{P}[e \in \pi_x(A)] \geq \mathbb{P}[e \in \pi_x(B)]$.*

Proof. Let $A \subseteq \text{supp}(x)$ and $e \in A$. If $|A| \leq k$, then $\mathbb{P}[e \in \pi_x(A)] = 1$, and the theorem trivially holds. We therefore suppose that $|A| > k$. In order to prove the theorem, it is clearly enough to show that for any $f \in \text{supp}(x) \setminus A$,

$$
\mathbb{P}[e \in \pi_x(A)] \geq \mathbb{P}[e \in \pi_x(A \cup \{f\})]. \tag{2.18}
$$

We show the difference of those two terms is greater than 0 by using Lemma 2.2 for both terms:

$$
\begin{aligned}
\mathbb{P}[e \in \pi_x(A)] - \mathbb{P}[e \in \pi_x(A \cup \{f\})] &= \frac{k - x_e}{|A|} + \frac{x(A \setminus e)}{|A|(|A|-1)} - \frac{k - x_e}{|A|+1} - \frac{x(A \setminus e) + x_f}{(|A|+1)|A|} \\
&= \frac{k - x_e}{|A|} - \frac{k - x_e}{|A|+1} - \frac{x_f}{(|A|+1)|A|} + x(A \setminus e)\left(\frac{1}{|A|(|A|-1)} - \frac{1}{(|A|+1)|A|}\right) \\
&= \frac{(|A|+1)(k - x_e) - |A|(k - x_e) - x_f}{|A|(|A|+1)} + \frac{2x(A \setminus e)}{(|A|^2-1)|A|} \\
&= \frac{k - x_e - x_f}{|A|(|A|+1)} + \frac{2x(A \setminus e)}{(|A|^2-1)|A|} \geq 0.
\end{aligned}
$$

The last inequality holds because since $x \in P_{\mathcal{I}} = \{x \in [0,1]^N \mid x(N) \leq k\}$, we have $x_e + x_f \leq k$, and all the other terms are positive. We have thus shown (2.18) which is enough to prove the theorem. □

2.7 Extension to partition matroids

A CR scheme for uniform matroids can be naturally extended to a CR scheme for *partition matroids*. This is not surprising since partition matroids can be seen as a direct sum of uniform matroids —

see Example 1.2. For completeness, in this section we discuss how the results from Section 2 lead to an optimal CR scheme for partition matroids. This is encapsulated in the following two results.

Proposition 2.3. *Let $\mathcal{M} = (N, \mathcal{I})$ be a partition matroid given by $\mathcal{I} = \{A \subseteq N : |A \cap D_i| \leq d_i,\ \forall i \in \{1, \ldots, k\}\}$. If there is a (monotone) $\alpha(k, n)$-balanced CR scheme for the uniform matroid U_n^k, then there is a (monotone) α-balanced CR scheme for $\mathcal{M}$, where $\alpha = \min_{i \in [k]} \alpha(d_i, |D_i|)$.*

Proof. For each $i \in [k]$, let π^i be an $\alpha(d_i, |D_i|)$-balanced CR scheme for the uniform matroid $U_{|D_i|}^{d_i}$. Let P_i denote the matroid polytope of the uniform matroid $U_{|D_i|}^{d_i}$, and $P_{\mathcal{I}}$ denote the matroid polytope of the partition matroid $\mathcal{M} = (N, \mathcal{I})$. Given any $x \in P_{\mathcal{I}}$, let $x^i \in [0,1]^{D_i}$ denote the restriction of x to D_i. Since $x \in P_{\mathcal{I}}$, it is clear that $x^i \in P_i$.

Consider the CR scheme π defined as follows, $\pi_x(A) = \biguplus_{i \in [k]} \pi^i_{x^i}(A \cap D_i)$. That is, we run the CR schemes π^i independently in each partition D_i and take the (disjoint) union of their outputs. Let $e \in N$ be such that $e \in D_i$. Then, $\mathbb{P}[e \in \pi_x(R(x))] = \mathbb{P}[e \in \pi^i_{x^i}(R(x) \cap D_i)] \geq \alpha(d_i, |D_i|) \cdot x_e$. Hence, it follows that π is an α-balanced CR scheme for $\mathcal{M}$, where $\alpha = \min_{i \in [k]} \alpha(d_i, |D_i|)$.

For the monotonicity part, assume that the CR schemes π^i defined above are all monotone. Let $x \in P_{\mathcal{I}}$, $e \in A \subseteq B$, and $i \in [k]$ be the unique index such that $e \in D_i$. Then, $\mathbb{P}[e \in \pi_x(A)] = \mathbb{P}[e \in \pi^i_{x_i}(A \cap D_i)] \geq \mathbb{P}[e \in \pi^i_{x_i}(B \cap D_i)] = \mathbb{P}[e \in \pi_x(B)]$. Hence π is also monotone. □

Proposition 2.4. *Let $\mathcal{M} = (N, \mathcal{I})$ be a partition matroid given by $\mathcal{I} = \{A \subseteq N : |A \cap D_i| \leq d_i,\ \forall i \in \{1, \ldots, k\}\}$. If there is no $\alpha(k, n)$-balanced CR scheme for the uniform matroid U_n^k, then there is no α-balanced CR scheme for $\mathcal{M}$, where $\alpha = \min_{i \in [k]} \alpha(d_i, |D_i|)$.*

Proof. Assume such a CR scheme π exists, and let $j \in \text{argmin}_{i \in \{1, \ldots, k\}} \alpha(d_i, |D_i|)$. Let P denote the matroid polytope of the uniform matroid $U_{|D_j|}^{d_j}$, and let $P_{\mathcal{I}}$ denote the matroid polytope of the partition matroid $\mathcal{M} = (N, \mathcal{I})$. Then, for any $\bar{x} \in P$, let $x \in [0,1]^N$ be defined as $x_e = \bar{x}_e$ if $e \in D_j$, and $x_e = 0$ otherwise. Clearly, $x \in P_{\mathcal{I}}$ since $\bar{x} \in P$. Hence, $\mathbb{P}[e \in \pi_x(R(x)) \mid e \in R(x)] \geq \alpha = \alpha(d_j, |D_j|)$. But this contradicts the assumption that there is no $\alpha(d_j, |D_j|)$-balanced CR scheme for the uniform matroid $U_{|D_j|}^{d_j}$. □

3 Conclusion

Contention resolution schemes are a general and powerful tool for rounding a fractional point in a relaxation polytope. It is known that matroids admit $(1 - 1/e)$-balanced CR schemes, and that this is the best possible. This impossibility result is in particular true for uniform matroids of rank one. For uniform matroids of rank k (i.e., cardinality constraints), one can get a $(1 - e^{-k} k^k / k!)$-balanced CR scheme by combining a reduction from [5] and a recent result from [3]. The main drawback of this approach, however, is its lack of simplicity. In this work, we provide an explicit and much simpler scheme with a balancedness of $c(k,n) := 1 - \binom{n}{k} \left(1 - \frac{k}{n}\right)^{n+1-k} \left(\frac{k}{n}\right)^k$. In particular, $c(k,n) > 1 - e^{-k} k^k / k!$ for every n, and $c(k,n)$ converges to $1 - e^{-k} k^k / k!$ as n goes to infinity. Our balancedness is therefore better for every fixed n, and achieves $1 - e^{-k} k^k / k!$ asymptotically. We also show optimality and monotonicity of our scheme, and discuss how it naturally extends to an optimal CR scheme for partition matroids.

We believe that finding other classes of matroids where the $1 - 1/e$ balancedness factor can be improved is an interesting direction for future work. Moreover, while this work focused on the offline setting, it also seems interesting to study the optimal balancedness for uniform matroids when the elements of $R(x)$ arrive in an online fashion (e.g., in the case of random or online contention resolution schemes).

Acknowledgements

We thank Chandra Chekuri and Vasilis Livanos for useful feedback, and for pointing out the connection with the work of [3].

References

[1] Marek Adamczyk and Michał Włodarczyk. Random order contention resolution schemes. In *2018 IEEE 59th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 790–801. IEEE, 2018.

[2] Saeed Alaei. Bayesian combinatorial auctions: Expanding single buyer mechanisms to many buyers. *SIAM Journal on Computing*, 43(2):930–972, 2014.

[3] Siddharth Barman, Omar Fawzi, Suprovat Ghoshal, and Emirhan Gürpınar. Tight approximation bounds for maximum multi-coverage. *Mathematical Programming*, pages 1–34, 2021.

[4] Simon Bruggmann and Rico Zenklusen. An optimal monotone contention resolution scheme for bipartite matchings via a polyhedral viewpoint. *Mathematical Programming*, pages 1–51, 2020.

[5] Chandra Chekuri, Jan Vondrák, and Rico Zenklusen. Submodular function maximization via the multilinear relaxation and contention resolution schemes. *SIAM Journal on Computing*, 43(6):1831–1879, 2014.

[6] Uriel Feige and Jan Vondrak. Approximation algorithms for allocation problems: Improving the factor of 1-1/e. In *2006 47th Annual IEEE Symposium on Foundations of Computer Science (FOCS'06)*, pages 667–676. IEEE, 2006.

[7] Uriel Feige and Jan Vondrák. The submodular welfare problem with demand queries. *Theory of Computing*, 6(1):247–290, 2010.

[8] Moran Feldman, Ola Svensson, and Rico Zenklusen. Online contention resolution schemes. In *Proceedings of the twenty-seventh annual ACM-SIAM symposium on Discrete algorithms*, pages 1014–1033. SIAM, 2016.

[9] Guru Guruganesh and Euiwoong Lee. Understanding the correlation gap for matchings. In *37th IARCS Annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS 2017)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018.

[10] Euiwoong Lee and Sahil Singla. Optimal online contention resolution schemes via ex-ante prophet inequalities. In *26th Annual European Symposium on Algorithms (ESA 2018)*. Schloss Dagstuhl-Leibniz-Zentrum fuer Informatik, 2018.

[11] László Lovász. Submodular functions and convexity. In *Mathematical Programming The State of the Art*, pages 235–257. Springer, 1983.

A Appendix

Proof of Lemma 2.2

Proof of Lemma 2.2.

$$
\begin{aligned}
\mathbb{P}[e \in \pi_x(A)] &= \sum_{\substack{B \subseteq A \\ |B|=k \\ e \in B}} q_A(B) = \sum_{\substack{B \subseteq A \setminus e \\ |B|=k-1}} q_A(B \cup e) \\
&= \sum_{\substack{B \subseteq A \setminus e \\ |B|=k-1}} \frac{1}{\binom{|A|}{k}} \left(1 + \frac{x(A \setminus e) - x(B)}{|A|-k} - \frac{x(B) + x_e}{k}\right) \\
&= \sum_{\substack{B \subseteq A \setminus e \\ |B|=k-1}} \frac{1}{\binom{|A|}{k}} \left(1 - \frac{x_e}{k} + \frac{x(A \setminus e)}{|A|-k} - x(B)\left(\frac{1}{|A|-k} + \frac{1}{k}\right)\right) \\
&= \sum_{\substack{B \subseteq A \setminus e \\ |B|=k-1}} \frac{1}{\binom{|A|}{k}} \left(\frac{k - x_e}{k} + \frac{x(A \setminus e)}{|A|-k} - x(B)\frac{|A|}{k(|A|-k)}\right). \qquad \text{(A.1)}
\end{aligned}
$$

We now use Equation (2.2) in a slightly modified form:

$$
\sum_{\substack{B \subseteq A \setminus e \\ |B|=k-1}} x(B) = \binom{|A|-2}{k-2} x(A \setminus e). \qquad \text{(A.2)}
$$

The only part in the sum (A.1) that depends on B is the last term with $x(B)$. Hence, by plugging-in (A.2) into (A.1), we get:

$$
\binom{|A|}{k}\mathbb{P}[e \in \pi_x(A)] = \binom{|A|-1}{k-1}\left(\frac{k - x_e}{k}\right) + \binom{|A|-1}{k-1}\frac{x(A \setminus e)}{|A|-k} - \binom{|A|-2}{k-2}\frac{|A|}{k(|A|-k)} x(A \setminus e). \qquad \text{(A.3)}
$$

We now use the formula $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$ to remove all the binomial coefficients from (A.3). We get

$$
\begin{aligned}
\frac{|A|}{k}\mathbb{P}[e \in \pi_x(A)] &= \frac{k - x_e}{k} + \frac{x(A \setminus e)}{|A|-k} - \frac{k-1}{|A|-1}\,\frac{|A|}{k(|A|-k)} x(A \setminus e) \\
&= \frac{k - x_e}{k} + \frac{x(A \setminus e)}{|A|-k}\left(1 - \frac{|A|(k-1)}{(|A|-1)k}\right) = \frac{k - x_e}{k} + \frac{x(A \setminus e)}{k(|A|-1)}.
\end{aligned}
$$

This implies the desired result:

$$
\mathbb{P}[e \in \pi_x(A)] = \frac{k - x_e}{|A|} + \frac{x(A \setminus e)}{|A|(|A|-1)}.
$$

□

Proof of Lemma 2.4

Proof of Lemma 2.4. Notice that for $i \in A$, $p_S(A)\,(1 - x_i) = p_S(A \setminus i)\, x_i$. Then

$$
h_S^k(x) = \sum_{\substack{A \subseteq S \\ |A|=k}} p_S(A) \sum_{i \in A} (1 - x_i) = \sum_{\substack{A \subseteq S \\ |A|=k}} \sum_{i \in S} p_S(A)(1 - x_i)\mathbf{1}_{\{i \in A\}}
$$

$$= \sum_{i\in S} \sum_{\substack{A\subseteq S\\|A|=k}} x_i\, p_S(A\setminus i)\,\mathbf{1}_{\{i\in A\}} = \sum_{i\in S} \sum_{\substack{B\subseteq S\setminus i\\|B|=k-1}} x_i\, p_S(B)$$

$$= \sum_{i\in S} x_i \sum_{\substack{A\subseteq S\setminus i\\|A|=k-1}} p_S(A) = \sum_{i\in S} x_i \left(\sum_{\substack{A\subseteq S\\|A|=k-1}} p_S(A) - \sum_{\substack{A\subseteq S\\|A|=k-1}} p_S(A)\mathbf{1}_{\{i\in A\}} \right)$$

$$= x(S)Q_S^{k-1}(x) - \sum_{\substack{A\subseteq S\\|A|=k-1}} p_S(A)x(A). \tag{A.4}$$

Notice that by definition of $h_S^k(x)$ we have

$$h_S^{k-1}(x) = \sum_{\substack{A\subseteq S\\|A|=k-1}} p_S(A)(k-1-x(A)) = (k-1) \sum_{\substack{A\subseteq S\\|A|=k-1}} p_S(A) - \sum_{\substack{A\subseteq S\\|A|=k-1}} p_S(A)x(A)$$

$$= (k-1)\, Q_S^{k-1}(x) - \sum_{\substack{A\subseteq S\\|A|=k-1}} p_S(A)x(A). \tag{A.5}$$

Substracting (A.5) from (A.4) we get

$$h_S^k(x) - h_S^{k-1}(x) = Q_S^{k-1}(x)\Big(x(S) - (k-1)\Big).$$

We can rewrite this recursive formula as

$$h_S^{i+1}(x) - h_S^i(x) = Q_S^i(x)\Big(x(S) - i\Big).$$

By summing both sides from 0 to $k-1$ and noticing that $h_S^0(x) = 0$, we get the desired result. $\square$

Proof of Lemma 2.5

Proof of Lemma 2.5. First, notice that the function $g(x) := \left(\frac{x-1}{x}\right)^x$ is strictly increasing for $x \geq 1$. Indeed, by using the strict inequality $\log(1+x) < x$ for any $x > 0$, we see that the derivative of $\log(g(x))$ is strictly positive:

$$\frac{d}{dx}\log(g(x)) = \frac{d}{dx} x \log\left(\frac{x-1}{x}\right) = \log\left(\frac{x-1}{x}\right) + x\frac{x}{x-1}\frac{1}{x^2} = \log\left(\frac{x-1}{x}\right) + \frac{1}{x-1}$$
$$= \frac{1}{x-1} - \log\left(\frac{x}{x-1}\right) = \frac{1}{x-1} - \log\left(1 + \frac{1}{x-1}\right) > 0.$$

We first prove (2.13). If $k = 1$, then $\alpha(k-1, n-1) = 0$ and the statement clearly holds. We may thus assume $k > 1$. Then,

$$\frac{\alpha(k,n)}{\alpha(k-1,n-1)} = \frac{k}{k-1}\, \frac{\binom{n}{k}}{\binom{n-1}{k-1}}\, \left(\frac{n-k}{n}\right)^{n+1-k}\, \left(\frac{k}{n}\right)^k \left(\frac{n-1}{n-k}\right)^{n+1-k}\, \left(\frac{n-1}{k-1}\right)^{k-1}$$

$$= \frac{k}{k-1}\ \frac{n}{k}\ \frac{k^k\ (n-1)^n}{n^{n+1}(k-1)^{k-1}} = \left(\frac{n-1}{n}\right)^n \left(\frac{k}{k-1}\right)^k = \frac{g(n)}{g(k)} > 1.$$

We now prove (2.14). If $k = n-1$, then $\alpha(k, n-1) = 0$ and the statement clearly holds. We may thus assume $k < n-1$. Then,

$$\frac{\alpha(k,n)}{\alpha(k,n-1)} = \frac{\binom{n}{k}}{\binom{n-1}{k}} \left(\frac{n-k}{n}\right)^{n+1-k} \left(\frac{k}{n}\right)^k \left(\frac{n-1}{n-1-k}\right)^{n-k} \left(\frac{n-1}{k}\right)^k$$

$$= \frac{n}{n-k}\ \frac{(n-k)^{n+1-k}\ (n-1)^n}{n^{n+1}\ (n-1-k)^{n-k}} = \left(\frac{n-1}{n}\right)^n \left(\frac{n-k}{n-k-1}\right)^{n-k} = \frac{g(n)}{g(n-k)} > 1. \quad \square$$

Proof of local maximality in Proposition 2.2

Proof. We want to show that the point $(k/n, \ldots, k/n)$ is a local maximum. We do that by computing the Hessian matrix $H(x)$ and showing that $H(k/n, \ldots, k/n)$ is negative definite. Note that $H(x)$ is a $(n-1) \times (n-1)$ matrix defined by:

$$H(x)_{i,j} = \frac{\partial^2 h_S^k(x)}{\partial x_i \partial x_j}.$$

By some simple computations, we have

$$\frac{\partial^2 h_S^k}{\partial x_i^2}(k/n, \ldots, k/n) = -2\binom{n-2}{k-1}\left(\frac{k}{n}\right)^{k-1}\left(\frac{n-k}{n}\right)^{n-k-1} \qquad \forall i \in S \tag{A.6}$$

and

$$\frac{\partial^2 h_S^k}{\partial x_i \partial x_j}(k/n, \ldots, k/n) = -\binom{n-2}{k-1}\left(\frac{k}{n}\right)^{k-1}\left(\frac{n-k}{n}\right)^{n-k-1} \qquad \text{for } i \neq j \tag{A.7}$$

Therefore,

$$H(k/n, \ldots, k/n) = -c \begin{pmatrix} 2 & 1 & 1 & \ldots & 1 \\ 1 & 2 & 1 & \ldots & 1 \\ \vdots & & & & \vdots \\ 1 & 1 & 1 & \ldots & 2 \end{pmatrix} =: -c\, A \tag{A.8}$$

where

$$c := \binom{n-2}{k-1}\left(\frac{k}{n}\right)^{k-1}\left(\frac{n-k}{n}\right)^{n-k-1} > 0.$$

Our goal is to show that $H(k/n, \ldots, k/n) \in \mathbb{R}^{(n-1)\times(n-1)}$ is negative-definite. Notice that λ is an eigenvalue of $H(k/n, \ldots, k/n)$ with corresponding eigenvector $v \in \mathbb{R}^{n-1}$ if and only if $-\lambda/c$ is an eigenvalue of A with the same eigenvector $v \in \mathbb{R}^{n-1}$. It is thus enough to show that A is positive-definite, i.e., all the eigenvalues of A are positive.

Notice that $A = I_{n-1} + J_{n-1}$, where I_{n-1} and J_{n-1} are respectively the identity matrix and the all-ones matrix of size $(n-1) \times (n-1)$. In particular, we may rewrite this as

$$A = I_{n-1} + e\, e^T \tag{A.9}$$

where $e \in \mathbb{R}^{n-1}$ is the all-ones vector.

Let μ be an eigenvalue of A with corresponding eigenvector v. Then

$$\begin{aligned} Av = \mu v &\iff v + (e^T v)e = \mu v \\ &\iff (e^T v)e = (\mu - 1)v. \end{aligned}$$

- If $\mu = 1$, the corresponding eigenspace is $\{v \in \mathbb{R}^{n-1} \mid e^T v = 0\}$. This eigenspace is a hyperplane of dimension $n-2$, which means that there exists $n-2$ linearly independent eigenvectors corresponding to the eigenvalue $\mu = 1$.

- If $\mu \neq 1$, then we see that e and v are collinear, which means that e is an eigenvector corresponding to μ. We compute the value of μ:

$$Ae = \mu e \iff e + (e^T e)e = \mu e \iff e + (n-1)e = \mu e \quad \iff \mu = n.$$

Hence, the spectrum of A is equal to $\{1, n\}$, where the multiplicity of the eigenvalue 1 is $n-2$, whereas the multiplicity of the eigenvalue n is 1. We have therefore just proven that A is positive-definite, which, by (A.8), implies that $H(k/n, \ldots, k/n)$ is negative-definite and concludes the proof. $\square$