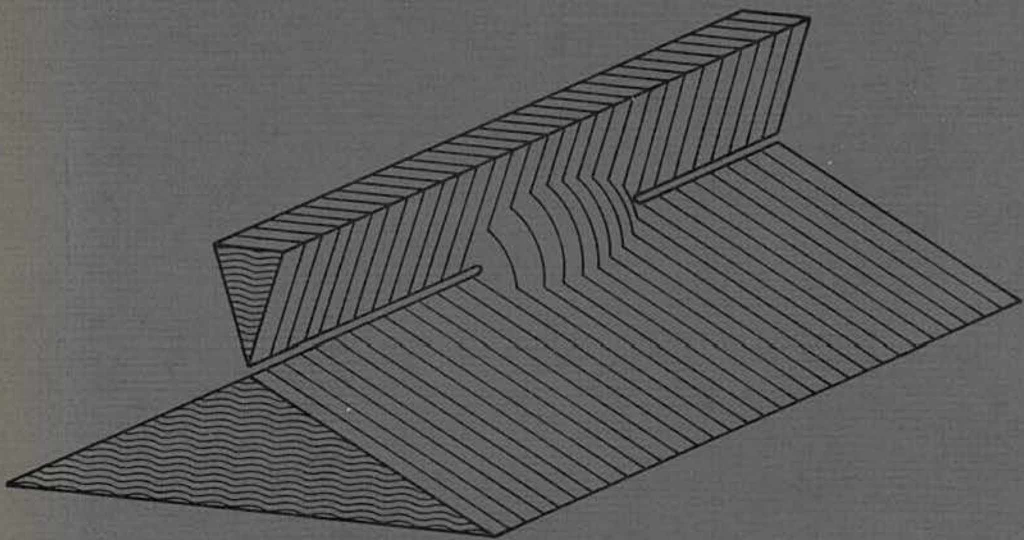
COMPLEX FOURIER TRANSFORMATION AND ANALYTIC FUNCTIONALS WITH UNBOUNDED CARRIERS



J.W. DE ROEVER

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CONTENTS

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INTRODUCTION

In distribution theory the Paley-Wiener-Schwartz theorem is well known. It describes the Fourier transforms of distributions g with compact support as a certain class of entire functions f. Here, distributions with compact support in $\mathbb{R}^n$ are continuous, linear functionals on the space $\mathcal{E}$ of C^∞-testfunctions in $\mathbb{R}^n$. Distributions with unbounded support can be defined if the testfunctions are submitted to growth conditions at infinity. For example, tempered distributions are obtained in this way as weak derivatives of continuous functions of polynomial growth. The Paley-Wiener-Schwartz theorem can easily be generalized for tempered distributions g with unbounded support. Then the function f is holomorphic only in a subdomain of $\mathbb{C}^n$ determined by the directions in which the support of g is unbounded. Similar to $\mathcal{E}'$ analytic functionals with compact carriers in $\mathbb{C}^n$ are defined as continuous, linear functionals on the space of entire functions in $\mathbb{C}^n$. The Ehrenpreis-Martineau theorem describes the Fourier transforms $\mathcal{F}\mu$ of analytic functionals μ with compact carriers as the class of entire functions of exponential type. Martineau has dealt with analytic functionals with bounded carriers in [48], but analytic functionals with unbounded carriers have never been studied extensively. It is our aim to fill up this gap in the theory and to extend the Ehrenpreis-Martineau theorem to analytic functionals with unbounded carriers.

The extension of the Paley-Wiener-Schwartz theorem to distributions with unbounded support does not give rise to any new problems, cf. [68, § 26.2, th. 2]. In the proof the possibility of having testfunctions with compact support is used. Since there are no such analytic testfunctions the proof of the Ehrenpreis-Martineau theorem cannot proceed along the same lines. For carriers which are polydiscs the proof is not very hard, cf. [65, th. 2.22 & 2.23] or [73, § 26], but it is the precise correspondence between an arbitrary, convex, compact carrier of an analytic functional μ and the exponential type of $\mathcal{F}\mu$ which complicates the proof. Polya has shown the theorem for $n = 1$, cf. [3, ch. 5] or [30, th. 4.5.3]; using quite different methods Ehrenpreis and Martineau proved it for the higher dimensional cases, cf. [15], [16, th. 5.21] and [48]. Later Hörmander applied his existence theorems for the Cauchy-Riemann operator to give another proof, cf. [30, th. 4.5.3].

The generalization of the Ehrenpreis-Martineau theorem is not straightforward and causes new difficulties: the proof that different analytic functionals with unbounded carriers yield different Fourier transforms is not trivial. One has to derive Ehrenpreis' fundamental principle for spaces of non-entire functions. This principle, first announced in [15], extends a given function f on a lower dimensional subset $\mathcal{W}$ of $\mathbb{C}^n$ to an entire function F satisfying certain bounds at infinity and also it describes the entire functions vanishing on $\mathcal{W}$. The principle is only valid if the bounds satisfy certain conditions. In order to derive it in [16] Ehrenpreis first extended f to a collection of holomorphic functions in neighborhoods of all the points of $\mathbb{C}^n$ and then he showed that these functions could be changed without changing the values on $\mathcal{W}$ so that they can be glued together to one global function F.

For our purpose we will use Ehrenpreis' local theory, but for the piecing together process we will use another method based on the L^2-estimates for the Cauchy-Riemann operator given by Hörmander in [30]. Furthermore, we will extend f to a function F holomorphic only in a subdomain Ω of $\mathbb{C}^n$ and satisfying bounds also at the boundary of Ω. In our case the conditions on the bounds are rather weak, but this is paid by the fact that a single f on $\mathcal{W}$ will be extended to different global functions each satisfying one bound, whereas in [16] f has been extended to one function F satisfying all the bounds simultaneously. In [56] Palamodov has derived a fundamental principle in the same weak form as our version. It is valid for functions holomorphic in convex tube domains Ω, but Palamodov's method does not yield estimates near the boundary of Ω. Therefore, although his work contains a generalization of the Ehrenpreis-Martineau theorem [56, VI, § 4.4^0, cor. 3], we cannot use it for our purposes.

The Paley-Wiener-Schwartz theorem for distributions with unbounded support is very useful in quantum field theory, where physicists are concerned with distributions g in p-space with support contained in a convex cone (the dual of the light cone). They search for properties of the Fourier transforms f in x-space. In particular they are interested in the holomorphic function f itself and not so much in its boundary value f^* on $\mathbb{R}^n$ or in the spaces of testfunctions on which f^* is a continuous, linear functional. The distribution f^* is tempered if g is. However, in [33] Jaffe remarks that it would be desirable to have distributions g which are weak derivatives of continuous functions G growing faster than polynomials. Then it turns out that f^* is a continuous, linear functional on a space of ultra-differentiable testfunctions; f^* is called an ultradistribution. Ultradifferentiable functions form a transition between ordinary C^∞ functions and analytic functions. If G grows too fast there are no longer testfunctions in x-space with a compact support. A field, defined on testfunctions in x-space which may have a compact support, is called strictly localizable. This is a desirable property in quantum field theory that, however, restricts the growth at infinity of the functions G in p-space. Similarly, a faster growth at infinity of the distributions in x-space would make the testfunctions in p-space ultradifferentiable or even analytic. So one might need a Paley-Wiener theorem for continuous, linear functionals with unbounded carriers defined on analytic testfunction spaces.

For example, it looks reasonable to consider distributions defined on Gauss-functions. Since these distributions and their Fourier transforms are in fact functionals on a space of entire functions, their carriers can be any subset of $\mathbb{C}^n$. But then another difficulty arises. Unlike supports of distributions analytic functionals do not have uniquely defined carriers and, worse, the intersection of carriers need not be a carrier. Hence it seems hopeless to try to generalize the notion of strictly localizable field for this case. To overcome this difficulty the best one can do is to content oneself with distributions in x-space and p-space which are weak derivatives of continuous functions growing slower than any exponential. For in that case their Fourier transforms have real, unbounded, carriers and a real-carried analytic functional μ does have an uniquely defined, smallest carrier, which therefore is called the support of μ. Fields of this type are called localizable, cf. [69].

Properties of real-carried analytic functionals have been studied by Martineau in [47] for bounded carriers and by Kawai in [38] for Fourier hyperfunctions. These are real-carried analytic functionals on the space of exponentially decreasing analytic testfunctions. We will derive the same properties for analytic functionals with unbounded, real carrier on spaces of slower decreasing analytic testfunctions. We will treat all cases between tempered distributions and Fourier hyperfunctions, i.e., all distributions and ultradistributions whose Fourier transforms are real-carried analytic functionals.

In chapter I the Paley-Wiener theorem will be applied in quantum field theory. We shall not choose a particular testfunction space using only the properties of boundary values of holomorphic functions. For these properties the Edge of the Wedge theorem is essential. We shall discuss problems arising from causality and localizability. It is known that particles cannot be localized in a bounded volume, cf. [28]; here it will be shown that they cannot even be absent in a bounded volume. Furthermore, we will prove that under a reasonable condition, the expectation value of any measurement is an anlytic function of space and time. So it certainly cannot be localized and if it is known in one space-time region it is known everywhere. Those interested in physics only may read merely this chapter and perhaps also section II.3.i, where a short proof of the Edge of the Wedge theorem for distributions is given. Others, not interested in physics, may skip this chapter.

In chapter II properties of analytic functionals μ with real, unbounded carrier will be discussed. Furthermore, analytic representations, i.e., sums of boundary values of holomorphic functions, of μ and of $F\mu$ will be treated. In particular Paley-Wiener theorems for ultradistributions with unbounded, convex support are studied in many details. It is our opinion that ultradistributions cannot be seen isolated from distributions and hyperfunctions, as they form a natural transition between these two. Chapter II concludes with an easy proof of the Edge of the Wedge theorem for distributions based on Fourier transformation which will be extended to the case of ultradistributions.

Chapter III deals with Fourier transforms of analytic functionals μ with complex, unbounded carriers as a generalization of the Ehrenpreis-Martineau theorem. It treats the precise correspondence between the carrier of μ and the exponential type of $F\mu$ in the directions determined by those in which the carrier of μ is unbounded. Particular cases yield Paley-Wiener type theorems that express a distribution or ultradistribution, which is the Fourier transform of an anlytic functional with a certain unbounded, convex carrier, as a boundary value of a holomorphic function. This chapter is more or less self-contained, except for the solutions of some problems which can be found in chapter VI.

In chapter IV the fundamental principle of Ehrenpreis and Palamodov will be discussed and moreover, it will be generalized so that it holds in spaces of non-entire functions. For entire functions there are actually three fundamental principles, as the conditions on the bounds in [16], [56] and here are not comparable and they supplement each other. The fundamental principle for non-entire functions is in fact a rewriting of the problems of chapter III in a more general frame. However, the contents of chapter III will not be needed for the understanding of chapter IV and those who are interested in the fundamental principle only may start reading at chapter IV.

In chapter V we will use the fundamental principle of chapter IV in a Fourier representation of all weak solutions in a certain space W of a homogeneous system of partial differential equations with constant coefficients. The spaces W are duals of spaces whose Fourier transforms consist of non-entire functions. Chapter III gives many examples of such spaces W. Also non-homogeneous systems are discussed. The Fredholm alternative, or if you like a generalized Poincaré lemma, will be derived for solutions in our spaces W. For other spaces this has been shown by Ehrenpreis, Malgrange, Hörmander and Komatsu, cf. [1]. Finally, we will indicate how the generalized Ehrenpreis-Martineau theorems of chapter III can be used to derive the Newton interpolation series for non-entire functions of exponential type of several variables. For a long time this series has been known to hold for exponential type functions of one variable holomorphic in a half-plane, cf. [55]. Recently, the series has been derived rigorously for entire functions of exponential type of several variables by Kioustelidis in [39].

Chapter VI will be devoted to the proofs left over from chapter IV. We will generalize the existence theorem [30, th. 4.4.2] for the Cauchy-Riemann operator of Hörmander slightly and derive cohomology with bounds in an arbitrary pseudoconvex domain.

In chapter VII we will prove an assertion made in chapter II in order to show the support property of real-carried analytic functionals. By functional analytic methods the existence theorem [30, th. 4.4.2] of Hörmander will be further generalized, so that it holds for functions satisfying countably many bounds. However, no uniform bounds will be obtained. The generalized existence theorem enables us to derive a stronger form of the fundamental principle than in chapter IV for certain spaces of non-entire functions.

CHAPTER I

CONNECTIONS WITH THEORETICAL PHYSICS

It is well known (cf. [37]) that the assumption of free particles being localized in a certain volume leads to inconsistencies in the mathematical description of this phenomenon. For a bounded volume this is clearly and shortly illustrated in [28]. We will show that under the same general conditions as in [28] even the assumption that a particle is absent in a bounded volume yields difficulties. For that purpose it is useful to consider functions or tempered distributions and their Fourier transforms as boundary values of analytic functions. This technique (see [49]) is essentially the basis for the more general theory of hyperfunctions (see [31] or [43]). In recent years this theory has been used in theoretical physics at several places, cf. [31], [32] and [52].

For simplicity, we will first show that no positive energy solutions in the space S' of tempered distributions of the Klein-Gordon and Dirac equations exist which vanish in a bounded space volume at some time t. Then the same technique reveals that any measurement of a positive observable cannot be zero in one space-time region while, if translated to another, it is positive. We will formulate this result in the theory of quantized fields (see [36] or [64]) and under a reasonable condition we will even obtain that the measurement of any observable yields a real analytic function of these translations. Finally, we will breifly discuss the localization problem of tachyons.

Fields satisfying the Gårding-Wightman axioms [71] are defined on a certain space of testfunctions, which themselves have no physical meaning. Therefore, the choice of the testfunction space is not forced by nature. The simplest choice is the space S of rapidly decreasing C^{∞}-functions, but smaller spaces of testfunctions with a larger class of distributions are also possible. Then one may ask for which testfunction spaces our reasoning yielding the above mentioned results remains valid. Very naturally, this leads to problems of purely mathematical nature concerning Fourier transforms of distributions, ultradistributions and analytic functionals. The remaining of this thesis deals with these problems put in a more general form than the special cases to which a physical sense might be ascribed. On the other hand, recent developments show that the mathematical generalizations may be applied to physics again; see [33] and [11] for ultradifferentiable testfunction spaces and [10], [63] and [52] for spaces of analytic testfunctions.

Not only the above discussed impossibility of localization, but many more physical properties such as local commutativity of microscopic causality (see [68, 29.6]) and the analytic continuation of the Wightman-functions (see [36] or [64]) depend on the way the occurring distributions are written as hyperfunctions. In fact, it seems that all physically interesting cases may fit in the frame of Fourier hyperfunctions [38]. A survey of the various cases is given in [69] and although not mentioned Fourier hyperfunctions actually enter at several places. Later, this has been made explicite and a Fourier hyperfunction quantum field theory has been formulated in [52].

Maybe the results of this chapter are not new to all physicists. For, the technique we use are so closely related to those of quantum field theory, for example exposed in [72] and [4], that it is hard to beleive that the conclusions have not been drawn. However, as in [28] we apply these techniques to relativistic quantum mechanics and we do not use the cyclic vacuum state which plays such a central role in quantum field theories.

I.1. CAUSALITY

The formulation and measurement of causality is closely related to the possibility of localization of a particle. Causality expresses the physical law of special relativity that no particle or signal can travel faster than light.

Let V be a space volume (an open set in $\mathbb{R}^3$), then for $t > 0$ we denote by V + ct the larger volume

$$V\text{+ct} \overset{\text{def}}{=\!=} \{\vec{y} \mid \|\vec{y}-\vec{x}\| \leq \text{ct for some } \vec{x}\in V\}.$$

Causality implies that a particle being in V at time 0 must be in V + ct at time $t > 0$ (cf. the definition of causality in [28]). For this characterization of causality the possibility of localization is necessary. However, if the volume V is bounded and if the above given formulation of causality is valid, a particle can never be localized, cf. [28]. Hence this formulation of causality is senseless.

The next step is to assume that it might be possible that a particle is absent in a bounded volume V. For $t > 0$ we denote by $V - ct$ the largest volume V' such that

$$V' + ct \subset V$$

Causality implies that a particle being absent in V at time 0 must be absent in $V - ct$ at time $t > 0$. However, we will show that, if this formulation of causality is valid, a particle can never be absent in any space volume. Hence, in order to give a meaningful formulation of causality, the above given characterizations need to be generalized.

In fact, what is needed is a flow of an observable quantity S and by causality this flow cannot go faster than light. To measure this it would be desirable if no part of S is destroyed or created during the observation time. Therefore, we assume that the density j^0 of S is the zero'th component of a Lorentz-four-vector j^μ which satisfies the continuity equation

$$(1.1) \qquad \partial_\mu j^\mu = 0$$

where

$$(\partial_0, \partial_1, \partial_2, \partial_3) \overset{\text{def}}{=\!=} \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right)$$

$$(\partial^0, \partial^1, \partial^2, \partial^3) \overset{\text{def}}{=\!=} \left(\frac{\partial}{\partial t}, \frac{-\partial}{\partial x_1}, \frac{-\partial}{\partial x_2}, \frac{-\partial}{\partial x_3} \right)$$

and where $\cdot_\mu \cdot^\mu$ means the summation over $\mu = 0,1,2,3$. Formula (1.1) expresses the property that during any time interval the change of the density j^0 in a certain volume is due to what flows in and out of that volume. Furthermore, if S, in principle, can attain every real value, it is impossible to say whether an increase of S in a volume V is due to a flow of a positive part of S into V or to a flow of a negative part of S out of V. Therefore, we assume that S attains only nonnegative values, i.e., for any space-time point $x = (t, \vec{x})$

$$(1.2) \qquad j^0(x) \geq 0.$$

We now define causality by the (equivalent) requirements (see [24]):

for any space volume V, any time t and any amount of time τ

$$\int_{V-c\tau} j^0(t\pm\tau,\vec{x})\,d\vec{x} \le \int_V j^0(t,\vec{x})\,d\vec{x}$$

(1.3)

$$\int_V j^0(t,\vec{x})\,d\vec{x} \le \int_{V+c\tau} j^0(t\pm\tau,\vec{x})\,d\vec{x}\ .$$

It is clear that (1.3) expresses causality only if j^0 is nonnegative, for the part of S that is in V at time t has to be in $V + c\tau$ at time $t \pm \tau$, but perhaps due to a flow into $V + c\tau$ from the outside during the time between t and $t + \tau$ there is more in $V + c\tau$ at time $t + \tau$ only if $j^0 \ge 0$, or if a surplus in $V + c\tau$ flows to the outside during the time between $t - \tau$ and t there was more in $V + c\tau$ at time $t - \tau$ only if the surplus was positive. Hence for a non-definite density causality cannot be defined in this way. Thus it is a meaningless to say that such a density (for example the charge density) propagates acausally and it is not true that causality implies the nonnegativity of the density as is pretended in [24].

In [24] it is shown that a density satisfying (1.1) and (1.2) necessarily satisfies (1.3). For example, any probability density which is the zero'th component of a current density satisfying (1.1) is causal. If it were possible to localize a particle in a bounded volume or the complement of a bounded volume, the earlier given characterizations of causality follow from (1.3) by taking for $j^0(x)$ the probability of finding the particle at x and by taking V bounded:

$$1 = \int_V j^0(t,\vec{x})\,d\vec{x} \le \int_{V+c\tau} j^0(t+\tau,\vec{x})\,d\vec{x}$$

and

$$\int_{V-c\tau} j^0(t+\tau,\vec{x})\,d\vec{x} \le \int_V j^0(t,\vec{x})\,d\vec{x} = 0, \tag{1.4}$$

respectively. It follows that the right hand side of the first formula equals 1 and that the left hand side of (1.4) equals 0.

We remark that the assumption of a probability density which satisfies (1.1) does not lead to acausal situations as in [28]. Another observable S suitable for describing causality is the energy because it is always nonnegative. In general the energy does not satisfy (1.1), but in [25] and [26] this condition has been weakened so that also energy propagates causally.

I.2. LOCALIZATION OF WAVE FUNCTIONS

We will consider free particles whose properties are determined by solutions of the Klein-Gordon or the Dirac equation. We only consider the positive frequency parts of these solutions (i.e., the energy remains positive) and we first investigate the localization of such solutions.

Let Ψ be a complex function (or more general a tempered distribution) of the real parameters $x = (x_0,x_1,x_2,x_3) = (t,\vec{x}) \in \mathbb{R}^4$ indicating the time and space variables and let $\bar{\Psi}$ be its complex conjugate. Furthermore, let Ψ be a solution of the Klein-Gordon equation

$$(1.5) \qquad (\partial_\mu \partial^\mu + m^2)\Psi = 0.$$

For each t Ψ is a tempered distribution in $\mathbb{R}^3$ and Ψ defines a continuous map from $\mathbb{R}$ into $S'(\mathbb{R}^3)$, (this can be seen by inspection of the Pauli-Jordan propagator Δ, see [34, formula (5.10)]). Ψ determines uniquely two tempered distributions ψ_1 and ψ_2 in $\mathbb{R}^3$ such that symbolically

$$(1.6) \qquad \begin{cases} \Psi(0,\vec{x}) = \psi_1(\vec{x}) \\ \dfrac{\partial\Psi}{\partial t}(0,\vec{x}) = \psi_2(\vec{x}) \end{cases}$$

and conversely, since Δ belongs to $S'(\mathbb{R}^4)$ each ψ_1 and ψ_2 determines a solution which is a tempered distribution in $\mathbb{R}^4$.

From (1.5) a first order equation, the Dirac equation, can be derived:

$$(1.7) \qquad (\gamma^\mu i\partial_\mu - mI)\Psi = 0.$$

Here the coefficients γ^μ and I are elements of a non-commutative group with unit I satisfying

$$(1.8) \qquad \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}I$$

where

$$(g^{\mu\nu}) \underset{\text{def}}{=} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Now Ψ is no longer a single distribution, but it belongs to a certain linear space in which the γ's act as linear transformations. For example, if the coefficients γ^μ are represented as certain $k \times k$-matrices, Ψ consits of k components $\Psi = (\Psi_1, \ldots, \Psi_k)$, where each Ψ_j is a tempered distribution satisfying the Klein-Gordon equation. For, in any representation of the γ's we have

$$(-\gamma^\nu i\partial_\nu - mI)(\gamma^\mu i\partial_\mu - mI)\Psi = 0$$

and hence by (1.8)

$$(\gamma^\nu\gamma^\mu\partial_\nu\partial_\mu + m^2 I)\Psi = (\partial_\mu\partial^\mu + m^2)\Psi = 0.$$

We can write (1.7) as

$$\text{(1.9)} \qquad \frac{\partial\Psi}{\partial t} = -im\gamma^0\Psi - \sum_{k=1}^{3} \gamma^0\gamma^k \frac{\partial\Psi}{\partial x_k} \ .$$

Hence if $\Psi(0,\vec{x})$ is given, $\frac{\partial\Psi}{\partial t}(0,\vec{x})$ is uniquely determined and the solution of the Dirac equation equals the solution of the Klein-Gordon equation with these initial values. Therefore, we only have to consider the initial value problem (1.5) and (1.6) and in particular we will consider only those solutions belonging to positive energy.

The energy p_0 and impulse $\vec{p}$ are real parameters arising as the variables in the dual $\mathbb{R}_4$ of the $(t,\vec{x})$-space $\mathbb{R}^4$. Hence Fourier transformation of a tempered distribution in x-space yields a tempered distribution in p-space. Thus the fact that we consider solutions Ψ in S' agrees with the fact that x and p must be real.

The Fourier transform $\Phi \in S'(\mathbb{R}_4)$ of a solution $\Psi \in S'(\mathbb{R}^4)$ of (1.1) satisfies

$$\text{(1.10)} \qquad (p_0^2 - \vec{p}^2 - m^2)\Phi(p) = 0.$$

The general solution in $S'(\mathbb{R}_4)$ of this equation determines two distributions ϕ_1 and ϕ_2 in $S'(\mathbb{R}_3)$, one corresponding to $p_0 > 0$ and one to $p_0 < 0$, and conversely, any two ϕ_1 and ϕ_2 in $S'(\mathbb{R}_3)$ determine a solution Ψ of (1.5) in the following symbolical way

$$(1.11) \qquad \Psi(t,\vec{x}) = F^{-1}\left[\frac{e^{-i\sqrt{\vec{p}^2+m^2}\,t}\phi_1(\vec{p})}{\sqrt{\vec{p}^2+m^2}}\right](\vec{x}) + F^{-1}\left[\frac{e^{i\sqrt{\vec{p}^2+m^2}\,t}\phi_2(\vec{p})}{\sqrt{\vec{p}^2+m^2}}\right](\vec{x})$$

where F^{-1} denotes the inverse Fourier transformation. The initial functions (or distributions) satisfy symbolically

$$\Psi(0,\vec{x}) = F^{-1}\left[\frac{\phi_1(\vec{p})+\phi_2(\vec{p})}{\sqrt{\vec{p}^2+m^2}}\right](\vec{x})$$

and

$$\frac{\partial\Psi}{\partial t}(0,\vec{x}) = F^{-1}\left[-i\phi_1(\vec{p})+i\phi_2(\vec{p})\right](\vec{x}) \ .$$

For a positive energy solution Ψ of (1.5) we require that $\phi_2 = 0$. Instead of (1.6) the initial values now have to satisfy symbolically

$$\frac{\partial\Psi}{\partial t}(0,\vec{x}) = \frac{1}{(2\pi)^3}\int\int e^{-i\langle\vec{p},\vec{x}-\vec{\xi}\rangle}(-i)\sqrt{\vec{p}^2+m^2}\,\Psi(0,\vec{\xi})\,d\vec{\xi}\,d\vec{p},$$

where only $\Psi(0,x)$ can be chosen arbitrarily in $S'(\mathbb{R}^3)$. Now Ψ is the inverse Fourier transform of a distribution in $S'(\mathbb{R}_4)$ with support in the cone $\overline{\Gamma^*} = \{(p_0,\vec{p}) \mid p_0 \geq \|\vec{p}\|\} \subset \mathbb{R}_4$. Then Ψ can be written as a boundary value in $S'(\mathbb{R}^4)$ of a function f holomorphic in $\mathbb{R}^4 + i\Gamma$, where Γ is the interior of the lightcone in $\mathbb{R}^4$, i.e., for every $\phi \in S(\mathbb{R}^4)$

$$\langle\Psi,\phi\rangle = \lim_{\substack{y\to 0\\ y\in C'\subset\subset\Gamma}} \int f(x+iy)\phi(x)\,dx.$$

Here Γ^* is the dual cone of the open cone $\Gamma \subset \mathbb{R}^4$:

$$\Gamma^* = \{p \mid \langle p,x\rangle > 0, x \in \Gamma\} \subset \mathbb{R}_4.$$

Roughly, this can be seen as follows: let g be a distribution in $S'(\mathbb{R}_n)$ which can be written as a certain derivative of a measure μ with support in a closed cone $\overline{C^*} \subset \mathbb{R}_n$ satisfying

$$\int_{\overline{C^*}} \frac{d|\mu(\xi)|}{(1+\|\xi\|^2)^k} < \infty$$

for some $k > 0$. Then for some multiindex α

$$f(z) \overset{\text{def}}{=} F[e^{-<\xi,y>}g(\xi)](x) = \int_{\overline{C^*}} (iz)^\alpha e^{i<\xi,x>-<\xi,y>} d\mu(\xi)$$

exists if $-<\xi,y> \le -\delta_y \|\xi\|$ for some $\delta_y > 0$ depending on y, thus for $y \in C$ if C^* is the dual of the open cone $C \subset \mathbb{R}^n$. Then

$$F[g](x) = \lim_{\substack{y\to 0 \\ y\in C' \subset\subset C}} f(x+iy) = f(x+i0)$$

in $S'(\mathbb{R}^n)$, see [12] or [68].

Now let f^+ be holomorphic in $\mathbb{R}^n + iC$ and f^- in $\mathbb{R}^n - iC$ for C an open cone in $\mathbb{R}^n$, such that $f^+(x+i0)$ and $f^-(x-i0)$ exist in $S'(\mathbb{R}^n)$. Furthermore, let the distributions $f^+(x+i0)$ and $f^-(x-i0)$, considered as distributions in $\mathcal{D}'(U)$ for some open set $U \subset \mathbb{R}^n$, be equal. Then f^+ is the analytic continuation of f^-. This theorem is the celebrated "Edge of the Wedge" theorem, see [64], [68] or for a simple proof Ch.II § 3.i of this thesis. In particular it follows by choosing $f^- \equiv 0$ that, if $f^+(x+i0) \equiv 0$ in U, then $f^+ \equiv 0$.

Thus every positive energy solution Ψ of the Klein-Gordon equation cannot vanish identically in any open space-time region without vanishing everywhere. In particular, the initial values ψ_1 and ψ_2 cannot vanish identically in the same open set in $\mathbb{R}^3$. For, if they do it follows from the fact that Ψ satisfies the hyperbolic differential equation (1.5), that then Ψ would vanish identically in some open set in $\mathbb{R}^4$. Similarly, the initial values of the Dirac equation cannot vanish identically in an open set in $\mathbb{R}^3$. For (1.9) implies that $\frac{\partial\Psi}{\partial t}(0,x)$ would vanish together with $\Psi(0,x)$ in the same open set in $\mathbb{R}^3$.

In the above we have shown some mathematical properties of solutions of certain differential equations. Only a few of the used mathematical concepts have also relation to physical phenomena. These phenomena cannot be seen directly, but only by means of measurements of observable concepts which are supposed to be influenced by them. Therefore, it may be disputable to conclude that free particles cannot be absent in any space volume at any time. However, the argument is quite fundamental as it applies under very general assumptions as in [28]. The same reasoning even implies that a measurement of a nonnegative observable cannot yield zero in one space-time region while, if translated to another, it is positive. In the next sections we will prove this for observable concepts described by densities which are bilinear forms on the space of wave functions Ψ.

I.3 LOCALIZATION OF PARTICLES

In the last section we have shown some mathematical properties of the solutions of the Klein-Gordon or the Dirac equation. Let us now show how these properties react in quantities which may have a physical interpretation.

In section I.1 we have seen how causality is related to a current density j^{μ} of a nonnegative observable S. In order to define the current density we assume that the space of solutions of the Klein-Gordon or the Dirac equation can be transformed into a Hilbert space, cf.[35] for other, more fundamental reasons why a Hilbert space is chosen. Let q^{μ} be a bilinear form defined on a dense subspace D of H and let for $\Psi \in H$ Ψ_x be defined by

$$\Psi_x(y) \overset{\text{def}}{=\!=} \Psi(y-x).$$

D must be such that $\Psi \in D$ implies $\Psi_x \in D$ for each $x \in \mathbb{R}^4$. For $\Psi \in D$ with $\|\Psi\| = 1$ a current density j^{μ} can be defined by

$$j^{\mu}(x) = q^{\mu}(\Psi_x, \Psi_x) \tag{1.12}$$

provided that q^{μ} is such that j^{μ} transforms as a Lorentz-four-vector.

If S is a bounded observable (for example if j^0 is a probability density), for each t and some constant $K > 0$ we have

$$\left| \int_{\mathbb{R}^3} j^0(t,\vec{x})d\vec{x} \right| \leq K.$$

Hence for each volume V in $\mathbb{R}^3$

$$S_V(t) \overset{\text{def}}{=\!=} \int_V j^0(t,\vec{x})\,d\vec{x}$$

is a bounded bilinear form defined everywhere on H. If S is not bounded, we moreover assume that for each volume $V \subset \mathbb{R}^3$ and for each t $S_V(t)$ is a closed bilinear form on $D \subset H$. This means that, if $S_V(t)$ is defined on $\{\phi_m\}_{m=1}^{\infty}$, if $\phi_n \to \phi$ in H and if $S_V(t)(\phi_k-\phi_m,\phi_k-\phi_m) \to 0$ as $k,m \to \infty$, then $S_V(t)$ is also defined on ϕ and $S_V(t)(\phi_m-\phi,\phi_m-\phi) \to 0$.

Before continuing with the general situation we will show by an explicit example that such current densities j^μ exist. We first consider the Dirac equation. Let for each $x \in \mathbb{R}^4$ $\Psi(x)$ (or actually, for each $\phi \in S(\mathbb{R}^4)$ $\langle\Psi,\phi\rangle$) belong to a certain Hilbert space on which the γ's act as a linear transformation. Usually the anti-linear functional associated to $\Psi(x)$ is denoted by $\Psi^\dagger(x)$ and the inner product of $\Psi(x)$ by itself is then written as $\Psi^\dagger(x)\Psi(x)$. Let moreover for each t $\Psi^\dagger(t,\vec{x})\Psi(t,\vec{x})$ be a L^1-function of $\vec{x} \in \mathbb{R}^3$, then the inner product in H is defined by

$$(\Phi,\Psi) \overset{\text{def}}{=\!=} \int_{\mathbb{R}^3} \Phi^\dagger(t,\vec{x})\Psi(t,\vec{x})\,d\vec{x} \ .$$

That this is independent of t follows from (1.7) and (1.8). In a k-dimensional representation $\Psi(x)$ belongs to the Hilbert space $\mathbb{C}^k$ and for every t each Ψ_j is a L^2-function on $\mathbb{R}^3$, $j=1,\ldots,k$. A bounded current density satisfying (1.1) (in distributional sense) can be defined by

$$j^\mu \overset{\text{def}}{=\!=} \Psi^\dagger\gamma^0\gamma^\mu\Psi \tag{1.13}$$

and clearly (1.2) is satisfied, too.

Thus the density (1.13) with $\mu = 0$ is always causal, i.e., it satisfies (1.3). j^0 equals $\Psi^\dagger\Psi$ and in the last section it has been shown that this density can never vanish in an open set V of $\mathbb{R}^3$ at any time t if Ψ is a positive energy solution of (1.7). j^0 can be interpreted as the probability density of some (bounded) observable S. Then at any time there is always a positive chance of finding S in any space volume.

Let us now turn to the Klein-Gordon equation. The Hilbert space is defined by the inner product

$$(\Phi,\Psi) \overset{\text{def}}{=\!=} \frac{i}{2}\int_{\mathbb{R}^3} \{\bar{\Phi}(t,\vec{x})\ \frac{\partial\Psi}{\partial t}\ (t,\vec{x}) - \frac{\partial\bar{\Phi}}{\partial t}\ (t,\vec{x})\Psi(t,\vec{x})\}d\vec{x}$$

which is independent of t, provided that the solutions Φ and Ψ of (1.5) are functions for which the above written integral exists. It should be remarked that this is an innerproduct only in the space of positive energy solutions, in which case $(\psi,\psi) \geq 0$. Indeed

$$(\Psi,\Psi) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}_3} \frac{|\phi(\vec{p})|^2}{\sqrt{\vec{p}^2+m^2}} d\vec{p} \geq 0,$$

where ϕ is an L^2-function on $\mathbb{R}_3$ with respect to the measure $(\vec{p}^2+m^2)^{-\frac{1}{2}} d\vec{p}$ so that by (1.11)

$$(1.14) \qquad \Psi(t,\vec{x}) = F^{-1}\left[\frac{e^{-i\sqrt{\vec{p}^2+m^2}\,t}\phi(\vec{p})}{\sqrt{\vec{p}^2+m^2}}\right](\vec{x}) \; . \qquad 1)$$

Thus the condition on the solution of (1.5) is that in (1.6) $\Psi(0,\vec{x})$ must belong to the Sobolev space $H^{\frac{1}{2}}(\mathbb{R}^3)$ and $\frac{\partial\Psi}{\partial t}(0,\vec{x})$ to $H^{-\frac{1}{2}}(\mathbb{R}^3)$. A current density satisfying (1.1) can be defined by

$$j^{\mu} \overset{\text{def}}{=\!=} \frac{i}{2}\{\bar{\Psi}\partial^{\mu}\Psi-(\partial^{\mu}\bar{\Psi})\Psi\}.$$

It is well known that for general solutions Ψ of (1.5) j^0 does not satisfy (1.2) and it is less known that the same is true for positive frequency solutions Ψ, see [22]. However, in [23] current densities are constructed which do satisfy (1.1) and (1.2), where in (1.2) even the > sign holds.

We will show that, in the general case for any current density, not identically zero, arising from a bilinear form on the Hilbert space of positive frequency solutions of the Klein-Gordon or the Dirac equations satisfying (1.1) and (1.2), (1.2) cannot hold with the = sign for $\vec{x}$ in any space volume V and for any t. This follows from the causality of the current density and from the fact that $S_V(t)$ cannot be zero for all t with $0 < t < \tau$ for any $\tau > 0$ and any V. This fact will be proved in the next section. For that purpose we have to rewrite the setting of this section so that the formalism of the next section can be applied to it.

---

1) Here there is a little ambiguity in the Fourier transformation F. In (1.11) F transforms tempered distributions in the $\vec{x}$-space $\mathbb{R}^3$ into tempered distributions in the $\vec{p}$-space $\mathbb{R}_3$, which is defined by Parsevals relation if F is a map from $S(\mathbb{R}_3)$ onto $S(\mathbb{R}^3)$. However, in (1.14) F should be understood in L^2-sense, which can be defined by completion if F is a map from $S(\mathbb{R}^3)$ onto $S(\mathbb{R}_3)$, cf.II § 2.i.

We have considered nonnegative densities of the form $j^0(x) = q(\Psi_x,\Psi_x)$ such that $\int_V j^0(t,\vec{x})d\vec{x}$ is a closed bilinear form. For the moment we do not bother whether this is the zero's component of a four-vector or not. Let V_0 be a fixed space volume and let

$$S_0(x) \overset{\text{def}}{=\!=} S_{V_0+\vec{x}}(t)$$

where $V_0 + \vec{x}$ is the over x translated volume V_0. According to [58,th.VIII.15] $S_0(x)$ can be written as

$$S_0(x) = (\Psi_x, T\Psi_x)$$

for some selfadjoint positive operator T. We define

$$T_x \overset{\text{def}}{=\!=} U^{-1}(x)T\ U(x)$$

where U(x) is the unitary operator with

$$U(x)\Psi = \Psi_x \ .$$

Since

$$U(t,\vec{x})\Psi(y) = \int e^{i\sqrt{\vec{p}^2+m^2}\,(t-y_0) + i<\vec{p},\vec{x}-\vec{y}>}\phi(\vec{p})\ \frac{d\vec{p}}{\sqrt{\vec{p}^2+m^2}}$$

where ϕ is determined by Ψ according to (1.14), U(x) has a spectral measure contained in $\{p \mid p_0 = \sqrt{p^2+m^2}\}$.

If in theorem 1.2 of the next section we replace T(f) by T (in fact, here the testfunction f is the characteristic function of V_0), this theorem shows that $S_0(x) = (\Psi,T_x\Psi)$ cannot vanish for $\|x\| < \varepsilon$ for every $\varepsilon > 0$. Actually the theorem gives more precise information where $S_0(x)$ can vanish. If now $S_V(t) = 0$ for $0 < t < \tau$, we choose $V_0 \subset\subset V$ and theorem 1.2 shows that $S_V(t) = 0$ for all t and all V, hence that $j^0 \equiv 0$. We summarize the foregoing in the following theorem.

<u>THEOREM</u> 1.1. *Let* H *be the Hilbert space of positive frequency solutions* Ψ *of the Klein-Gordon equation or the Dirac equation. Let* $q(\Psi,\Psi)$ *be a non-vanishing bilinear form on a dense subspace* D *of* H *such that for all* $x \in \mathbb{R}^4$

$$j(x) \stackrel{\text{def}}{=} q(\Psi_x, \Psi_x) \geq 0$$

and such that for all t *and space volumes* $V \int_V j(t,\vec{x})d\vec{x}$ *is a closed bilinear form on* D. *Let* V_0 *be an arbitrary space volume and let*

$$S_0(t,\vec{x}) \stackrel{\text{def}}{=} \int_{V_0+\vec{x}} j(t,\vec{y})d\vec{y}.$$

Then for any $\varepsilon > 0$ $S_0(x)$ *cannot vanish identically for* $\|x\| < \varepsilon$.

In theorem 1.1 we do not assume that the nonnegative density is causal, but if it is, it follows that for each t $S_0(t,\vec{x})$ cannot vanish identically even for $\|\vec{x}\| < \varepsilon$. So also formula (1.4) cannot be used for defining causality. For if it holds, it can never occur. Nonnegative causal desities arise, for example, from a current density satisfying (1.1). In [25] and [26] nonnegative densities corresponding to the energy are discussed which do not satisfy (1.1) but still are causal. In [13] Dirac proposed a new wave equation yielding only positive energy solutions which satisfy the Klein-Gordon equation, too. Moreover, he has defined a current density as in (1.12) satisfying (1.1) and (1.2). Hence the zero's component of this density can never be localized, contrarily to what Dirac said in [14]. Perhaps, it is also possible to define noncausal nonnegative densities which then cannot satisfy (1.1), cf. [28].

The solutions of the Klein-Gordon or the Dirac equation are particular cases of quantized fields. Therefore, in the next section we will pass to the (mathematical) problem of localization of fields, although we do not use all the axioms defining these fields. We will select only those axioms which imply the result that $S_0(x)$ cannot vanish identically for $\|x\| < \varepsilon$.

I.4. ANALYTIC PROPERTIES OF EXPECTATION VALUES

In the theory of quantized fields satisfying the Gårding-Wightman axioms [71] we shall use the same principle as before in order to show that not both, the testfunctions and the field operators, are localizable (cf. [72] for a stronger result saying that the field operators are nowhere ordinary functions, which follows from more conditions than we assume here). We remark that from now on all concepts will have only a mathematical meaning and the physical interpretation, if there is any, will not be discussed.

We shall not give all axioms defining a quantized field but only those which are needed in this section. For example, we do not need the vacuum state which cannot be missed in defining the general theory and properties of quantized fields. Although we introduce them no proper use will be made of the testfunctions and therefore, our conditions are as general as in [28] and they apply to relativistic quantum mechanics as well. For simplicity we shall discuss the case of an observable scalar field; the case of vector and tensor fields is similar, see [71].

Let F be a nuclear, locally convex, topological vector space of C^{∞}-testfunctions defined in x-space or in a complexification of the x-space. We shall not specify F in this section; in [36] F equals the space $S(\mathbb{R}^4)$ and in [71] F equals $\mathcal{D}(\mathbb{R}^4)$ (cf. also [68, 29.6]); ultradifferentiable testfunctions are discussed in [33] and in [11], whereas in [10], [63] and [52] spaces F of analytic functions are considered. If there are testfunctions in F with compact support the field is called *strictly localizable*, see [33]. Furthermore, there is a complex Hilbert space H of states with inner product $\langle\ ,\ \rangle$. In order not to confuse this notation with the action $\langle p,x\rangle$ of $p \in \mathbb{R}_4$ to $x \in \mathbb{R}^4$, we shall here denote this action by $x\cdot p$.

Axiom I. *The field* T *is a linear map from* F *into linear operators in* H. *For all* $f \in F$ *the operators* $T(f)$ *and* $T(f)^*$ *possess a common dense domain* D *on which they are defined, such that for all* $\Phi,\Psi \in D$ $\langle\Phi,T(\cdot)\Psi\rangle$ *belongs to* F'. *Moreover, for all* $f \in F$ $T(f)D \subset D$.

Axiom II. *The translations over the four-vector* x *induce a continuous map* $\{x\}$ *from* F *into* F *by*

$$\{x\}f(y) \stackrel{\text{def}}{=} f(y-x),\ f \in F.$$

An unitary, continuous representation U *of the group of translations exists, such that for all* $f \in F$

$$U(x)^{-1}T(f)U(x) = T_x(f)$$

where

$$T_x(f) \stackrel{\text{def}}{=} T(\{x\}f).$$

Furthermore, $U(x)D \subset D$ *for all* $x \in \mathbb{R}^4$.

Axiom III. $U(x)$ *has a spectral decomposition*

$$U(x) = \int e^{ix\cdot p}dE(p)$$

where the support of E *is contained in the cone*

$$\overline{\Gamma^*} = \{p_0^2 \geq \|\vec{p}\|^2,\ p_0 \geq 0\}.$$

We show that a strictly localizable field satisfying only the above mentioned axioms, as an operator valued distribution, cannot have a support which is not $\mathbb{R}^4$. First, let us assume that the field is positive [1)], which means that for all $\Phi \in D$ $\langle\Phi,T(\cdot)\Phi\rangle$ is a positive distribution in F'. Thus for every real and nonnegative testfunction f the operator T(f) is positive, i.e., for all $\Phi \in D$ and for such an f

$$\langle\Phi,T(f)\Phi\rangle \geq 0.$$

Let us call such a field a positive field. Furthermore, let us call $x(s) = (t(s),\vec{x}(s))$ a time-like curve if t and $\vec{x}$ are continuously differentiable functions of the real variable s with

$$(t'(s),\vec{x}'(s)) \in \Gamma$$

where Γ is the open light cone. If moreover for each $\lambda = 0,1,2,3,x_\lambda$ is a real analytic function of s, we call the curve an analytic time-like curve.

<u>THEOREM 1.2</u>. *Let* T *be a positive field as defined by axioms* I, II *and* III, *let* f *be a real nonnegative testfunction in* F *and let* x(s) *be an analytic time-like curve for* $s \in \mathbb{R}$. *If for some* $\Phi \in D$ *and* $\varepsilon > 0$

$$\langle\Phi,T_{x(s)}(f)\Phi\rangle = 0 \tag{1.15}$$

for all $0 < s < \varepsilon$, *then* (1.15) *vanishes for all* $s \in \mathbb{R}$.

In particular, if $x(s) = (\tau s, s\vec{a})$ where $\vec{a}$ varies in the unit ball in $\mathbb{R}^3$ and τ in $(1,\infty)$, it follows that $S_0(x)$, defined in theorem 1.1, cannot vanish identically in an open set in $\mathbb{R}^4$.

---

1) For some fields this would be desirable, but unfortunately a strictly localizable field (as defined by more axioms than the above) is, in general, not positive, see [18].

PROOF. By Friedrichs extension theorem [58, th. X.23] the positive operator $T(f)$, defined on D, has a positive selfadjoint extension $\tilde{T}(f)$. By the spectral theorem there exists a positive selfadjoint operator $A(f)$ such that $A(f)^2 = \tilde{T}(f)$, which certainly holds on D. Since every translated f is real and nonnegative if f is, (1.15) implies

$$\langle\Phi, A(\{x(s)\}f)A(\{x(s)\}f)\Phi\rangle = \langle A(\{x(s)\}f)\Phi,\ A(\{x(s)\}f)\Phi\rangle = 0$$

for $0 < s < \varepsilon$. Hence $A(\{x(s)\}f)\Phi = 0$ and so

$$U(x(s))T_{x(s)}(f)\Phi = 0, \qquad 0 < s < \varepsilon. \tag{1.16}$$

Therefore, for any $\tau \in \mathbb{R}$ we have $I(\tau,s) = 0$ for $0 < s < \varepsilon$ where

$$I(\tau,s) \overset{\text{def}}{=} \langle U(x(\tau))\Phi,\ U(x(s))T_{x(s)}(f)\Phi\rangle.$$

According to axiom II $I(\tau,s)$ can be written as

$$I(\tau,s) = \langle U(x(\tau))\Phi,\ U(x(s))T_{x(s)}(f)U(x(s))^{-1}U(x(s))\Phi\rangle =$$
$$= \langle U(x(\tau))\Phi,\ T(f)U(x(s))\Phi\rangle$$

and by axiom III

$$I(\tau,s) = \int e^{ix(s)\cdot p} d\langle T(f)U(x(\tau))\Phi,\ E(p)\Phi\rangle.$$

Since E has its support in the cone $\overline{\Gamma^*}$ this integral, as a distribution of the variable $x = x(s) \in \mathbb{R}^4$, is the boundary value of a function G holomorphic in $\mathbb{R}^4 + i\Gamma$.

Let s be the real part of the complex variable $s + i\mu$ and let $u(s,\mu) \in \mathbb{R}^4$ and $v(s,\mu) \in \mathbb{R}^4$ be the real and imaginary parts of the analytic continuation of the function $x(s)$, thus $u(s,0) = s(x)$ and $v(s,0) = 0$. Then by the Cauchy-Riemann equations

$$\left(\frac{\partial v_0}{\partial \mu}(s,0), \ldots, \frac{\partial v_3}{\partial \mu}(s,0)\right) = x'(s) \in \Gamma,$$

hence for each $s \in \mathbb{R}$ $v(s,\mu) \in \Gamma'$ for some $\Gamma' \subset\subset \Gamma$ and for all $\mu > 0$ with

$|\mu|$ sufficiently small depending on s. Thus

$$I(\tau,s+i\mu) = G(u(s,\mu)+iv(s,\mu))$$

exists and is an analytic function of $s + i\mu$ for $\mu > 0$ and $|\mu|$ sufficiently small depending on s.[1)] Since $\lim I(\tau,s+i\mu) = 0$ as $\mu \downarrow 0$ for $0 < s < \varepsilon$, it follows that $I(\tau,s) \equiv 0$, in particular $I(\tau,\tau) = 0$. This yields

$$\langle U(x(\tau))\Phi,\ U(x(\tau))T_{x(\tau)}(f)\Phi\rangle = \langle\Phi,T_{x(\tau)}(f)\Phi\rangle = 0. \qquad \square$$

COROLLARY 1.3. *A nonvanishing, strictly localizable field* T *satisfying only the axioms* I, II *and* III *has support* $\mathbb{R}^4$.

For otherwise there is a testfunction f and $\varepsilon > 0$ such that for all $\Phi \in D$ $T_x(f)\Phi = 0$ for all $x \in \mathbb{R}^4$ with $\|x\| < \varepsilon$, so that (1.16) would hold.

We can drop the assumption of positivity of the field, if we impose a condition on the state Φ and then we get the stronger result that the expectation values are analytic functions of the translations in space and time. The condition implies that the high-energy contributions to the state may not be too strong. More precisely, let $U(x) = e^{ix\cdot P}$ and let P_0 be the zero'th component of the operator P. Then P_0 is a positive selfadjoint unbounded operator and we assume that the state Φ belongs to the domain of definition of the operator $e^{\delta P_0}$ for some $\delta > 0$. This property is equivalent to the following definition

DEFINITION. *A state* $\Phi \in H$ *is called analytic for the energy if* Φ *belongs to the domain of definition of any* P_0^m *and if*

$$\sum_{m=0}^{\infty} \frac{\|P_0^m\Phi\|}{m!}\, \delta^m < \infty$$

for some $\delta > 0$.

Nelson's analytic vector theorem tells us that there are many of such vectors (namely a dense subset of H) [58, IIth. X . 39].

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1) Actually, here we have the restriction of a distribution (hyperfunction) to an analytic curve defined by the restriction of its defining function, here G, see [31, lemma 2.1 p.50].

THEOREM 1.4. *Let T be a field defined by axioms I, II and III and let $\Phi \in D$ be an analytic vector for the energy. Then for any $f \in F$ the function*

$$\langle\Phi, T_x(f)\Phi\rangle$$

is analytic in $x \in \mathbb{R}^4$.

PROOF. Define the function G of $(x,\xi) \in \mathbb{R}^4 \times \mathbb{R}^4$ by

$$G(x,\xi) = \langle\Phi, U(x)^{-1}T(f)U(\xi)\Phi\rangle.$$

Since for all $f \in F$ we have $T(f)D \subset D$ the expression

$$\langle\Phi, T(\cdot)^* T(\cdot)\Phi\rangle$$

determines a separately continuous bilinear map on $F \times F$. By Schwartz' kernel theorem this map is continuous on $F \times F$. Hence for each $f \in F$

$$\|T_\xi(f)\Phi\| = \|T(f)U(\xi)\Phi\| = [\langle\Phi, T(\{\xi\}f)^* T(\{\xi\}f)\Phi\rangle]^{\frac{1}{2}}$$

is a continuous function of $\xi \in \mathbb{R}^4$. Also for $x,\xi \in \mathbb{R}^4$ $U(\xi)^{-1}U(x)\Phi$ varies continuously in H. Therefore G is a continuous function:

$$|\langle U(\xi)^{-1}U(x)\Phi, T_\xi(f)\Phi\rangle - \langle U(\eta)^{-1}U(y)\Phi, T_\eta(f)\Phi\rangle| \leq$$
$$\leq |\langle U(\xi)^{-1}U(x)\Phi, T(\{\xi\}f-\{\eta\}f)\Phi\rangle| + \|\{U(\xi)^{-1}U(x)-U(\eta)^{-1}U(y)\}\Phi\| \cdot$$
$$\cdot \|T_\eta(f)\Phi\|.$$

In particular G is measurable.

For fixed $\xi \in \mathbb{R}^4$ G can be extended as a holomorphic function of z in the tubular domain with base $(\delta,0,0,0)-\Gamma$ by

$$G(z,\xi) = \int e^{-iz\cdot p-\delta p_0}\, d\langle E(p)e^{\delta P_0}\Phi, T(f)U(\xi)\Phi\rangle$$

satisfying there

$$|G(z,\xi)| \leq \|e^{\delta P_0}\Phi\| \cdot \|T(f)U(\xi)\Phi\|.$$

Since $\|T(f)U(\xi)\Phi\|$ is continuous the right hand side is bounded if ξ varies in a bounded set in $\mathbb{R}^4$. On the other hand, for fixed $x \in \mathbb{R}^4$ G can be exended as a holomorphic function of ζ in the tubular domain with base $-(\delta,0,0,0) + \Gamma$ by

$$G(x,\zeta) = \int e^{i\zeta\cdot p-\delta p_0}\, d\langle T(f)^* U(x)\Phi, E(p)e^{\delta P_0}\Phi\rangle$$

satisfying there

$$|G(x,\zeta)| \leq \|T(f)^* U(x)\Phi\| \cdot \|e^{\delta P_0}\Phi\|.$$

Similarly to above, it follows that the right hand side is bounded if x varies in a bounded set in $\mathbb{R}^4$. Then it follows from Hartogs theorem for real-analytic functions (see [7], cf. also chapter II, § 3.i of this thesis) that G is an analytic function of $(x,\xi) \in \mathbb{R}^4 \times \mathbb{R}^4$. In particular $G(x,x)$ is an analytic function of $x \in \mathbb{R}^4$. □

Finally, we make some remarks concerning local commutativity, which expresses the fact that two space-like separated events cannot influence each other (sometimes also called microscopic causality). For strictly localizable fields the axiom of local commutativity is formulated as follows:
Axiom IV. *Let* f *and* g *in* F *have their supports such that any two points* x *in the support of* f *and* y *in the support of* g *are space-like separated, i.e.,* $|x_0-y_0| < \|\vec{x}-\vec{y}\|$, *then*

$$T(f)T(g) = T(g)T(f).$$

For the description of non-normalized interactions it is convenient to work with distributions growing faster than polynomials in p-space. Hence the functions in the Fourier transform of F must decrease more rapidly than functions in S. If they decrease too fast at infinity, the space F consists of non-localizable functions or even analytic functions. In the last case the expectation values are analytic functions anyhow (by axiom II). Theorem 1.4 reveals that this is not a rare phenomenon. Thus there would be no objection against analytic testfunctions. However, in that case the above given definition of local commutativity is impossible.

In [63] the space F is taken to be Z, the Fourier transform of $\mathcal{D}$, consisting of certain entire functions, and local commutativity is not required, but another way of defining microscopic causality is given. In [10] a condition for causality is given on non-localizable functions in F, namely that the distributions in p-space have a growth at infinity of order one and type zero, i.e., they are $\mathcal{O}(\exp \varepsilon\|p\|)$ for any $\varepsilon > 0$. In [69] such a field is called *localizable*. In chapter II we shall see that then the Fourier transforms in x-space are functionals on a space of real-analytic testfunctions. In spite of this such analytic functionals have a uniquely defined support (see chapter II, def. 2.6). As in [47] we will show (chapter II, th. 2.7) that an analytic functional T can be written as $\sum_{k=1}^{N} T_k$, where the analytic functionals T_k have their supports in a priori given closed sets U_k such that $\bigcup_{k=1}^{N} U_k = \mathbb{R}^4$. In a localizable, but non-strictly-localizable field T the space F consists of real-analytic testfunctions. Then local commutativity might be defined as follows:

For all $f,g \in F$ *and all decompositions* $T = T_1 + T_2 + T_3$ *where* T_1 *and* T_2 *have space-like separated supports,* $T_1(f)$ *and* $T_2(g)$ *commute.*

I.5. LOCALIZATION OF TACHYONS

In the description of tachyons (particles travelling faster than light) another application of the theory of functions of several complex variables can be made. As physics intend to study phenomena which take place outside the human mind, this section is perhaps more of mathematical interest than that it pretends to describe something of physical reality. Therefore, we shall not make the assumptions as general as possible, but we shall just study the solutions of the tachyonic Klein-Gordon equation. This enables us to explain a seeming contradiction between [66] and [50] concerning the existence of acausal solutions of certain wave equations corresponding to high-spin-particles. As to tachyons themselves there exists an extensive literature, see for example [51].

Let a superluminal state be described by a wave function Ψ satisfying the tachyonic Klein-Gordon equation

$$(1.17)\qquad (\partial_\mu \partial^\mu - m^2)\Psi = 0.$$

Since here positive and negative energy solutions can be transformed into each other, we allow states which are a mixture of positive and negative energy.

Let us investigate to which situation a solution leads, which is localized in a bounded volume V during some time interval $|t| < \tau$. Then also $\frac{\partial\Psi}{\partial t}(0,\vec{x}) = 0$ for $\vec{x} \notin V$. Hence, since Ψ satisfies a hyperbolic differential equation, for any t $\Psi(t,\vec{x})$ as a function or distribution in $\vec{x}$-space has a bounded support: the support grows to the future and to the past with velocity 1, which is the velocity of light, here. If we assume that Ψ belongs to $S'(\mathbb{R}^4)$, it follows that the Fourier transform Φ can be written as

$$\Phi(p) = F^+(p+i0) - F^-(p-i0),$$

where $F^{\pm}(p \pm i0)$ are the boundary values in $S'(\mathbb{R}_4)$ of holomorphic functions in $\mathbb{R}_4 \pm iC^*$ with $C^* = \{(q_0,\vec{q}) \mid q_0 > \|\vec{q}\|\}$, see [68]. Since Ψ satisfies (1.17) $\Phi(p)$ vanishes for $\|\vec{p}\| < m$ (in fact, similarly to (1.10) Φ is concentrated on the hyperboloide $p_0^2 = \vec{p}^2 - m^2$). The "Edge of the Wedge" theorem implies that F^+ and F^- are analytic continuations of each other.

Furthermore, it can be shown (see [68]) that any function F, which is holomorphic in $\{\mathbb{R}^n + iC\} \cup \{\mathbb{R}^n - iC\} \cup U \subset \mathbb{C}^n$, where $C = \{(y_0,\vec{y}) \mid y_0 > \alpha\|\vec{y}\|, \vec{y} \in \mathbb{R}^{n-1}\}$ for some $\alpha > 0$ and where U is an open neighborhood in $\mathbb{C}^n$ of $\{(x_0,\vec{x}) \mid \|\vec{x}\| < a\}$ for some $a > 0$, is an entire function. Hence in the above $F^+(p+i0) - F^-(p-i0)$ vanishes everywhere. Therefore Φ, and thus Ψ, is identically zero. The conclusion is that except zero no solution Ψ of (1.17) with a bounded support during some time interval belongs to $S'(\mathbb{R}^4)$. In particular, the fundamental solution belongs to $\mathcal{D}'(\mathbb{R}^4)$ and not to $S'(\mathbb{R}^4)$ and it does not correspond to real energy p_0 and impulse $\vec{p}$, cf. [19]. Therefore, not every pair of initial values ψ_0 and ψ_1 in $S'(\mathbb{R}^3)$ yields a solution corresponding to real p. Only those ψ_0 and ψ_1 in $S'(\mathbb{R}^3)$ whose Fourier transforms vanish for $\|\vec{p}\| < m$ yield a solution in $S'(\mathbb{R}^4)$, see formula (1.11) with m^2 replaced by $-m^2$. Hence, for any wave function Ψ describing a superluminal state, $\Psi(t,\vec{x})$ or $\frac{\partial\Psi}{\partial t}(t,\vec{x})$ cannot vanish identically for $\vec{x}$ outside a bounded volume at any time t.

Although equation (1.17) is supposed to describe a superluminal state, the characteristics show that any solution localized in a bounded space-volume cannot grow faster than with the speed of light, cf. the conclusion in [66]. However, this phenomenon can never be "observed", since localized solutions do not correspond to real values of energy and impulse, cf. the conclusion in [50] that an equation like (1.17) may describe superluminal procession.

Unlike subluminal free particles, it can happen that a solution Ψ of

(1.17) as well as its time derivative $\frac{\partial\Psi}{\partial t}$ vanishes in a bounded volume at some time t. Then such a "hole" would be filled with the speed of light. For, if $\Psi \in S'(\mathbb{R}^4)$ is written as $\Psi = \Psi^+ + \Psi^-$ where Ψ^+ corresponds to $p_0 \geq 0$ and Ψ^- to $p_0 < 0$, and if we require that for any t $\Psi^\pm(t,\vec{x})$ and $\frac{\partial\Psi^\pm}{\partial t}(t,\vec{x})$ are L^2-function of $\vec{x} \in \mathbb{R}^3$, then the question whether $\Psi(t,\vec{x})$ and $\frac{\partial\Psi}{\partial t}(t,\vec{x})$ can vanish in the same space-volume at the same time is equivalent to the following question:

Does there exist a function f in the Sobolev space $H^1(\mathbb{R}^3)$ such that both the function itself and its Fourier transform vanish identically in some open set in $\mathbb{R}^3$ and in $\mathbb{R}_3$, respectively?

It is very easy to see that the answer is affirmative if f is a tempered distribution, for example we can choose the fundamental solution g of the wave equation. Now let ϕ and ψ be C^∞-functions with small supports around the origin in $\mathbb{R}_3$ and $\mathbb{R}^3$, respectively. Then $\phi * Fg$ is a C^∞-function of polynomial growth and

$$\hat{f}(\xi) \stackrel{\text{def}}{=} F\psi(\xi)\cdot(\phi * Fg)(\xi)$$

is a function in $S(\mathbb{R}_3)$, which vanishes identically in some open set in $\mathbb{R}_3$ because Fg does. Also

$$f \stackrel{\text{def}}{=} F^{-1}\hat{f} = \frac{1}{(2\pi)^n}(g\cdot F^{-1}\phi) * \psi$$

vanishes identically in some open set in $\mathbb{R}^3$ because g does. Finally, f belongs to $H^1(\mathbb{R}^3)$ because it even belongs to $S(\mathbb{R}^3)$.

CHAPTER II

REAL-CARRIED ANALYTIC FUNCTIONALS AND BOUNDARY VALUES OF ANALYTIC FUNCTIONS

In [48] Martineau has discussed properties of analytic functionals with bounded carrier and their Fourier transforms. Here, we shall treat analytic functionals with unbounded carrier defined on spaces of analytic functions satisfying certain growth conditions at infinity. Unlike in the case of bounded carriers, these growth conditions are involved in the definition of unbounded carriers, and moreover, a class of neighborhoods has to be specified.

In section 1 properties of real-carried analytic functionals will be derived. We shall consider two types of analytic functionals, of which one belongs to a Frechet space. The properties are similar to those given in [47] for analytic functionals with bounded, real carriers. The proofs given here rely on [47] as long as we deal with Frechet spaces, while in the other case the proofs are suitably adapted.

Section 2 is concerned with Fourier transforms of real-carried analytic functionals defined on spaces Z_M which are subsets of Z, the space of Fourier transforms of $\mathcal{D}$. The spaces Z_M are determined by growth conditions in the real directions. As a limit case the space of exponentially decreasing real analytic functions arises and the dual of this space is just the set of Fourier hyperfunctions [38]. Since the space of Fourier transforms of elements in Z_M is a subset of $\mathcal{D}$, its dual contains more general objects (namely, ultra-distributions) than distributions in $\mathcal{D}'$. As has been done in [60] for distributions, here we shall represent such ultradistributions as boundary values of analytic functions. So they arise very naturally between distributions and hyperfunctions on the one hand. Being boundary values of analytic functions, too, their Fourier transforms form the transition from real-carried analytic functionals in Z' to Fourier hyperfunctions on the other hand. Since Fourier transformation is an isomorphism it is possible to define ultradistributions completely by studying their Fourier transforms which are the analytic functionals we are concerned with. However, for clarity we shall discuss ultradistributions and some properties directly, where for the proofs we refer to [42].

Finally, the "Edge of the Wedge" theorem for distributions and for ultradistributions as well will be the subject of section 3. We will give a simple proof by means fo Fourier transformation, which is based on techniques used in [4].

II.1 REAL-CARRIED ANALYTIC FUNCTIONALS

II.1.i THE SPACE Z'

We consider a familiar example of a space of analytic functionals. The Fourier transform of the space $\mathcal{D}$ of C^∞-testfunctions with compact support is the space Z of entire functions decreasing in the real directions faster than each negative power of $\|z\|$ and increasing exponentially in the imaginary directions. The dual space Z' is a space of analytic functionals and its Fourier transform is the space $\mathcal{D}'$ of distributions. Tempered distributions in $S'(\mathbb{R}^n)$ or distributions with compact support K in $\mathbb{R}^{2n} \cong \mathbb{C}^n$ are examples of elements of Z'. For an entire function f and for a multiindex α we have

$$\sup_{z\in K} |D^\alpha f(z)| \le \frac{\alpha!\sqrt{n}^{|\alpha|}}{\bar{\varepsilon}^{\alpha}} \sup_{z\in K(\varepsilon)} |f(z)|$$

for every $\varepsilon > 0$, where $K(\varepsilon)$ denotes the ε-neighborhood of K in $\mathbb{C}^n$ and $\bar{\varepsilon}$ the vector in $\mathbb{R}^n$ with components ε. Hence, for all $f \in Z$ and every $\varepsilon > 0$, a distribution T with support K satisfies

$$(2.1) \qquad |\langle T,f\rangle| \le M_\varepsilon \sup_{z\in K(\varepsilon)} |f(z)|$$

for some constants M_ε depending on ε and T. We may consider K as the support of the analytic functional T, but in general such a notion has properties different from supports of distributions. In [30, p.105] an example has been given of an analytic functional μ which satisfies (2.1) for all sets K in $\mathbb{C}^2$ of the form $K_\alpha = \{(z_1,z_2)\,|\,|z_1|\le\alpha, |z_2|\le\frac{1}{\alpha}\}$, but which does not satisfy (2.1) for $K = \bigcap_{\alpha>0} K_\alpha = \emptyset$ (μ is the Fourier transform of the distribution in $\mathbb{R}^2$ defined by the function $\cosh 2\sqrt{\xi_1\xi_2}$). Therefore a compact set $K \subset \mathbb{C}^n$

satisfying (2.1) for every $\varepsilon > 0$ is called the *carrier* of the analytic functional T. In Z' unbounded carriers can be defined, too. For that purpose we first analyze the topology of the space Z.

Let Z(a) be the Frechet space $\operatorname{proj\,lim}_{m\to\infty} Z(a)_m$, where $Z(a)_m$ is the space of entire functions endowed with the norm

$$(2.2)\qquad \|f\|_m \stackrel{\text{def}}{=} \sup_{z\in\mathbb{C}^n} (1+\|z\|)^m e^{-a\|y\|}|f(z)|.$$

Then $Z = \operatorname{ind\,lim}_{a\to\infty} Z(a)$. Elements $\mu \in Z'(a)$ can be written as $\langle\mu,f\rangle = \int h(x)f(x)dx$ for some entire function h [21, III § 2.3]. Hence μ is a functional on the space of restrictions to $\mathbb{R}^n$ of functions in Z(a). In general, this is no longer true for $\mu \in Z'$. For example the Fourier transform of the infinite order distribution $\sum_m \delta^{(m)}(\xi-m)$ is defined by $\sum_m \int (ix)^m e^{imx} f(x)dx$ for $f \in Z$.

DEFINITION. *An analytic functional* $\mu \in Z'$ *is carried by the closed set* $\Omega \subset \mathbb{C}^n$ *with respect to the decreasing sequence* $\{\Omega_k\}_{k=1}^{\infty}$ *of neighborhoods of* Ω*, if for every* k μ *is already a functional on the space* $Z|_{\Omega_k}$ *of restrictions to* Ω_k *of functions in* Z*, where* $Z|_{\Omega_k}$ *carries the topology induced by* Z*, i.e., in (2.2) the supremum should be taken over all* $z \in \Omega_k$.

If the neighborhoods Ω_k are the set of 1/k-neighborhoods

$$\Omega(1/k) \stackrel{\text{def}}{=} \{z \,|\, \|z-z'\|\le 1/k, z'\in\Omega\}$$

we will just say that μ is carried by Ω.

According to [16, th.5.13*] a fundamental system of neighborhoods of zero in Z is given by

$$V(K,\alpha) \stackrel{\text{def}}{=} \{f\in Z \,|\, |f(z)|\le \alpha K(z)\},$$

where $\alpha > 0$ and where K is a positive, continuous function of the following form: let $\{a_j\}$ be a strictly increasing sequence of integers with $a_0 = a_1 = a_2 = 0$, $a_{j+2} > 2a$, and let ℓ be a positive integer; set $K(z) = (1+\|x\|)^{-\ell^1} \times (1+\|y\|)^{-\ell^{j+2}} \exp((j-2)\|y\|)$ for $a_j(1+\log(1+\|x\|)) \le \|y\| \le \frac{1}{2} a_{j+1}(1+\log(1+\|x\|))$; the definition of K is completed by requiring that K is a function of $\|x\|$, $\|y\|$ which is continuous and such that, for fixed $\|x\|$, $\log K(\|x\|,\|y\|) + \ell[\log(1+\|x\|) + \log(1+\|y\|)]$ is linear in $\|y\|$ in the regions in which it is not already defined above. Then a fundamental system of neighborhoods of zero in $Z|_{\Omega_k}$ is obtained by $\{f\in Z|_{\Omega_k} \,|\, |f(z)|\le\alpha K(z), z\in\Omega_k\}$. Now the Hahn-Banach theorem and Reisz' representation theorem imply that for every k an analytic functional μ carried by Ω with respect to $\{\Omega_k\}$ can be represented as a measure μ_k on Ω_k satisfying

$$\int_{\Omega_k} K_k(z)\,|d\mu_k(z)| \le M_k,$$

where K_k is a function as described above depending on k.

In chapter III we shall investigate the Fourier transforms of analytic functionals carried by convex sets $\Omega \subset \mathbb{C}^n$. In this chapter we restrict ourselves to the case where Ω is contained in $\mathbb{R}^n = \{z|z=x+iy, y=0, x\in\mathbb{R}^n\}$. In this case the spaces

$$Z_F \overset{\text{def}}{=\!=} \operatorname*{proj\,lim}_{m\to\infty}\ \operatorname*{ind\,lim}_{a\to\infty} Z(a)_m$$

and

$$Z \overset{\text{def}}{=\!=} \operatorname*{ind\,lim}_{a\to\infty}\ \operatorname*{proj\,lim}_{m\to\infty} Z(a)_m$$

induce the same topology on $Z|_{\Omega(\varepsilon)}$. Indeed, according to [76, th.5.10] a fundamental system of neighborhoods of zero in Z_F is given by $V(K',\alpha)$, where now $K'(z) = (1+\|z\|)^{-m} K_1'(y)$ with $m \ge 0$ and with K_1' a positive, continuous function dominating every $\exp a\|y\|$, $a > 0$. Z_F is the Fourier transform of $\mathcal{D}_F$, the test space for the finite-order-distributions. Hence the (inverse) Fourier transforms of all elements μ in Z' carried by the real set Ω are finite-order-distributions and, moreover, for every $\varepsilon > 0$ these μ satisfy

$$|\langle\mu,f\rangle| \le M_\varepsilon \sup_{z\in\Omega(\varepsilon)} [(1+\|x\|)^{m(\varepsilon)}|f(z)|], \qquad f \in Z,$$

with M_ε and $m(\varepsilon)$ depending on ε and μ. The above given representation yields that for every $\varepsilon > 0$ μ can be represented as a measure μ_ε on $\Omega(\varepsilon)$ satisfying

$$\int_{\Omega(\varepsilon)} \frac{|d\mu_\varepsilon(z)|}{(1+\|x\|)^{m(\varepsilon)}} \le M_\varepsilon.$$

II.1.ii. GENERAL SPACES OF REAL-CARRIED ANALYTIC FUNCTIONALS

We introduce real-carried analytic functionals in spaces defined in a more general way of which the real-carried elements of Z' are only an example. Real-carried analytic functionals, originally defined on some space H of entire functions f, can be extended to the space A of restrictions of f to ε-neighboorhoods of $\mathbb{R}^n$ by the Hahn-Banach theorem, where A carries the topology induced by H. This extension is unique if H is dense in A. We shall not treat this question, but we shall merely start with spaces A consisting of all funcitons analytic in ε-neighborhoods of $\mathbb{R}^n$, which satisfy certain growth conditions at infinity. We shall consider two types of such spaces A.

Let $\{\phi_j\}_{j=1}^{\infty}$ be an increasing or a decreasing sequence of continuous functions defined on $\mathbb{R}^n$, and let Ω_j be the open $1/j$-neighborhood in $\mathbb{C}^n$ of the closed set Ω in $\mathbb{R}^n$. Let $A_m(\Omega_k)$ be the Banach space of analytic functions f in Ω_k with

$$(2.3) \qquad \|f\|_{m,k} \overset{\text{def}}{=\!=} \sup_{z\in\Omega_k} |f(z)\exp - \phi_m(x)| < \infty.$$

If $\{\phi_j\}$ is an increasing sequence, define $A(\Omega)$ by

$$(2.4) \qquad A(\Omega) \overset{\text{def}}{=\!=} \operatorname{ind}\lim_{k\to\infty} A_k(\Omega_k)$$

and if $\{\phi_j\}$ is decreasing by

$$(2.5) \qquad A(\Omega) \overset{\text{def}}{=\!=} \operatorname{ind}\lim_{k\to\infty} \operatorname{proj}\lim_{m\to\infty} A_m(\Omega_k),$$

where all needed injections are defined by restriction. If $\Omega = \mathbb{R}^n$ we shall just write A.

Real-carried analytic functionals in Z' are defined on a space $Z(\mathbb{R}^n)$ of the second type with $\phi_m(x) = -m \log(1+\|x\|)$. In section II.2 the functions ϕ_j will be negative with order of growth between $-j \log(1+\|x\|)$ and $-1/j\|x\|$. The limits of the spaces they define are on the one side $Z(\mathbb{R}^n)$ and on the other side the space of the first type (2.4) defined by $\phi_k(x) = -1/k\|x\|$. The duals of these limit spaces consist of Fourier transforms of certain distributions and, by definition [38], of Fourier hyperfunctions, respectively. The cases in between correspond to Fourier transforms of certain ultradistributions of Roumieu type or of Beurling type, depending on the respective cases (2.4) and (2.5) (cf. section II.2.iii).

A $\mu \in A'$ carried by Ω can be extended to an element of $A(\Omega)'$ with the same carrier. This extension is unique if A is dense in $A(\Omega)$ and then every $\mu \in A(\Omega)'$ is uniquely determined by its action on functions of A. Again, as we are here interested in elements of A' only, we do not bother about the question whether A is dense in $A(\Omega)$. [1)]

II.1.iii. PROPERTIES OF REAL-CARRIED ANALYTIC FUNCTIONALS

First we shall show that every analytic functional in A' has a, uniquely defined, smallest carrier which joins some properties of supports of distributions. In order to do so we have to make some assumptions implying the triviality of a cohomology group which will be shown in chapter VI for spaces A of type (2.4) and in chapter VII (cor.7.5) for spaces A of type (2.5). The result is that for each $f \in A(\Omega_1 \cap \Omega_2)$ there are $f_j \in A(\Omega_j)$, $j = 1,2$, such that

(2.6) $$f = f_2 - f_1$$

The proof uses the possibility of rewriting the spaces A in a different form. Essentially, it is based on the following property of closed sets Ω in $\mathbb{R}^n$.

<u>LEMMA 2.1.</u> (see chapter V, lemma 5.1). *For any* $1/k$-*neighborhood* $\Omega(1/k)$ *of* Ω *there is an open pseudoconvex neighborhood* Ω_k *with* $\Omega(1/2k) \subset \Omega_k \subset \Omega(1/k)$.

Hence formula's (2.4) and (2.5) with pseudoconvex sets Ω_k define the

1) This happens certainly if Ω is compact, because each compact set in $\mathbb{R}^n$ is polynomially convex (cf. chapter V, lemma 5.1), hence for $f \in A(\Omega)$ the function $f(z)\exp z^2$ can be approximated in every Ω_k by polynomials P_k and then f is approximated by $P_k(z)\exp - z^2 \in A$. It follows from results obtained in the following chapters (th.4.1 and cor.7.4, cf. also cor.3.1) that A is dense in $A(\Omega)$ if Ω is convex and if $\{\phi_j\}$ satisfies the conditions of theorem 2.4 below. In [38, th.2.2.1] it is shown that A is dense in every $A(\Omega)$, if $A(\Omega)$ is a space of type (2.4) with $\phi_k(x) = -1/k\|x\|$ and with certain neighborhoods Ω_k, larger than ε-neighborhoods.

spaces A just as well. Furthermore, the spaces A should not change if the weight functions ϕ_j of x are changed into plurisubharmonic functions ψ_j of z and if moreover the differences of the functions ϕ_j are not too small. More precisely, the following condition must be satisfied: there is an α-neighborhood $\mathbb{R}^n(\alpha)$ in $\mathbb{C}^n$ of $\mathbb{R}^n$ and, if $\{\phi_j\}$ is increasing, for every j there exist a plurisubharmonic function $\psi = \psi_j$ on $\mathbb{R}^n(\alpha)$ and, for every $N \geq 0$, moreover an $m = m(j,N) \geq j$ and $C = C(j,N) \geq 0$, or if $\{\phi_j\}$ is decreasing, for every m there exist a plurisubharmonic function $\psi = \psi_m$ on $\mathbb{R}^n(\alpha)$ and, for every $N \geq 0$, moreover a $j = j(m,N) \geq m$ and $C = C(m,N) \geq 0$, such that

$$\phi_j(x) \leq \psi(z) + N \log(1+\|z\|^2) \leq \phi_m(x) + C, \quad \|y\| < \alpha. \tag{2.7}$$

In lemma 5.2 it will be shown that the spaces of the next section satisfy this condition.

According to [73, cond. HS_1 and HS_2, p.15] it follows from condition (2.7) that A can be written with the L^2-norms

$$\{\int_{\Omega_k} |f(z)|^2 \exp - 2\psi_m(z) d\lambda(z)\}^{\frac{1}{2}} , \tag{2.8}$$

where $\lambda(z)$ denotes the Lebesgue measure in $\mathbb{C}^n$, instead of the sup-norms (2.3). We denote by $H(\Omega_k;\psi_m)$ the Hilbert space of holomorphic functions in Ω_k with inner product induced by the norm (2.8).

Furthermore, let $\Omega_k^{(1/m)}$ be the open (ε_k/m)-shrinking of Ω_k, where $\varepsilon_k > 0$ is such that the ε_k-shrinking of Ω_k contains Ω_{k-1}. This is possible because we deal with ε-neighborhoods of closed sets in $\mathbb{R}^n$. Moreover, it is clear that (2.5) does not change if the functions in $A_m(\Omega_k)$ have only finite norms on $\Omega_k^{(1/m)}$. Finally, since in (2.4) and (2.5) only restrictions of functions in Ω_k to Ω_{k-1} or to $\Omega_k^{(1/m)}$, respectively, are important, we may change the functions ψ_j of condition (2.7) near the boundary of Ω_k. So we have obtained the following lemma.

<u>LEMMA</u> 2.2. *Let condition* (2.7) *be satisfied. Then the space* $A(\Omega)$ *given by* (2.4) *can also be written as*

$$A(\Omega) = \operatorname*{ind\,lim}_{k\to\infty} H(\Omega_k;\psi_k) = \operatorname*{ind\,lim}_{k\to\infty} H(\Omega_k;\psi_k(z) + \log(1+\|z\|^2) + \log(1+d(z,\Omega_k^c)^{-1}))$$

and the space $A(\Omega)$ *given by* (2.5) *as* $A(\Omega) = \operatorname{ind}\lim_{k\to\infty} B(\Omega_k)$ *with*

$$(2.9) \qquad B(\Omega_k) \overset{\text{def}}{=\!=} \operatorname{proj}\lim_{m\to\infty} H(\Omega_k^{(1/m)};\psi_m(z)) = \operatorname{proj}\lim_{m\to\infty} H(\Omega_k^{(1/m)};\psi_m(z) + \log(1+\|z\|^2) + \log(1+d(z,\Omega_k^c)^{-1})),$$

where the sets $\{\Omega_k\}$ *are pseudoconvex and where* $d(z,\Omega_k^c)$ *denotes the distance from* z *to the boundary of* Ω_k.

Now bearing in mind that intersections of pseudoconvex sets are again pseudoconvex and using lemma 2.1, we can choose in lemma 2.2 pseudoconvex neighborhoods $\{(\Omega_1 \cup \Omega_2)_k\}$, $\{(\Omega_1)_k\}$ and $\{(\Omega_2)_k\}$ of $\Omega_1 \cup \Omega_2$, Ω_1 or Ω_2, respectively, which satisfy

$$(2.10) \qquad (\Omega_1 \cup \Omega_2)_k = (\Omega_1)_k \cup (\Omega_2)_k.$$

For the spaces of type (2.4) formula (2.6) now follows from lemma 2.2 (cf. cor. 7.5 with $\Omega^k = \Omega^{k+1}$ and $\phi^k = \phi^{k+1}$, $k = 1,2,\ldots$).

<u>LEMMA</u> 2.3.i. *Let* Ω_1 *and* Ω_2 *be closed sets in* $\mathbb{R}^n$ *with non-empty intersection and let condition* (2.7) *be satisfied. Furthermore, let* $A(\Omega_1)$, $A(\Omega_2)$ *and* $A(\Omega_1 \cap \Omega_2)$ *be given by* (2.4), *then for any* $f \in A(\Omega_1 \cap \Omega_2)$ *there are* $f_j \in A(\Omega_j)$, $j = 1,2$, *such that* (2.6) *holds.*

For spaces of type (2.5) this result is more difficult to prove and a further condition (cf. cond. (7.3)) is needed, which implies that the differences of the functions ψ_m may not be too large: for every p and m with $p \geq m$ there exists a holomorphic function $g_{p,m}$ is an α-neighborhood of $\mathbb{R}^n$ in $\mathbb{C}^n$ and, for every k, moreover a constant $K = K(p,m,k,)$ such that

$$(2.11) \qquad 0 < |g_{p,m}(z)| \leq K \exp -k\{\psi_m(z) - \psi_p(z)\},\ \|y\| < \alpha,\ k = 1,2,\ldots$$

For the spaces of the next section it suffices to take $g_{p,m}(z) = \exp -z^2$, but if, for example, $\phi_m(x) = \exp(1/m \exp x^2)$ condition (2.11) cannot be satisfied. Now corollary 7.5 yields (2.6) for the spaces $B(\Omega_k)$ given by (2.9), because for the function σ in condition (4.22) of the corollary we can take $\sigma(z) = -\log d(z,\Omega_k^c)$ which is plurisubharmonic [30, th. 2.6.7].

LEMMA 2.3.ii. *Let* Ω_1 *and* Ω_2 *be as in lemma 2.3.i and let conditions (2.7) and (2.11) be satisfied. Let the pseudoconvex neighborhoods* $\{(\Omega_1)_k\}$ *and* $\{(\Omega_2)_k\}$ *of* Ω_1 *and* Ω_2 *be such that also the neighborhoods* $\{(\Omega_1)_k \cup (\Omega_2)_k\}$ *of* $\Omega_1 \cup \Omega_2$ *are pseudoconvex. Then for* $k = 1,2,\ldots$ *and for any* $f \in B((\Omega_1)_k \cap (\Omega_2)_k)$ *there are* $f_j \in B((\Omega_j)_k)$, $j = 1,2$, *such that (2.6) holds in* $(\Omega_1)_k \cap (\Omega_2)_k$.

THEOREM 2.4. (cf. [47, prop 1]). *Let A be given by (2.4) or (2.5) and let condition (2.7) be satisfied. If A is of type (2.5), let moreover condition (2.11) be satisfied. If* $\mu \in A'$ *is carried by the closed sets* Ω_1 *and* Ω_2 *in* $\mathbb{R}^n$ *with* $\Omega_1 \cap \Omega_2 \neq \emptyset$, *then* μ *is already carried by* $\Omega_1 \cap \Omega_2$.

PROOF. Since by lemma 2.1 $\Omega_1 \cup \Omega_2$, Ω_1 and Ω_2 have pseudoconvex neighborhood bases which moreover satisfy (2.10), lemma 2.3.i and ii shows that any function $f \in A(\Omega_1 \cap \Omega_2)$ can be written as (2.6) with $f_j \in A(\Omega_j)$, $j = 1,2$. Hence, the following continuous map I is surjective

$$(2.12) \qquad I: A(\Omega_1) \times A(\Omega_2) \to A(\Omega_1 \cap \Omega_2)$$

with $I(f_1,f_2) = f_2 - f_1$. The kernel of I is just $\{(f,f)\,|\,f \in A(\Omega_1 \cup \Omega_2)\}$.

Furthermore, we assert that I is an open map. Let us first show this for spaces $A(\Omega)$ of type (2.4). It follows from lemma (2.2) that such spaces are inductive limits of Hilbert spaces, hence DFS^*-spaces [40] and thus duals of reflexive Frechet spaces. Since such spaces are Ptak spaces [61, IV. § 8 ex. 2, p.162] the open mapping theorem [61, IV. § 8.3, cor 1] implies that I is an open map. If the spaces $A(\Omega)$ are of type (2.5), we have the more precise result (lemma 2.3.ii) that even for every k the map I_k, defined similarly to I, is a surjective map between the Frechet spaces

$$I_k: B((\Omega_1)_k) \times B((\Omega_2)_k) \to B((\Omega_1)_k \cap (\Omega_2)_k)$$

where $B(\Omega)$ is given by (2.9). Hence the ordinary open mapping theorem implies that I_k is open. The maps $\{I_k\}$ commute with the restriction maps, and so lemma 2.2 and the definition of open sets in an inductive limit (cf. the characterization of a 0-neighborhood base in [20, § 23, 3.14]) imply that I is open.

Now we first extend μ to an element of $A(\Omega_1 \cup \Omega_2)'$ and then to elements

$\mu_1 \in A(\Omega_1)'$ and $\mu_2 \in A(\Omega_2)'$. Define $\tilde{\mu} \in A(\Omega_1 \cap \Omega_2)'$ by

$$<\tilde{\mu},f> \overset{\text{def}}{=\!=} <\mu_2,f_2> - <\mu_1,f_1>$$

for some $(f_1,f_2) \in I^{-1}(f)$. Since μ_1 equals μ_2 on $A(\Omega_1 \cup \Omega_2)$ $\tilde{\mu}$ is independent of the representant in $I^{-1}(f)$. Furthermore, since μ_1 and μ_2 are continuous, they are bounded on some neighborhood of zero in $A(\Omega_1)$ and $A(\Omega_2)$, respectively. The fact that I is an open map implies that $\tilde{\mu}$ is bounded on some neighborhood of zero in $A(\Omega_1 \cap \Omega_2)$, hence that it is continuous. Finally, for any $f \in A$ we have

$$<\tilde{\mu},f> = <\mu_2,f_2+h> - <\mu_1,h> = <\mu_2,f> = <\mu,f>$$

for some $h \in A(\Omega_1 \cup \Omega_2)$. □

COROLLARY 2.5. *Let the conditions of theorem 2.4 be satisfied. If* μ *is carried by two disjunct closed sets in* $\mathbb{R}^n$ *then* $\mu = 0$.

PROOF. By enlarging the carriers of μ suitably theorem 2.4 yields that there is a ball S in $\mathbb{R}^n$ such that μ is carried by any closed set in S. We may assume that $S = \{x | \|x\| \le 1\}$. For any multiindex α we have

$$<\mu,z^\alpha> = D^\alpha f(0)$$

where

$$f(\zeta) \overset{\text{def}}{=\!=} <\mu_z,e^{z\cdot\zeta}>.$$

f is an entire function and since μ is carried by any closed subset of the unitsphere, there are $K > 0$ and $\varepsilon > 0$ with

$$|f(\zeta)| \le K \exp\{-\frac{1}{2}\|\xi\|+\varepsilon\|\eta\|\}.$$

Hence the Fourier transform of f is, on the one hand, real-analytic and, on the other hand, by the Paley-Wiener theorem a C^∞-function with compact support, thus $f \equiv 0$. Hence $<\mu,z^\alpha> = 0$ for all α. Since the polynomials are dense in the functions holomorphic in the origin and since μ is also carried by the origin, it follows that $\mu = 0$. □

Now we are able to define the support [1)] of $\mu \in A'$.

DEFINITION 2.6. *Let the conditions of theorem 2.4 be satisfied. Then the intersection of all the carriers of an analytic functional $\mu \in A'$ is called the support of μ.*

REMARK. In the example of [30] given earlier the set

$$\{(z_1,z_2) \mid |z_1|\le 2,\ |z_2|\le \tfrac{1}{2} \quad \text{or} \quad |z_1|\le \tfrac{1}{2},\ |z_2|\le 2\} \subset \mathbb{C}^2$$

is not pseudoconvex. For its holomorphically convex hull equals its logarithmic convex hull $\{(z_1,z_2) \mid |z_1|\le 2,\ |z_2|\le 2,\ |z_1||z_2|\le 1\}$, see [68]. The intersection of carriers is no carrier and hence the support cannot be defined.

Next we shall prove that (real) carriers can be localized, a property which is easy to show for supports of distributions (the property that for any finite collection of closed sets $\{U_k\}_{k=1}^N$ covering $\mathbb{R}^n$ every distribution g can be written as $g = \sum_{k=1}^N g_k$ where g_k has its support in U_k).

THEOREM 2.7. (cf. [47, prop 2] and [60, proof of th. 4.2]). *For any finite collection of closed sets $\{U_k\}_{k=1}^N$ in $\mathbb{R}^n$ with union $\mathbb{R}^n$, each $\mu \in A'$ can be written as $\mu = \sum_{k=1}^N \mu_k$ where $\mu_k \in A(U_k)'$.*

PROOF. Define the continuous map

$$I: A \to \prod_{k=1}^N A(U_k)$$

by restriction. Its transposed I^t between the duals

$$I^t: \prod_{k=1}^N A(U_k)' \to A'$$

---

1) The support of a (ultra) distribution g, defined on a space $\mathcal{W}$ of C^∞ testfunctions, is defined as the smallest closed set U in $\mathbb{R}^n$ such that any $x_0 \notin U$ has an open neighborhood V_0 with $\langle g,\phi\rangle = 0$ for every $\phi \in \mathcal{W}$ with $\phi(x) = 0$ if $x \notin V_0$. Since there are no analytic functions $\phi \not\equiv 0$ satisfying this, this definition of support is impossible for an analytic functional. The reason for calling the smallest carrier the support of the analytic functional is that this concept has similar properties to the support of a distribution, unlike the carrier of an analytic functional (cf. the earlier mentioned example of [30]).

is given by $I^t(\mu_1,\ldots,\mu_N) = \sum_{k=1}^{N} \mu_k$, for

$$\langle I^t(\mu_1,\ldots,\mu_N),f\rangle = \sum_{k=1}^{N} \langle \mu_k,(If)_k\rangle = \sum_{k=1}^{N} \langle \mu_k,f\rangle = \langle \sum_{k=1}^{N} \mu_k,f\rangle.$$

Clearly, I is an injective and open map from A into Im I, when Im I carries the topology induced by $\Pi A(U_k)$ (this can be seen by inspection of the open sets in the spaces A). Then according to [65, prop. 35.4 and lemma 37.7] I^t is surjective (if the duals of the spaces A are reflexive Frechet spaces, this can be seen also by [65, th. 37.2] since clearly I has closed image, cf. [47]). □

In general, a distribution in $\mathcal{D}'(U)$ where U is an open set in $\mathbb{R}^n$ cannot be extended to a distribution in $\mathcal{D}'(\mathbb{R}^n)$. We shall now show that this property does hold for real carried analytic functionals.[1)] Before formulating this we introduce the concept of local equality of real-carried analytic functionals, see [47].

If $\mu \in A'$ with A satisfying the conditions of theorem 2.4, according to theorem 2.7, can be written as $\mu = \sum_{k=1}^{N} \mu_k$ and as $\mu = \sum_{j=1}^{M} \tilde{\mu}_j$, we have

$$\sum_{k=1}^{N} \mu_k - \sum_{j=1}^{M} \tilde{\mu}_j = 0.$$

Hence for any $x \in \mathbb{R}^n$

$$\sum_{\{k|x \in \text{carrier of } \mu_k\}} \mu_k - \sum_{\{j|x \in \text{carrier of } \tilde{\mu}_j\}} \tilde{\mu}_j = - \sum_{\{\text{remaining } k\}} \mu_k + \sum_{\{\text{remaining } j\}} \tilde{\mu}_j.$$

By theorem 2.4 the left hand side and the right hand side have their support contained in the intersection of their carriers, so that x does not belong to the support of the left hand side. We now consider, more generally, infinite sums of analytic functionals with bounded carriers U_k. Therefore, no weightfunctions ϕ_j occur in the definition of $A(U_k)$ and theorem

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1) This may be expressed by saying that the sheaf of real-carried analytic functionals, and by consequence [47] the sheaf of hyperfunctions, is flabby.

2.4 is valid without its conditions on the weight functions, cf. [47,prop 1].

Let $\{U_k\}$ and $\{\tilde{U}_k\}$ be locally finite coverings, consisting of compact sets, of the open set U in $\mathbb{R}^n$ and let $\{\mu_k\}$ and $\{\tilde{\mu}_k\}$ be analytic functionals carried by U_k or $\tilde{U}_k$, respectively. Then we define $\mu = \sum_k \mu_k$ and $\tilde{\mu} = \sum_k \tilde{\mu}_k$ to be locally equal if each $x \in U$ does not belong to the support of the analytic functional

$$\sum_{\{k \mid x \in U_k\}} \mu_k - \sum_{\{k \mid x \in \tilde{U}_k\}} \tilde{\mu}_k .$$

In general, $\mu = \sum_k \mu_k$ is not an element of A'. However, we shall show that there exists an element $\nu \in A'$ which is locally equal to μ.

<u>THEOREM</u> 2.8. (cf. [47, prop. 3]). *Let* $\{U_k\}_{k=1}^{\infty}$ *be a locally finite covering of the open set* $U \subset \mathbb{R}^n$ *consisting of compact sets and let* $\mu = \sum_{k=1}^{\infty} \mu_k$, *where* μ_k *is an analytic functional carried by* U_k, $k = 1,2,\ldots$ *.Furthermore, let* A *be given by (2.4) or (2.5) where condition (2.7) is satisfied. Then there exists a* $\nu \in A'$ *carried by* $\bar{U}$ *which is locally equal to* μ *in* U.

<u>PROOF</u>. It is convenient to have Frechet spaces of analytic functionals. If $A(\Omega)$ is given by (2.4), as in the proof of theorem 2.4, lemma 2.2 implies that $A(\Omega)$ is a DFS$\overset{*}{-}$space [40] so that the strong dual $A(\Omega)'$ is a Frechet space. If $A(\Omega)$ is given by (2.5), for any fixed m we will find a $\nu \in A(\Omega)'_m$ with the required properties, where

$$A(\Omega)_m \overset{\text{def}}{=\!=} \operatorname*{ind\,lim}_{k \to \infty} H(\Omega_k;\psi_m)$$

Here $H(\Omega_k;\psi_m)$ is the space whose definition preceeds lemma 2.2. Since for every $k = 2,3,\ldots$ and any m $B(\Omega_k)$ defined by (2.9) is mapped by restriction into $H(\Omega_{k-1};\psi_m)$, by lemma 2.2 $\nu \in A(\Omega)'_m$ certainly belongs to $A(\Omega)'$. But now, as before $A(\Omega)'_m$, as the strong dual of an inductive limit of Hilbert spaces, is a Frechet space.

In order to contain both cases, we denote by $A(\Omega)_{(m)}$ the space $A(\Omega)$ if $A(\Omega)$ is of type (2.4) and the space $A(\Omega)_m$ if A is of type (2.5). Thus now $A(\Omega)'_{(m)}$ is a Frechet space and it suffices to find $\nu \in A(\bar{U})'_{(m)}$ which is locally equal to μ in U.

In virtue of theorem 2.7 μ is locally equal to a sum $\sum_{k=1}^{\infty} \tilde{\mu}_k$ where $\tilde{\mu}_k$ is carried by $\overline{V_k \backslash V_{k-1}}$ and where $\{V_k\}_{k=0}^{\infty}$ are compact sets such that

$V_0 = \emptyset$, $V_k \subset \text{int } V_{k+1}$, $\bigcup_k V_k = U$ and $\overline{U\backslash V_k}$ only contains unbounded components or components intersecting ∂U. Since $A(\overline{U\backslash V_k})_{(m)}$ is mapped injectively by restriction into $A(\partial U)_{(m)}$ (here we define the class of neighborhoods of ∂U as the ε-neighborhoods in $\mathbb{C}^n$ of the complements in $\bar{U}$ of compact sets in U), $A(\partial U)'_{(m)}$ is dense in $A(\overline{U\backslash V_k})'_{(m)}$. Now $A(\overline{U\backslash V_k})'_{(m)}$ is a Frechet space, thus there is a distance d_k to the origin defining its topology. Furthermore, $A(\overline{U\backslash V_k})'_{(m)}$ can be continuously mapped into $A(\overline{U\backslash V_j})'_{(m)}$ for $k \geq j$ and therefore, for each k there exists an element $\nu_k \in A(\partial U)'_{(m)}$ with

$$d_j(\tilde{\mu}_k - \nu_k) \leq 2^{-k}, \qquad 0 \leq j \leq k-1.$$

Then

$$\nu \stackrel{\text{def}}{=\!=} \sum_{k=1}^{\infty} (\tilde{\mu}_k - \nu_k)$$

is an element of $A(\bar{U})'_{(m)}$, because its distance $d_0(\nu)$ to the origin is finite. Moreover, for every j we have

$$\nu = \sum_{k=1}^{j} \tilde{\mu}_k - \sum_{k=1}^{j} \nu_k + \sum_{k=j+1}^{\infty} (\tilde{\mu}_k - \nu_k),$$

where the last term converges in $A(\overline{U\backslash V_j})'_{(m)}$ and where the second term is carried by the complement of V_j in U. Hence ν is locally equal to μ in the interior of each V_j, thus in U. □

As an example we consider distributions in $\mathcal{D}'(\mathbb{R}^n)$. First, let T be a distribution with compact support $K \subset \mathbb{R}^n$ (hence T can be defined on C^{∞}-functions). By restriction to analytic functions T can be considered as an element of A(K)' and the support of T as analytic functional is the same as the support K of T as distribution, see [42, lemma 7.4]. Any $g \in \mathcal{D}'(\mathbb{R}^n)$ is a locally finite sum of distributions with compact support. Hence, for any $g \in \mathcal{D}'$ there is a real-carried analytic functional in Z' which is locally equal to g, but it is difficult to write down an explicite, non-trivial, example.

II.2. FOURIER TRANSFORMS OF REAL-CARRIED ANALYTIC FUNCTIONALS.

II.2.i. FOURIER TRANSFORMATION AND BOUNDARY VALUES OF ANALYTIC FUNCTIONS.

We shall define the Fourier transformation of analytic functionals defined on a subset Z_M of Z. For a C^∞ function ϕ with compact support in $\mathbb{R}_n$, the dual of $\mathbb{R}^n$, the Fourier transform $F\phi$ is defined by

$$F\phi(x) \stackrel{\text{def}}{=} \int_{\mathbb{R}_n} \phi(\xi) \exp i <\xi,x> d\xi. \tag{2.13}$$

Then $F\phi$ is a function on $\mathbb{R}^n$ which can be extended to an entire function belonging to $Z(\mathbb{C}^n)$. If ϕ belongs to a certain, locally convex, topological vector space $\mathcal{D}_M$ of C^∞-functions with compact support, the image Z_M of F in Z is given the topology such that F becomes a topological isomorphism from $\mathcal{D}_M(\mathbb{R}_n)$ onto $Z_M(\mathbb{C}^n)$. The transposed map F^t of F defines an isomorphism from $Z_M(\mathbb{C}^n)'$ onto $\mathcal{D}_M(\mathbb{R}_n)'$. We may restrict F^t to $Z_M(\mathbb{C}^n)$ or to $\mathcal{D}_M(\mathbb{R}^n)$ and we may identify a $\xi \in \mathbb{R}_n$ with an n-dimensional vector $(\xi_1,\ldots,\xi_n)$ in $\mathbb{R}^n$ so that $<\xi,x>$ becomes

$$<\xi,x> = x\cdot\xi \stackrel{\text{def}}{=} x_1\xi_1+\ldots+x_n\xi_n.$$

Then the maps

$$F^t|_{Z_M} : Z_M(\mathbb{C}^n) \to \mathcal{D}_M(\mathbb{R}_n)$$

and

$$F^t|_{\mathcal{D}_M} : \mathcal{D}_M(\mathbb{R}^n) \to Z_M(\mathbb{C}_n)$$

are also given by (2.13) due to Parseval's relation

$$<\psi,F\phi> = \int \psi(x)\{\int e^{ix\cdot\xi}\phi(\xi)d\xi\}dx = \int \phi(\xi)\{\int e^{i\xi\cdot x}\psi(x)dx\}d\xi =$$

$$= <F^t\psi,\phi>.$$

Hence we shall call also F^t Fourier transformation and denote it by

$$F: Z_M(\mathbb{C}^n)' \to \mathcal{D}_M(\mathbb{R}_n)'. \tag{2.14}$$

The transposed of the maps $F^t|_{Z_M}$ and $F^t|_{\mathcal{D}_M}$ are isomorphisms

$$(F^t|_{Z_M})^t : \mathcal{D}_M(\mathbb{R}_n)' \to Z_M(\mathbb{C}^n)'$$

$$(F^t|_{\mathcal{D}_M})^t : Z_M(\mathbb{C}_n)' \to \mathcal{D}_M(\mathbb{R}^n)'$$

and again, restricted to L^1-functions ϕ, these maps are given by (2.13). Finally, the transposed of the restriction to $Z_M(\mathbb{C}_n)$ of one of these maps yields the isomorphism

$$((F^t|_{Z_M})^t|_{Z_M})^t : \mathcal{D}_M(\mathbb{R}^n)' \to Z_M(\mathbb{C}_n)',$$

which for an L^1-function ϕ is also given by (2.13). Hence from (2.13) several maps arise which we will call Fourier transformation and denote by F. Thus, although we intended to deal with the Fourier transformation (2.14) only, this map cannot be defined in this way without introducing naturally the other maps

$$\begin{aligned} F&: Z_M(\mathbb{C}_n)' \to \mathcal{D}_M(\mathbb{R}^n)' \\ F&: \mathcal{D}_M(\mathbb{R}^n)' \to Z_M(\mathbb{C}_n)' \\ F&: \mathcal{D}_M(\mathbb{R}_n)' \to Z_M(\mathbb{C}^n)'. \end{aligned} \tag{2.15}$$

As we will see, these definitions have the advantage that, as soon as $\mu \in Z_M(\mathbb{C}_n)'$ also belongs to the dual of a space of analytic functions of ζ of which $\exp i \langle\zeta,z\rangle$ is one for z in a certain open set in $\mathbb{C}^n$, F given by (2.15) can be written as the boundary value in some sense of the function

$$\hat{\mu}(z) \overset{\text{def}}{=\!=} \langle\mu_\zeta, e^{i\langle\zeta,z\rangle}\rangle,$$

cf. lemma 2.26. We shall call the function $\hat{\mu}$ the Fourier transform [1)] of μ and $\hat{\mu}(z)$ will be denoted as $F\mu(z)$.

With the aid of Fourier transformation it will be shown that real-carried analytic functionals in Z_M' can be written as sum of boundary values

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1) Sometimes F is called Fourier-Laplace transformation [68], Fourier-Borel transformation [48] or even Fourier-Laplace-Carleman-Sato transformation [43], but we shall call F merely Fourier transformation.

of functions holomorphic in tubular radial domains, i.e., in domains of the form $T^C \overset{\text{def}}{=\!=} \mathbb{R}^n + iC$ where C is an open convex cone in $\mathbb{R}^n$. The boundary value is defined as follows: let f be a holomorphic function in $T^C_r \overset{\text{def}}{=\!=} T^C \cap \{z \mid \|y\| < r\}$ such that, for all $y \in C$ with $\|y\| < r$, $\int f(x+iy)\psi(x)dx$ exists for every $\psi \in Z_M$; the boundary value f^* of f in Z'_M is defined by

$$(2.16) \qquad \langle f^*, \psi\rangle \overset{\text{def}}{=\!=} \lim_{\substack{y\to 0\\ y\in C}} \int_{\mathbb{R}^n} f(x+iy)\psi(x)dx$$

for $\psi \in Z_M$. This limit exists, since the integral is independent of Im x, so that for each $y_0 \in C$ with $\|y_0\| < r$

$$(2.17) \qquad \langle f^*, \psi\rangle = \lim_{\substack{y\to 0\\ y\in C}} \int f(x+iy_0+iy)\psi(x+iy_0)dx =$$

$$= \int_{\mathbb{R}^n} f(x+iy_0)\psi(x+iy_0)dx, \quad \psi \in Z_M.$$

Since the testfunction space

$$H(\mathbb{R}^n) \overset{\text{def}}{=\!=} \operatorname*{ind\,lim}_{\varepsilon \to 0} H(\mathbb{R}^n(\varepsilon); -\varepsilon\|x\|)$$

for Fourier hyperfunctions is contained in all the spaces consisting of restrictions to ε-neighborhoods $\mathbb{R}^n(\varepsilon)$ in $\mathbb{C}^n$ of $\mathbb{R}^n$ of functions in Z_M, all real-carried analytic functionals μ in Z' of Z'_M can be considered as Fourier hyperfunctions in $H(\mathbb{R}^n)'$. As the Fourier transform of $H(\mathbb{R}_n)$ is just $H(\mathbb{R}^n)$, the Fourier transforms $F\mu$ of real-carried analytic functionals in Z' or Z'_M, which are certain distributions or ultradistributions, are examples of Fourier hyperfunctions in $H(\mathbb{R}_n)'$. Thus the spaces of Fourier hyperfunctions form the limit case in which all the real-carried analytic functionals in Z' or Z'_M and their Fourier transforms as well are contained. The other limit case is the space of tempered distributions which is contained in all spaces of real-carried analytic functionals and their Fourier transforms.

Now a Fourier hyperfunction can be represented as sum of boundary values f^* (2.16) of analytic functions f in T^C_r satisfying for all $C' \subset\subset C$ and all $\varepsilon > 0$

$$(2.18) \qquad |f(z)| \le K(C',\varepsilon)e^{\varepsilon\|x\|}, \quad y \in C', \ \varepsilon < \|y\| < r - \varepsilon$$

where $K(C',\varepsilon)$ depends on C' and ε, see [38]. A tempered distribution g can be wriiten as sum of boundary values of analytic functions f satisfying for all $C' \subset\subset C$

$$(2.19) \qquad |f(z)| \le K(C')(1+\|x\|)^N \|y\|^{-N}, \qquad y \in C', \ \|y\| < r'$$

with $0 < r' < r$ and with N depending on g, see [49]. In the following sections we shall give analytic representations of real-carried analytic functionals μ in Z' or Z'_M and of $F\mu$ as boundary values of analytic functions f or h, respectively. So these functions certainly satisfy (2.18), whereas functions satisfying (2.19) are examples of such functions f and h.

II.2.ii. CHARACTERIZATION OF DISTRIBUTIONS WITH REAL-CARRIED FOURIER TRANSFORMS.

Let us consider the example of real-carried analytic functionals μ in the space Z'. Then μ is an element in the space A' where A is given by (2.5) with $\phi_m(x) = -m \log(1+\|x\|)$ and $F\mu$ is a distribution in $\mathcal{D}(\mathbb{R}_n)'$. Now μ is the sum of boundary values of analytic functions and actually the following theorem 2.9 holds [60]. Before formulating this theorem we introduce the dual [1)] C^* of an open convex cone C in $\mathbb{R}^n$ as the open convex cone

$$C^* \overset{\text{def}}{=\!=} \operatorname{int}\{\xi \mid \langle\xi,y\rangle > 0, \ y \in C\} = \operatorname{int}\{\xi \mid \langle\xi,y\rangle \ge 0, \ y \in \bar{C}\}$$

in $\mathbb{R}_n$. We identify the dual of $\mathbb{R}_n$ with $\mathbb{R}^n$ and then, if $C^* \ne \emptyset$, the dual of C^* equals C

$$(C^*)^* = C = \{x \mid \langle\eta,x\rangle > 0, \ \eta \in C^*\}$$

because C is open and convex.

<u>THEOREM</u> 2.9. *For $\mu \in Z'$ the following four statements are equivalent:*

(1) *μ is carried by $\mathbb{R}^n$*

(2) *For any $\varepsilon > 0$, $F\mu \in \mathcal{D}'$ can be represented as $F\mu = \sum_{|\alpha| \le m(\varepsilon)} D^\alpha G_{\alpha,\varepsilon}$, where $G_{\alpha,\varepsilon}$ are continuous functions on $\mathbb{R}_n$ satisfying*

1) In [68] C^* stands for $\{\xi \mid \xi_1 y_1 + \ldots + \xi_n y_n \ge 0, y \in C\}$ and then $(C^*)^*$ is the closed convex hull of C.

$$|G_{\alpha,\varepsilon}(\xi)| \leq K_\alpha(\varepsilon) \exp \varepsilon\|\xi\|$$

(3) *μ is the sum of boundary values in* Z' *of functions* f_j *holomorphic in* $\mathbb{R}^n + i\,C_j$ *satisfying for any* $C_j' \subset\subset C_j$ *and any* $\varepsilon > 0$

$$|f_j(z)| \leq K(C_j',\varepsilon)(1 + \|z\|)^{N(C_j',\varepsilon)}, \quad y \in C_j',\ \|y\| > \varepsilon$$

for $j = 1,\ldots,k$, *where* $\{C_j\}_{j=1}^k$ *are open convex cones in* $\mathbb{R}^n$ *such that the closure of their duals cover* $\mathbb{R}_n$.

(4) $F\mu \in \mathcal{D}'$ *is the sum of boundary values in* $\mathcal{D}'$ *of functions* h_j *holomorphic in* $\mathbb{R}_n + i\,C_j^*$ *satisfying for any* $C_j^{*\prime} \subset\subset C_j^*$ *and any* $\varepsilon > 0$

$$|h_j(\zeta)| \leq K(C_j^{*\prime},\varepsilon)(1 + \|\eta\|^{-m(\varepsilon)})e^{\varepsilon\|\xi\|}, \quad \eta \in C_j^{*\prime}$$

for $j = 1,\ldots,p$, *where* $\{C_j^*\}_{j=1}^p$ *are open convex cones in* $\mathbb{R}_n$ *such that the closure of their duals cover* $\mathbb{R}^n$.

This theorem deals with boundary values in Z' in several dimensions and in this way it generalizes the one dimensional case discussed in [46].

II.2.iii. ULTRADISTRIBUTIONS

In the following section we will pay attention to spaces A defined by weight functions ϕ_j with an order of growth between $-j \log(1 + \|x\|)$ and $-1/j\ \|x\|$. Then the Fourier transforms of elements in A' are certain ultradistributions of Roumieu type if A is of type (2.4) and of Beurling type if A is of type (2.5). In section 2.iv we will give characterizations of these ultradistributions similar to (2), (3) and (4) of theorem 2.9. Ultradistributions are continuous, linear functionals on spaces of ultradifferentiable testfunctions. It follows the lines of this chapter if ultradifferentiable functions ϕ are defined by growth conditions on their Fourier transforms. No direct information about ϕ is obtained in this way, and therefore in this section we will also give a direct definition. Furthermore, some properties of ultradistributions will be mentioned whose proofs can be found in [42].

Throughout this and the following chapter M will stand for a continuous increasing piecewise differentiable function on $[0,\infty)$ with $M(0) = 0$, $M(\infty) = \infty$, such that M' is strictly decreasing and $\rho M'(\rho)$ is increasing to ∞ and such that

$$(2.20) \qquad \int_1^\infty \frac{M(\rho)}{\rho^2}\, d\rho < \infty$$

and for some constants $\tau > 1$ and $K > 0$

$$(2.21) \qquad 2M(\rho) \le M(\tau\rho) + K.$$

DEFINITION 2.10.i. *Let* f *be an entire function such that for every positive* m *there is a* $K > 0$ *(there are positive constants* m *and* K*) with*

$$(2.22) \qquad |f(z)| \le K \exp\{-M(m\|z\|) + a\|y\|\}$$

for some $a > 0$. *Then the inverse Fourier transform* ϕ *of* f *is an ultradifferentiable function with support in the ball with radius* a *of class* M *of Beurling type (of Roumieu type), or shortly of class* (M) *(of class* {M}*).*

Let $\{M_p\}_{p=0}^{\infty}$ be an increasing sequence of positive numbers satisfying the following properties (called M.1, M.2 and M.3 in [42]): for some positive K and h

$$M_p^2 \le M_{p-1}\, M_{p+1}, \qquad p = 1,2,\ldots$$

$$M_p \le K\, h^p \min_{0\le q\le p} M_q\, M_{p-q}, \qquad p = 0,1,\ldots$$

$$\sum_{q=p+1}^{\infty} M_{q-1}/M_q \le K\, p\, M_p/M_{p+1}, \qquad p = 1,2,\ldots$$

An equivalent, direct definition is obtained as follows:

DEFINITION 2.10.ii. *Let the sequence* $\{M_p\}_{p=0}^{\infty}$ *satisfy the above given properties. Then a* C^∞*-function* ϕ *with compact support* S *is called ultradifferentiable of class* M_p *of Beurling type (of Roumieu type), if its derivatives can be estimated as follows: for every* $\varepsilon > 0$ *there is a* $K > 0$ *(there are positive* ε *and* K*) with*

$$(2.23) \qquad |D^\alpha \phi(\xi)| \le K\, \varepsilon^p\, M_p, \quad \xi \in S, \quad |\alpha| = p, \quad p = 0,1,\ldots .$$

In [42] ϕ is called an ultradifferentiable function of class (M_p) (of class $\{M_p\}$). The sequence $\{M_p\}_{p=0}^{\infty}$ and the function M determine each other according to

$$(2.24)\qquad \begin{cases} M(\rho) = \sup\limits_{p} \log \dfrac{\rho^p M_0}{M_p} \\ M_p = M_0 \sup\limits_{\rho} \dfrac{\rho^p}{\exp M(\rho)} \end{cases}$$

and this implies the equivalence of definition 2.10 i and ii [42, th. 9.1]. The properties of the sequence $\{M_p\}_{p=0}^{\infty}$ are equivalent to those of the function M.

As in the case of the space $\mathcal{D}$ of all C^∞ functions with compact support, the spaces $\mathcal{D}_M$ of ultradifferentiable functions of class M_p with compact support in $\mathbb{R}_n$ can be given locally convex topologies such that their Fourier transforms $Z_M = \mathcal{F}\mathcal{D}_M$ have the following topologies: in case of Beurling type ultradifferentiable testfunctions Z_M is defined by

$$Z_{(M)} \overset{\text{def}}{=} \operatorname*{ind\,lim}_{a \to \infty} \operatorname*{proj\,lim}_{m \to \infty} H_\infty(\mathbb{C}^n;\ -M(m\|z\|) + a\|y\|)$$

and in case of Roumieu type ultradifferentiable testfunctions Z_M is defined by

$$Z_{\{M\}} \overset{\text{def}}{=} \operatorname*{ind\,lim}_{a \to \infty} \operatorname*{ind\,lim}_{k \to \infty} H_\infty(\mathbb{C}^n;\ -M(\|z\|/k) + a\|y\|),$$

where $H_\infty(\Omega;\phi(z))$ denotes the Banach space of holomorphic function f in Ω with the finite norm

$$\sup_{z\in\Omega} |f(z)| \exp - \phi(z).$$

DEFINITION 2.11.i. *An ultradistribution of class* (M) *(of class* {M}*) is the Fourier transform of an analytic functional in* $Z_{(M)}'$ *(in* $Z_{\{M\}}'$*).*

DEFINITION 2.11.ii. *An ultradistribution of class* (M) *(of class* {M}*) is an element in the dual of* $\mathcal{D}_{(M)}$ *(of* $\mathcal{D}_{\{M\}}$*).*

Just as a distribution can be locally written as a finite sum of derivatives of a continuous function, an ultradistribution is locally an infinite sum of derivatives of a continuous function. To explain this we introduce differential operators of infinite order:

DEFINITION 2.12. *An operator of the form*

$$P(D) \overset{\text{def}}{=} \sum_{|\alpha|=0}^{\infty} a_\alpha D^\alpha$$

is called an ultradifferentiable operator of class (M) *(of class* {M}*) if there are constants* L *and* K *(for every* L *there is a* K*) with*

$$|P(z)| = |\sum_{|\alpha|} a_\alpha z^\alpha| \le K \exp M(L\|z\|), \quad z \in \mathbb{C}^n. \tag{2.25}$$

LEMMA 2.13. [42, th. 2.12]. *An ultradifferentiable operator* P *of class* M *maps* $\mathcal{D}_M$ *continuously into itself.*

LEMMA 2.14. [42, th. 10.3]. *Every ultradistribution of class* M *can locally be written as* P(D)G *for some continuous function* G *and for some ultradifferentiable operator* P(D) *of the same class.*

Ultradifferentiable operators satisfying an additional property exist. Before formulating this we define the following concept which plays a role in the Roumieu type case.

DEFINITION 2.15. *A positive, increasing function* η *on* $[0,\infty)$*, with* $\eta(0) = 0$ *and with* $\eta(\rho)/\rho \to 0$ *as* $\rho \to \infty$*, is called a subordinate function.*

LEMMA 2.16. *For every* $m > 0$ *there exists an ultradifferentiable operator* $P_m(D)$ *of class* (M) *with*

$$|P_m(iz)| \ge \exp M(m\|z\|), \quad \|y\| < 1. \tag{2.26.i}$$

and for every subordinate function η *there exists an ultradifferentiable operator* $P_\eta(D)$ *of class* {M} *with*

$$|P_\eta(iz)| \ge \exp M(\eta\|z\|), \quad \|y\| < 1. \tag{2.26.ii}$$

PROOF. The existence of the operators $P_m(D)$ and $P_\eta(D)$ follows from [42, proof of th. 10.1] where it is shown that the entire functions h_m and h_η in $\mathbb{C}$, whose Hadamard factorizations are,

$$h_m(w) \overset{\text{def}}{=} \prod_{p=1}^{\infty} \left(1 + \frac{\ell w}{m_p}\right)$$

for some $\ell > 0$ depending on m and

$$h_\eta(w) \overset{\text{def}}{=\!=} \prod_{p=1}^{\infty} (1 + \frac{\ell_p w}{m_p})$$

for some sequence $\{\ell_p\}_{p=1}^{\infty}$ of positive numbers depending on η with $\ell_p \to 0$, where $m_p \overset{\text{def}}{=\!=} M_p/M_{p-1}$ for M_p given by (2.24), satisfy

$$(2.27.\text{i}) \qquad \left|\prod_{j=1}^{n} h_m(z_j)\right| \geq \exp M(m\|z\|), \quad \operatorname{Re} z_j \geq 0$$

and

$$(2.27.\text{ii}) \qquad \left|\prod_{j=1}^{n} h_\eta(z_j)\right| \geq \exp M(\eta(\|z\|)), \quad \operatorname{Re} z_j \geq 0.$$

In [42, prop. 4.5 & 4.6, cf. remark on p. 60] it is shown that $h_m(D)$ and $h_\eta(D)$ are ultradifferentiable operators of class (M) and {M}, respectively. □.

Distributions can be written as sums of boundary values of analytic functions of algebraic growth in $1/\|\operatorname{Im} \zeta\|$ for $\|\operatorname{Im} \zeta\|$ small. Ultradistributions can be represented in a similar way. For that purpose we introduce a function M^* associated to M: it follows from (2.20) that for each $\sigma > 0$

$$(2.28) \qquad M^*(\sigma) \overset{\text{def}}{=\!=} \max_{\rho>0} \{M(\rho) - \sigma\rho\}$$

exists. M^* is a convex function on $(0,\infty)$ with $M^*(0) = \infty$ and $M^*(\infty) = 0$. If M^* is a function with this properties, a function M can be associated to M^*, which equals M in (2.28) if this formula defines M^*, by

$$(2.29) \qquad M(\rho) = \min_{\sigma>0} \{M^*(\sigma) + \rho\sigma\}.$$

Indeed, for almost every $\rho > 0$ and all $\sigma > 0$

$$M(\rho) \leq \max_{\tau>0} \{M(\tau) - \sigma(\tau - \rho)\} = M^*(\sigma) + \rho\sigma$$

and hence

$$M(\rho) \leq \min_{\sigma>0} \{M^*(\sigma) + \rho\sigma\} \leq \max_{\tau>0} \{M(\tau) - M'(\rho)(\tau - \rho)\},$$

where in the right hand side we have taken $\sigma = M'(\rho)$. There the maximum is attained for τ satisfying $M'(\tau) = M'(\rho)$, thus since M' is monotonous, for $\tau = \rho$. Then the right hand side equals $M(\rho)$ and by continuity (2.29) holds everywhere.

LEMMA 2.17. [42, th. 11.5]. *Let f be a function holomorphic in* $\mathbb{R}_n + i C^*$ *for some open convex cone* C^* *in* $\mathbb{R}_n$ *such that for every compact set S in* $\mathbb{R}_n$ *and for every* $C' \subset\subset C^*$ *there are positive constants* $t = t(S,C')$ *and* $K = K(S,C')$ *(for every* $t > 0$ *there is a* $K = K(S,C',t) > 0$*) with*

$$\sup_{\xi \in S} |f(\xi + i\eta)| \le K \exp M^*(t\|\eta\|), \quad \eta \in C', \quad \|\eta\| < \delta \tag{2.30}$$

where $\delta > 0$ *may depend on S and C'. Then there is an ultradistribution* f^* *of class* (M) *(of class* {M}*) which is the boundary value of f as* $\eta \to 0$, $\eta \in C' \subset\subset C^*$, *where M is given by (2.29), i.e., for each* $\phi \in \mathcal{D}_M$

$$\langle f^*, \phi \rangle = \lim_{\substack{\eta \to 0 \\ \eta \in C'}} \int_{\mathbb{R}_n} f(\xi + i\eta)\phi(\xi)d\xi.$$

REMARK. It is already sufficient for (2.30) to hold if it holds for η only on a ray in C^* [42, prop. 11.6].

The converse of lemma 2.17 is

LEMMA 2.18. [42, th. 11.7]. *Let* f^* *be an ultradistribution of class M and let* $\{C_j^*\}_{j=1}^k$ *be open, convex cones in* $\mathbb{R}_n$ *such that the closure of their duals cover* $\mathbb{R}^n$. *Then for each bounded open set S in* $\mathbb{R}_n$ *there is a function f holomorphic in* $\bigcup_{j=1}^k \{S + i C_j^*\}$ *which satisfies (2.30) where* $C' = \bigcup_{j=1}^k C_j'$ *with* $C_j' \subset\subset C_j^*$, *such that in S*

$$f^* = \sum_{j=1}^k \lim_{\substack{\eta \to 0 \\ \eta \in C_j'}} f(\xi + i\eta).$$

(In [42] M^* is defined in a different way and it corresponds to our function M^* if in the right hand side of (2.28) σ is replaced by $1/\sigma$).

Similarly to finite-order-distributions, ultradistributions of "finite order" can be defined by global versions of lemma 2.14 or lemma 2.18.

DEFINITION 2.19.i. *An ultradistribution is called of "finite order" if lemma 2.14 holds globally, i.e., if it can be written as* P(D)G *globally.*

DEFINITION 2.19.ii. *An ultradistribution is called of "finite order" if it can be represented globally as in lemma 2.18, where, in the Beurling type case, (2.30) holds for* t *independent of* S *and where, in the Roumieu type case, (2.30) holds with* K(S,C',t) *replaced by a constant of the form* $K_1(S)K_2(C',t)$ *for* $K_1(S) > 0$ *depending on* S *and for* $K_2(C',t) > 0$ *depending on* C' *and* t.

The equivalence of these definitions follows from the proofs in [42, § 10 and § 11].

We remark that due to the fact that $\rho M'(\rho)$ is increasing and to (2.21) the functions M and M^* satisfy:
for each $m > 0$ and each $t > 0$ there is a $t' = t'(m,t) \geq t$ and a constant $K = K(m,t) > 0$, and for each $m > 0$ and each $t' > 0$ there is a positive $t = t(m,t') \leq t'$ and a constant $K = K(m,t') > 0$, such that for $\rho \geq 1$ and for $0 < \sigma \leq 1$

$$(2.31) \qquad \begin{cases} M(\rho/t') + m \log \rho \leq M(\rho/t) + K \\ M^*(t'/\sigma) + m \log 1/\sigma \leq M^*(t\sigma) + K. \end{cases}$$

Hence M does not increase too slowly, while by (2.20) it does not increase too rapidly.

Condition (2.20) assures that there are ultradifferentiable functions with compact support (Denjoy-Carlman-Mandelbrojt, cf. [42, th. 4.2]). For example, if (2.22) is satisfied only for $\|y\| < 1$ with $M(\rho) = \rho$, then (2.20) is not satisfied and ϕ is analytic in the tube $\{\zeta \mid \|\eta\| < m\}$ or, correspondingly if in (2.23) we set $M_p = p!$ then ϕ is analytic in the ε-neighborhood of $\mathbb{R}_n$.

Furthermore, it is necessary that for each $\varepsilon > 0$ there is a $K(\varepsilon) > 0$ such that for $\rho \geq 0$

$$(2.32) \qquad M(\rho) \leq \varepsilon\rho + K(\varepsilon),$$

but this is not sufficient for (2.20) to hold. Finally, condition (2.21) will be used in lemma 5.2 to allow the replacement of $M(\|x\|)$ by $M(|x_1|)+\ldots+M(|x_n|)$ in the definition of the spaces A by (2.4) or (2.5).

II.2.iv. CHARACTERIZATION OF ULTRADISTRIBUTIONS WITH REAL-CARRIED FOURIER TRANSFORMS.

The Fourier transform of an ultradistribution of class M is an analytic functional on the space Z_M and conversely, the Fourier transforms of such analytic functionals are ultradistributions. Now, similarly to theorem 2.9, we shall characterize those ultradistributions g which are the Fourier transforms of real-carried analytic functionals μ and then, both g and μ, can be written as sum of boundary values of analytic functions. As in the case of distributions, such ultradistributions g are of "finite order", cf. definition 2.19 i and ii.

Let here $A_t(k)$ be the Banach space of functions ψ, holomorphic in the open 1/k-neighbourhood of $\mathbb{R}^n$ in $\mathbb{C}^n$ and continuous on the closure, such that $|\psi(z)| \exp M(\|x\|/t) \to 0$ as $z \to \infty$ while $\|y\| \le 1/k$, with the norm $\|\psi\| \stackrel{\text{def}}{=} \sup_{\|y\|\le 1/k} |\psi(z)| \exp M(\|x\|/t)$. Then real-carried analytic functionals in $Z'_{(M)}$ (in $Z'_{\{M\}}$) can be extended to elements of A', where

$$A \stackrel{\text{def}}{=} \operatorname*{ind\,lim}_{k\to\infty} \operatorname*{proj\,lim}_{t\downarrow 0} A_t(k) \tag{2.33}$$
$$\Big(A \stackrel{\text{def}}{=} \operatorname*{ind\,lim}_{k\to\infty} A_k(k)\Big).$$

<u>THEOREM</u> 2.20. *The following four statements are equivalent:*

(1) $\mu \in A'$, *where* A *is given by (2.33), and* $g = F\mu$, *i.e., the ultradistribution* g *of class* M *is the Fourier transform of a real-carried analytic functional* μ *in* Z'_M.

(2) g *is an ultradistribution of class* (M) *(of class* {M}*), which for every* $\varepsilon > 0$ *can be represented as* $g = P_\varepsilon(D)G_\varepsilon$, *where* $P_\varepsilon(D)$ *is an ultradifferential operator of class* (M) *(of class* {M}*) and where the continuous function* G_ε *on* $\mathbb{R}_n$ *satisfies*

$$|G_\varepsilon(\xi)| \le K(\varepsilon)e^{\varepsilon\|\xi\|}$$

(3) μ *is the sum of boundary values in* A' *of functions* f_j *holomorphic in* $\mathbb{R}^n + iC_j$, *such that for every* $C' \subset\subset C_j$ *and every* $\varepsilon > 0$ *there are* $K = K(C'_j,\varepsilon) > 0$ *and* $t = t(C'_j,\varepsilon) > 0$ *(for every* $t > 0$ *there is a* $K = K(C'_j,\varepsilon,t) > 0$*) with*

$$|f_j(z)| \le K \exp M(t\|z\|), \quad y \in C'_j,\ \|y\| > \varepsilon$$

for $j = 1,\dots,k$, *where* $\{C_j\}_{j=1}^k$ *are open, convex cones in* $\mathbb{R}^n$ *such that the closure of their duals cover* $\mathbb{R}_n$.

(4) g *is the sum of boundary values of functions* h_j *holomorphic in* $\mathbb{R}_n + i\,C_j^*$, *such that for every* $C_j^{*}{}' \subset\subset C_j^*$ *and every* $\varepsilon > 0$ *there are positive numbers* $t = t(C_j^{*}{}',\varepsilon)$ *and* $K = K(C_j^{*}{}',\varepsilon)$ *(for every* $t > 0$ *there is a* $K = K(C_j^{*}{}',\varepsilon,t)>0$*)* *with*

$$|h_j(\zeta)| \le K \exp\{M(t\|\eta\|) + \varepsilon\|\xi\|\}, \qquad \eta \in C_j^{*}{}', \tag{2.34}$$

for $j = 1,\dots,p$, *where the open, convex cones* C_j^* *in* $\mathbb{R}_n$ *are such that the closure of the duals cover* $\mathbb{R}^n$ *and where* M^* *is determined by* M *according to (2.28).*

<u>PROOF</u>. (1) ⇒ (2). On any ε-nieghborhood $\Omega(\varepsilon)$ of $\mathbb{R}^n$ in $\mathbb{C}^n$ there exists a measure μ_ε which represents μ on $\operatorname{proj}\lim_{t\downarrow 0} A_t(1/\varepsilon)$ and which satisfies

$$\int_{\Omega(\varepsilon)} \exp - M(m(\varepsilon)\|x\|)\,|d\mu_\varepsilon(z)| \le K(\varepsilon) \tag{2.35.i}$$

for some positive numbers $K(\varepsilon)$ and $m(\varepsilon)$ depending on ε.
(Let μ satisfy for all $\varepsilon > 0$ and $t = 1,2,\dots$

$$|\langle\mu,\psi\rangle| \le K_\varepsilon(t) \sup_{\|y\|\le\varepsilon} |\psi(z)| \exp M(\|x\|/t), \quad \psi \in \operatorname{ind}\lim_{t\to\infty} A_t(1/\varepsilon)$$

for some $K_\varepsilon(t) > 0$ depending on ε and t with $K_\varepsilon(t+1) > K_\varepsilon(t)$ for every $\varepsilon > 0$ and $t = 1,2,\dots$. For each $\varepsilon > 0$ we define a subordinate function η_ε (cf. definition 2.15) by

$$M(\eta_\varepsilon(\rho)) \overset{\text{def}}{=\!=} \inf_t\{M(\rho/t) + \log(K_\varepsilon(t)/K_\varepsilon(1))\};$$

that $\eta_\varepsilon(\rho)/\rho \to 0$ as $\rho \to \infty$ follows as in [42, after lemma 9.5]. Then for each $\varepsilon > 0$ μ satisfies

$$|\langle\mu,\psi\rangle| \le K_\varepsilon(1) \sup_{\|y\|\le\varepsilon} |\psi(z)| \exp M(\eta_\varepsilon(\|x\|)), \quad \psi \in \operatorname{ind}\lim_{t\to\infty} A_t(1/\varepsilon).$$

Hence for every $\varepsilon > 0$ μ can be expressed as a measure μ_ε on $\Omega(\varepsilon)$ which satisfies

$$\int_{\Omega(\varepsilon)} \exp - M(\eta_\varepsilon(\|x\|))\,|d\mu_\varepsilon(z)| \le K(\varepsilon) \tag{2.35.ii}$$

for some $K(\varepsilon) > 0$ depending on ε.)

Now for any $\varepsilon > 0$, let $P_\varepsilon = P_{m(\varepsilon)}$, where $m(\varepsilon)$ is determined by (2.35.i) and $P_{m(\varepsilon)}$ by lemma 2.16 (let $P_\varepsilon = P_{\eta_\varepsilon}$, where η_ε is determined by (2.35.ii) and P_{η_ε} by lemma 2.16). Then $P_\varepsilon(D)$ is an ultradifferentiable operator of class (M) (of class {M}). For every $\phi \in \mathcal{D}_M$ and for every $\varepsilon > 0$, we get with $\hat{\phi} = \mathcal{F}\phi$

$$<\mu,\hat{\phi}> = <\mu, \int e^{i<\xi,z>}\phi(\xi)d\xi> =$$

$$= \int_{\Omega(\varepsilon)} \{\phi(\xi)P_\varepsilon(D_\xi)e^{i<\xi,z>}d\xi\} \frac{d\mu_\varepsilon(z)}{P_\varepsilon(iz)} =$$

$$= \int \{P_\varepsilon(-D)\phi(\xi)\} \int_{\Omega(\varepsilon)} \frac{e^{i<\xi,z>}}{P_\varepsilon(iz)} d\mu_\varepsilon(z)d\xi.$$

Hence for every $\varepsilon > 0$ $g = \mathcal{F}\mu = P_\varepsilon(D)G_\varepsilon$, where

$$G_\varepsilon \stackrel{\text{def}}{=\!=} \int_{\Omega(\varepsilon)} \frac{e^{i<\xi,z>}}{P_\varepsilon(iz)} d\mu_\varepsilon(z)$$

is a continuous function on $\mathbb{R}_n$ which according to (2.26.i) and (2.26.ii) satisfies

$$|G_\varepsilon(\xi)| \le K(\varepsilon)e^{\varepsilon\|\xi\|}.$$

(2) $\Rightarrow$ (3). Let U be the closure of an open set in $\mathbb{R}_n$ and let $\varepsilon > 0$. If $\phi \in \mathcal{D}_{(M)}$ ($\phi \in \mathcal{D}_{\{M\}}$), for every t (for some t) the following norm is finite

$$\|\phi\|_{U,\varepsilon,t} \stackrel{\text{def}}{=\!=} \sup_{\substack{\xi\in U\\ \alpha}} e^{\varepsilon\|\xi\|} \frac{|D^\alpha\phi(\xi)|}{t^{|\alpha|}M_{|\alpha|}}, \tag{2.36}$$

where the supremum is taken over all nonnegative n-dimensional multiindices α and where $M_{|\alpha|}$ is determined by the function M according to (2.24). Let $E_{\varepsilon,t}(U)$ denote the completion in this norm of the set of such functions ϕ and let

$$E(U) \stackrel{\text{def}}{=\!=} \operatorname*{ind\,lim}_{\varepsilon \downarrow 0}\ \operatorname*{proj\,lim}_{t \downarrow 0} E_{\varepsilon,t}(U)$$

$$(E(U) \stackrel{\text{def}}{=\!=} \operatorname*{ind\,lim}_{\varepsilon \downarrow 0}\ \operatorname*{ind\,lim}_{t \to \infty} E_{\varepsilon,t}(U)).$$

The restriction map from $E(\mathbb{R}_n)$ into $\prod_{j=1}^{k} E(U_j)$ is injective and open, when $\bigcup_{j=1}^{k} U_j = \mathbb{R}_n$. So, as in the proof of theorem 2.7, its transposed is surjective.

If g satisfies condition (2) of the theorem it belongs to $E(\mathbb{R}_n)'$. Indeed, for every $\varepsilon > 0$ there are $t = t(\varepsilon) > 0$ and $K = K(\varepsilon) > 0$ (for each t there is a $K = K(\varepsilon,t) > 0$) with

$$(1+\|x\|)^{n+1} \quad |P_{1/3\varepsilon}(iz)| \leq K \exp M(\|z\| / \sqrt{n}t).$$

Hence for $\phi \in \mathcal{D}_M$, using (2.24) and the fact that for each $z \in \mathbb{C}^n$ and multiindex α there is another multiindex β with $|\beta| = |\alpha|$ and $(\|z\|/\sqrt{n})^{|\alpha|} \leq |z^\beta|$, we get

$$|\langle g,\phi\rangle| \leq K' \sup_{\xi} e^{2/3\varepsilon\|\xi\|} |P_{1/3\varepsilon}(-D)\phi(\xi)| \leq$$

$$\leq K' \sup_{\xi} e^{2/3\varepsilon\|\xi\|} \{ \inf_{\|y\|\leq 2/3\varepsilon} \frac{1}{(2\pi)^n} \int |P_{1/3\varepsilon}(iz) e^{-i\langle\xi,z\rangle} \hat{\phi}(z)|dx\} \leq$$

$$\leq K'' \sup_{\xi}\{ \inf_{\|y\|\leq 2/3\varepsilon} \exp[\frac{2}{3}\varepsilon\|\xi\| + \langle\xi,y\rangle + M(\|z\| / \sqrt{n}t)] |\hat{\phi}(z)|\} \leq$$

$$\leq M_0 K'' \sup_{\substack{\alpha \\ \|y\|\leq 2/3\varepsilon}} \frac{\|z\|^{|\alpha|}}{(\sqrt{n}t)^{|\alpha|} M_{|\alpha|}} |\hat{\phi}(z)| \leq M_0 k'' \sup_{\substack{\alpha \\ \|y\|\leq 2/3\varepsilon}} \frac{|z^\alpha|}{t^{|\alpha|} M_{|\alpha|}} |\hat{\phi}(z)| \leq$$

$$\leq M_0 K'' \sup_{\substack{\alpha \\ \|y\|\leq 2/3\varepsilon}} \frac{1}{t^{|\alpha|} M_{|\alpha|}} \left| \int e^{i\langle\xi,z\rangle} D^\alpha \phi(\xi) d\xi \right| \leq$$

$$\leq M_0 K'' \sup_{\substack{\alpha \\ \xi\in\mathbb{R}_n}} e^{\varepsilon\|\xi\|} \frac{|D^\alpha\phi(\xi)|}{t^{|\alpha|} M_{|\alpha|}} \int \exp -\frac{1}{3}\varepsilon\|\xi'\| d\xi' \leq K''' \|\phi\|_{\mathbb{R}_n,\varepsilon,t}.$$

Conversely, the restriction to $E(\mathbb{R}_n)$ of an element $g \in E(U)'$ satisfies condition (2) of the theorem. For F^{-1} maps A continuously into $E(\mathbb{R}_n)$, because for $\psi \in A$, by (2.24), we have

$$\|F^{-1}\psi\|_{\mathbb{R}_n,\varepsilon,t'} \leq \sup_{\xi,\alpha}\{ \frac{1}{(2\pi)^n} e^{\varepsilon\|\xi\|} \frac{1}{t'^{|\alpha|} M_{|\alpha|}} \inf_{\|y\|\leq\varepsilon}$$

$$\int \|z\|^{|\alpha|} |e^{-i\langle\xi,z\rangle} \psi(z)|dx\} \leq$$

$$\leq \frac{1}{(2\pi)^n M_0} \sup\{\inf_{\xi} \; \inf_{\|y\|\leq\varepsilon} e^{\varepsilon\|\xi\|+<\xi,y>}\} \sup_{\|y\|\leq\varepsilon} \int |\psi(z)|\exp M(\|z\|/_{t'})dx \leq$$

$$\leq K \int (1+\|x\|)^{-(n+1)}dx \sup_{\substack{\|y\|\leq\varepsilon \\ x\in\mathbb{R}^n}} |\psi(z)|\exp M(\|x\|/_t),$$

where, according to (2.31) with m = n+1, t' determines t (t determines t'). Hence F^{-1} g belongs to A' and in the proof of (1) ⇒ (2) it has been shown already that then g satisfies (2).

Now choose open, convex cones $C_j \subset \mathbb{R}^n$, j = 1,...,k such that $\bigcup_{j=1}^k \overline{C_j^*} = \mathbb{R}_n$ and let $g = \sum_{j=1}^k g_j$ with $g_j \in E(-\overline{C_j^*})'$. In lemma 2.23 it will be shown that for $\psi \in A$ and $y \in C_j$

$$<F^{-1} g_j,\psi> = \frac{1}{(2\pi)^n} <F(g_j)_{-\xi},\psi> = \int f_j(z)\psi(z)dx,$$

where f_j is the function

$$f_j(z) \overset{\text{def}}{=\!=\!=} \frac{1}{(2\pi)^n} <(g_j)_{-\xi}, e^{i<\xi,z>}> \overset{\text{def}}{=\!=\!=} <(g_j)_{\xi}, \frac{e^{-i<\xi,z>}}{(2\pi)^n}>$$

which is holomorphic in $\mathbb{R}^n + iC_j$. For each ε > 0 and $C_j' \subset\subset C_j$ there is a $\delta = \delta(\varepsilon,C_j') > 0$ such that $<\xi,y> \leq -\delta\|\xi\|$ if $\xi \in -C_j^*$ and $y \in C_j'$ with $\|y\| \geq \varepsilon$. Then for every ε > 0 and for every C_j' there are $K = K(\varepsilon,C_j') > 0$ and $t = t(\varepsilon,C_j') > 0$ (for every t > 0 there is a $K = K(\varepsilon,C_j',t) > 0$) such that for $y \in C_j'$ with $\|y\| > \varepsilon$

$$|f_j(z)| \leq K\|e^{i<\xi,z>}\|_{\overline{-C_j^*},\delta,1/t} \leq K \sup_{\substack{\xi\in\overline{-C_j^*} \\ \alpha}} e^{\delta\|\xi\|+<\xi,y>} \frac{(t\|z\|)^{|\alpha|}}{M_{|\alpha|}} \leq$$

$$\leq K/_{M_0} \exp M(t\|z\|)$$

according to (2.24). Thus g satisfies condition (3) of the theorem.

(3) ⇒ (1). It is obvious that a sum of boundary values as in (3) determines an analytic functional in A': for $\psi \in A'$

$$|\sum_{j=1}^k \int_{\mathbb{R}^n} f_j(x+iy^j)\psi(x+iy^j)dx| \leq$$

$$\leq K' \sum_{j=1}^k \int_{\mathbb{R}^n} \frac{\exp\{M(t'\|z^j\|) + (n+1)\log(1+\|x\|)\}}{(1+\|x\|)^{n+1}} |\psi(z^j)|dx \leq$$

$$\leq K \sup_{\substack{x\in \mathbb{R}^n \\ \|y\|\leq\varepsilon}} |\psi(z)| \exp M(t\|x\|),$$

which holds for each $\varepsilon > 0$ by choosing $y^j \in C_j'$ with $\|y^j\| = \varepsilon$ and for t', hence t by (2.31), and K depending on ε (for each $t > 0$, by choosing t' according to (2.31) and for K depending on ε and t).

(1) ⇒ (4). According to theorem 2.7 $\mu \in A'$ can be written as $\mu = \sum_{j=1}^{p} \mu_j$ with $\mu_j \in A(\bar{C}_j)'$, where the closures of the open, convex cones $C_j \subset \mathbb{R}^n$ cover $\mathbb{R}^n$. The same proof of theorem 2.7 applies if we had taken the closed neighborhoods $\Omega_j(\varepsilon) \overset{\text{def}}{=\!=} \{z | x\in\bar{C}_j, \|y\|\leq\varepsilon\}$ instead of the open ε-neighborhoods of $\bar{C}_j$ in $\mathbb{C}^n$. (Then a space of analytic functions in $\bar{\Omega}$ is defined by functions holomorphic in the interior and continuous on the closure of Ω.) Thus assume that μ_j is an analytic functional with respect to these neighborhoods. In lemma 2.26 (which actually deals with the map (2.15) instead of the map (2.14) we have here) it will be shown that the Fourier transform of such an analytic functional is the boundary value of the function

$$h_j(\zeta) \overset{\text{def}}{=\!=} <(\mu_j)_z, e^{i<\zeta,z>}>$$

which is holomorphic in $\mathbb{R}_n + i C_j^*$. For every $\varepsilon > 0$ there is a $K = K(\varepsilon) > 0$ and for every $C_j^{*\prime} \subset\subset C_j^*$ there is moreover a positive $t = t(\varepsilon, C_j^{*\prime})$ (for every $t > 0$ there is a $K = K(\varepsilon, C_j', t) > 0$) with

$$|h_j(\zeta)| \leq K \sup_{\substack{x\in\bar{C}_j \\ \|y\|\leq\varepsilon}} \exp\{-<\xi,y> - <\eta,x> + M(t'\|x\|)\} \leq$$

(2.37)

$$\leq K \exp\{\varepsilon\|\xi\| + \sup_{\rho\geq 0}[M(t'\rho) - \delta\rho\|\eta\|]\} \leq K \exp\{M^*(t\|\eta\|) + \varepsilon\|\xi\|\}, \quad \eta \in C_j^{*\prime},$$

for t' depending on ε (for every t'), δ depending on $C_j^{*\prime}$ and with $t = \delta/t'$, where for the last inequality (2.28) has been used.

(4) ⇒ (7). This in fact will be shown in chapters III and VI. There the function h, holomorphic in $\mathbb{R}_n + i C^*$, satisfies

$$|h(\zeta)| \leq K \exp(M^*(t\|\eta\|) + \varepsilon\|\zeta\|), \quad \eta \in C^{*\prime}$$

which is more general than (2.34) and its boundary value is the Fourier transform of an analytic functional μ carried by $\bar{C}$ with respect to neighborhoods larger than ε-neighborhoods, namely with respect to the neighborhoods

$$\Omega(\varepsilon,C^{*\prime}) \overset{\text{def}}{=\!=} \{z \mid -\langle\xi,y\rangle - \langle\eta,x\rangle < \varepsilon\|\zeta\|, \eta\in C^{*\prime}, \xi\in\mathbb{R}_n\}.$$

Such an analytic functional μ certainly belongs to A'. □

Note that in condition (4) of theorem 2.9 $m(\varepsilon)$ depends on ε only, whereas in (2.34) in the Beurling type case t depends on both $C_j^{*\prime}$ and ε. This is due to the different behaviour of the function M in case of distributions, where $M(t\rho)$ has to be replaced by $t\log(1+\rho)$ and where for $M^*(\sigma)$ the function $\log\sigma^{-1}$, $\sigma \leq 1$, can be choosen. Then M^* satisfies $M^*(\delta\sigma) \leq M^*(\sigma) + K$ where K depends on δ (cf. the use of M^* in (2.37)).

REMARK. In [60] in the proof of theorem 2.9 the implication (4) ⇒ (2) instead of (4) ⇒ (1) is shown, which is performed by integration of the functions h. Then we get no information about the carrier of $F^{-1}h$ and in the above theorem no such information is needed. A direct proof of the implication (4) ⇒ (2) in theorem 2.20, is quite complicated and might be performed along the lines of [42, proof of th. 11.5].

II.2.v. PALEY-WIENER THEOREMS FOR ULTRADISTRIBUTIONS.

In the proof of theorem 2.20 a certain correspondence turned up between the boundary value of an analytic function of exponential type and the support or carrier of its Fourier transform. We shall make this correspondence more explicit. Let C be an open, convex cone in $\mathbb{R}^n$ and let a be a convex function on C, homogeneous of degree one. The pair (a,C) determines uniquely a closed convex set U(a,C), not containing a straight line, in $\mathbb{R}_n$ by

$$U(a,C) \overset{\text{def}}{=\!=} \{\xi \mid -\langle\xi,y\rangle \leq a(y), y\in C\}. \tag{2.38}$$

Conversely, each closed, convex set U in $\mathbb{R}_n$, which does not contain a straight line, determines uniquely an open, convex cone C in $\mathbb{R}^n$ and a homogeneous, convex function a on C such that U = U(a,C) according to (2.38), see [60].

The following theorems (th. 2.21 and th. 2.24) give the above mentioned correspondence explicitly. They are more general than the corresponding theorems for tempered distributions in [68, th. 26.2], but as soon as the occurring concepts are introduced, the proofs are very similar. They may be considered as a version of the real Paley-Wiener theorem for ultradistributions, whereas in chapter III complex Paley-Wiener theorems will be discussed which, actually, may be considered as versions of the Ehrenpreis-Martineau theorem.

First we state the theorem for distributions in $\mathcal{D}'$, whose proof can be found in [60, th. 4.1], and then we prove the theorem for ultradistributions.

THEOREM 2.21.i. *Let* C *be an open, convex cone in* $\mathbb{R}^n$, *let* a *be a convex function on* C, *homogeneous of degree one, let* U(a,C) *be the convex set in* $\mathbb{R}_n$ *given by (2.38) and let moreover* f *be a holomorphic function in* $\mathbb{R}^n + iC$ *which satisfies: for every* $\varepsilon > 0$ *and* $C' \subset\subset C$ *there is a* $m = m(\varepsilon, C') > 0$ *and for every* $\varepsilon > 0$ *there is moreover a positive number* $K = K(\varepsilon, C', \sigma)$ *such that*

$$|f(z)| \le K(1+\|z\|)^m \exp\{a(y)+\sigma\|y\|\}, \quad y \in C', \ \|y\| \ge \varepsilon.$$

Then $f(z) = \mathcal{F}[e^{-<\xi,y>} g_\xi](x)$ *for some distribution* $g \in \mathcal{D}'$ *with support in* U(a,C) *satisfying condition (2) of theorem 2.9 and the boundary value of* f *in* Z' *equals* $\mathcal{F}g$.

THEOREM 2.21.ii. *Let* C, a, U(a,C) *and* f *be as in theorem 2.21.i, but let* f *now satisfy: for every* $\varepsilon > 0$ *and* $C' \subset\subset C$ *there is a* $t = t(\varepsilon, C') > 0$ *and for every* $\sigma > 0$ *there is moreover a positive number* $K = K(\varepsilon, C', \sigma)$ *(for every* $\varepsilon > 0$, $\sigma > 0$, $C' \subset\subset C$ *and* $t > 0$ *there is a* $K = K(\varepsilon, \sigma, C' t) > 0$) *such that*

$$|f(z)| \le K \exp\{M(t\|z\|) + a(y) + \sigma\|y\|\}, \quad y \in C', \ \|y\| \ge \varepsilon. \tag{2.39}$$

Then $f(z) = \mathcal{F}[e^{-<\xi,y>} g_\xi](x)$ *for some ultradistribution* g *of class* (M) *(of class* {M}*) with support in* U(a,C) *satisfying condition (2) of theorem 2.20 and the boundary value of* f *equals* $\mathcal{F}g$.

PROOF. In the proof of (3) ⇒ (1) of theorem 2.20 the behaviour of f only for $\|y\|$ small has been used. Hence it follows from this and from (1) ⇒ (2) that the inverse Fourier transform g of the boundary value of f satisfies condition (2) of theorem 2.20. For $\phi \in \mathcal{D}_M$ g is defined by $<g,\phi> = \int f(z)\psi(z)dx$ where $\psi = \mathcal{F}^{-1}\phi$, and the integral is independent of $y \in C$. The function $\xi \to \exp -<\xi,y>$ is analytic and therefore a multiplier in any space of ultradistributions. So, for $y \in C$ we get

$$<g, e^{-<\xi,y>}, \phi> = <g, e^{-<\xi,y>}\phi> = \int f(z)\psi(x)dx,$$

hence $f(z) = \mathcal{F}[e^{-<\xi,y>} g_\xi](x)$ and it remains to prove the support property of g.

Let ξ_0 be a point in $\mathbb{R}_n \setminus U(a,C)$, hence there is an $y_0 \in C$ with $\|y_0\| = 1$ and with $-<\xi_0,y_0> > a(y_0)$. Furthermore, let $\eta > 0$ be so small that

$$-<\xi_0,y_0> \geq a(y_0) + 2\eta$$

and let $\phi_0 \in \mathcal{D}_M$ has its support in $\{\xi \,|\, \|\xi-\xi_0\| \leq \eta\}$. Then ϕ_0 has its support in $\mathbb{R}_n \setminus U(a,C)$, because for ξ in the support of ϕ_0 we have

$$(2.40) \qquad <\xi,y_0> = <\xi_0,y_0> + <\xi-\xi_0,y_0> \leq -a(y_0) - 2\eta + \eta = -a(y_0) - \eta < -a(y_0).$$

Let $C' \subset\subset C$ be such that $y_0 \in C'$ and let $\sigma = \frac{1}{4}\eta$. Then according to lemma 2.16 there is an ultradifferentiable operator P(D) of class (M) (of class {M}, where the construction is performed after the definition of a suitable subordinate function as in the proof of (1) ⇒ (2) of theorem 2.20 using the constants $K(\varepsilon,\sigma,C',t)$ in (2.39) for $\varepsilon = 1$, $\sigma = \frac{1}{4}\eta$ and C' fixed), such that

$$(2.41) \qquad \int \left| \frac{f(x+iy)}{P(ix)} \right| dx \leq K \exp\{M(t\|y\|) + a(y) + \sigma\|y\|\}$$

for some K and t and for all $y \in C'$ with $\|y\| \geq 1$. Then we have

$$(2.42) \qquad <g,\phi_0> = \int_{\mathbb{R}^n} \frac{f(x+iy)}{P(ix)} \left\{ \int_{\mathbb{R}_n} e^{-i<\xi,x>} P(D)[e^{<\xi,y>}\phi_0(\xi)] \frac{d\xi}{(2\pi)^n} \right\} dx.$$

Furthermore there are t' and K' depending on P (depending on ϕ_0) with

$$|P(D)e^{<\xi,y>}\phi_0(\xi)| \leq \left| \frac{P(D)}{(2\pi)^n} \int e^{-i<\xi,z>}\hat{\phi}_0(x)dx \right| \leq$$

$$\leq e^{<\xi,y>} \int \left| \frac{P(-iz)}{(2\pi)^n} \hat{\phi}_0(x) \right| dx \leq e^{<\xi,y>} K' \int |\hat{\phi}_0(x)| \exp M(t'\|z\|)dx \leq$$

$$\leq e^{<\xi,y>} K' e^{M(t'\|y\|)} \int_{\mathbb{R}^n} |\hat{\phi}_0(x)\ e^{M(t'\|x\|)} dx.$$

Now we take $y = \lambda y_0$, $\lambda > 1$ in (2.42) and taking into account (2.40) and (2.41) we find

$$|<g,\phi_0>| \leq K(\phi_0)\exp\{M(t\lambda) + a(y) + \frac{1}{4}\eta\lambda + M(t'\lambda) - a(y) - \eta\lambda\}.$$

Using (2.32) two times with $\varepsilon = \frac{1}{4}\,\eta/t$ and $\varepsilon = \frac{1}{4}\,\eta/t'$, successively, and taking the limit for $\lambda \to \infty$ we finally get $\langle g,\phi_0\rangle = 0$. □

In [60] and [68] it is shown that a distribution g (occurring in [68, th. 26.2] and [60, th. 4.1]) with convex (or more general, regular) support is a sum of derivatives of measures on its support. This is proved with the aid of Whitney's extension theorem, which says that the restriction map from $C^\infty(L)$ into $C^\infty(K)$ is surjective if K is closed, convex (or regular) and contained in the interior of L. For ultradifferentiable function spaces there is no such theorem, except in the one-dimensional case, see [9], but "it is quite plausible that this result can be extended to the higher dimensional case", see [42] (indeed, cf. foot note [2]). Then we would be able to prove a sharper theorem than just the converse to theorem 2.21, so that the estimate (2.39) would be improved, see corollary 2.25 (cf. [60] for distributions in $\mathcal{D}'$).

The above mentioned results on distributions with bounded regular support have already been mentioned in [62] and for tempered distributions with unbounded regular support in [67]. However, at some places, mostly oriented to physics (see for example [12] and [58]) a particular [1] case of this result is used which has been proved later [5]. It is called the lemma of Bros - Epstein - Glaser and it says that tempered distributions with support in a convex cone can be written as a higher order derivative of a continuous function with support in the cone. Fortunately, it is this result that can be generalized here, so that we are able to derive a converse to theorem 2.21 which is similar to the one for distributions, cf. [60]. Therefore, we state the following lemma, which is a generalization of the Bros - Epstein - Glaser lemma. [2]

---

[1] Indeed, if the support is a convex cone it is easy to see that the fact, that a distribution is the sum of derivatives of measures on the cone, implies that it is also the derivative of a continuous function with support in the cone. The particularity lies in the fact that it only applies to some particular, unbounded sets and not to general, regular sets.

[2] On the other hand, with the aid of this lemma it can be shown that indeed the restriction map from $C_M^\infty(L)$ into $C_M^\infty(K)$ is surjective in both cases (M) and {M}, if $K \subset \text{int}\, L$ is closed and satisfies some conditions, not as general as regular, but more general than convex.

LEMMA 2.22. *Let* U *be the closure of an open set in* $\mathbb{R}_n$ *such that there is a fixed, convex, open cone* C^* *with the property that for each* $\xi \notin U$ *the set* $\{\xi - C^*\} \cap U$ *is empty and let* g *be an ultradistribution of class* (M) *(of class* {M}*) which satisfies condition (2) of theorem 2.20 and which has its support in* U. *Then condition (2) of theorem 2.20 is satisfied for continuous functions* G_ε *which have their supports in* U.

PROOF. Let C be the dual cone of C^*, then it is possible to choose a base $\{e_1,\dots,e_n\}$ in $\mathbb{R}^n$ such that $\bar{C} \subset \Gamma$, where Γ is the open, convex cone $\{y \mid y = \sum_{j=1}^{n} y_j e_j, y_j > 0\}$. Then we have $\Gamma^* \subset C^*$. Every $z \in \mathbb{C}^n$ can be written uniquely as $z = x + iy = \sum_{j=1}^{n} x_j e_j + i \sum_{j=1}^{n} y_j e_j$ and we use these $(x_1,\dots,x_n)$ as coordinates for $\mathbb{R}^n$ and $\{z_j = x_j + iy_j\}_{i=1}^{n}$ as coordinates for $\mathbb{C}^n$.

According to theorem 2.20 g is the Fourier transform of a real-carried analytic functional μ. As in the proof of (1) $\Rightarrow$ (2) of theorem 2.20, let μ be represented by measures μ_ε satisfying (2.35.i) for some $m(\varepsilon) > 0$ depending on ε and μ ((2.35.ii) for some subordinate function η_ε depending on ε and μ). Let

$$P_\varepsilon(z) \overset{\text{def}}{=\!=} \prod_{j=1}^{n} (z_j + 1)^2 \, h_\varepsilon(2z_j + 2),$$

where $h_\varepsilon \overset{\text{def}}{=\!=} h_{m(\varepsilon)}$ ($h_\varepsilon \overset{\text{def}}{=\!=} h_{\eta_\varepsilon}$) is determined in the proof of lemma 2.16. Then $P_\varepsilon(D)$ is an ultradifferentiable operator of class (M) (of class {M}), $\exp M(m(\varepsilon)\|x\|) / P_\varepsilon(-ix)$ ($\exp M(\eta_\varepsilon(\|x\|)) / P_\varepsilon(-ix)$) is an L^1-function and $1/P_\varepsilon(-iz)$ is holomorphic in any α-neighborhood of $\mathbb{R}^n$ in $\mathbb{C}^n$ with $\alpha < 1$ and in $\mathbb{R}^n + i\Gamma$, where by (2.27.i) (by (2.27.ii)) it satisfies an even stronger estimate than (2.39) with $a = 0$. According to [42, lemma 3.3] the function

$$\lambda_\varepsilon(\xi) \overset{\text{def}}{=\!=} F^{-1} \left[\frac{1}{P_\varepsilon(-ix)} \right] (\xi)$$

is ultradifferentiable on $\mathbb{R}_n$ and according to theorem 2.21 λ_ε has its support in $\overline{\Gamma^*}$. We will see that λ_ε is "sufficiently ultradifferentiable" such that g can be applied to it. Another property of λ_ε is that $P_\varepsilon(D)\lambda_\varepsilon = \delta$, where δ is the Dirac-δ-function.

Now let

$$G_\varepsilon(\xi) \overset{\text{def}}{=\!=} g * \lambda_\varepsilon(\xi) \overset{\text{def}}{=\!=} \langle g_\eta, \lambda_\varepsilon(\xi - \eta) \rangle$$

which exists because $1/P_\varepsilon(iz)$ is holomorphic in $\Omega(\varepsilon)$ so that we have

$$|G_\varepsilon(\xi)| = |<\mu_z, \frac{e^{i<\xi,z>}}{P_\varepsilon(iz)}>| = |\int_{\Omega(\varepsilon)} \frac{e^{i<\xi,z>}}{P_\varepsilon(iz)} d\mu_\varepsilon(z)| \le$$

$$\le \begin{cases} Ke^{\varepsilon\|\xi\|} \int e^{-M(m(\varepsilon)\|x\|)} |d\mu_\varepsilon(z)| \le K(\varepsilon)e^{\varepsilon\|\xi\|} \\ (Ke^{\varepsilon\|\xi\|} \int e^{-M(\eta_\varepsilon(\|x\|))} |d\mu_\varepsilon(z)| \le K(\varepsilon)e^{\varepsilon\|\xi\|}) \end{cases}$$

by (2.35.i) (by (2.35.ii)). Furthermore G_ε, as the Fourier transform of a bounded measure, is a continuous function on $\mathbb{R}_n$ which has its support in U, because if $\xi \notin U$ the set $\{\xi - \overline{\Gamma^*}\} \cap U$ is empty since $\overline{\Gamma^*} \subset C^*$. Finally we have

$$P_\varepsilon(D)G_\varepsilon = g * P_\varepsilon(D)\lambda_\varepsilon = g * \delta = g. \qquad \square$$

The condition on the set U is satisfied by the set U(a,C) given by (2.38) if C is an open, convex cone not containing a straight line, or equivalently, if $C^* \neq \emptyset$. In case we have a cone $\tilde{C}$ with $\tilde{C}^* = \emptyset$, for example if $\tilde{C} = \mathbb{R}^n$, and hence $U(a,\tilde{C})$ is a bounded, convex set, we must think of $U(a,\tilde{C})$ to be contained in a larger set U(a,C), where C is an open, convex subcone of $\tilde{C}$ containing no straight lines.

Let g be an ultradistribution of class M with support in the set U(a,C), which satisfies condition (2) of theorem 2.20. It is shown in the proof of (2) ⇒ (3) of that theorem that g belongs to $\mathcal{E}(\mathbb{R}_n)'$ and the last lemma shows that g can be considered as an element of $\mathcal{E}(U(a,C))'$. Furthermore the function $\xi \to e^{i<\xi,z>}$ belongs to $\mathcal{E}(U(a,C))$ if $y \in C$. Keeping these remarks in mind we can interprete the following lemma which characterizes the Fourier transform of g.

<u>LEMMA</u> 2.23. *Let* C, a *and* U(a,C) *be as in theorem 2.21 and let* g *be as in lemma 2.22 with* U = U(a,C). *Then*

$$\mathcal{F}[e^{-<\xi,y>}g_\xi](x) = <g,e^{i<\xi,z>}>$$

and this is a function holomorphic in $\mathbb{R}^n + iC$ *whose boundary value equals* $\mathcal{F}g$.

PROOF. Let $\psi \in Z_M$, $y \in C$ and if $C^* = \emptyset$ instead of C we take a subcone, also denoted by C, containing y and no straight lines. Then using lemma 2.22 we have

$$<\mathcal{F}e^{-<\xi,y>}g,\psi> = <g, \int_{\mathbb{R}^n} e^{i<\xi,z>}\psi(x)dx> =$$

$$= \int_{U(a,C)} G_\varepsilon(\xi)P_\varepsilon(-D_\xi) \int_{\mathbb{R}^n} e^{i<\xi,z>}\psi(x)dxd\xi =$$

$$= \int_{\mathbb{R}^n} \int_{U(a,C)} G_\varepsilon(\xi)P_\varepsilon(-D_\xi)e^{i<\xi,z>}d\xi\psi(x)dx =$$

$$= \int_{\mathbb{R}^n} <g,e^{i<\xi,z>}>\psi(x)dx$$

where $\varepsilon > 0$ is choosen depending on y such that the integrals exist. It is clear that

$$f(z) \overset{\text{def}}{=\!=} <g,e^{i<\xi,z>}>$$

is holomorphic in $\mathbb{R}^n + iC$ and furthermore, a similar procedure to above, shows that for $y \in C$

$$<\mathcal{F}g,\psi> = <g, \int_{\mathbb{R}^n} e^{i<\xi,z>}\psi(z)dx> = \int_{\mathbb{R}^n} f(z)\psi(z)dx.$$

Hence $\mathcal{F}g$ is the boundary value of f in Z'_M. □

Now we are able to prove a stronger theorem than just the converse to theorem 2.21.ii. Again, first we mention the theorem for distributions in $\mathcal{D}'$ given in [60, th. 4.2] and then we prove the theorem for ultradistributions.

THEOREM 2.24.i. *Let C, a and U(a,C) be as in theorem 2.21 and let g be a distribution in $\mathcal{D}'$ with support in U(a,C) satisfying condition (2) of theorem 2.9. Then the function* $f(z) \overset{\text{def}}{=\!=} \mathcal{F}[e^{-<\xi,y>}g_\xi](x)$, *whose boundary value equals* $\mathcal{F}g$, *satisfies: for every* $\varepsilon > 0$ *and* $C' \subset\subset C$ *there are* $N = N(\varepsilon,C') > 0$ *and* $K = K(\varepsilon,C') > 0$ *such that*

$$|f(z)| \le K(1+\|z\|)^N e^{a(y)}, \quad y \in C', \ \|y\| \ge \varepsilon.$$

THEOREM 2.24.ii. *Let C, a and g be as in lemma 2.23. Then the function* $f(z) \overset{\text{def}}{=\!=} F[e^{-\langle\xi,y\rangle} g_\xi](x)$, *whose boundary value equals* Fg, *satisfies: for every* $\varepsilon > 0$ *and* $C' \subset\subset C$ *there are* $t = t(\varepsilon,C') > 0$ *and* $K = K(\varepsilon,C') > 0$ *(for every* $\varepsilon > 0$, $C' \subset\subset C$ *and* $t > 0$ *there is* $K = K(\varepsilon,C',t) > 0$*) such that*

$$(2.43) \qquad |f(z)| \le K \exp\{M(t\|z\|) + a(y)\}, \qquad y \in C', \ \|y\| \ge \varepsilon.$$

PROOF. According to lemma 2.23 we have to estimate the $\|\cdot\|_{U(a,C),\varepsilon,t}$ norms of the function $e^{i\langle\cdot,z\rangle}$, defined in (2.36). For $t > 0$ we get

$$|D^\alpha e^{i\langle\xi,z\rangle}| \le |z^\alpha| e^{-\langle\xi,y\rangle} \le \frac{1}{M_0}\frac{M_{|\alpha|}}{t^{|\alpha|}} e^{-\langle\xi,y\rangle} \sup_{p=0,1,\ldots} \frac{(t\|z\|)^p M_0}{M_p} \le$$

$$\le \frac{1}{M_0}\frac{M_{|\alpha|}}{t^{|\alpha|}} \exp\{M(t\|z\|) - \langle\xi,y\rangle\}.$$

Let $C' \subset\subset C$ and in case C^* is empty let C_j, $j = 1,\ldots,\ell$ be subcones of C with $C_j^* \ne \emptyset$ covering C and such that there are $C_j' \subset\subset C_j$ which cover C', and let $C_j' \subset\subset C_j'' \subset\subset C_j$. Then there is a $\delta = \delta(C_j') > 0$ with $-\langle\xi,y\rangle \le -\delta\|y\|\|\xi\|$ if $y \in C_j'$ and $\xi \in C_j''^*$. For each $\eta > 0$ there are $t' = t'(\eta)$ and $K' = K'(\eta)$ (for every $t' > 0$ there is a $K' = K'(\eta,t')$) with for $\phi \in \mathcal{D}_M$

$$|\langle g,\phi\rangle| \le K'\|\phi\|_{U(a,C_j),\eta,t'}, \qquad j = 1,\ldots,\ell.$$

It is possible that $a(y) < 0$ for some y, so in the following $\alpha \overset{\text{def}}{=\!=} \min\{a(y)\,|\,y \in C', \|y\| = 1\}$ might be negative. Now in the above we choose $\eta = \frac{1}{2}\delta\varepsilon$ and $t' = \frac{1}{t}$. If ξ ranges in $C_j''^*$ while $\|\xi\| \ge -2\frac{\alpha}{\delta}$ we estimate for $y \in C_j'$ with $\|y\| \ge \varepsilon$

$$\eta\|\xi\| - \langle\xi,y\rangle \le \frac{1}{2}\delta\varepsilon\|\xi\| - \frac{1}{2}\delta\varepsilon\|\xi\| - \frac{1}{2}\delta\|\xi\|\|y\| \le \alpha\|y\| \le a(y).$$

The remaining of $U(a,C_j)$ is compact and there by (2.38) we have

$$\exp\{\eta\|\xi\| - \langle\xi,y\rangle\} \le K'' \exp a(y),$$

where $K'' \ge 1$. Hence, for $y \in C'$ with $\|y\| \ge \varepsilon$

$$|\langle g, e^{i\langle\xi,z\rangle}\rangle| \le \frac{K'K''}{M_0} \exp\{M(t\|z\|) + a(y)\}. \qquad \square$$

COROLLARY 2.25. *A holomorphic function* f*, which satisfies (2.39), satisfies already (2.43), i.e., in (2.39)* K *is independent of* σ *and we may take* $\sigma = 0$.

Whether the ultradistributions g of theorem 2.24 are defined on certain ultradifferentiable testfunctions in $\mathbb{R}_n$ or in real ε-neighborhoods of $U = U(a,C)$ makes no difference due to the existence of ultradifferentiable functions λ which are identically one on U and zero outside an ε-neighborhood of U. So we can say that the Fourier transform F is a bijective map from the dual of a certain space, say $S(U)$, of ultradifferentiable functions defined on real ε-neighborhoods of the convex, real set $U(a,C)$ onto a certain space H of functions holomorphic in $\mathbb{R}^n + iC$ and of exponential type a in Im z. Thus shortly

$$FS(U)' \cong H.$$

In the next section we will discuss the case where U is replaced by a complex, convex set Ω in $\mathbb{C}_n$ and then g becomes an analytic functional μ defined on a space of functions holomorphic in complex neighborhoods of Ω.

II.2.vi. THE CASE OF COMPLEX DOMAINS

We consider the following question. Let Γ be an open, convex cone in $\mathbb{C}^n$ and let a be a convex function on Γ, homogeneous of degree one, let $\Omega = \Omega(a,\Gamma)$ be the closed, convex set in $\mathbb{C}_n$ given by

$$(2.44) \qquad \Omega(a,\Gamma) = \{\zeta \mid -\operatorname{Im} \langle\zeta,z\rangle \le a(z),\ z \in \Gamma\},$$

and finally, let $A(\Omega)$ be a space of analytic functions defined on certain neighborhoods of Ω in $\mathbb{C}_n$ whose growth at infinity is determined by the weight-functions $\exp M(t\|\zeta\|)$, and let $H(\Gamma)$ be a space of analytic functions in Γ of exponential type a for $\|z\|$ large whose behaviour at the vertex of Γ (i.e., for $\|z\|$ small) is determined by the function M. Then one may ask whether it is possible to find such conditions that the Fourier transformation F is a bijective map from $A(\Omega)'$ onto $H(\Gamma)$, or shortly, whether

$$FA(\Omega)' \cong H(\Gamma)$$

In chapters III and IV this question is solved affirmative. In case there exist testfunctions with compact support the injectivity and the surjectivity of $\mathcal{F}$ present no problems (cf. the proof of theorem 2.21). In $A(\Omega)$, however, no such testfunctions exist and the proofs are very complicated. Actually, using a generalization of Ehrenpreis' fundamental priciple (see chapter IV) we will return to a situation where we do have C^∞ functions on real domains. For that purpose we have to identify $\mathbb{C}^n$ with $\mathbb{R}^n \times \mathbb{R}^n = \mathbb{R}^{2n}$ $z = x + iy \Leftrightarrow (x,y)$ and $\mathbb{C}_n$ with $\mathbb{R}_n \times \mathbb{R}_n = \mathbb{R}_{2n}$ by $\zeta = \xi + i\eta \Leftrightarrow (\eta,\xi)$. Then we will deal with distributions defined on a C^∞ testfunction space in a neighborhood of the, now real, domain $\Omega \subset \mathbb{R}_{2n}$ and with functions holomorphic in $\mathbb{R}^{2n} + i\Gamma \subset C^{2n}$. In the following section we will give a lemma concerning this situation, similarly to theorems 2.21 and 2.24.

Of particular interest is the case where Γ is a tubular radial domain, i.e., a domain of the form $T^C = \mathbb{R}^n + iC$ with C an open convex cone in $\mathbb{R}^n$, and where $f \in H(\Gamma)$ has ultradistributional boundary values on $\mathbb{R}^n$. Then, if we interchange the variables z and ζ in theorem 2.20 (1) and (4) the surjectivity of $\mathcal{F}$ yields the proof of (4) $\Rightarrow$ (1) of that theorem. If a, defined on T^C, can be continued to a continuous function on $\mathbb{R}^n + iC'$, with $C' \subset\subset C$, i.e., if $\lim a(x,y) = a(x,0)$ exists as $y \to 0$ while $y \in C'$, then

$$\Omega(a,T^C) = \{\zeta \,|\, -\langle\eta,x\rangle - \langle\xi,y\rangle \le a(x,y),\ x \in \mathbb{R}^n,\ y \in C\},$$

given by (2.44), is bounded in the imaginary directions, namely

$$\Omega(a,T^C) \subset \{\zeta \,|\, \|\eta\| \le \max_{\|x\|=1} a(x,0)\}.$$

Also, it may happen that Ω is not bounded in the imaginary directions and then we give $A(\Omega)$ the topology induced by Z_M, so that the functions $\psi \in A(\Omega)$ have to satisfy

$$(2.45) \qquad |\psi(\zeta)| \le K \exp\{-M(t\|\xi\|) + \ell\|\eta\|\}$$

on a neighborhood of Ω, for some $\ell > 0$ depending on ψ. Since $e^{i\langle\zeta,z\rangle}$ satisfies this condition for each $z \in T^C$, we can characterize the Fourier transform of an element $\mu \in A(\Omega)'$, considered as an analytic functional in Z'_M carried by Ω, as in lemma 2.23.

<u>LEMMA</u> 2.26. *Let* C, a, $\Omega = \Omega(a,T^C)$ *and* $A(\Omega)$ *be as above and let* $\mu \in A(\Omega)'$. *Then the Fourier transform of* μ *is the boundary value in* $\mathcal{D}'_M$ *as* $y \to 0$, *while*

$y \in C' \subset\subset C$, *of the function*

$$(2.46) \qquad f(z) \stackrel{\text{def}}{=\!=} \langle\mu_\zeta, e^{i\langle\zeta,z\rangle}\rangle,$$

which is holomorphic in $\mathbb{R}^n + iC$.

PROOF. For $\phi \in \mathcal{D}_M$ and $y \in C$ let

$$\psi_y(\zeta) \stackrel{\text{def}}{=\!=} \int e^{i\langle\zeta,z\rangle}\phi(x)\,dx.$$

The limit of Riemann sums converges in the topology of the space $A(\Omega)$ and furthermore $\psi_n \to \psi_0$ in $A(\Omega)$ as $y \to 0$ while $y \in C' \subset\subset C$, because $-\langle\xi,y\rangle \le \le a(0,y)$ for all $\zeta \in \Omega$. Therefore, we may write

$$\langle F\mu,\phi\rangle = \langle\mu, \int e^{i\langle\zeta,x\rangle}\phi(x)\,dx\rangle =$$

$$= \lim_{\substack{y\to 0\\ y\in C'}} \langle\mu, \int e^{i\langle\zeta,z\rangle}\phi(x)\,dx\rangle = \lim_{\substack{y\to 0\\ y\in C'}} \int \langle\mu_\zeta, e^{i\langle\zeta,z\rangle}\rangle\phi(x)\,dx. \qquad \square$$

In view of this lemma in chapter III we will define the Fourier transform of μ by formula (2.46) also in the general case where Γ is not a tubular radial domain. There we will treat F as a topological isomorphism and therefore, it is more convenient to consider L^2-norms instead of sup-norms, because the strong dual of a projective (inductive) limit of Hilbert spaces can be written as the inductive (projective) limit of the duals, see [40]. Using Sobolev embedding theorems, see [73], one can pass from the one norm to the other.

II.2.vii. A PALEY-WIENER TYPE THEOREM.

In chapter III we will need the lemma given in this section. It is a Paley-Wiener type theorem treating various, rather technical, cases which will become clear in chapter III. We will prove only the case exposing the most typical features. This section has little connection with the other sections of this chapter and we place it here because the proof of the lemma proceeds along the lines of theorem 2.21 and 2.24.

First we introduce some notations and definitions whose meaning will be made clear in chapter III. If a is a convex function on the convex, open cone Γ in $\mathbb{C}^n$ which is homogeneous of degree one, we mean by $a + \varepsilon$ the function on Γ given by

$$(a+\varepsilon)(z) \overset{\text{def}}{=\!=} a(z) + \varepsilon\|z\|.$$

$\{\Gamma_k\}_{k=1}^{\infty}$ denotes a sequence of open, relatively compact subcones of Γ such that $\Gamma_k \subset\subset \Gamma_{k+1} \subset\subset \Gamma$ and $\bigcup_{k=1}^{\infty} \Gamma_k = \Gamma$, and

$$(2.47) \qquad \Gamma(k) \overset{\text{def}}{=\!=} \{z \,|\, z \in \Gamma_k,\ \|z\| > \tfrac{1}{k}\}.$$

Then the neighborhoods (cf. formula (2.44))

$$(2.48.\text{i}) \qquad \Omega_\varepsilon^k \overset{\text{def}}{=\!=} \Omega(a + \tfrac{1}{k}, \Gamma)$$

are the $\frac{1}{k}$-neighborhoods in $\mathbb{C}_n$ of $\Omega = \Omega(a,\Gamma)$, $k = 1,2,\ldots$, whereas the neighborhoods

$$(2.48.\text{ii}) \qquad \Omega_c^k \overset{\text{def}}{=\!=} \Omega(a + \tfrac{1}{k}, \Gamma_k)$$

are larger neighborhoods. The subscript ε expresses that we deal with ε-neighborhoods and the subscript c denotes the case of conic neighborhoods. If not a particular case is meant we will denote these two cases by a subscript α. For the case $\alpha = \varepsilon$ we will need the following set

$$(2.49) \qquad \frac{1}{k} z_0 + \Gamma \overset{\text{def}}{=\!=} \{z \,|\, z = \frac{1}{k} z_0 + z',\ z' \in \Gamma\}$$

where $z_0 \in \text{pr}\,\Gamma_1$ is fixed.

In particular we can choose $\Gamma = T^C$ where C is an open, convex cone in $\mathbb{R}^n$. This is of interest because then one might consider holomorphic functions in T^C having boundary values on $\mathbb{R}^n$ in some sense. We will now introduce the above given concepts for this case. For Γ_k we will choose

$$(2.50) \qquad (T^C)_k \overset{\text{def}}{=\!=} \{z \,|\, y \in C_k,\ \|x\| < k\|y\|\}$$

where $\{C_k\}_{k=1}^{\infty}$ is a sequence exhausting C, and

$$(2.51) \qquad (T^C)(k) \overset{\text{def}}{=\!=} \{z \,|\, z \in (T^C)_k,\ \|y\| > \tfrac{1}{k}\}.$$

Furthermore, let $y_0 \in \mathrm{pr}\, C_1$ be fixed and then let

$$(T^C)^k_\varepsilon \overset{\text{def}}{=\!=} T^{1/k\, y_0 + C} \cup \{z \mid \|x\| < k,\ y \in C_k\} \tag{2.52.i}$$

and

$$(T^C)^k_c \overset{\text{def}}{=\!=} \mathrm{ch}[(T^C)_k \cup \{z \mid \|x\| < k,\ y \in C_k\}] \tag{2.52.ii}$$

where ch means the convex hull. For a domain $B \subset \mathbb{C}^n$ we define the tube domain $T(B) \subset \mathbb{C}^n \times \mathbb{C}^n \cong \mathbb{C}^{2n}$ by

$$T(B) \overset{\text{def}}{=\!=} \{(\theta^1, \theta^2) \mid \mathrm{Im}\, \theta^1 + i\ \mathrm{Im}\, \theta^2 \in B\}. \tag{2.53}$$

Moreover, if a is a homogeneous, convex function on T^C such that $a(x,0)$ becomes unbounded, we change the function a into functions $\tilde{a}_k$ on T^C such that for each k $\tilde{a}_k$ is a convex function satisfying

$$\tilde{a}_k(x,y) \overset{\text{def}}{=\!=} \tilde{a}_k(z) = a(z), \qquad z \in T^C,\ \|y\| \geq 1/2k$$

and for $k = 1,2,\ldots$

$$\tilde{a}_k(z) \leq K_k, \qquad y \in C_k,\ \|y\| \leq 1/k,\ \|x\| \leq k$$

where K_k is a positive constant depending on k and a. For then the growth of a function f satisfying $|f(z)| \leq K_k \exp\{M^*(t\|y\|) + \tilde{a}_k(z)\}$ for $\|y\|$ small and $\|x\| \leq k$ is determined completely by the factor $\exp M^*(t\|y\|)$, while we need the growth $\exp a(z)$ of f only on rays $\{\lambda z \mid \lambda > 0\}$ for λ large and $z \in \mathrm{pr}\, T^C$. If $\lim a(x,y)$ exists as $y \to 0$, $t \in C_k$ then a will not be changed and, for convenience, in that case we denote

$$\tilde{a}_k \overset{\text{def}}{=\!=} a, \quad k = 1,2,\ldots .$$

We now define the functions

$$a^k_\varepsilon(z) \overset{\text{def}}{=\!=} a(x, y - \frac{1}{2k}\, y_0), \qquad y \in \frac{1}{k}\, y_0 + C \tag{2.54.i}$$

where a^k_ε should be continued as a convex function on $\overline{T^C}$, just as $\tilde{a}_k$ on $\overline{T^{C_k}}$, and

(2.54.ii) $\quad a_c^k(z) \overset{\text{def}}{=\!=} \tilde{a}_k(z), \quad z \in (T^C)_c^k.$

Finally, if Ω is the closure of a domain in $\mathbb{R}^n$ and M a continuous function on Ω, let $W_2^m(\Omega;M(u))$ denote the space of measurable functions f in Ω for which the weak derivatives $D^\alpha f$ exist for $|\alpha| \le m$ as measurable functions such that the norm

$$\Big[\sum_{|\alpha|\le m} \int_\Omega \{|D^\alpha f(u)|\exp M(u)\}^2 du\Big]^{\frac{1}{2}}$$

is finite. If Ω is a domain in $\mathbb{C}^n$ and M a continuous function on Ω, let $H_\infty(\Omega;M(z))$ denote the space of holomorphic functions f in Ω such that the norm

$$\sup_{z\in\Omega} |f(z)| \exp -M(z) \tag{2.55}$$

is finite.

Besides the cases $\alpha = \varepsilon$ and $\alpha = c$, in chapter III we will consider four other cases, namely ultradistributional boundary values of class (M) and {M}, distributional boundary values and boundary values in the sense of Fourier hyperfunctions. Depending on these various cases we introduce the following spaces: If $\Gamma = T^C$ in the definition (2.47) and (2.48) of Ω_α^k, let

$$\begin{cases} S_\alpha(m,k,t) \overset{\text{def}}{=\!=} W_2^m(\Omega_\alpha^k; -M(\|\xi\|/t) + k\|\eta\| - m\log(1+\|\zeta\|)) \\ H_\alpha(m,k,t) \overset{\text{def}}{=\!=} H_\infty(T((T^C)_\alpha^k); M^*(t\|\mathrm{Im}\ \theta^2\|) + a_\alpha^k(\mathrm{Im}\ \theta) + \frac{1}{k}\|\mathrm{Im}\ \theta\| + \\ \qquad\qquad + m\log(1+\|\theta\|)) \end{cases} \tag{2.56}$$

and let

$$\begin{cases} S_\alpha(k,m) \overset{\text{def}}{=\!=} W_2^m(\Omega_\alpha^k; -m\log(1+\|\zeta\|) + k\|\eta\|) \\ H_\alpha(k,m) \overset{\text{def}}{=\!=} H_\infty(T((T^C)_\alpha^k); \log(1+\|\mathrm{Im}\ \theta^2\|^{-m}) + a_\alpha^k(\mathrm{Im}\ \theta) + \frac{1}{k}\|\mathrm{Im}\ \theta\| + \\ \qquad\qquad + m\log(1+\|\theta\|)) \end{cases}$$

for $\alpha \in \{\varepsilon, c\}$. If Γ is an open, convex cone in $\mathbb{C}^n$, let

$$(2.57)\quad \begin{cases} S_\alpha(m,k) \overset{\text{def}}{=\!=} W_2^m(\Omega_\alpha^k;\, -\frac{1}{k}\,\|\zeta\| - m\,\log(1+\|\zeta\|)) \\ H_\varepsilon(m,k) \overset{\text{def}}{=\!=} H_\infty(T(\frac{1}{k}\,z_0 + \Gamma);\, a(\operatorname{Im}\theta^1 - \frac{1}{2k}x_0,\ \operatorname{Im}\theta^2 - \frac{1}{2k}y_0) + \\ \qquad\qquad + \frac{1}{k}\,\|\operatorname{Im}\theta\| + m\,\log(1+\|\theta\|)) \\ H_c(m,k) \overset{\text{def}}{=\!=} H_\infty(T(\Gamma(k));\, a(\operatorname{Im}\theta) + \frac{1}{k}\,\|\operatorname{Im}\theta\| + m\,\log(1+\|\theta\|)). \end{cases}$$

In the above defined S-spaces the set Ω_α^k has to be considered as a closed set in $\mathbb{R}_{2n}$.

If we take the projective limit of the S-spaces for $m \to \infty$, we get FS^*-spaces (cf. [40], weakly compact, projective sequences) which have nice properties, for example they are reflexive. If we would have S-spaces defined with sup-norms instead of L^2-norms, due to the fact that Ω_α^k is convex these projective limits would even be FS-spaces (compact, projective sequences) which, of course, have nicer properties. But the properties of FS^*-spaces are all we need and so we don't have to show that in the sup-norm case we get FS-spaces. As a matter of fact it doesn't change much whatever norm we have, L^2-norm or sup-norm. This follows from certain Sobolev embedding theorems: let $W_{\infty,0}^m(\Omega;M(u))$ denote the space of C^m-functions f on the closed set Ω (in the sense of Whitney) with the finite sup-norm

$$\sup_{\substack{u\in\Omega \\ |\alpha|\le m}} |D^\alpha f(u)|\exp - M(u)$$

such that moreover $|D^\alpha f(u)|\exp - M(u) \to 0$ as $u \to \infty$ in Ω for $|\alpha| \le m$; (by Riesz' theorem the dual of such a space consists of weak derivatives of measures on Ω); let Ω' be a closed convex set such that an ε-neighborhood of Ω' is contained in Ω, then according to [73, p.11 condition HS_1 and p.14 condition HS_2] the embedding maps

$$W_{\infty,0}^{m+n+1}(\Omega;M(u) - (m+n+1)\log(1+\|u\|)) \to W_2^m(\Omega;M(u) - m\,\log(1+\|u\|))$$

$$W_2^{m+n+1}(\Omega;M(u) - (m+n+1)\log(1+\|u\|)) \to W_{\infty,0}^m(\Omega';M(u) - m\,\log(1+\|u\|))$$

are continuous.

Now similarly to theorems 2.21 and 2.24 we will obtain the following Paley-Wiener type theorem.

LEMMA 2.27. *Let the functions* M *and* M^* *satisfy (2.31), where* M *and* M^* *are related to each other by (2.28) and (2.29). For every* m *and* k, *and for each* t *there is a* $t' = t'(m,k,t) \geq t$ *and for each* t' *there is a positive* $t = t(m,k,t') \leq t'$, *such that* $\mathcal{F}$ *and* $\mathcal{F}^{-1}$ *are continuous maps*

$$\mathcal{F}: S_\alpha(m,k+1,t')' \to H_\alpha(m+n+1,k,t)$$
$$\mathcal{F}^{-1}: H_\alpha(m,k+1,t') \to S_\alpha(m+2n+2,k,t')'.$$

Moreover, the maps

$$\mathcal{F}: S_\alpha(k+1,m)' \to H_\alpha(k,m+n+1)$$
$$\mathcal{F}^{-1}: H_\alpha(k+1,m) \to S_\alpha(k,m+2n+2)'$$

are continuous and for each k *there is a* $p > k$ *such that*

$$\mathcal{F}: S_\alpha(m,p)' \to H_\alpha(m+n+1,k)$$
$$\mathcal{F}^{-1}: H_\alpha(m,k+1) \to S_\alpha(m+2n+2,k)'$$

are continuous maps for $\alpha \in \{\varepsilon,c\}$. *In all these cases* $\mathcal{F}$ *can be represented as in lemma 2.23.*

PROOF. We only prove the first pair, the other cases are similar. We embed the space $S_\alpha(m,k+1,t')'$ into the dual of the space $W^{m+n+1}_{\infty,0}(\Omega^{k+1}_\alpha;-M(\|\xi\|/t') + +(k+1)\|\eta\| - (m+n+1)\log(1+\|\zeta\|))$. Then as in the proof of theorem 2.24 we have to estimate

$$(2.58) \qquad \sup_{\zeta\in\Omega^{k+1}_\alpha} -\langle\eta,x\rangle - \langle\xi,y\rangle + M(\|\xi\|/t') - (k+1)\|\eta\| + (m+n+1)\log(1+\|\zeta\|)$$

for $z \in (T^C)^k_\alpha$, where $z = (x,y)$ has to be considered as the imaginary part of θ. Let $t'' < t'$ be such that according to (2.31)

$$M(\rho/t') + (m+n+1)\log(1+\rho) \leq M(\rho/t'') + K'(m,t')$$

and let C'_k be such that $C_k \subset\subset C'_k \subset\subset C_{k+1}$. Then there is a $\delta_k > 0$ such that for $y \in C_k$ and $\xi \in C'^*_k$

$$-\langle\xi,y\rangle \leq -\delta_k\|y\|\,\|\xi\|.$$

We first estimate (2.58) if $y \in C_k$, $\|y\| \le 1$ and $\|x\| \le k$. If ξ varies only in $C_k'^{*}$ we estimate (2.58) by

$$-\langle\xi,y\rangle + M(\|\xi\|/t") - \langle\eta,x\rangle - k\|\eta\| - \|\eta\| + (m+n+1)\log(1+\|\eta\|) + K' \le$$

$$\le \sup_{\rho>0} \{-\delta_k t"\|y\|\rho + M(\rho)\} + K \le M^*(t\|y\|) + K(m,t')$$

where $t = \delta_k t"$. If ζ varies in the remaining part of Ω_α^{k+1} then $\|\xi\|$ is bounded by a constant d_k depending on k and also $\|\eta\|$ is bounded, namely

$$\|\eta\| \le \sup_{\|x\|=1} a(x,y_0) + \frac{\sqrt{2}}{k+1} + d_k .$$

Hence then (2.58) can be estimated by a constant depending on m,t'(or t") according to (2.31) and on k, while t depends on k and on t" and t" on m and on t' (or t' depends on m and on t" and t" on k and t).

Now let z be a point in the remaining of $(T^C)_\alpha^k$; hence for $\alpha = \varepsilon$ $z \in T^{1/k\, y_0+C}$ and for $\alpha = c$ there is a $p > k$ depending on k with $y \in C_k$, $\|y\| \ge 1$ and $\|x\| \le p\|y\|$. Then in both cases for sufficiently small ε_1 and $0 < \varepsilon_2 \le \varepsilon_1$

$$(x, y - \varepsilon_2\, y_0) \in U_\alpha^k$$

where

$$U_\alpha^k \overset{\text{def}}{=\!=} T^{1/2k\, y_0+C}$$

$$U_c^k \overset{\text{def}}{=\!=} (T^C)_{p+1} .$$

In the $\alpha = \varepsilon$ case we take $\varepsilon_2 = 1/2k$ and for $z \in T^{1/k\, y_0+C}$ we estimate (2.58) by

$$\begin{aligned}
&-\langle\eta,x\rangle - \langle\xi,y-\varepsilon_2 y_0\rangle - \varepsilon_2\langle\xi,y_0\rangle + M(\|\xi\|/t") + K"(m,t',k) \le \\
&\le a(x,y-\varepsilon_2 y_0) + \|z\|/k+1 - \varepsilon_2\delta_k\|\xi\| + M(\|\xi\|/t") + K" \le \\
&\le a(x,y-1/2k\ y_0) + \|z\|/k + M^*(1/2k\ \delta_k t") + K' \le \\
&\le a(x,y-1/2k\ y_0) + \|z\|/k + K,
\end{aligned} \tag{2.59.i}$$

where K depends on t', t" (or only t'), m and k.

If $\alpha = c$ we proceed as follows: since a is uniformly continuous on

$U_c^k \cap \{z|\|z\|=1\}$, for each $\delta > 0$ there is an ε_2 with $0 < \varepsilon_2 \le \varepsilon_1$, depending on δ and on k, such that

$$a(x,y-\varepsilon_2 y_0) \le a(\tilde{z}) + \delta$$

where $\tilde{z}$ denotes $z/\|z\|$. Hence for all $z \in (T^C)_p \cap \{z|\|y\|\ge 1\}$

$$\begin{aligned} &a(x,y-\varepsilon_2 y_0) \le a(\tilde{z})\|(x,y-\varepsilon_2 y_0)\| + \delta\|(x,y-\varepsilon_2 y_0)\| \le \\ &\le a(z) + \delta\|z\| + \varepsilon_2\delta + \varepsilon_2 \max_{z\in(T^C)_p} |a(\tilde{z})| \le a(z) + \delta\|z\| + K''(k). \end{aligned} \tag{2.60}$$

Let $\delta = 1/k - 1/k+1$ then we estimate (2.58) by

$$\begin{aligned} &a(x,y-\varepsilon_2 y_0) + \|z\|/k+1 - \varepsilon_2\delta_k\|\xi\| + M(\|\xi\|/t'') + K' \le \\ &\le a(z) + \delta\|z\| + K'' + \|z\|/k+1 + M^*(\varepsilon_2\delta_k t'') + K' \le a(z) + \|z\|/k + K \end{aligned} \tag{2.59.ii}$$

where again K depends on t', t" (or only t"), m and k.

For the proof of the continuity of F^{-1} we proceed as in the proof of theorem 2.21. Each $f \in H_{\mathfrak{C}}^{\alpha}(m,k+1,t')$ is a tempered distribution in the variable Re θ for every $\text{Im}\,\theta \in (T^C)_\alpha^{k+1}$; denoting the inverse Fourier transform of this tempered distribution by $F_{S'}^{-1}[f(\text{Re}\,\theta + i\,\text{Im}\,\theta)]_{\eta,\xi}$ we get

$$(F^{-1}f)_{\eta,\xi} = \exp\{<(\eta,\xi),\text{Im}\,\theta>\}F_{S'}^{-1}[f(\text{Re}\,\theta + i\,\text{Im}\,\theta)]_{\eta,\xi}$$

and this is a distribution in $\mathcal{D}'_{\eta,\xi}$. For a C^∞-function ϕ with compact support in $\mathbb{R}_n \times \mathbb{R}_n$ and for $\alpha = \varepsilon$ we have

$$\begin{aligned} <F^{-1}f,\phi> = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} f(\theta^1,\theta^2 + \frac{i\,y_0}{2k+2}) \Big\{ \int_{\mathbb{R}_n}\int_{\mathbb{R}_n} \phi(\eta,\xi) \\ \exp[-i<(\eta,\xi),\text{Re}\,\theta> + <\eta,\text{Im}\,\theta^1> + <\xi,\text{Im}\,\theta^2 + \frac{y_0}{2k+2}>]d\eta d\xi \Big\} d\,\text{Re}\,\theta \end{aligned} \tag{2.61.i}$$

whereas for $\alpha = c$ we have

$$\begin{aligned} <F^{-1}f,\phi> = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}^{2n}} f(\theta) \Big\{ \int_{\mathbb{R}_n}\int_{\mathbb{R}_n} \phi(\eta,\xi)\exp[-i<(\eta,\xi),\text{Re}\,\theta> + \\ + <(\eta,\xi),\text{Im}\,\theta>]d\eta d\xi \Big\} d\,\text{Re}\,\theta. \end{aligned} \tag{2.61.ii}$$

The integrals exist and are independent of $\mathrm{Im}\,\theta \in (T^C)^{k+1}_\alpha$ because $F^{-1}[\phi](\theta)$ is an entire function which is rapidly decreasing in Re θ for each Im θ in a compact set in $\mathbb{R}^{2n}$. As in the proof of theorem 2.21 we use the growth of $|f(\theta)|$, either for $\|\mathrm{Im}\,\theta\|$ large in the set $\{(x,y)\,|\,y - y_0/k+1 \in C,\ x \in \mathbb{R}^n\}$ if $\alpha = \varepsilon$ in which case $|f(\theta^1, \theta^2 + iy_0/2k+2)|$ is $O(\exp a(\mathrm{Im}\,\theta))$ for $\mathrm{Im}\,\theta \to \infty$ on any ray in T^C, or for $\|\mathrm{Im}\,\theta\|$ large in the set $\{(x,y)\,|\,y \in C_{k+1},\ \|y\| \geq 1/2k+2,\ \|x\| \leq (k+1)\|y\|\}$ if $\alpha = c$, to show that $F^{-1}f$ has its support in Ω^{k+1}_α.

In order to find the growth at infinity of the C^∞_- functions ϕ on which $F^{-1}f$ can be defined, we write (2.61) in a different way. Let $\gamma = \gamma(k)$ be so large that

$$|\gamma + \sum_{j=1}^{2n} \theta_j^2| \geq 1 + \|\mathrm{Re}\,\theta\|^2$$

for

$$\mathrm{Im}\,\theta \in B_k \overset{\text{def}}{=\!=} \{(x,y)\,|\,y \in C_{k+1},\ \|y\| \leq 1,\ \|x\| \leq k+1\}.$$

Then for such Im θ we can write (2.61) as

$$<F^{-1}f,\phi> = \frac{1}{(2\pi)^{2n}} \int_{\mathbb{R}_n} \int_{\mathbb{R}_n} \left\{ \int_{\mathbb{R}^{2n}} \frac{f(\theta)\exp - i<(\eta,\xi),\theta>}{(\gamma + \Sigma\theta_j^{\,2})^{\ell}}\, d\,\mathrm{Re}\,\theta \right\}$$

$$(\gamma - \Delta_{\eta,\xi})^{\ell}\phi(\eta,\xi)\,d\eta d\xi,$$

where we have set $\ell = [(m+n)/2] + 1$. The third integral is independent of $\mathrm{Im}\,\theta \in B_k$. Hence $F^{-1}f$, which is itself independent of k, is a sum (depending on k) of derivatives up to order 2ℓ of a continuous function G (depending on k) which for each $(x,y) \in B_k$ satisfies

$$|G(\eta,\xi)| \leq K(f)K\exp\{M^*(t'\|y\|) + <\eta,x> + <\xi,y>\} \leq$$

$$\leq K(f)K\exp\{M^*(t'\|y\|) + \|y\|\|\xi\| + <\eta,x>\},$$

where K(f) denotes

$$K(f) \overset{\text{def}}{=\!=} \sup_{\mathrm{Im}\theta\in B_k} |f(\theta)|\exp\{-m\log(1+\|\theta\|) - M^*(t'\|\mathrm{Im}\,\theta^2\|)\}.$$

By (2.29) we can choose $(x,y) \in B_k$ suitably with $x = -(k+1)\tilde{\eta}$, so that for

$\|\xi\|$ sufficiently large

$$|G(\eta,\xi)| \le K(f)K \exp\{M(\|\xi\|/t') - (k+1)\|\eta\|\}.$$

Thus if we consider the space of all ϕ with ϕ defined in the ε-neighborhood of Ω_α^{k+1} where $\varepsilon = 1/k - 1/k+1$ and with

$$|D^\alpha\phi(\zeta)| \le K \exp\{-M(\|\xi\|/t') + (k+1)\|\eta\| - (n+1)\log(1+\|\zeta\|)\}, |\alpha| \le 2\ell$$

for some $K \ge 0$, then $F^{-1}f$ is defined and continuous on this space. Embedding into this space the space $W_2^{m+2n+2}(\Omega_\alpha^k; -M(\|\xi\|/t') + k\|\eta\| - (m+2n+2)\log(1+\|\zeta\|))$ we find that F^{-1} is continuous from $H_\alpha(m,k+1,t')$ into $S_\alpha(m+2n+2,k,t')'$ for $\alpha \in \{\varepsilon,c\}$. □

II.3. THE EDGE OF THE WEDGE THEOREM

In this section we shall give a short proof of the edge of the wedge theorem for distributions and we shall extend it so that it applies to ultra-distributions, too. We will be concerned with the general situation, cf, [17], where the two cones need not be opposite each other. Our proof also applies to the case of the Malgrange-Zerner theorem, cf. [49], where the functions are holomorphic only in lower dimensional regions. Usually, the known proofs of the edge of the wedge theorem are more complicated and use some functional analysis (Schwartz' kernel theorem), see for example [64] or [8], whereas our proof is based on Fourier transformation.

II.3.i. THE EDGE OF THE WEDGE THEOREM FOR DISTRIBUTIONS.

We shall derive the local version from a global one by a transformation as performed by Borchers in the proof of [4, lemma 8]. In fact, [4, lemma 8] contains already the edge of the wedge theorem for functions with continuous boundary values, cf. for example [64, th. 2.14], which is usually needed in the proof of the general case, cf. [64, th. 2.16]. Moreover, [4, lemma 8] is of the type of the Malgrange-Zerner theorem, cf. [44, th. 3] or [49, p. 286-287], i.e., it gives the analytic continuation of a separately holomorphic function defined, if n = 2, on

$$\{(z_1,z_2) \mid |z_1|<1, y_1>0, |x_2|<1, y_2=0\} \cup$$

$$\cup \{(z_1,z_2) \mid |x_1|<1, y_1=0, |z_2|<1, y_2>0\},$$

where this function has equal continuous boundary values for $y_1 \downarrow 0$ and for $y_2 \downarrow 0$. We shall extend the method of [4] so that we get the result for distributional boundary values and even for ultradistributional boundary values.

It should be remarked that [4, lemma 8], as a particular case, yields the Cameron-Storvick theorem, cf. [44, th. 4], i.e., the analytic continuation into the domain

$$\{(z_1,z_2) \mid |z_1|<\kappa, |z_2|<\kappa\}$$

of a function which is separately holomorphic, if n = 2, in

$$\{(z_1,z_2) \mid |z_1|<1, |x_2|<1, y_2=0\} \cup \{(z_1,z_2) \mid |x_1|<1, y_1=0, |z_2|<1\},$$

where $\kappa = \sqrt{2}-1$. This is a better constant than $\kappa = 1-1/\sqrt{2}$ of [4, th. 4] which on its turn is better than the original $\kappa = 2/(5+2\sqrt{2})$ of Cameron-Storvick, cf. [44].

For our proof of the edge of the wedge theorem we need lemma's usually preceding it, cf. [64]. In particular, we mention the following lemma's whose proofs can be obtained from those in [64], cf. also the next section.

<u>LEMMA</u> 2.28. ([64, th. 2.6 & 2.10]). *Let* C *be a convex cone in* $\mathbb{R}^n$ *(not necessarily open) and let* $C_r \overset{\text{def}}{=} \{y \mid y \in C, \|y\| < r\}$. *Let* f *be a holomorphic function in an open neighborhood in* $\mathbb{C}^n$ *of* $\mathbb{R}^n + iC_r$ *satisfying*

$$(2.62) \qquad |f(z)| \le M(r')(1+\|x\|)^m \|y\|^{-m}, \quad y \in C_{r'},$$

where $M(r')$ *may depend on* r' *for* $0 < r' < r$, *and let* f^* *be the boundary value in* S' *of* f *as* $y \to 0$, $y \in C$. *Then* $f^* \in S'$ *is such that for each* $y \in C_r \cup \{0\}$

$$(2.63) \qquad e^{-\langle\xi,y\rangle} F^{-1}[f^*]_\xi \in S'_\xi \ .$$

LEMMA 2.29. ([64, th. 2.6 & 2.10]). *Let* $f^* \in S'$ *be a tempered distribution satisfying (2.63) for* $y \in (\bar{C})_r$ *where* C *is an open convex cone. Then* $F[e^{-<\xi,y>}F^{-1}[f^*]_\xi](x)$ *is a holomorphic function of* $z = x + iy$ *in* $\mathbb{R}^n + iC_r$, *which tends to* $F[e^{-<\xi,y>}F^{-1}[f^*]_\xi]_x$ *in* S'_x *on* $(\partial C)_r$ *and to* f^* *in* S' *as* $y \to 0$, $y \in C$.

LEMMA 2.30. ([64, th. 2.5]). *Let* $f_\xi \in \mathcal{D}'_\xi$ *be a distribution such that* $e^{-<\xi,y>}f_\xi \in S'_\xi$ *for* $y \in B$, *where* B *is some set in* $\mathbb{R}^n$. *Then also* $e^{-<\xi,y>}f_\xi \in S'_\xi$ *for each* y *in the convex hull* ch B *of* B.

THEOREM 2.31. (Edge of the wedge theorem for distributions). *Let* U *be a domain in* $\mathbb{R}^n$, *let* C^1 *and* C^2 *be two open, connected cones in* $\mathbb{R}^n$ *and let* $r_1 > 0$ *and* $r_2 > 0$. *If two functions* f_1 *and* f_2, *holomorphic in* $U + iC^1_{r_1}$ *and* $U + iC^2_{r_2}$, *respectively, have the same distributional boundary value* f^* *in* $\mathcal{D}(U)'$, *then* f^* *is the boundary value in* $\mathcal{D}(U)'$ *of a function holomorphic in* $\Omega \cap \mathbb{R}^n + i\,\mathrm{ch}(C^1 \cup C^2)$, *which coincides with* f_1 *and* f_2 *on their common domains of definition, where* Ω *is a certain open neighborhood of* U *in* $\mathbb{C}^n$ *not depending on* f_1 *and* f_2.

PROOF. Let $y_0 \in \mathrm{ch}(C^1 \cup C^2)$ and first assume that $y_0 \neq 0$. Let $y_1,\dots,y_n \in C^1 \cup C^2$ be linear independent vectors such that $y_0 \in \mathrm{ch}\{y_1,\dots,y_n\}$. Since analytic continuation is unique, it is sufficient to show that f_1 and f_2 can be continued analytically into $\Omega \cap \mathbb{R}^n + i[\mathrm{int\ ch}\{0,y_1,\dots,y_n\}]$. We choose $y_1,\dots,y_n$ as the new coordinate directions of $\mathbb{R}^n$, so that by a change of coordinates (cf. [64, th. 2.15]) we may assume that

$$f^*_x = \lim_{y_j \downarrow 0} f_j(x_1,\dots,x_j + iy_j,\dots,x_n)$$

in distributional sense in $\{x \mid |x_1| < 1,\dots,|x_n| < 1\}$, where the n functions f_j are holomorphic in a neighborhood in $\mathbb{C}^n$ of

$$\{z \mid |x_1| < 1, y_1=0,\dots,|z_j| < 1, y_j > 0,\dots,|x_n| < 1, y_n=0\}, \tag{2.64}$$

and that for some $M > 0$ and $m > 0$ there

$$|f^j(x_1,\dots,x_j + iy_j,\dots,x_n)| \le M|y_j|^{-m}$$

for $j = 1,\dots,n$, cf. [49]. Let

$$\tilde{f}^j(u_1,\dots,w_j,\dots,u_n) \overset{\text{def}}{=\!=} f^j\left(\frac{e^{u_1}-1}{e^{u_1}+1},\dots,\frac{e^{w_j}-1}{e^{w_j}+1},\dots,\frac{e^{u_n}-1}{e^{u_n}+1}\right).$$

Then $\tilde{f}^j$ is holomorphic in a neighborhood in $\mathbb{C}^n$ of

$$\{w \mid w = u + iv,\ u \in \mathbb{R}^n,\ v_1 = 0,\dots,0 < v_j < \pi/2,\dots,v_n = 0\}$$

and it satisfies there for some $K > 0$ and $k > 0$

$$|\tilde{f}^j(u_1,\dots,w_j,\dots,u_n)| \le K \frac{e^{k\|u\|}}{|v_j|^k}.$$

Every $\tilde{f}^j$ has the same boundary value in $\mathcal{D}'_u$ and the functions

$$h^j(w) \overset{\text{def}}{=\!=} e^{-w^2}\tilde{f}^j(w)$$

satisfy (2.62). Hence they have the same boundary value h^* in S'_u, cf. (2.19). By lemma 2.28

$$e^{-<\xi_j,v_j>}F^{-1}[h^*]_\xi \in S'_\xi,\quad 0 < v_j < \pi/2,\quad j = 1,\dots,n$$

and by lemma 2.30

$$e^{-<\xi,v>}[F^{-1}h^*]_\xi \in S'_\xi,\quad v \in B \overset{\text{def}}{=\!=} \{v \mid v_j \ge 0, j = 1,\dots,n,\ v_1+\dots+v_n < \pi/2\}.$$

According to lemma 2.29 h^* is the boundary value of a holomorphic function in $\mathbb{R}^n + i\,\text{int}\,B$ which coincides with the functions h^j on the parts of the boundary of $\mathbb{R}^n + i\,B$ where these are defined, because $h^j(u_1,\dots,w_j,\dots,u_n) = F[e^{-<\xi_j,v_j>}F^{-1}[h^*]_\xi](u)$. Since $\tilde{f}^j(w) = e^{w^2}h^j(w)$ and since e^{w^2} is entire, it follows that the functions $\tilde{f}^j$ can be continued analytically to the same holomorphic function in $\mathbb{R}^n + i\,\text{int}\,B$. By transforming back, we find that f^* is the boundary value of a holomorphic function in $\Omega \cap \mathbb{R}^n + i\{y \mid y_j > 0, j=1,\dots,n\}$ coinciding with f^j on the boundary, where Ω is determined by the transformation of the domain $\mathbb{R}^n + i\,\text{int}\,B$.

Finally, if $y_0 = 0$, we choose n vectors $y_1,\dots,y_n \in \text{ch}\,C^1$ such that $-y_1,\dots,-y_n \in \text{ch}\,C^2$ and we perform the same steps as above such that now B becomes $\{v \mid |v_1|+\dots+|v_n| < \pi/2\}$. Then f_1 and f_2 can be continued analytically into a neighborhood of U in $\mathbb{C}^n$ and f^* is a holomorphic function there. □

REMARK. It follows from the proof that the domain into which a function, which is separately holomorphic in the regions (2.64) for j = 1,...,n and which has the same boundary value for every $y_j \downarrow 0$, can be continued contains (cf. [4])

$$\bigcup_{\substack{\lambda_j>0 \\ \lambda_1+\ldots+\lambda_n=1}} \{z \mid z_j \in C_j^+(\lambda_1,\ldots,\lambda_n)\}$$

where $C_j^+(\lambda_1,\ldots,\lambda_n)$ is the intersection of the upper half-plane with the open circle with center $-i\rho$ and with radius $\sqrt{1+\rho^2}$ where $\rho \overset{\text{def}}{=\!=} (\text{tg } 1/2\lambda_j\pi)^{-1}$. This yields the constant $\kappa = \sqrt{2} - 1$ in the Cameron-Storvick theorem, cf. [44, th.4].

II.3.ii. THE EDGE OF THE WEDGE THEOREM FOR ULTRADISTRIBUTIONS.

The proof of th. 2.31 relies on the fact that we can suppress the growth at infinity of the functions $\tilde{f}^j$ by a function holomorphic in a tube, namely by e^{-w^2}. Now, if f^* is an ultradistribution in $\mathcal{D}_M(U)'$, the functions $\tilde{f}^j$ have boundary values in $\mathcal{D}_M'$, because the growth of f_1 and f_2 for $\|y\|$ small is the same as the growth of $\tilde{f}^j$ for v_j small, but $\tilde{f}^j(u_1,\ldots,u_j+iv_j,\ldots,u_n)$ grows faster than exponentially for $\|u\| \to \infty$. Then we do not have a function like e^{-w^2}, holomorphic in a tube, which suppresses this growth. Therefore, we have to generalize the lemma's 2.28, 2.29 and 2.30 such that they hold for ultradistributions f^* in $\mathcal{D}_M'$ and analytic functionals $F^{-1}[f^*]$ in Z_M'. The proof of the generalization, lemma 2.32, of lemma 2.28 requires some invention, while the proofs of lemma's 2.33 and 2.34 are similar to those of lemma's 2.29 and 2.30.

If $\mu \in Z_M'$ we mean by $e^{-\langle\zeta,y_0\rangle}\mu_\zeta \in Z_M'$ that μ_ζ can be applied to entire functions of the form $e^{-\langle\zeta,y_0\rangle}\psi(\zeta)$ with $\psi \in Z_M$ and that $|\langle\mu_\zeta, e^{-\langle\zeta,y_0\rangle}\psi(\zeta)\rangle| \le K\|\psi\|_\alpha$ for some $K > 0$ where $\|\cdot\|_\alpha$ is one of the half norms defining the topology of Z_M.

LEMMA 2.32. *Let C and C_r be as in lemma 2.28. Let f be a holomorphic function in an open neighborhood in $\mathbb{C}^n$ of $\mathbb{R}^n + iC_r$ with a boundary value f^* in $\mathcal{D}_M'$ as $y \to 0$, $y \in C$. Then $\mu \overset{\text{def}}{=\!=} F^{-1}[f^*] \in Z_M'$ is such that*

$$e^{-\langle\zeta,y\rangle}\mu_\zeta \in Z_M'$$

for every $y \in C_r \cup \{0\}$.

PROOF. Let $\{K_k\}_{k=1}^{\infty}$ be an increasing sequence of convex, compact sets with union $\mathbb{R}^n + iC_r$. Let H_k be the space of analytic functionals carried by K_k provided with the FS-space topology defined by duals of sup-norms and finally, let $H \overset{\text{def}}{=} \operatorname{ind}\lim_{k\to\infty} H_k$, where the injection maps are obtained as transposed of restriction maps. Then f is an element of the dual H' of H. Now the Ehrenpreis -Martineau theorem, [16, th. 5.21] or [30, th. 4.5.3], describes the space A of Fourier transforms of elements of H very well: A consists of entire functions h with the order of growth at infinity

$$\exp(\varepsilon\|\xi\| + k\|\eta\| + \sup_{y\in S_k} -\langle\xi,y\rangle)$$

for all $\varepsilon > 0$ and for some k depending on h, where $\{S_k\}_{k=1}^{\infty}$ is an increasing sequence of compact subsets of C_r with union C_r. We give A the topology which turns the Fourier transformation into a topological isomorphism. Then there is an element μ in the dual A' of A with

$$\langle\mu_\zeta, e^{i\langle\zeta,z\rangle}\rangle = f(z), \quad z \in \mathbb{R}^n + iC_r.$$

If $y_0 \in C_r$ and $\psi \in Z_M$ the function $\zeta \to e^{-\langle\zeta,y_0\rangle}\psi(\zeta)$ belongs to A and, in fact, it is the Fourier transform of the analytic functional defined by $\hat{\psi}_x\delta(y_0)_y$ where $\hat{\psi} \in \mathcal{D}_M$ is the inverse Fourier transform of ψ and where $\delta(y_0)$ is the Dirac-delta function concentrated in the point y_0. Hence

$$\langle\mu_\zeta, e^{-\langle\zeta,y_0\rangle}\psi(\zeta)\rangle = \int f(x+iy_0)\hat{\psi}(x)\,dx.$$

Furthermore, μ is also a continuous linear functional on Z_M by means of the following definition

$$\langle\mu,\psi\rangle \overset{\text{def}}{=} \lim_{\substack{y\to 0\\ y\in C}} \langle\mu_\zeta, e^{-\langle\zeta,y\rangle}\psi(\zeta)\rangle = \lim_{\substack{y\to 0\\ y\in C}} \int f(x+iy)\hat{\psi}(x)\,dx, \quad \psi \in Z_M.$$

That the limit exists and indeed defines an element in Z_M' follows from the last equality and the data of the lemma. Thus we have $\mu = \mathcal{F}^{-1}[f^*]$ and since for $y_0 \in C_r$ the space $e^{-\langle\zeta,y_0\rangle}Z_M$ (i.e., the space of all entire functions $\phi(\zeta) = e^{-\langle\zeta,y_0\rangle}\psi(\zeta)$ with $\psi \in Z_M$ provided with the half norms $\|\phi\| \overset{\text{def}}{=} \|e^{\langle\zeta,y_0\rangle}\phi\|_\alpha$ where $\|\cdot\|_\alpha$ are the half norms defining the topology of Z_M) can be continuously embedded into A, it follows that $e^{-<\zeta,y>}\mu_\zeta \in Z'_M$ for $y \in C_r$. □

LEMMA 2.33. *Let* $\mu \in Z'_M$ *be such that* $e^{-<\zeta,y>}\mu_\zeta \in Z'_M$ *for each* y *in the closure of an open, convex cone* C *with* $\|y\| < r$. *Then* $F[e^{-<\zeta,y>}\mu_\zeta](x)$ *is a holomorphic function of* z *in* $\mathbb{R}^n + iC_r$, *which tends to* $F[e^{-<\zeta,y>}\mu_\zeta]$ *in* $\mathcal{D}'_M$ *on the boundary of* C *and to* $F[\mu]$ *in* $\mathcal{D}'_M$ *as* $y \to 0$, $y \in C$.

PROOF. The space Z_M is defined as the space of all entire functions with certain finite, weighted, sup-norms. Let $C(Z_M)$ be the space of all continuous functions with the same finite, weighted, sup-norms. Let $\tilde{\mu}$ be an extension of μ to $C(Z_M)'$. Then by Riesz' theorem for each testfunction $\tilde{\mu}$ can be represented as a measure $\tilde{\mu}(\zeta)$ on $\mathbb{C}_n$. Furthermore, let $y_0 \in C_r$. Then as in [64, proof of th. 2.6, formula 2.70] it is shown that there is an $\varepsilon > 0$ such that

$$e^{\varepsilon\sqrt{1+\|\xi\|^2}}\, e^{-<\zeta,y>}\tilde{\mu}_\zeta = \sum_{j=1}^{k} \tilde{\mu}^j_\zeta$$

for y in a neighborhood $U(y_0)$ of y_0 contained in C_r and for some elements $\tilde{\mu}^j \in C(Z_M)'$ depending on y. Then for $y \in U(y_0)$

$$f(z) \overset{\text{def}}{=\!=} \int e^{i<\zeta,z>} d\tilde{\mu}(\zeta) = \sum_{j=1}^{k} \int_{\mathbb{C}_n} \exp(i<\zeta,x> - \varepsilon\sqrt{1+\|\xi\|^2})\, d\tilde{\mu}^j(\zeta)$$

exists and is holomorphic in $\mathbb{R}^n + iU(y_0)$. By analytic continuation we get a function f which is holomorphic in $\mathbb{R}^n + iC_r$. Now Fubini's theorem shows that $F[e^{-<\zeta,y>}\mu_\zeta](x) = f(z)$. Furthermore, let $y_1 \in (\partial C)_r$, let $y_0 = 0$ and let $y_2,\ldots,y_n \in C_r$ such that the convex hull B of $\{y_0,\ldots,y_n\}$ has a non-empty interior. Then as in [64, proof of th. 2.6, formula 2.68] we can write

$$e^{-<\zeta,y>}\tilde{\mu}_\zeta = \sum_{j=0}^{n} a(y,\xi)\, e^{-<\xi,y_j>}\tilde{\mu}_\zeta e^{-i<\eta,y>}$$

for $y \in B$, where $a(y,\xi)$ is a continuous function, bounded uniformly for all $\xi \in \mathbb{R}_n$ and $y \in B$, cf. the proof of the next lemma. Therefore, $e^{-<\zeta,y>}\tilde{\mu}_\zeta$ tends to $e^{-<\zeta,y_1>}\tilde{\mu}_\zeta$ in $C(Z_M)'$ as $y \to y_1$, $y \in B$ or to $\tilde{\mu}_\zeta$ in $C(Z_M)'$ as $y \to 0$, $y \in B$. Hence the statements of the lemma follow. □

LEMMA 2.34. *Let* $\mu \in Z'_M$ *be such that* $e^{-<\zeta,y>}\mu_\zeta \in Z'_M$ *for* y *in some set* B *in* $\mathbb{R}^n$. *Then also* $e^{-<\zeta,y>}\mu_\zeta \in Z'_M$ *for all* $y \in \mathrm{ch}\,B$.

PROOF. It is sufficient to show that for $y_1, y_2 \in B$ and $y = ty_1 + (1-t)y_2$, $0 \le t \le 1$, $e^{-<\xi,y>}\mu_\zeta \in Z_M'$. Let $\tilde{\mu} \in C(Z_M)'$ be an extension of μ, then also $e^{-<\xi,y_1>}\tilde{\mu}_\zeta$ and $e^{-<\xi,y_2>}\tilde{\mu}_\zeta$ belong to $C(Z_M)'$. The continuous function $\xi \to$

$$a(y,\xi) \overset{\text{def}}{=\!=} \frac{e^{-<\xi,y>}}{e^{-<\xi,y_1>} + e^{<\xi,y_2>}}$$

is bounded in $\mathbb{R}_n$ (see [64, proof of th. 2.5]). Accordingly

$$e^{-<\xi,y>}\tilde{\mu}_\zeta = a(y,\xi)e^{-<\xi,y_1>}\tilde{\mu}_\zeta + a(y,\xi)e^{-<\xi,y_2>}\tilde{\mu}_\zeta \in C(Z_M)',$$

so that also $e^{-<\zeta,y>}\tilde{\mu}_\zeta \in C(Z_M)'$. Therefore, its restriction to Z_M, which equals $e^{-<\zeta,y>}\mu_\zeta$, belongs to Z_M'. □

Now the proof of the edge of the wedge theorem for ultradistributions is obtained similarly to that of theorem 2.31 using the above given lemma's instead of the lemma's of the last section. So we have got the following theorem.

THEOREM 2.35. (Edge of the wedge theorem for ultradistributions). *Let* C_1, C_2, f_1 *and* f_2, U, r_1 *and* r_2 *be as in theorem 2.31, where now* f_1 *and* f_2 *have the same ultradistributional boundary value* f^* *in* $\mathcal{D}_M(U)'$. *Then the conclusion of theorem 2.31 holds in* $\mathcal{D}_M(U)'$ *instead of* $\mathcal{D}(U)'$.

REMARK. More general edge of the wedge theorems exist, where f^* is a sum of boundary values of more than two functions, see for example [31] and [43, p. 40-81]. If distributional boundary values are concerned, this theorem has been shown by Martineau in [49] and an easy proof by induction has been given by Bros & Iagolnitzer in [6, section 7], where first the notion of essential support is introduced by means of a generalized Fourier transformation. This method might be extendable to ultradistributions, but a forthcoming paper on this subject, announced in [6] and in [31], has not yet appeared.

CHAPTER III

FOURIER TRANSFORMS OF ANALYTIC FUNCTIONALS WITH COMPLEX, UNBOUNDED, CONVEX CARRIERS

The theorems of this chapter describe the Fourier transformation $\mathcal{F}$ as a topological isomorphism between spaces of analytic functionals μ carried by closed, convex sets $\Omega \subset \mathbb{C}_n$ and spaces of holomorphic functions f of exponential type in open, convex cones $\Gamma \subset \mathbb{C}^n$. The functionals μ are carried with respect to some class of open neighborhoods of Ω and to some class of weight functions on these neighborhoods. This determines the behaviour of f near the vertex of Γ and conversely. The convex set Ω itself determines the cone Γ and the type a(z) of f, and conversely. These theorems generalize the Ehrenpreis-Martineau theorem, [16, th. 5.21] or [30, th. 4.5.3], where Ω is bounded and $\Gamma = \mathbb{C}^n$, and the one dimensional version due to Polya, [3, ch. 5].

In [65, th. 2.22 & 2.23] the Ehrenpreis-Martineau theorem is given for polydiscs Ω and in [73] $\mathcal{F}$ is treated as a topological isomorphism for this case. Then the proof can be given directly, but for general, bounded, convex sets Ω the proof is more complicated. The proof given by Ehrenpreis in [16] is based on the case of polydiscs, which by the Oka embedding can be extended to convex polyhedrons, using the fact that a bounded, convex set can be approximated arbitrarily close from the inside by convex polyhedrons. This is no longer true for general, unbounded, convex sets. Hörmander's method which uses an existence theorem for the $\bar{\partial}$-operator, see [30, ch. 4], applies directly to general, unbounded, convex sets Ω. Therefore, in case Ω is unbounded we will follow the method of [30, ch. 4] for proving our theorems, but since we deal with non-entire functions f we have to pay attention to the growth of f near the boundary of Γ.

Unlike in the case where Ω is bounded the proof of the injectivity of $\mathcal{F}$ is not trivial if Ω is unbounded. In this chapter we shall reduce the proof of the bijectivity of $\mathcal{F}$ to two problems, which will be solved in chapter VI by a generalization of Hörmander's method of [30, ch. 7]. On the other hand, this is, in fact, just a version of Ehrenpreis' fundamental principle with non-entire functions and looking at it in this way, our proof follows Ehrenpreis' method. The generalization of Ehrenpreis fundamental principle to non-entire functions will be treated in chapter IV, where also the two problems of this chapter will be reformulated in a more general form.

In particular, it is interesting if Γ is the open cone $T^C \overset{\text{def}}{=\!=} \mathbb{R}^n + iC$ where C is an open, convex cone in $\mathbb{R}^n$. Then functions f, holomorphic in T^C, may have ultradistributional boundary values on $\mathbb{R}^n$ (or in the limiting cases, on the one side distributional boundary values and on the other side boundary values in the sense of Fourier hyperfunctions). They are the Fourier transforms of analytic functionals in Z_M' carried by certain, convex sets Ω which may be unbounded in the imaginary directions. Then a more complicated aspect of the topology of Z_M arises and the testfunctions ψ on which the analytic functionals act satisfy (2.45) on a neighborhood of Ω. This actually expresses the fact that we deal with ultradistributions defined on ultra-differentiable testfunctions with compact support, which is so if M satisfies (2.20). However, in this chapter we shall not need this property and our theorems remain valid for ultradistributions defined on quasi-analytic test-functions. Then, if Ω is unbounded in the imaginary directions, there is perhaps no other reason for requiring the analytic testfunctions to satisfy (2.45) on neighborhoods of Ω than that the theorems are true as they are stated here. Anyhow, we shall not deal with the ultradistributions as boundary values themselves, but we shall define the Fourier transformation F merely by formula (2.46), which in case M satisfies (2.20) is justified by lemma 2.26.

III.1. ANALYTIC FUNCTIONALS ON EXPONENTIALLY DECREASING TESTFUNCTIONS; FOURIER TRANSFORMATION AS A SURJECTION.

In this section we consider functions f, holomorphic in a cone Γ in $\mathbb{C}^n$, of exponential type a(z) for $\|z\|$ large, which do not satisfy growth conditions near the vertex of Γ. Such functions turn out to be Fourier transforms of analytic functionals with unbounded carrier $\Omega(a,\Gamma)$, cf. (2.44). We shall discuss two cases: one, denoted by the index ε, corresponds to analytic functionals with carriers with respect to ε-neighborhoods, i.e., with respect to the neighborhoods $\{\Omega(a+1/k,\Gamma)\}_{k=1}^{\infty}$, cf. (2.48.i), and the other, denoted by the index c, corresponds to conic neighborhoods, i.e., neighborhoods of $\Omega(a,\Gamma)$ of the form $\Omega(a+1/k,\Gamma_k)$, cf. (2.48.ii). If $\Gamma = T^C$ the case of conic neighborhoods is perhaps more suitable for describing quantum field theory, cf. [53].

Let $\Gamma \subset \mathbb{C}^n$ be an open, convex cone, a a convex function on Γ which is homogeneous of degree one, $\{\Gamma_k\}_{k=1}^{\infty}$ an increasing sequence of open, convex cones exhausting Γ and let $z_0 \in \Gamma_1$ be fixed with $\|z_0\| = 1$. Then the collection $\{1/k\ z_0 + \Gamma\}_{k=1}^{\infty}$ given by (2.49) exhausts Γ. In the case denoted by ε, let the convex function a_k^{ε} on $1/k\ z_0 + \Gamma$ be defined by

$$a_k^{\varepsilon}(z) \stackrel{\text{def}}{=\!=} \max_{\|w\| \le \delta_k^{\varepsilon}} a(z+w) \tag{3.1.i}$$

where $\delta_k^{\varepsilon} > 0$ is so small that $z + w \in 1/_{k+1}\ z_0 + \Gamma$ for $z \in 1/_k\ z_0 + \Gamma$ and $\|w\| \le \delta_k^{\varepsilon}$. Then after a detailed inspection one can see that for each k there are $q \ge p \ge k$ and a constant $K_k > 0$ such that for $z \in 1/k\ z_0 + \Gamma$

$$a(z - 1/_{2q}\ z_0) \le a_p^{\varepsilon}(z) \le a(z - 1/_{2k}\ z_0) + (1/k - 1/p)\|z\| + K_k.$$

Hence we have the following equality of spaces

$$\begin{aligned} \text{Exp}_{\varepsilon} &\stackrel{\text{def}}{=\!=} \operatorname*{proj\,lim}_{k \to \infty} H_{\infty}(1/k\ z_0 + \Gamma; a_k^{\varepsilon}(z) + 1/k\|z\|) = \\ &= \operatorname*{proj\,lim}_{k \to \infty} H_{\infty}(1/k\ z_0 + \Gamma; a(z - 1/_{2k}\ z_0) + 1/k\|z\|), \end{aligned} \tag{3.2.i}$$

where the space $H_{\infty}(\Omega; M(z))$ has been defined in section II.2.vii by means of the norm (2.55). According to [73, cond. HS_1 and HS_2] Exp_{ε} is a nuclear FS-space (it can also be written as projective limit of Hilbert spaces). If a is a bounded function on pr Γ, the space Exp_{ε} may also be written as

$$\text{Exp}_{\varepsilon} = \operatorname*{proj\,lim}_{k \to \infty} H_{\infty}(1/k\ z_0 + \Gamma; a(z) + 1/k\|z\|), \tag{3.3}$$

cf. (2.60).

In the case denoted by c we exhaust Γ by the sequence $\{\Gamma(k)\}_{k=1}^{\infty}$ given by (2.47). For each k let $\delta_k^c > 0$ be so small that for $z \in \Gamma(k)$ and for $\|w\| \le \delta_k^c$ we have $z + w \in \Gamma(k+1)$ and $a(z+w) \le a(z) + (1/_k - 1/_{k+1})\|z\| + K_k$ for some $K_k > 0$, cf. (2.60). Then we define for $z \in \Gamma(k)$

$$a_k^{c}(z) \stackrel{\text{def}}{=\!=} \max_{\|w\| \le \delta_k^{c}} a(z+w) \tag{3.1,ii}$$

and we have the following equality of spaces

$$(3.2.ii)\qquad \mathrm{Exp}_c \overset{\mathrm{def}}{=\!=} \operatorname{proj}\lim_{k\to\infty} H_\infty(\Gamma(k); a(z) + 1/k\|z\|) =$$
$$= \operatorname{proj}\lim_{k\to\infty} H_\infty(\Gamma(k); a_k^c(z) + 1/k\|z\|).$$

Furthermore, let for $\alpha = \varepsilon$ or c

$$(3.4)\qquad A_\alpha^k \overset{\mathrm{def}}{=\!=} H_\infty(\Omega_\alpha^k; -1/k\|\zeta\|)$$

where Ω_α^k is given by (2.48) and let

$$(3.5)\qquad A_\alpha \overset{\mathrm{def}}{=\!=} \operatorname{ind}\lim_{k\to\infty} A_\alpha^k.$$

According to [73, cond. HS_1 & HS_2] A_α is a nuclear DFS-space (it can also be written as inductive limit of Hilbert spaces), hence the strong dual A'_α is a nuclear FS-space. In particular A'_α is bornologic.

For both $\alpha = \varepsilon$ and $\alpha = c$ the set

$$L \overset{\mathrm{def}}{=\!=} \{e^{i\langle\zeta,z\rangle} \mid z \in \Gamma\}$$

is a subset of A_α and it follows from an easy estimate (as in the proof of lemma 2.27, formula (2.59)) that the map

$$(3.6)\qquad F: A'_\alpha \to \mathrm{Exp}_\alpha$$

is bounded, hence continuous, where F is defined by

$$(3.7)\qquad F(\mu)(z) \overset{\mathrm{def}}{=\!=} \langle\mu_\zeta, e^{i\langle\zeta,z\rangle}\rangle, \quad \mu \in A'.$$

F is sometimes called the Fourier-Laplace or Fourier-Borel transform if the factor i is omitted, but we merely call F Fourier transform and we shall see later that there is an analogue with the Paley-Wiener theorem if we maintain the factor i in (3.7) as we do here. In the next section we shall pay attention to the injectivity of F and here we shall show that F is surjective. Then it follows from the open mapping theorem that the inverse F^{-1} of F is continuous.

If for each $p = 1,2,\ldots$ $\delta_p > 0$ is such that for $z \in \Gamma_p$ and $\zeta \in \Gamma^*_{p+1}$ $\mathrm{Im}\langle\zeta,z\rangle \geq \delta_p \|\zeta\|\|z\|$, then for $k \geq \max(p+2, p/\delta_p)$ we have

$$(3.8.\mathrm{i}) \qquad e^{i\langle\zeta,z\rangle} \in A^k_c, \quad z \in \Gamma(p).$$

Similarly, for each p there is a $k > p$ such that

$$(3.8.\mathrm{ii}) \qquad e^{i\langle\zeta,z\rangle} \in A^k_\varepsilon, \quad z \in 1/p\ z_0 + \Gamma.$$

Denote

$$\Gamma^k_\varepsilon \overset{\mathrm{def}}{=\!=} 1/k\ z_o + \Gamma,\ \Gamma^k_c \overset{\mathrm{def}}{=\!=} \Gamma(k).$$

Now in view of (3.8) for every $f \in \mathrm{Exp}_\alpha$ we have to find for each k a continuous linear functional μ^k_α on A^k_α with

$$(3.9) \qquad f(z) = \langle(\mu^k_\alpha)_\zeta, e^{i\langle\zeta,z\rangle}\rangle, \quad z \in \Gamma^p_\alpha.$$

Indeed, let $\tilde{A}^k_\alpha$ be the closed subspace of A^k_α defined by completion of the set $\{e^{i\langle\zeta,z\rangle} | z \in \Gamma^p_\alpha\}$ in A^k_α, where p is determined by k according to (3.8), then the closed subspace $\tilde{A}_\alpha$ of A_α, defined by completion of the set L in A_α, can be written as

$$\tilde{A}_\alpha = \operatorname*{ind\,lim}_{k\to\infty} \tilde{A}^k_\alpha$$

cf. [20, § 25.13] or [40, th. 7']. By (3.9) we have

$$\mu^{k+1}_\alpha \big|_{\tilde{A}^k_\alpha} = \mu^k_\alpha \big|_{\tilde{A}^k_\alpha}$$

so that $\{\mu^k_\alpha\}^\infty_{k=1}$ determines an element $\tilde{\mu} \in \tilde{A}'_\alpha$ with $F(\tilde{\mu}) = f$. Finally, according to the Hahn-Banach theorem and to definition (3.7) there is a $\mu \in A'_\alpha$ with $F(\mu) = f$.

As in the proof of the theorem with entire functions in [30] we try to extend f as a holomorphic function F in 2n complex variables θ satisfying a certain growth condition and we apply the Paley-Wiener theorem of lemma 2.27. If we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$, we will write Γ for both, cones in $\mathbb{C}^n$ or in $\mathbb{R}^{2n}$. Now assume that for each k we have found a function F^k_α of the complex variables $\theta = (\theta^1,\theta^2) \in \mathbb{C}^n \times \mathbb{C}^n = \mathbb{C}^{2n}$ holomorphic in $\mathbb{R}^{2n} + i\Gamma_\alpha^{k+2}$, which satisfies for some $M_k > 0$ and $m_k > 0$

$$(3.10) \qquad |F_\alpha^k(\theta^1,\theta^2)| \le M_k(1+\|\theta\|)^{m_k}\exp\{a_{(k+2)}^\alpha(\operatorname{Im}\theta) + 1/_{k+2}\|\operatorname{Im}\theta\|\},$$
$$\operatorname{Im}\theta \in \Gamma_\alpha^{k+2} \subset \mathbb{R}^{2n}$$

and

$$(3.11) \qquad F_\alpha^k(iz,z) = f(z),\ z \in \Gamma_\alpha^p \subset \mathbb{C}^n$$

where we take $a_{(k+2)}^\alpha$ different from a only if $\alpha = \varepsilon$ and a is not bounded on pr Γ, in which case $a_{(k+2)}^\varepsilon(z) \stackrel{\text{def}}{=} a(z - 1/_{k+2}\, z_0)$, cf. (3.2.i), (3.3) and (3.2.ii). Then F_α^k belongs to the space $H_\alpha(m,k+2)$ defined by (2.57). From lemma 2.27 it follows that F_α^k can be written as

$$(3.12) \qquad F_\alpha^k(\theta) = \langle(\mu_\alpha^k)_{\eta,\xi},\ e^{i\langle\eta,\theta^1\rangle + i\langle\xi,\theta^2\rangle}\rangle,\quad \operatorname{Im}\theta \in \Gamma_\alpha^p$$

for some $\mu_\alpha^k \in S_\alpha(m+2n+2,k+1)'$, cf. (2.57). From (3.11) formula (3.9) follows and using [73, cond. HS_1] for $\phi \in A_\alpha^k$ we get

$$|\langle\mu_\alpha^k,\phi\rangle| \le K_k'' \sum_{|\ell|\le m_k+2n+2} [\int_{\Omega_\alpha^{k+1}} |D^\ell\phi(\zeta)|^2\{\exp \tfrac{2}{k+1}\|\zeta\|\} +$$
$$+ (1+\|\zeta\|)^{2m_k+4n+4} d\eta d\xi]^{\frac12} \le$$
$$\le K_k' \sum_{|\ell|\le m_k+2n+2} \sup_{\zeta\in\Omega_\alpha^{k+1}} |D^\ell\phi(\zeta)|\exp 1/k\|\zeta\| \le$$
$$\le K_k \sup_{\zeta\in\Omega_\alpha^k} |\phi(\zeta)|\exp 1/k\,\|\zeta\|$$

because an ε-neighborhood of Ω_α^{k+1} is contained in Ω_α^k and for any m

$$(3.13) \qquad A_\alpha^k \subset S_\alpha(m,k+1).$$

Hence μ_α^k determines a conintuous linear functional in $(A_\alpha^k)'$ and (3.9) is valid, whenever we can find functions F_α^k satisfying (3.10) and (3.11) for $f \in Exp_\alpha$. Then the map (3.6) would be surjective.

Since Exp_α can also be written as projective limit of Hilbert spaces and since the function $a^\alpha_{(k)}$ may be changed into a^α_k given by (3.1.i) and (3.1.ii), cf. (3.2.i) and (3.2.ii), it is sufficient if (3.10) is satisfied with an L^2-norm instead of a sup-norm and with weight functions $\exp - a^\varepsilon_k(z)$ instead of $\exp - a^\varepsilon_{(k)}(z)$. Precisely, this means that (3.10) may be replaced by

$$\int_{\mathbb{R}^{2n}+i\,\mathrm{ch}\,\Gamma^k_\alpha} \frac{|F^k_\alpha(\theta^1,\theta^2)|^2 \exp -2\{a^\alpha_k(\mathrm{Im}\theta)+1/k\|\mathrm{Im}\theta\|\}}{(1+\|\theta\|)^{m_k}}\, d\lambda(\theta) \le M_k$$

for some (other) positive numbers M_k and m_k depending on k, where $\lambda(\theta)$ denotes the Lebesgue measure in $\mathbb{C}^{2n}$. Then the extensions F^k_α of f follow exactly from the following theorem, if we choose there $\Omega = \mathbb{R}^{2n} + i\Gamma \subset \mathbb{C}^{2n}$, $\Omega_1 = \mathbb{R}^{2n} + i\,\mathrm{ch}\,\Gamma^k_\alpha$, $\Omega_2 = \mathbb{R}^{2n} + i\,\mathrm{ch}\,\Gamma^{k+1}_\alpha$, $s_1 = i\,\theta_{n+1},\dots,s_n = i\,\theta_{2n}$ and $\phi(\theta) = 2a(\mathrm{Im}\,\theta) + 2/k\|\mathrm{Im}\,\theta\|$ or in the $\alpha = \varepsilon$ case where moreover a is not bounded on $\mathrm{pr}\,\Gamma$, $\phi(\theta) = 2a(\mathrm{Im}\,\theta^1 - \eta x_0, \mathrm{Im}\,\theta^2 - \eta y_0) + 2/k\|\mathrm{Im}\,\theta\|$ with $\eta < \delta^\varepsilon_k$, cf. (3.1.i), so that these functions ϕ are convex, hence certainly plurisubharmonic.

<u>THEOREM</u> 3.1. *Let a n-k dimensional hyperplane in $\mathbb{C}^n$ be given by the linear functions*

$$\theta_1 = s_1(\theta_{k+1},\dots,\theta_n)$$
$$\vdots$$
$$\theta_k = s_k(\theta_{k+1},\dots,\theta_n)$$

or shortly $w = s(z)$ with $w \in \mathbb{C}^k$, $z \in \mathbb{C}^{n-k}$. Let $\Omega_1 \subset \Omega_2 \subset \Omega$ be pseudoconvex domains in $\mathbb{C}^n$ such that an ε-neighborhood of Ω_1, with respect to closed polydiscs in the first k coordinates, is contained in Ω_2, i.e.,

$$(3.14)\qquad \{\theta \,|\, |\theta_j - \theta^0_j| \le \varepsilon \text{ for } j = 1,\dots,k; \theta_j = \theta^0_j \text{ for } j = k+1,\dots,n; \theta^0 \in \Omega_1\} \subset \Omega_2.$$

Furthermore, let ϕ be a plurisubharmonic function on Ω and for $\theta \in \Omega_1$ let

$$\phi_\varepsilon(\theta) \overset{\mathrm{def}}{=} \max\{\phi(\theta_1 + w_1,\dots,\theta_k + w_k, \theta_{k+1},\dots,\theta_n) \,|\, |w_j| \le \varepsilon,\ j = 1,\dots,k\}.$$

Finally let $\Omega' \overset{\text{def}}{=} \{z \mid (s(z),z) \in \Omega\} \subset \mathbb{C}^{n-k}$ *and* $\Omega'_j \overset{\text{def}}{=} \{z \mid (s(z),z) \in \Omega_j\}$, $j = 1,2$, *and let* ϕ' *be the function in* Ω' *given by* $\phi'(z) \overset{\text{def}}{=} \phi(s(z),z)$. *Then for a given function* f, *holomorphic in* Ω', *there exists a function* F, *holomorphic in* Ω_1, *which satisfies*

$$F(s(z),z) = f(z), \quad z \in \Omega'_1 \tag{3.15}$$

and for some $K > 0$, *depending only on* k *and* s_j, $j = 1,\dots,k$,

$$\int_{\Omega_1} \frac{|F(\theta)|^2 \exp - \phi_\varepsilon(\theta)}{(1+\|\theta\|^2)^{3k}} \, d\lambda(\theta) \le K\, \varepsilon^{-2k} \int_{\Omega'_2} |f(z)|^2 \exp - \phi'(z)\, d\lambda(z) \tag{3.16}$$

(where $\lambda(\theta)$ *and* $\lambda(z)$ *denote the Lebesgue measures in* $\mathbb{C}^n$ *or* $\mathbb{C}^{n-k}$ *respectively), if* f *is such that the right hand side is finite.* F *depends besides on* f *also on* Ω_1, ε *and* ϕ.

<u>PROOF</u>. Let ψ be a C^2-function in $\mathbb{C}$ with values between 0 and 1, which is equal to 1 in the disc with radius $1/2\ \varepsilon$, which vanishes outside the disc with radius ε and which satisfies

$$\left| \frac{\partial \psi}{\partial \bar{p}}(p) \right| \le \frac{K}{\varepsilon}, \quad p \in \mathbb{C}$$

for some $K > 0$. Define the $(0,1)$ - form $\psi'(p) \overset{\text{def}}{=} \partial\psi/\partial\bar{p}\,(p)\,d\bar{p}$ and let for $j = 1,\dots,k$

$$p_j = p_j(\theta_j;z) \overset{\text{def}}{=} \theta_j - s_j(z), \quad z \in \mathbb{C}^{n-k}$$

then $d\bar{p}_j = d\bar{\theta}_j - \sum_{\ell=k+1}^{n} \partial\bar{s}_j/\partial\bar{z}_\ell \ \ d\bar{z}_\ell$. We define the function F as follows

$$F(\theta) \overset{\text{def}}{=} \sum_{j=1}^{k} \psi(p_j(\theta_j;\theta_{k+1},\dots,\theta_n)) f(\theta_{k+1},\dots,\theta_n) - \sum_{j=1}^{k} \{ \prod_{m=j+1}^{k}$$

$$\psi(p_m(\theta_m;\theta_{k+1},\dots,\theta_n))\} p_j(\theta_j;\theta_{k+1},\dots,\theta_n) U_j(\theta_1,\dots,\theta_j;\theta_{k+1},\dots,\theta_n)$$

for certain functions U_j of $n-k+j$ complex variables, where an empty product is defined as 1. For $\theta \in \Omega_1$ $F(\theta)$ is defined, because then $\prod_{j=1}^{k} (p_j(\theta_j;z)) = 0$ for $z \notin \{z \mid \exists w \in \mathbb{C}^k, |w_j - s_j(z)| < \varepsilon$ for $j = 1,\dots,k,$

$(w,z) \in \Omega_1\} \subset \Omega_2'$. If $\theta_j = s_j(\theta_{k+1},\dots,\theta_n)$, i.e., if $p_j = 0$, for $j = 1,\dots,k$, we get (3.15).

Now we will choose the functions U_j with a suitable bound such that F is holomorphic in Ω_1, that is such that $\bar\partial F = 0$ there. First we write F in a different form, namely denote

$$\theta[j] \overset{\text{def}}{=\!=} (\theta_1,\dots,\theta_j;z) \in \mathbb{C}^{j+n-k}$$

for $z \in \mathbb{C}^{n-k}$, let

$$G_0(\theta[0]) = G_0(z) \overset{\text{def}}{=\!=} f(z)$$

and let

$$G_j(\theta[j]) \overset{\text{def}}{=\!=} \psi(p_j(\theta_j;z))G_{j-1}(\theta[j-1]) - p_j(\theta_j;z)U_j(\theta[j])$$

for $j = 1,\dots,k$ successively, then

$$G_k = F.$$

G_j is defined in

$$\Omega[j] \overset{\text{def}}{=\!=} \{\theta[j] \,|\, \exists w \in \mathbb{C}^{k-j}, |w_m - s_m(z)| < \varepsilon \text{ for } m = j+1,\dots,k$$
$$\text{and } (\theta_1,\dots,\theta_j,w_{j+1},\dots,w_k;z) \in \Omega_1\} \subset \mathbb{C}^{j+n-k}$$

if G_{j-1} is defined in $\Omega[j-1]$.

The sets $\Omega[j]$ are in general not pseudoconvex, so we will define pseudoconvex, open sets $\tilde\Omega[j]$ containing $\Omega[j]$, such that G_j is defined in $\tilde\Omega[j]$ if G_{j-1} is defined in $\tilde\Omega[j-1]$. For that purpose we first note that

$$\Omega[j] = \{\theta[j] \,|\, (\theta_1,\dots,\theta_j,s_{j+1}(z),\dots,s_k(z);z) \in \Omega_1^{(j+1,\dots,k)}\}$$

where $\Omega_1^{(j+1,\dots,k)}$ denotes the ε-neighborhood of Ω_1 with respect to open polydiscs in the $(\theta_{j+1},\dots,\theta_k)$-space, i.e.,

$$\Omega_1^{(j+1,\dots,k)} \overset{\text{def}}{=\!=} \{\theta \,|\, \theta_m = \theta_m^0 \text{ for } m=1,\dots,j,k+1,\dots,n \text{ and}$$
$$|\theta_m - \theta_m^0| < \varepsilon \text{ for } m=j+1,\dots,k \text{ with } \theta^0 \in \Omega_1\}.$$

In general $\Omega_1^{(j+1,\ldots,k)}$ is not pseudoconvex and we denote by $H(\Omega^{(j+1,\ldots,k)})$ the smallest, open, pseudoconvex set containing it. Then we define

$$\tilde{\Omega}[j] \overset{\text{def}}{=\!=} \{\theta[j] \,|\, (\theta_1,\ldots,\theta_j,s_{j+1}(z),\ldots,s_k(z);z) \in H(\Omega_1^{(j+1,\ldots,k)})\},$$

which according to [30, th. 2.5.14] is pseudoconvex. If we show that under the projection $\pi_j\colon \theta[j] \to \theta[j-1]$

$$\pi_j(\tilde{\Omega}[j] \cap \{\theta[j] \,|\, |\theta_j - s_j(z)| < \varepsilon\}) \subset \tilde{\Omega}[j-1] \tag{3.17}$$

the stated conjecture follows.

Now

$$\pi_j(\tilde{\Omega}[j] \cap \{\theta[j] \,|\, |\theta_j - s_j(z)| < \varepsilon\}) = \{\theta[j-1] \,|\, (\theta_1,\ldots,\theta_{j-1}, s_j(z),\ldots,s_k(z);z) \in (H(\Omega_1^{(j+1,\ldots,k)}))^{(j)}$$

where $\Omega^{(j)}$ denotes the open ε-neighborhood of a domain Ω with respect to discs in the θ_j-plane. Let $\Omega_{(j)}$ denote the open ε-shrinking of Ω with respect to discs in the θ_j-plane, i.e.,

$$\Omega_{(j)} \overset{\text{def}}{=\!=} \{z \in \Omega \,|\, (z_1,\ldots,z_j + w_j,\ldots,z_n) \in \Omega \text{ if } |w_j| \le \varepsilon\}.$$

If Ω is pseudoconvex $\Omega^{(j)}$, in general, is not, but $\Omega_{(j)}$ is pseudoconvex (a similar proof to that of [57, p.97, Satz 7] shows that $\Omega_{(j)}$ is pseudoconvex in every direction and according to [57, p.111-112 Korollar 14.1] $\Omega_{(j)}$ is pseudoconvex). Thus $(H(\Omega_1^{(j,\ldots,k)}))_{(j)}$ is pseudoconvex and clearly $\Omega_1^{(j+1,\ldots,k)} \subset (\Omega_1^{(j,\ldots,k)})_{(j)} \subset (H(\Omega_1^{(j,\ldots,k)}))_{(j)}$. Accordingly $H(\Omega_1^{(j+1,\ldots,k)}) \subset (H(\Omega_1^{(j,\ldots,k)}))_{(j)}$ and hence

$$(H(\Omega_1^{(j+1,\ldots,k)}))^{(j)} \subset ((H(\Omega_1^{(j,\ldots,k)}))_{(j)})^{(j)} \subset H(\Omega_1^{(j,\ldots,k)}), \tag{3.18}$$

which implies (3.17). Therefore, G_j is defined in $\tilde{\Omega}[j]$ if G_{j-1} is defined in $\tilde{\Omega}[j-1]$.

By (3.14) we have $\Omega[0] \subset \Omega_2'$ and since Ω_2' is pseudoconvex, we get $\tilde{\Omega}[0] \subset \Omega_2'$. Therefore, G_0 is holomorphic in $\tilde{\Omega}[0]$. Thus G_j is holomorphic in $\tilde{\Omega}[j]$ if G_{j-1} is holomorphic in $\tilde{\Omega}[j-1]$ and if U_j satisfies

$$(3.19) \qquad \bar{\partial} U_j(\theta[j]) = g_j(\theta[j]) \overset{\text{def}}{=\!=} G_{j-1}(\theta[j-1])\psi'(p_j(\theta_j;z))/p_j(\theta_j;z)$$

in $\tilde{\Omega}[j]$. Then F is holomorphic in $\tilde{\Omega}[k] = \Omega[k] = \Omega_1$. Since by assumption G_{j-1} is holomorphic in $\tilde{\Omega}[j-1]$, $1/p$ is holomorphic outside any neighborhood of zero, $\psi'(p) = 0$ in a neighborhood of zero and since $\bar{\partial}\psi'(p_j(\theta_j;z)) = \bar{\partial}\bar{\partial}\psi(p_j(\theta_j;z)) = 0$ (because ψ is a C^2-function), we get $\bar{\partial} g_j = 0$ in $\tilde{\Omega}[j]$. Furthermore, let u_j be the analytic map of $\mathbb{C}^{j+n-k}$ into $\mathbb{C}^n$ given by

$$u_j(\theta[j]) \overset{\text{def}}{=\!=} (\theta_1 + w_1, \ldots, \theta_j + w_j, s_{j+1}(z), \ldots, s_k(z); z)$$

for some $w \in \mathbb{C}^j$ with $|w_m| \le \varepsilon$, $m = 1,\ldots,j$. Then by (3.18) $u_j(\tilde{\Omega}[j]) \subset \subset H(\Omega_1^{(1,\ldots,k)}) \subset \Omega_2$ and therefore a function ϕ_j can be defined on $\tilde{\Omega}[j]$ by

$$\phi_j(\theta[j]) \overset{\text{def}}{=\!=} \max\{\phi(u_j(\theta[j])) \mid |w_m| \le \varepsilon,\ m = 1,\ldots,j\}.$$

For each $w \in \mathbb{C}^j$ with $|w_m| \le \varepsilon$ for $m = 1,\ldots,j$ the function $\phi(u_j(\theta[j]))$ is plurisubharmonic in $\tilde{\Omega}[j]$, cf. [30, th. 2.6.4] and if we show that ϕ_j is upper semicontinuous, it follows from [30, th. 1.6.2] that ϕ_j is plurisubharmonic in $\tilde{\Omega}[j]$. Assuming this for the moment we continue the proof of theorem 3.1.

All the conditions of [30, th. 4.4.2] are satisfied now and this theorem gives a solution U_j of (3.19) in $\tilde{\Omega}[j]$ with

$$\int_{\tilde{\Omega}[j]} |U_j(\theta[j])|^2 \frac{\exp - \phi_j(\theta[j])}{(1+\|\theta[j]\|^2)^{3j-1}}\, d\lambda(\theta[j]) \le$$

$$\le \int_{\tilde{\Omega}[j]} |g_j(\theta[j])|^2 \frac{\exp-\phi_j(\theta[j])}{(1+\|\theta[j]\|^2)^{3(j-1)}}\, d\lambda(\theta[j]).$$

Next we estimate G_j in terms of G_{j-1}, using $(a+b)^2 \le 2a^2 + 2b^2$, $|p_j(\theta_j;z)|^2/(1+\|\theta[j]\|^2) \le M$ depending on s_j and $\phi_j(\theta[j]) \ge \phi_{j-1}(\theta[j-1])$ for every θ_j with $|\theta_j - s_j(z)| < \varepsilon$:

$$\int_{\tilde{\Omega}[j]} |G_j(\theta[j])|^2 \frac{\exp - \phi_j(\theta[j])}{(1+\|\theta[j]\|^2)^{3j}}\, d\lambda(\theta[j]) \le$$

$$\leq 2\pi\varepsilon^2 \int_{\tilde{\Omega}[j-1]} |G_{j-1}(\theta[j-1])|^2 \frac{\exp - \phi_{j-1}(\theta[j-1])}{(1+\|\theta[j-1]\|^2)^{3(j-1)}} d\lambda(\theta[j-1]) +$$

$$+ 2M \int_{\tilde{\Omega}[j]} |g_j(\theta[j])|^2 \frac{\exp-\phi_j(\theta[j])}{(1+\|\theta[j]\|^2)^{3(j-1)}} d\lambda(\theta[j]) \leq$$

$$\leq \frac{8M\pi K^2+2\pi\varepsilon^4}{\varepsilon^2} \int_{\tilde{\Omega}[j-1]} |G_{j-1}(\theta[j-1])|^2 \frac{\exp - \phi_{j-1}(\theta[j-1])}{(1+\|\theta[j-1]\|^2)^{3(j-1)}} d\lambda(\theta[j-1]).$$

Since $G_k = F$, $\tilde{\Omega}[k] = \Omega[k] = \Omega_1$, $G_0 = f$ and $\tilde{\Omega}[0] \subset \Omega_2'$, (3.16) follows. □

We still have to show the following lemma.

LEMMA 3.2. *Let ϕ be an upper semicontinuous function in a domain $\Omega \subset \mathbb{R}^n$. Let S be a compact neighborhood of the origin in $\mathbb{R}^n$ and let $\Omega_1 \subset \Omega$ be a domain such that $\{x | x = x_1 + w, x_1 \in \Omega_1, w \in S\} \subset \Omega$. Then the function ϕ_1 on Ω_1 given by*

(3.20) $$\phi_1(x) \stackrel{\text{def}}{=} \max_{w\in S} \phi(x+w)$$

is upper semicontinuous.

PROOF. First we show that an upper semicontinuous function f in a domain U attains a maximum on a compact set $K \subset U$. Let $M \stackrel{\text{def}}{=} \sup_{x\in K} f(x)$ and let $\{M_k\}_{k=1}^{\infty}$ be an increasing sequence with $M_k \uparrow M$. The sets $U_k \stackrel{\text{def}}{=} \{x \in U | f(x) < M_k\}$ are open and if there is no $x_0 \in K$ with $f(x_0) = M$ we have $K \subset \bigcup_{k=1}^{\infty} U_k$. Since K is compact, there is a number m with $K \subset \bigcup_{k=1}^{m} U_k$. This implies $f(x) < M_m < M$ for $x \in K$, contrarily to the definition of M. Thus there is $x_0 \in K$ with $f(x_0) = M$. Hence definition (3.20) (and also the definiiton of ϕ_ε in theorem 3.1) is a good definition.

Now let $x_0 \in \{x | \phi_1(x) < c\} \cap \Omega_1$, then $\phi(x_0 + x) < c$ for $x \in S$. Since ϕ is upper semicontinuous, there is an open neighborhood U of S with $\phi(x_0+x) < c$ for $x \in U$. In particular, since S is compact, there is $\varepsilon > 0$ such that $\phi(x_0+x+w) < c$ for $w \in S$ and $\|x\| < \varepsilon$. Since an upper semicontinuous function attains a maximum on a compact set, it follows from (3.20) that the set $\{x \in \Omega_1 | \phi_1(x) < c\}$ is open and thus ϕ_1 is upper semicontinuous in Ω_1. □

Applying theorem 3.1 for obtaining (3.10) and (3.11) we get the following result.

THEOREM 3.3. *Let for* $\alpha = \varepsilon$ *and* $\alpha = c$ *the space* A_α *of holomorphic functions in the unbounded convex neighborhoods* Ω_α^k *of* $\Omega(a,\Gamma)$ *be defined by (3.5) and let* Exp_α *be defined by (3.2.i) and (3.2.ii). Then the map (3.6)* $\mathcal{F}:A'_\alpha \to \mathrm{Exp}_\alpha$*, given by (3.7), is surjective for* $\alpha \in \{\varepsilon, c\}$.

III.2. ANALYTIC FUNCTIONALS ON EXPONENTIALLY DECREASING TESTFUNCTIONS; FOURIER TRANSFORMATION AS AN INJECTION.

In this section we state the problem whose solution implies the injectivity of the map (3.6).

In formula (3.13) we have embedded A_α^k into the space

$$(3.21) \qquad S_\alpha^{k+1} \stackrel{\text{def}}{=\!=\!=} \operatorname*{proj\,lim}_{m\to\infty} S_\alpha(m,k+1)$$

cf. (2.57), which is a weakly compact projective sequence. Another possibility is to take instead of A_α^k, defined by (3.4), the subspace $\hat{A}_\alpha^k$ of S_α^k consisting of those elements $\phi \in S_\alpha^k$ with $\bar{\partial}\phi = 0$, where $\bar{\partial}$ is the Cauchy-Riemann operator. Then any element $\mu^k \in (S_\alpha^k)'$ that satisfies $\mu^k = \bar{\partial}^t \vec{\sigma}^k$ for some $\vec{\sigma}^k \in ((S_\alpha^k)')^n$ vanishes on $\hat{A}_\alpha^k$. Therefore we define equivalent classes of sequences $\{\mu^k\}$ with $\mu^k \in (S_\alpha^k)'$ where two sequences $\{\mu_1^k\}$ and $\{\mu_2^k\}$ are equivalent if for every k there is $\vec{\sigma}^k \in ((S_\alpha^k)')^n$ with $\mu_1^k - \mu_2^k = \bar{\partial}^t \vec{\sigma}^k$. Since also

$$(3.22) \qquad A_\alpha = \operatorname*{ind\,lim}_{k\to\infty} \hat{A}_\alpha^k$$

where A_α is defined by (3.5), the elements of A'_α can be identified with the equivalent classes of such sequences $\{\mu^k\}$ that for any k and p there is a $\vec{\sigma}^{k,p} \in ((S_\alpha^m)')^n$ with $\mu^k - \mu^p = \bar{\partial}^t \vec{\sigma}^{k,p}$ in $(S_\alpha^m)'$ where $m = \min(k,p)$.

The space (3.22) is defined by a weakly compact, injective sequence because an open set in $\hat{A}_\alpha^k$ is bounded in $\hat{A}_\alpha^{k+1}$ and hence relatively weakly compact, for the space (3.21) is reflexive, cf. [65, th. 36.3]. Therefore, cf. [40, th. 12] the strong dual of (3.22) equals

$$(3.23) \qquad A'_\alpha = \operatorname*{proj\,lim}_{k\to\infty} (\hat{A}_\alpha^k)'.$$

By [40, th. 13] we have

$$(\hat{A}^k_\alpha)' = (S^k_\alpha)'/(\hat{A}^k_\alpha)^0$$

where $(\hat{A}^k_\alpha)^0$ denotes the annihilator of $\hat{A}^k_\alpha$. Furthermore, $\hat{A}^k_\alpha$ is the kernel of the continuous map $\bar{\partial}_k \overset{\text{def}}{=\!=} (\partial/\partial\bar{\zeta}_1,\dots,\partial/\partial\bar{\zeta}_n)$

$$\bar{\partial}_k\colon S^k_\alpha \to (S^k_\alpha)^n,$$

so that according to [65, prop. 35.4] $(\hat{A}^k_\alpha)^0$ is the weak* closure (cf. footnote on page 85) in $(S^k_\alpha)'$ of the range of the transposed map $\bar{\partial}^t_k$ of $\bar{\partial}_k$. Since S^k_α is reflexive the weak* closure of this range equals the closure in the strong topology, cf. [65, prop. 35.2]. We denote the closure in $(S^k_\alpha)'$ of the range of the map

$$T_k \overset{\text{def}}{=\!=} \bar{\partial}^t_k\colon ((S^k_\alpha)')^n \to (S^k_\alpha)'$$

by $\overline{R(T_k)}$. Hence we have

$$(\hat{A}^k_\alpha)' = (S^k_\alpha)'/\overline{R(T_k)}\ . \tag{3.24}$$

According to lemma 2.27 for every k there is a p > k such that the following maps are continuous

$$\begin{cases} F\colon (S^p_\alpha)' \to H^k_\alpha \\ F^{-1}\colon H^{k+1}_\alpha \to (S^k_\alpha)', \end{cases} \tag{3.25}$$

where

$$H^k_\alpha \overset{\text{def}}{=\!=} \operatorname{ind}\lim_{m\to\infty} H_\alpha(m,k)$$

with $H_\alpha(m,k)$ defined by (2.57), and where F is defined by a formula like (3.12). Let $P \overset{\text{def}}{=\!=} (\theta_1 - i\theta_{n+1},\dots,\theta_n - i\theta_{2n})$ and let $P\cdot\vec{H}^k_\alpha$ be the subspace of H^k_α consisting of functions F which can be written as

$$F(\theta) = \sum_{j=1}^{n} (\theta_j - i\theta_{n+j})G_j(\theta)$$

with $G_j \in H^k_\alpha$, j = 1,...,n. Then

$$(3.26) \qquad F\colon \overline{R(T_p)} \to \overline{P\cdot\vec{H}^k_\alpha}, \quad F^{-1}\colon \overline{P\cdot\vec{H}^{k+1}_\alpha} \to \overline{R(T_k)}.$$

Now by (3.23), (3.24), (3.25) and (3.26) the maps (3.25) induce an isomorphism F between

$$(3.27) \qquad F\colon A'_\alpha \to \operatorname*{proj\,lim}_{k\to\infty} (H^k_\alpha/\overline{P\cdot\vec{H}^k_\alpha}).$$

Furthermore, for each k there is a $p > k$ such that

$$\overline{P\cdot\vec{H}^p_\alpha} \subset P\cdot\vec{H}^k_\alpha$$

is a continuous injection, for let $F_\beta \in P\cdot\vec{H}^p_\alpha$ be a Cauchy net converging to $F \in H^p_\alpha$. Then $F_\beta = P\cdot\vec{G}_\beta$ with $\vec{G}_\beta \in (H^p_\alpha)^n$, so that F_β, and hence F, vanishes on the set

$$V^p_\alpha \overset{\text{def}}{=\!=} \{\mathbb{R}^{2n} + i\Gamma^p_\alpha\} \cap \{\theta \,|\, \theta_j - i\theta_{n+j} = 0,\ j = 1,\dots,n\}.$$

The inclusion follows if we have solved the following problem.

<u>PROBLEM</u> 3.1. For each k there is a $p > k$ such that a function $F \in H^p_\alpha$ vanishing on V^p_α can be written as

$$F(\theta) = P\cdot\vec{G}(\theta), \quad \theta \in \mathbb{R}^{2n} + i\Gamma^k_\alpha$$

with $\vec{G} \in (H^k_\alpha)^n$.

Assuming that this problem has been solved we have the following commutative diagram of continuous maps

$$\begin{array}{ccc} H^p_\alpha/\overline{P\cdot\vec{H}^p_\alpha} & \longrightarrow & H^k_\alpha/\overline{P\cdot\vec{H}^k_\alpha} \\ & \searrow \quad \nearrow & \\ & H^k_\alpha/P\cdot\vec{H}^k_\alpha & \end{array} \quad ;$$

here the upper spaces are Hausdorff spaces, but in the lower space we do not have to bother about the closure. Anyhow, this implies that

$$(3.28)\qquad H_\alpha \overset{\text{def}}{=\!=} \operatorname*{proj\,lim}_{k\to\infty} (H^k_\alpha/\overline{P\cdot\vec{H}^k_\alpha}) = \operatorname*{proj\,lim}_{k\to\infty} (H^k_\alpha/P\cdot\vec{H}^k_\alpha)$$

and this is always a Hausdorff space. Its elements can be described as follows, cf. [20, §6.2]: define equivalence classes of sequences $\{F^k\}$ with $F^k \in H^k_\alpha$, where $\{F^k\} \sim \{H^k\}$ if $F^k(\theta) - H^k(\theta) = P(\theta)\cdot\vec{G}^k(\theta)$ for $\theta \in \mathbb{R}^{2n} + i\Gamma^k_\alpha$ and for $\vec{G}^k \in \vec{H}^k_\alpha$; then the elements of H_α are the equivalence classes of such sequences $\{F^k\}$ that for every k and p there is a $\vec{G}^{k,p} \in \vec{H}^m_\alpha$ with

$$(3.29)\qquad F^k(\theta) - F^p(\theta) = P(\theta)\cdot G^{k,p}(\theta),\quad \theta \in \mathbb{R}^{2n} + i\Gamma^m_\alpha,\quad m = \min(k,p).$$

We have to solve problem 3.1 anyway, so we don't pay attention to the closure of $P\cdot\vec{H}^k_\alpha$ in H^k_α and (3.28) is valid. Since $P\cdot\vec{H}^k_\alpha$ vanishes on V^k_α we can define continuous restriction maps I^k

$$I^k\colon H^k_\alpha/P\cdot\vec{H}^k_\alpha \to H^k_\alpha|V^k_\alpha.$$

Here $H^k_\alpha|V^k_\alpha$ is the space of restrictions of functions in H^k_α to V^k_α with the topology induced by H^k_α. Then I^k is surjective. Furthermore, there is a natural continuous injection J^k

$$J^k\colon H^k_\alpha|V^k_\alpha \to H_\infty(\Gamma^k_\alpha; a^\alpha_{(k)}(z) + 1/_{k-1}\|z\|)$$

defined by $(J^kF)(z) \overset{\text{def}}{=\!=} F(iz,z)$. Hence we can complete (3.27) as

$$(3.30)\qquad A'_\alpha \xrightarrow{F} H_\alpha \xrightarrow{I} \operatorname*{proj\,lim}_{k\to\infty} (H^k_\alpha|V^k_\alpha) \xrightarrow{J} \mathrm{Exp}_\alpha,$$

so that $J\circ I\circ F$ is the map F defined by (3.7). Indeed, by (3.29) if $\{F^k\} \in H_\alpha$ then for $p \geq k$ and for $\theta \in V^k_\alpha$ we have $F^p(\theta) = F^k(\theta)$. Hence the elements of $\operatorname*{proj\,lim}_{k\to\infty} (H^k_\alpha|V^k_\alpha)$ are just those functions f on

$$V \overset{\text{def}}{=\!=} \bigcup_k V^k_\alpha = \{\mathbb{R}^{2n} + i\Gamma\} \cap \{\theta\,|\,\theta_j - i\theta_{n+j} = 0,\ j = 1,\dots,n\}$$

such that for any k there is a $F^k \in H^k_\alpha$ with

$$(3.31)\qquad F^k(\theta^1,\theta^2) = f(\theta^2),\quad (\theta^1,\theta^2) \in V^k_\alpha.$$

Thus J is defined similarly to J^k and J is injective.

Theorem 3.1 shows that the map J is surjective. However, the by $\{I^k\}$ induced map I is a priori not surjective, although each I^k is surjective. We have the following commutative diagram

$$\begin{array}{ccc} H^p_\alpha / P\cdot\vec{H}^p_\alpha & \xrightarrow{\quad I^p \quad} & H^p_\alpha | \mathcal{V}^p_\alpha \\ \Big\downarrow \alpha_{p,k} & & \Big\downarrow \beta_{p,k} \\ H^k_\alpha / P\cdot\vec{H}^k_\alpha & \xrightarrow[\quad I^k \quad]{} & H^k_\alpha | \mathcal{V}^k_\alpha \end{array}$$

where $\alpha_{p,k}$ and $\beta_{p,k}$ denote the restriction maps. Hence the range of I in $\operatorname{proj}\lim_{k\to\infty} (H^k_\alpha | \mathcal{V}^k_\alpha)$ consists of those f on $\mathcal{V}$ which, besides (3.31) for $F^k \in H^k_\alpha$, moreover satisfy (3.29). The solution of problem 3.1 implies that I is injective and surjective (actually it says that $\operatorname{Ker} I^p \subset \operatorname{Ker} \alpha_{p,k}$)[1].

$\mathcal{V}$ is defined as the simultaneous zero-set of the polynomials $p_j \overset{\text{def}}{=\!=} \theta_j - i\theta_{n+j}$, $j = 1,\dots,n$. These polynomials generate a prime ideal in any point of a pseudoconvex, open set $\Omega \subset \mathbb{C}^{2n}$. Therefore, according to Hilbert's Nullstellensatz, see [27, ch.III. A], every holomorphic function f in Ω vanishing on $\mathcal{V}$ can locally, that is in a neighborhood ω of any point in Ω, be written as

$$(3.32) \qquad f = P\cdot\vec{g}_\omega, \quad \vec{g}_\omega \in A(\omega)^n,$$

where $A(\omega)$ is the set of holomorphic functions in ω. With the aid of Cartan's theorem B it can be shown, see for example [27] or [30, th. 7.2.9 & th. 7.4.3],

---

1) If we do not assume that problem 3.1 has been solved, it still might happen that I is surjective without its injectivity being established and this is actually the case here. Indeed, in section III.1 we have shown that for any $f \in \operatorname{proj}\lim_{k\to\infty} (H^k_\alpha | \mathcal{V}^k_\alpha)$ there is a $\mu \in A'_\alpha$ with $\mathcal{F}(\mu) = Jf$, where $\mathcal{F}$ is given by (3.7). But if we apply the maps $\mathcal{F}$ and I in (3.30) successively, we get $f = I\bullet\mathcal{F}\,\mu \in R(I)$. Hence I is surjective. This means that for any sequence $\{\tilde{F}^k\}$ with $\tilde{F}^k \in H^k_\alpha$ and $\tilde{F}^p - \tilde{F}^k = 0$ on $\mathcal{V}^k_\alpha$ for all k and $p \geq k$, there exists another sequence $\{F^k\}$ with $F^k \in H^k_\alpha$ satisfying (3.29) and with $F^k - \tilde{F}^k = 0$ on $\mathcal{V}^k_\alpha$. However, here we are not interested in the surjectivity of I, i.e., in the above solved statement, but in the injectivity of I, i.e., in problem 3.1.

that $f \in A(\Omega)$ satisfying (3.32) can be written globally as

$$f = P \cdot \vec{g}, \quad \vec{g} \in A(\Omega)^n.$$

Problem 3.1 asks for a function $\vec{G}$ which satisfies almost the same growth conditions as F, so it is the analogue with estimates of the above mentioned problem. If $\Omega = \mathbb{C}^n$ this problem is solved in [30, th. 7.6.11] and in chapter VI we will perform the same method of proof, but there we have to take care of the estimates near the boundary of Ω. For the general case, as in theorem 3.1, all conditions, besides the one that ϕ is plurisubharmonic in the density $\exp - \phi$, will be discussed precisely in the next chapter.

Since problem 3.1 implies the injectivity of F, its definition (3.7) implies the following corollary.

COROLLARY 3.4. *The set* $\{e^{i<\zeta,z>} | z \in \Gamma\}$ *is dense in the spaces* A_α *given by (3.5) for* $\alpha = \varepsilon$ *or* $\alpha = c$.

REMARK. Since F is surjective, F^t: $\mathrm{Exp}'_\alpha \to A_\alpha$ is injective, where F^t is given by

$$(F^t\sigma)(\zeta) = <\sigma_z, e^{i<\zeta,z>}>, \quad \sigma \in \mathrm{Exp}'_\alpha \quad ,$$

because for $\mu \in A'_\alpha$

$$<\mu, F^t\sigma> = <\sigma, F\mu> = <\sigma_z, <\mu_\zeta, e^{i<\zeta,z>}>> = <\mu_\zeta, <\sigma_z, e^{i<\zeta,z>}>>$$

by Fubini's theorem. Hence also the set $\{e^{i<\zeta,z>} | \zeta \in \Omega(a,\Gamma)\}$ is dense in Exp_α for both $\alpha = \varepsilon$ and $\alpha = c$.

So finally, we have obtained the following theorem.

THEOREM 3.5. *The map* F *of theorem 3.3 is also injective.*

REMARK. Theorems 3.3 and 3.5 state that the map (3.6) is bijective. This fact can be considered as a generalization of the Ehrenpreis-Martineau theorem, which gives the isomorphism (3.6) for $\alpha = \varepsilon$ if Ω is compact and $\Gamma = \mathbb{C}^n$, just as the Paley-Wiener theorems of chapter II, cf. also [68, § 26.4, th. 2], can be considered as a generalization of the original Paley-Wiener-Schwartz theorem for distributions with compact support.

III.3. PALEY-WIENER THEOREMS FOR FOURIER HYPERFUNCTIONS.

In this section we treat the particular case of theorems 3.3 and 3.5 where $\Gamma = T^C$ with C an open, convex cone in $\mathbb{R}^n$. Again as a particular case of this situation we may consider functions a(z) which are only functions of y = Im z. Then $\Omega(a,T^C)$ is a subset of $\mathbb{R}_n$ and a function in Exp_ε determines a Fourier hyperfunction.

Let $(T^C)_k$ and $(T^C)(k)$ be given by (2.50) and (2.51), respectively. If in (3.2.i), (3.2.ii) and (3.5) $\Gamma = T^C$, we get the spaces

$$(3.33)\quad \begin{cases} \mathrm{Exp}_\varepsilon[a(z),T^C] \overset{\text{def}}{=\!=} \operatorname{proj}\lim\limits_{k\to\infty} H_\infty(T^{1/ky_0+C}; a(x,y-1/_{2k}y_0) + 1/k\|z\|) \\ A_\varepsilon(a,T^C) \overset{\text{def}}{=\!=} \operatorname{ind}\lim\limits_{k\to\infty} H_\infty(\Omega(a+1/k,T^C); -1/k\|\zeta\|) \end{cases}$$

where $y_0 \in \mathrm{pr}\, C_1$ is fixed, and

$$(3.34)\quad \begin{cases} \mathrm{Exp}_c[a(z),T^C] \overset{\text{def}}{=\!=} \operatorname{proj}\lim\limits_{k\to\infty} H_\infty((T^C)(k); a(z)+1/k\|z\|) \\ A_c(a,T^C) \overset{\text{def}}{=\!=} \operatorname{ind}\lim\limits_{k\to\infty} H_\infty(\Omega(a+1/k,(T^C)_k); -1/k\|\zeta\|). \end{cases}$$

By theorems 3.3 and 3.5, in both pairs of spaces Fourier transformation is an isomorphism from the strong dual of the second space onto the first space. Similarly, the same statement can be derived for the following pair of spaces, where we have a mixture of the two foregoing cases, namely analytic functionals carried by $\Omega(a,T^C)$ with respect to ε-neighborhoods in the imaginary directions and to conic neighborhoods in the real directions:

$$(3.35)\quad \begin{cases} \mathrm{Exp}_{\varepsilon,c}[a(z),T^C] \overset{\text{def}}{=\!=} \operatorname{proj}\lim\limits_{k\to\infty} H_\infty(T^{1/k\,y_0+C_k}; a(x,y-1/_{2k}y_0) + \\ \qquad\qquad + 1/k\|z\|) \\ A_{\varepsilon,c}(a,T^C) \overset{\text{def}}{=\!=} \operatorname{ind}\lim\limits_{k\to\infty} H_\infty(\Omega(a+1/k,T^{C_k}); -1/k\|\zeta\|). \end{cases}$$

Thus we obtain the following theorem.

<u>THEOREM</u> 3.6. *In the pairs of spaces (3.33), (3.34) and (3.35) the strong dual of the second space is topologically isomorphic to the first space by means of the map $\mathcal{F}$ defined by (3.7).*

The pair (3.33) will be used in chapter V to derive the Newton interpolation series for functions in $\mathrm{Exp}_\varepsilon[a(z),T^C]$, if lim a(x,y) as $y \to 0$, $y \in C_k$

exists for every k, i.e., if $\Omega(a,T^C)$ is bounded in the imaginary directions.

If the convex, homogeneous function a is only a function of $y \in C$, i.e., if $a(z) = a(y)$ then

$$\Omega(a,T^C) \subset \{\zeta \mid \zeta = \xi + i\eta, \eta = 0\}.$$

In that case for each k every function f in $\mathrm{Exp}_\varepsilon[a(y),T^C]$ or in $\mathrm{Exp}_{\varepsilon,c}[a(y),T^C]$ satisfies

$$|f(z)| \le K_k \exp 1/k\|x\|, \quad y \in C_k, \quad 1/k \le \|y\| \le k$$

for some positive constants K_k depending on k and f. Hence it determines a Fourier hyperfunction, see [38]. Then theorem 3.6 is the Paley-Wiener theorem for Fourier hyperfunctions:

i. The elements of $\mathrm{Exp}_{\varepsilon,c}[a(y),T^C]$ are just the Fourier hyperfunctions which are the Fourier transforms of the Fourier hyperfunctions with support in $\Omega(a,T^C)$, where the support is defined as the smallest carrier with respect to conic neighborhoods $\Omega(a+1/k,T^{C_k})$ in the real directions, which is done in [38].

ii. The elements of $\mathrm{Exp}_\varepsilon[a(y),T^C]$ may be considered as the Fourier transforms of the Fourier hyperfunctions with support in $\Omega(a,T^C)$, where this kind of support with respect to ε-neighborhoods is defined by means of definition 2.6.

iii. In [53] analytic functionals carried by real sets with respect to conic neighborhoods in $\mathbb{C}^n$ are mentioned. They are called Fourier hyperfunctions of the second kind and they seem to be more useful for describing quantum field theory. In this view the elements of $\mathrm{Exp}_c[a(y),T^C]$ are the Fourier hyperfunctions of the second kind which are the Fourier transforms of the Fourier hyperfunctions of the second kind with support in the set $\Omega(a,T^C)$, where this kind of support is defined with the aid of conic neighborhoods.

III.4. ANALYTIC FUNCTIONALS IN $Z'_{\{M\}}$; FOURIER TRANSFORMATION AS A BIJECTION; PALEY-WIENER THEOREMS FOR ULTRADISTRIBUTIONS OF ROUMIEU TYPE.

In this section we shall mention the problems which have to be solved in order that the Ehrenpreis-Martineau theorem can be extended to analytic functionals in $Z'_{\{M\}}$ carried by unbounded, convex sets with respect to various classes of neighborhoods. Now we no longer exhaust an open, pseudoconvex set Γ by sets $\{\Gamma_\alpha^k\}_{k=1}^\infty$ such that an ε-neighborhood of Γ_α^k is contained in Γ_α^{k+1} as in problem 3.1. In this section we shall get problems similar to theorem 3.1 and problem 3.1, but with estimates extending to the boundary of the domain.

As in section II.2.iii we require that M is a continuous, increasing, piecewise differentiable function on $[0,\infty)$ with $M(0) = 0$, $M(\infty) = \infty$, such that M' is strictly decreasing. Furthermore, in this and the following section we only require that (2.31) is valid. Then M^*, defined by (2.28), is a convex function on $(0,\infty)$ with $M^*(0) = \infty$ and $M^*(\infty) = 0$, satisfying (2.29) and (2.31). Briefly, the following formula's hold:

$$M^*(\sigma) = \max_{\rho>0} \{M(\rho) - \sigma\rho\} \tag{3.36}$$

$$M(\rho) = \min_{\sigma>0} \{M^*(\sigma) + \rho\sigma\}; \tag{3.37}$$

$\forall t > 0$, $\forall m > 0$, $\exists t' \geq t$, $\exists K > 0$ and $\forall t' > 0$, $\forall m > 0$, $\exists t$ with $0 < t \leq t'$, $\exists K > 0$

such that for $\rho \geq 1$ and $0 < \sigma \leq 1$

$$\begin{cases} M(\rho/t') + m\log\rho \leq M(\rho/t) + K \\ M^*(t'\sigma) + m\log 1/\sigma \leq M^*(t\sigma) + K. \end{cases} \tag{3.38}$$

We shall first describe the analogue of sections III.1 and III.2, but now with $\Gamma = T^C$. This will yield the most general setting of the problems to be solved. Next we shall state the Paley-Wiener type theorems and, for arbitrary cones Γ, the Ehrenpreis-Martineau theorem. Let C be an open, convex cone in $\mathbb{R}^n$, let for $\alpha = \varepsilon$ and $\alpha = c$ $(T^C)_\alpha^k$ be given by (2.52.i) and (2.52.ii), Ω_α^k by (2.48.i) with Γ replaced by T^C and by (2.48.ii) with Γ_k replaced by $(T^C)_k$, defined in (2.50), and let a_α^k be given by (2.54.i) and (2.54.ii), respectively. Then we define the following pair of spaces

$$\begin{cases} \mathrm{Exp}_\alpha[a,T^C;M^*] \overset{\mathrm{def}}{=\!=} \operatorname{proj}\lim_{k\to\infty} H_\infty((T^C)_\alpha^k; a_\alpha^k(z) + 1/k\|z\| + M^*(k\|y\|)) \\ A_\alpha(a,T^C;M) \overset{\mathrm{def}}{=\!=} \operatorname{ind}\lim_{k\to\infty} H_\infty(\Omega_\alpha^k; -M(\|\xi\|/k) + k\|\eta\|). \end{cases} \tag{3.39}$$

By lemma 2.17 each $f \in \mathrm{Exp}_\alpha[a,T^C;M^*]$ determines an ultradistribution of Roumieu type.

As in section III.2 formula (3.21), here too we introduce an S-space of C^{∞}-functions. In this section for $\alpha \in \{\varepsilon, c\}$ we denote by S_{α}^{k} the space

$$S_{\alpha}^{k} \overset{\text{def}}{=\!=} \underset{m \to \infty}{\text{proj lim}}\ S_{\alpha}(m,k,k)$$

where $S_{\alpha}(m,k,k)$ is defined by (2.56) and again we write the strong dual of $A_{\alpha}(a,T^{C};M)$ as

$$A_{\alpha}(a,T^{C};M)' = \underset{k \to \infty}{\text{proj lim}}\ (S_{\alpha}^{k})'/\overline{R(T_{k})}$$

where T_k is the transposed of the Cauchy-Riemann operator. Let us now denote by H_{α} the space

$$H_{\alpha} \overset{\text{def}}{=\!=} \underset{k \to \infty}{\text{proj lim}}\ (H_{\alpha}^{k}/\overline{P \cdot H_{\alpha}^{k}})$$

where $H_{\alpha}^{k} \overset{\text{def}}{=\!=} \underset{m \to \infty}{\text{ind lim}}\ H_{\alpha}(m,k,k)$, cf. (2.56). Then by lemma 2.27 the Fourier transformation $\mathcal{F}$ is an isomorphism

$$\mathcal{F}: A_{\alpha}(a,T^{C};M)' \to H_{\alpha}.$$

As before, the maps I and J are introduced

$$H_{\alpha} \xrightarrow{\ I\ } \underset{k \to \infty}{\text{proj lim}}\ (H_{\alpha}^{k}|V_{\alpha}^{k}) \xrightarrow{\ J\ } \mathrm{Exp}_{\alpha}[a,T^{C};M^{*}].$$

We shall investigate which problems have to be solved in order that I is bijective and J surjective.

The bijectivity of I will follow from a problem similar to problem 3.1. It asks for a function $\vec{g} \in A(\Omega)^{n}$ with $P \cdot \vec{g} = f$ if (3.32) is satisfied, where now $\vec{g}$ is holomorphic in the same pseudoconvex domain Ω as f and satisfies some estimates. This is only possible if some conditions are imposed on the densities in the estimates. Therefore, we have to introduce the following concepts. Let Ω be a pseudoconvex domain and let ϕ be a function in Ω such that for each N there exists a plurisubharmonic function $\tilde{\phi}_N$ in Ω which satisfies

$$(3.40) \qquad K + \tilde{\phi}_N(z) \geq \phi_N(z) \overset{\text{def}}{=\!=} \max\{\phi(z') + N\log(1 + \|z'\|^{2}) + \log(1+d(z',\Omega^{C})^{-N}) \mid \\ \mid \|z - z'\| \leq \min[N, (e^{N}-1)d(z,\Omega^{C}), (e^{N}-1)d(z',\Omega^{C})]\}$$

for some $K > 0$ depending on ϕ and N, where $d(z,\Omega^c)$ denotes the distance from z to the complement of Ω. Furthermore, we define the plurisubharmonic function $\hat{\phi}$ by

$$(3.41)\qquad \hat{\phi}(z) \overset{\text{def}}{=\!=} \tilde{\phi}_N(z) + N\log(1+\|z\|^2) + \log(1+d(z,\Omega^c)^{-N}).$$

Then $\hat{\phi}$ satisfies the following inequalities

$$\phi \le \phi_N \le \tilde{\phi}_N + K \le \hat{\phi} + K.$$

Let

$$\psi^{k,m}(\theta) \overset{\text{def}}{=\!=} M^*(k\|\operatorname{Im}\theta^2\|) + a_\alpha^k(\operatorname{Im}\theta) + 1/k\|\operatorname{Im}\theta\| + m\log(1+\|\theta\|^2)$$

if $\alpha = c$ for $\theta \in T((T^C)_c^k)$ or if $\alpha = \varepsilon$ for $\theta \in T((T^C)_\varepsilon^k)$, in which case we complete $\psi^{k,m}$ arbitrarily to the remaining of $T(T^C)$, cf. (2.53) for the definition of $T(B)$. Then in virtue of (3.38) for each q and N there are $p > q$ and $K_q > 0$ such that for $\alpha = \varepsilon$ or $\alpha = c$

$$(\psi^{p,m})_N(\theta) \le \psi^{q,m+N} + K_q, \qquad \theta \in T((T^C)_\alpha^q).$$

For a fixed $\xi_0 \in \operatorname{pr} C^*$ there is $\delta > 0$ such that $\delta\|y\| \le \langle\xi_0,y\rangle \le \|y\|$ for $y \in C$ and therefore, for each k there is a $q > k$ with

$$M^*(q\langle\xi_0,y\rangle) \le M^*(k\|y\|), \qquad y \in C.$$

But now $M^*(q\langle\xi_0,\operatorname{Im}\theta^2\rangle)$ is convex, hence plurisubharmonic, in $T(T^C)$. Hence for each k there is a $p > k$ such that by a suitable choice of $\widehat{(\psi^{p,m})}_N$ we get

$$(3.42)\qquad \widehat{\psi^{p,m}} \le \psi^{k,m+2N}$$

in $T((T^C)_\alpha^k)$.

In the $\alpha = \varepsilon$ case an extra complication arises by the fact that the domain $T((T^C)_\varepsilon^k)$ is not pseudoconvex, because by Bochners theorem its pseudoconvex hull $H(T((T^C)_\varepsilon^k))$ equals $T(T^C)$. Hence every $F \in H_\varepsilon^k$ is holomorphic in $T(T^C)$ and if F vanishes on $\mathcal{V}_\varepsilon^k$, it vanishes on $\mathcal{V}$. Each $F \in H_\varepsilon^p$ satisfies for some m and K

$$|F(\theta)| \le K \exp \psi^{p,m}(\theta), \qquad \theta \in T((T^C)^p_\varepsilon)$$

$$|F(\theta)| \le \exp(\log|F(\theta)|), \qquad \theta \in T(T^C).$$

Then with $\psi(\theta) \overset{\text{def}}{=\!=} \max\{\log|F(\theta)|, \psi^{p,m}(\theta)\}$ for $\theta \in T(T^C)$ F satisfies

$$(3.43) \qquad |F(\theta)| \le K \exp \psi(\theta).$$

Furthermore, we make the restriction that $\psi^{p,m}$ on $T((T^C)^p_\varepsilon)$ has been extended to $T(T^C)$ in such a way that (3.40) can be satisfied for the function ψ of formula (3.43). If $\alpha = c$ and $F \in H^p_c$, we set $\psi = \psi^{p,m}$ for some m depending on F and (3.43) is satisfied for $\theta \in T((T^C)^p_c)$, which is a pseudoconvex domain.

Now assume that for $\alpha = \varepsilon$ and $\alpha = c$ every $F \in H^p_\alpha$ vanishing on V if $\alpha = \varepsilon$ or on V^p_c if $\alpha = c$ and satisfying (3.43) can be written as $F = P\cdot\vec{G}$ for holomorphic functions G_j in $T(T^C)$ if $\alpha = \varepsilon$ or in $T((T^C)^p_c)$ if $\alpha = c$ which satisfy there $G_j(\theta) \le K \exp \hat{\psi}(\theta)$, $j = 1,\dots,n$, where $\hat{\psi}$ is obtained from ψ as in (3.41) for some N. Then if p is sufficiently large there is a k such that in view of (3.42) G_j would belong to H^k_α. If this can be done for every k, the bijectivity of the map I would be implied. Taking into account (3.32) and the embedding maps between spaces with L^2-norms and sup-norms (cf. [73]), we really get the foregoing if the following problem is solved.

<u>PROBLEM</u> 3.2. Let Ω be a pseudoconvex domain, let ϕ be a function in Ω such that (3.40) can be satisfied for every N and let P be a vector of polynomials. If a holomorphic function f in Ω can locally, i.e., in a neighborhood ω of each point in Ω, be written as $f = P\cdot\vec{g}_\omega$ with $\vec{g}_\omega \in \vec{A}(\omega)$, then

$$f(z) = P(z)\cdot\vec{g}(z), \qquad z \in \Omega$$

for some $\vec{g} \in \vec{A}(\Omega)$ satisfying for some K independent of f

$$\int_\Omega \|\vec{g}(z)\|^2 \exp - \hat{\phi}(z)d\lambda(z) \le K \int_\Omega |f(z)|^2 \exp - \phi(z)d\lambda(z)$$

where $\|\vec{g}(z)\|^2 = \Sigma|g_j(z)|^2$ and where $\hat{\phi}$ is given by (3.41) for some N independent of f, provided that f is such that the right hand side is finite.

Since in problem 3.1 an ε-neighborhood of $T(\Gamma^k_\alpha)$ is contained in $T(\Gamma^p_\alpha)$ and since the equalities (3.2.i) and (3.2.ii) hold, problem 3.1 follows from problem 3.2. Furthermore, problem 3.2 implies that (cf. (3.28) where the spaces H^k_α are different from the H^k_α of this section)

$$H_\alpha \overset{\text{def}}{=\!=} \operatorname*{proj\,lim}_{k\to\infty} (\overline{H^k_\alpha/P\cdot\vec{H}^k_\alpha}) = \operatorname*{proj\,lim}_{k\to\infty} (H^k_\alpha/P\cdot\vec{H}^k_\alpha),$$

hence we don't need to pay attention to the closure of $P\cdot\vec{H}^k_\alpha$ in H^k_α.

We will now state the problem whose solution implies the surjectivity of the map J. Theorem 3.1 yields local extensions $\{F_\omega | \omega \subset\subset \Omega\}$ of f with $F_\omega(iz,z) = f(z)$ and problem 3.3 will state that the functions F_ω can be changed and glued together to one global function F in Ω with $F(iz,z) = f(z)$ and with good bounds. The conditions on the bounds will be the same as those of problem 3.2.

Let ω be a pseudoconvex open set with $\omega \subset\subset T((T^C)^p_c)$ if $\alpha = c$ or $\omega \subset\subset T(T^C)$ if $\alpha = \varepsilon$ and let

$$\omega' \overset{\text{def}}{=\!=} \{\theta \,|\, \|\theta - \theta'\| \le \min[1, \tfrac{1}{2}\, d(\theta',\Omega^c)], \Omega = T(T^C), \theta' \in \omega\}.$$

Then for some $q > p$ and for $\omega \subset T((T^C)^p_\alpha)$

$$\omega' \subset T((T^C)^q_\alpha).$$

Let $f \in Exp_\alpha[a,T^C;M^*]$ and let the convex function ϕ_q be defined by

$$\phi_q(z) \overset{\text{def}}{=\!=} M\,(q\langle\xi_0,y\rangle) + a^q_\alpha(z) + 1/q\|z\|, \quad z \in (T^C)^q_\alpha,$$

where in case $\alpha = \varepsilon$ ϕ_q is extended to a convex function on T^C such that for some $K > 0$

$$|f(z)| \le K \exp \phi_q(z)$$

for $z \in T^C$. If $\alpha = c$ this formula holds for $z \in (T^C)^q_c$. Let $H(T((T^C)^q_\varepsilon)) = T(T^C)$ and $H(T((T^C)^q_c)) = T((T^C)^q_c)$, which in both cases is a pseudoconvex domain in $\mathbb{C}^{2n}$. The function $\theta \to \phi_q(\operatorname{Im}\theta)$ is a convex, hence plurisubharmonic, function on $H(T((T^C)^q_\alpha))$. Hence we can apply theorem 3.1 and for each ω we obtain a holomorphic function F_ω in ω with $F_\omega(iz,z) = f(z)$ for $z \in \{z | (iz,z) \in \omega\}$ which, in view of (3.40) and (3.42), for some m and K satisfies

$$\int_{\omega} |F_\omega(\theta)|^2 \exp - 2\psi^{p,m}(\theta)\,d\lambda(\theta) \le K \int_{\{z|(iz,z)\in\omega'\}} |f(z)|^2 \frac{\exp - 2\phi_q(z)}{(1+\|z\|^2)^{\ell}}\,d\lambda(z)$$

where $\ell = [n/2]+1$ and where the extension of $\psi^{p,m}$ on $T((T^C)^p_\varepsilon)$ to $T(T^C)$ is determined by ϕ_q. We select a collection $\mathcal{U}$ of sets ω with the property that each point in $H(T((T^C)^p_\alpha))$ is contained in at least one set $\omega \in \mathcal{U}$ and each point in $H(T((T^C)^q_\alpha))$ in not more than L sets ω' for a fixed L. In section VI.1 it will be shown that such a covering exists. Then with $\psi \overset{\text{def}}{=\!=} 2\psi^{p,m}$ we get

$$\|\{F_\omega\}\| \overset{\text{def}}{=\!=} \sum_{\omega\in\mathcal{U}} \int_{\omega} |F_\omega(\theta)|^2 \exp - \psi(\theta)\,d\lambda(\theta) \le KL \int_{H(T((T^C)^q_\alpha))} |f(z)|^2 \frac{\exp - 2\phi_q(z)}{(1+\|z\|^2)^{\ell}}\,d\lambda(z) < \infty.$$

It is sufficient if we can find a holomorphic function F in $H(T((T^C)^p_\alpha))$ with $F - F_\omega = 0$ on $\omega \cap V$ and with

$$\int_{H(T(T^C)^p_\alpha))} |F(\theta)|^2 \exp - \hat{\psi}(\theta)\,d\lambda(\theta) \le K\|\{F_\omega\}\|$$

for some K, where $\hat{\psi}$ is obtained from ψ according to (2.41) for some N. For by (2.42) if p is sufficiently large we would have $F \in H^k_\alpha$.

For two sets ω_1 and ω_2 in $\mathcal{U}$ $F_{\omega_1} - F_{\omega_2}$ vanishes on $V \cap \omega_1 \cap \omega_2$, hence $F_{\omega_1} - F_{\omega_2} = P\cdot\vec{G}_{12}$ in $\omega_1 \cap \omega_2$ for some $\vec{G}_{12}$ holomorphic in $\omega_1 \cap \omega_2$. Now if the following problem is solved, we can find a function F as above and the map J would be surjective.

<u>PROBLEM</u> 3.3. Let Ω, P, ϕ and $\hat{\phi}$ be as in problem 3.2 and let $\mathcal{U}$ be the covering of Ω specified in section VI.1. Furthermore, let $\{f_j | \omega_j \in \mathcal{U}\}$ be a collection of holomorphic functions f_j in ω_j such that for each ω_j and ω_k in $\mathcal{U}$ $f_j - f_k = P\cdot\vec{g}_{j,k}$ for some $\vec{g}_{j,k}$ holomorphic in $\omega_j \cap \omega_k$. Then there is a holomorphic function f in Ω with for each $\omega_j \in \mathcal{U}$ $f - f_j = P\cdot\vec{g}_j$ for some $\vec{g}_j$ holomorphic in ω_j such that

$$\int_{\Omega} |f(z)|^2 \exp - \hat{\phi}(z)\,d\lambda(z) \le K \sum_{\omega_j\in\mathcal{U}} \int_{\omega_j} |f_j(z)|^2 \exp - \phi(z)\,d\lambda(z)$$

for some K and N independent of $\{f_j | \omega_j \in U\}$, provided that the collection $\{f_j\}$ is such that the right hand side is finite.

<u>REMARK</u>. If $\alpha = \varepsilon$, $T(T^C) = \bigcup_{p=1}^{\infty} T((T^C)^p_\varepsilon)$ and the densities on $T((T^C)^p_\varepsilon)$ had first to be extended to all of $T(T^C)$ before applying problems 3.2 and 3.3. These extensions depended on the particular holomorphic function F or f one was dealing with. Therefore in the $\alpha = \varepsilon$ case we may get estimates with K depending on F or f, although in problems 3.2 and 3.3 K is independent of f or $\{f_j\}$, respectively. However, the open mapping theorem helps us to overcome the difficulty of not getting uniform bounds. In the next chapter we will treat the case of holomorphic functions f in $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ which are bounded with respect to some density on each Ω_k, uniformly in f. But the condition, cf. (4.22), which must be satisfied then, is not valid for $\Omega = T(T^C) = \bigcup_{k=1}^{\infty} T((T^C)^k_\varepsilon)$ of this chapter.

In chapter IV problems 3.2 and 3.3 will be reformulated and in chapter VI they will be solved. Therefore, the Fourier transformation F is a topological isomorphism from $A_\alpha(a,T^C;M)'$ onto $\mathrm{Exp}_\alpha[a,T^C;M^*]$ for $\alpha = \varepsilon$ or $\alpha = c$, where the spaces are determined by (3.39). Similarly, the same can be derived for the following pair of spaces, which is a mixture of ε- and conic neighborhoods,

$$(3.44)\quad \begin{cases} \mathrm{Exp}_{\varepsilon,c}[a,T^C;M^*] \overset{\mathrm{def}}{=\!=} \operatorname*{proj\,lim}_{k\to\infty} H_\infty(T^{1/ky_0+C_k} \cup \{z \,|\, \|x\| < k, y \in C_k\}; \\ \qquad\qquad a^k_\varepsilon(z) + 1/k\|z\| + M^*(k\|y\|)) \\ A_{\varepsilon,c}(a,T^C;M) \overset{\mathrm{def}}{=\!=} \operatorname*{ind\,lim}_{k\to\infty} H_\infty(\Omega(a+1/k,T^{C_k}); -M(\|\xi\|/k) + k\|\eta\|) \end{cases}$$

and if $\alpha = \varepsilon$ or $\alpha = c$ for the pair

$$(3.45)\quad \begin{cases} \mathrm{Exp}_\alpha[a,\Gamma;M^*] \overset{\mathrm{def}}{=\!=} \operatorname*{proj\,lim}_{k\to\infty} H_\infty(\Gamma^k_\alpha; a^k_\alpha(z) + 1/k\|z\| + M^*(k\|z\|)) \\ A_\alpha(a,\Gamma;M) \overset{\mathrm{def}}{=\!=} \operatorname*{ind\,lim}_{k\to\infty} H_\infty(\Omega^k_\alpha; -M(\|\zeta\|/k)), \end{cases}$$

where Γ is an open, convex cone in $\mathbb{C}^n$ with $\Gamma^k_\varepsilon \overset{\mathrm{def}}{=\!=} \Gamma_k \cup \{1/k\, z_0 + \Gamma\}$ and $\Gamma^k_c \overset{\mathrm{def}}{=\!=} \Gamma_k$, where $a^k_\varepsilon(z) \overset{\mathrm{def}}{=\!=} a(z - 1/_{2k}\, z_0)$ for $z \in 1/k\, z_0 + \Gamma$ and a^k_ε must be continued as a convex function on $\bar{\Gamma}$, where $a^k_c \overset{\mathrm{def}}{=\!=} a$ and where Ω^k_α is given by (2.48.i) and (2.48.ii). The last pair yields the Ehrenpreis-Martineau theorem for analytic functionals carried by arbitrary unbounded, convex sets in $\mathbb{C}^n$ with respect to ε- or conic neighborhoods and to the class of weightfunctions $\{\exp M(\|\zeta\|/k)\}_{k=1}^{\infty}$.

Summarizing we get the following theorem.

THEOREM 3.7. *If (3.38) is satisfied, in the pairs (3.39), (3.44) and (3.45) the strong dual of the second space is topologically isomorphic to the first space by means of the map $\mathcal{F}$ defined by (3.7).*

If lim $a(x,y)$ exists as $y \to 0$, $y \in C_k$ the set $\Omega(a,T^C)$ is bounded in the imaginary directions in $\mathbb{C}_n$. Then in (3.39) for $\alpha = \varepsilon$ and in (3.44) the restriction $\|x\| < k$ in the definition of the first space and the term $k\|\eta\|$ in the definition of the second space can be omitted. In both cases functions in $\mathrm{Exp}_{\varepsilon}[a,T^C;M^*]$ and in $\mathrm{Exp}_{\varepsilon,c}[a,T^C;M^*]$ determine ultradistributions of Roumeiu type of "finite order", cf. definition 2.19.ii. Hence we obtain

COROLLARY 3.8. *Fourier transforms of "infinite order" ultradistributions of Roumieu type can never have a carrier with respect to neighborhoods which are bounded in the imaginary directions.*

If $a(x,0)$ exists, as in (3.3) $\mathrm{Exp}_{\varepsilon,c}$ becomes

$$\mathrm{Exp}_{\varepsilon,c}[a,T^C;M^*] = \operatorname{proj}\lim_{k\to\infty} H_{\infty}(T^{C_k};a(z) + 1/k\|z\| + M^*(k\|y\|))$$

and if $a(z) = 0$ for all z we get the particular case which yields the proof of (4) $\Rightarrow$ (1) of theorem 2.20.

III.5. PALEY-WIENER THEOREMS FOR ULTRADISTRIBUTIONS OF BEURLING TYPE.

As in section III.4 it can be derived that the Fourier transformation $\mathcal{F}$ is an isomorphism between a space of analytic functionals with a fixed carrier onto a space of functions, holomorphic in a certain tubular cone and of certain exponential type, which have ultradistributional boundary values of Beurling type. However, the topologies of the occurring spaces become more complex, especially we don't get a space of analytic functionals which has the topology of the strong dual of a certain space of analytic functions. Therefore, we only state the Fourier transformation $\mathcal{F}$ as a bijection. Spaces of a more simple topological structure arise if we consider Fourier transforms of analytic functionals such that sufficiently small conic neighborhoods of their carriers are contained in a given, open, convex set. In this form we shall give extensions of the Ehrenpreis-Martineau theorem and of the Paley-Wiener theorem for ultradistributions of Beurling type.

Let now $\alpha = 1,2,3$ denote the cases of analytic functionals carried with respect to ε-neighborhoods, conic neighborhoods or a mixture of these neighborhoods, respectively. So here we denote

$$(T^C)_1^k \overset{\text{def}}{=\!=} T^{1/k\, y_0+C}$$
$$(T^C)_2^k \overset{\text{def}}{=\!=} (T^C)_k$$
$$(T^C)_3^k \overset{\text{def}}{=\!=} T^{1/k\, y_0+C_k}$$

and furthermore, cf. (2.54) $a_2^k(z) \overset{\text{def}}{=\!=} \tilde{a}_k(z)$ and $a_1^k(z) \overset{\text{def}}{=\!=} a_3^k(z) \overset{\text{def}}{=\!=} a(z-1/_{2k}\, y_0)$ in $(T^C)_1^k$ or $(T^C)_3^k$, respectively and these functions must be continued as convex functions on $\overline{T^C}$. Let f be a holomorphic function in T^C, which for every k and for some positive K_k and m_k depending on k satisfies

$$(3.46) \qquad |f(z)| \le K_k \exp\{M^*(\|y\|/m_k) + a_\alpha^k(z) + 1/k\|z\|,$$
$$z \in \{z \,|\, \|x\| \le k, y \in C_k\} \cup (T^C)_\alpha^k$$

for $\alpha = 1,2$, or 3. According to lemma 2.17 f uniquely determines an ultradistribution of Beurling type. Now we begin with a formula like (3.23) and we don't have to show that it is the dual of some space of holomorphic functions as the space (3.23) is of the space (3.22). Then by the same procedure as before lemma 2.27, problem 3.2 and 3.3 show that f can be written as

$$(3.47) \qquad f(z) = \langle \mu_\zeta, e^{i\langle \zeta, z\rangle} \rangle$$

where μ is an anlytic functional in $Z'_{(M)}$ uniquely determined by f which is carried by $\Omega(a,T^C)$ with respect to neighborhoods of the form

$$(3.48) \qquad \begin{cases} \Omega_1^k \overset{\text{def}}{=\!=} \Omega(a+1/k, T^C), \\ \Omega_2^k \overset{\text{def}}{=\!=} \Omega(a+1/k, (T^C)_k) \\ \Omega_3^k \overset{\text{def}}{=\!=} \Omega(a+1/k, T^{C_k}) \end{cases}$$

for $\alpha = 1,2$ or 3, respectively. Thus μ can be uniquely extended such that it acts on functions ϕ which are holomorphic in these neighborhoods and satisfy there

$$|\phi(\zeta)| \le K_m \exp\{-M(m\|\xi\|) + k\|\eta\|\}$$

for some k depending on ϕ, for every $m > 0$ and for $K_m > 0$ depending on m. So (3.47) is defined. Furthermore, there are positive K_k and m_k depending on k and μ such that for such ϕ μ satisfies

$$(3.49) \qquad |\langle\mu,\phi\rangle| \le K_k \sup_{\zeta\in\Omega_\alpha^k} |\phi(\zeta)|\exp\{M(m_k\|\xi\|) - k\|\eta\|\}$$

for $\alpha = 1,2$ or 3, respectively. Thus the following Paley-Wiener theorem for ultradistributions of Beurling type holds.

THEOREM 3.9. *If* M *satisfies (3.8) and* f *(3.46), then (3.47) holds for a unique analytic functional* $\mu \in Z'_{(M)}$ *which satisfies (3.49).*

If $a(x,0)$ exists, $\Omega(a,T^C)$ is bounded in the imaginary directions and for $\alpha = 1$ and 3 the condition $\|x\| \le k$ in (3.46) and the term $-k\|\eta\|$ in (3.49) can be omitted. Then f determines an ultradistribution of Beurling type of "finite order", cf. definition 2.19.ii.

COROLLARY 3.10. *Fourier transforms of "infinite order" ultradistributions of Beurling type can never have a carrier with respect to neighborhoods which are bounded in the imaginary directions.*

If $\alpha = 3$ and $a(z) = 0$ for all z, we get the particular case which yields the proof of (4) $\Rightarrow$ (1) of theorem 2.20 for ultradistributions of Beurling type.

We will now define topological spaces of holomorphic functions and we will treat F as a topological isomorphism from the strong dual of an A-space onto an Exp-space. Let $\{\Gamma^m\}_{m=1}^\infty$ and $\{C^m\}_{m=1}^\infty$ be a decreasing sequence of convex cones in $\mathbb{C}^n$ or $\mathbb{R}^n$ with intersection Γ or C, respectively, and with $\Gamma \subset\subset \Gamma^m$, $C \subset\subset C^m$ and let $\{a_m\}_{m=1}^\infty$ be an increasing sequence of convex functions, homogeneous of degree one, each a_m defined on Γ^m or T^{C^m} with $a_m(z) + \varepsilon_m \le a_{m+1}(z)$, $z \in \mathrm{pr}\,\Gamma^{m+1}$ or $\mathrm{pr}\,T^{C^{m+1}}$ for some $\varepsilon_m > 0$, converging in any point of Γ or T^C to the convex, homogeneous function a. Define

$$(3.50) \qquad \begin{cases} \mathrm{Exp}_C(a,\Gamma;M^*) \overset{\mathrm{def}}{=\!=} \operatorname{ind\,lim}_{m\to\infty} H_\infty(\Gamma^m;M^*(\|z\|/m) + a_m(z)) \\ A_C[a,\Gamma;M] \overset{\mathrm{def}}{=\!=} \operatorname{proj\,lim}_{m\to\infty} H_\infty(\Omega(a_m,\Gamma^m); -M(m\|\zeta\|)). \end{cases}$$

In virtue of (3.38) and [73, conditions HS_1 and HS_2] the first space is a nuclear DFS-space and the second a nuclear FS-space. The generalization of the Ehrepreis-Martineau theorem states in this case that the dual of the second space is topologically isomorphic to the first space by means of Fourier transformation. We shall also give a Palèy-Wiener version for ultra-distributions of Beurling type. For simplicity we assume that for each m $a_m(x,0)$ exists, so that each $\Omega(a_m,T^{C^m})$ is bounded in the imaginary directions. Define

$$\text{(3.51)}\qquad \begin{cases} \operatorname{Exp}_C(a,T^C;M^*) \overset{\text{def}}{=\!=} \operatorname{ind}\lim\limits_{m\to\infty} H_\infty(T^{C^m};M^*(\|y\|/m)+a_m(z)) \\ A_C[a,T^C;M] \overset{\text{def}}{=\!=} \operatorname{proj}\lim\limits_{m\to\infty} H_\infty(\Omega(a_m,T^{C^m});-M(m\|\xi\|)). \end{cases}$$

Again $\operatorname{Exp}_C(a,T^C;M^*)$ is a nuclear DFS-space and $A_C[a,T^C;M]$ a nuclear FS-space. It follows from an estimate as we have already met several times that for each m and $\ell > m$ the collection $\{e^{i<\zeta,z>} \mid z \in \Gamma^\ell \text{ or } z \in T^{C^\ell}\}$ of functions of ζ is a subset of $H_\infty(\Omega(a_m,\Gamma^m);-M(m\|\zeta\|))$ or $H_\infty(\Omega(a_m,T^{C^m});-M(m\|\xi\|))$, respectively. Therefore, the Fourier transformation can be defined by (3.7) and it follows from the injectivity of $\mathcal{F}$ that these subsets are dense. Hence the projective limits in (3.50) and (3.51) are strict, cf. [20, § 26.1] so that there strong duals can be represented as inductive limits of strong dual spaces. In the same way as the other theorems of this chapter are derived and by the fact that the open mapping theorem also holds for duals of reflexive Frechet spaces, cf. [61, IV, § 8.3, cor. 1 and ex. 2, p. 162], the following theorem is derived

THEOREM 3.11. *If M satisfies (3.38), in the pairs (3.50) and (3.51) the strong dual of the second space is topologically isomorphic to the first space by means of the map $\mathcal{F}$ defined by (3.7).*

Note that the strong dual of $A_C[a,T^C;M]$, and hence $\operatorname{Exp}_C(a,T^C;M^*)$, carries a finer topology than the one induced by $Z'_{(M)}$ or $\mathcal{D}'_{(M)}$, respectively.

III.6. PALEY-WIENER THEOREMS FOR DISTRIBUTIONS IN $\mathcal{D}'$.

The same ramarks made for ultradistributions of Beurling type can be made for distributions in $\mathcal{D}'$. Instead of (3.36) and (3.37) here we have

$$M^*(\sigma) \overset{\text{def}}{=\!=} \log(1+\sigma^{-1}), \quad M(\rho) \overset{\text{def}}{=\!=} \log(1+\rho).$$

Let f be a holomorphic function in T^C which for every k satisfies

$$(3.52) \qquad |f(z)| \le K_k(1+\|y\|^{-m_k})\exp\{a_\alpha^k(z)+1/k\|z\|\},$$
$$z \in \{z \,|\, \|x\| \le k, y \in C_k\} \cup (T^C)_\alpha^k,$$

where $(T^C)_\alpha^k$ and a_α^k for $\alpha = 1,2$ or 3 are as in section III.5. Then f determines uniquely a distribution in $\mathcal{D}'$. Lemma 2.27 and problems 3.2 and 3.3 show that f can be written as (3.47) for some unique, analytic functional $\mu \in Z'$ carried by $\Omega(a,T^C)$ with respect to the neighborhoods Ω_α^k defined by (3.48). Thus μ can be uniquely extended to an analytic functional acting on functions ϕ which are holomorphic in these neighborhoods and which satisfy there

$$|\phi(\zeta)| \le K_m \frac{\exp k\|\eta\|}{(1+\|\xi\|)^m}$$

for some k depending on ϕ and for every positive m and some positive K_m depending on m and ϕ. Furthermore, for such a ϕ μ satisfies

$$(3.53) \qquad |\langle\mu,\phi\rangle| \le K_k \sup_{\zeta\in\Omega_\alpha^k} |\phi(\zeta)|(1+\|\xi\|)^{m_k} e^{-k\|\eta\|}$$

for $\alpha = 1,2$ or 3, where the positive numbers K_k and m_k depend on k and μ. Now the following Paley-Wiener theorem for distributions in $\mathcal{D}'$ is valid.

THEOREM 3.12. *Let f satisfy (3.52), then f is the Fourier transform of a unique analytic functional $\mu \in Z'$ carried by $\Omega(a,T^C)$, i.e., (3.53) holds.*

If $\Omega(a,T^C)$ is bounded in the imaginary directions, the condition $\|x\| \le k$ in (3.52) and the factor $\exp -k\|\eta\|$ in (3.53) can be omitted if $\alpha = 1$ or 3. Then f determines a distribution of finite order.

COROLLARY 3.13. *The Fourier transform of a distribution of infinite order can never have a carrier with respect to neighborhoods which are bounded in the imaginary directions.*

REMARK. The Fourier transform of any distribution can always be represented as a sum of analytic functionals which are carried by the 3^n sets of the form

$$(3.54)\qquad \{\zeta \mid \xi_j = 0 \text{ or } \zeta_j \in \Omega(a,\mathbb{C}^{\pm}), j = 1,\dots,n\}$$

where $\mathbb{C}^{\pm}$ are the upper and lower halfplane and where a is a convex, homogeneous function on $\mathbb{C}^+$ which is unbounded on pr $\mathbb{C}^+$, or the convex, homogeneous function on $\mathbb{C}^-$ given by $a(z) = a(\bar{z})$, so that $\Omega(a,\mathbb{C}^{\pm}) \subset \mathbb{C}_1$ is not bounded in the imaginary direction. The analytic functionals are carried with respect to any class of neighborhoods and, a fortiori, they can be represented as measures on the sets (3.54), see [16, th. 5.24, where these sets are shown to be sufficient for $\mathcal{D}'$].

A theorem similar to theorem 3.12 can be derived for functions f which are holomorphic in a cone $\Gamma \subset \mathbb{C}^n$, but we merely state the theorem with analytic functionals such that sufficiently small, conic neighborhoods of their carriers are contained in a fixed, open, convex set. Let the notations be as in (3.50) and (3.51) and let

$$(3.55)\qquad \begin{cases} \mathrm{Exp}_C(a,\Gamma) \overset{\text{def}}{=\!=} \operatorname*{ind\,lim}_{m\to\infty} H_\infty(\Gamma^m; \log(1+\|z\|^{-m}) + a_m(z)) \\ A_C[a,\Gamma] \overset{\text{def}}{=\!=} \operatorname*{proj\,lim}_{m\to\infty} H_\infty(\Omega(a_m,\Gamma^m);\ -m\log(1+\|\zeta\|)), \end{cases}$$

and

$$(3.56)\qquad \begin{cases} \mathrm{Exp}_C(a,T^C) \overset{\text{def}}{=\!=} \operatorname*{ind\,lim}_{m\to\infty} H_\infty(T^{C^m}; \log(1+\|y\|^{-m}) + a_m(z)) \\ A_C[a,T^C] \overset{\text{def}}{=\!=} \operatorname*{proj\,lim}_{m\to\infty} H_\infty(\Omega(a_m,T^{C^m});\ -m\log(1+\|\xi\|)). \end{cases}$$

The first space in each pair is a nuclear DFS-space and the second a nuclear, strict FS-space. For these pairs the Ehrenpreis-Martineau theorem can be generalized, where in the second pair it might be considered as an extension of the Paley-Wiener theorem:

<u>THEOREM</u> 3.14. *In the pairs (3.55) and (3.56) the strong dual of the second space is topologically isomorphic to the first space by means of the Fourier transformation $\mathcal{F}$ given by (3.7).*

We conclude this chapter with the remark that in (3.56) the isomorphism $\mathcal{F}$ acts between spaces with a finer topology than the ones induced by Z' and $\mathcal{D}'$.

CHAPTER IV

THE FUNDAMENTAL PRINCIPLE

In [16] Ehrenpreis and in [56] Palamodov proved, independently, a fundamental principle in the theory of systems of linear partial differential equations with constant coefficients. This principle completes the theory of those systems in a very natural way, but the proof is very hard. Let W' be a locally convex topological vector space such that the space H of Fourier transforms of elements of W' consists of entire functions whose growth conditions at infinity satisfy certain properties, and let W be the dual of W'. Briefly, the fundamental principle says that all weak solutions in W of the homogeneous system can be represented as Fourier transforms of finite sums of weak derivatives of measures concentrated in the zero set of the Fourier transform of the transposed differential operator. If there is only one ordinary linear differential equation with constant coefficients this is just the usual representation of Euler. In [16] a space W for which the fundamental principle is valid is called localizable. In the last chapter we have studied spaces W (namely the Exp- and A-spaces) with $H = \mathcal{F}W'$, or equivalently $W = \mathcal{F}H'$ [1)] such that the elements of H are non-entire functions. In this chapter the fundamental principle will be generalized so that it applies to spaces W which are the Fourier transforms of the duals of spaces

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1) As in the foregoing sections the following definition is used: when $\mathcal{F}$ is a topological isomorphism between the spaces B and $\mathcal{F}B = A$, then the Fourier transform of an element f in the dual A' of A is the element $\mathcal{F}f$ of B' defined by

$$\langle \mathcal{F}f, \psi \rangle_B = \langle f, \mathcal{F}\psi \rangle_{A'}, \qquad \psi \in B.$$

By use of this definition the ambiguity mentioned in [16, p.140] is avoided. Of course, as in [16], this definition corresponds to the following action of a function f, regarded as a distribution in $\mathcal{D}'$, to testfunctions $\phi \in \mathcal{D}$

$$\langle f, \phi \rangle = \int f(x)\phi(x)dx.$$

H consisting of functions holomorphic in pseudoconvex domains Ω, not necessarily $\mathbb{C}^n$.

For a vector P of complex polynomials, in [16] Ehrenpreis has defined a multiplicity variety $\mathcal{W}$ in the set where all the components of P vanish. Let $H(\mathcal{W})$ be the space of restrictions to $\mathcal{W}$ of all entire functions satisfying on $\mathcal{W}$ the same growth conditions as the entire functions of H. Then for deriving the fundamental principle Ehrenpreis showed that H modulo $P\cdot\vec{H}$ is isomorphic to $H(\mathcal{W})$. In order to prove this isomorphism he first constructed a local and a semilocal (i.e., in an a priori given covering of $\mathbb{C}^n$ consisting of bounded sets) theory and then the extended the semilocal results to global results. The same can be done if P is a matrix of polynomials and if $\vec{\mathcal{W}}$ is an associated vector multiplicity variety. For our purpose the local and semilocal theory remains unchanged (except for the a priori given covering of Ω), but we will use a different method for getting global results. If then in particular $\Omega = \mathbb{C}^n$ we will obtain a weaker form of the isomorphism than in [16]. The difference is that in [16] one globally defined function, whose restriction to $\mathcal{W}$ has been given, is obtained that satisfies all the bounds required in H, while in this chapter for every bound a different global function will be constructed. As to this the fundamental principle obtained by Palamodov in [56] is similar. On the other hand, here often less restrictive conditions on the bounds are required then in [16], so that for example the space of C^{∞} functions in an open, convex set is localizable here as well as in [56], where in [16] it is in general not.

Compared with [56] our conditions are simpler, although if $\Omega = \mathbb{C}^n$ the method of Hörmander in [30] we will use cannot be applied to the space Z because the function $\log(1+\|z\|^2)^{-1}$ is not plurisubharmonic in $\mathbb{C}^n$, while Z satisfies the conditions of both [16] and [56]. If Ω is a convex tube domain $(\neq \mathbb{C}^n)$ this objection is disposed of (cf. lemma 5.2) and our treatment of this case is much more general than in [56]. Moreover, we will derive the isomorphism H mod $P\cdot\vec{H} \leftrightarrow H(\mathcal{W} \cap \Omega)$ for general pseudoconvex domains Ω, where in [56] it is essential that Ω is a convex tube domain.

Sections 1 and 2 of this chapter will give an introduction along the lines of [16] to the problems without growth conditions. In section 3 Ehrenpreis' and Palamodov's formulations of the fundamental principle will be discussed. The remaining part of this chapter will be devoted to derive the weak form of the above mentioned isomorphism for spaces of non-entire functions. In chapter V we will show that this implies the representation of solutions of homogeneous systems of partial differential equations with constant coefficients and in chapter VII we will make some remarks concerning the strong form of the isomorphism for certain spaces of non-entire functions.

IV.1. LOCAL THEORY

In this section we will discuss Ehrenpreis' generalization of Hilbert's Nullstellensatz.

Let $z \in \mathbb{C}^n$ and let A_z be the ring of germs at z of holomorphic functions in a neighborhood of z. Consider an ideal J_z in A_z generated by the germs $(h_1)_z,\dots,(h_q)_z$ at z of functions $h_1,\dots,h_q$ in a neighborhood ω of z. We define the analytic variety

$$V \overset{\text{def}}{=\!=} \{w \mid h_1(w) = 0,\dots,h_q(w) = 0,\ w \in \omega\} \tag{4.1}$$

and let V_z be the equivalent class of V under the equivalence relation $V \sim W$ if there is a neighborhood of z in which they are equal. V_z is called the germ at z of V. It is clear that the ideal J_z is not trivial only if $h_1(z) = \dots = h_q(z) = 0$. When $f_z \in A_z$ we will denote by f a holomorphic function in a neighborhood of z such that f_z is the germ of f at z. Then for any $f_z \in J_z$, $z \in V$, there is a neighborhood ω of z with

$$f(w) = 0, \quad w \in V \cap \omega. \tag{4.2}$$

Conversily, consider the ideal I_z in A_z of all the germs at z of holomorphic functions vanishing on V_z, i.e.,

$$I_z \overset{\text{def}}{=\!=} \{f_z \mid \text{there is a neighborhood } \omega \text{ of z such that } f|_{V\cap\omega} = 0\}. \tag{4.3}$$

It is clear that I_z is an ideal and by (4.2) $J_z \subset I_z$.

Hilbert's Nullstellensatz says that for $f_z \in I_z$ there is a positive integer m with $(f_z)^m \in J_z$, or

$$I_z = \text{rad } J_z \overset{\text{def}}{=\!=} \{f_z \mid (f_z)^m \in J_z \text{ for some m depending on } f_z\},$$

see [27, II.E. th. 20]. Obviously, when J_z is a prime ideal this yields [27, III.A. 7]

$$(4.4) \qquad \mathcal{J}_z = \mathcal{I}_z,$$

i.e., any f_z can be written as, cf. (3.32),

$$f(w) = \sum_{k=1}^{q} g_k(w)h_k(w)$$

for w in a neighborhood ω of z and for some $g_k \in A(\omega)$, $k = 1,\dots,q$.

Ehrenpreis generalized this result in such a way that (4.4) always holds if in (4.3) V_z is replaced by the germ $\mathcal{W}_z$ of a certain local multiplicity variety $\mathcal{W}$ depending on the functions $h_1,\dots,h_q$ and z. In general a *local analytic multiplicity variety* $\mathcal{W}$ in a point $z \in \mathbb{C}^n$ is defined as a finite collection $\mathcal{W} = \{V_1,\partial_1;\dots;V_r,\partial_r\}$ of pairs (V_j,∂_j), where the V's are analytic varieties in a neighborhood of z (i.e., V_j is defined by (4.1) in a neighborhood ω of z for certain holomorphic functions h_k^j in ω depending on z and j for $k = 1,\dots,q^j$, where the number q^j of functions also may depend on j and z) and where ∂_j is a differential operator with coefficients holomorphic in a neighborhood of z for $j = 1,\dots,r$. If for each $z \in \mathbb{C}^n$ all the defining functions h_k^j, $k = 1,\dots,q^j$, $j = 1,\dots,r$ are the same polynomials for every z and if the coefficients of the differential operators ∂_j are the same polynomials, $\mathcal{W}$ is called a *polynomial multiplicity variety* in $\mathbb{C}^n$. In this case for $\omega \subset \mathbb{C}^n$, $\mathcal{W} \cap \omega$ is the restriction of $\mathcal{W}$ to the points of ω. Let f_z be the germ of a holomorphic function at z, then $f_z|_{\mathcal{W}_z}$, the *restriction of* f_z *to* $\mathcal{W}_z$, is defined as the collection of functions $\{f_j\}_{j=1}^r$, where each f_j is defined on V_j in a neighborhood ω of z, by

$$(4.5) \qquad f_j \overset{\text{def}}{=\!=} \partial_j f|_{V_j \cap \omega}$$

Conversely, a collection of functions $\{f_j\}_{j=1}^r$ with f_j defined on V_j in a neighborhood of z is called a *holomorphic function on* $\mathcal{W}_z$ if there exists a holomorphic function f in a neighborhood ω of z with $f|_{\mathcal{W}\cap\omega} = \{f_j\}_{j=1}^r$.

LEMMA 4.1 [16, th. II.2.4]. *Let* $\{h_k\}_{k=1}^q$ *be a q-tuple of holomorphic functions in* ω. *Then it is possible for each* $z \in \omega$ *to define the germ* $\mathcal{W}_z$ *at z of a local analytic multiplicity variety, such that for each* $z \in \omega$ *the germ at z of every function f, holomorphic in a neighborhood of z in* ω, *vanishes on* $\mathcal{W}_z$ *if and only if it can be written as*

$$f(w) = \sum_{k=1}^{q} h_k(w)g_k(w)$$

for w *in a neighborhood of* z *in* ω *and for functions* g_k *holomorphic there,* $k = 1,\dots,q$.

Thus for any vector $\vec{h}_z \in A_z^q$ there exists the germ $\mathcal{W}_z$ of a multiplicity variety such that the subset I_z of A_z of germs of functions vanishing on $\mathcal{W}_z$ is always an ideal which satisfies (4.4). It should be remarked that $\mathcal{W}$ is not uniquely determined by the functions $h_1,\dots,h_q$. Instead of proving lemma 4.1 we shall give some examples of polynomial multiplicity varieties.

(i) For $n = 2$, $q = 1$ and $h(z) = z_1^m(z_1 - z_2)^\ell$ both the multiplicity varieties
$\mathcal{W}' \overset{\text{def}}{=} \{z_1 = 0, \text{identity};\dots;z_1 = 0, \partial^{m-1}/\partial z_1^{m-1};z_1 = z_2,\text{id};\dots;z_1 = z_2, \partial^{\ell-1}/\partial z_1^{\ell-1}\}$
and
$\mathcal{W} \overset{\text{def}}{=} \{z_1 = z_2 = 0,\text{id};\dots;z_1 = z_2 = 0, \partial^{m+\ell-1}/\partial z_1^{m+\ell-1};z_1 = 0,\text{id};\dots; z_1 = 0, \partial^{m-1}/\partial z_1^{m-1};z_1 = z_2,\text{id};\dots;z_1 = z_2, \partial^{\ell-1}/\partial z_1^{\ell-1}\}$
are such that, if they replace V in (4.3), then (4.4) is satisfied for each $z \in \mathbb{C}^n$, cf. [16, ch II, § 2, ex. 3].

(ii) Let $n = 2$, $q = 2$, $h_1(z) = z_2^2 - z_1$ and $h_2(z) = z_1^2$. Then we may take cf. [16, ch. II, § 2, ex. 4]
$\mathcal{W} \overset{\text{def}}{=} \{z_1 = z_2 = 0,\text{id.};z_1 = z_2 = 0, \partial/\partial z_2;z_1 = z_2 = 0, \partial/\partial z_1 + \frac{1}{2}\partial^2/\partial z_2^2; z_1 = z_2 = 0, \partial^2/\partial z_1\partial z_2 + \frac{1}{6}\partial^3/\partial z_2^3\}$,
because obviously for every $z \in \mathbb{C}^n$ and $f_z \in A_z$ $h_1 f\big|_{\mathcal{W}\cap\omega} = 0$ and $h_2 f\big|_{\mathcal{W}\cap\omega} = 0$ for some neighborhood ω of z, and if $f\big|_{\mathcal{W}\cap\omega} = 0$, we first expand f in a power series

$$f(z_1,z_2) = \sum f_{ij}\, z_1^i\, z_2^j.$$

Since $f(0,0) = 0$ we have $f_{00} = 0$, since $\partial f/\partial z_2(0,0) = 0$ we have $f_{01} = 0$, since $\partial f/\partial z_1(0,0) + \frac{1}{2}\partial^2 f/\partial z_1^2(0,0) = 0$ we have $f_{10} + f_{02} = 0$ and finally since $\partial^2 f/\partial z_1\partial z_2(0,0) + \frac{1}{6}\partial^3 f/\partial z_2^3(0,0)$ we have $f_{11} + f_{03} = 0$.
Next writing

$$f(z_1,z_2) = z_1^2 \sum_{\substack{i\ge 2\\ j\ge 0}} f_{ij}\, z_1^{i-2}\, z_2^j + z_1 \sum_{j\ge 0} f_{1j}\, z_2^j + \sum_{j\ge 0} f_{0j}\, z_2^j$$

and using

$$z_1 z_2^2 = z_1(z_2^2 - z_1) + z_1^2 \in \vec{h}\cdot\vec{A}_z$$

$$z_2^2 = (z_2^2 - z_1) + z_1 \equiv z_1 \mod \vec{h}\cdot\vec{A}_z$$

$$z_2^3 = z_2(z_2^2 - z_1) + z_1 z_2 \equiv z_1 z_2 \mod \vec{h}\cdot\vec{A}_z$$

$$z_2^4 = (z_2^2 + z_1^2)(z_2^2 - z_1^2) + z_1^4 \in \vec{h}\cdot\vec{A}_z$$

by the above we get

$$f(z_1,z_2) = f_{10}z_1 + f_{11}z_1z_2 + f_{00} + f_{01}z_2 + f_{02}z_1 + f_{03}z_1z_2 \mod \vec{h}\cdot\vec{A}_z \equiv$$
$$\equiv 0 \mod \vec{h}\cdot\vec{A}_z .$$

(iii) Finally we give an example which shows that the differential operators do not necessarily have constant coefficients. Let $n = 3$, $h_1(z) = z_2 - z_1 z_3$ and $h_2(z) = z_2^2$, cf. [16, II exercise 2.2]. Then as in example (ii) one can check that the polynomial multiplicity variety

$$\mathcal{W} \stackrel{\text{def}}{=\!=} \{z_2 = z_3 = 0, \text{id}; z_2 = z_3 = 0, z_1\partial/\partial z_2 + \partial/\partial z_3; z_1 = z_2 = 0, \text{id.};$$
$$z_1 = z_2 = 0, \partial/\partial z_1 + z_3\partial/\partial z_2\}$$

satisfies the required properties. To see how the multiplicity variety $\mathcal{W}$ could be obtained one first determines a multiplicity variety $\mathcal{W}_1$ belonging to the polynomial $z_2 - z_1 z_3$. For that purpose, we introduce the change of variables $u = z_1 + z_3$, $v = z_2$ and $w = z_1 - z_3$ so that any holomorphic function $f(z_1,z_2,z_3)$ can be written as

$$\tilde{f}(u,v,w) = f(\frac{u+w}{2}, v, \frac{u-w}{2})$$

and so that the polynomial $z_2 - z_1 z_3$ multiplied by 4 becomes

$$w^2 - u^2 + 4v,$$

which now is a distinguished polynomial in w. A multiplicity variety belonging to it is

$$\tilde{\mathcal{W}}_1 \stackrel{\text{def}}{=\!=} \{w^2 - u^2 + 4v = 0, \text{id.}; w = u^2 - 4v = 0, \partial/\partial w\},$$

which in the original coordinates is

$$\mathcal{W}_1 \stackrel{\text{def}}{=\!=} \{z_2 - z_1 z_3 = 0, \text{id.}; z_1 - z_3 = z_2 - z_1^2 = 0, \partial/\partial z_1 - \partial/\partial z_3\}.$$

Now we write an analytic function $\tilde{f}(u,v,w)$ as

$$\tilde{f}(u,v,w) \equiv K_0(u,v) + w\, K_1(u,v) \mod (w^2 - u^2 + 4v),$$

where $K_0(u,v)$ and $K_1(u,v)$ are computed by the values of $\tilde{f}$ on the variety $w^2 - u^2 + 4v = 0$ above the point (u,v), if $u^2 - 4v \neq 0$. Precisely, since $\tilde{f}(u,v,w) = K_0(u,v) + wK_1(u,v)$ for $w = \pm\sqrt{u^2 - 4v}$ we get two equations with two unknowns yielding the solution

$$K_0(u,v) = \frac{\tilde{f}(u,v,\sqrt{u^2-4v}) + \tilde{f}(u,v,-\sqrt{u^2-4v})}{2}$$

$$K_1(u,v) = \frac{\tilde{f}(u,v,\sqrt{u^2-4v}) - \tilde{f}(u,v-\sqrt{u^2-4v})}{2\sqrt{u^2-4v}}$$

if $u^2-4v \neq 0$, while for $u^2=4v$ we have the equations
$\tilde{f}(u,v,0) = K_0(u,v)$ & $\partial\tilde{f}/\partial w\ (u,v,0) = K_1(u,v)$, $u^2=4v$.
Hence the functions K_0 and K_1 can be continued analytically over the variety $u^2-4v=0$. Furthermore, the multiplicity variety belonging to the polynomial v^2 is
$\tilde{\mathcal{W}}_2 \overset{\text{def}}{=\!=} \{v=0,\text{id.};v=0,\partial/\partial v\}$.
So we write K_0 and K_1 as

$$K_0(u,v) \equiv K_{00}(u) + v\,K_{01}(u) \mod v^2$$

$$K_1(u,v) \equiv K_{10}(u) + v\,K_{11}(u) \mod v^2$$

and compute $K_{ij}(u)$ by the values of K_0 and K_1 on the variety $v = 0$, which yields

$$K_{00}(u) = K_0(u,0)$$

$$K_{10}(u) = K_1(u,0)$$

$$K_{01}(u) = \partial K_0/\partial v\ (u,0)$$

$$K_{11}(u) = \partial K_1/\partial v\ (u,0)$$

Using the expressions for K_0 and K_1 we find

$$K_{00}(u) = \frac{\tilde{f}(u,0,u) + \tilde{f}(u,0,-u)}{2} = \frac{f(u,0,0) + f(0,0,u)}{2}$$

$$K_{10}(u) = \frac{f(u,0,u) - f(u,0,-u)}{2u} = \frac{f(u,0,0) - f(0,0,u)}{2u}$$

Defining
$\mathcal{W}' \overset{\text{def}}{=\!=} \{z_2=z_3=0,\text{id.};z_1=z_2=0,\text{id.}\}$
by a power series expansion of f we see that K_{00} and K_{10} can be expressed in terms of the restriction of f to $\mathcal{W}'$. The expressions for K_{01} and K_{11} become

$$K_{01}(u) = \frac{1}{2}\frac{\partial\tilde{f}}{\partial v}(u,0,u) - \frac{1}{u}\frac{\partial\tilde{f}}{\partial w}(u,0,u) + \frac{1}{2}\frac{\partial\tilde{f}}{\partial v}(u,0,-u) +$$
$$+ \frac{1}{u}\frac{\partial\tilde{f}}{\partial w}(u,0,-u) = \frac{1}{2}\frac{\partial f}{\partial z_2}(u,0,0) - \frac{1}{2u}\frac{\partial f}{\partial z_1}(u,0,0) +$$
$$+ \frac{1}{2u}\frac{\partial f}{\partial z_3}(u,0,0) + \frac{1}{2}\frac{\partial f}{\partial z_2}(0,0,u) + \frac{1}{2u}\frac{\partial f}{\partial z_1}(0,0,u) -$$
$$- \frac{1}{2u}\frac{\partial f}{\partial z_3}(0,0,u)$$

and

$$K_{11}(u) = \frac{1}{u}\{\frac{1}{2}\frac{\partial\tilde{f}}{\partial v}(u,0,u) - \frac{1}{u}\frac{\partial\tilde{f}}{\partial w}(u,0,u) - \frac{1}{2}\frac{\partial\tilde{f}}{\partial v}(u,0,-u) -$$
$$- \frac{1}{u}\frac{\partial\tilde{f}}{\partial w}(u,0,-u)\} + \frac{1}{u^3}\{\tilde{f}(u,0,u) - \tilde{f}(u,0,-u)\} =$$
$$= \frac{1}{u}\{\frac{1}{2}\frac{\partial f}{\partial z_2}(u,0,0) - \frac{1}{2u}\frac{\partial f}{\partial z_1}(u,0,0) + \frac{1}{2u}\frac{\partial f}{\partial z_3}(u,0,0) -$$
$$- \frac{1}{2}\frac{\partial f}{\partial z_2}(0,0,u) - \frac{1}{2u}\frac{\partial f}{\partial z_1}(0,0,u) + \frac{1}{2u}\frac{\partial f}{\partial z_3}(0,0,u)\} +$$
$$+ \frac{1}{u^3}\{f(u,0,0) - f(0,0,u)\}.$$

Finally, expressing $u\,K_{01}(u) \pm u^2\,K_{11}(u)$ in terms of f and bearing in mind that K_{01} and K_{11} are analytic, we see that K_{01} and K_{11} can be expressed in terms of $f|_{W'}$ and the restriction of f to the multiplicity variety

$W'' \stackrel{def}{=} \{z_2 = z_3 = 0, z_1\ \partial/\partial z_2 + \partial/\partial z_3;\ z_1 = z_2 = 0, z_3\ \partial/\partial z_2 + \partial/\partial z_1\}$.

Thus any f_z can be expressed modulo $\vec{h}\cdot\vec{A}_z$ in terms of the restriction of f to $W \stackrel{def}{=} W' \cup W''$ and clearly $\vec{h}\cdot\vec{A}_z$ vanishes on W for each z.

Furthermore, [16, th. 2.5] determines a procedure (called parametrization) which extends the restriction to the germ of a local multiplicity variety W of the germ of an analytic function f to the germ of an unique analytic function $\hat{f}$; if f_z vanishes on W_z then always $\hat{f}_z \equiv 0$. Moreover, this procedure is linear in the following sense: for $a,b \in \mathbb{C}$ we have $\widehat{(a f + b g)}_z = a\,\hat{f}_z + b\,\hat{g}_z$. In example (iii) the extension of $f|_W$ is $K_{00}(z_1+z_3) + z_2K_{01}(z_1+z_3) + (z_1-z_3)K_{10}(z_1+z_3) + (z_1-z_3)z_2K_{11}(z_1+z_3)$.

The case of modules in A_z^p generated by a p×q-matrix $H = (h_{jk})$ of holomorphic functions is more delicate. The difficulty is that we want to solve a matrix equation $H\cdot\vec{g}=\vec{f}$ in a ring. In this and the next section lemma's 4.2 and 4.3 will express the following facts:

(1) *Any submodule* M *of* A_z^p *is* $\mathbb{C}$*-linearly isomorphic to a direct sum of* p *ideals* $I_z^1,\dots,I_z^p$ *in the ring* A_z *and moreover, there exists a* $\mathbb{C}$*-linear bijective map* $\sigma\colon A_z^p \to A_z^p$ *such that* M *is mapped onto* $\overset{p}{\underset{j=1}{\oplus}} I_z^j$. That such a map exists can be seen by induction. For $p=1$ it is trivial. Let the A_z-module homomorphism $\phi\colon A_z^p \to A_z^p$ be defined by $\phi(f_1,\dots,f_p) \overset{\text{def}}{=\!=} f_1$. Then A_z^{p-1} can be identified with $\text{Ker}\,\phi = (0,A_z^{p-1})$. Furthermore, let M_0 be the module $M \cap \text{Ker}\,\phi$ and let the ideal $I_z^1 \subset A_z$ be the image of M under ϕ. If A and $A \cup B$ are Hamel bases of M_0 and M, respectively, this determines a linear direct decomposition $M = M_1 \oplus M_0$, where M_1 is a linear space which is mapped by ϕ linearly and bijectively onto I_z^1. Moreover, by using completions of A to a Hamel basis $A \cup C$ of $(0,A_z^{p-1})$ and of $A \cup B \cup C$ to a Hamel basis of A_z^p we find that M_1 is a linear subspace of a linear space $N_1 = (A_z,\tilde{N})$ with $\tilde{N} \subset A_z^{p-1}$, such that A_z^p is linearly decomposed as $A_z^p = N_1 \oplus (0,A_z^{p-1})$, where M_0 can be considered as a submodule of A_z^{p-1}. By the inductive hypothesis there exists a linear bijection $\sigma_0\colon A_z^{p-1} \to A_z^{p-1}$ which maps M_0 onto a direct sum of ideals. Let P_1 be the projection of A_z^p onto N_1, then we define $\sigma \overset{\text{def}}{=\!=} \sigma_0 \circ (1-P_1) + \phi \circ P_1$.

(2) *If* M *is generated by the vectors* $\{\vec{h}^k\}_{k=1}^q$ *of germs at* z *of holomorphic vector functions, the ideals* I_z^ℓ *can depend on these vectors by*

$$I_z^\ell = \{ \sum_{k=1}^q g_k\, h_\ell^k \,|\, \vec{g} \in A_z^q, \textit{ with } \sum_{k=1}^q g_k\, h_j^k = 0 \textit{ for } j=1,\dots,\ell-1 \textit{ if } \ell>1\}.$$

This follows from (1) where $I_z^1 = \{\Sigma g_k\, h_1^k \,|\, \vec{g} \in A_z^q\}$ and $M_0 = \{\Sigma g_k\, \vec{h}^k \,|\, \vec{g} \in A_z^q$ with $\Sigma g_k\, h_1^k = 0\}$. Note that any module in A_z^p is finitely generated because the ring A_z is Noetherian [30, lemma 6.3.2 & th. 6.3.3].

(3) *According to lemma 4.1 to the vector* $\vec{I}_z = (I_z^1,\dots,I_z^p)$ *of ideals there is accociated the germ* $\vec{W}_z = (W_z^1,\dots,W_z^p)$ *at* z *of a vector of local multiplicity varieties, such that* $\vec{I}_z$ *consists of the vector functions vanishing on* $\vec{W}_z$.

The need of Hamel bases in (1) makes it impossible to obtain ideals of functions satisfying growth conditions. Therefore, with the aid of parametrization (see p.122) in the proof of lemma 4.2 we will perform the steps of (1) in a more constructive way. However, in order to get bounds later, we will keep some freedom in the definition of the map there. The result will be a map $\rho_z : A_z^p \to A_z^p$ which depends on z and is only $\mathbb{C}$-linear from A_z^p onto $A_z^p / \bigoplus_{j=1}^{p} I_z^j$. As (1) also holds for sections over a domain, in lemma 4.3 it will be shown that the freedom in the definition of ρ_z will not prevent us from obtaining sections on the multiplicity varieties $\mathcal{W}^j$.

For a p×q-matrix H of holomorphic functions we will denote the module in A_z^p of germs at z of functions $\vec{f} = H\cdot\vec{g}$ with $\vec{g} \in A_z^q$ by $\vec{J}_z$.

LEMMA 4.2. *For each* p×q-*matrix* $H = (h_{jk})$ *of holomorphic functions* $h_{jk} \in A(\omega)$ *and for each* $z \in \omega$, *there exist a local vector multiplicity variety* $\vec{\mathcal{W}}_z$ *and a linear, surjective map* ρ_z *from* A_z^p *onto* $A_z^p/\vec{I}_z$ *whose kernel is just* $\vec{J}_z$, *where* $\vec{I}_z$ *is the module associated to* $\vec{\mathcal{W}}_z$.

PROOF. For each $z \in \omega$ define $\mathcal{W}_z^1$ as the analytic multiplicity variety belonging to the functions $h_{11},\dots,h_{1k},\dots,h_{1q}$ by lemma 4.1. Let $\mathcal{M}_z^\ell$ be the sheaf of relations at z of the first ℓ rows of H, i.e., $\vec{g}_z \in \mathcal{M}_z^\ell$ if and only if

$$(4.6)\qquad \sum_{k=1}^{q} (h_{jk})_z (g_k)_z = 0, \qquad j = 1,\dots,\ell.$$

Now by Oka's theorem [30, th. 7.1.5] $\mathcal{M}_z^\ell$ is locally finitely generated, hence the functions $\sum_k h_{\ell+1\,k}\, g_k$ with $\vec{g}$ satisfying (4.6) determine the germ $\mathcal{W}_z^{\ell+1}$ at z of an analytic multiplicity variety according to lemma 4.1. Thus $f \in A_z$ vanishes on $\mathcal{W}_z^{\ell+1}$ (i.e., $f \in I_z^{\ell+1}$) if and only if

$$(4.7)\qquad f_z = \sum_{k=1}^{q} (h_{\ell+1\,k})_z (g_k)_z \text{ for a } \vec{g} \text{ satisfying (4.6).}$$

Now we will define the map ρ_z for $\vec{f}_z \in A_z^p$: $(\rho_z\vec{f}_z)_1$ is given by

$$(\rho_z\vec{f}_z)_1 \overset{\text{def}}{=\!=} (f_1)_z.$$

Let $(\hat{f}_1)_z$ be the extension of $f_1|_{\mathcal{W}^1}$ at z and let $\vec{g}_z^1$ be such that

$$(4.8)\qquad \sum_{k=1}^{q} h_{1k}\, g_k^1 = f_1 - \hat{f}_1.$$

According to lemma 4.1 it is always possible to find such $\vec{g}_z^1$. Then we define

$$(4.9)\qquad (\rho_z\vec{f}_z)_2 \overset{\text{def}}{=\!=} (f_2)_z - \sum_{k=1}^{q} (h_{2k}g_k^1)_z.$$

Successively for $\ell = 2,\dots,p-1$ let $\hat{f}_\ell$ be the extension of the restriction at z to W^ℓ of $f_\ell - \sum_{j=1}^{\ell-1}\sum_{k=1}^{q} h_{\ell k} g_k^j$, let $\vec{g}_z \in M_z^{\ell-1}$ be such that

$$(4.10)\qquad \sum_{k=1}^{q} h_{\ell k}\, g_k^\ell = f_\ell - \sum_{j=1}^{\ell-1}\sum_{k=1}^{q} h_{\ell k}\, g_k^j - \hat{f}_\ell$$

and define

$$(4.11)\qquad (\rho_z \vec{f}_z)_{\ell+1} \overset{\text{def}}{=\!=} (f_{\ell+1})_z - \sum_{j=1}^{\ell}\sum_{k=1}^{q} (h_{\ell+1\,k}\, g_k^j)_z .$$

The functions $\vec{g}_z^\ell$ are not uniquely determined, since an arbitrary element of M_z^ℓ can be added to $\vec{g}_z^\ell$. This changes $(\rho_z\vec{f}_z)_{\ell+1}$, although $(\rho_z\vec{f}_z)_{\ell+1}|_{W_z^{\ell+1}}$ and $(\rho_z\vec{f}_z)_{\ell+j}$, $j \geq 2$, are not altered (see next section proof of lemma 4.3)[1]. So ρ_z is determined by the choices of $\vec{g}_z^\ell$ and we may choose suitable $\vec{g}_z^\ell$ depending on $z \in \omega$ to be determined later. Therefore, we get a map ρ_z from A_z^p into A_z^p which can depend on z. It is clear that ρ_z is surjective from A_z^p onto A_z^p. Furthermore, it follows from the linearity of the map $f_z|_{W_z} \to \hat{f}_z$ and from the fact that a different choice of $\vec{g}_z^\ell$ for $\ell = 1,\dots,p-1$ has the effect of addition of an element of $\vec{I}_z$ to $\rho_z\vec{f}_z$, that the map ρ_z is linear from A_2^p into $A_z^p/\vec{I}_z$.

Let $\vec{f}_z \in \vec{J}_z$, thus $f_j = \sum_k h_{jk}\, g_k'$ for some $\vec{g}' \in A_z^q$. Then $(\rho_z\vec{f}_z)_1$ vanishes on W_z^1, hence $\hat{f}_1 \equiv 0$ and $\vec{g}_z^1 = \vec{g}_z' - \vec{m}_z^1$ for some $\vec{m}_z^1 \in M_z^1$ depending on the choice of $\vec{g}_z^1$. This implies that $(\rho_z\vec{f}_z)_2 = \sum_k h_{2k}\, m_k^1$ which vanishes on W^2 in a neighborhood of z. Successively for $\ell = 2,\dots,p-1$ we find that $\hat{f}_\ell \equiv 0$, that $\vec{g}_z^\ell = \vec{m}_z^{\ell-1} - \vec{m}_z^\ell$ for some $\vec{m}_z^\ell \in M_z^\ell$ and that $(\rho_z\vec{f}_z)_{\ell+1} = \sum_k h_{\ell+1\,k}\, m_k^\ell$ which vanishes on $W^{\ell+1}$ in a neighborhood of z by (4.7). Thus $\rho_z\vec{f}_z \in \vec{I}_z$.

Conversely, if $\rho_z\vec{f}_z \in \vec{I}_z$, thus if $\rho_z\vec{f}_z$ vanishes on $\vec{W}_z$, then $f_1 = \sum_k h_{1k}\, g_k^1$ for some $\vec{g}_z^1 \in A_z^q$ by lemma 4.1. Since $\hat{f}_j \equiv 0$ for $j = 1,\dots,p-1$, by (4.10) we get for $\ell = 1,\dots,p-1$

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[1] At this point [16] is a little puzzling. On page 49 it is remarked that $(\rho_z\vec{f}_z)_{\ell+2}|_{W^{\ell+2}}$ does change by a different choice of $\vec{g}_z^\ell$. On the other hand this should not be true if one wants to obtain global sections on $\vec{W}$ (see next section), which is really the case in [16, p.100-105, especially p.104, proof of b, shows that one is concerned with global sections]. The key lies perhaps in the fact that systematically the wrong formula has been used in [16], where in the formula's (2.19), (2.20), (2.58), (2.59) and (3.44) $F_{i+1,j}$ should be replaced by $F_{t+1,j}$, $F_{t+1,j}$, $F_{k+1,j}$, $F_{k,j}$ or $F_{k,j}$, respectively.

$$f_\ell - \sum_{j=1}^{\ell-1} \sum_{k=1}^{q} h_{\ell k}\, g_k^j = \sum_{k=1}^{q} h_{\ell k}\, g_k^\ell$$

with $\vec{g}_z^{\,\ell} \in M_z^{\ell-1}$ and this holds also for $\ell = p$ for some $\vec{g}_z^{\,p} \in M_z^{p-1}$, because (4.11) vanishes on W_z^p if $\ell = p-1$ there. Thus since $\sum_k h_{\ell k}\, \vec{g}_k^{\,j} = 0$ for $j > \ell$, f can be written as

$$f_\ell = \sum_{k=1}^{q} h_{\ell k}\, (g_k^1 + \ldots + g_k^p),$$

i.e., $f_z \in \vec{J}_z$. □

<u>REMARK</u>. If the map $f_z\big|_{W_z} \to \hat{f}_z$ would be multiplicative, ρ_z would be multiplicative. It is possible, cf. [16, th. 2.5 & lemma 2.14] to give a rule of multiplication by an element of A_z in $A_z^p/\vec{I}_z$ such that ρ_z becomes a homomorphism of A_z-modules.

IV.2. GLOBAL THEORY.

We will study the global analog of the foregoing with sections over a pseudoconvex domain Ω instead of germs at a point z.

Let J be a sheaf of ideals generated in each point of Ω by holomorphic functions $\vec{h} = (h_1,\ldots,h_q)$ in Ω. Their simultaneous zero set defines a global analytic variety $V = \bigcup_{z\in\Omega} V_z$ in Ω (at points z where some $h_k(z) \neq 0$ V_z is empty). We will define the sheaf of analytic functions on V. Let I be the sheaf on Ω

$$I \overset{\text{def}}{=\!=} \bigcup_{z\in\Omega} I_z$$

where I_z is defined by (4.3); let $I_z \overset{\text{def}}{=\!=} A_z$ when $z \in \Omega\backslash V$. We define a sheaf F on Ω by

$$F_z \overset{\text{def}}{=\!=} A_z/I_z, \quad z \in \Omega, \tag{4.12}$$

so that the following sequence is exact

$$0 \to I \to A \to F \to 0.$$

For $z \in \Omega \backslash V$ $\mathcal{I}_z = \mathcal{A}_z$, thus $\mathcal{F}_z = 0$. Hence $\mathcal{F}$ is only non-trivial in points of V, thus we may just as well consider the restriction $\mathcal{F}'$ of $\mathcal{F}$ to V

$$\mathcal{F}' \stackrel{\text{def}}{=\!=} \bigcup_{z \in V} \mathcal{F}_z$$

which is a sheaf on V. By definition a section f in $\Gamma(V,\mathcal{F}')$ is a holomorphic function in V; considered as a section f_1 in $\Gamma(\Omega,\mathcal{F})$ we would have $f_1(z) = f(z)$ for $z \in V$ and $f_1(z) = 0$ for $z \in \Omega \backslash V$. So, it makes no essential difference if we regard the sections in $\Gamma(\Omega,\mathcal{F})$ as the holomorphic functions on V.

Finally, let $\mathcal{R}$ be the sheaf of relations of $\vec{h}$, so that we have the exact sequence

$$0 \to \mathcal{R} \to \mathcal{A}^q \xrightarrow{h} \mathcal{J} \to 0.$$

By [27, IV. D.2] the sheaf $\mathcal{I}$ is coherent and by Oka's thoerem [30, th. 7.1.5] or [27, IV. B.8 and IV. C.1] also $\mathcal{R}$ is coherent. Hence we can apply Cartan's theorem B [27, VIII. A.14] or [30, th. 7.4.3], which says that the first cohomology groups $H^1(\Omega,\mathcal{I})$ and $H^1(\Omega,\mathcal{R})$ vanish. This means that the following sequences of sections over Ω are exact

$$(4.13) \qquad 0 \to \Gamma(\Omega,\mathcal{I}) \to \Gamma(\Omega,\mathcal{A}) \to \Gamma(\Omega,\mathcal{F}) \to H^1(\Omega,\mathcal{I}) = 0$$

$$(4.14) \qquad \Gamma(\Omega,\mathcal{A}^q) \xrightarrow{\vec{h}} \Gamma(\Omega,\mathcal{J}) \to H^1(\Omega,\mathcal{R}) = 0.$$

(4.13) means that the restriction map from $\Gamma(\Omega,\mathcal{A}) = A(\Omega)$ to V is a surjection and if (4.4) holds for all $z \in \Omega$, for example if $\mathcal{J}_z$ is a prime ideal for each $z \in \Omega$ (cf. chapter III), by (4.14) we find that in

$$\Gamma(\Omega,\mathcal{A}) / \vec{h}\cdot\Gamma(\Omega,\mathcal{A})^q \to \Gamma(\Omega,\mathcal{A}) / \Gamma(\Omega,\mathcal{I}) \to \Gamma(\Omega,\mathcal{F}')$$

both maps are isomorphisms. Thus any holomorphic function on V is the restriction of a holomorphic function in Ω and any function f in $A(\Omega)$ vanishing on V can be written as

$$f(z) = \sum_{k=1}^{q} h_k(z) g_k(z), \qquad z \in \Omega$$

for some $g_k \in A(\Omega)$, $k = 1,\dots,q$.

Now we will study the sheaf of modules $\vec{J}$ in A^p generated by a matrix H of holomorphic functions h_{jk} in Ω. The difference with the above is that for $p > 1$ $\vec{J}$ is not equal to the sheaf $\vec{I}$ of vector functions vanishing on an associated vector multiplicity variety $\vec{W}$, but the maps ρ_z of lemma 4.2 determine a bijection between $A^p/\vec{J}$ and $A^p/\vec{I}$. The multiplicity varieties W^ℓ_z, $\ell = 1,\dots,p$ were defined locally according to lemma 4.1. In the overlap of two neighborhoods ω_1 of z_1 and ω_2 of z_2 in Ω where $W^\ell_{z_1}$ and $W^\ell_{z_2}$ are defined they can be choosen to coincide, so that $\vec{W} = \bigcup_{z\in\Omega} \vec{W}_z$ is a global, analytic vector multiplicity variety in Ω. Moreover, in lemma 4.3 we will show that $\rho_z\vec{f}_z$ is the germ of a section in $\Gamma(\omega, A^p/\vec{I})$ if $\vec{f}_z$ is a germ of a section $f \in \Gamma(\omega, A^p) = A(\omega)^p$. This means that ρ_z determines a sheaf homomorphism between sheafs of linear spaces, so that the following sequence is exact

$$0 \to \vec{J} \to A^p \overset{\rho}{\to} F \to 0.$$

where, as before, we may consider

$$F \overset{\text{def}}{=\!=} \bigcup_{z\in\Omega} A^p_z/\vec{I}_z$$

as the sheaf of holomorphic functions on $\vec{W}$. As in (4.14), it follows that the map $H\colon \Gamma(\Omega, A^q) \to \Gamma(\Omega, J)$ is surjective. So finally, since $H^1(\Omega, J) = 0$, we obtain an isomorphism ρ^L between linear spaces, defined by the map ρ followed by restriction to $\vec{W}$

$$(4.15) \qquad \rho^L\colon \Gamma(\Omega, A^p) \big/ _{H\cdot\Gamma(\Omega, A^q)} \to \Gamma(\Omega, A(\vec{W})),$$

where $A(\vec{W})$ is the sheaf of holomorphic functions on $\vec{W}$.

<u>LEMMA</u> 4.3. [16, th. 2.6]. *For any matrix* H *of holomorphic functions in* Ω, *there exist an analytic vector multiplicity variety* $\vec{W}$ *and a local restriction map* ρ^L *such that (4.15) is an isomorphism between linear spaces.*

<u>PROOF</u>. We will show that $\rho_z\vec{f}_z$ is the germ of a section over ω in $A^p/\vec{I}$ if $\vec{f} \in A(\omega)^p$. We may assume that ω is pseudoconvex. That $(\rho_z\vec{f}_z)_1$ is the germ of a section in $A(\omega)$ follows immediately from the definition. Since $(\hat{f}_1)_z$ is uniquely determined by $f_1|W^1$ it follows from (4.14) that

$$f_1(z) - \hat{f}_1(z) = \sum_{k=1}^{q} h_{1k}(z)\tilde{g}_k(z)$$

for a section $\vec{\tilde{g}} \in A(\omega)^q$. Thus in (4.8) $\vec{g}^1_z = \vec{\tilde{g}}(z) - \vec{m}^1_z$ for some $\vec{m}^1_z \in M^1_z$ and (4.9) becomes

$$(\rho_z \vec{f}_z)_2 = f_2(z) - \sum_{k=1}^{q} h_{2k}(z)\tilde{g}_k(z) + \sum_{k=1}^{q} h_{2k}(z)(m^1_k)_z,$$

which is a section in $\Gamma(\Omega, A/I^2)$, because the last term belongs to I^2_z. Let M be a locally finitely generated subsheaf of $\vec{A}$ over ω, let $\vec{h}$ be a vector of holomrphic functions in ω and let F be the sheaf $\vec{h}\cdot M$, i.e., the sequence

$$0 \to R \to M \xrightarrow{\vec{h}} F \to 0$$

is exact for some coherent analytic sheaf R, cf. [30, th. 7.1.5] & [30, th. 7.1.7] or [27, IV. B.13]. Hence as in (4.14) the map $\vec{h}:\Gamma(\omega,M) \to \Gamma(\omega,F)$ is surjective. For a function $k \in A(\omega)$ $k|W^\ell$ determines uniquely a function $\hat{k}^\ell \in A(\omega)$, hence $k - \hat{k}^\ell$ is a section in $\Gamma(\omega,F)$ where F is determined as above with $M = M^{\ell-1}$ and $\vec{h} = (h_{\ell 1},\dots,h_{\ell q})$. Therefore, $k - \hat{k}^\ell = \sum_k h_{\ell k}\tilde{m}^{\ell-1}_k$ for some vector function $\vec{\tilde{m}}^{\ell-1} \in A(\omega)^q$ satisfying (4.6) (with g_k replaced by $\tilde{m}^{\ell-1}_k$). Thus for $\ell = 2,\dots,p-1$, successively, we find that there is some global function $\vec{\tilde{m}}^{\ell-1} \in A(\omega)^q$ with

$$f_\ell(z) - \sum_{j=1}^{\ell-1}\sum_{k=1}^{q} h_{\ell k}(z)(g^j_k)_z - \hat{f}_\ell(z) =$$

$$= \sum_{k=1}^{q} h_{\ell k}(z)\{\tilde{m}^{\ell-1}_k(z) + (m^{\ell-1}_k)_z\},$$

hence by (4.10) that $\vec{g}^\ell_z = \vec{\tilde{m}}^{\ell-1}(z) + \vec{m}^{\ell-1}_z - \vec{m}^\ell_z$ for some $\vec{m}^\ell_z \in M^\ell_z$, and by (4.11) that

$$(\rho_z \vec{f}_z)_{\ell+1} = f_{\ell+1}(z) - \sum_{j=2}^{\ell}\sum_{k=1}^{q} h_{\ell+1k}(z)\{\tilde{m}^{j-1}_k(z) + (m^{j-1}_k)_z - (m^j_k)_z\} -$$

$$- \sum_{k=1}^{q} h_{\ell+1k}(z)\{\tilde{g}_k(z) - (m^1_k)_z\} =$$

$$= f_{\ell+1}(z) - \sum_{j=1}^{\ell-1}\sum_{k=1}^{q} h_{\ell+1\,k}(z)\,\tilde{m}^j_k(z) -$$

$$- \sum_{k=1}^{q} h_{\ell+1\,k}(z)\,\tilde{g}_k(z) + \sum_{k=1}^{q} h_{\ell+1\,k}(z)(m^\ell_k)_z$$

determines a section in $A/I^{\ell+1}$, because the last term vanishes on $W^{\ell+1}$.

From the last formula is can also be seen that a change of $\vec{g}^{\ell}$ does not alter $(\rho_z \vec{f}_z)_{\ell+j}$ for $j \geq 2$, because the choice of $\vec{m}_z^{\ell}$ determines the germ $\vec{g}_z^{\ell}$.

□

Thus any holomorphic function in $\Gamma(\Omega, A(\vec{W}))$ is the image under ρ^L of a holomorphic vector function in Ω and any holomorphic vector function $\vec{f} \in A(\Omega)^p$ vanishing under ρ^L on $\vec{W}$ can be written as $\vec{f} = H \cdot \vec{g}$ for some $\vec{g} \in A(\Omega)^q$.

REMARK. It follows that the holomorphic functions f on a vector multiplicity variety $\vec{W}$ are defined as restrictions of a collection $\{f^{\omega} | \omega \subset\subset \Omega\}$ of locally defined holomorphic functions, i.e., by (4.5) for all $\omega \subset\subset \Omega$ we have, if $f = \{f_1, \ldots, f_r\}$,

$$f_j(z) = \partial_j f^{\omega}(z), \quad z \in V_j \cap \omega.$$

Only if $p = 1$, a holomorphic function on W is also the restriction of an entire function, where restriction is defined in (4.5) which in this case defines the map ρ^L, too.

IV.3. EHRENPREIS' AND PALAMODOV'S FUNDAMENTAL PRINCIPLE.

In this section we will mention the fundamental principle with spaces of entire functions satisfying certain growth conditions, formulated by Ehrenpreis in [16] and by Palamodov in [56]. We shall not discuss all these conditions in full detail, but in the next section we shall give alternative conditions, which enables us to generalize the principle. The only purpose of this section is to relate our work to that of Ehrenpreis and Palamodov.

If $\Omega = \mathbb{C}^n$, H is a matrix of polynomials and if all the functions in (4.15) are bounded with respect to certain weighted sup-norms, then the fact that ρ^L is a topological isomorphism is sometimes also called the fundamental principle. This is formulated by Ehrenpreis in [16, th. 4.2] and by Palamodov in [56, IV, § 5. th. 2] and the difference between these two are the conditions on the bounds. The need for bounds makes it necessary to consider matrices P of polynomials with associated polynomial vector multiplicity varieties $\vec{W}$, instead of matrices H of arbitrary entire functions. Our discussion will mainly follow the lines of [16], but at the end of this section we will make some remarks on Palamodov's formulation, which holds in convex tube domains Ω, too.

Firstly, we remark that the sheaf of relations between a finite number of polynomials is globally finitely generated by polynomilas [30, lemma 7.6.3]. Hence the vector multiplicity variety $\vec{W}$ of lemma 4.3 will be a polynomial vector multiplicity variety. Furthermore, there are only finitely many possible polynomial vector multiplicity varieties to choose $\vec{W}$ from. Unfortunately, for obtaining bounds one cannot use the same multiplicity variety at each place. This difficulty can be overcome by taking for $\vec{W}$ the union of all the possibilities, so that at every place the bounds hold for at least one multiplicity variety. That this yields no more complications, has been shown in [16, proof of (4.9), p. 102-105]. Moreover, the choice of the functions $\vec{g}$ at every place in the definition of the map ρ (cf. (4.11)) can be done in such a way that we obtain good bounds. Due to this the functions $\vec{g}_z$ depend on the place z (actually, $\vec{g} = \{\vec{g}_\omega\}$ depends on a priori given bounded sets ω of a covering of $\mathbb{C}^n$), but in the proof of lemma 4.3 we have seen that this produces no problems for obtaining sections on $\vec{W}$. We only remark that the map ρ^L has been defined by restricting the entire functions to any set ω of the covering, next by applying the map ρ_z with the $\vec{g}_\omega$'s belonging to that ω and finally by restriction to $\vec{W}$. This yields a section on $\vec{W}$ which is defined by a collection of semi-local functions.

In order to discuss the conditions on the bounds, we describe the general structure of the allowed spaces H of entire functions. An *analytical uniform structure* K on H is a collection of continuous positive functions k on $\mathbb{C}^n$, such that for each $F \in H$ and each $k \in K$

$$F(z)/k(z) \to 0 \quad \text{as } \|z\| \to \infty$$

and such that the sets

$$\{F \in H | F(z) \leq k(z), z \in \mathbb{C}^n\}$$

form a fundamental system of neighborhoods of zero in H. Then the space $W = \mathcal{F}H'$, the Fourier transform of the dual H' of H, is called an analytically uniform space, AU-space, cf [16, p. 9, (a), (b) & (c)] or [2, p. 7 (1) (iii)].

The set K is not uniquely determined by H. We require that [16, p. 96 (a) & (b)] or [2, p. 8 (iv)]

(i) any entire function which is $\mathcal{O}(k(z))$ for all $k \in K$ is in H

(ii) for any $N > 0$, if we replace the analytically uniform structure $K = \{k\}$ by $K_N = \{k_N\}$ where

$$k_N(z) \overset{\text{def}}{=\!=} \max_{\|z-z'\| \le N} k(z')(1+\|z'\|)^N, \tag{4.16}$$

then K_N is again an analytically uniform structure for W.

The AU-structure K provides the space $H(\mathcal{W})$ of restrictions to $\mathcal{W}$ satisfying the bounds induced by K with a topology in a very natural way: from (4.16) it follows that together with F also all its derivatives belong to H; let $\mathcal{W} = \{V_1, \partial_1; \ldots; V_r, \partial_r\}$ and let $g = (g_1, \ldots, g_r)$ be a section on $\mathcal{W}$, i.e., in the bounded sets ω in $\mathbb{C}^n$ with $\omega \cap V_j \neq \emptyset$ for some $j \in \{1,\ldots,r\}$ there is a holomorphic function h^ω with $\partial_j h^\omega | V_j = g_j$, $j = 1,\ldots,r$, cf. (4.5); then the space $H(\mathcal{W})$ is defined as the set of all sections g on $\mathcal{W}$ satisfying for every $k \in K$

$$|g_j(z)| \big/ k(z) \le C, \quad z \in V_j, \quad j = 1,\ldots,r \tag{4.17}$$

for some $C \ge 0$ depending on k; with $C > 0$ and $k \in K$ fixed condition (4.17) determines an open set of a 0-neighborhood base of the topology of $H(\mathcal{W})$.

<u>LEMMA</u> 4.4. (Ehrenpreis' fundamental principle) *Let H be a space of entire functions with an AU-structure satisfying certain conditions discussed below. Then to any matrix P of polynomials there is associated a polynomial vector multiplicity variety $\vec{\mathcal{W}}$, such that the map ρ^L, determined by lemma's 4.2 and 4.3, is a topological isomorphism from $\vec{H}/P\cdot\vec{H}$ onto $H(\vec{\mathcal{W}})$.*

An example shows that indeed further conditions are required.

<u>EXAMPLE</u>. Let H be the space of entire functions F in $\mathbb{C}^2$ satisfying for every $\varepsilon > 0$

$$|F(\theta)| \le M_\varepsilon (1+\|\theta\|)^m \exp \varepsilon \|\operatorname{Im}\theta\| ,$$

where m depends on F. Let $\mathcal{W} \overset{\text{def}}{=\!=} (\{(\theta_1,\theta_2) | \theta_2 - i\theta_1 = 0\}, \text{id.})$, then the growth conditions of H yield the space $H(\mathcal{W})$ of entire functions f in $\mathbb{C}$ satisfying for every $\varepsilon > 0$

$$|f(z)| \le M_\varepsilon \exp \varepsilon |z|.$$

However, it is not true that any function in $H(\mathcal{W})$ can be extended to a function in H. For example, the function

$$f(z) \overset{\text{def}}{=\!=} \oint \exp(iz\zeta + 1/\zeta)d\zeta \in H(\mathcal{W})$$

cannot be written as $f(z) = F(z,iz)$ with $F \in H$, since all functions in H are polynomials, see [68, 29.1], while f is not.

An AU-space W is called *localizable*, LAU-space, if H satisfies such conditions that lemma 4.4 holds. In order to let W be localizable in [16, p. 96(c)] or [2, p. 8(v)] the following condition has been imposed: there is a family M (BAU-structure) of continuous positive functions m on $\mathbb{C}^n$ with for every $m \in M$ and $k \in K$ $m(z) = \mathcal{O}(k(z))$ such that the bounded sets

$$\{F \in H \,|\, |F(z)| \le \alpha\, m(z), z \in \mathbb{C}^n\}, \quad \alpha > 0,\ m \in M$$

define a fundamental system of bounded sets in H; moreover, the functions $k \in K$ and $m \in M$ can be written as a product of functions k_i and m_i, respectively, of the variable z_i, $i = 1,\dots,n$ and these functions must satisfy certain conditions [16, (4.3) & (4.4)] or [2, p. 21(vii) & (viii)], among others [2, (viii)]: for every $\varepsilon > 0$ and for every $m = m_1 \dots m_n \in M$ there is $m^* = m_1^* \dots m_n^* \in M$ such that for every $j = 1,\dots,n$ and any $z_0 = x_0 + iy_0 \in \mathbb{C}^1$ there exists an entire function ϕ in $\mathbb{C}^1$ for which

$$(4.18) \qquad m_j(z_0)\,|\phi(z)| \Big/ \min_{|\zeta - z_0| \le \varepsilon} |\phi(\zeta)| \le m_j^*(z), \quad z \in \mathbb{C}^1.$$

If these conditions are satisfied the space W is called *product localizable*, PLAU-space, cf. [16].

In the example we have defined the space H by the PLAU- structure

$K = \{k \,|\, k(\theta) = k_1(\mathrm{Re}\,\theta_1)\, k_2(\mathrm{Im}\,\theta_1)\, k_1(\mathrm{Re}\,\theta_2)\, k_2(\mathrm{Im}\,\theta_2)$, k_1 is a continuous function dominating all polynomials and $k_2(y) = \exp \varepsilon|y|$, $\varepsilon > 0\}$.

Another possible PLAU-structure would be

$K' = \{k \,|\, k(\theta) = k_1(|\theta_2|)\, k_1(|\theta_2|)$, k_1 is a continuous function dominating all polynomials$\}$.

A BAU-structure M belonging to K is

$$M = \{m|m(\theta) = m_1(\mathrm{Re}\,\theta_1)\, m_2\,(\mathrm{Im}\,\theta_1)\, m_1\,(\mathrm{Re}\,\theta_2)\, m_2(\mathrm{Im}\,\theta_2), m_1(x) = \alpha(1+|x|)^{\ell},\ \alpha > 0,\ \ell > 0 \text{ and } m_2(y) \text{ is a continuous, positive function which is dominated by every function } \exp \varepsilon|y|, \varepsilon > 0\}$$

and a BAU-structure M' belonging to both K and K' is

$$M' = \{m|m(\theta) = m_1(|\theta_1|)\, m_1\,(|\theta_2|),\ m_1(x) = \alpha(1+|x|)^{\ell}, \alpha > 0, \ell > 0\}.$$

M' satisfies condition (4.18), but M does not satisfy it, because m_2 is allowed to be a function that itself dominates all polynomials. In the example K defined the PLAU-structure and the growth conditions of $H(\mathcal{W})$. Hence the BAU-structure, which completes the conditions for product localizablity, must be M'. However, M' does not induce a BAU-structure on $H(\mathcal{W})$. A BAU-structure on $H(\mathcal{W})$ would be the one induced by M.

Besides condition (4.18), the condition that M induces a BAU-structure on $H(\mathcal{W})$ is used to extend a collection of semilocally defined functions satisfying the bounds on $\mathcal{W}$ to a globally defined function in $\mathbb{C}^n$ satisfying the right bounds. Thus in the example this condition is not satisfied.

Now there are two ways to get rid of the problems exposed by the example. Either, if one wants to define $H(\mathcal{W})$ by one of the AU-structures K on H, cf. [2], one moreover has to require that the BAU-structure M on H, belonging to K and satisfying the conditions for PLAU-structure (among others condition (4.18)), induces also a BAU-structure on $H(\mathcal{W})$. This assumption has been omitted in [2]. Or, the space $H(\mathcal{W})$ should be defined as the one induced by *all* the possible AU-structures on H, cf. [16]. The special condition is satisfied then, but one has to know all the possible AU-structures on H.

REMARK. In the following sections we will present the fundamental principle in a different way using the L^2-estimates for the Cauchy-Riemann operator given by Hörmander in [30]. Then the above mentioned problems are avoided and less involved conditions will be required on the growth conditions for the functions in H. These conditions and those of [16] are not always comparable. For example, the space $\mathcal{D}'$ of distributions is LAU in the sense of [16], but our method does not work for the space H = Z. On the other hand, the approach followed here enables us to derive the principle for the space

$\mathcal{E}(U)$ of C^∞ functions in a convex set $U \subset \mathbb{R}^n$, while the methods of [16] only yields that $\mathcal{E}(U)$ is PLAU when U is a cube or that $\mathcal{E}(U)$ is LAU when U is a convex polyhedron, cf. [16, remark 4.5]. As far as the Ehrenpreis-Martineau theorem [16, th. 5.21] is concerned the fact that U must be a polyhedron is not serious, because between any two ε-neighborhoods of a bounded, convex set in $\mathbb{R}^n$ there lies a convex polyhedron P and the theorem follows by application of the fundamental principle to the space $\mathcal{E}(P)$. However, in chapter III we discussed a similar theorem for analytic functionals carried by unbounded convex sets with respect to ε-neighborhoods and in general no polyhedra lie between two such neighborhoods. The Fourier transforms of these analytic functionals are no longer entire functions and we need the fundamental principle for spaces H consisting of functions holomorphic in some pseudoconvex domain and satisfying certain growth conditions there.

For some parts of our needs the fundamental principle of Palamodov in [56] suffices. For, he does not necessarily deal with entire functions, as the theorems of [56] are valid for functions holomorphic in convex tube domains. More, precisely he considered an increasing sequence of majorants M_α of the form

$$M_\alpha(z) = R_\alpha(z) \exp I_\alpha(y)$$

[56, III. § 1.1^0 & 4^0]. Here R_α is an everywhere finite and positive function in $\mathbb{C}^n$ and I_α is a convex function which need only to be defined in a convex set U_α in $\mathbb{R}^n$ with the property that an ε_α-neighborhood of $U_{\alpha+1}$ is contained in U_α. Furthermore, the functions $\{R_\alpha\}_{\alpha=1}^\infty$ and $\{I_\alpha\}_{\alpha=1}^\infty$ have to satisfy a condition similar to (4.16), namely for $y \in U_{\alpha+1}$

$$(1+\|z\|)R_\alpha(z) \le K_\alpha R_{\alpha+1}(z), \quad (1+\|y\|)\exp I_\alpha(y) \le K_\alpha \exp I_{\alpha+1}(y)$$

$$\sup_{\|z-z'\|\le\varepsilon_\alpha} R_\alpha(z') \le K_\alpha R_{\alpha+1}(z), \quad \sup_{\|y-y'\|\le\varepsilon_\alpha} \exp I_\alpha(y') \le$$

$$\le K_\alpha \exp I_{\alpha+1}(y)$$

and a condition somewhat similar to (4.18) but less involved. The fundamental theorem in [56, IV. § 5, th. 2], the isomorphism (4.15), has a weaker form with respect to the bounds than in [16].

<u>LEMMA</u> 4.5. (Palamodov's fundamental principle). *For any matrix* P *of polynomials there is associated a polynomial vector multiplicity variety* $\vec{\mathcal{W}}$, *such that any holomorphic function in* $\vec{\mathcal{W}} \cap (\mathbb{R}^n + i\,U_\alpha)$, *which is bounded in absolute value by* M_α *on* $\vec{\mathcal{W}}$, *can be extended under* $(\rho^L)^{-1}$ *to a function holomorphic in* $\mathbb{R}^n + i\,U_{\alpha+m}$ *and bounded there in absolute value by* KM_α, *for some* $K > 0$ *and positive integer* m. *Moreover, any holomorphic function* $\vec{f}$ *in* $(\mathbb{R}^n + i\,U_\alpha)^p$, *bounded in absolute value by* M_α *there and vanishing under* ρ^L *on* $\vec{\mathcal{W}} \cap (\mathbb{R}^n + i\,U_\alpha)$, *can be written as*

$$\vec{f} = P\cdot\vec{g} \tag{4.19}$$

for some $\vec{g}$ *holomorphic in* $(\mathbb{R}^n + i\,U_{\alpha+m})^q$ *and bounded there in absolute value by* $K\,M_\alpha$.

If $\Omega = \mathbb{C}^n$ we have $U_\alpha = \mathbb{R}^n$ for every α. Then the difference with [16] is that in [16] a holomorphic function in $H(\vec{\mathcal{W}})$ has been extended under $(\rho^L)^{-1}$ to one function satisfying all the bounds and if $\vec{f}$ vanishes on $\vec{\mathcal{W}}$ it can be written as (4.19) where $\vec{g}$ also satisfies all the bounds.

Now problem 3.1 of the last chapter can be solved by lemma 4.5 and indeed it is contained in [56, III, § 5, theorem and 9^0], but problems 3.2 and 3.3 cannot be solved in this way. Palamodov applied the fundamental principle to the Cauchy-Riemann equations in [56, VI, § 4, 4^0, cor. 3] which contains the Ehrenpreis-Martineau theorem. From this corollary the theorems of chapter III.3 can be derived [1)], but we can not apply it to obtain the remaining theorems of chapter III. The reason is that we are concerned with holomorphic functions in the tube domains $\{\mathbb{R}^n + i\Gamma^k\}_{k=1}^\infty$, where the convex sets $\Gamma^k \subset \Gamma^{k+1} \subset \Gamma$ do not have the property that an ε_k-neighborhood of Γ^k is contained in Γ^{k+1}.

In the next section we will discuss different conditions on the bounds and the fundamental priciple (in a similar weak form as in [56]) for functions holomorphic in tube domains $\Omega \neq \mathbb{C}^n$ will be considerably more general than in [56]. For $\Omega = \mathbb{C}^n$ one has in fact three fundamental principles, which supplement each other.

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1) Actually, due to condition [56, (5.3) p. 240] one has to assume that $\Omega(a,\Gamma)$ contains a neighborhood of the origin, i.e., a is a positive function on Γ.

IV.4. THE FUNDAMENTAL PRINCIPLE FOR SPACES OF NON-ENTIRE FUNCTIONS.

In this section we will formulate the fundamental principle for spaces H of non-entire functions. As in [16] we will express the topology of H by projective limits, i.e., H will have an AU-structure. As far as the fundamental principle (the isomorphism ρ^L) is concerned this will not be necessary, as the principle essentially follows from the semilocal theory of [16, ch. III] and from theorems 4.11 and 4.12 of section 6 of this chapter, but in chapter V it will be convenient to have spaces H whose topology is defined by a projective limit, although an extra condition is needed then.

We will assume that the growth conditions on the functions of H can be expressed by L^p-norms with respect to weight functions of the form $\exp -\phi^\alpha$ for $\alpha \in A$, where A is a directed set and where $\{\phi^\alpha\}_{\alpha\in A}$ is a decreasing net of plurisubharmonic functions in a pseudoconvex domain $\Omega \subset \mathbb{C}^n$. Furthermore, let $\{\Omega_k\}_{k=1}^\infty$ be an increasing sequence of relatively closed subsets of Ω with union Ω. Denote for $p = 1,2,\ldots$ and for a function f

$$(4.20) \qquad \|f\|_{\alpha,k}^{(p)} \overset{\text{def}}{=\!=} \{ \int_{\Omega_k} |f(z)|^p \exp - p\phi^\alpha(z)d\lambda(z)\}^{1/p}$$

where $\lambda(z)$ is de Lebesgue measure in $\mathbb{C}^n$, and for $p = \infty$

$$\|f\|_{\alpha,k}^{(\infty)} \overset{\text{def}}{=\!=} \sup_{z\in\Omega_k} |f(z)|\exp - \phi^\alpha(z);$$

when $p = 2$ we will write $\|\cdot\|_{\alpha,k}$ instead of $\|\cdot\|_{\alpha,k}^{(2)}$. If f is bounded with respect to the norm

$$\|f\|_{\alpha}^{(p)} \overset{\text{def}}{=\!=} \{ \int_{\Omega} |f(z)|^p \exp - p\phi^\alpha(z)d\lambda(z)\}^{1/p}$$

for $p = 1,2,\ldots$ or

$$\|f\|_{\alpha}^{(\infty)} \overset{\text{def}}{=\!=} \sup_{z\ \Omega} |f(z)| \exp - \phi^\alpha(z)$$

for $p = \infty$, we will sometimes express this by saying that the sequence $\{\|f\|_{\alpha,k}^{(p)}\}_{k=1}^\infty$ is bounded. For $p = 1,2,\ldots,\infty$ let

$$H_p(\Omega_k;\phi^\alpha)$$

be the Banach space of functions holomorphic in int Ω_k, and in case $p = \infty$ also continuous on Ω_k, such that the norm (4.20) is finite, and let

$$H_p[\Omega;\phi^\alpha] \overset{\text{def}}{=\!=} \operatorname*{proj\,lim}_{k\to\infty} H_p(\Omega_k;\phi^\alpha)$$

where in the projective limit the restriction maps from Ω_{k+1} to Ω_k are intended. When $p = 2$ we will just write $H[\Omega;\phi^\alpha]$.

If all the sets Ω_k are different, the following conditions are imposed:

$$\forall k,\ \exists \ell > k:\ \forall z \in \Omega_k,\ \forall z' \in B(z;1/2,1) \Rightarrow z' \in \Omega_\ell, \tag{4.21}$$

where for $0 \le \delta < 1$ and $K \ge 0$

$$B(z;\delta,K) \overset{\text{def}}{=\!=} \{z' \mid \|z'-z\| \le \min[K,\delta d(z,\Omega^c)]\};$$

here $d(z,\Omega^c)$ denotes the distance from z to the complement of Ω, i.e.,

$$d(z,\Omega^c) \overset{\text{def}}{=\!=} \inf_{z'\in\Omega^c} \|z-z'\|.$$

There must exist a plurisubharmonic function σ in Ω with

$$\Omega_k = \{z \mid z \in \Omega, \sigma(z) \le k\}. \tag{4.22}$$

For compact sets Ω_k (4.22) is not a special condition on Ω, cf. [30, th. 2.6.7.ii], but we have in mind unbounded sets Ω_k.

Finally, we have to make an assumption on the net $\{\phi^\alpha\}$. Although it is not necessary, the proof of theorem 6.4 will be simpler if we would have neighborhoods $B(z;\delta,K)$ of z with the property that the neighborhood

$$\cup\{B(z';\varepsilon,L) \mid z' \in B(z;\delta,K)\}$$

of z itself is contained in a neighborhood $B(z;\eta,M)$ of z for some η and M. Since this is not true for the neighborhoods B we will define quite similar neighborhoods S which do have this property. Let for $\varepsilon \ge 0$ and $K \ge 0$

$$D(z;\varepsilon,K) \overset{\text{def}}{=} \{z' \mid z' \in \Omega, \|z'-z\| \le \min[\varepsilon d(z,\Omega^c), \varepsilon d(z',\Omega^c), K]\}.$$

Then

(4.23) $$B(z;\delta,K) \subset D(z;\delta/(1-\delta),K)$$

and

$$\cup\{D(z';\varepsilon,L) \mid z' \in D(z;\delta,K)\} \subset D(z;\varepsilon+\varepsilon\delta+\delta,K+L).$$

So if for positive K we define the neighborhood of z

(4.24) $$S(z;K) = D(z;e^K-1,K),$$

then

(4.25) $$\cup\{S(z';K) \mid z' \in S(z;L)\} \subset S(z;K+L).$$

For a function ϕ in Ω and for $N,M,K \ge 0$ define, cf. (3.40),

(4.26) $$\phi_{N,M,K}(z) = \max\{\phi(z') + N\log(1+\|z'\|^2) + \log(1+d(z',\Omega^c)^{-M}) \mid z' \in S(z,K)\}.$$

If $N = M = K$ we will just write ϕ_N and if for $p = 2$ in the norm (4.20) ϕ^α is replaced by $\phi^\alpha_{N,M,K}$ or ϕ^α_N we will denote that norm by $\|\cdot\|^{N,M,K}_{\alpha,k}$ or $\|\cdot\|^N_{\alpha,k}$, respectively. The functions $\log(1+\|z\|^2)$ and $\log(1+d(z,\Omega^c)^{-M})$ are plurisubharmonic in Ω, [30, (4.4.6) and th. 2.6.2] and [30, th. 2.6.7 (i) and cor. 1.6.8]. For $\Omega = \mathbb{C}^n$ we have $S(z;K) = \{z' \mid \|z-z'\| \le K\}$ and then, as in the proof of theorem 3.1, [30, th. 1.6.2] and lemma 3.2 imply that $\phi_{N,M,K}$ (which in this case does not depend on M) is plurisubharmonic if ϕ is. Due to property (4.25) for $N_1,N_2 \ge 0$ and for a function ϕ in Ω we have

(4.27) $$(\phi_{N_1})_{N_2} \le \phi_{N_1+N_2}.$$

Our final requirement is that for every $N \ge 0$ and $\alpha \in A$ there is a $\alpha' \ge \alpha$ and a positive constant $C_{\alpha,N}$ with

(4.28) $\qquad \phi_N^{\alpha'} \le \phi^\alpha + C_{\alpha,N}.$

We now define the space H. Condition (4.28) implies that for every $N \ge 0$

$$(4.29) \qquad H \overset{\text{def}}{=\!=} \operatorname*{proj\,lim}_{\alpha \in A} H_p[\Omega;\phi^\alpha] = \operatorname*{proj\,lim}_{\alpha \in A} H_p[\Omega;\phi_N^\alpha],$$

where the identity maps from $H_p[\Omega;\phi^{\alpha'}]$ into $H_p[\Omega;\phi^\alpha]$, $\alpha' \ge \alpha$, determine the projective limit. Conditions (4.21) and (4.28) imply that H is independent of $p \in \{1,2,\dots,\infty\}$, cf. [73, cond. HS_1 & HS_2, p. 15], and that moreover for $f \in H$, $\alpha \in A$ and every k

$$(4.30) \qquad |f(z)|\exp - \phi^\alpha(z) \to 0 \quad \text{as} \quad z \to \partial\Omega \quad \text{or} \quad \|z\| \to \infty \quad \text{in} \quad \Omega_k.$$

If $\Omega = \mathbb{C}^n$ and $k = \exp\phi$, then (4.26) yields that $k_N = \exp\phi_N$, where k_N is given by (4.16) and the condition on the AU-structure of H given there is just our condition (4.29).

Let P be a p×q-matrix of polynomials and let $\vec{\mathcal{W}}$ be an associated polynomial vector multiplicity variety. We define the Frechet space $H_\infty[\vec{\mathcal{W}} \cap \Omega; \log k]$ as the space of sections $\vec{g}$ on $\vec{\mathcal{W}} \cap \Omega$ such that for each component $g = \{g_1,\dots,g_r\}$ of $\vec{g}$ (4.17) holds only for $z \in \Omega_\ell \cap \vec{\mathcal{W}}$ and for C depending on ℓ, provided with the semi-norms obtained by taking from all the components g of $\vec{g}$ the largest supremum of the left hand side of (4.17) over $z \in \Omega_\ell \cap V_j$, $j = 1,\dots,r$. Again if $\Omega_\ell = \Omega$ for all ℓ we will write $H_\infty(\vec{\mathcal{W}} \cap \Omega; \log k)$ instead of $H_\infty[\vec{\mathcal{W}} \cap \Omega; \log k]$ and then this is a Banach space.

The fundamental principle proved in this chapter (the completions of the proofs will be given in chapter VI) says that the map ρ^L

$$(4.31) \qquad \operatorname*{proj\,lim}_{\alpha \in A} \left\{ H[\Omega;\phi^\alpha]^p \big/ H[\Omega;\phi^\alpha]^p \cap P\cdot H[\Omega;\phi^\alpha]^q \right\} \xrightarrow{\rho^L} \operatorname*{proj\,lim}_{\alpha \in A} H_\infty[\vec{\mathcal{W}} \cap \Omega;\phi^\alpha]$$

is a toplogical isomorphism between linear spaces. Here ρ^L is defined by restriction if $p = 1$ and (only semilocally) by lemma's 4.2 and 4.3 if $p > 1$. In section 6, formula (4.44) we will show that the space on the left hand side remains the same if we replace $H[\Omega;\phi^\alpha]^p \cap P\cdot H[\Omega;\phi^\alpha]^q$ in the denominator by its closure in $H[\Omega;\phi^\alpha]^p$. Hence the left hand side of (4.3) is a Hausdorff space; its elements can be described as follows: for $\vec{f}^\alpha \in H[\Omega;\phi^\alpha]^p$ let $[\vec{f}^\alpha]$

denote the equivalence class of $\vec{f}^{\alpha}$, where $\vec{f}^{\alpha} \sim \vec{h}^{\alpha}$ if $\vec{f}^{\alpha} - \vec{h}^{\alpha} = P\cdot\vec{g}^{\alpha}$ for some $\vec{g}^{\alpha} \in H[\Omega;\phi^{\alpha}]^{q}$; then the elements of the space on the left hand side of (4.31) can be identified with such nets $\{[\vec{f}^{\alpha}]\}_{\alpha\in A}$ of equivalence classes, where $\vec{f}^{\alpha} \in H[\Omega;\phi^{\alpha}]^{p}$ for every $\alpha \in A$, that for every α and β in A with $\beta \geq \alpha$ there is a $\vec{g}^{\alpha,\beta} \in H[\Omega;\phi^{\alpha}]^{q}$ with

$$\vec{f}^{\alpha} - \vec{f}^{\beta} = P\cdot\vec{g}^{\alpha,\beta}.$$

If $\Omega_k = \Omega$ for every k, we define a space H with the only requirement that for every $N \geq 0$ H can be written as

$$\text{(4.32)} \qquad H \overset{\text{def}}{=\!=} \operatorname*{proj\,lim}_{\alpha\,\in\,A} H_p(\Omega;\phi^{\alpha}) = \operatorname*{proj\,lim}_{\alpha\,\in\,A} H_p(\Omega;\phi_N^{\alpha}).$$

Finally, if $\{\Omega_{\ell}\}_{\ell=1}^{\infty}$ is a decreasing sequence of pseudoconvex domains and if $\{\phi^{\alpha}\}$ is a decreasing net of plurisubharmonic functions in Ω_1, it is possible to consider the following space H, which for every $N \geq 0$ by assumption can be written as

$$\text{(4.33)} \qquad H \overset{\text{def}}{=\!=} \operatorname*{ind\,lim}_{\ell\to\infty}\ \operatorname*{proj\,lim}_{\alpha\,\in\,A} H_p(\Omega_{\ell};\phi^{\alpha}) = \operatorname*{ind\,lim}_{\ell\to\infty}\ \operatorname*{proj\,lim}_{\alpha\,\in\,A} H_p(\Omega_{\ell};\phi_N^{\alpha}),$$

where ϕ_N^{α} is defined by (4.26) with Ω replaced by Ω_1. Also here the spaces (4.32) and (4.33) are independent of $p \in \{1,2,\ldots,\infty\}$, provided that in the last case

$$\text{(4.34)} \qquad \forall\ell,\ \exists k > \ell,\ \exists\delta > 0:\ \forall z \in \Omega_k\ \&\ \|z-z'\| \leq \min\{1,\delta d(z,\Omega_1^{c})\} \Rightarrow z' \in \Omega_{\ell}.$$

For the spaces H given by (4.32) or (4.33) the fundamental principle yields the isomorphisms ρ^{L}

$$\text{(4.35)} \qquad \operatorname*{proj\,lim}_{\alpha\,\in\,A}\left\{ H(\Omega;\phi^{\alpha})^{p} \Big/ H(\Omega;\phi^{\alpha})^{p} \cap P\cdot H(\Omega;\phi^{\alpha})^{q} \right\} \xrightarrow{\rho^{L}} \operatorname*{proj\,lim}_{\alpha\in\,A} H_{\infty}(\vec{\mathcal{W}} \cap \Omega;\phi^{\alpha})$$

and

$$\text{(4.36)} \qquad \operatorname*{ind\,lim}_{\ell\to\infty}\ \operatorname*{proj\,lim}_{\alpha\,\in\,A}\left\{ H(\Omega_{\ell};\phi^{\alpha})^{p} \Big/ H(\Omega_{\ell};\phi^{\alpha})^{p} \cap P\cdot H(\Omega_{\ell};\phi^{\alpha})^{q} \right\} \xrightarrow{\rho^{L}}$$

$$\xrightarrow{\rho^L} \operatorname*{ind\,lim}_{\ell \to \infty} \; \operatorname*{proj\,lim}_{\alpha \in A} H_\infty(\vec{W} \cap \Omega\,;\phi^\alpha),$$

respectively.

THEOREM 4.6 (fundamental principle). *Let Ω be a pseudoconvex domain and let $\{\phi^\alpha\}$ be a decreasing net of plurisubharmonic functions in Ω. To any* p×q-*matrix* P *of polynomials there are associated a polynomial vector multiplicity variety $\vec{W}$ and a restriction map ρ^L, such that (4.35) is a topological isomorphism between linear spaces, provided that condition (4.32) is satisfied. If moreover, $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ satisfies (4.21) and (4.22), the map ρ^L in (4.31) is a topological isomorphism provided that (4.29) holds. Finally, if $\{\Omega_\ell\}_{\ell=1}^{\infty}$ is a decreasing sequence of pseudoconvex domains satisfying (4.34) and if $\{\phi^\alpha\}$ is a decreasing net of plurisubharmonic functions in Ω_1, the map ρ^L in (4.36) is a toplogical isomorphism, provided that (4.33) is valid.*

In chapter VII, cor. 7.4, we will supplement this theorem.

PROOF. That ρ^L in (4.36) is an isomorphism follows from (4.33), (4.34) and the fact that ρ^L in (4.35) is an isomorphism. The remaining two sections of this chapter, as well as chapter VI, will be devoted to the proof of the assertion that the maps (4.31) and (4.35) are topological isomorphisms. □

REMARK. Let W' be a locally convex space whose Fourier transform is topologically isomorphic to one of the spaces H given by (4.29), (4.32) or (4.33) and let W be the dual of W'. Then, as in [16], in view of theorem 4.6 we might call W localizable. In most examples it is obvious how the Fourier transformation F is defined. In general, since the δ-functions in the points $z_0 \in \Omega$ belong to H', their Fourier transforms $e^{i\langle \cdot, z_0\rangle}$ belong to W. Then we can define the Fourier transform $f \in H$ of $\phi \in W'$ by

$$f(z) = (F\phi)(z) \overset{\text{def}}{=\!=} \langle e^{i\langle \zeta, z\rangle}, \phi_\zeta \rangle,$$

cf. (2.46). Here ζ varies in a certain set Ω^* in $\mathbb{C}_n$ and W consists of objects (such as functions or distributions) in Ω^*. From the requirement that F is a topological isomorphism from W' onto H it follows that the set $\{e^{i\langle \zeta, z_0\rangle} \mid z_0 \in \Omega\}$ of functions of ζ must at least be weakly* dense in W. Furthermore, if besides this set W contains all other holomorphic functions of $\zeta \in \Omega^*$ which are bounded in absolute value by $|\exp i\langle\zeta,z_0\rangle|$ with $z_0 \in \Omega$, it follows from the fact, that the geometric mean is smaller than the arithmetic mean, that for $z_1, z_2 \in \Omega$ and $0 \le t \le 1$ also

$$\exp i\langle\zeta, tz_1 + (1-t)z_2\rangle \in W.$$

Hence then the set Ω would be convex. On the other hand, it may happen that the set $\{e^{i\langle\zeta_0,z\rangle} | \zeta_0 \in \Omega^*\}$ of functions of z is contained in H, cf. the A- and Exp-spaces of chapter III. Then Ω^* is convex, too and the set $\{e^{i\langle\zeta_0,z\rangle} | \zeta_0 \in \Omega^*\}$ is dense in H. However, all these properties will not be used to derive the fundamental principle of theorem 4.6, as they are only needed when Fourier transformation comes in.

IV.5. SEMILOCAL THEORY.

In this section we shall mention the semilocal theory of [16] and we shall indicate the differences with the theory we need.

Let $\mathcal{U} = \{U_i\}_{i=1}^{\infty}$ be a certain open covering of Ω with $U_i \subset\subset \Omega$ and let $\mathcal{U}^{(1)}$ be a certain open shrinking of $\mathcal{U}$. Then the proof in [16, proof of c, p. 104] shows that any $f \in \operatorname{proj}\lim_{\alpha \in A} H_\infty[\vec{\mathcal{W}} \cap \Omega; \phi^\alpha]$ can be extended to a collection of functions c_i holomorphic in U_i and satisfying good bounds. In fact, a method similar to theorem 3.1 can be applied, see [2]. Only now one has to take into account coinciding roots of a polynomial. The procedure followed in [16], [56] or [2] uses the Weierstraß division theorem and the Lagrange interpolation formula, cf. [2, IV lemma's 1-4].

Define $C^p[\mathcal{U}, \mathcal{F}, \phi^\alpha]$ as the Hilbert space of all alternating p-cochains c on the covering $\mathcal{U}$ with values in the analytic sheaf $\mathcal{F}$ that satisfy for every k

$$(4.37) \qquad \|c\|_{\alpha,k} \overset{\text{def}}{=\!=} \left\{ \sum_{|s|=p+1} \int_{U_s \cap \Omega_k} \|c_s(z)\|^2 \exp - 2\phi^\alpha(z) d\lambda(z) \right\}^{\frac{1}{2}} < \infty,$$

where $\|f(z)\|^2 \overset{\text{def}}{=\!=} |f_1(z)|^2 + \ldots + |f_q(z)|^2$ if $f = (f_1, \ldots, f_q)$ is a vector-function. The coverings $\mathcal{U}$ and $\mathcal{U}^{(1)}$ have to satisfy certain properties listed in chapter VI, section 1, in order that the estimates can be carried over to globally defined functions and conversely.

Let $\mathcal{A}$ be the sheaf in Ω of germs of holomorphic functions and let $\mathcal{F}$

be the image under P of the sheaf $\mathcal{A}^q$, thus $F \overset{\text{def}}{=\!=} P\cdot\mathcal{A}^q \subset \mathcal{A}^p$. Finally, let $C^t[\mathcal{U},\mathcal{A}^p,\phi;P]$ be the set of t-cochains $c \in C^t[\mathcal{U},\mathcal{A}^p,\phi]$ with

$$\delta c \in C^{t+1}(\mathcal{U},F)$$

where δ is the coboundary operator.

LEMMA 4.7. *For any* p×q-*matrix* P *of polynomials and associated polynomial vector multiplicity variety* $\vec{W}$ *the map*

$$\operatorname*{proj\,lim}_{\alpha\in A} C^0[\mathcal{U},\mathcal{A}^p,\phi^\alpha;P] \Big/ \operatorname*{proj\,lim}_{\alpha\in A} C^0[\mathcal{U},\mathcal{A}^p,\phi^\alpha;P] \cap P\cdot \operatorname*{proj\,lim}_{\alpha\in A} C^0[\mathcal{U}^{(1)},\mathcal{A}^q,\phi^\alpha] \longrightarrow$$

$$\longrightarrow \operatorname*{proj\,lim}_{\alpha\in A} H_\infty[\vec{W}\cap\Omega;\phi^\alpha]$$

given by lemma 4.3 is a topological isomorphism.

PROOF. We shall not give all the details, because these can be found in [16]. There a function $f \in \operatorname*{proj\,lim}_{\alpha\in A} H_\infty[\vec{W}\cap\Omega;\phi^\alpha]$ has been extended to a collection of functions $\{c_s\}_{s=1}^\infty$ with c_s holomorphic in U_s. Firstly, in [16, proof of c, p. 104] for each s f is extended to a finite collection of functions holomorphic in finitely many very small sets covering U_s, whose differences in the overlaps are sections in F. Then one has to apply a piecing together process of this collection of functions to one function c_s in U_s. As is remarked in [16] this process follows the same lines as the proof of the similar statements for the map λ we will define in the next section and even it is simpler, because U_s is a bounded set so that no convergence factors such as ϕ arising in condition (4.18) are needed. We have not assumed this condition, so that the proof of [16] is valid here, too. Of course, one can also follow the piecing together process we will perform in chapter VI.

Let us briefly mention the differences with [16] arising from the sizes of the sets of the covering of Ω we have here. In [16] all the sets of the covering of $\mathbb{C}^n$ have the same size. There each set U_s is covered in such a way that the bounds for c_s depend on the bounds for f on $V_s \cap \vec{W}$, where V_s is the enlargement by a factor 2 of U_s the center z_s kept fixed.

Furthermore, the minimal size of the sets that cover U_s is proportional to a power of $(1+\|z_s\|)^{-1}$ and to a power of the size β_s of U_s. Also, the maximal number of sets covering U_s is proportional to a power of $1+\|z_s\|$ and to β_s^{-1}. However, these powers do not depend on s, see [16, ch. III]. It follows from the piecing together process of chapter VI or of [16] that c_s satisfies for some N and K independent of s

$$\left\{\int_{U_s} \|c_s(z)\|^2 d\lambda(z)\right\}^{\frac{1}{2}} \le K\left(\frac{1+\|z_s\|^2}{\beta_s}\right)^N \sup_{z\in V_s\cap \vec{\mathcal{W}}} \|f(z)\| .$$

where $\|f(z)\|$ here denotes the maximum of $f_j^{\ell}(z)$ for $\ell = 1,\dots,p, j = 1,\dots,r_{\ell}$ if $f^{\ell} = (f_1^{\ell},\dots,f_{r_\ell}^{\ell})$ is the section on $\mathcal{W}^{\ell}$ determined by f. Actually, in [16] c_s is bounded in sup-norm, but [73, cond. HS_1, p. 15] shows that this implies the estimate we have here, because the sizes of the sets U_s will be bounded.

The sets U_s will be such that they have a fixed size if they are far enough from $\partial\Omega$ or that the size is proportional to d_s, where d_s is the distance from U_s to $\partial\Omega$. Therefore, since by (4.24) for sufficiently large N we have $z_s \in S(z;N)$ if $z \in U_s$ and $V_s \subset S(z_s;N)$, for every $\alpha \in N$ we get

$$\left\{\int_{U_s} \|c_s(z)\|^2 \exp - 2\phi_N^{\alpha}(z)d\lambda(z)\right\}^{\frac{1}{2}} \le K \sup_{z\in V_s\cap \vec{\mathcal{W}}} \|f(z)\| \exp - \phi^{\alpha'}(z)$$

where α' is determined by (4.28). Since the sets U_s will be chosen such that every $z \in \Omega$ is contained is not more than L different sets V_s and since V_s will be contained in Ω_ℓ if $U_s \cap \Omega_k \neq \emptyset$ for some $\ell > k$, in virtue of (4.29) for every k and $\alpha \in A$ we get

$$(4.38) \qquad \|c\|_{\alpha,k} \le LK \sup_{z\in\Omega_\ell\cap\vec{\mathcal{W}}} \|f(z)\| \exp - \phi^{\alpha'}(z).$$

A similar procedure, now with respect to the covering $\mathcal{U}^{(1)}$, shows that the map of the lemma is injective. Finally, (4.38) implies that its inverse is continuous. □

If we want to derive the strong version of the fundamental principle (i.e., all the bounds are satisfied simultaneously) as in chapter VII, we should apply this lemma together with the strong versions of theorems 4.11 and 4.12 below, cf. corollary 7.4. But for the weak form treated in this chapter it is convenient to have the following isomorphism.

LEMMA 4.8. *Let* E *denote the space on the left hand side of the isomorphism of lemma 4.7 and let*

$$F^{\alpha} \overset{\text{def}}{=\!=} C^{0}[\mathcal{U}^{(1)}, A^{p}, \phi^{\alpha}; P]$$

and

$$M^{\alpha} \overset{\text{def}}{=\!=} F^{\alpha} \cap P\cdot C^{0}(\mathcal{U}^{(1)}, A^{q}).$$

Then there is a topological isomorphism between

$$E \to \operatorname*{proj\,lim}_{\alpha \in A} (F^{\alpha}/M^{\alpha}).$$

PROOF. We define the map by restriction. That it is injective can be seen as follows: any $c \in \operatorname*{proj\,lim}_{\overrightarrow{\alpha \in A}} C^{0}[\mathcal{U}, A^{p}, \phi^{\alpha}; P]$ that can be written as $c = P\cdot g$ with $g \in C^{0}(\mathcal{U}^{(1)}, A^{q})$ vanishes on $\Omega \cap \mathcal{W}$, because also $\mathcal{U}^{(1)}$ is a covering of Ω, so that by lemma 4.7 c can be written as $c = P\cdot g$ with $g \in \operatorname*{proj\,lim}_{\alpha \in A} C^{0}[\mathcal{U}^{(1)}, A^{q}, \phi^{\alpha}]$. Similarly, it follows that M^{α} is a closed subspace of F^{α}. Hence the space F^{α}/M^{α} is a Frechet space, thus bornologic. In order to conclude the continuity of the inverse of the map we need to know that the bounded sets in F^{α}/M^{α} arise from bounded sets in F^{α}. Let us assume this for the moment. Then the method (as in the proof of lemma 4.7) of proving that the map of the lemma is surjective shows that its inverse is continuous (here each set $U_s \in \mathcal{U}$ is covered by finitely many sets from $\mathcal{U}^{(1)}$, the number and size depending only on the size of U_s). □

It remains to prove the following lemma.

LEMMA 4.9. *Let* F^{α} *and* M^{α} *be as in lemma 4.8. Then the bounded sets in* F^{α}/M^{α} *arise from bounded sets in* F^{α}.

PROOF. Let a bounded set B in F^{α}/M^{α} be determined by cochains $f \in F^{\alpha}$ which for all k satisfy

$$\inf_{P\cdot g \in M^{\alpha}} \| f + P\cdot g \|_{\alpha,k} \le K_k .$$

This means that for arbitrary k_1 there are functions $g_s^1 \in A(U_s)^q$ for every $U_s \in \mathcal{U}^{(1)}$ with $U_s \cap \Omega_{k_1} \neq \emptyset$ such that

$$\| f + P\cdot g^1 \|_{\alpha,k_1} \leq K_{k_1} + 1.$$

Let k_1 be so large that each set $U_s \in \mathcal{U}^{(1)}$ with $U_{s_1} \cap \Omega_1 \neq \emptyset$ is contained in Ω_{k_1}, define $g_s^0 \in A(U_s)^q$ if $U_s \cap \Omega_1 \neq \emptyset$ by $g_s^0 \overset{\text{def}}{=\!=} g_s^1$ and set $k_{-1} = k_0 = 1$. Assume that a cochain g^m has been defined on the union of all sets $U_s \in \mathcal{U}^{(1)}$ with $U_s \cap \Omega_{k_m} \neq \emptyset$ satisfying

$$\| f + P\cdot g^m \|_{\alpha,k_m} \leq C_m$$

for some positive C_m and that $g_s^m = g_s^{m-1}$ if $U_s \cap \Omega_{k_{m-2}} \neq \emptyset$. Let $k_{m+1} > k_m$ be so large that each set $U_s \in \mathcal{U}^{(1)}$ with $U_s \cap \Omega_{k_m} \neq \emptyset$ is contained in $\Omega_{k_{m+1}}$, and for $U_s \in \mathcal{U}^{(1)}$ with $U_s \cap \Omega_{k_{m+1}} \neq \emptyset$ let $\tilde{g}_s^{m+1} \in A(U_s)^q$ be functions which satisfy

$$\| f + P\cdot \tilde{g}^{m+1} \|_{\alpha,k_{m+1}} \leq K_{k_{m+1}} + 1.$$

Now we define $g_s^{m+1} \overset{\text{def}}{=\!=} g_s^m$ if $U_s \subset \Omega_{k_m}$ and $g_s^{m+1} \overset{\text{def}}{=\!=} \tilde{g}_s^{m+1}$ for the remaining s. Then g^{m+1} is defined on the union of all sets $U_s \in \mathcal{U}^{(1)}$ with $U_s \cap \Omega_{k_{m+1}} \neq \emptyset$, $g_s^{m+1} = g_s^m$ if $U_s \cap \Omega_{k_{m-1}} \neq \emptyset$, and

$$\| f + P\cdot g^{m+1} \|_{\alpha,k_{m+1}} \leq C_m + K_{k_{m+1}} + 1.$$

So we obtain a cochain $g \in C^0(\mathcal{U}^{(1)}, A^q)$ with for all $m = 0,1,2,\ldots$

$$\| f + P\cdot g \|_{\alpha,k_m} \leq \sum_{j=1}^{m+2} K_{k_j} + m+2.$$

This determines a bounded set in F^α whose image in F^α/M^α contains B. □

In case $\Omega_k = \Omega$ for every k, as in lemma's 4.7 and 4.8 there is a topological isomorphism between

$$(4.39) \qquad \operatorname*{proj\,lim}_{\alpha \in A} \left\{ C^0(\mathcal{U}^{(1)}, A^p, \phi^\alpha; P) \middle/ C^0(\mathcal{U}^{(1)}, A^p, \phi^\alpha; P) \cap P\cdot C^0(\mathcal{U}^{(1)}, A^q) \right\} \longrightarrow$$

$$\longrightarrow \operatorname*{proj\,lim}_{\alpha \in A} H_\infty(\vec{\mathcal{W}} \cap \Omega; \phi^\alpha),$$

where $C^0(\mathcal{U}^{(1)},\mathcal{A}^p,\phi^\alpha;P)$ denotes the space of those $c \in F^\alpha$ with the norms (4.37) bounded by a constant independent of k, i.e., instead of (4.37) we have

$$\|c\|_\alpha \overset{\text{def}}{=\!=} \{\sum_s \int_{U_s} \|c_s(z)\|^2 \exp - 2\phi^\alpha(z)\,d\lambda(z)\}^{\frac{1}{2}} < \infty. \tag{4.40}$$

IV.6. TRANSITION FROM SEMILOCAL TO GLOBAL RESULTS.

In this section we will formulate the two theorems which together with lemma's 4.7 and 4.8 and formula (4.39) imply theorem 4.6. Besides, these theorems, especially the second whose formulation is not concerned with cochains, may be of interest by themselves, cf. chapter V.4. The main problem is to extend the semilocally defined functions to a globally defined function.

<u>LEMMA</u> 4.10. *Let the conditions of theorem 4.6 be satisfied and let F^α and M^α be as in lemma 4.8. Then there is a topological isomorphism λ:*

$$\operatorname*{proj\,lim}_{\alpha \in A} \left\{ H[\Omega;\phi^\alpha]^p \Big/ H[\Omega;\phi^\alpha]^p \cap P\cdot H[\Omega;\phi^\alpha]^q \right\} \longrightarrow \operatorname*{proj\,lim}_{\alpha \in A} (F^\alpha/M^\alpha). \tag{4.41}$$

A similar isomorphism exists if $\Omega_k = \Omega$ for every k.

Let us decompose the map λ into a collection of continuous restriction maps λ_α. Then denoting

$$H^\alpha \overset{\text{def}}{=\!=} H[\Omega;\phi^\alpha]^p$$

and

$$T^\alpha \overset{\text{def}}{=\!=} H^\alpha \cap P\cdot H[\Omega;\phi^\alpha]^q$$

we have to show for each β there is an $\alpha \geq \beta$ and a continuous map $\mu_{\alpha,\beta}$ such that the following diagram is commutative:

$$\begin{array}{ccc} H^{\alpha}/T^{\alpha}. & \xrightarrow{\quad I_{\alpha,\beta}\quad} & .\ H^{\beta}/T^{\beta} \\ \Big\downarrow \lambda_{\alpha} & \nearrow \mu_{\alpha,\beta} & \Big\downarrow \lambda_{\beta} \\ F^{\alpha}/M^{\alpha}. & \xrightarrow[\quad I'_{\alpha,\beta}\quad]{} & .\ F^{\beta}/M^{\beta} \end{array}$$

where the maps $I_{\alpha,\beta}$ and $I'_{\alpha,\beta}$ are determined by the identity maps. We will define the maps $\mu_{\alpha,\beta}$ by means of the following theorems.

For a positive number N and a function ϕ^{α} in Ω let $\tilde{\phi}^{\alpha}_{N}$ be a plurisubharmonic function in Ω such that for some positive C_N

$$\phi^{\alpha}_{N} \le \tilde{\phi}^{\alpha}_{N} + C_N \tag{4.42}$$

where $\phi_N = \phi_{N,N,N}$ is defined by (4.26), cf. (3.40). This might not be possible for an arbitrary function ϕ^{α}, but if we refer to (4.22) we will always mean that ϕ^{α} is such that there exists a plurisubharmonic function $\tilde{\phi}^{\alpha}_{N}$ satisfying (4.42) (for example, by (4.28) this is true if ϕ^{α} belongs to the set $\{\phi^{\alpha}\}_{\alpha\in A}$ in the conditions of theorem 4.6).

THEOREM 4.11. *Let $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ be a pseudoconvex domain satisfying (4.21) and (4.22), let the covering $\mathcal{U}^{(1)}$ of Ω be given as in section VI.1 and let ϕ^{α} be a function on Ω such that (4.42) can be satisfied for every* N. *Then for any* p×q-*matrix* P *of polynomials there is a positive number* N *and moreover for each sequence $\{K_k\}_{k=1}^{\infty}$ of positive numbers there is another sequence $\{M_k\}_{k=1}^{\infty}$ of positive numbers, such that for every $h \in C^0[\mathcal{U}^{(1)}, A^p, \phi^{\alpha}; P]$ with $\|h\|_{\alpha,k} \le K_k$, $k = 1,2,\dots$, there is a function $v \in A(\Omega)^p$ and a $g \in C^0(\mathcal{U}^{(1)}, A^q)$ with*

$$v\big|_{U_s} - h_s = P\cdot g_s, \quad U_s \in \mathcal{U}^{(1)}, \tag{4.43}$$

and with

$$\left\{ \int_{\Omega_k} \|v(z)\|^2 \exp - 2\phi^{\beta}(z)\, d\lambda(z) \right\}^{\frac{1}{2}} \le M_k, \quad k = 1,2,\dots,$$

where the plurisubharmonic function ϕ^β *is given by*

$$\phi^\beta \overset{\text{def}}{=\!=} \tilde{\phi}^\alpha_N + N\log(1+\|z\|^2) + \log(1+d(z,\Omega^c)^{-N})$$

for $\tilde{\phi}^\alpha_N$ *determined by (4.42); thus* $v \in H[\Omega;\phi^\beta]^p$. *If* $h \in C^0(\mathcal{U}^{(1)},A^p,\phi^\alpha;P)$, *i.e., if* $\{K_k\}_{k=1}^\infty$ *is bounded, (4.21) and (4.22) need not be satisfied and* $\{M_k\}_{k=1}^\infty$ *is bounded, too, i.e.,* $v \in H(\Omega;\phi^\beta)^p$.

THEOREM 4.12. *Let* Ω *and* ϕ^α *be as in theorem 4.11. Then for any* p×q*-matrix* P *of polynomials there is a positive number* N *and moreover for each sequence* $\{K_k\}_{k=1}^\infty$ *of positive numbers there is another sequence* $\{M_k\}_{k=1}^\infty$ *of positive numbers, such that every* $f \in H[\Omega;\phi^\alpha]^p$ *with* $\|f\|_{\alpha,k} \le K_k$, $k = 1,2,\ldots$, *which can locally be written as* $f = P\cdot g^\omega$, $g^\omega \in A(\omega)^q$, $\omega \subset\subset \Omega$, $\cup\omega = \Omega$, *can be written globally as* $f = P\cdot v$ *for some* $v \in H[\Omega;\phi^\beta]^q$ *with* $\|v\|_{\beta,k} \le M_k$, $k = 1,2,\ldots$, *where* ϕ^β *is determined by* ϕ^α *and* N *as in theorem 4.11. Moreover, if* $h \in H(\Omega;\phi^\alpha)^p$ *i.e., if* $\{K_k\}_{k=1}^\infty$ *is bounded, then (4.21) and (4.22) need not be satisfied and* $\{M_k\}_{k=1}^\infty$ *is bounded, i.e.,* $v \in H(\Omega;\phi^\beta)^q$.

In chapter VI we will give the covering $\mathcal{U}^{(1)}$ and we will prove these theorems (if $\Omega = \mathbb{C}^n$, theorem 4.12 follows from [30, th. 7.6.11]). It is clear from (3.40) and (3.41) that problem 3.2 follows from theorem 4.12 and problem 3.3 from theorem 4.11. The map $\mu_{\alpha,\beta}$ can now be defined by means of theorems 4.11 and 4.12.

PROOF OF LEMMA 4.10. According to (4.28) for each $\beta \in A$ and $N \ge 0$ there is a $\alpha \in A$ with $\alpha \ge \beta$ such that in (4.42) we can choose $\tilde{\phi}^\alpha_N = \phi^\beta$; hence for each $\beta \in A$ there is a $\alpha \in A$, $\alpha \ge \beta$, such that theorems 4.11 and 4.12 hold with the functions ϕ^α and ϕ^β belonging to the set $\{\phi^\alpha\}_{\alpha\in A}$. Now for each $\beta \in A$ let $\gamma \in A$, $\gamma \ge \beta$, be such that theorem 4.12 holds if ϕ^α is replaced by ϕ^γ there, and let $\alpha \in A$, $\alpha \ge \gamma$, be such that theorem 4.11 holds if ϕ^β is replaced by ϕ^γ there. Then for $h \in F^\alpha$ we define

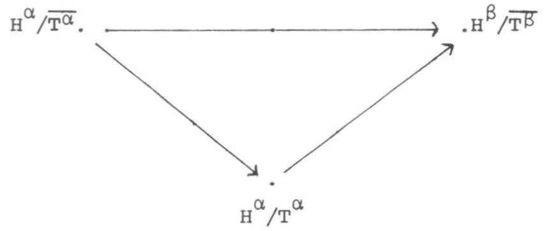
$$\mu_{\alpha,\beta}(h) = I_{\gamma,\beta}v$$

where $v \in H^\gamma$ is determined by h according to theorem 4.11. If $h \in M^\alpha$ then by (4.43) $v|_{U_s} = P\cdot g_s$ for some $g_s \in A(U_s)^q$, $U_s \in \mathcal{U}^{(1)}$, hence according to theorem 4.12 v is mapped by $I_{\gamma,\beta}$ into T^β. Thus $\mu_{\alpha,\beta}$ is well defined.

Moreover, it follows from lemma 4.9 and from theorem 4.11 that $\mu_{\alpha,\beta}$

is a bounded, hence continuous, map. Furthermore, that $I_{\alpha,\beta} = \mu_{\alpha,\beta} \circ \lambda_\alpha$ follows from (4.43) and theorem 4.12, whereas (4.43) alone implies that $I'_{\alpha,\beta} = \lambda_\beta \circ \mu_{\alpha,\beta}$. Hence the diagram is commutative, so that the maps $\{\lambda_\alpha\}_{\alpha\in A}$ determine the map λ and the maps $\mu_{\alpha,\beta}$ its inverse. □

Finally, we show that the space on the left hand side of (4.41) is well behaved. Let $\{f_m\}_{m=1}^{\infty} \subset T^\alpha$ be a Cauchy sequence which converges in H^α to a function f. Then f vanishes on $\vec{W} \cap \Omega$, hence satisfies the conditions of theorem 4.12. Therefore f can be written as $f = P\cdot g$ with $g \in H^\beta$. Thus for each $\beta \in A$ there is $\alpha \in A$ with $\alpha \geq \beta$ such that the following diagram is commutative:



Therefore, the space on the left hand side of (4.41), or (4.31), is a Hausdorff space and equals (cf. (3.28))

$$(4.44) \qquad \operatorname*{proj\,lim}_{\alpha\,\in\,A} H^\alpha/T^\alpha = \operatorname*{proj\,lim}_{\alpha\,\in\,A} H^\alpha/\overline{T^\alpha} \quad .$$

REMARK. In our notation Ehrenpreis formulation of the fundamental principle has the form

$$(4.45) \qquad \operatorname*{proj\,lim}_{\alpha\,\in\,A} H(\mathbb{C}^n;\phi^\alpha)^p \Big/ P\cdot \operatorname*{proj\,lim}_{\alpha\,\in\,A} H(\mathbb{C}^n,\phi^\alpha)^q \xrightarrow{\rho^L} \operatorname*{proj\,lim}_{\alpha\,\in\,A} H_\infty(\vec{W};\phi^\alpha) .$$

Thus a function on $\vec{W}$ satisfying the bounds is extended to one global function satisfying all the bounds simultaneously. In this chapter there is no problem in the semilocal extension, but the transition from semilocal results to global results yields different global functions for the different bounds. Ehrenpreis requires more conditions and , in fact, his result is too strong, as the weaker fundamental principle, formulated here and in [56], satisfies quite as well, i.e., it implies the Fourier representation of all solutions of homogeneous systems of differential equations, see chapter V.3. For example, in our formulation and in that of Palamodov the example given in section IV.3 presents no problems, since the weightfunctions are of the required type. Also, this example exposes the impossibility of getting global extensions satisfying all the bounds simultaneously without further conditions.[1)] In chapter VII, corollary 7.4, we will give such conditions for spaces of non-entire functions. There we will improve theorems 4.11 and 4.12 so that they hold for functions v satisfying all the bounds. Then it follows from lemma 4.7 that we would get a strong fundamental principle like (4.45). However, in that case we will not get uniform bounds as in theorems 4.11 and 4.12. Therefore, we will have to use the open mapping theorem for the conclusion that the inverse of the map (4.41) is continuous.

[1)] This example leads to a family of majorants with non-trivial cohomology which seems to fit a similar condition to that discussed in [56, p. 121] for the case where the bounds must be satisfied only separately.

CHAPTER V

EXAMPLES AND APPLICATIONS

In chapter III we have introduced certain spaces of analytic functions in pseudoconvex domains. In this chapter we will show that these spaces W are localizable. This means that they are duals of spaces W' whose Fourier transforms H satisfy theorem 4.6. Here the Fourier transformation F has been given in chapter III as a generalization of the Ehrenpreis-Martineau theorem. In the proof we have used theorem 4.6. So the fundamental principle helps us to find new examples of localizable spaces W such that $H = FW'$ consists of non-entire functions. We will show that in such spaces the Fourier representation of all weak solutions of a homogeneous system of partial differential equations, mentioned in the last chapter, is valid. This representation is sometimes called the fundamental principle, too. For applications of this principle we refer the reader to [16]. Furthermore, we will give the Fredholm alternative for non-homogeneous systems in localizable spaces. In particular these theorems are valid in spaces of (ultra) distributions which are the boundary values of functions of exponential type, holomorphic in tubular cones. Finally, we will indicate how the theorems of chapter III can be used to derive the Newton interpolation series for non-entire functions of several complex variables.

V.1. TWO LEMMA'S ON PSEUDOCONVEX DOMAINS AND PLURISUBHARMONIC FUNCTIONS.

In chapter II we have considered spaces of holomorphic functions in ε-neighborhoods in $\mathbb{C}^n$ of closed sets S in $\mathbb{R}^n$. In lemma 5.1 we will show that such sets have a neighborhood base of pseudoconvex sets equivalent to the neighborhood base of ε-neighborhoods, a result which we have used in lemma 2.1. In chapter II and III we had weight functions of the form $\exp M(t\|x\|)$, which are not plurisubharmonic. In lemma 5.2 it will be shown that these weight functions can be changed into plurisubharmonic functions without damaging the spaces they define. This is needed in order to satisfy the conditions of theorem 4.6.

Two systems $\{\Omega_k\}$ and $\{\Omega'_k\}$ of neighborhoods are said to be equivalent if for each k there is an ℓ such that $\Omega_k \subset \Omega'_\ell$ and $\Omega'_k \subset \Omega_\ell$. Then both systems determine the same spaces A (2.4) or (2.5) and the same space H (4.29).

<u>LEMMA</u> 5.1. *Let S be a closed set in* $\mathbb{R}^n$ *and let* Ω_1 *be an* ε*-neighborhood of S in* $\mathbb{C}^n$. *Then there is an open pseudoconvex set* Ω *with* $\Omega_2 \subset \Omega \subset \Omega_1$, *where* Ω_2 *is the* $\frac{1}{2}\varepsilon$*-neighborhood of S in* $\mathbb{C}^n$.

<u>PROOF</u>. Define Ω as the holomorphic envelope of

$$\Omega'_2 \overset{\text{def}}{=\!=} \bigcup_{x^0 \in S} \{z \,|\, \|x-x^0\| + \|y\| < \varepsilon/\sqrt{2}\}.$$

It is clear that $\Omega_2 \subset \Omega$. If we show that

$$\Omega \subset \bigcup_{x^0 \in S} \{z \,|\, \|x-x^0\| < \varepsilon/\sqrt{2},\ \|y\| < \varepsilon/\sqrt{2}\}$$

it follows that $\Omega \subset \Omega_1$.

Ω is contained in the $\varepsilon/\sqrt{2}$-neighborhood in $\mathbb{C}^n$ of $\mathbb{R}^n$ because this is pseudoconvex. Furthermore, let $\tilde{z} = \tilde{x} + i\tilde{y}$ with $\tilde{x} \notin \Omega'_2 \cap \mathbb{R}^n$. Then the function

$$F(z) \overset{\text{def}}{=\!=} \exp - (z-\tilde{x})\cdot(z-\tilde{x})$$

is holomorphic in Ω_2 and satisfies $|F(\tilde{z})| \geq 1$ and $|F(z)| < 1$ for $z \in \Omega'_2$. Hence $\tilde{z} \notin \Omega$, because every holomorphic function in Ω'_2 attains the same values in its holomorphic envelope Ω, see [68, § 20.3]. □

In order to show that the spaces of chapters II and III do not alter by a change of the weight functions into a sequence of plurisubharmonic functions we define the equivalence of two sequences of weight functions, cf. (2.7). Two increasing or decreasing sequences $\{\phi_j\}$ and $\{\psi_j\}$ of weight functions on the set Ω are equivalent if for each j there is an m, or for each m an index j, depending on whether the sequences are increasing or decreasing, respectively, and a positive number C such that

$$\phi_j(z) \leq \psi_m(z) + C \quad \text{and} \quad \psi_j(z) \leq \phi_m(z) + C, \qquad z \in \Omega.$$

It is clear that the spaces (2.4) and (2.5) are the same if they are defined by $\{\phi_j\}$ or by $\{\psi_j\}$.

LEMMA 5.2. *The sequences* $\{-j \log(1+\|x\|)\}$, $\{-1/j\ \|x\|\}$, $\{-M(1/j\ \|x\|)\}$ *and* $\{-M(j\|x\|)\}$ *in an* ε*-neighborhood* Ω *of* $\mathbb{R}^n$ *in* $\mathbb{C}^n$ *are equivalent to sequences of plurisubharmonic functions, where* M *is a function as in section II.2.iii.*

PROOF. It is clear that the sequence $\{-j \log(1+\|x\|)\}_{j=1}^{\infty}$ in Ω is equivalent to $\{\log|\alpha^2+z\cdot z|^{-j}\}_{j=1}^{\infty}$ if $\alpha > \varepsilon$, and the sequence $\{-1/j\ \|x\|\}_{j=1}^{\infty}$ to $\{\log |\exp - 1/j\ \sqrt{\alpha^2+z\cdot z}|\}_{j=1}^{\infty}$. These sequences consist of plurisubharmonic functions, because $\log|f|$ is plurisubharmonic if f is holomorphic, see [30, cor. 1.6.6].

In case we deal with $\{-M(1/j\|x\|)\}_{j=1}^{\infty}$ or $\{-M(j\|x\|)\}_{j=1}^{\infty}$ we replace $-M(t\|x\|)$ by the function

$$g_t(z) \overset{\text{def}}{=\!=} \sum_{k=1}^{n} h_t(z_k),$$

where

$$h_t(w) \overset{\text{def}}{=\!=} \max\{\log|\exp - \sqrt{\alpha^2+w^2}| + C,\ -M(t|u|)\}$$

for $\alpha > \varepsilon$ and for C so large that $\log|\exp - \sqrt{\alpha^2+w^2}|+C > -M(t|u|)$ in an open neighborhood in $\mathbb{C}^1$ of $\{w|w = u+iv, u = 0, |v| < \alpha\}$. Since $-M(t|u|)$ is a convex function in the sets $\{w|\,|v| < \varepsilon, \pm u > 0\}$, the function h_t is plurisubharmonic in the strip $\{w|\,|v| < \varepsilon\}$. Hence the function g_t is plurisubharmonic in Ω.

Furthermore, the properties of M imply that

$$M(\|x\|) \le M(|x_1|+\ldots+|x_n|) \le M(|x_1|)+\ldots+M(|x_n|) \le n\ M(\|x\|).$$

An n times repeated application of property (2.21) yields that the last inequality can be further estimated by

$$2^n\ M(\|x\|) \le M(\tau^n\|x\|) + (2^n - 1)K.$$

Finally, this together with the fact, that $-M(t\rho)$ dominates $-\rho$ by (2.32), yields that the sequences $\{g_{1/j}(z)\}_{j=1}^{\infty}$ and $\{g_j(z)\}_{j=1}^{\infty}$ are equivalent to $\{-M(1/j\|x\|)\}_{j=1}^{\infty}$ and to $\{-M(j\|x\|)\}_{j=1}^{\infty}$ in Ω, respectively. □

For spaces H of holomorphic functions defined in tubular cones T^C and bounded with respect to sup-norms with densities $\exp -M^*(t\|y\|)$, $t > 0$, cf. chapter III, we can harmless change these densities into $\exp -M^*(t<\xi_0,y>)$ for some fixed $\xi_0 \in C^*$ with $\|\xi_0\| = 1$, because there is a $\delta > 0$ such that

$$\delta\|y\| \le <\xi_0,y> \le \|y\|.$$

Now the functions $M^*(t<\xi_0,y>)$ are convex in T^C, hence plurisubharmonic. In case the topology of H is given by an inductive limit, $H = \text{ind} \lim_{m \to \infty} H_\infty[\Omega;\phi_m]$, as in [16] this can be changed into a projective limit, $H = \text{proj} \lim_\alpha H_\infty[\Omega;\phi^\alpha]$, where $\{\phi^\alpha\}$ is the collection of convex functions dominating every ϕ_m, $m = 1,2,\ldots$.

Finally, let us make some remarks concerning condition (4.22) in the space H given by (4.29). In particular this condition implies that each set int Ω_k is pseudoconvex, see [68, 12.9]. So not all the Exp-spaces of chapter III satisfy this condition, for example the space $\text{Exp}_\varepsilon[a,T^C;M^*]$ given by (3.39) does not satisfy it. In the other cases it is not difficult to see that a plurisubharmonic, even convex function σ exists such that the sets $\{\Omega_k\}$ determined by condition (4.22) are equivalent to the sets in the definition of the Exp- and A-spaces of chapter III.

V.2. EXAMPLES OF LOCALIZABLE SPACES.

We say that a space W is *localizable* if it is the dual of a space W' whose Fourier transform H can be written as (4.29), (4.32) or (4.33), where the conditions of theorem 4.6 are satisfied and where moreover H is dense in each $H[\Omega;\phi^\alpha]$ or in $H(\Omega;\phi^\alpha)$, or $\text{proj} \lim_{\alpha \in A} H(\Omega_\ell;\phi^\alpha)$ in each $H(\Omega_\ell;\phi^\alpha)$, respectively. Some spaces W such that $H = \mathcal{F}W'$ consists of entire functions are localizable here, but not in the sense of [16], cf. example 4, while others, such as $\mathcal{D}'$, are localizable in [16] but not here. That $\mathcal{D}'$ is not localizable here is due to the fact that $-\log(1+\|\zeta\|^2)$ is not plurisubharmonic in $\mathbb{C}_n$. Below we will see that there are subsets of $\mathcal{D}'$ (with a finer topology than the one induced by $\mathcal{D}'$) which are localizable in our sense. These are the spaces of distributions in $\mathcal{D}'$ whose inverse Fourier transforms have their carrier contained in some unbounded, convex, open set.

EXAMPLE 1. Spaces of Fourier hyperfunctions, ultradistributions of Roumieu type and of Beurling type, and distributions, which are the boundary values of functions of exponential type, holomorphic in tubular radial domains T^C.

These are precisely the Exp-spaces of chapter III defined in (3.33), (3.34), (3.35), (3.39), (3.44), (3.51) and (3.56). The spaces H are given by the corresponding A-spaces. Also the Exp-spaces (3.2.i & ii), (3.45), (3.50) and (3.55) are examples of localizable spaces.

EXAMPLE 2. Spaces of analytic functions in convex sets decreasing at infinity. These are exactly the A-spaces of chapter III defined in (3.5), (3.33), (3.34), (3.35), (3.39) for $\alpha = c$, (3.45) for $\alpha = c$, (3.50), (3.51), (3.55) and (3.56). The spaces H are given by the corresponding Exp-spaces.

EXAMPLE 3. Spaces of C^∞ functions in convex sets decreasing at infinity. These are essentially the S-spaces of lemma 2.27. Precisely, they are the spaces of C^∞ functions which are the duals of the spaces of distributions $\operatorname{proj\,lim}_{k\to\infty} \operatorname{ind\,lim}_{m\to\infty} S_c(m,k,k)'$, $S_c(k,m)'$ and $S_\alpha(m,k)'$. The spaces H are determined by lemma 2.27. Also spaces of C^∞ functions in a fixed, open, convex set decreasing at infinity can be localizable. For example, the spaces $\operatorname{proj\,lim}_{m\to\infty} W_2^m(\Omega(a_m,\Gamma^m);-M(m\|\zeta\|))$ and $\operatorname{proj\,lim}_{m\to\infty} W_2^m(\Omega(a_m,\Gamma^m);-m\log(1+\|\zeta\|))$, cf. (3.50) and (3.55) are localizable. The spaces H are determined as in lemma 2.27.

EXAMPLE 4. The spaces of C^∞ functions in an open, convex set U. The space H is given by $H = \operatorname{ind\,lim}_{k\to\infty} H_\infty(\mathbb{C}^n; k\log(1+\|z\|^2) + \sup\{-\langle\xi,y\rangle \mid \xi \in U_k\})$, where $\{U_k\}$ is an increasing sequence of compact, convex subsets of U exhausting U. If W is the space of C^∞ functions in the compact set $\bar{U}$, in the above we set $U_k = \bar{U}$ for every k. Cf. the remark in the next section.

V.3. REPRESENTATIONS OF SOLUTIONS OF HOMOGENEOUS SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS.

In this section we will show that the exponential representation of [16, th. 7.1], [56, VI § 4] or [2, (9), p. 93] of all solutions of a homogeneous system of partial differantial equations with constant coefficients remains valid in localizable spaces W as defined in the last section. This representation follows immediately from theorem 4.6 and therefore it is also called the fundamental principle.

THEOREM 5.3. *Let* $T \in W$ *be a weak solution of the system*

$$\vec{P}(D)T = \vec{0} \tag{5.1}$$

in the localizable space W, *where* $\vec{P} = (P_1,\dots,P_q)$ *is a vector of complex polynomials and*

$$D = (-i\frac{\partial}{\partial\xi_1},\dots,-i\frac{\partial}{\partial\xi_n}) .$$

Let $\mathcal{W} = (V_1,\partial_1;\dots;V_r,\partial_r)$ *be a polynomial multiplicity variety associated to the vector of polynomials* $\vec{P}(z)$ *according to lemma 4.1 and let* W *be the dual of* W' *whose Fourier transform* H *is given by (4.29). Then there are an index* k, *an index* $\alpha_0 \in A$ *and bounded measures* μ_j *on* $V_j \cap \Omega_k$, $j = 1,\dots,r$, *such that symbolically*

$$T(\xi) = \sum_{j=1}^{r} \int_{V_j\cap\Omega_k} \{\partial_j \exp i\langle\xi,z\rangle\} \exp -\phi^{\alpha_0}(z)\,d\mu_j(z), \tag{5.2}$$

i.e., for $\psi \in W'$

$$\langle T,\psi\rangle = \sum_{j=1}^{r} \int_{V_j\cap\Omega_k} e^{-\phi^{\alpha_0}(z)}(\partial_j F\psi)(z)\,d\mu_j(z). \tag{5.3}$$

Conversely, if $T \in W$ *is determined by (5.3) then it satisfies (5.1). If* H *is given by (4.32) we just set* $\Omega_k = \Omega$ *in (5.2) and (5.3), and if* H *is given by (4.33), for every* $\ell = 1,2,\dots$ *there are an index* $\alpha_\ell \in A$ *and bounded measures* $(\mu^\ell)_j$ *on* $V_j \cap \Omega_\ell$, $j = 1,\dots,r$, *such that any weak solution of (5.1) in* W *can be represented symbolically as*

$$T(\xi) = \sum_{j=1}^{r} \int_{V_j\cap\Omega_\ell} \{\partial_j \exp i\langle\xi,z\rangle\} \exp -\phi^{\alpha_\ell}(z)\,d(\mu^\ell)_j(z)$$

for every $\ell = 1,2,\dots,$ *and conversely as above.*

PROOF. As in section IV.6 we denote

$$H^\alpha \overset{\text{def}}{=\!=} H[\Omega;\phi^\alpha]$$

and

$$T^{\alpha} \stackrel{\text{def}}{=} H^{\alpha} \cap \vec{P}\cdot\vec{H}^{\alpha}.$$

If H is given by (4.29) each $T \in W$ can be written as $T = \mathcal{F}\mu$ for some $\mu \in (H^{\beta})'$ for a certain $\beta \in A$. That T satisfies (5.1) means that for all $\vec{\phi} \in (W')^{q}$

$$(5.4) \qquad \langle T, \vec{P}(-D)\cdot\vec{\phi}\rangle = 0,$$

and moreover this holds for all $\vec{\phi}$ such that $\mathcal{F}\vec{\phi} \in \vec{H}^{\alpha_0}$, because H is dense in H^{α_0} and $\vec{P}\colon \vec{H}^{\alpha_0} \to H^{\beta}$ is continuous for some $\alpha_0 \geq \beta$.

Let $f^{\alpha_0} \in H^{\alpha_0}$ be such that $f^{\alpha_0}(z) = \vec{P}(z)\cdot\vec{g}^{\alpha_0}(z)$ for some $\vec{g}^{\alpha_0} \in \vec{H}^{\alpha_0}$. Then

$$\langle \mu, f^{\alpha_0}\rangle = \langle \mu, \vec{P}(z)\cdot\vec{g}^{\alpha_0}(z)\rangle = \langle T, \vec{P}(-D)\cdot\mathcal{F}^{-1}\vec{g}^{\alpha_0}\rangle =$$

$$= \langle \vec{P}(D)T, \mathcal{F}^{-1}\vec{g}^{\alpha_0}\rangle = 0.$$

Hence in fact

$$(5.5) \qquad \mu \in \{H^{\alpha_0}/\overline{T^{\alpha_0}}\}'.$$

Conversely, if (5.5) holds, then

$$\langle \vec{P}(D)T, \mathcal{F}^{-1}\vec{g}^{\alpha_0}\rangle = \langle \mu, f^{\alpha_0}\rangle = 0$$

for all $\vec{g}^{\alpha_0} \in \vec{H}^{\alpha_0}$ with $f^{\alpha_0} \stackrel{\text{def}}{=} \vec{P}\cdot\vec{g}^{\alpha_0} \in \vec{H}^{\alpha_0}$, so certainly for all $\vec{g} \in \vec{H}$. Hence (5.4) holds.

Now the representation (5.3) follows from (4.44), the isomorphism (4.31) and the Riesz representation theorem, where property (4.30) and the fact that Ω_k is relatively closed in Ω are used.

The case where H is given by (4.32) is similar and if H is given by (4.33) for $T \in W$ we have $T = \mathcal{F}\mu$ with $\mu \in H(\Omega_{\ell};\phi^{\alpha_{\ell}})'$ for every $\ell = 1,2,\ldots$ and a certain sequence $\{\alpha_{\ell}\}_{\ell=1}^{\infty} \subset A$. Then similarly to above we find that for every ℓ

$$\mu \in \left\{ {}^{H(\Omega_\ell;\phi^{\alpha_\ell})} \Big/ {}_{H(\Omega_\ell;\phi^{\alpha_\ell}) \,\cap\, \vec{P}\cdot H(\Omega_\ell;\phi^{\alpha_\ell})^q} \right\}'$$

and the theorem follows from the isomorphism (4.36). □

For a system of differential equations we use the local restriction map ρ^L determined by lemma's 4.2 and 4.3 and similarly to above we get the following theorem, cf. [16, th. 7.3].

THEOREM 5.4. *For a* q×p-*matrix* P *of polynomials let* $\vec{T} \in W^p$ *be a weak solution of*

$$P(D)\cdot\vec{T} = \vec{0}$$

in the localizable space W. *Let* $\vec{\mathcal{W}}$ *be a vector of polynomial multiplicity varieties* $\mathcal{W}^m = (V_1^m, \partial_1^m; \ldots; V_{r_m}^m, \partial_{r_m}^m)$, $m = 1,\ldots,p$, *with the local restriction map* ρ^L *associated to the* p×q-*matrix* ${}^tP(z)$ *of polynomials according to lemma's 4.2 and 4.3, and let* H *be given by (4.29). Then there are an index* k, *an index* $\alpha_0 \in A$ *and bounded measures* μ_j^m *on* $V_j^m \cap \Omega_k$, $m = 1,\ldots,p$, $j = 1,\ldots, r_m$, *such that for* $\vec{\psi} \in (W')^p$

$$(5.6)\qquad \langle\vec{T},\vec{\psi}\rangle = \sum_{m=1}^{p}\sum_{j=1}^{r_m}\int_{V_j^m\cap\Omega_k} \exp - \phi^{\alpha_0}(z)\,\partial_j^m(\rho_z F\vec{\psi})_m(z)\,d\mu_j^m(z).$$

Conversely, if $\vec{T}$ *is determined by (5.6), it satisfies* $\vec{P}(D)\cdot\vec{T} = \vec{0}$. *If* H *is given by (4.32) we just set* $\Omega_k = \Omega$ *in (5.6), and if* H *is given by (4.33), for every* ℓ *there are an index* $\alpha_\ell \in A$ *and bounded measures* $(\mu^\ell)_j^m$ *on* $V_j^m \cap \Omega_\ell$ *such that (5.6) becomes*

$$\langle\vec{T},\vec{\psi}\rangle = \sum_{m=1}^{p}\sum_{j=1}^{r_m}\int_{V_j^m\cap\Omega_\ell} \exp - \phi^{\alpha_\ell}(z)\,\partial_j^m(\rho_z F\vec{\psi})_m(z)\,d(\mu^\ell)_j^m(z)$$

for every $\ell = 1,2,\ldots$, *and conversely as above.*

Note that, by construction of the map ρ_z, there is no 1-1 correspondence between $T^m \in W$ and the measure μ^m on $\mathcal{W}^m$, but T^m is determined by all the measures μ^k on $\mathcal{W}^k$ for $k = m,m+1,\ldots,p$.

<u>REMARK</u>. In [16] W is provided with the strong dual topology and there it is shown that the integrals in (5.3) and (5.6) converge in this topology. Here we have considered W with its weak* topology. Moreover, our condition that H is dense in each H^α is not required in [16]. This condition restricts the possible AU-structures. For example, the AU-structure K of the example in section IV.3 does not satisfy it. It should be remarked that this condition is only required if the topology of H is written as a projective limit. In some of the examples of the last section H has been given as an inductive limit. It is true that in these cases H can be written as a projective limit such that H is dense in each H^α. For instance, in example 4 this follows roughly from the fact that the intersection of all classes of ultradistributions with compact support is the set of distributions with compact support (because any C^∞ function is ultradifferentiable of some type in a compact set) and from the fact that the space of distributions with compact support is dense in any space of ultradistributions with compact support (which on its turn follows from the injectivity of the embedding of the space of ultradifferentiable functions into the space of C^∞-functions). However, in these cases theorems 5.3 and 5.4 can be proved for spaces H which are inductive limits directly along the same lines as the proof of theorem 5.3, cf. [56, VI. § 4]. So it was right to give H as an inductive limit in example 4. The only reason for writing H as a projective limit is to give a uniform treatment of all the examples of section 2.

V.4. INHOMOGENEOUS SYSTEMS.

In the last section we have studied the kernel of the map

$$W^p \xrightarrow{P(D)} W^q.$$

here we will discuss its image. We will show that for certain spaces W the obviously necessary - so called compatibility - conditions are also sufficient. For LAU-spaces W this result has been shown by Ehrenpreis in [16, th. 6.1]; similar results have been obtained by Malgrange, Hörmander in [30, th. 7.6.13] and Komatsu in [41], cf. also [1, ch. 3]. Our spaces W are duals of spaces the Fourier transforms of which consist of non-entire functions, such as the examples of section 2. In particular, we get the result for spaces of analytic functions in convex sets satisfying certain growth conditions, whereas in [41, th. 2] it has been shown without growth conditions.

The following theorem is valid for all the examples of section 2. It can be seen as the Fredholm alternative for systems P of partial differential equations with constant coefficients: $P\cdot\vec{u} = \vec{v}$ has a solution $\vec{u}$ if and only if v is "orthogonal" to the null space of the adjoint of P.

THEOREM 5.5. *Let* W *be a localizable space, let* P *be a* q×p*-matrix of polynomials and let* $D = -i\,\partial/\partial\xi$. *Then for* $\vec{v} \in W^q$ *the equation*

$$P(D)\cdot\vec{u} = \vec{v}$$

has a weak solution $\vec{u} \in W^p$ *if and only if* $\vec{v}$ *satisfies*

$$\vec{Q}(D)\cdot\vec{v} = 0$$

weakly for all polynomials q*-vectors* $\vec{Q}$ *with*

$${}^tP(z)\cdot\vec{Q}(z) = \vec{0}.$$

PROOF. It is clear that the condition $\vec{Q}(D)\cdot\vec{v} = 0$ is necessary. Now let $\vec{v} \in W^q$ satisfy this condition. We want to solve $P(D)\cdot\vec{u} = \vec{v}$ weakly, i.e., for all $\vec{\psi} \in (W')^q$

$$\langle\vec{u}, {}^tP(-D)\vec{\psi}\rangle = \langle\vec{v},\vec{\psi}\rangle.$$

Let $\vec{u} = F\vec{\mu}$ and $\vec{v} = F\vec{\sigma}$ for some $\vec{\mu} \in (H')^p$ and $\vec{\sigma} \in (H')^q$ with $\vec{Q}(z)\cdot\vec{\sigma}_z = 0$ weakly. Let H be given by (4.29). Since H is dense in H^γ, we may assume that $\vec{\sigma} \in (\vec{H}^\gamma)'$ for some $\gamma \in A$ and, as in the proof of theorem 5.3, that σ vanishes on $\vec{H}^\gamma \cap \vec{Q}H^\gamma$. We want to find an index $\alpha \geq \gamma$ and $\vec{\mu} \in (\vec{H}^\alpha)'$ such that for all $\vec{g} \in \vec{H}$

$$\langle\vec{\mu}_z, {}^tP(z)\cdot\vec{g}(z)\rangle = \langle\vec{\sigma}_z,\vec{g}(z)\rangle. \tag{5.8}$$

Thus $\vec{\mu}$ is already defined on the subspace M of $\vec{H}^\alpha$ consisting of all $\vec{f}$ for which there is a $\vec{g} \in \vec{H}^\beta$ with ${}^tP(z)\cdot\vec{g}(z) = \vec{f}(z)$, where $\alpha \geq \beta \geq \gamma$ are sufficiently large. If we show that $\vec{\mu}$ is continuous on M, then by the Hahn-Banach theorem we can extend $\vec{\mu}$ to all of $\vec{H}^\alpha$ and $\vec{u} = F\vec{\mu}$ is the required solution.

It is clear that an arbitrary element of the kernel of ${}^tP$ may be added to $\vec{g}$ without changing $\vec{f}$. By [30, lemma 7.6.3] this kernel is generated by finitely many (say r) polynomial q-vectors. So there is a r×q-matrix Q of polynomials such that in the following sequences, where the matrices ${}^tP$ and ${}^tQ$ determine densely defined closed operators,

$$(\vec{H}^{\alpha})' \xrightarrow{P(z)} (\vec{H}^{\beta})' \xrightarrow{Q(z)} (\vec{H}^{\gamma})'$$

$$(H^{\alpha})^p \xleftarrow{{}^tP(z)} (H^{\beta})^q \xleftarrow{{}^tQ(z)} (H^{\gamma})^r$$

the image of one map is contained in the kernel of the other. Here the first sequence is dual to the second and we have to show that it is exact. Theorem 4.12 implies that Ker ${}^tP = R({}^tQ)$ if $\beta \geq \gamma$ is sufficiently large, i.e., the second sequence is exact. Denoting the range $R({}^tP)$ of ${}^tP$ by M we get the following inverse map

$$M \xrightarrow{({}^tP)^{-1}} (H^{\beta})^q \big/ R({}^tQ) \ . \tag{5.9}$$

We have to show that the map (5.9) is continuous and because M, as a subspace of a Frechet space, is bornologic, it is sufficient to show that $({}^tP)^{-1}$ is a bounded map. So let $\vec{f} \in \vec{H}^{\alpha}$ with $\vec{f} = {}^tP\cdot\vec{g}$ for some $\vec{g} \in \vec{H}^{\beta}$ satisfy $\|\vec{f}\|_{\alpha,k} \leq K_k$, where this norm is defined in (4.20). According to theorem 4.12 there is a $\vec{g}' \in \vec{H}^{\beta}$ with ${}^tP\cdot\vec{g}' = \vec{f}$ and with $\|\vec{g}'\|_{\beta,k} \leq M_k$, $k = 1,2,\dots,$ where $\{M_k\}$ depends on $\{K_k\}$ but not on $\vec{f}$, if $\alpha \geq \beta$ is sufficiently large. Hence the map (5.9) is continuous.

Finally, since $\vec{\sigma}$ vanishes on $\vec{H}^{\gamma} \cap {}^tQ\cdot\vec{H}^{\gamma}$, it certainly vanishes on $R({}^tQ) \subset \vec{H}^{\beta}$. Therefore, we may consider $\vec{\sigma}$ as an element of $\{(H^{\beta})^q/R({}^tQ)\}'$. Thus the functional $\vec{\mu}$ on M satisfying (5.8) is given by

$$\langle\vec{\mu},\vec{f}\rangle = \langle\vec{\sigma},({}^tP)^{-1}\cdot\vec{f}\rangle, \qquad \vec{f} \in M,$$

and this determines a continuous linear functional on M. Therefore, $\vec{\mu}$ can be extended to an element of $(\vec{H}^{\alpha})'$.

If H is given by (4.32) or (4.33) the proof is similar. In the last case M is also bornologic, because an inspection of a 0-neighborhood base (cf. [20, § 23.3.14]) shows that $\operatorname{ind}\lim_{\ell\to\infty} H(\Omega_\ell;\phi^{\alpha\ell})$ induces on its subspace M an inductive limit topology. □

It follows from the proof that there are only finitely many conditions on $\vec{v}$.

REMARK. The condition that H is dense in H^α is not required for a strong fundamental principle as (4.45) of [16]. In chapter VII a similar strong isomorphism will be derived. Therefore theorems 5.3, 5.4 and 5.5 are also valid in spaces W such that H satisfies the conditions of corollary 7.4.

V.5. THE NEWTON INTERPOLATION SERIES.

In [39] Kioustelidis has derived the Newton interpolation series for entire functions of exponential type in $\mathbb{C}^n$. This generalizes the one dimensional case only partially, because in one dimension the Newton series also holds for functions holomorphic in a half-plane, see [55]. Kioustelidis used the Ehrenpreis-Martineau theorem for entire functions. As we have generalized this theorem in chapter III, we are able to derive the Newton series in several variables also for non-entire functions of exponential type. In this section we will mention the results, where for the details we refer to [59].

Let f be an entire function. For $h \in \mathbb{C}^n$ define the operator

$$\Delta_{ih} f(z) \stackrel{\text{def}}{=} f(z+ih)-f(z),$$

so that

$$\Delta_{ih}^k f(z) = \sum_{m=0}^{k} \binom{k}{m} (-1)^{k-m} f(z+imh).$$

The Newton series expresses the value of f in an arbitrary point in terms of the values of f at equidistant points. Precisely, for $s \in \mathbb{C}$

$$f(z+ish) = \sum_{k=0}^{\infty} \binom{s}{k} \Delta_{ih}^k f(z). \qquad (5.10)$$

The polynomials $\binom{s}{k} = s(s-1)\dots(s-k+1)/k!$ are the Newton polynomials $p_k(s)$. Usually, the factor i is omitted, but here it will appear to be convenient to use formula (5.10) for the Newton interpolation series.

Inverse Fourier transformation of (5.10) yields formally

$$(5.11) \qquad e^{-s<\zeta,h>}\hat{f}_\zeta = \hat{f}_\zeta \sum_{k=0}^{\infty} \binom{s}{k} (e^{-<\zeta,h>}-1)^k.$$

It is clear that (5.11) can only hold if $\hat{f}$ is concentrated in the set where the series converges. Denoting $-<\zeta,h> = u+iv \in \mathbb{C}$ for this set we find the condition (cf. [39] or [59, section 9])

$$u < \log(2 \cos v).$$

The component of this set containing the origin is a unbounded, convex set in $\mathbb{C}$ which is bounded in the imaginary directions. Hence the domain of convergence of (5.11) is an unbounded, convex set Ω in $\mathbb{C}_n$ depending on the region in which h may vary. In chapter III we have seen that functions f, which are the Fourier transforms of analytic functionals carried by unbounded subsets of Ω, are functions of exponential type holomorphic in cones in $\mathbb{C}^n$. In [39] only those f have been considered which are the Fourier transforms of analytic functionals with bounded carrier in Ω. So in [39] the functions f for which the series (5.10) is valid are entire, while here we get the result for non-entire functions.

In [59, section 9] it has been shown that (5.10) can be generalized to non-entire functions only if h varies in a subset of $\mathbb{C}^n$ of real dimension n. So we may take h real and in particular we will require that

$$h \in \bar{C}_b = \{h \mid h \in \bar{C}, \|h\| \le b\}$$

where $b > 0$ and C is an open, convex cone in $\mathbb{R}^n$. Let Ω be the component containing the origin of the set

$$\{\zeta \mid |e^{-<\zeta,h>}-1| < 1,\ h \in \bar{C}_b\} \subset \mathbb{C}_n.$$

The other components will not give a series (5.10) for non-entire functions, cf. [59]. Since Ω is a convex set in $\mathbb{C}_n$ which is bounded in the imaginary directions, a function $a-\eta$ on T^C can be defined by

$$(5.12) \qquad (a - \eta)(z) \stackrel{\text{def}}{=} \sup_{\zeta\in\Omega} \{-\mathrm{Im}\langle\zeta,z\rangle\} - \eta\|z\|,$$

where $\eta > 0$ is small. The Newton series will be valid for functions f of exponential type $a - \eta$ and holomorphic in T^C. Moreover in [59, section 9] it has been shown that if $\mathrm{Re}\, s > p \geq 0$ the series (5.10) does not depend on the values of f at the points $z + imh$, where $m = 0,1,\ldots,p$. Hence the series will be valid also for certain points z not in T^C.

According to [59, lemma 9.1 and p. 78], for $h \in \bar{C}_b$ and for $s \in \mathbb{C}$ and $z \in \mathbb{C}^n$ such that $z + ish \in T^C$, the series

$$e^{i\langle\zeta,z\rangle} \sum_{k=0}^{N}{}^{*} \binom{s}{k} (e^{-\langle\zeta,h\rangle}-1)^k \to e^{i\langle\zeta,z+ish\rangle}$$

converges for $N \to \infty$ in the space $A_\varepsilon(a - \eta, T^C)$ given by (3.33), where $\eta > 0$ is so small that this space is defined and where Σ^* means that the terms with $e^{-m\langle\zeta,h\rangle}$ for $m = 0,1,\ldots,p$ should be taken zero if $\mathrm{Re}\, s > p \geq 0$. Hence the following theorem can be derived, see [59, th. 9.1 & 9.1*].

<u>THEOREM</u> 5.6. *Let* C *be an open, convex cone in* $\mathbb{R}^n$, *let* $b > 0$ *and let* $a - \eta$ *be given by (5.12) for* $\eta > 0$ *so small that the spaces* $\mathrm{Exp}_\varepsilon[a - \eta, T^C]$ *and* $A(a - \eta, T^C)$ *can be defined by (3.33). Then for any* $h \in \bar{C}_b$, $s \in \mathbb{C}$ *and* $z \in \mathbb{C}^n$ *such that* $z + ish \in T^C$ *the series (5.10) is valid for functions* $f \in \mathrm{Exp}_\varepsilon[a-\eta, T^C]$, *where-if* $\mathrm{Re}\, s > p \geq 0$*-in the points* $\{z + imh \mid m = 0,1,\ldots,p\}$, *at which* f *is singular or undefined, we take zero instead of* $f(z + imh)$.

The series (5.10) converges uniformly for z in a compact set $K \subset \mathbb{C}^n$ such that $K + ish \subset T^C$, and even in [59] a more precise result on the convergence has been given. The series remains valid for functions in the other Exp-spaces of chapter III, but since this would mainly change the rate of convergence, we will not state the precise results here.

In [55, p. 237, first example 123] the Newton series (without the factor i) in one variable has been given for the function $f(z) = 1/z$ and for $h = 1$. It has been shown there that (in our notation) (5.10) converges if $z + is \in \mathbb{C}^+$, where $\mathbb{C}^+$ is the open upper half-plane. So obviously theorem 5.6 is the generalization to several dimensions of this one dimensional case.

The above formalism has the disadvantage that one cannot see directly what the type of f should be in order that the series (5.10) is valid if h varies in a given domain (for a detailed study of the correspondence between h and the type in case of entire functions f and complex h, see [39]).

Another approach would be to start with an $f \in Exp_{\varepsilon}[a,T^C]$ for a given type a and to find out what the domain of h is such that (5.10) is valid. Then it turns out that the bounds for $\|h\|$ will not be the same in every direction in $\bar{C}$. For a precise result, which is however not as best as possible, see [59, cor. 9.1 & 9.2]. Here we shall only mention the case where $a(z) = a\|z\|$ for a positive number $a > 0$. Then (5.10) holds for $f \in Exp_{\varepsilon}[a-\eta,T^C]$ if $z + ish \in T^C$ and if

$$h \in \bar{C},\ \|h\| \le \frac{\log 2}{a}$$

For $n = 1$ this condition for $\|h\|$ is exactly the one given in [55, p. 237].

CHAPTER VI

PROOFS OF THEOREMS 4.11 AND 4.12

In this chapter we shall prove theorems 4.11 and 4.12. Since problems 3.2 and 3.3 follow immediately from these theorems, in this chapter the proofs of theorem 2.20 and of the theorems in chapter III are completed. Our method uses the L^2-estimates for the Cauchy-Riemann operator given by Hörmander in [30]. In [30, ch. 7.6] cohomology with bounds in $\mathbb{C}^n$ has been derived. Along the same lines we shall derive cohomology with bounds in an arbitrary, open, pseudoconvex set Ω. It relies on appropriate coverings of Ω which will be constructed in section 1. In [54] cohomology with bounds in a bounded, pseudoconvex set Ω has been treated also based on the method of [30]. There the same growth conditions at the boundary of Ω appear as we will get here.

VI.1. COVERINGS

We construct open coverings $\mathcal{U}^{(\lambda)} = \{U_i^{(\lambda)}\}_{i\in I_\lambda}$, $\lambda = 0,1,2,\ldots$ of the pseudoconvex open set Ω that satisfy the following properties:

(6.1) (i) every $U_i^{(\lambda)}$ is pseudoconvex and $U_i^{(\lambda)} \subset\subset \Omega$;

(ii) there is a positive integer L such that more than L distinct sets in $\mathcal{U}^{(\lambda)}$ have empty intersection;

(iii) the size of $U_i^{(\lambda)}$ satisfies

$$\operatorname{diam} U_i^{(\lambda)} \le \min[b4^{-\lambda}d_i, B4^{-\lambda}],$$

where d_i is the distance from $U_i^{(\lambda)}$ to $\partial\Omega$, and $U_i^{(\lambda)}$ contains a cube whose side for any $z \in U_i^{(\lambda)}$ satisfies

$$\text{side} \ge \min[a4^{-\lambda}d(z,\Omega^c),\ A4^{-\lambda}],$$

for some constants $a < b$ and $A < B$;

(iv) for each μ $\mathcal{U}^{(\mu+1)}$ is a refinement of $\mathcal{U}^{(\mu)}$ and, moreover, each

$U_i^{(\mu)} \in \mathcal{U}^{(\mu)}$ enlarged $2^{\mu-\lambda}$ times with respect to some center in $U_i^{(\mu)}$ is contained in some $U_{j_i}^{(\lambda)} \in \mathcal{U}^{(\lambda)}$ for every $\lambda = 0,1,\ldots,\mu-1$; denote the map ρ between I_μ and I_λ with $\rho(i) = j_i$ by $\rho_{\lambda,\mu}$;

(v) there are positive integers $L_{\lambda,\mu}$ depending on λ and μ $(\mu > \lambda)$ such that for each $j \in I_\lambda$ there are at most $L_{\lambda,\mu}$ indices $i_k \in I_\mu$ with $\rho_{\lambda,\mu}(i_k) = j$, $k = 1,2,\ldots,L_{\lambda,\mu}$.

When $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ satisfies (4.21) it follows from property (iii) that

(vi) every set in $\mathcal{U}^{(\lambda)}$ that intersects Ω_k is contained in some Ω_ℓ, where $\ell = \ell(k) > k$ depends on k.

The essential idea for the construction of $\mathcal{U}^{(0)}$ has already been used in [70], and it can be found in [29] too.

Divide $\mathbb{C}^n$ into a collection of closed cubes with side 1 (such that the vertices form a retangular lattice), select those cubes in Ω whose distances to Ω^c are larger than the length $\sqrt{2n}$ of their diagonal and call this collection $\mathcal{U}_0$. Divide the remaining cubes into a collection of cubes of side $\frac{1}{2}$, select those cubes in Ω whose distances to Ω^c are larger than $\frac{1}{2}\sqrt{2n}$ and call this collection $\mathcal{U}_1$. Generally, when the collections $\mathcal{U}_0,\mathcal{U}_1,\ldots,\mathcal{U}_{k-1}$ have been defined let $\mathcal{U}_k$ consist of those closed cubes with side $\frac{1}{2}^k$ that are not contained in the union of the cubes of $\bigcup_{\ell=0}^{k-1} \mathcal{U}_\ell$, but that are contained in Ω and whose distances to Ω^c are larger than $\sqrt{2n}/2^k$. Then $\mathcal{U}_0' \overset{\text{def}}{=\!=} \bigcup_{k=0}^{\infty} \mathcal{U}_k$ covers Ω and a cube in $\mathcal{U}_k$ can intersect cubes of $\mathcal{U}_\ell$ only if $\ell = k-1$, k or k+1. Hence $\mathcal{U}_0'$ satisfies property (ii) (with $L = 2^{2n}$) and property (iii) (with $\lambda = 0$, $A = 1$, $B = \sqrt{2n}$, $a = 1/(4\sqrt{2n})$ and $b = 1$).

Define a map α on $\mathcal{U}_0'$ by mapping $U_i' \in \mathcal{U}_0'$ to the enlargement of the interior of U_i' with a factor 3/2, the center kept fixed. Finally, define

$$\mathcal{U}^{(0)} \overset{\text{def}}{=\!=} \{U_i^{(0)} \mid U_i^{(0)} = \alpha U_i',\ U_i' \in \mathcal{U}_0'\}.$$

It is still true that $U_i^{(0)} \cap U_j^{(0)} \neq \emptyset$ if and only if $\alpha^{-1}U_i^{(0)} \cap \alpha^{-1}U_j^{(0)} \neq \emptyset$. Hence, the open covering $\mathcal{U}^{(0)}$ of Ω satisfies properties (i), (ii) (with $L = 2^{2n}$) and (iii) (with $A = 3/2$, $B = \sqrt{2n}\ 3/2$, $a = 1/(3\sqrt{2n})$ and $b = 2$) for $\lambda = 0$.

Now let $\mathcal{U}^{(0)},\ldots,\mathcal{U}^{(\lambda-1)}$ be defined with the properties (i), (ii), (iii), (iv) and (v) satisfied and let each $\mathcal{U}^{(\mu)}$ consist of open cubes $U_i^{(\mu)}$, such that the collection $\mathcal{U}_\mu'$ of the closed cubes $\alpha^{-1}U_i^{(\mu)}$ covers Ω, $\mu = 0,1,\ldots,\lambda-1$.

Define $\mathcal{U}'_\lambda$ as the collection of all closed cubes obtained by dividing each cube in $\mathcal{U}'_{\lambda-1}$ into 4^{2n} closed cubes. Then define

$$\mathcal{U}^{(\lambda)} \overset{\text{def}}{=\!=} \{U_i^{(\lambda)} \mid U_i^{(\lambda)} = \alpha U_i',\ U_i' \in \mathcal{U}'_\lambda\}.$$

It is clear that $\mathcal{U}^{(\lambda)}$ satisfies properties (i), (ii) and (iii) and it satisfies (iv), since 2 times a cube $U_i^{(\lambda)} \in \mathcal{U}^{(\lambda)}$ is contained in the cube $U_j^{(\lambda-1)} \in \mathcal{U}^{(\lambda-1)}$, when $\alpha^{-1}U_i^{(\lambda)}$ is one of the 4^{2n} cubes $\alpha^{-1}U_j^{(\lambda-1)}$ had been divided in. Hence (v) is satisfied with $L_{\lambda,\lambda-1} = 4^{2n}$, so that $L_{\lambda,\mu} = (4^{2n})^{\mu-\lambda}$.

If $\Omega = \mathbb{C}^n$ we just get the usual coverings of $\mathbb{C}^n$ given in [30, p. 188].

VI.2. COHOMOLOGY WITH BOUNDS IN AN OPEN, PSEUDOCONVEX SET.

In this section we will prove a theorem B with bounds in an open, pseudoconvex set Ω, just as [30, th. 7.6.10] for $\Omega = \mathbb{C}^n$. The following lemma is an extension of [30, th. 4.4.2].

<u>LEMMA</u> 6.1. *Let Ω be an open pseudoconvex set, let $\{\Omega_k\}_{k=1}^\infty$ be an increasing sequence of subsets of Ω satisfying (4.22) and let ϕ be a plurisubharmonic function on Ω. For any sequence $\{K_k\}_{k=1}^\infty$ there is a sequence $\{M_k\}_{k=1}^\infty$ such that for every $(0,q+1)$-form g with locally square integrable coefficients and with $\bar\partial g = 0$ there is a $(0,q)$-form u in Ω with locally square integrable coefficients, so that $\bar\partial u = g$ and for every $k = 1,2,\ldots$*

$$\int_{\Omega_k} \|u(z)\|^2 \frac{\exp -2\phi(z)}{(1+\|z\|^2)^2}\, d\lambda(z) \le M_k^2,$$

provided that for each k

$$\int_{\Omega_k} \|g(z)\|^2 \exp -2\phi(z)\, d\lambda(z) \le K_k^2.$$

Here $\bar\partial$ acts in distributional sense. We remark that u will depend on the sequence $\{K_k\}_{k=1}^\infty$, too. In the above formulation [30, th. 4.4.2] says that $\{M_k\}_{k=1}^\infty$ is bounded when $\{K_k\}_{k=1}^\infty$ is bounded, while (4.22) need not be satisfied (in fact, if $K_k = K$, then $M_k = K$ for $k = 1,2,\ldots$).

PROOF. Let χ be a convex majorant of the nonnegative function $\tilde{\chi}$

$$\tilde{\chi}(t) \overset{\text{def}}{=} \begin{cases} 0 & \text{for } t < 1 \\ \max[0, \log(2^{k+1} K_{k+1}^2)] & \text{for } k \le t < k+1, \quad k = 1,2,\ldots . \end{cases}$$

Then $\psi(z) \overset{\text{def}}{=} \chi(\sigma(z)) \ge 0$ is plurisubharmonic in Ω, so that we may apply [30, th. 4.4.2] in the domain Ω with the plurisubharmonic function $2\phi + \psi$. This yields a $(0,q)$-form u in Ω with $\bar{\partial}u = g$ and with for each k

$$\int_{\Omega_k} \|u(z)\|^2 \frac{\exp -2\phi(z)}{(1+\|z\|^2)^2}\, d\lambda(z) \le$$

$$\le e^{\chi(k)} \int_{\Omega_k} \|u(z)\|^2 \frac{\exp\{-2\phi(z)-\psi(z)\}}{(1+\|z\|^2)^2}\, d\lambda(z) \le$$

$$\le e^{\chi(k)} \int_{\Omega} \|u(z)\|^2 \frac{\exp\{-2\phi(z)-\psi(z)\}}{(1+\|z\|^2)^2}\, d\lambda(z) \le$$

$$\le e^{\chi(k)} \int_{\Omega} \|g(z)\|^2 \exp\{-2\phi(z)-\psi(z)\} d\lambda(z) \le$$

$$\le e^{\chi(k)} \left\{ \int_{\Omega_m} + \sum_{\ell=m}^{\infty} \int_{\Omega_{\ell+1}\setminus\Omega_\ell} \right\} \|g(z)\|^2 \exp\{-2\phi(z)-\psi(z)\} d\lambda(z) \le$$

$$\le e^{\chi(k)} \left\{ K_m^2 + \sum_{\ell=m}^{\infty} 1/2^{\ell+1} \right\} = e^{\chi(k)} \{K_m^2 + 1/2^m\}$$

for arbitrary $m \in \{1,2,\ldots\}$. So we may take $M_k = [e^{\chi(k)}(K_1^2 + 1/2)]^{\frac{1}{2}}$. □

It also follows that, if $\{g_n\}_{n=1}^{\infty}$ is a sequence converging in every norm $\|\cdot\|_k$ to zero, $\{u_n\}_{n=1}^{\infty}$ converges in every norm to zero. This follows from the continuity of a bounded map from a bornological space (here a Fréchet space) into another locally convex space, too.

REMARK. If g is such that every L^2-norm on Ω_k with respect to a different density $\exp -2\phi^k$ is finite and if the u of lemma 6.1 would have the same property (cf. chapter VII), then the following lemma's and theorems could be changed in such a way that theorems 4.11 and 4.12 would hold with one global function v satisfying all these bounds.

The following lemma is an extension of [30, prop. 7.6.1]. The proof follows the same lines, only here one has to look more carefully to the estimates near the boundary of Ω.

LEMMA 6.2. *For every λ and for each sequence $\{K_k\}_{k=1}^{\infty}$ there is a sequence $\{M_k\}_{k=1}^{\infty}$ such that every cocykel $c \in C^p[\mathcal{U}^{(\lambda)},A,\phi^{\alpha}]$, $p \geq 1$, with $\|c\|_{\alpha,k} \leq K_k$ can be written as $c = \delta c'$ for some $c' \in C^{p-1}[\mathcal{U}^{(\lambda)},A,\phi_{N,M,0}]$ with $\|c'\|_{\alpha,k}^{N,M,0} \leq M_k$ for every k, when $\|\cdot\|_{\alpha,k}^{N,M,0}$ denotes the L^2-norm with respect to the density $\exp - 2\phi_{N,M,0}^{\alpha}$ with*

$$\phi_{N,M,0}^{\alpha}(z) \overset{\text{def}}{=\!=} \phi^{\alpha}(z) + N\log(1+\|z\|^2) + \log(1+d(z,\Omega^c)^{-M}),$$

where $N = M = \min[p,n]$, *when the pseudoconvex open set $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ satisfies (4.21) and (4.22) and when the function ϕ^{α} is plurisubharmonic in Ω. Moreover, when $\{K_k\}_{k=1}^{\infty}$ is bounded, (4.21) and (4.22) need not be satisfied and $\{M_k\}_{k=1}^{\infty}$ is bounded.*

PROOF. Let L_q be the sheaf of germs of (0,q)-forms with locally square integrable coefficients and let Z_q be the subsheaf of $\bar{\partial}$-closed forms of type (0,q). Here $\bar{\partial}$ acts in distributional sense. By [30, th. 4.2.5] and the Sobolev embedding theorem $\bar{\partial}c = 0$, weakly, for an L^2_{loc}-function c implies that c is a C^1-function, hence a holomorphic function. Thus a section $c \in \Gamma(\Omega,Z_0)$ is a holomorphic function $c \in A(\Omega)$. For $c \in C^p[\mathcal{U}^{(\lambda)},Z_q,\phi^{\alpha}]$ with $\delta c = 0$ and $\|c\|_{\alpha,k} \leq K_k$ we want to find a $c' \in C^{p-1}[\mathcal{U}^{(\lambda)},Z_q,\phi_{N,M,0}^{\alpha}]$ such that $\delta c' = c$ and $\|c'\|_{\alpha,k}^{N,M,0} \leq M_k$, when $q = 0$. Assume that this has already been proved for smaller values of p and all q, when $p > 1$, $N = M = p$ and when $\{M_k\}_{k=1}^{\infty}$ depends moreover on p.

We construct a partition $\{\phi_i\}_{i\in I_\lambda}$ of unity subordinate to the covering $\mathcal{U}^{(\lambda)}$ of Ω satisfying for some constant C_λ

$$\|\bar{\partial}\sqrt{\phi_i(z)}\|^2 \leq \frac{C_\lambda^2}{\min[1,d(z,\Omega^c)^2]}, \tag{6.2}$$

where

$$\|\bar{\partial}\phi(z)\|^2 \overset{\text{def}}{=\!=} \sum_{j=1}^{n} |\partial/\partial\bar{z}_j\ \phi(z)|^2.$$

For example, let χ be a nonnegative C^{∞}-function on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ equal to 1 in the closed cube with center 0 and sides 1 and with its support contained in the open concentric cube with sides 3/2. Let the length of the side of $U_i^{(\lambda)} \in \mathcal{U}^{(\lambda)}$ be $3/2\beta_i$ and let the center of $U_i^{(\lambda)}$ be z_i, then define

$$\chi_i(z) \stackrel{\text{def}}{=} \chi\left(\frac{z-z_i}{\beta_i}\right)$$

and let

$$\phi_i(z) \stackrel{\text{def}}{=} \frac{\chi_i(z)^2}{\sum_{j\in I_\lambda} \chi_j(z)^2} .$$

By property (6.1) (ii) for each z not more than L terms in the denominator differ from zero and since $\mathcal{U}_\lambda'$ covers Ω at least one term equals 1. Hence, (6.2) follows from this and from property (6.1) (iii). Furthermore, ϕ_i has its support contained in $U_i^{(\lambda)}$.

For $s \in I_\lambda^p$ we set

$$g_s \stackrel{\text{def}}{=} \sum_{i\in I_\lambda} \phi_i \, c_{is}$$

when $c \in C^p[\mathcal{U}^{(\lambda)}, \mathcal{Z}_q, \phi^\alpha]$. Using $\Sigma_i \, \phi_i = 1$, by computing we find $\delta g = c$, if $\delta c = 0$. Furthermore, writing $\phi_i = \sqrt{\phi_i}\,\sqrt{\phi_i}$ and using Cauchy-Schwartz and again $\Sigma_i \, \phi_i = 1$, for any function ψ we find

$$(\|g_s\|_{\psi,k})^2 \stackrel{\text{def}}{=} \int_{U_s^{(\lambda)}\cap\Omega_k} \|g_s(z)\|^2 \exp - 2\psi(z) \, d\lambda(z) \le$$

$$\le \sum_{i\in I_\lambda} \int_{U_s^{(\lambda)}\cap\Omega_k} \phi_i(z) \, \|c_{is}(z)\|^2 \exp - 2\psi(z) \, d\phi(z) \le$$

$$\le \sum_{i\in I_\lambda} \|c_{is}\|^2_{\psi,k}.$$

By summing up for each k we get

$$(6.3) \qquad \|g\|_{\psi,k} \le \|c\|_{\psi,k}$$

for ψ such that the right hand side is finite, hence $g \in C^{p-1}[\mathcal{U}^{(\lambda)}, \mathcal{L}_q, \psi]$.

Let $\bar{\partial}g = f$ be defined by

$$f_s \stackrel{\text{def}}{=\!=} \bar{\partial}g_s = \sum_{i \in I_\lambda} \bar{\partial}\phi_i \wedge c_{is} = 2 \sum_{i \in I_\lambda} \sqrt{\phi_i}\ (\bar{\partial}\sqrt{\phi_i} \wedge c_{is}), \qquad s \in I_\lambda^p.$$

This yields

$$\|f_s\|_{\alpha,k}^{0,1,0} \le 2\left\{\sum_{i \in I_\lambda}\left(\|\bar{\partial}\sqrt{\phi_i} \wedge c_{is}\|_{\alpha,k}^{0,1,0}\right)^2\right\}^{\frac{1}{2}}$$

and by summing up, in virtue of (6.2) for every k we find

$$\|f\|_{\alpha,k}^{0,1,0} \le 2C_\lambda \|c\|_{\alpha,k} \le 2C_\lambda\, K_k,$$

so that $f \in C^{p-1}[\mathcal{U}^{(\lambda)}, \mathcal{Z}_{q+1}, \phi_{0,1,0}^{\alpha}]$.

Now $\delta f = \bar{\partial}\delta g = \bar{\partial}c = 0$. If $p > 1$, by the inductive hypothesis (note, that $\phi_{N,M,0}^{\alpha}$ is plurisubharmonic because Ω is pseudoconvex) there is a cochain $f' \in C^{p-2}[\mathcal{U}^{(\lambda)}, \mathcal{Z}_{q+1}, \phi_{p-1,p,0}^{\alpha}]$ with $\delta f' = f$ and with for every k

$$\|f'\|_{\alpha,k}^{p-1,p,0} \le M_k' \quad ,$$

where the sequence $\{M_k'\}_{k=1}^{\infty}$ depends on $\{2C_\lambda K_k\}_{k=1}^{\infty}$, hence on $\{K_k\}_{k=1}^{\infty}$. By lemma 6.1 second part (actually [30, th. 4.4.2]) and by property (6.1) (i) for every $s \in I_\lambda^{p-1}$ there is a $(g')_s \in \Gamma(U_s^{(\lambda)}, \mathcal{L}_q)$ so that $\bar{\partial}(g')_s = (f')_s$ in $U_s^{(\lambda)}$ and

$$\|(g')_s\|_{\alpha}^{p,p,0} \le \|(f')_s\|_{\alpha}^{p-1,p,0} \quad . \tag{6.4}$$

By summing up by property (6.1) (vi) we get

$$\|g'\|_{\alpha,k}^{p,p,0} \le \|f'\|_{\alpha,\ell(k)}^{p-1,p,0} \le M'_{\ell(k)},$$

so that $g' \in C^{p-2}[\mathcal{U}^{(\lambda)}, \mathcal{Z}_q, \phi_{p,p,0}^{\alpha}]$.

Finally, set $c' \stackrel{\text{def}}{=\!=} g - \delta g'$, then for every $k = 1,2,\ldots$ (6.3), property (6.1) (ii) and the above estimate yield

$$\|c'\|^{p,p,0}_{\alpha,k} \le \|c\|^{p,p,0}_{\alpha,k} + p\sqrt{L-p+1}\,\|g'\|^{p,p,0}_{\alpha,k} \le$$

$$\le M_k \overset{\text{def}}{=\!=} K_k + p\sqrt{L-p+1}\,M'_{\ell(k)} \,.$$

Furthermore, $\delta c' = \delta g = c$ and $\bar\partial c' = f - \delta\bar\partial g' = f - \delta f' = f - f = 0$, hence $c' \in C^{p-1}[\mathcal{U}^{(\lambda)}, \mathcal{I}_q, \phi^{\alpha}_{p,p,0}]$.

It remains to consider the case $p = 1$. The fact that $\delta f = 0$ then means that f defines uniquely a $(0,q+1)$-form f in Ω with $\bar\partial f = 0$. By lemma 6.1 there is a $\tilde g \in \Gamma(\Omega, L_q)$ with $\bar\partial\tilde g = f$ and a sequence $\{M'_k\}_{k=1}^{\infty}$ depending on $\{2C\,K_k\}_{k=1}^{\infty}$ with

$$\int_{\Omega_k} \|\tilde g(z)\|^2 \frac{\exp - 2\phi(z)}{(1+\|z\|^2)^2(1+d(z,\Omega^c)^{-2})}\, d\lambda(z) \le M'^2_k, \qquad k = 1,2,\ldots\,.$$

Setting $(c')_i \overset{\text{def}}{=\!=} g_i - \tilde g|_{U_i^{(\lambda)}}$ we obtain a cochain with the required properties (using property (6.1) (ii) in the estimate for the cochain $\{\tilde g|_{U_i^{(\lambda)}}\}_{i\in I_\lambda}$).

In fact, there are not more than n induction steps, because all $(0,n)$-forms g satisfy $\bar\partial g = 0$. Therefore, the estimates hold already when p is replaced by $\min[p,n]$ and the sequence $\{M_k\}_{k=1}^{\infty}$ may be taken independent of p.

The second part follows from the second part of lemma 6.1 in case $p = 1$. □

The following lemma is a rewriting of [30, prop. 7.6.5] with L^2-norms instead of sup-norms

<u>LEMMA</u> 6.3. *Let P be a matrix of polynomials, ϕ a weight function, for some λ let $V_i \in \mathcal{U}^{(\lambda)}$ and let $u \in A(V_i)^q$. Then there are $\mu > \lambda$ and positive numbers N and $C(\lambda)$ such that for $U_j \in \mathcal{U}^{(\mu)}$ with $\rho_{\lambda,\mu}(j) = i$ there is a $v \in A(U_j)^q$ with*

$$P(w)v(w) = P(w)u(w), \qquad w \in U_j$$

and with

$$\int_{U_j} \|v(w)\|^2 \exp - 2\phi_N(w)\,d\lambda(w) \le C(\lambda) \int_{V_i} \|P(w)u(w)\|^2 \exp - 2\phi(w)\,d\lambda(w)\,,$$

where ϕ_N *is determined by* ϕ *according to (4.26).*

PROOF. In [30, prop. 7.6.5] (or [16, th. III. 3.4. (3) when $p = q = 1$, cf. also th. 1.4, and the general case is contained in th. III. 3.6]) it is shown that for each $p \times q$-matrix P with polynomial entries there are a number $0 < \delta < 1$ and constants C, $\tilde{N}$ and N' such that, when S denotes the unit cube (actually in [30] the unit ball is used, but this only changes the constants), for every $0 < \varepsilon \le 3/2$ and for every $u \in A(S+z/\varepsilon)^q$ there is a $v \in A(\delta S+z/\varepsilon)^q$ with

$$P(\varepsilon w)v(w) = P(\varepsilon w)u(w), \quad w \in \delta S + z/\varepsilon,$$

and with

$$\sup_{w \in \delta S+z/\varepsilon} \|v(w)\| \le C\varepsilon^{-N'}(1+\|z/\varepsilon\|)^{\tilde{N}} \sup_{w \in S+z/\varepsilon} \|P(\varepsilon w)u(w)\|.$$

In fact this is [30, formula (7.6.5)] and it follows from the proof given in [30], that the constants δ, C, N' and $\tilde{N}$ can be taken independent of ε, if we write $C\varepsilon^{-N'}$ in the above estimate. Therefore, by shrinking the variable w with a factor ε, we find again constants C, $t > 1$, $\tilde{M}$ and $\tilde{N}$ such that for $0 < \eta < 3/2t^{-1}$ and for every $u \in A(t\eta S+z)^q$ there is a $v \in A(\eta S+z)^q$ with

$$P(w)v(w) = P(w)u(w), \quad w \in \eta S + z$$

and with

$$\sup_{w \in \eta S+z} \|v(w)\| \le C\eta^{-\tilde{M}}(1+\|z\|)^{\tilde{N}} \sup_{w \in t\eta S+z} \|P(w)u(w)\|.$$

Now we change this estimate into one with L^2-norms. Let $V_i \in \mathcal{U}^{(\lambda)}$, choose $\mu > \lambda$ so that $2^{\mu-\lambda} \ge t+1$ and let $U_j \in \mathcal{U}^{(\mu)}$ be such that $\rho_{\lambda,\mu}(j) = i$. We write U_j with center z_j and sides η_j as $U_j = \eta_j S+z_j$. Since by the construction of $\mathcal{U}^{(\mu)}$ $\alpha^{-1}U_j \subset \alpha^{-1}V_i$ we have $tU_j = t\eta_j S + z_j \subset \{z \mid \|z-z'\| \le$ $\le \tfrac{1}{4}\text{diam}\, \alpha^{-1}V_i + \text{diam}\, U_j\}$ for any $z' \in U_j$ and by property (6.1) (iii) $tU_j \subset$ $\subset \{z \mid \|z-z'\| \le (\tfrac{1}{4}^{\lambda+1} + \tfrac{1}{4}^{\mu})\min[bd(z',\Omega^c), B]\}$. Therefore, in view of (4.23), $b = 2$, $B = \sqrt{2n}\, 3/2$, $\lambda \ge 0$ and $\mu \ge 2$ we take $\tilde{K} \overset{\text{def}}{=} \max[\log 8/3, 15/32 \sqrt{2n}]$ obtaining

$$tU_j \subset \{z \mid z \in S(z';\tilde{K})\}, \quad z' \in U_j,$$

where $S(z;K)$ is given by (4.24). Also, for $z \in (t+1)U_j$ there is a $z' \in tU_j$ with $\|z-z'\| \le \text{diam } U_j$, hence similarly to above

$$(t+1)U_j \subset \bigcup_{z' \in tU_j} S(z',\bar{K})$$

with $\bar{K} = \max[\log 8/7,\ 3/32\sqrt{2n}]$. Now for a weight function ϕ and for $N \overset{\text{def}}{=} \max\{\tilde{N}/2 + (n+1)/4,\ \tilde{M}+n,\ \tilde{K}+\bar{K}\}$ define the plurisubharmonic function ϕ_N by (4.26). In virtue of [73, conditions HS_1 and HS_2, p. 15] property (6.1)(iii) and (4.27) we get

$$\left[\int_{U_j} \|v(w)\|^2 \exp - 2\phi_N(w)\,d\lambda(w)\right]^{\frac{1}{2}} \le$$

$$\le C_1 \left(\frac{4^{\mu}\eta_j}{a}\right)^{\tilde{M}+n} \sup_{w \in U_j} \|v(w)\| \exp - \phi_{\tilde{N}/2,0,\tilde{K}+\bar{K}}(w) \le$$

$$\le C_2(\lambda)\eta_j^n \sup_{w \in \eta_j S + z_j} \left(\frac{1+\|z_j\|}{1+\|w\|}\right)^{\tilde{N}} \sup_{w \in t\eta_j S + z_j} \|P(w)u(w)\| \exp - \phi_{0,0,\bar{K}}(w) \le$$

$$\le C(\lambda) \left[\int_{V_i} \|P(w)u(w)\|^2 \exp - 2\phi(w)\,d\lambda(w)\right]^{\frac{1}{2}},$$

where in [73, cond. HS_2, p. 15] the radius d_z of the polydisc $D(z,d_z)$ is taken $d_z = \eta_j$ for every $z \in t\eta_j S + z_j$, so that the constant there depends on η_j^{-n} and where

$$\{w \mid w \in D(z,\eta_j), z \in t\eta_j S + z_j\} \subset (t+1)\eta_j S + z_j \subset V_i. \qquad \square$$

The next theorem is Cartan's theorem B with bounds in an open, pseudoconvex set Ω. It is an extension of [30, th. 7.6.10]. Let $\mathcal{F}$ be either the sheaf of relations of P on Ω, thus $\mathcal{F} = \mathcal{R}_P$ or the image under P of the sheaf A^q, thus $\mathcal{F} = PA^q$.

<u>THEOREM</u> 6.4. *For all polynomial matrices P there is a positive N, for all nonnegative integers λ there is a $\mu > \lambda$ (depending moreover on P) and for*

each sequence $\{K_k\}_{k=1}^{\infty}$ *a sequence* $\{M_k\}_{k=1}^{\infty}$ *(depending moreover on* λ *and* P*), such that every cocykel* $f \in C^p[\mathcal{U}^{(\lambda)},\mathcal{F},\phi^{\alpha}]$, $p \geq 1$, *with* $\|f\|_{\alpha,k} \leq K_k$ *can be written as* $\delta f' = \rho^*_{\lambda,\mu} f$ *(i.e.,* $(\delta f')_s = f_{s'}$ *with* $s' = \rho_{\lambda,\mu}(s)$ *for* $s \in I^{p+1}_{\mu}$*) for some* $f' \in C^{p-1}[\mathcal{U}^{(\mu)},\mathcal{F},\phi^{\beta}]$ *with* $\|f'\|_{\beta,k} \leq M_k$, *when the pseudoconvex open set* $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ *satisfies (4.21) and (4.22) and when* ϕ^{β} *if the plurisubharmonic function determined by* ϕ^{α} *and* N *as in theorem 4.11. Moreover, when* $\{K_k\}_{k=1}^{\infty}$ *is bounded, (4.21) and (4.22) need not be satisfied and* $\{M_k\}_{k=1}^{\infty}$ *is bounded.*

PROOF. Conversely to lemma 6.2 this theorem is proved by induction for decreasing p, since the theorem is true for $p \geq L$ (see property (6.1) (ii)), because there are no non-zero cochains $f \in C^L[\mathcal{U}^{(\lambda)},\mathcal{F},\phi^{\alpha}]$. Thus assume that the theorem has been proved for all matrices P, when p is replaced by p+1 and when the constants N, μ and $\{M_k\}_{k=1}^{\infty}$ depend moreover on p.

In case $\mathcal{F} = \mathcal{R}_P$ there is a polynomial matrix Q, such that $\mathcal{F} = Q\mathcal{A}^r$ in virtue of [30, lemma 7.6.3] and we can write $f \in C^p[\mathcal{U}^{(\lambda)},\mathcal{F},\phi^{\alpha}]$ as $f_s = Qg_s$ where $g \in C^p(\mathcal{U}^{(\lambda)},\mathcal{A}^r)$, cf. [30, lemma 7.6.4] or (4.14) where the fact, that every $U_i^{(\lambda)} \in \mathcal{U}^{(\lambda)}$ is pseudoconvex, has been used. In case $\mathcal{F} = P\mathcal{A}^q$ we write $Q = P$ and $r = q$, then also $f = Qg$ with $g \in C^p(\mathcal{U}^{(\lambda)},\mathcal{A}^r)$, cf. [30, th. 7.2.9] or again (4.14). According to lemma 6.3 there are $\nu > \lambda$, $N_1 > 0$ and a cochain $\tilde{g} \in C^p[\mathcal{U}^{(\nu)},\mathcal{A}^r,\phi^{\alpha}_{N_1}]$ with $Q\tilde{g}_s = Qg_{s'} = f_{s'}$ where $s' = \rho_{\lambda,\nu}(s)$, hence $\rho^*_{\lambda,\nu} f = Q\tilde{g}$ and with

$$\|\tilde{g}_s\|_{\beta}^{N_1} \leq C(\lambda)\, \|f_{s'}\|_{\alpha}.$$

Since (4.21) holds property (6.1) (vi) is satisfied and it follows from this property and from property (6.1) (v) that for every k there is an $\ell(k) > k$ with

$$\|\tilde{g}\|_{\alpha,k}^{N_1} \leq K_{k'} \overset{\text{def}}{=\!=} (L_{\lambda,\nu})^{p+1} C(\lambda) \|f_{s'}\|_{\alpha,\ell(k)}.$$

When $\delta f = 0$, $\delta Q\tilde{g} = Q\delta\tilde{g} = 0$, whence $\delta\tilde{g} = c$ is a cocykel in $C^{p+1}[\mathcal{U}^{(\nu)},\mathcal{R}_Q,\phi^{\alpha}_{N_1}]$. In view of (4.27) for $N' \geq 0$ we have $(\phi^{\alpha}_{N_1})_{N'} \leq \phi^{\alpha}_{N_1+N'}$.

By the inductive hypothesis we can find $\mu > \nu$, a positive N', a sequence $\{M_k'\}_{k=1}^{\infty}$ (belonging to $\{(p+2)\sqrt{L-p-1}\, K_k'\}_{k=1}^{\infty}$) and a cochain $c' \in C^p[\mathcal{U}^{(\mu)},\mathcal{R}_Q,\phi^{\alpha}_{N',N,0}]$ with $\delta c' = \rho_{\nu,\mu} c$ and $\|c'\|^{N',N',0}_{\beta,k} \leq M_k'$, where the plurisubharmonic function ϕ^{β} is determined by (4.42): $\phi^{\beta} \overset{\text{def}}{=\!=} \tilde{\phi}^{\alpha}_{N_1+N'}$.

We set $g_0 \overset{\text{def}}{=\!=} \rho^*_{\nu,\mu}\, \tilde{g} - c' \in C^p[\mathcal{U}^{(\mu)},\mathcal{A}^r,\phi^{\beta}_{N',N',0}]$ so that $\delta g_0 =$

$= \rho^*_{\nu,\mu}c - \rho^*_{\nu,\mu}c = 0$. According to lemma 6.2 there is a sequence $\{M''_k\}_{k=1}^{\infty}$ belonging to $\{(L_{\nu,\mu})^{p+1} K'_k + M'_k\}_{k=1}^{\infty}$ and a cochain $g' \in C^{p-1}[\mathcal{U}^{(\mu)}, A^r, \phi^{\beta}_{N_2,N_2,0}]$ with $\delta g' = g_0$ and $\|g'\|^{N_2,N_2,0}_{\beta,k} \leq M''_k$ for some $N_2 > N'$.

Finally define $f' \overset{\text{def}}{=\!=} Qg' \in C^{p-1}[\mathcal{U}^{(\mu)}, \mathcal{F}, \phi^{\beta}_{N_2+N_3,N_2,0}]$, where N_3 depends on Q. Then $\delta f' = Q\delta g' = Qg_0 = \rho^*_{\nu,\mu}Q\tilde{g} = \rho^*_{\nu,\mu}\rho^*_{\lambda,\mu}f = \rho^*_{\lambda,\mu}f$. Furthermore, let N denote $N_2 + N_3$, then for every k and some C' depending on Q we get

$$\|f'\|^{N,N,0}_{\beta,k} \leq C'\|g'\|^{N_2,N_2,0}_{\beta,k} \leq M_k \overset{\text{def}}{=\!=} C'M''_k.$$

Here $\{M_k\}_{k=1}^{\infty}$ depends on Q, λ, ν, μ, p and $\{K_k\}_{k=1}^{\infty}$, but ν depends on λ and P (since t in the proof of lemma 6.3 depends on P) and μ on ν; N_3 depends on Q; N_2 depends on p by the inductive hypothesis and on P, since the constants $\tilde{N}$ and $\tilde{M}$ in the proof of lemma 6.3 depend on P; Q depends on P; C' depends on Q; and finally $\{K'_k\}_{k=1}^{\infty}$ depends on P and on $\{\|f\|_{\alpha,\ell(k)}\}_{k=1}^{\infty}$. However, there are only finitely many induction steps, so that we can take the largest of all the constants. Therefore, the theorem is true for all p with constants $\{M_k\}_{k=1}^{\infty}$ depending on P, λ and $\{K_k\}_{k=1}^{\infty}$; N depending on P; μ depending on λ and P.

Moreover, when $\{K_k\}_{k=1}^{\infty}$ is bounded, so that in the above proof we do not use (4.21) and $\{K'_k\}_{k=1}^{\infty}$ is bounded, it follows that $\{M'_k\}_{k=1}^{\infty}$ is bounded and by lemma 6.2 (4.21) and (4.22) need not be satisfied and $\{M''_k\}_{k=1}^{\infty}$ is bounded. Hence (4.21) and (4.22) need not be satisfied and $\{M_k\}_{k=1}^{\infty}$ is bounded. □

VI.3. PROOF OF THEOREM 4.11.

Let $\mathcal{F}$ be the sheaf PA^q. We can estimate the cocykel $f = \delta h$ in terms of h, then $\|f\|_{\alpha,k} \leq \sqrt{L-1}\,K_{\ell(k)}$ and $f \in C^1[\mathcal{U}^{(1)}, \mathcal{F}, \phi^{\alpha}]$. According to theorem 6.4 there is a cochain $f' \in C^0[\mathcal{U}^{(\mu)}, \mathcal{F}, \phi^{\beta}]$ with $\delta f' = \rho^*_{1,\mu}f$ and a sequence $\{M_k\}_{k=1}^{\infty}$ with $\|f'\|_{\beta,k} \leq M'_k$ for some μ and for some plurisubharmonic function ϕ^{β} determined by ϕ^{α} and by a positive integer N as in theorem 4.11.

For every $i \in I_{\mu}$ and $z \in U_i^{(\mu)}$ let

$$v_i(z) \overset{\text{def}}{=\!=} h_j(z) - f'_i(z)$$

where $j = \rho_{1,\mu}(i)$. Then $\delta v = \rho^*_{1,\mu}\delta h - \delta f' = \rho^*_{1,\mu}f - \rho^*_{1,\mu}f = 0$, thus $\{v_i \mid i \in I_{\mu}\}$ determines a function $v \in A(\Omega)^P$. Furthermore, using property (6.1) (v) for every k we obtain

$$\left[\int_{\Omega_k} \|v(z)\|^2 \exp - 2\phi^\beta(z)\,d\lambda(z) \right]^{\frac{1}{2}} \leq$$

$$\leq \|v\|_{\beta,k} \leq L_{1,\mu}\|h\|_{\beta,k} + M'_k \leq M_k \overset{\text{def}}{=\!=} L_{1,\mu}\,K_k + M'_k.$$

Moreover, if $\{K_k\}_{k=1}^\infty$ is bounded, (4.21) and (4.22) need not be satisfied and $\{M'_k\}_{k=1}^\infty$ is bounded, so that $\{M_k\}_{k=1}^\infty$ is bounded, too.

For $s \in I_1$, let $I'(s) \in I_\mu$ be the set of those $i \in I_\mu$ with $V_i \overset{\text{def}}{=\!=} U_i^{(\mu)} \cap U_s^{(1)} \neq \emptyset$. For each $i \in I'(s)$ and $z \in V_i$ we have

$$v(z) - h_s(z) = h_j(z) - f'_i(z) - h_s(z), \quad j = \rho_{1,\mu}(i).$$

This is a holomorphic function in $U_s^{(1)}$ and since $h_j - h_s \in \Gamma(U_j^{(1)} \cap U_s^{(1)}, F)$ and also $f'_i \in \Gamma(U_i^{(\mu)}, F)$, we obtain

$$v\Big|_{U_s^{(1)}} - h_s \in \Gamma(U_s^{(1)}, F).$$

Since the sets V_i and $U_s^{(1)}$ are pseudoconvex (property (6.1) (i)), Cartan's theorem B yields, cf. (4.14),

$$v\Big|_{U_s^{(1)}} - h_s \in P\cdot\Gamma(U_s^{(1)}, A^q),$$

that is $v\Big|_{U_s^{(1)}} - h_s = P\cdot g_s$ for some $g \in C^0(\mathcal{U}^{(1)}, A^q)$. □

VI.4. PROOF OF THEOREM 4.12.

From Cartan's theorem, namely from (4.14), it follows that for every $i \in I_0$ $f = Pg_i$ in $U_i^{(0)} \in \mathcal{U}^{(0)}$ with $g \in C^0(\mathcal{U}^{(0)}, A^q)$. According to lemma 6.3 there are positive integers ν and N_1 and a cochain $\tilde{g} \in C^0[\mathcal{U}^{(\nu)}, A^q, \phi_{N_1}^\alpha]$ with $P\tilde{g}_j = f$ in $U_j^{(\nu)}$ for each $j \in I_\nu$ and with

$$\|\tilde{g}_j\|_\alpha^{N_1} \leq C(0)\|f_{\rho_{0,\nu}(j)}\|_\alpha,$$

where f is regarded as a cocykel in $C^0[\mathcal{U}^{(0)},\mathcal{A}^p,\phi^\alpha]$. Summing over j and using properties (6.1) (ii) and (vi) for each k we get an $\ell(k) > k$ with

$$\|\tilde{g}\|^{N_1}_{\alpha,k} \le C(0)L_{0,\nu}\|f\|_{\alpha,\ell(k)} \le K'_k \overset{\text{def}}{=} C(0)L_{0,\nu}K_{\ell(k)}.$$

Consider the differences c of the functions $\tilde{g}_j$ in the overlaps of the sets $U^{(\nu)}_j$ for $j \in I_\nu$, i.e., $c = \delta\tilde{g}$. Then

$$\|c\|^{N_1}_{\alpha,k} \le 2\sqrt{L-1}\ K'_k$$

and $Pc = P\delta\tilde{g} = \delta f = 0$ and also $\delta c = 0$, hence c is a cocykel in $C^1[\mathcal{U}^{(\nu)},\mathcal{R}_P,\phi^\alpha_{N_1}]$.

According to theorem 6.4 and (4.27) there are $\mu > \nu$, a sequence $\{M'_k\}^\infty_{k=1}$ (depending on $\{2\sqrt{L-1}\ K'_k\}^\infty_{k=1}$), a plurisubharmonic function ϕ^β, which satisfies the condition of theorem 4.11 for some $N > N_1$, and a cochain $c' \in C^0[\mathcal{U}^{(\mu)}, \mathcal{R}_P,\phi^\beta]$ with $\delta c' = \rho_{\nu,\mu}c$ and with

$$\|c'\|_{\beta,k} \le M'_k.$$

Finally, for every $s \in I_\mu$ we set $v_s(z) \overset{\text{def}}{=} \tilde{g}_{s'}(z) - c'_s(z)$ for $z \in U^{(\mu)}_s$, where $s' = \rho_{\nu,\mu}(s)$, which defines a function $v \in A(\Omega)^q$, because $\delta v = \rho^*_{\nu,\mu}\delta\tilde{g} - \rho^*_{\nu,\mu}c = 0$, that satisfies for every k

$$\left[\int_{\Omega_k} \|v(z)\|^2 \exp - 2\phi^\beta(z)\,d\lambda(z)\right]^{\frac{1}{2}} \le \|v\|_{\beta,k} \le L_{\nu,\mu}\|\tilde{g}\|_{\beta,k} + M'_k \le$$

$$\le M_k \overset{\text{def}}{=} L_{\nu,\mu}\ K'_k + M'_k.$$

If $\{K_k\}^\infty_{k=1}$ is bounded, (4.21) need not be satisfied and $\{K'_k\}^\infty_{k=1}$ is bounded, hence also (4.22) need not be satisfied and $\{M'_k\}^\infty_{k=1}$ is bounded, so that $\{M_k\}^\infty_{k=1}$ is bounded.

Furthermore, for every $s \in I_\mu$ in $U^{(\mu)}_s$ we have

$$Pv = Pv_s = P\tilde{g}_{s'} - Pc'_s = f. \qquad \square$$

CHAPTER VII

A COHOMOLOGY VANISHING THEOREM

In chapter II we had assumed that the map (2.12) was surjective. In fact, this expresses the triviality of the first Čech-cohomology group of a covering consisting of two open, pseudoconvex sets with values in the sheaf of germs of holomorphic functions satisfying countably many bounds. Explicitely, let $\Omega = \Omega^1 \cup \Omega^2$, where Ω, Ω^1 and Ω^2 are open, pseudoconvex sets in $\mathbb{C}^n$, let a set of countably many growth conditions in Ω be given and let f be a holomorphic function in $\Omega^1 \cap \Omega^2$ satisfying these growth conditions there. Then the question is whether there exist holomorphic functions f_1 and f_2 in Ω^1 and Ω^2 satisfying the growth conditions in Ω^1 and Ω^2, respectively, such that $f = f_2 - f_1$ in $\Omega^1 \cap \Omega^2$. We will solve this problem with functions bounded with respect to countably many, weighted L^2-norms instead of sup-norms. However, the conditions imposed in chapter II are such that this makes no essential difference. In chapter II the above mentioned result was also needed for functions satisfying only one growth condition and, actually, this is lemma 6.2. As is noticed in the remark after lemma 6.1, lemma 6.2 holds with functions satisfying countably many bounds if lemma 6.1 does. Then a theorem B with functions satisfying countably many bounds can be derived and the stronger version of the fundamental principle can be given. In this chapter we will improve lemma 6.1 by functional analytic methods.

Let $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ be an open, pseudoconvex domain in $\mathbb{C}^n$ with $\Omega_k \subset \Omega_{k+1} \subset \Omega$. Furthermore, let for some integer q with $0 \le q \le n-1$ and for $j = 1,2$ $H_j^k(\Omega_m)$ be the Hilbert space of $(0,q+j-1)$-forms in Ω_m with square integrable coefficients with respect to the density

$$(7.1) \qquad \exp - 2\{\phi^k(z) + (2-j)\log(1+\|z\|^2)\},$$

where $\{\phi^k\}_{k=1}^{\infty}$ is a decreasing sequence of plurisubharmonic functions with ϕ^k defined on Ω. Then the restriction map $\pi_{k+1,k}^j$ from $H_j^{k+1}(\Omega_{k+1})$ into $H_j^k(\Omega_k)$ is continuous, so that the projective limits can be defined

(7.2) $$H_j \overset{\text{def}}{=\!=} \operatorname*{proj\,lim}_{k \to \infty} H_j^k(\Omega_k), \quad j = 1,2,\dots$$

Often we shall write H_j^k instead of $H_j^k(\Omega_k)$.

Let $f \in H_1^k$ be such that $\bar\partial f \in H_2^k$, where $\bar\partial$ is defined in distributional sense. We denote the operator which assigns to such f the (0,q+1)-form $\bar\partial f$ by T_k. Then T_k is a closed, densely defined operator

$$T_k : H_1^k \to H_2^k, \quad k = 1,2,\dots$$

That T_k is closed follows from the continuity of $\bar\partial$ in distribution theory. This also implies that the sets

$$F \overset{\text{def}}{=\!=} \{g \in H_2 \mid \bar\partial g = 0 \text{ in distributional sense}\}$$
$$F_k \overset{\text{def}}{=\!=} \{g \in H_2^k \mid \bar\partial g = 0 \text{ in distributional sense}\}$$

are closed subspaces of H_2 and H_2^k, respectively. For $p > k$ we have

$$\pi^2_{p,k}\, T_p = T_k\, \pi^1_{p,k}$$

so that $\{T_k\}$ determines a closed, densely defined operator T from H_1 into H_2. That T is densely defined follows from the fact that the space of (0,q)-forms with C^∞-coefficients with compact support in Ω lies in D_T and is dense in H_1 by Lebesgue's theorem. The following diagram is commutative

$$\begin{array}{ccc}
H_1 & \xrightarrow{\quad T \quad} & H_2 \\
\downarrow{\scriptstyle \pi^1_p} & & \downarrow{\scriptstyle \pi^2_p} \\
H_1^p & \xrightarrow{\quad T_p \quad} & H_2^p \\
\downarrow{\scriptstyle \pi^1_{p,k}} & & \downarrow{\scriptstyle \pi^2_{p,k}} \\
H_1^k & \xrightarrow{\quad T_k \quad} & H_2^k
\end{array}$$

Since also

$$F = \{g \in H_2 \mid \pi^2_k\, g \in F_k,\ k = 1,2,\dots\}$$

we have $R(T) \subset F$. We want that $R(T) = F$, but by [30, th. 4.4.2] (lemma 6.1) we only know that $R(T_k) = F_k$ for every k. In particular, $R(T_k)$ is closed in H_2^k.

We will use that the range $R(T)$ of a closed, densely defined operator $T: E \to F$ is closed if and only if $R(T^*)$ is weakly* closed [1] in E' provided that E and F are Frechet spaces. This follows from [61, IV 7.3], cf, also [65, lemma 37.4], [61, IV 7.4] or [65, lemma 37.6] and the open mapping theorem for closed operators [61, IV 8.4], see also [40, th. 19(i)]. If moreover E is reflexive the weak* topology on E' equals the weak topology and accordingly [65, prop. 35.2] in that case $R(T^*)$ is closed in the strong topology of E', because $R(T^*)$ is convex.

LEMMA 7.1. *Let* $T: E \to F$ *be a closed, densely defined operator from the reflexive Frechet space* E *into the Frechet space* F, *then the following three statements are equivalent:*

(1) $R(T)$ *is closed in* F

(2) $R(T^*)$ *is weakly** *closed in* E'

(3) $R(T^*)$ *is strongly closed in* E'.

For the improvement of lemma 6.1 we will apply a similar trick as Kawai has done in [38, lemma 2.1.2]. Besides condition (4.22) on the domains $\{\Omega_k\}$ we impose the following condition on the weight functions $\{\phi^k\}$ in Ω: *for every* k *and every* $p > k$ *there exists a holomorphic function* $\psi^{k,p}$ *in* Ω *and moreover for every* $m = 1,2,\ldots$ *a positive number* $K(k,p,m)$ *such that*

$$(7.3) \qquad 0 < |\psi^{k,p}(z)| \le K(k,p,m)\exp - m\{\phi^k(z) - \phi^p(z)\}, \qquad z \in \Omega,\ m = 1,2,\ldots$$

and such that $\log \psi^{k,p}$ *is holomorphic in* Ω. Since $\phi^k \ge \phi^p$ for $p \ge k$ it follows that this condition cannot be satisfied if $\Omega = \mathbb{C}^n$, unless all the functions $\{\phi^k\}$ are equal. Hence (7.3) is a condition on the domain Ω, too.

Our stronger version of lemma 6.1 is based on the following lemma, cf. [38, lemma 2.1.2].

---

1) The weak* topology on the dual H' of a locally convex space H, sometimes denoted by the $\sigma(H',H)$-topology, is the one induced by the polars of finite subsets of H. The weak topology on H', sometimes denoted by $\sigma(H',H'')$, is induced by the sets in H' on which a finite number of strongly continuous functionals are bounded. If H is reflexive the weak* and weak toplogies on H' coincide.

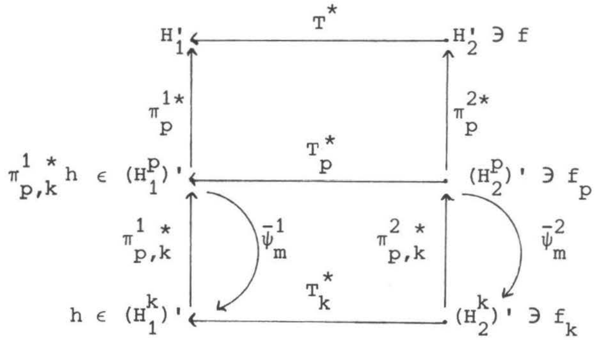
<u>LEMMA</u> 7.2. *Let Ω be a pseudoconvex domain and $\{\phi^k\}$ be a decreasing sequence of plurisubharmonic functions in Ω satisfying condition (7.3). Furthermore, let* H_j *be given by (7.2) with* $\Omega_k = \Omega$ *for* $j = 1,2$. *If for* $f \in D_{T^*} \subset H_2'$ *we have* $T^*f \in \pi_k^{1*}(H_1^k)'$ *, then there is an* $f_k \in D_{T_k^*} \subset (H_2^k)'$ *with*

$$\pi_k^{1*}\, T_k^*\, f_k = T^* f.$$

<u>PROOF</u>. Let $H_j^k = H_j^k(\Omega)$. If $p > k$, let $\psi_m(z) \overset{\text{def}}{=\!=} (\psi^{k,p}(z))^{1/m}$; by (7.3) these functions satisfy

$$|\psi_m(z)| \le K(k,p,m)^{1/m} \exp-\{\phi^k(z) - \phi^p(z)\}, \qquad z \in \Omega,\ m = 1,2,\dots .$$

Hence multiplication of each coefficient of a $(0,q+j-1)$-form in Ω by ψ_m defines a continuous map from H_j^k into H_j^p; we denote these map by ψ_m^j. Its adjoint (multiplication by $\bar\psi_m$) is a continuous map from $(H_j^p)'$ into $(H_j^k)'$ which we denote by $\bar\psi_m^j$. We have the following diagram



Here all maps π and π^* are identity maps, because $\Omega_k = \Omega$ for every k.

Since ψ_m is holomorphic in Ω, for all $u \in D_{T_k}$ we have in distributional sense

$$\bar\partial \psi_m\, u = \psi_m \bar\partial u = \psi_m^2\, T_k\, u \in H_2^p.$$

Thus $\psi_m^1\, u \in D_{T_p}$ and $\psi_m^2\, T_k\, u = T_p \psi_m^1\, u$. Therefore, if $g \in D_{T_p}^*$ we get

$$\langle \bar\psi_m^2 g, T_k u\rangle = \langle g, \psi_m^2 T_k u\rangle = \langle g, T_p \psi_m^1 u\rangle = \langle T_p{}^* g, \psi_m^1 u\rangle = \langle \bar\psi_m^1 T_p^*\, g, u\rangle.$$

This means that $\bar{\psi}_m^2 g \in D_{T_k^*}$ and that

$$T_k^* \bar{\psi}_m^2 = \bar{\psi}_m^1 T_p^* \quad \text{on } D_{T_p^*} \quad .$$

Now let $p > k$ and $f_p \in D_{T_p^*}$ be such that $f = \pi_p^{2*} f_p$, and let $T_p^* f_p = \pi_{p,k}^{1\ *} h$ for some $h \in (H_1^k)'$. Then in the above we take this p and we find

$$T_k^* \bar{\psi}_m^2 f_p = \bar{\psi}_m^1 T_p^* f_p = \bar{\psi}_m^1 \pi_{p,k}^{1\ *} h.$$

Furthermore, by Lebesgue's theorem $\bar{\psi}_m^1 \pi_{p,k}^{1\ *} h \to h$ as $m \to \infty$ in $(H^k)'$. Since by lemma 7.1 T_k^* has closed range in $(H_1^k)'$, it follows that there exists an $f_k \in D_{T_k^*}$ with $T_k^* f_k = h$. Hence

$$\pi_k^{1*} T_k^* f_k = \pi_k^{1*} h = \pi_p^{1*} \pi_{p,k}^* h = \pi_p^{1*} T_p^* f_p = T^* \pi_p^{2*} f_p = T^* f.$$

□

Now using lemma 6.1 we can easily prove its following extension, cf. [38, lemma 2.1.1].

<u>THEOREM</u> 7.3. *Let* $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ *satisfy (4.22) for a plurisubharmonic function* σ *in the pseudoconvex domain* Ω, *let* $\{\phi^k\}$ *be a decreasing sequence of plurisubharmonic functions in* Ω *satisfying condition (7.3) and let* H_j *be given by (7.2) for* $j = 1,2$. *Then for each* $g \in H_2$ *with* $\bar{\partial} g = 0$ *there is an* $u \in H_1$ *with* $\bar{\partial} u = g$ *in distributional sense.*

<u>PROOF</u>. Let $g \in F$ be fixed. Then there are positive numbers K_k with

$$\int_{\Omega_k} \|g(z)\|^2 \exp - 2\phi^k(z) d\lambda(z) \le K_k, \quad k = 1,2,...$$

As in the proof of lemma 6.1 the function σ and the numbers $\{K_k\}$ determine a plurisubharmonic function ψ. For g we get the estimates

$$\int_{\Omega} \|g(z)\|^2 \exp\{-2\phi^k(z)-\psi(z)\} d\lambda(z) \le$$

$$\le \left\{ \int_{\Omega_k} + \sum_{\ell=k}^{\infty} \int_{\Omega_{\ell+1}\setminus\Omega_\ell} \right\} \|g(z)\|^2 \exp\{-2\phi^k(z)-\psi(z)\} d\lambda(z) \le$$

$$\leq K_k^2 + \sum_{\ell=k}^{\infty} (2^{\ell+1} K_{\ell+1}^2)^{-1} \int_{\Omega_{\ell+1}} \|g(z)\|^2 \exp - 2\phi^k(z)d\lambda(z) \leq K_k^2 + 2^{-k} < \infty,$$

because $\{\phi^k\}$ is decreasing so that

$$\int_{\Omega_{\ell+1}} \|g(z)\|^2 \exp - 2\phi^k(z)d\lambda(z) \leq \int_{\Omega_{\ell+1}} \|g(z)\|^2 \exp - 2\phi^{\ell+1}(z)d\lambda(z).$$

For $j = 1,2$, let now H_j be the space (7.2) with $\Omega_k = \Omega$ and with in (7.1) ϕ^k replaced by $\phi^k + 1/2\psi$, $k = 1,2,\ldots$. The above estimates show that g belongs to this space H_2. Assume that the theorem has been proved for spaces (7.2) with $\Omega_k = \Omega$ for every k. This would yield an u in the above given H_1 with $\bar{\partial}u = g$ and so (cf. the proof of lemma 6.1)

$$\int_{\Omega_k} \|u(z)\|^2 \frac{\exp-2\phi^k(z)}{(1+\|z\|^2)^2} \, d\lambda(z) \leq e^{\chi(k)} \int_{\Omega} \|u(z)\|^2 \frac{\exp\{-2\phi^k(z)-\psi(z)\}}{(1+\|z\|^2)^2} \, d\lambda(z) < \infty$$

for every k. Thus u would satisfy the conclusion of the theorem. It remains to prove the theorem for spaces H_j with $\Omega_k = \Omega$ for every k.

So in the remaining we assume that $H_j^k = H_j^k(\Omega)$.

(i) R(T) *is dense in* F.

Let $f \in H_2'$ with $<f,Tu> = 0$ for all $u \in D_T \subset H_1$, hence $f \in D_{T^*}$ and $<T^*f,u> = 0$. Since D_T is dense in H_1, we get $T^*f = 0$. There are k and $f_k \in D_{T_k^*}$ with $f = \pi_k^{2*} f_k$ and $T_k^* f_k = 0$. Now let $g \in F$, then $\pi_k^2 g \in F_k$. According to [30, th. 4.4.2] (lemma 6.1) $\pi_k^2 g = T_k u_k$ for some $u_k \in D_{T_k}$. So we have

$$<f,g> = <\pi_k^{2*} f_k, g> = <f_k, \pi_k^2 g> = <f_k, T_k u_k> = <T_k^* f_k, u_k> = 0.$$

This implies that R(T) is dense in F.

(ii) R(T) *is closed in* H_2.

The spaces H_1 and H_2 are reflexive Frechet spaces, namely they are FS^*-spaces see [40]. Therefore, by lemma 7.1 it is sufficient to show that $R(T^*)$ is weakly* closed in H_1'. According to the theorem of Banach-Dieudonné [65, th. 37.1], [45, § 21, 10(5)] or [61, IV. 6.4, where it is called the Krein-Šmulian theorem] it suffices to prove that $R(T^*) \cap B$ is weakly closed in H_1' for every bounded, convex, weakly* closed subset B of H_1'. Bearing in mind that H_1' is a DFS*-space, hence reflexive so that the weak* and weak topologies on H_1' coincide, by [40, th. 6] there is a k such that B is weakly homeomorphic with a bounded, convex, weakly closed set in $(H_1^k)'$. Thus there is a bounded set $B_k \subset (H_1^k)'$ with $\pi_k^{1*} B_k = R(T^*) \cap B$, where π_k^{1*} is a weak homeomorphism. Since B_k is convex its weak closure equals its strong closure in $(H_1^k)'$. Thus we have to show that B_k is closed in $(H_1^k)'$.

Let $h^m \to h$ as $m \to \infty$ in $(H_1^k)'$ with $h^m \in B_k$. Thus for each m there is an $f^m \in D_{T^*} \subset H_2'$ with $\pi_k^{1*} h^m = T^* f^m$. According to lemma 7.2 for each m there is a $f_k^m \in D_{T_k^*} \subset (H_2^k)'$ with $T_k^* f_k^m = h^m$. Since by lemma 7.1 $R(T_k^*)$ is closed in $(H_1^k)'$, there is an $f_k \in D_{T_k^*}$ with $T_k^* f_h = h$. Hence $\pi_k^{1*} h \in R(T^*)$ and thus B_k is closed in $(H_1^k)'$. This implies that $R(T^*) \cap B$ is weakly* closed in H_1' for every bounded, convex, weakly* closed subset B of H_1'. Therefore R(T) is closed in H_2. □

<u>REMARK</u>. Unlike lemma 6.1 theorem 7.3 does not give uniform bounds. The only thing which can be said is that, in virtue of the open mapping theorem, T is an open map, i.e.,

$$T^{-1}: F \to H_1/\text{Ker } T \quad \text{is continuous.}$$

As is remarked after lemma 6.1 using theorem 7.3 instead of leamm 6.1 one could obtain a theorem B with countably many bounds. However, there remains one difficulty. Since theorem 7.3 does not give uniform bounds, in the proof of lemma 6.2 formula (6.4) becomes

$$\| (g')_s \|_k^{p,p,0} < \infty, \quad k = 1,2,\ldots$$

only, and we cannot sum over s for getting $\|g'\|_{k,k}^{p,p,0} < \infty$, k = 1,2,... . We solve this problem by a direct proof of the existence of $u \in \text{proj}\lim_{k\to\infty} C^p[\mathcal{U}^{(\lambda)}, \mathcal{L}_q, \phi_{1,0,0}^k]$ with $\bar\partial u = g$ for a given $g \in \text{proj}\lim_{k\to\infty} C^p[\mathcal{U}^{(\lambda)}, \mathcal{Z}_{q+1}, \phi^k]$. The proof is exactly that of theorem 7.3; we only have to take for H_2^k the Hilbert space of cochains c with norm $\|c\|_{k,k}$ given by (4.37). In lemma 7.2, which is needed in this proof, H_2^k should be the Hilbert space of cochains c with norm $\|c\|_k$ given by (4.40). In both cases, the replacement of ϕ^k by $\phi_{1,0,0}^k$ yields the space H_1^k.

Thus if condition (7.3) holds, theorems 3.1, 4.11 and 4.12 could be derived for functions satisfying countably many bounds and we get the

$\delta c' = c$.This means that on $U_s^j \cap U_t^j$ we have $c'(U_s^j) = c'(U_t^j)$ for j = 1,2 so that c' determines two holomorphic functions f_j in Ω_j, j = 1,2, with $f_2 - f_1 =$ $= c'(U^2) - c'(U^1) = f$ on $U^1 \cap U^2$ for all s,t ∈ I. Hence $f_2 - f_1 = f$ in $\Omega^1 \cap \Omega^2$

strong version of the fundamental principle. The continuity of $(\rho^L)^{-1}$ in this case follows from the open mapping theorem, because we deal with Frechet spaces.

<u>COROLLARY</u> 7.4. *Let* $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ *be a pseudoconvex domain satisfying (4.21) and (4.22) and let* $\{\phi^k\}_{k=1}^{\infty}$ *be a decreasing sequence of plurisubharmonic functions in* Ω *satisfying condition (7.3). Furthermore, for every* k *and* $N \geq 0$ *let there be a* $p \geq k$ *and a* $C_{k,N} \geq 0$ *with*

$$\phi_N^p(z) \leq \phi^k(z) + C_{k,N}, \qquad z \in \Omega_k.$$

Then for each p×q*-matrix* P *with polynomial entries and associated vector multiplicity variety* $\vec{\mathcal{W}}$ *the map* ρ^L*, defined by lemma 4.3,*

$$\{\operatorname*{proj\,lim}_{k\to\infty} H(\Omega_k;\phi^k)\}^p \Big/ P\cdot\{\operatorname*{proj\,lim}_{k\to\infty} H(\Omega_k;\phi^k)\}^q \xrightarrow{\rho^L} \operatorname*{proj\,lim}_{k\to\infty} H_\infty(\vec{\mathcal{W}}\cap\Omega_k;\phi^k)$$

is a topological isomorphism between linear spaces.

For the spaces in chapter II and III in condition (7.3) we may choose

$$\psi^{k,p}(z) = \exp - z^2,$$

because Ω is bounded in the imaginary directions or Ω is a conic neighborhood in $\mathbb{C}^n$ of a real domain, and $\phi^k = -M(k\|x\|)$. Here M satisfies (2.32) so that for some $K \geq 0$ and $\varepsilon > 0$ we have

$$-\varepsilon\|x\| \leq -\{\phi^k(z) - \phi^p(z)\} + K.$$

Moreover, lemma 5.2 shows how the difficulty that $-M(\|x\|)$ is not plurisubharmonic can be overcome. For example, the A-spaces in (3.51) or (3.56) satisfy the conditions of corollary 7.4, because for σ we can even find a convex function.

In chapter II the domains Ω were bounded in the imaginary directions, so that any holomorphic function $g_{p,m}$ satisfying (2.11) is such that $\log g_{p,m}$ is holomorphic in Ω. In lemma's 2.3.i and 2.3.ii we have used the following corollary, which solves the problem discussed at the beginning of this chapter.

<u>COROLLARY</u> 7.5. *Let* $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ *be a pseudoconvex domain satisfying (4.22) and let* $\{\phi^k\}_{k=1}^{\infty}$ *be a decreasing sequence of plurisubharmonic functions in* Ω *satisfying condition (7.3). Let moreover* Ω^1 *and* Ω^2 *be pseudoconvex open sets with* $\Omega = \Omega^1 \cup \Omega^2$ *such that for some positive* ε *with* $\varepsilon < 1$ *and for each* $z \in \Omega^1 \cap \Omega^2$ *there is a* $z' \in \Omega^1 \cap \Omega^2$ *with* $\|z-z'\| < \varepsilon(z') \overset{\text{def}}{=\!=} \varepsilon \min\{1, d(z',\Omega^c)\}$ *and with*

$$(7.4) \qquad \|z'-w\| < \varepsilon(z') \Rightarrow w \in \Omega^1 \cap \Omega^2.$$

Then for every holomorphic function $f \in \operatorname{proj}\lim_{k\to\infty} H(\Omega^1 \cap \Omega^2 \cap \Omega_k; \phi^k)$ *there are holomorphic functions* $f_j \in \operatorname{proj}\lim_{k\to\infty} H(\Omega^j \cap \Omega_k; \phi^k_{1,1,0})$ *for* $j = 1,2$ *with* $f(z) = f_2(z) - f_1(z)$ *for* $z \in \Omega^1 \cap \Omega^2$*, where*

$$\phi^k_{1,1,0}(z) \overset{\text{def}}{=\!=} \phi^k(z) + \log(1+\|z\|^2) + \log(1+d(z,\Omega^c)^{-1}).$$

<u>PROOF</u>. The proof will be that of lemma 6.2. Let for $j = 1,2$

$$\mathcal{U}_j \overset{\text{def}}{=\!=} \{U^j_s \mid U^j_s \overset{\text{def}}{=\!=} U_s \cap \Omega^j,\ U_s \in \mathcal{U}^{(\lambda)}\}$$

for some λ and let $\mathcal{U} \overset{\text{def}}{=\!=} \mathcal{U}_1 \cup \mathcal{U}_2$ be a covering of Ω, where $\mathcal{U}^{(\lambda)}$ is the covering constructed in section VI.1. Due to (7.4) for λ sufficiently large there is an embedding τ of $\mathcal{U}^{(\lambda)}$ into $\mathcal{U}$ given by $\tau U_s = U^1_s$ if $U_s \subset \Omega^1$ and $\tau U_s = U^2_s$ for the remaining $U_s \in \mathcal{U}^{(\lambda)}$. Hence the partition of unity subordinate to the covering $\mathcal{U}^{(\lambda)}$, constructed in the proof of lemma 6.2, induces a partition of unity subordinate to the covering $\mathcal{U}$ of Ω. We let c be the 1-cocykel defined by $c = 0$ on every set $U^j_s \cap U^j_t$ for $j = 1,2$ and $c = f$ on every set $U^1_s \cap U^2_t$ for all $s,t \in I_\lambda$. In the proof of lemma 6.2 with $p = 1$ and with $\mathcal{U}$ as the covering of Ω, we take the above given partition of unity and we apply theorem 7.3 instead of lemma 6.1. So we find a 0-cochain c' satisfying good bounds (note that for $p = 1$ property (4.21) is not necessary) with $\delta c' = c$.This means that on $U^j_s \cap U^j_t$ we have $c'(U^j_s) = c'(U^j_t)$ for $j = 1,2$ so that c' determines two holomorphic functions f_j in Ω_j, $j = 1,2$, with $f_2 - f_1 = c'(U^2_t) - c'(U^1_s) = f$ on $U^1_s \cap U^2_t$ for all $s,t \in I_\lambda$. Hence $f_2 - f_1 = f$ in $\Omega^1 \cap \Omega^2$

and the bounds of c' imply that $f_j \in \text{proj} \lim_{k \to \infty} H(\Omega^j \cap \Omega_k;\phi^k)$ for j = 1,2. □

This corollary concludes all the promised proofs of the assertions in chapter II.

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SUBJECT INDEX

SAMENVATTING

Fourier getransformeerden van distributies met begrensde drager zijn gekarakteriseerd in de stelling van Paley-Wiener-Schwartz. Deze stelling geldt ook voor distributies met onbegrensde drager. Een soortgelijke stelling van Ehrenpreis en Martineau karakteriseert de Fourier getransformeerden van analytische functionalen met begrensde steunsels. Maar onbegrensd gesteunde analytische functionalen zijn niet eerder diepgaand bestudeerd. In dit proefschrift wordt in deze leegte voorzien en wordt de stelling van Ehrenpreis-Martineau uitgebreid tot het geval van onbegrensd gesteunde analytische functionalen.

De generalisaties van de stelling van Ehrenpreis-Martineau worden behandeld in hoofdstuk III. De bewijzen zijn veel lastiger en langer dan die van de stelling van Paley-Wiener-Schwartz. Het komt erop neer het fundamentele principe van Ehrenpreis af te leiden voor ruimtes van niet-gehele functies. Aangezien dit principe moeilijk te begrijpen is, wordt in hoofdstuk IV enige aandacht besteed aan het uitleggen van zowel Ehrenpreis' als Palamodov's versie. De generalisatie tot niet-gehele functies wordt in hoofdstuk VI bewezen met behulp van technieken die door Hörmander ontwikkeld zijn voor L^2-schattingen voor de Cauchy-Riemann operator.

De Paley-Wiener-Schwartz stelling voor distributies met onbegrensde drager heeft zijn nut bewezen in de quantum veldentheorie. Wil men deze uitbreiden dan zijn ook Paley-Wiener stellingen voor onbegrensd gesteunde analytische functionalen wenselijk. Om iets als localiseerbaarheid te behouden dient men te volstaan met analytische functionalen die gesteund worden door reële verzamelingen. Reële steunsels hebben namelijk prettige eigenschappen die overeenkomen met die van dragers van distributies. Dit wordt uitvoering besproken in hoofdstuk II voor reëel gesteunde analytische functionalen waarvan de Fourier getransformeerden distributies en ultradistributies zijn. Zulke ultradistributies vormen een natuurlijke schakel tussen tamme distributies en Fourier hyperfuncties.

In hoofdstuk I treft men beschouwingen aan omtrent causaliteit en localiseerbaarheid in de quantum veldentheorie, waar het gebruik van holomorfe functies in meer veranderlijken geïllustreerd wordt. Hierin speelt de "kant van de rand" stelling een belangrijke rol. Een eenvoudig bewijs van deze stelling is te vinden aan het slot van hoofdstuk II.

Hoofdstuk V geeft enkele toepassingen van het fundamentele principe uit hoofdstuk IV op stelsels partiële differentiaalvergelijkingen met constante coëfficiënten. Verder wordt aangegeven hoe de stellingen uit hoofdstuk III gebruikt kunnen worden om de Newton interpolatiereeks af te leiden in zijn meest algemene vorm, namelijk in meer veranderlijken voor niet-gehele functies van exponentiëel type. Tenslotte wordt in hoofdstuk VII cohomologie met aftelbaar veel grenzen afgeleid om een nog onbewezen uitspraak uit hoofdstuk II te staven.

STELLINGEN

I

Het vermoeden van Komatsu, dat ultradifferentiëerbare functies op een reguliere compacte verzameling $V \subset \mathbb{R}^n$, met niet-leeg inwendige, voortzetbaar zijn tot $\mathbb{R}^n$ als ultradifferentiëerbare functies van dezelfde klasse, is juist als V convex is of een C^1-rand heeft.

H. Komatsu, *Ultradistributions, I, Structure theorems and a characterization,* J. Fac. Sci. Univ. Tokyo, Sec. IA, 20 (1973), p. 45.

II

Zij $\mathcal{F}: S \to S$ de Fouriertransformatie in de ruimte S van snel-dalende C^∞-functies in $\mathbb{R}^n$. Voor iedere begrensde open verzameling $V \subset \mathbb{R}^n$ is er een $\phi \in S$ met drager buiten V zodat ook $\mathcal{F}\phi$ zijn drager buiten V heeft.

III

Zij $T: S' \to S'$ de pseudodifferentiaaloperator in de ruimte S' van getemperde distributies in $\mathbb{R}^n$ bepaald door het symbool $\sqrt{1+\xi^2}$. Voor iedere open verzameling $V \subset \mathbb{R}^n$ en voor iedere distributie $f \in S'$ met drager buiten V geldt dat V bevat is in de drager van Tf.

IV

Laat $\mathcal{O}$ de verzameling van equivalentieklassen zijn van functies van $x \in \mathbb{R}^+$ die monotoon zijn voor grote x, onder de equivalentierelatie: $f \sim g$ d.e.s.d.a. er een positieve K is zodat $f(x) - K \le g(x) \le f(x) + K$ voor grote x. Onder de partiële ordening

$$f \le g \Leftrightarrow \exists K, \exists M: f(x) \le g(x) + K,\ x \ge M$$

wordt $\mathcal{O}$ een distributief tralie met machtigheid $2^{\aleph_0}$, bestaande uit positieve en negatieve elementen plus nul, waarin geen aftelbare deelverzameling cofinaal is en dat zelfs voldoet aan:
voor elk tweetal totaal geordende deelverzamelingen $A, B \subset \mathcal{O}$ met $|A| \le \aleph_0$, $|B| \le \aleph_0$ en

$\forall f \in A, \forall g \in B: \quad f < g$

is er een $h_0 \in \mathcal{O}$ met

$\forall f \in A, \forall g \in B: \quad f < h_0 < g.$

V

Zij f een holomorfe functie in het gebied $\mathbb{R}^n + iC_r$, waarbij $C \subset \mathbb{R}^n$ een open convexe kegel is met $(1,0,\ldots,0) \in C$ en $C_r = \{y \mid y \in C, \|y\| < r\}$. Laat verder een van de volgende eigenschappen gelden

a) f heeft een distributionele randwaarde op $\mathbb{R}^n$

b) f heeft een ultradistributionele randwaarde van klasse M

c) f heeft geen randwaarde.

Zij tenslotte ϕ een $\overset{\infty}{C}$-functie in $\mathbb{R}^1$ zodanig dat

$$G_y(x') \overset{\text{def}}{=\!=} \int_{\mathbb{R}} f(x+iy,x')\phi(x)\,dx, \quad x' \in \mathbb{R}^{n-1}$$

bestaat voor $0 < y < r$. Dan is lim G_y voor $y \downarrow 0$ in geval a) een $\overset{\infty}{C}$-functie in $\mathbb{R}^{n-1}$, in geval b) een ultradifferentiëerbare functie van klasse M mits ϕ dat ook is, en in geval c) een analytische functie mits ϕ dat ook is.

VI

De fundamentele oplossingen van geïtereerde golfvergelijkingen zijn eenvoudiger te vinden door distributies als hyperfunties te schrijven dan door gebruik te maken van Fouriertransformatie.

J.W. de Roever, *Boundary values of holomorphic functions and the iterated wave equation*, in *Conference on the theory of ordinary and partial differential equations*, Lect. Notes in Math., no. 280, Springer, Berlin (1972), p. 325-329.

D.W. Bresters, *Initial value problems for iterated wave operators*, Thesis, Enschede, 1969.

VII

De bewering van Jauch dat op grond van localiseerbaarheid de logica van de quantummechanica geen modulair, niet-atomair, orthocomplementeerbaar tralie kan zijn (zodat het tralie van de gesloten lineaire deelruimten van een Hilbertruimte overblijft) is onjuist.

J.M. Jauch, *Foundations of quantum mechanics*, Addison-Wesley, Reading, Menlo Park, London, Don Mills, 1968.

VIII

Betreffende de vraag, o.a. gesteld door Hegerfeldt, of de kans een deeltje buiten een bepaald volume aan te treffen willekeurig klein kan zijn, kan het volgende gezegd worden: er bestaan oplossingen van de Diracvergelijking behorende bij positieve energie, waarvan de dichtheid buiten een gegeven ruimtevolume op een gegeven tijdstip, weliswaar niet nul, maar wel willekeurig klein kan zijn.

G.C. Hegerfeldt, *Remark on causality and particle localization*, Phys. Rev. D., 10 (1974), p. 3320-3321.

IX

De bewering van Westerskov dat alle kleinst waterhoenen *(porzana pusilla)* olijfgroene poten hebben, is weerlegd door Oreel, de Roever, e.a. door het signaleren van enkele exemplaren met vleeskleurige poten. Dit feit is niet voor ieder even overtuigend.

K.E. Westerskov, *Leg and foot colour of the marsh crake* (porzana pusilla) Notornis 17 (1970), p. 324-330.

G.J. Oreel, *Letter (on Leg and foot colour of the marsh crake)*, Notornis 19 (1972), p. 93-94.

S.D. Ripley, *Rails of the world*, M.F. Feheley Publ., Toronto (1977), p. 242.

X

Grossman & Hamlet hebben 73 kleuren bruin gedefiniëerd en van Engelse namen voorzien; Smithe deed hetzelfde voor 86 kleuren, waaronder 23 kleuren bruin. Er zouden gestandaardiseerde kleurtabellen moeten komen met o.a. Nederlandse namen.

M.L. Grossman & J. Hamlet, *Birds of prey of the world*, Cassell, London, 1964.

F.B. Smithe, *Naturalist's color guide*, Am. Mus. Nat. Hist., New York, 1975.